

mentum effects. It may be that we shall be driven to assume that the electron here functions as a dipole (magnetic), evidence for which has already been given by Compton, and that when moving in different directions in the non-isotropic force field of a crystal it assumes different orientations.

<sup>1</sup> W. Voigt, *Lehrbuch der Krystallophysik*, Teubner, Leipzig, 1910, p. 369.

<sup>2</sup> W. Thomson, *Edin. Proc.*, 3, 255, 1854; *Edin. Trans.*, 21, 153, 1857; *Phil. Mag.* (4) 11, 379 and 433, 1856.

<sup>3</sup> W. Voigt, *l. c.*, p. 534.

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## ON THE QUANTUM THEORY OF THE POLARIZATION OF RESONANCE RADIATION IN MAGNETIC FIELDS

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*Introduction.*—The theory of the polarization of resonance radiation in magnetic fields strong enough for spacial quantization was developed independently and simultaneously by Pringsheim,<sup>1</sup> Breit,<sup>2</sup> Joos<sup>3</sup> and Gaviola and Pringsheim,<sup>4</sup> following to some extent previous suggestions of Hanle<sup>5</sup> and of Foote, Ruark and Mohler.<sup>6</sup> It is a direct consequence of the quantum theory of the Zeeman effect, which may be either normal or anomalous. The polarization of an individual Zeeman component<sup>\*</sup> is linear or circular according as the change in the magnetic quantum number is zero or unity in the quantum transition with which the component is associated. In the experiments of Wood,<sup>7</sup> Ellett<sup>8</sup> and others the Zeeman components are unresolved and so the observed polarization is the resultant polarization resulting from the superposition of the linear and circular components. This resultant polarization is the quantity in which we are here interested and can be computed when we know the relative intensities of the Zeeman components; i.e., the relative probabilities of the various transitions.

The present paper aims to call attention to two interesting consequences of this quantum theory of polarization: (1) the quantum theory gives the same angle  $54.7^\circ$  of no polarization as does the classical theory provided the hypothesis of spectroscopic stability (explained below) is valid; (2) the recent experiments of Ellett and others on the amount of polarization of sodium *D* resonance radiation are in reasonably good accord with the theory for determining transition probabilities developed by Ornstein and Burger,<sup>9</sup> and extended independently by Hönl,<sup>10</sup> Goudsmit and Kronig,<sup>11</sup> Heisenberg (unpublished) and H. N. Russell.<sup>12</sup> Only the original rules

of Ornstein and Burger and not the extensions are directly involved in the present paper, for the former suffice to determine relative intensities in the Zeeman pattern of the sodium *D* lines.

*Spectroscopic Stability and the Angle of No Polarization.*—Let us first suppose that the primary radiation which excites resonance is unpolarized and isotropic. By the term “isotropic” we mean that the incident rays are equally intense in all directions. This will be the case when atoms are excited by temperature radiation. Then the resonance radiation is obviously unpolarized in the absence of a magnetic field, as one direction is as good as another. According to the so-called hypothesis of spectroscopic stability discussed by Bohr,<sup>13</sup> Heisenberg<sup>14</sup> and others, the resultant polarization due to the superposition of the various components is also zero in the presence of a magnetic field (neglecting unimportant gradual changes proportional to the strength of the field), provided that the primary radiation remains unpolarized and isotropic. Under this isotropic excitation in a magnetic field there is twice as much energy available for absorption of circularly polarized components as for the absorption of the linearly polarized components, since circular vibrations may be decomposed into two perpendicular linear vibrations.

Ordinarily resonance is excited by a directed beam of light. This type of excitation is that which we shall consider henceforth. The primary radiation is no longer isotropic, and the resonance radiation is then usually polarized. The amount of polarization depends on the distribution of electrons among excited orbits. This in turn depends on the relative absorption of the linearly and circularly polarized components associated with transitions to the excited states. Now if the electric vector  $E$  of a plane-polarized primary beam makes an angle  $\theta$  with the direction of the applied field  $H$ , the energies  $\rho_c$  and  $\rho_l$  available for the absorption of circularly and linearly polarized components are, respectively, proportional to  $E^2 \sin^2 \theta$  and  $E^2 \cos^2 \theta$ . Consequently for  $\theta = \tan^{-1} \sqrt{2} = 54.7^\circ$  the ratio  $\rho_c/\rho_l$  has the same value 2 as in isotropic radiation. Hence if spectroscopic stability is granted, the resonance radiation is unpolarized when the angle between  $E$  and  $H$  is  $54.7^\circ$ . The same result has been previously obtained by Eldridge<sup>15</sup> and others,<sup>16</sup> but on the basis of the classical theory of scattering by linear oscillators.

