A combination of electron bombardment with high pressure should give larger displacements for lines which are strengthened by strong electrical conditions. The following evidence bears this out: (*a*) The lines which are strong in the arc relatively to the furnace have large pressure displacements. (*b*) Enhanced lines as a class are found to be displaced by pressure more than arc lines. (*c*) For a given pressure the enhanced lines are displaced more in the spark than in the arc.

The details of this investigation, with corroborative evidence from other spectra, will be published in the *Astrophysical Journal.*

ON THE FACTORIZATION OF VARIOUS TYPES OF EXPRESSIONS

By Henry Blumberg

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA

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Not a few mathematicians have dealt with the problem of setting up criteria by means of which the irreducibility of certain expressions in certain domains may be seen at a glance from the character of the expressions. Gauss, Kronecker, Schoenemann, Eisenstein, Dedekind, Floquet, Koenigsberger, Netto, Perron, M. Bauer, and Dumas have written on the subject.[1] The work of these authors may be said to center around the Schoenemann-Eisenstein theorem, which, however, is an exceedingly special case of various theorems obtained (for example, of theorem IV when interpreted for situation 1). With the exception of Floquet and Koenigsberger, the authors mentioned deal exclusively with the case where the expressions are polynomials, and chiefly with the case of polynomials whose coefficients belong to the domain of rational numbers. Floquet and Koenigsberger have extended the investigation to the case of linear homogeneous differential expressions.

A variety of methods have been employed. Thus the theory of algebraic fields, the theory of algebraic functions, the character of the solutions of linear homogeneous differential equations, and even geometric representation (Dumas, loc. cit.) have been used. Elementary methods, requiring no such means, have succeeded in yielding only the less general results.

One of our objects is to show that elementary and comparatively short considerations may be made to yield results more general than any hitherto obtained. In fact, for a complete comprehension of the proofs, little more is required in the way of specific knowledge than an understanding of the definition of the various expressions considered

Moreover, our results have a most intimate connection with the work of every author cited, and, in most cases, the results bearing on the problem before us that are contained in the articles quoted subordinate themselves as special cases to the theorems obtained here.[2]

A sharp distinction has been made in the literature (as regards polynomials) between those investigations that are based on the *divisibility* properties of the coefficients and those that proceed from the consideration of the *magnitude* of the coefficients.[3] One of the interesting results is that the gap between these two types of investigation may, in a certain sense, be bridged. As a consequence, it is possible to show how the various theorems obtained flow from certain general considerations as a common source, and thus a surprising unification of the material is achieved.[4]

Our work deals with polynomials (for the various kinds of coefficients considered, see situations 1–5 incl.), linear homogeneous differential expressions, linear homogeneous difference expressions (for the first time as regards our problem, as far as the writer knows), and more general expressions. There is no better way of making what is essential in the proofs come to the front than by treating the subject in abstract, postulational fashion. Thereby also the interconnection between the results and the unification above referred to are made manifest. Thus, it is easy to see, as Koenigsberger has pointed out by an example, that the Schoenemann-Eisenstein theorem cannot be directly extended to the case of linear homogeneous differential expressions; but our abstract treatment furthermore lays bare the underlying reason—by no means evident otherwise—why it breaks down, and at once indicates what analogous theorem may replace it.[5] The abstract treatment is, moreover, especially fitting here because a small number of simple assumptions is sufficient for the foundations of the theory.

We start[6] with any *aggregate* $\mathfrak{S}$ whatsoever and shall deal with finite, ordered subaggregates $E = (e_0, e_1, e_2, \cdots, e_m)$ of $\mathfrak{S}$, e_ν $\{\nu = 0, 1, 2, \cdots, m\}$ being an element of $\mathfrak{S}$.

