MORE COMPLETE DISCUSSION OF THE TIME-DEPENDENCE OF THE NON-STATIC LINE ELEMENT FOR THE UNIVERSE

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§1. *Introduction.*—In a previous article,[1] I have shown that a continuous transformation of matter into radiation, occurring throughout the universe, as postulated by the astrophysicists, would necessitate a non-static line element for the universe, and have shown that the non-static character thus introduced might provide an explanation of the red shift in the light from the extra-galactic nebulae. In the present article, I wish to discuss the form of dependence of the line element on the time more completely than was possible on the previous occasion. This is a matter of considerable importance, since changes in the approximations which must be introduced to obtain a usable result affect to quite a different extent the expressions for the relation between red shift and distance and for the rate of annihilation of matter. Indeed, the possibility arises of slight changes from the treatment previously given which would leave the theoretical relation between red shift and distance still approximately linear, as observationally found, and yet produce a very considerable change in the calculated rate for the annihilation of matter.

§2. *The General Form of the Line Element.*—As shown in the article already mentioned, a satisfactory non-static line element for the universe can be derived in the general form

$$ds^2 = -\frac{e^{g(t)}}{\left[1 + \frac{r^2}{4R^2}\right]^2}(dx^2 + dy^2 + dz^2) + dt^2, \tag{1}$$

where r is an abbreviation for $\sqrt{x^2 + y^2 + z^2}$, R is a constant, and the time appears in the gravitational potentials only in the function $g(t)$ as shown in the equation.

The derivation of this line-element can be rigorously obtained from the principles of general relativity on the basis of the following five assumptions: (*a*) The line element shall have spatial spherical symmetry in agreement with the approximately equal distribution of nebulae in different directions observationally found. (*b*) The velocity of light shall be the same in opposite directions, say, *for example*, $+dx/dt$ and $-dx/dt$, in agreement with the approximately uniform distribution of nebulae observationally found.[2] (*c*) Nebulae which are stationary in the system of coördinates chosen shall remain stationary and not be subject to accelerations, in agreement with the necessities for a stable model for the universe. (*d*) The percentage rate for the increase with time in the proper volume bounded by a system of nebulae shall be independent of position. (*e*) And finally, the average density of matter shall be independent of position, both of the two last requirements again being in agreement with the observational uniform density of nebular distribution.

Since these assumptions appear reasonable, the status of the above line element may also be regarded as reasonably satisfactory. Furthermore, after having published a derivation of this line element, I learned from a discussion with Professor H. P. Robertson that he had already previously published[3] the derivation of an entirely equivalent line element, based on very general geometrical considerations as to the homogeneity and isotropicity of space-time and its separability into space and time. I think that the agreement of the results of the two methods of attack is very satisfactory.

Before leaving the subject of the validity of the general form of the non-static line element for the universe, some remarks should be made, however, concerning the continuous and uniform distribution of matter which has been tacitly or explicitly assumed in its derivations. With regard to the continuity of distribution, it is evident that the derivation of this line element, as of previous cosmological line elements, is dependent on a neglect of the local gravitational fields due to the immediate neighborhood of stars or stellar systems. And at a later stage of theoretical development it may be necessary to make corrections for this, since, if we exclude some slight evidence for the existence of intergalactic dust, the matter that we can actually observe in the universe appears to be concentrated in widely separated nebulae which are pouring radiation into the intervening space. With regard to the assumed uniformity of distribution, this is in agreement with the average approximate uniformity in the distribution of nebulae found to exist to those distances to which the Mount Wilson instruments have been able to penetrate.[4] It would, however, be quite out of keeping with a desire to correlate known experimental facts with the help of accepted theory to demand that this approximate uniformity should necessarily continue to all distances. The above line element agrees with the approximately uniform distribution to the distances that we now know and its form can presumably be subjected to reasonable modification to allow for deviations from uniformity that may later be established.

§3. *Application of the Laws of Mechanics.*—As a preliminary to our discussion of the form of $g(t)$, it will first be desirable to consider the application of relativistic mechanics to the above line element.

