INTERPOLATION AND AN ANALOGUE OF THE LAURENT DEVELOPMENT

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In a recent note[1] the writer has pointed out the close analogy between approximation in the sense of least squares by polynomials on the unit circle $C: |z| = 1$ to an arbitrary function $f(z)$ continuous on C, and interpolation by polynomials in the nth roots of unity to this same function. We have the following two theorems:[2]

I. *Let $f(z)$ be analytic for $|z| \leqq R > 1$, let $P_n(z)$ denote the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least squares [that is, $P_n(z)$ is the sum of the first $n + 1$ terms of the Maclaurin development of $f(z)$], and let $p_n(z)$ denote the polynomial of degree n determined by interpolation to $f(z)$ in the $(n + 1)$st roots of unity. Then we have*

$$\lim_{n \to \infty} p_n(z) = f(z), \text{ for } |z| < R, \text{ uniformly for } |z| \leqq R_1 < R. \qquad (1)$$

Moreover we have

$$\lim_{n \to \infty} [P_n(z) - p_n(z)] = 0, \text{ for } |z| < R^2, \text{ uniformly for } \qquad (2)$$

$$|z| < R_1^2 < R^2.$$

It will be noted that many results concerning the convergence of the sequence $p_n(z)$ can be read off directly from (2)—Abel's theorem and its modified converse, divergence of the sequence $p_n(z)$ for $R < |z| < R^2$ if $f(z)$ has a singularity for $|z| = R$, etc.

An illuminating special case of the following is due to Méray; the general theorem is to be found in the reference:[1]

II. *Let $f(z)$ be continuous for $|z| = 1$ and let $p_n(z)$ denote the polynomial of degree n found by interpolation to $f(z)$ in the $(n + 1)$ st roots of unity. Then we have*

$$\lim_{n \to \infty} p_n(z) = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{t - z}, \text{ for } |z| < 1, \text{ uniformly for } |z| \leqq r < 1. \qquad (3)$$

The right-hand member of (3) is also the limit for $|z| < 1$ of the sequence of polynomials $P_n(z)$ of degree n found by best approximation to $f(z)$ on C in the sense of least squares. If, however, a sequence of polynomials $P_n'(z)$ of degree n is found say by best approximation to $f(z)$ on C in the sense of Tchebycheff, then the limit of $P_n'(z)$ for $|z| < 1$ is not in general the right-hand member of (3).

It is the object of the present note to point out the close analogy *for expansions in rational functions* between interpolation in roots of unity and approximation on $C: |z| = 1$ in the sense of least squares. We prove the two following theorems, which correspond closely to I and II. These new theorems bear the relation to the Laurent series that I and II bear to the Taylor series.

III. *Let C denote the unit circle, $f(z)$ a function analytic for $1/R < |z| < R > 1$:*

$$\begin{aligned} f(z) &= f_1(z) + f_2(z), z \text{ on } C, \\ f_1(z) &= a_0 + a_1 z + a_2 z^2 + \ldots, \\ f_2(z) &= a_{-1} z^{-1} + a_{-2} z^{-2} + \ldots, \end{aligned} \tag{4}$$

and $r_{2n}(z)$ the function

$$\begin{aligned} r_{2n}(z) &= r_n'(z) + r_n''(z), \\ r_n'(z) &= a_{0n} + a_{1n} z + \ldots + a_{nn} z^n, \\ r_n''(z) &= a_{-1n} z^{-1} + a_{-2n} z^{-2} + \ldots + a_{-nn} z^{-n}, \end{aligned} \tag{5}$$

which coincides with $f(z)$ in $2n + 1$ points equally spaced on C. Then we have

$$f(z) = \lim_{n \to \infty} r_{2n}(z), \text{ for } 1/R < |z| < R, \text{ uniformly for } 1/R_1 \leqq |z| \leqq R_1 < R$$
$$f_1(z) = \lim_{n \to \infty} r_n'(z), \text{ for } |z| < R, \text{ uniformly for } |z| \leqq R_1 < R,$$
$$f_2(z) = \lim_{n \to \infty} r_n''(z), \text{ for } |z| > 1/R, \text{ uniformly for } |z| \geqq 1/R_1 > 1/R.$$

IV. *Let C denote the unit circle and $f(z)$ a function continuous on C. Introduce the notation*

$$\begin{aligned} f_1(z) &= \frac{1}{2\pi i} \int_C \frac{f(t)dt}{t - z}, \quad |z| < 1, \\ f_2(z) &= \frac{1}{2\pi i} \int_C \frac{f(t)dt}{t - z}, \quad |z| > 1, \end{aligned} \tag{6}$$