Such a finite, ordered subaggregate E we shall call a '*parenthesis*' of $\mathfrak{S}$; m will be called the '*order*' of E. We assume that the '*product*' $A \cdot B = (a_0, a_1, a_2, \cdots, a_r) \cdot (b_0, b_1, b_2, \cdots, b_s)$ of any two parentheses of $\mathfrak{S}$ is equal to a parenthesis $C = (c_0, c_1, c_2, \cdots, c_n)$ of $\mathfrak{S}$ for which $n = r + s$. We assume furthermore that with every element e of $\mathfrak{S}$ there is associated a single number η, called the '*rank* of e,' η being an integer or $-\infty$ (never $+\infty$), such that when $A \cdot B = C$ one of the following 3 sets of relations holds (see situations 1–10 that make either I or II or III valid):

$$\text{I} \quad a) \quad \gamma_\nu \leqq \max_{\lambda+\mu=\nu} (\alpha_\lambda + \beta_\mu), \quad \{\nu = 0, 1, 2, \cdots, n\},$$

$$b) \quad \gamma_\nu = \max_{\lambda+\mu=\nu} (\alpha_\lambda + \beta_\mu), \quad \{\nu = 0, 1, 2, \cdots, n\},$$

if $\alpha_\lambda + \beta_\mu$ attains its maximum, λ and μ varying so as to satisfy the relation $\lambda + \mu = \nu$, for a single pair of values of λ and μ. Here $\alpha_\lambda, \beta_\mu, \gamma_\nu =$ rank[7] of a_λ, b_μ, c_ν.

$$\text{II} \quad a) \quad \gamma_\nu \leqq \max_{\lambda+\mu\leqq\nu} (\alpha_\lambda + \beta_\mu), \quad \{\nu = 0, 1, 2, \cdots, n\},$$

$$b) \quad \gamma_\nu = \max_{\lambda+\mu=\nu} (\alpha_\lambda + \beta_\mu), \quad \{\nu = 0, 1, 2, \cdots, n\},$$

if $\alpha_\lambda + \beta_\mu$ attains its maximum M_ν (λ and μ varying so as to satisfy $\lambda + \mu = \nu$) for a single pair of values of λ and μ, and if $M_\nu \geqq \alpha_\lambda + \beta_\mu$ whenever $\lambda + \mu < \nu$.

$$\text{III} \quad a) \quad \gamma_\nu \leqq \max_{\lambda+\mu+\sigma=\nu} (\alpha_\lambda + \beta_\mu + \sigma), \quad \{\nu = 0, 1, 2, \cdots, n\},$$

σ being an integer such that $0 \leqq \sigma \leqq \nu$.

$$b) \quad \lambda_\nu = \max_{\lambda+\mu=\nu} (\alpha_\lambda + \beta_\mu), \quad \{\nu = 0, 1, 2, \cdots, n\},$$

if $\alpha_\lambda + \beta_\mu$ attains its maximum M_ν (λ and μ varying so as to satisfy $\lambda + \mu = \nu$) for a single pair of values of λ and μ, and if $M_\nu > \alpha_\lambda + \beta_\mu + \sigma$ for $\lambda + \mu + \sigma = \nu$ and $\sigma > 0$.

It is to be noted that I implies II and that II implies III. *Hence every theorem proved for all* $\mathfrak{S}^{\text{III}}$ (*i.e., for all* $\mathfrak{S}$ *having property* III) *is valid for every* $\mathfrak{S}^{\text{II}}$ *and* $\mathfrak{S}^{\text{I}}$; *and every theorem proved for all* $\mathfrak{S}^{\text{II}}$ *holds for every* $\mathfrak{S}^{\text{I}}$.

We shall now describe various important situations where I or II or III holds. For this purpose, we must define in each case the *aggregate* $\mathfrak{S}$, the *parentheses* of $\mathfrak{S}$, the *product* of two parentheses and the *rank* of every element of $\mathfrak{S}$.