As shown in the previous article, values for the energy-momentum tensor corresponding to the line element (1) can be calculated from the principles of general relativity, and treating the material in the universe as a perfect fluid, can be written in the form:

$$\begin{aligned}
8\pi T_1^1 &= 8\pi T_2^2 = 8\pi T_3^3 = -8\pi p_0 = \frac{1}{R^2}e^{-g} + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \\
8\pi T_4^4 &= 8\pi\rho_{00} = \frac{3}{R^2}e^{-g} + \frac{3\dot{g}^2}{4} - \Lambda \\
8\pi T_\pi^\nu &= 0 \; (\mu \neq \nu) \\
8\pi T &= 8\pi\rho_0 = \frac{6}{R^2}e^{-g} + 3\ddot{g} + 3\dot{g}^2 - 4\Lambda
\end{aligned} \tag{2}$$

where p_0 is the proper pressure, ρ_{00}, the proper macroscopic density, ρ_0, the proper density of matter, Λ, the cosmological constant, and the dots indicate differentiation with respect to the time.

With the help of these expressions for the components of the energy momentum tensor we can now easily apply the principles of relativistic mechanics in the well-known form

$$\frac{\partial \mathfrak{T}_\mu^\nu}{\partial x_\nu} - \frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\mu} = 0 \tag{3}$$

With $\mu = 1, 2, 3$, we merely obtain identities, but substituting in this equation for the case $\mu = 4$, we easily obtain the important result

$$\frac{d}{dt}\left(\rho_{00}e^{\frac{3\mu}{2}}\right) + p_0\frac{de^{\frac{3\mu}{2}}}{dt} = 0 \tag{4}$$

where for brevity we have set, as in the previous article,

$$e^\mu = \frac{e^g}{\left[1 + \frac{r^2}{4R^2}\right]^2} \tag{5}$$

Since $e^{\frac{3\mu}{2}}dxdydz$ is evidently the proper volume associated with the coördinate range $dxdydz$, equation (4) has the simple physical interpretation that the rate of increase in the proper energy in any coördinate range is the negative of the rate of expenditure of work on the surroundings. This is a very satisfactory and illuminating result.

§4. *The Annihilation of Matter Requires That g(t) Cannot Be a Constant.*—We are now ready to commence the investigation of the form of $g(t)$. Since the quantity of matter in any given coördinate range $dxdydz$ is evidently $\rho_0 e^{\frac{3\mu}{2}}\,dxdydz$, we at once see, on examining the expressions for ρ_0 and μ given by equations (2) and (5), that the amount of matter in the universe can be changing with the time, through annihilation (i.e., transformation of matter into radiation), only if g itself is changing with the time. This simple qualitative result is already of considerable interest, since it will evidently make the velocity of light in the universe dependent on the time and thus at least lead to a shift in the wave-length of light from distant objects, even though we cannot yet say whether this shift would be toward the red or violet.

§5. *An Additional Condition for the Stability of the Model Requires That $g(t)$ Increase with the Time.*—We can now show, however, that a further consideration of the requirements for the stability of our model of the universe presents a certain measure of preliminary argument which would lead us to expect in the actual universe a red rather than a violet shift in the light from distant objects. In deriving the line element (1), it was assumed as a necessary condition for the stability of our model that nebulae which are stationary in the coördinates chosen should not be subject to acceleration. Let us now examine the effect of adding to this the further requirement that nebulae which are not stationary in our system of coördinates shall tend to become so with the time.

From the form of the line element

$$ds^2 = -e^{\mu}(dx^2 + dy^2 + dz^2) + dt^2 \tag{6}$$

we can evidently write for the square of the spatial "velocity" of a nebula

$$\left(\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right) = e^{-\mu}\left[\left(\frac{dt}{ds}\right)^2 - 1\right] \tag{7}$$

and can hence conclude that the spatial "velocity" will be approaching zero if the quantity $(dt/ds)^2$ is approaching unity.