A conclusive experimental confirmation of a depolarizing angle of  $54.7^\circ$  would be an interesting check on the much-mooted hypothesis of spectroscopic stability. For the mercury line  $2536\text{\AA}$  the angle of no polarization observed by Hanle<sup>17</sup> is  $54.7^\circ$  within the experimental error, but Ellett<sup>8</sup> in experiments which are a refinement of previous work by Wood and Ellett<sup>7</sup> obtains an angle of  $45^\circ \pm 3^\circ$  for sodium *D* radiation.

If the primary beam is unpolarized instead of plane-polarized, it is found by reasoning similar to that given above that with spectroscopic stability

the resonance radiation should be unpolarized when the applied magnetic field makes an angle of  $54.7^\circ$  with the direction of the incident beam instead of an angle  $54.7^\circ$  with the electric vector  $E$ . An experimental distinction between the depolarizing angles in the plane-polarized and circularly polarized cases does not appear to have been made and may reveal why Ellett gets  $45^\circ$  instead of  $55^\circ$  in the plane-polarized case.

*Mathematical Theory.*—To get the polarization for an arbitrary angle  $\theta$  between  $E$  and  $H$  we must make a calculation similar to that of Breit<sup>2</sup> or Gaviola and Pringsheim,<sup>4</sup> which we shall give only briefly. Let  $a_i$  be the transition probability of the linearly polarized Zeeman component originating at the excited state  $i$ , and let  $A_i$  be the sum of the corresponding probabilities for the two circularly polarized components. Very often some of these linear and circular components are lacking because of the absence of appropriate final states, and we then take the corresponding probabilities to be zero. The number of electrons reaching the excited state  $i$  is proportional to  $E^2(a_i \cos^2\theta + \frac{1}{2}A_i \sin^2\theta)$ . This follows<sup>18</sup> since absorption and emission probabilities are proportional to one another in non-degenerate systems, and since the plane-polarized primary beam under consideration is only half as effective for producing circularly polarized absorption as is the isotropic radiation assumed in defining absorption probabilities. Of the electrons reaching the state  $i$  a fraction  $a_i/(a_i + A_i)$  return via the linearly polarized route and a fraction  $A_i/(a_i + A_i)$  via the circularly polarized one. Consequently the total visible intensities  $I_l$  and  $I_c$  of the linearly and circularly polarized components are, respectively,

$$I_l = C \sum_i \frac{a_i}{a_i + A_i} (a_i \cos^2\theta + \frac{1}{2}A_i \sin^2\theta) E^2, \quad (1)$$

$$I_c = \frac{1}{2} C \sum_i \frac{A_i}{a_i + A_i} (a_i \cos^2\theta + \frac{1}{2}A_i \sin^2\theta) E^2, \quad (2)$$

where  $C$  is a constant and the summation is to be taken over all the excited states  $i$  which are optically unresolved. We assume that the direction of observation is normal to the magnetic field, so that only half of the circularly polarized light can be seen, thus giving the factor  $\frac{1}{2}$  in (2).

According to the hypothesis of spectroscopic stability we have  $\sum A_i = 2 \sum a_i$  and it can be verified from (1) and (2) that then, and only then we have  $I_l = I_c$  when  $\tan\theta = \sqrt{2}$ . This agrees with the depolarizing angle obtained above by very simple reasoning.