I *holds in the following situations* (1–7 *incl.*):

1. $\mathfrak{S}$ consists of the set of rational numbers. We understand by the parenthesis $(e_0, e_1, e_2, \cdots, e_m)$ of $\mathfrak{S}$ the rational polynomial $e_0y^m + e_1y^{m-1} + \cdots + e_m$ in the letter y. The product of two parentheses $(a_0, a_1, \cdots, a_r) \cdot (b_0, b_1, \cdots, b_s)$ is defined, as usual, to be equal to the parenthesis $(c_0, c_1, \cdots, c_n)$, where $n = r + s$ and $c_0 = a_0b_0$, $c_1 = a_0b_1 + a_1b_0, \cdots,$ $c_n = a_rb_s$. The rank of an element $e = e'/e''$ (where e' and e'' are integers) is defined with reference to a fixed prime p. First let $e \neq 0$; let e' be divisible by $p^{d'}$ but not by $p^{d'+1}$; e'', by $p^{d''}$ but not by $p^{d''+1}$. We define the rank of e by the equation $\eta = d'' - d'$. Moreover, we (naturally) define the rank of 0 to be $-\infty$.

2. $\mathfrak{S}$ consists of the class of the Hensel p-adic numbers. The parentheses of $\mathfrak{S}$ and the product of two parentheses are defined as in 1. The rank of the p-adic number e is the negative of Hensel's[8] 'order' of e with respect to p.

3. $\mathfrak{S}$ consists of the class of rational fractions $e = e'(x_1, \cdots, x_K)/e''(x_1, \cdots, x_k)$ in k letters $x_1, x_2, \cdots, x_k$, e' and e'' being polynomials with

arbitrary complex numerical coefficients.[9] The parentheses and the product of two parentheses are defined as in 1. The rank η of an element e ($\neq 0$) is defined as $d'-d''$, where d' and d'' represent respectively the degrees (in the usual sense) of e' and e'' in the k letters $x_1, \cdots, x_k$. The rank of 0 is defined to be $-\infty$.

4. $\mathfrak{S}$ consists of the collectivity of elements $e = \sum_{\lambda=0}^{\infty} c_\lambda x^{\eta-\lambda}$, where the c's are arbitrary complex numbers, that is, e is a Laurent series having only a finite number of terms with positive exponents.[10] The parentheses and the products of parentheses are defined as in 1. The rank of e ($\neq 0$) is defined to be the exponent of x in the first non-zero term of the development of e. The rank of 0 is (naturally) defined to be $-\infty$.

5. $\mathfrak{S}$ consists of the collectivity of fractions $e = e'/e''$, where e' and e'' are power series in k letters $x_1, x_2, \cdots, x_k$ with arbitrary complex coefficients:

$$e' = \sum_{\lambda_1, \lambda_2, \cdots, \lambda_k = 0}^{\infty} c'_{\lambda_1, \lambda_2, \cdots, \lambda_k} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}, \quad e'' = \sum_{\lambda_1, \lambda_2, \cdots, \lambda_k = 0}^{\infty} c''_{\lambda_1, \lambda_2, \cdots, \lambda_k} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}.$$

The rank of e ($\neq 0$) is defined as $d''-d'$, where d' and d'' represent the lowest degrees (in the usual sense) in the k letters $x_1, x_2, \cdots, x_k$ of a non-zero term of e' and e'' respectively. The rank of 0 is $-\infty$.

6. $\mathfrak{S}$ consists of all rational fractions $e(x) = e'(x)/e''(x)$ in x ($e'(x)$ and $e''(x)$ being polynomials) with arbitrary complex coefficients. The parenthesis $(e_0, e_1, \cdots, e_m)$ of $\mathfrak{S}$ is the linear homogeneous difference expression $e_0 y_{x+m} + e_1 y_{x+m-1} + \cdots + e_m y_x$. The product $A \cdot B = (a_0, a_1, \cdots, a_r) \cdot (b_0, b_1, \cdots, b_s)$ of two parentheses of $\mathfrak{S}$ is the ordinary *symbolic* product of the linear difference expressions A and B and is equal to $(c_0, c_1, \cdots, c_n)$ where $n = r + s$ and

$$c_\nu = \sum_{\lambda=0}^{\nu} a_\lambda(x)\, b_{\nu-\lambda}(x+r-\lambda), \qquad \{\nu = 0, 1, 2, \cdots, n\}.$$

The rank of $e(x)$ is what is ordinarily called the degree of $e(x)$, i.e., $d'-d''$, where d' = degree of $e'(x)$ and d'' = degree of $e''(x)$.