On the other hand, the behavior of the quantity (dt/ds) for a particle such as a nebula will be determined in accordance with the principles of relativity by the geodesic equation

$$\frac{d^2x_\alpha}{ds^2} + \{\rho\sigma,\alpha\}\frac{dx_\rho}{ds}\frac{dx_\sigma}{ds} = 0 \tag{8}$$

and applying this to the case $\alpha = 4$, making use of the expressions for the Christoffel symbols given by equations (19) in the previous article, we easily obtain

$$\frac{d^2t}{ds^2} + \frac{1}{2}e^{\mu}\frac{\partial\mu}{\partial t}\left(\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right) = 0 \tag{9}$$

or substituting equation (7) and inserting $\dot{g}$ in place of its equivalent $(\partial\mu/\partial t)$ we can write

$$\frac{d}{ds}\left(\frac{dt}{ds}\right) = \frac{\dot{g}}{2}\left[1 - \left(\frac{dt}{ds}\right)^2\right] \tag{10}$$

Hence, since in accordance with equation (7) the quantity (dt/ds) cannot be less than unity, we can at once conclude that (dt/ds) will either be unity or approaching unity, provided $\dot{g}$ is positive. In other words, a positive value for $\dot{g}$ is the necessary condition in order that nebulae which are not stationary in our coördinates should tend to become so. On the other hand, as will be remembered from the previous article and as will be shown more clearly later on, a positive value for $\dot{g}$ is also the condition for a red rather than a violet shift in the light from distant objects. This result, combined with the evidences for actual stability in the arrangement of the nebulae, gives a certain preliminary theoretical support to the expectation that we shall find a red rather than a violet shift in the actual universe.

Since a positive value of $\dot{g}$ leads to a positive rate of increase in the proper volume bounded by a system of nebulae, we can say, in accordance with equation (10), that in an "expanding" universe there is a tendency for the nebulae to "keep step" since their individual "proper" motions tend to decrease. On the other hand, if we insert a negative value of $\dot{g}$ in equation (10), we see that in a "contracting" universe there would be a tendency for the "proper" motions of the nebulae to increase, and we could not expect that line element (1) would remain permanently applicable. The tendency under these circumstances for the "proper" motions of the nebulae to increase with time can perhaps be interpreted as corresponding to a tendency for the nebulae to clump together in a "contracting" universe.

§6. *Relation between $g(t)$ and the Rate of Annihilation of Matter.*—We must now turn to a more quantitative consideration of the form of $g(t)$, by obtaining those expressions for rate of annihilation of matter and for red shift which correspond to the general form of the line element (1). In contrast to the treatment given in the previous article, we shall first endeavor to obtain these expressions in their exact form and postpone until later the introduction of approximations, to which we shall give a separate treatment in §8.

To obtain an expression for the rate of annihilation of matter we must first consider the distinction between the two different kinds of proper density indicated by the symbols ρ_{00} and ρ_0.

The quantity ρ_{00}, which may be called the proper *macroscopic* density, is the density of energy as it would be measured by an observer, using Galilean coördinates which are stationary with respect to the macroscopic motion at the point of interest, and looking at the matter and radiation present from a coarse-grained point of view which does not distinguish between individual atoms and light quanta. It hence corresponds to the total energy density of matter and radiation as it would be measured in ordinary laboratory experiments.

On the other hand, the quantity ρ_0, which may be called the proper *microscopic* density, is the density of energy as it would be measured by an observer, using Galilean coördinates which are stationary with respect to the precise microscopic motion at the point of interest, and looking at the matter or radiation present from a completely fine-grained point of view. Hence, since the rest-mass of a light quantum is zero, ρ_0 corresponds to the density of matter *alone*, as measured by an observer with respect to which the matter is completely stationary. Thus, this is the quantity which will interest us in calculating the rate of annihilation of matter.[5]

Since the system of coördinates used in our line element (1) has been purposely chosen so that matter will be stationary in those coördinates, except for relatively negligible "proper" motions, it is now evident that the quantity of matter in any range of coördinates $\int\int\int dxdydz$ can be calculated by multiplying the proper density of matter ρ_0 by the proper volume $\int\int\int e^{\frac{3\mu}{2}}dxdydz$ corresponding to that range. Hence, choosing for convenience unit coördinate range $\int\int\int dxdydz = 1$, we can now write for the quantity of matter in that range

$$M = \rho_0 e^{\frac{3\mu}{2}} \tag{11}$$

and since matter, being stationary in our coördinates, will not be moving across the coördinate boundary, a calculation of the percentage rate at which M is decreasing with the time will give us the percentage rate at which matter is being annihilated.