*Sodium D Radiation.*—There the possible changes of the magnetic quantum number are as follows:

$$\begin{array}{lll} D_2 & \frac{3}{2} \longrightarrow \frac{1}{2} (A_1); & \frac{1}{2} \longrightarrow \frac{1}{2} (a_2); & \frac{1}{2} \longrightarrow -\frac{1}{2} (A_2); \\ D_1 & & \frac{1}{2} \longrightarrow \frac{1}{2} (a_3); & \frac{1}{2} \longrightarrow -\frac{1}{2} (A_3). \end{array}$$

There is also a similar set of transitions which is obtained by reversing the sign of the magnetic quantum number throughout, but this, because of the symmetry, can be omitted from consideration. We have placed in parentheses after each transition the designations which we shall use for the corresponding transition probabilities. As only one circularly polarized component originates at each excited state for  $D$  radiation, the  $A_i$  here denotes a single transition probability rather than the sum of two as in the general case discussed above. We have not introduced a probability  $a_1$ , as there is no final state of magnetic quantum number  $3/2$ , and hence we may take  $a_1 = 0$  in (1), (2), etc.

Spectroscopic stability alone does not suffice to determine the transition probabilities for  $D_2$ , but according to the rules developed by Ornstein and Burger<sup>9</sup> we have

$$A_1:a_2:A_2:a_3:A_3 = 3:2:1:1:2. \quad (3)$$

The transition probabilities used by Breit and also those proposed by Gaviola and Pringsheim do not conform to (3) and are probably not tenable. This is not surprising, for their papers anteceded recent theoretical developments on the intensities of Zeeman components. Breit's probabilities, to be sure, are based upon experimental data on the intensities of Zeeman components given in Paschen and Götze's tables. This data, however, is not sufficiently accurate, for it gives a depolarizing angle of only  $31^\circ$ —not to mention the fact that recent accurate experiments of Ornstein, Burger and van Geel<sup>19</sup> made on Zeeman components in zinc indicate that the rules of Ornstein and Burger are correct.

Joos<sup>3</sup> employed the transition probabilities (3), but as intimated by Gaviola and Pringsheim,<sup>4</sup> his calculations are marred by a numerical error because he did not include the factor  $1/2$  by which  $A_i$  is multiplied in our Eqs. (1) and (2). (For this reason he gets a depolarizing angle of  $45^\circ$  instead of  $54.7^\circ$ ). Gaviola and Pringsheim revised Joos' calculations on the assumption of no  $D_1$  radiation. We shall extend the results by calculating the percentage polarization for an arbitrary intensity-ratio  $q$  of  $D_1$  to  $D_2$  in the incident beam.

The fractional amount,  $p$ , of polarization is obtained by substituting (3) into (1) and (2). We thus get after some simplification

$$p = \frac{I_l - I_c}{I_l + I_c} = \frac{9\cos^2\theta - 3}{7 + 4q + 3\cos^2\theta}. \quad (4)$$

Here we have assumed that the primary beam contains  $q$  times as much energy of frequency  $D_1$  as of frequency  $D_2$ , so that the factor  $E^2$  in (1) and (2) is  $q$  times as large for  $D_1$  (i.e., for the term  $i = 3$ ) as for  $D_2$  ( $i = 1, 2$ ). We suppose the direction of observation normal to the field. The

maximum electric intensity in the resonance radiation is parallel or perpendicular to the magnetic field according as  $p$  is positive or negative.

Unless otherwise stated, throughout the balance of the article we shall take  $q = 1/2$  since  $D_1$  is half as intense as  $D_2$  when emitted under equilibrium conditions.

Let the direction of observation be the  $z$ -axis, and the direction of the primary beam the  $x$ -axis. Various simple cases of the above formula will then be considered below.

(a) Let the applied magnetic field be parallel to the  $x$ -axis, i.e., longitudinal with respect to the primary beam. Then  $\theta = \pi/2$  and  $p = -0.33$ . Ellett<sup>8</sup> finds  $p = -0.28$ , while experiments by Gaviola and Pringsheim<sup>4</sup> give  $p = -0.25$ . In this case (a), it is immaterial whether or not the primary beam is polarized.

(b) Let the incident beam be plane-polarized with its electric vector  $E$  parallel to the  $z$ -axis, and let the applied field be anywhere in the  $x$ - $y$  plane. Then the results are as above, as we still have  $\theta = \pi/2$ .