7. $\mathfrak{S}$ and rank are defined as in 4; parentheses and products of parentheses, as in 6.

II *holds in the following situations* (8–9 *incl.*):

8. $\mathfrak{S}$ and rank are defined as in 6. The parenthesis $(e_0, e_1, \cdots, e_m)$ of $\mathfrak{S}$ is the linear homogeneous differential expression

$$e_0(x)\frac{d^m y}{dx^m} + e_1(x)\frac{d^{m-1} y}{dx^{m-1}} + \cdots + e_m(x)\, y.$$

The product $A \cdot B = (a_0, a_1, \cdots, a_r) \cdot (b_0, b_1, \cdots, b_s)$ of two parentheses is the ordinary *symbolic* product of the linear differential expressions A and B and is equal to $(c_0, c_1, \cdots, c_n)$, where $n = r + s$ and

$$c_\nu = \sum_{\mu=0}^{\nu} \sum_{\lambda=0}^{\nu-\mu} \binom{r-\lambda}{\nu-\mu-\lambda} a_\lambda \frac{d^{\nu-\mu-\lambda} b_\mu}{dx^{\nu-\mu-\lambda}},$$

$$\left\{ \binom{r-\lambda}{\nu-\mu-\lambda} = \frac{(r-\lambda)!}{(\nu-\mu-\lambda)!\,(r-\nu+\mu)!} \right\}.$$

9. $\mathfrak{S}$ and rank are defined as in 4; parentheses and products of parentheses, as in 8.

III *holds in the following situation*:

10. $\mathfrak{S}$ consists of the collectivity of fractions $e = e'/e''$, where e' and e'' are power series in $x - h$ $(= x_1$, cf. 5) with arbitrary complex coefficients. The rank of e is defined as in 5. The parentheses of $\mathfrak{S}$ and products of parentheses are defined as in 8.

We shall now proceed to the statement of certain consequences of the assumptions made that relate to factorization properties of a given parenthesis $C = (c_0, c_1, \cdots, c_n)$ of $\mathfrak{S}$. We base all further considerations on the tentative assumption that C may be expressed as a product $A \cdot B = (a_0, a_1, \cdots, a_r)\ (b_0, b_1, \cdots, b_s)$ of two parentheses, where $r \geqq 1$ and $s \geqq 1$. When other assumptions to be made in the theorems contradict this assumption of 'reducibility,' it must be that C is under those later assumptions incapable of being expressed as a product of two parentheses whose orders are at least 1. C is then said to be 'irreducible.' (In general, the terminology employed for our abstract situation is parallel to that for the ordinary concrete situations.) *We assume furthermore throughout in what follows that* γ_0, *the rank of* c_0, *is finite* (i.e., $\neq -\infty$). We introduce the following notations (partly for the purpose of indicating our method of investigation and partly for the purpose of simplifying certain future statements):

$$\begin{cases} \Delta_0 = \alpha_0 - \alpha_0 = 0, & \Delta_1 = \alpha_1 - \alpha_0, \cdots, \quad \Delta_r = \alpha_r - \alpha_0\,. \\ \Delta_0' = \beta_0 - \beta_0 = 0, & \Delta_1' = \beta_1 - \beta_0, \cdots, \quad \Delta_s' = \beta_s - \beta_0\,. \end{cases}$$

$$\begin{cases} G = \max_{1 \leqq \nu \leqq r} (\Delta_\nu);\ \Delta_\tau = \text{last } \Delta_\nu\ \{\nu = 1, 2, \cdots, r\} \text{ equal to } G. \\ G' = \max_{1 \leqq \nu \leqq s} (\Delta_\nu');\ \Delta_{\tau'}' = \text{last } \Delta_\nu'\ \{\nu = 1, 2, \cdots, s\} \text{ equal to } G'. \end{cases}$$