To calculate this quantity we can now return to equation (4), obtained from the principles of relativistic mechanics, and substituting the well-known relation between the two expressions for density

$$\rho_{00} = \rho_0 + 3p_0 \tag{12}$$

may write

$$\frac{d}{dt}\left(\rho_0 e^{\frac{3\mu}{2}}\right) + 4p_0 \frac{de^{\frac{3\mu}{2}}}{dt} + 3e^{\frac{3\mu}{2}} \frac{dp_0}{dt} = 0, \tag{13}$$

or substituting the expressions for M given by equation (11) and for μ given by (5), this can be rewritten in the form

$$-\frac{1}{M}\frac{dM}{dt} = 6\frac{p_0}{\rho_0}\dot{g} + \frac{3}{\rho_0}\frac{dp_0}{dt}. \tag{14}$$

For purposes of later use we can advantageously transform the last term in this equation, by substituting the expressions for ρ_0 and p_0 given by equations (2), in such a way as to obtain

$$-\frac{1}{M}\frac{dM}{dt} = 6\frac{p_0}{\rho_0}\dot{g} + \left(1 + 4\frac{p_0}{\rho_0}\right)\frac{\frac{3}{R^2}e^{-g} - 3\frac{\dddot{g}}{\dot{g}} - \frac{9}{2}\ddot{g}}{\frac{2}{R^2}e^{-g} - \ddot{g}}\dot{g}. \quad (15)$$

Furthermore, in discussing the form of $g(t)$ it will also sometimes be convenient to express this function in the form of a series

$$g = 2(kt + lt^2 + mt^3 + \ldots) \quad (16)$$

where the factor 2 has been introduced to avoid later fractions, and the term in t^0 has been set equal to zero, since this merely has the effect of determining the choice of units used in laying off the system of spatial coördinates in such a way that the line element will reduce to the special relativity form at $t = 0$ and in the neighborhood of $r = 0$. Substituting this expression in equation (15), we can then write at the time $t = 0$, which we shall regard as the time of observation,

$$\left[-\frac{1}{M}\frac{dM}{dt}\right]_{t=0} = 12\frac{p_0}{\rho_0}k + \left(1 + \frac{4p_0}{\rho_0}\right)\frac{\frac{3}{R_2} - \frac{18m}{k} - 18l}{\frac{1}{R^2} - 2l}k. \quad (17)$$

Equations (14), (15) and (17) give expressions for the rate of annihilation of matter which correspond exactly to our general line element (1). Their further discussion, however, will be postponed until we have also obtained an exact expression for the relation between red shift and distance.

§7. *Relation between $g(t)$ and the Wave-Length of Light from Distant Objects.*—To obtain an expression for the wave-length of light from the nebulae, we note in accordance with the general form of the line element (1), that the proper period of emission ds_0 from atoms situated on objects such as the nebulae which have no coördinate velocity would everywhere be the same as the coördinate period dt. Hence to calculate the period and wave-length of the light on reception at the origin of coördinates we shall only need to consider the velocity with which light impulses are transmitted.

Setting $ds = 0$ in equation (1), we obtain for the radial velocity of light

$$\frac{dr}{dt} = \pm e^{-\frac{g}{2}}\left[1 + \frac{r^2}{4R^2}\right], \quad (18)$$

where r as before is merely an abbreviation for $\sqrt{x^2 + y^2 + z^2}$. Separating variables and integrating we can then write

$$\int_0^r \frac{dr}{1+\frac{r^2}{4R^2}} = \int_{t_1}^{t_2} e^{-\frac{g}{2}}\,dt \tag{19}$$

as an expression connecting the coördinate distance r from the origin to a given nebula, with the time t_1 at which a given light impulse leaves the nebula and the time t_2 at which it arrives at the origin.