(c) Let both  $E$  and  $H$  be parallel to the  $y$ -axis. Then  $\theta = 0$  and  $p = 0.50$ . Ellett observes 45% polarization.

(d) Let  $H$  be parallel to the  $y$ -axis and the incident beam be unpolarized. As the two orthogonal components of unpolarized light are equal, the effect is the same as though we had linearly polarized light with  $\theta = \pi/4$ . This gives<sup>20</sup>  $p = 0.14$ , which is just the amount of polarization observed by Ellett.

(e) Let  $H$  be parallel to the  $z$ -axis. This case is not covered by Eq. (4), as the field is parallel rather than normal to the direction of observation. The polarization is, however, obviously zero, for the linear components are invisible and the circularly polarized components project equally onto two perpendicular axes for resolving the polarization. A vanishing polarization is observed experimentally provided the field is strong enough for spacial quantization, as assumed throughout the article.

The greater the amount of  $D_1$ , the less the polarization since  $D_1$  radiation by itself ( $q = \infty$ ) is unpolarized. To illustrate this quenching effect of  $D_1$  on the polarization, calculations are included in the following table for  $q = 0$  (i.e., no  $D_1$ ) and for  $q = 1$ . The results obtained by Breit and by Gaviola and Pringsheim with other transition probabilities (as listed) are also given.

	PROBABILITIES	$D_1/D_2$	PERCENTAGE POLARIZATION			
	$A_1 : a_2 : A_3 : a_4 : A_5$	$q$	CASES (a)-(b)	(c)	(d)	(e)
Experiment		$\frac{1}{2}$ ?	28%	45%	14.2%	0%
Eq. (4) (Revised Joos)	$\left\{ \begin{array}{l} 3 : 2 : 1 : 1 : 2 \\ 3 : 2 : 1 : 1 : 2 \\ 3 : 2 : 1 : 1 : 2 \end{array} \right.$	0	43	60	17.6	0
		$\frac{1}{2}$	33	50	14.3	0
		1	27	43	12.5	0
Breit	29 : 15 : 17 : 9 : 20	1	30	14	....	0
Gav. & Prings.	1 : 1 : 1 : 1 : 1	0	20	33	9.1	0

The observations agree much better with Eq. (4) than with the

probabilities of Breit or of Gaviola and Pringsheim, thus confirming the correctness of Ornstein and Burger's method of determining transition probabilities. Breit and Gaviola and Pringsheim make their calculations for  $q = 1$  and  $q = 0$ , respectively. Their results could readily be extended to the more probable case  $q = \frac{1}{2}$ . This would make Gaviola and Pringsheim's results even lower and would not raise Breit's results sufficiently to avoid the bad disagreement with experiment in case (c).

The observed results are intermediate between the results obtained with Eq. (4) for  $q = \frac{1}{2}$  and  $q = 1$ . This is not unreasonable, as there may be more than half as much  $D_1$  as  $D_2$  in the incident beam since  $D_2$  is absorbed more strongly than  $D_1$ , thus making the strength of  $D_1$  relative to  $D_2$  somewhat greater than at the source.<sup>21</sup> Another possibility is that while  $q = \frac{1}{2}$  the experimental values are slightly low because of unavoidable impurities or stray fields.

*Polarization in the Absence of a Magnetic Field.*—Heisenberg<sup>14</sup> has recently suggested that the polarization of resonance radiation excited by a plane-polarized primary beam is the same in the absence of a field as when a magnetic field is applied parallel to the electric vector of the incident beam (case c). Ellett observes 16% in the absence of the field which is quite different from the observed value 45% in case (c). Breit<sup>2</sup> has suggested that the polarization in the absence of a field is the same as the average for a random distribution of fields. If a formula given by Breit on p. 839 of his paper<sup>2</sup> is used but modified in accordance with the transition probabilities (3) and with the assumption  $q = \frac{1}{2}$ , the calculated polarization is 11.5%, which is not a great deal too low.

When the primary beam is unpolarized instead of plane-polarized, it is easily found by extending Breit's method of calculation that the percentage polarization should be 6.2%, whereas 9% is observed by Ellett. Heisenberg's hypotheses would lead us to expect the same polarization as in case (a), where 28% is observed. Possibly Heisenberg's hypotheses are really correct but there are impurities or stray fields which make Breit's method more nearly applicable.

