

# Theorems about the divisors of numbers contained in the form $paa \pm qbb^*$

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In the following theorems, the letters  $a$  and  $b$  designate arbitrary relatively prime integers, that is, which aside from unity have no other common divisor.

## Theorem 1.

All the prime divisors of numbers contained in the form  $aa + bb$  are either 2 or are numbers of the form  $4m + 1$ .

## Theorem 2.

All prime numbers of the form  $4m + 1$  in turn are contained in the form of numbers  $aa + bb$ .

## Theorem 3.

Thus the sum of two squares, that is a number of the form  $aa + bb$ , is never able to be divided by any number of the form  $4m - 1$ .

## Theorem 4.

All the prime divisors of numbers contained in the form  $aa + 2bb$  are either 2, or numbers contained in the form  $8m + 1$  or in the form  $8m + 3$ .

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### Theorem 5.

All prime numbers contained in the forms  $8m + 1$  or  $8m + 3$  in turn are numbers of the form  $aa + 2bb$ .

### Theorem 6.

No number of the form  $aa + 2bb$  is able to be divided by a number of the forms  $8m - 1$  or  $8m - 3$ .

### Theorem 7.

All the prime divisors of numbers contained in the form  $aa + 3bb$  are either 2 or 3, or are contained in one of the forms  $12 + 1, 12m + 7$ .

### Theorem 8.

All prime numbers contained in either of the forms  $12m + 1$  or  $12m + 7$ , that is in the one form  $6m + 1$ , are numbers of the form  $aa + 3bb$ .

### Theorem 9.

No number of the form  $12m - 1$  or of the form  $12m - 7$ , that is no number of the form  $6m - 1$ , is a divisor of any number contained in the form  $aa + 3bb$ .

### Theorem 10.

All the prime divisors of numbers contained in the form  $aa + 5bb$  are either 2, or 5, or are contained in one of 4 forms  $2m + 1, 20m + 3, 20m + 7, 20m + 9$ .

### Theorem 11.

If a number  $20m + 1, 20m + 3, 20m + 9, 20m + 7$  were prime, then it will follow that

$$\begin{array}{ll} 20m + 1 = aa + 5bb; & 2(20m + 3) = aa + 5bb \\ 20m + 9 = aa + 5bb; & 2(20m + 7) = aa + 5bb \end{array}$$

### Theorem 12.

No number contained in one of the following forms  $20m - 1; 20m - 3; 20m - 9; 20m - 7$  is able to be a divisor of any number of the form  $aa + 5bb$ .

### Theorem 13.

All the prime divisors of numbers contained in the form  $aa + 7bb$  are either 2 or 7 or are contained in one of the following six forms

$$\begin{array}{cc|l} 28m + 1 & 28m + 11 & \text{that is in one of these} \\ 28m + 9 & 28m + 15 & 14m + 1 \\ 28m + 25 & 28m + 23 & 14m + 9 \\ & & 14m + 11 \end{array}$$

### Theorem 14.

If a prime number were contained in one of the forms  $14 + 1, 14m + 9, 14m + 11$ , then at once it would be contained in the form  $aa + 7bb$ .

### Theorem 15.

No number of the form  $aa + 7bb$  is able to be divided by any number which is contained in one of the following six forms

$$\begin{array}{cc|l} 28m + 3, & 28m + 5 & \text{that is of these three} \\ 28m + 13, & 28m + 17 & 14m + 3 \\ 28m + 19, & 28m + 27 & 14m + 5 \\ & & 14m + 13 \end{array}$$

### Theorem 16.

All the prime divisors of numbers contained in the form  $aa + 11bb$  are either 2 or 11, or are contained in one of the following

$$\begin{array}{cc|l} \text{10 forms} & & \text{5 forms} \\ 44m + 1 & 44m + 3 & 22m + 1 \\ 44m + 9 & 44m + 27 & 22m + 3 \\ 44m + 37 & 44m + 23 & 22m + 9 \\ 44m + 25 & 44m + 31 & 22m + 5 \\ 44m + 5 & 44m + 15 & 22m + 15 \end{array}$$

### Theorem 17.

If a prime number were contained in any of these ten or five forms, then at once either itself or four times it will be a number of the form  $aa + 11bb$ .

### Theorem 18.

No number of the form  $aa + 11bb$  is able to be divided by any number which is contained in one of the following

10 forms	5 forms
$44m + 7$	$22m + 7$
$44m + 13$	$22m + 13$
$44m + 17$	$22m + 17$
$44m + 19$	$22m + 19$
$44m + 21$	$22m + 21$

### Theorem 19.

All the prime divisors of numbers contained in the form  $aa + 13bb$  are either 2 or 3 or are contained in one of the following 12 formulas.