For a given nebula located at r, the value of the left-hand side of this equation will, however, evidently be independent of the time of departure t_1, so that by differentiating the right-hand side with respect to t_1, we can write

$$e^{-g_2/2}\frac{dt_2}{dt_1} - e^{-g_1/2} = 0, \tag{20}$$

where g_2 and g_1 are the values of g at times t_2 and t_1. And taking dt_1 as the period of the light on its emission and dt_2 its period on reception we can now write

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{dt_2}{dt_1} = e^{\frac{g_2-g_1}{2}}, \tag{21}$$

as an exact expression connecting the observed wave-length $(\lambda + \delta\lambda)$ of the light from the nebula with the wave-length λ of light from a similar terrestrial source.

Since it is customary, however, to correlate the actual observational data on red shift with the distance to the nebula involved, it will be advantageous to differentiate equation (21) with respect to r, the coördinate distance to the nebula. In doing so, we can regard g_2 as a constant since this is the value of g at the time of observation and we actually make observations all at the same time on the nebulae at different distances; furthermore, in agreement with our previous convention as stated in connection with equation (16), we can take g as equal to zero at the time of observation. On the other hand, the quantity g_1 will obviously change as we go from one nebula to a more distant one since the light will have to leave at an earlier time, and g is known to depend on the time. Taking the above into consideration we can now differentiate equation (21)and write

$$\frac{d}{dr}\left(\frac{\delta\lambda}{\lambda}\right) = -\frac{1}{2}e^{-g_1/2}\frac{dg_1}{dt}\frac{dt}{dr}. \tag{22}$$

The quantity dt occurring in this equation is the change in the time of light departure associated with the change to a nebula whose coördinate distance is greater by dr, and can hence be calculated from equation (18) for the velocity of light. Substituting the result in (22), we easily obtain, with some rearrangement,

$$\frac{d}{dr}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{1}{1+\dfrac{r^2}{4R^2}}\,\frac{\dot{g}_1}{2}, \tag{23}$$

where $\dot{g}_1$ is the time derivative of g at the time of departure t_1. Or substituting for g the previous expansion given by equation (16), we can write

$$\frac{d}{dr}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{1}{1+\dfrac{r^2}{4R^2}}\,(k + 2lt_1 + 3mt_1^2 + \ldots) \tag{24}$$

where t_1 is the time of departure from the nebula of the light which reaches the observer at the origin at time $t_2 = 0$.

Equations (23) and (24) give expressions for the shift in wave-length with distance which correspond exactly to our general line element (1).

§8. *Correlation with Observational Data.*—We now turn to the correlation of the actual observational material with our theoretical expressions for shift in wave-length and rate of annihilation. In particular, we must consider the different possibilities as to approximations which could be introduced into our theoretical expressions in order to make them usable at our present stage of incomplete observational knowledge.

(*a*) As our first approximations, which we shall use throughout the remaining discussion, we shall take r^2/R^2 and $\delta\lambda/\lambda$ as negligible compared with unity. This will introduce great simplification, since, as is perhaps obvious and as will be discussed more completely in a later article, this assumption will be sufficient to make the coördinate distance r agree with distances calculated in the usual manner from luminosities or from the angular extension of the nebulae. Nevertheless, it should be remarked that at least in the case of $\delta\lambda/\lambda$ the observational material is already pushed nearly far enough so as to permit a closer approximation.