<sup>1</sup> P. Pringsheim, *Die Naturwissenschaften*, 12, 247 (1924); *Zs. Physik*, 23, 324 (1924).

<sup>2</sup> G. Breit, *Phil. Mag.*, 47, 832 (1924).

<sup>3</sup> G. Joos, *Physik. Zs.*, 25, 130, 298, 400 (1924).

<sup>4</sup> E. Gaviola and P. Pringsheim, *Zs. Physik*, 25, 367 (1924).

<sup>5</sup> W. Hanle, *Die Naturwissenschaften*, 11, 690 (1923).

<sup>6</sup> P. D. Foote, A. Ruark and F. L. Mohler, *J. Opt. Soc. Amer.*, 7, 415 (1923).

<sup>7</sup> R. W. Wood and A. Ellett, *Proc. Roy. Soc. London*, 103, 396 (1923); *Physic. Rev.*, 24, 243 (1924).

<sup>8</sup> A. Ellett, *J. Opt. Soc. Amer.*, 10, 427 (1925).

<sup>9</sup> L. S. Ornstein and H. C. Burger, *Zs. Physik*, 28, 135; also especially 29, 241 (1924).

<sup>10</sup> H. Hönl, *Zs. Physik*, 31, 340 (1925).

<sup>11</sup> G. Goudsmit and R. de L. Kronig, *Versl. Akad. Amsterdam*, **34**, no. 2; also R. de L. Kronig, *Zs. Physik*, **31**, 885 (1925).

<sup>12</sup> H. N. Russell, *Proc. Nat. Acad. Sci.*, **11**, 314 (1925), especially p. 319.

<sup>13</sup> N. Bohr, *The Quantum Theory of Line Spectra*, Copenhagen, p. 85; *Zs. Physik*, **13**, 149 (1923); *Die Naturwissenschaften*, **12**, 1115 (1924).

<sup>14</sup> W. Heisenberg, *Zs. Physik*, **31**, 617 (1925).

<sup>15</sup> J. A. Eldridge, *Physic. Rev.*, **24**, 234 (1924).

<sup>16</sup> Hanle also predicts an angle of  $54.7^\circ$ , but it is not clear whether he uses the classical or quantum view-point.

<sup>17</sup> W. Hanle, *Zs. Physik*, **30**, 93 (1924).

<sup>18</sup> For greater detail see Gaviola and Pringsheim,<sup>4</sup> p. 372, ff., or Breit,<sup>2</sup> p. 835.

<sup>19</sup> L. S. Ornstein, H. C. Burger and W. C. van Geel, *Zs. Physik*, **32**, 681 (1925).

<sup>20</sup> The writer is unable to agree with a statement made by Ellett<sup>8</sup> that according to the quantum theory we may likely expect zero polarization in case (*d*).

<sup>21</sup> Cf. Gaviola and Pringsheim,<sup>4</sup> p. 372.

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## THE STRUCTURE OF THE HYDROGEN MOLECULE ION

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If the condition of dynamic stability is ignored, the steady state of lowest energy of the hydrogen molecule ion should be that in which the electron vibrates in a straight line perpendicular to the line joining the nuclei and through the middle point of this line. The energy of this steady state has been calculated using the following assumptions: the inverse square law of force is assumed to hold between the electron and the positive nuclei, the motion of the electron is described by classical mechanics and the steady states fixed by the Wilson-Sommerfeld quantum conditions, and finally the mean value of the component of the force of attraction between the electron and one nucleus is set equal to the force of repulsion between the two nuclei. This model is dynamically unstable with respect to a displacement of the electron in the direction of the line joining the nuclei at least when the nuclei are not rotating.

The results of the calculations can be briefly summarized and will be published in greater detail elsewhere. The phase integral has been taken over one complete oscillation of the electron, i.e., from a point farthest from the line joining the nuclei to the farthest point on the opposite side of this line and back to the original position. The energy for these steady states in which the electron is moving in a straight line perpendicular to the line joining the nuclei (hereafter referred to as the vibration orbits) is given by the relation