If $f(z)$ in (4) is the formal Laurent expansion of $f(z)$ on C, then the representations (4) and (6) of $f_1(z)$ for $|z| < 1$ and of $f_2(z)$ for $|z| > 1$ are known to agree. If now $r_{2n}(z)$ denotes the function as in (5) which coincides with $f(z)$ in the $(2n + 1)$st roots of unity, then we have

$$\begin{aligned} f_1(z) &= \lim_{n \to \infty} r_n'(z), \text{ for } |z| < 1, \text{ uniformly for } |z| \leqq r < 1, \\ f_2(z) &= \lim_{n \to \infty} r_n''(z), \text{ for } |z| > 1, \text{ uniformly for } |z| \geqq R > 1. \end{aligned} \tag{7}$$

In (6) the second integral is to be taken in the positive sense with respect to the region $|z| > 1$, that is, in the clockwise sense on C.

Let us give first a proof of IV, using direct methods; this proof is similar to the proof of II (loc. cit.).

Introduce the notation

$$\omega = e^{2\pi i/(2n+1)}.$$

Then, since $r_{2n}(z)$ is found by interpolation to $f(z)$ in the $2n + 1$ distinct points ω^k, the function $z^n r_{2n}(z)$ is a polynomial in z of degree $2n$ defined by interpolation to the function $z^n f(z)$ in the points ω^k, and is represented by Lagrange's interpolation formula. Lagrange's formula for the polynomial $p_m(z)$ of degree m which takes on the values $K_1, K_2, \ldots, K_{m+1}$ in the $m + 1$ distinct points $z_1, z_2, \ldots, z_{m+1}$ is

$$p_m(z) = \sum_{k=1}^{m+1} \frac{K_k}{p'(z_k)} \frac{p(z)}{z - z_k},$$

where $p(z) = (z - z_1)(z - z_2)\ldots(z - z_{m+1})$; the polynomial $p_m(z)$ is uniquely determined by these requirements. Under the circumstances of the present theorem we have

$$z^n r_{2n}(z) = \sum_{k=1}^{2n+1} f(\omega^k)\omega^{kn} \frac{\omega^k(z^{2n+1} - 1)}{(2n + 1)(z - \omega^k)},$$

$$r_{2n}(z) = \frac{z^{2n+1} - 1}{(2n + 1)z^n} \sum_{k=1}^{2n+1} \frac{f(\omega^k)\omega^{k(n+1)}}{z - \omega^k}.$$

We find $r_n'(z)$ by omitting here the negative powers of z. Clearly one has

$$\frac{z^{2n+1} - 1}{z^n(z - \omega^k)} = z^n + \omega^k z^{n-1} + \omega^{2k} z^{n-2} + \ldots + \omega^{2nk} z^{-n},$$

from which the non-negative powers of z are

$$z^n + \omega^k z^{n-1} + \ldots + \omega^{nk} = \frac{z^{n+1} - \omega^{k(n+1)}}{z - \omega^k}.$$

This yields

$$r_n'(z) = \sum_{k=1}^{2n+1} \frac{f(\omega^k)\omega^{k(n+1)}(z^{n+1} - \omega^{k(n+1)})}{(2n + 1)(z - \omega^k)}.$$

The expression on the right suggests the computation of $f_1(z)$ directly by means of the definite integral in (6), where the circumference C is divided into $2n + 1$ parts by the points ω^k:

$$f_1(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \sum_{k=1}^{2n+1} \frac{f(\omega^k)(\omega^{k+1} - \omega^k)}{\omega^k - z}, \quad |z| < 1; \qquad (8)$$

the limit on the right exists uniformly[3] for all $|z| \leqq r < 1$.

We find now

$$\lim_{n \to \infty} [f_1(z) - r_n'(z)]$$

$$= \lim_{n \to \infty} \sum_{k=1}^{2n+1} \left[\frac{1}{2\pi i} + \frac{\omega^{nk} z^{n+1} - 1}{(2n + 1)(\omega - 1)}\right] \frac{\omega^k(\omega - 1)f(\omega^k)}{\omega^k - z}. \qquad (9)$$

The quantity $(2n + 1)(\omega - 1)$ approaches as its limit $2\pi i$, for we have

$$\omega = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1},$$

$$\frac{(2n+1)(\omega - 1)}{2\pi i} = \frac{\cos\dfrac{2\pi}{2n+1} - 1}{\dfrac{2\pi i}{2n+1}} + \frac{\sin\dfrac{2\pi}{2n+1}}{\dfrac{2\pi}{2n+1}},$$

which approaches the limit unity. By the uniformity of the convergence in (8) as applied to the right-hand member of (9) and also by the uniform convergence to zero of z^{n+1} for $|z| \leqq r < 1$, we have now the first of the equations (7).