$$\begin{cases} M = \max_{1 \leqq \nu \leqq r} \left(\frac{\Delta_\nu}{\nu}\right);\ \frac{\Delta_t}{t} = \text{last } \frac{\Delta_\nu}{\nu}\ \{\nu = 1, 2, \cdots, r\} \text{ equal to } M. \\ M' = \max_{1 \leqq \nu \leqq s} \left(\frac{\Delta_\nu'}{\nu}\right);\ \frac{\Delta_{t'}'}{t'} = \text{last } \frac{\Delta_\nu'}{\nu}\ \{\nu = 1, 2, \cdots, s\} \text{ equal to } M'. \end{cases}$$

The following results (lemmas[11] I–VI *incl., theorems* I–X *incl.) are valid for every* $\mathfrak{S}^{\text{II}}$:

Lemma I. *If* $G \leqq 0$ *and* $G' \leqq 0$, $\gamma_\nu - \gamma_0 \leqq 0$ *for every* ν.

Lemma II. *If* $G \geqq 0$ *and* $G' \geqq 0$, $\gamma_{T+T'} - \gamma_0 = G + G' \geqq \gamma_\nu - \gamma_0$ *for every* ν.

Lemma III. *If* $M > M'$ *and* $M > 0$ $(M' > M$ *and* $M' > 0)$, $M = \dfrac{\gamma_t - \gamma_0}{t} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}\left(M' = \dfrac{\gamma_{t'} - \gamma_0}{t'} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}\right)$ *for every* $\nu \neq 0$, *the* $>$ *sign alone holding when* $\nu > t$ $(\nu > t')$.

Lemma IV. *If* $M > M' \geqq 0$ $(M' > M \geqq 0)$, *there is no fixed* k *such that* $\gamma_k > \gamma_\nu$ *and* $\dfrac{\gamma_k - \gamma_0}{k} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0, k$.

Lemma V. *If* $M > M' > 0$ $(M' > M > 0)$ *there is no fixed* k *such that* $\gamma_k \geqq \gamma_\nu$ *and* $\dfrac{\gamma_k - \gamma_0}{k} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu > 0$.

Lemma VI. *If* $M = M' > 0$, $\dfrac{\gamma_{t+t'} - \gamma_0}{t + t'} = M = M' \left(= \dfrac{\Delta_t}{t} = \dfrac{\Delta'_{t'}}{t'}\right) \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu > 0$, *the* $>$ *sign alone holding when* $\nu > t + t'$.

Theorem I. *If, for a fixed* k, $\gamma_k - \gamma_0 > 0$ and $\dfrac{\gamma_k - \gamma_0}{k} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n\}$, *at least one of the following* $n - k + 1$ *congruences holds:*

$$\gamma_k - \gamma_0 \equiv 0 \left(\text{mod}\, \frac{k}{(k, \sigma)}\right),$$

where σ *takes the successive values* $k - r, k - r + 1, \cdots, n - r$ *(or* $k - s, k - s + 1, \cdots, n - s$*) and* (k, σ) *represents, as usual, the G.C.D. of* k *and* σ.

Theorem II. *If, for a fixed* k, $\gamma_k - \gamma_0 > 0$, $\dfrac{\gamma_k - \gamma_0}{k} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n\}$ *and* $(\gamma_k - \gamma_0, k) = 1$, *the parenthesis* C *contains an irreducible factor of order* $\geqq k$.

Theorem III (a special case of I). *If* $\gamma_n - \gamma_0 > 0$ *and* $\dfrac{\gamma_n - \gamma_0}{n} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n - 1\}$, $\gamma_n - \gamma_0 \equiv 0 \left(\text{mod}\, \dfrac{n}{(n, r)}\right)$.

Theorem IV (a special case of III). *If* $\gamma_n - \gamma_0 > 0$, $\dfrac{\gamma_n - \gamma_0}{n} \geqq \dfrac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n - 1\}$ *and* $(\gamma_n - \gamma_0, n) = 1$, C *is irreducible.*

Theorems V–X inclusive materially extend and generalize Perron's theorem, *J. Math., Berlin,* **132,** 304 (1907), which Perron designates as 'ein sehr allgemeines Kriterium.'