(*b*) On the basis of the above approximations we shall now rewrite equation (24) for the red shift in the form

$$\frac{d}{dL}\left(\frac{\delta\lambda}{\lambda}\right) = k + 2lt_1 + 3mt_1^2 + \ldots, \tag{25}$$

where L is to be taken as agreeing with the observational distances of Hubble. Since Hubble and Humason actually find an approximately linear increase of red shift with distance, the higher order terms in this expression cannot be large compared with k out to the most distant nebula so far observed[6] for which the time of light departure is about

$t_1 = -10^8$ years. Hence, taking the observational value $k = 5.1 \times 10^{-10}$ (years)$^{-1}$, we can preserve the approximate linear relation provided $|l| \ll 5 \times 10^{-18}$ (years)$^{-2}$, and $|m| \ll 5 \times 10^{-26}$ (years)$^{-3}$.

(*c*) In the case of those objects in the universe which we can observe in the telescope, stars and nebulae, the pressure p_0 is certainly very small compared with the density of matter ρ_0. Hence, in the first of the following considerations we shall take $p_0 \ll \rho_0$.

(*d*) Nevertheless, we must also consider the possible contents of intergalactic space, radiation and dust, which cannot be observed in the telescope. Doing so we are forced to conclude that the temperature of intergalactic space would not have to be unreasonably high to give a radiation pressure that would be at least of the order of the averaged out density of the matter that can actually be observed in the stars. Moreover, we have no certain means at present of estimating how much dust[7] may be present in intergalactic space to reduce the ratio of p_0/ρ_0. Hence, we shall also give some consideration to the case $p_0 \gg \rho_0$. This would correspond to a universe so full of radiation as to make this the chief determinant of its properties.[8]

(*e*) Combining the equations for ρ_0 and p_0 given by (2) we can write for the time $t = 0$, $g = 0$, the expression $4\pi(\rho_0 + p_0) = 1/R^2 - \ddot{g}/2 = 1/R^2 - 2l$. For the case $p_0 \ll \rho_0$ and $|2l| \ll 1/R^2$ this reduces to the familiar Einstein relation between density and radius of the universe, and using the density of visible matter estimated by Hubble[9] we have $R = 8.5 \times 10^{28}$ cm. or changing units we obtain $1/R^2 \approx 10^{-22}$ (years)$^{-2}$. If there is a large density of dust in intergalactic space $1/R^2$ would become greater. If $p_0 \ll \rho_0$ and $|2l| \nless 1/R^2$ the estimate can be taken as applying directly to the whole quantity $(1/R^2 - 2l)$. If $p_0 \gg \rho_0$ we have no satisfactory basis for estimate.

We are now ready to consider the effect of introducing different possible approximations into the expressions for the red shift and the rate of annihilation of matter.

(*f*) (Case $p_0 \ll \rho_0$; l, m, etc., exactly zero.) We shall first consider the very simple case in which we take the average pressure in the universe to be negligible compared with the average density of matter, and take g exactly linear in t. This is the case that was considered in the previous article. It makes the percentage red shift as given by equation (25) exactly linear with the distance

$$\frac{d}{dL}\left(\frac{\delta\lambda}{\lambda}\right) = k, \tag{26}$$

and in accordance with equations (15) and (16) it gives for the rate of annihilation of matter

$$-\frac{1}{M}\frac{dM}{dt} = 3k, \qquad (27)$$

which holds exactly as long as $p_0 \ll \rho_0$.

Using the value of k determined from the red shift data, equation (27) requires, as shown in the previous article, an average rate of annihilation of matter which is of the order of that found in the most massive stars, much higher than that corresponding to a typical star such as the sun. Hence, the present treatment could hardly be correct unless the nuclei of the nebulae should have a very high rate of annihilation, or intergalactic space should contain dust having a high rate of annihilation. Neither of these suggestions, however, seems impossible. In particular, intergalactic dust might perhaps be expected to have the high rate of annihilation of an early type star and might be the source of the cosmic rays, which if radiational in character must certainly originate in very diffuse matter.[7]

The logarithmic decay of matter given by equation (27), which remains exact as long as $p_0 \ll \rho_0$, might be regarded as furnishing theoretical support for setting $l = m = \ldots = 0$, except for the complicated nature of the actual observational facts as to the rate of annihilation of matter.