We omit the details of the proof of the second of equations (7), for they are similar to the details already given.

It will be noted that in IV as in II the given function $f(z)$ need not be continuous on C provided it is integrable on C in the sense of Riemann.

Let us turn now to the proof of III. A proof can readily be given by the direct study of the functions involved, as suggested by the proof of IV just presented. A shorter proof of III is, however, the following.

If $R_2 < R$ is arbitrary, there exists M such that

$$\left| f(z) - [a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \ldots + a_0 + a_1 z + \ldots + a_n z^n] \right| \leqq \frac{M}{R_2^n}, \quad z \text{ on } C; \qquad (10)$$

the proof is immediate by the Cauchy inequality for the coefficients a_k as applied to $f_1(z)$ and $f_2(z)$. It will be noticed that the polynomial in z and $1/z$ which appears in (10) is a trigonometric polynomial of order n.

From (10) follows now from a very general theorem due to Jackson[4] the inequality

$$\left| f(z) - r_{2n}(z) \right| \leqq \frac{2BM\log n}{R_2^n}, \quad z \text{ on } C,$$

where B is an absolute constant. The conclusion of III is an immediate consequence of a theorem due to the present writer.[5]

One might suspect by analogy with I that in III the equation

$$\lim_{n\to\infty} [r_n'(z) - (a_0 + a_1 z + \ldots + a_n z^n)] = 0$$

could be established for certain values of z in modulus greater than R, at least in case $f(z)$ has no singularity for $|z| \geqq R$. That is not the case, however, for an arbitrary function $f(z)$, as we proceed to show by an example.

Choose $f(z) = 1/(z - \rho)$, $0 < \rho < 1$. We find in succession the equations (which may be verified directly)

$$\frac{z^n}{z - \rho} - z^n r_{2n}(z) = \frac{\rho^n(z^{2n+1} - 1)}{(\rho^{2n+1} - 1)(z - \rho)},$$

$$r_{2n}(z) = \frac{(\rho^{2n+1} - 1)z^n - \rho^n(z^{2n+1} - 1)}{(\rho^{2n+1} - 1)(z - \rho)z^n},$$

$$r_n'(z) = -\frac{\rho^n(z^n - \rho^n)}{(\rho^{2n+1} - 1)(z - \rho)}$$

We also find

$$a_0 + a_1 z + \ldots + a_n z^n = 0,$$

$$r_k'(z) - (a_0 + a_1 z + \ldots + a_n z^n) = -\frac{\rho^n(z^n - \rho^n)}{(\rho^{2n+1} - 1)(z - \rho)};$$

this right-hand member approaches zero ($|z| > 1$) if and only if we have $|z| < 1/\rho$.

The writer is not aware of any illustration other than $r_{2n}(z)$ of a sequence of functions found by interpolation *in a multiply-connected region* from an arbitrary function analytic in that region, which represents in that multiply-connected region the given analytic function.

[1] *Bull. Amer. Math. Soc.*, **38**, 289–294 (1932).

[2] I is due to Runge, *Theorie und Praxis der Reihen*, p. 137 (1904); Fejér, Göttinger Nachrichten, 319–331 (1918); (see also Kalmár, *Mathematikai és Physikai Lapok*, 120–149 (1926)); and Walsh, *Trans. Amer. Math. Soc.*, **34**, 22–74 (1932).

[3] Runge, *Acta Math.*, **6**, 229–244 (1885); Montel, *Séries de polynomes*, 51 (1910); or Osgood, *Funktionentheorie*, 579–581 (1928).

[4] *The Theory of Approximation* (1930), p. 121 Theorem I. To be sure, Jackson's theorem is there stated merely for real functions but follows immediately in the more general case.

[5] *Trans. Amer. Math. Soc.*, **30**, 838–847 (1928), Theorem II. Compare also de la Vallée Poussin, *Approximation des Fonctions*, Ch. VIII (1919), who does not bring out clearly however the convergence of the sequences $r_{2n}(z)$, $r_n'(z)$, $r_n''(z)$ as expressed in III.