Theorem V. *If* $G \geqq 0$, $G' \geqq 0$, $\gamma_k > \gamma_\nu$ *for every* $\nu \neq k$ *and* $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *the following relations hold:*

$$\begin{cases} 0 < M = M' = \dfrac{\gamma_k - \gamma_0}{k}\left(= \dfrac{\Delta_t}{t} = \dfrac{\Delta'_{t'}}{t'}\right); \\ k = t + t';\ t = \tau,\ t' = \tau';\ (\Delta_t = G;\ \Delta_{t'} = G'). \end{cases}$$

Theorem VI. *If* $G \geqq 0$, $G' \geqq 0$, $\gamma_k > \gamma_\nu$ *for every* $\nu \neq k$, $\frac{\gamma_k - \gamma_0}{k} > \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0, k$, *and* $(\gamma_k - \gamma_0, k) = 1$, *C is irreducible.*

Theorem VII. *If* $G > 0$, $G' > 0$, $\gamma_k \geqq \gamma_\nu$ *for every* ν *and* $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *the same conclusions may be drawn as in theorem* V *with the possible exception of* $t = \tau$ *and* $t' = \tau'$.

Theorem VIII. *If* $G > 0$, $G' > 0$, $\gamma_k \geqq \gamma_\nu$ *for every* ν, $\frac{\gamma_k - \gamma_0}{k} > \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0, k$, *and* $(\gamma_k - \gamma_0, k) = 1$, *C is irreducible.*

Theorem IX (generalizes VI). *If* $G \geqq 0$, $G' \geqq 0$, $\gamma_k > \gamma_\nu$ *for every* $\nu \neq k$, $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *and if furthermore every parenthesis* $(e_0, e_1, \cdots, e_m)$ *that may occur as a factor of C is such that* $\eta_\nu \geqq \eta_0$ *for at least one* $\nu > 0$, *every decomposition of C into a product of parentheses contains at most* $(\gamma_k - \gamma_0, k)$ *factors.*

Theorem X (generalizes VIII). *If* $G > 0$, $G' > 0$, $\gamma_k \geqq \gamma_\nu$ *for every* ν, $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *and if furthermore every parenthesis* $(e_0, e_1, \cdots, e_m)$ *that may occur as a factor of C is such that* $\eta_\nu > \eta_0$ *for at least one* ν, *every decomposition of C into a product of parentheses contains at most* $(\gamma_k - \gamma_0, k)$ *factors.*

The lemmas leading up to theorems XI and XII are omitted.

Theorem XI. *Theorem* I *holds for every* $\mathfrak{S}^{\mathrm{I}}$ *if the inequality* $\gamma_k - \gamma_0 > 0$ *is replaced by* $\gamma_k - \gamma_0 < 0$.

Theorem XII. *Theorem* I *holds for every* $\mathfrak{S}^{\mathrm{III}}$ *if the inequality* $\gamma_k - \gamma_0 > 0$ *is replaced by* $\frac{\gamma_k - \gamma_0}{k} > 1$.

The results here outlined will be offered for publication *in extenso* to the *Transactions of the American Mathematical Society.*

[1] Gauss, *Disquisitiones arithmeticae* (1801) Art. 341; Kronecker, *J. Math., Berlin*, **29**, 280 (1845), **100**, 79 (1887), *J. Math., Paris*, **19**, 177 (1854) and ser. 2, **1**, 399 (1856); Schoenemann, *J. Math., Berlin*, **32**, 100 (1846) and **40**, 188 (1850); Eisenstein, *J. Math., Berlin*, **39**, 160 (1850); Dedekind, *J. Math., Berlin*, **54**, 27 (1857); Floquet, *Ann. Sci. Ec. norm., Paris*, (1879), cited by Koenigsberger; Koenigsberger, *J. Math., Berlin*, **115**, 53 (1895), **121**, 320 (1900), *Math. Ann., Leipzig*, **53**, 49 (1900); Netto, *Math. Ann.*, **48**, 81 (1897); Perron, *Math. Ann.*, **60**, 448 (1905), *J. Math., Berlin*, 132, 288 (1907); M. Bauer, *J. Math., Berlin*, 128, 87 and 298, (1905), **132**, 21 (1907), **134**, 15 (1908); Dumas, *J. Math., Paris*, ser. 6, **2**, 191 (1906).