(*g*) (Case $p_0 \ll \rho_0$; l, m, etc., not exactly zero.) We next consider the possibility of dropping the assumption that g is exactly linear in t, keeping l, m, etc., small enough, however, as discussed in paragraph *b*, not to disagree with the observational linearity of red shift with distance. Equation (17) for the rate of annihilation of matter then reduces so that it can be written in the form

$$\left[-\frac{1}{M}\frac{dM}{dt}\right]_{t=0} = \frac{3\left(\frac{1}{R^2} - 2l\right) - 18\frac{m}{k} - 12l}{\left(\frac{1}{R^2} - 2l\right)}k \qquad (28)$$

Referring to paragraph *e*, however, if we take Hubble's estimate of the density of matter in the universe, we have $(1/R^2 - 2l) \approx 10^{-22}$ (years)$^{-2}$. And on the other hand, from paragraph *b*, we only need $|l| \ll 5 \times 10^{-18}$ (years)$^{-2}$ and $|m/k| \ll 10^{-16}$ (years)$^{-2}$, in order not to disagree with the observational linearity of red shift with distance.

Hence, the possibility very definitely arises that the actual values of m or l might be large enough to produce a large effect on the calculated rate of annihilation of matter, without causing a disagreement with the observational relation between red shift and distance. More specifically, a small positive value of l, which would reduce the fractional rate of annihilation far below $3k$, would not be impossible. At the present stage of our knowledge this possibility alone is sufficient to keep us from discarding the general theory on the basis that it gives too high rates of annihilation.

(h) (Case $p_0 \gg \rho_0$; l, m, etc., exactly zero.) The treatment of cases, in which the pressure is not assumed negligible compared with the density of matter, is not easy as we have little to guide us. We restrict ourselves to a universe which is so full of radiation that $p_0 \gg \rho_0$, and assume g exactly linear with respect to t. Equation (17) for the rate of annihilation of matter then evidently reduces to

$$\left[-\frac{1}{M}\frac{dM}{dt}\right]_{t=0} = 24\frac{p_0}{\rho_0}k, \tag{29}$$

and with p_0 very large compared with ρ_0 this gives a rate of annihilation of matter very high to appear probable. But of course this one case does not exhaust the possibilities of universes containing appreciable radiation.

In conclusion, it is merely desired to emphasize the possible flexibility as to the calculated rate of annihilation which would arise as shown above if g were not exactly linear in t, even though the exactly linear relation still appears very attractive. The desirability of further observational determinations of the relation between red shift and distance, in order to determine the actual relation between g and t as accurately as possible, is of course obvious.

[1] These PROCEEDINGS, 16, 320 (1930). The present article is part of the further investigation of the effect of the mathematical approximations mentioned in the previous article, p. 336. Computations for a less unduly simplified model have not yet been made.

[2] The effect of this restriction is to make $g_{14} = g_{24} = g_{34} = 0$. In the previous article this result was obtained by requiring that the line element should be symmetrical with respect to past and future time.

[3] Robertson, these PROCEEDINGS, 15, 822 (1929).

[4] Hubble, *Astrophys. J.*, 64, 369 (1926).

[5] For details as to this distinction between ρ_{00} and ρ_0, see Eddington *The Mathematical Theory of Relativity*, Cambridge, 1923, pp. 117, 121, 122. In the present stage of physics, arguments might be brought against the definition of ρ_0 given by Eddington and in the text above, on the ground that the uncertainty principle would rule out the possibility of attaching any meaning to such a microscopic quantity. Such arguments would perhaps be valid. For the purposes of the present article, however, we could regard ρ_0 as defined by $\rho_{00} - 3p_0$ which under our circumstances is obviously the total macroscopic energy density minus that for radiation alone.

[6] Data of Hubble and Humason not yet published.

[7] According to a private communication from Dr. Hubble, the total mass of dust spread out through intergalactic space could be several thousand times as great as the mass concentrated in the nebulae without being detectable by present astronomical methods.

[8] This possibility has already had some previous consideration. See Silberstein, *Phil. Mag.*, 9, 50 (1930). Also Tolman, these PROCEEDINGS, 15, 303 (1929).

[9] See reference 4.