[2] Thus the work on the subject before the publication of the Schoenemann-Eisenstein theorem is summarized and generalized by that theorem, which, as previously indicated, is a very special case of theorem IV. In fact, theorem I alone, for example, includes as special cases a great bulk of the results heretofore published, and, in particular, the following: the theorem of Floquet—quoted by Koenigsberger—, all the results contained in the 1895 paper of Koenigsberger, nearly all contained in his 1900 *Math. Ann.* paper, almost all the results of Netto and almost all the results in the 1905 paper of Perron. Cf. also the paragraph preceding the statement of theorem V.

[3] Perron, *J. Math., Berlin*, **132**, 288, and Loewy, in Pascal's *Repertorium* (1910) Analysis I_1, 292 and 293.

[4] I may perhaps be permitted to remark, for the purpose of indicating that the results given are less artificial than one would at first suppose, that I had obtained my chief results with little knowledge of the literature. Their intimate connection with results already obtained points to a degree of 'naturalness' of these results that one would hardly attribute to them in the absence of such a connection.

[5] As a matter of fact, two distinct theorems obtained may be properly regarded as (highly) generalized Schoenemann-Eisenstein theorems for the case of linear homogeneous differential expressions: theorem I for situation 8 and theorem XII for situation 10. Curiously, Koenigsberger himself has obtained theorems—special cases of XII—for differential expressions that may be properly regarded as generalized Schoenemann-Eisenstein theorems, without his having noticed this relation.

[6] It is possible to build up just as general a theory as ours by dealing exclusively with 'parentheses' whose elements are 'ranks'—see below in the same paragraph—and hence always integers or $-\infty$. The reader who prefers a more concrete, tho necessarily less general, discussion may, for example, at once interpret $\mathfrak{S}$, '*parenthesis*,' '*product*' and '*rank*'—see below in the same paragraph—as in situation 8. In that case, II will hold.

[7] In general, we denote the rank of an element represented by a Latin letter by the corresponding Greek letter. By $\max\limits_{\lambda+\mu=\nu}(\alpha_\lambda+\beta_\mu)$, $\max\limits_{\lambda+\mu\leqq\nu}(\alpha_\lambda+\beta_\mu)$, $\max\limits_{\lambda+\mu+\sigma=\nu}(\alpha_\lambda+\beta_\mu+\sigma)$—see II and III below—we naturally understand the largest value attained by the numbers of the set $\alpha_\lambda+\beta_\mu$, $\alpha_\lambda+\beta_\mu$, $\alpha_\lambda+\beta_\mu+\sigma$, λ, μ and σ varying so as to satisfy the relations $\lambda+\mu=\nu$, $\lambda+\mu\leqq\nu$, $\lambda+\mu+\sigma=\nu$, and in addition, of course, $0\leqq\lambda\leqq r$ and $0\leqq\mu\leqq s$.

[8] Hensel, *J. Math., Berlin*, 127, 51–84, §2 (1904) or Zahlentheorie (1913), chaps. 3 and 6.

[9] More generally, the numerical coefficients may belong to any abstract system $(K, +, \times)$, where K is a class, such that if a and b are elements of K both $a+b$ and $a\times b$ are elements of K. This remark applies just as well to the numerical coefficients in situations 4–10 incl.

[10] The question of convergence does not enter here because the formal character of the series is sufficient for our purpose. More generally, we may have such series in two or more variables, the required change in the definition of rank being evident.

[11] These lemmas lead up to the theorems I–X and are given partly for the purpose of indicating the nature of the proofs and partly because they are believed to be of interest in themselves.