$52m + 1$	$52m + 7$
$52m + 49$	$52m + 31$
$52m + 9$	$52m + 11$
$52m + 25$	$52m + 19$
$52m + 29$	$52m + 47$
$52m + 17$	$52m + 15$

### Theorem 20.

All prime numbers which are contained in the first of the above columns of formulas are at once numbers of the form  $aa + 13bb$ . On the other hand, twice the prime numbers which are contained in the second column of formulas are numbers of the form  $aa + 13bb$ .

### Theorem 21.

No number of the form  $aa + 13bb$  is able to be divided by any number which is

contained in one of the following formulas

$52m + 3$	$52m + 35$
$52m + 5$	$52m + 37$
$52m + 21$	$52m + 41$
$52m + 23$	$52m + 43$
$52m + 27$	$52m + 45$
$52m + 33$	$52m + 51$

### Theorem 22.

All the prime divisors of numbers contained in the form  $aa + 17bb$  are either 2 or 17 or are contained in one of the following forms

$68m + 1$	$68m + 3$
$68m + 9$	$68m + 27$
$68m + 13$	$68m + 39$
$68m + 49$	$68m + 11$
$68m + 33$	$8m + 31$
$68m + 25$	$68m + 7$
$68m + 21$	$68m + 63$
$68m + 53$	$68m + 23$

### Theorem 23.

All the prime numbers which are contained in the first of the above columns of formulas are either themselves or nine times them numbers of the form  $aa + 17bb$ . On the other hand, three times the prime numbers in the other column are numbers of the form  $aa + 17bb$ .

### Theorem 24.

No number of the form  $aa + 17bb$  is able to be divided by any number which is

contained in one of the following formulas

$68m - 1$	$68m - 3$
$68m - 9$	$68m - 27$
$68m - 13$	$68m - 39$
$68m - 49$	$68m - 11$
$68m - 33$	$68m - 31$
$68m - 25$	$68m - 7$
$68m - 21$	$68m - 63$
$68m - 53$	$68m - 23$

### Theorem 25.

All the prime divisors of numbers contained in the form  $aa + 19bb$  are either 2, or 19, or are contained in one of the following

18 formulas		9 formulas
$76m + 1$	$76m + 5$	$38m + 1$
$76m + 25$	$76m + 49$	$38m + 5$
$76m + 17$	$76m + 9$	$38m + 7$
$76m + 45$	$76m + 73$	$38m + 9$
$76m + 61$	$76m + 7$	$38m + 11$
$76m + 35$	$76m + 23$	$38m + 17$
$76m + 39$	$76m + 43$	$38m + 23$
$76m + 63$	$76m + 11$	$38m + 25$
$76m + 55$	$76m + 47$	$38m + 35$

### Theorem 26.

All prime numbers which are contained in one of these forms are either themselves or four times them numbers of the form  $aa + 19bb$ .

### Theorem 27.

No number of the form  $aa + 19bb$  is able to be divided by any number which

may be contained in any of the following 9 formulas

$$\begin{aligned}
 &38m - 1 \\
 &38m - 5 \\
 &38m - 7 \\
 &38m - 9 \\
 &38m - 11 \\
 &38m - 17 \\
 &38m - 23 \\
 &38m - 25 \\
 &38m - 35
 \end{aligned}$$

Thus the character of the forms  $aa + qbb$  is contained in these theorems; if  $q$  is a prime number, we see first for all the prime divisors of these forms to be 2 or  $q$ , or to be able to be expressed in the form  $4qm + \alpha$ , so that no divisor is not contained in this form, and indeed also that each prime number  $4qm + \alpha$  is at once a divisor of such a form  $aa + qbb$ . Then moreover it can be deduced that that if a prime number of the form  $4qm + \alpha$  were a divisor of some number  $aa + qbb$ , then no number of the form  $4qm - \alpha$  can be a divisor of the expression  $aa + qbb$ . It is clear from this therefore that as  $4mq + 1$  is always contained among the divisors of the form  $aa + qbb$ , no number  $aa + qbb$  is able to be divided by any number of the form  $4mq - 1$ . Indeed it is clear with attention that were  $q$  a prime number of the form  $4n - 1$  for the forms of the divisors to be able to be reduced to less than twice the number, so that they can be reduced to the formulas  $2qm + \alpha$ , and that this cannot be done if  $q$  were a prime number of the form  $4n + 1$ . If therefore  $4(4n + 1)m + \alpha$  were a divisor for the form  $aa + (4n + 1)bb$ , then no number of the form  $4(4n + 1)m + 2(4n + 1) + \alpha$  will be able to be a divisor of the expression  $aa + (4n + 1)bb$ . We will presently give a number of notes about the forms  $aa + qbb$ , in which we shall contemplate when  $q$  is not a prime number.

### Theorem 28.

All the prime divisors of numbers contained in the form  $aa + 6bb$  or in the form  $2aa + 3bb$  are either 2 or 3 or are contained in one of the following formulas

$$\begin{array}{ll}
 24m + 1 & 24m + 7 \\
 24m + 5 & 24m + 11
 \end{array}$$

### Theorem 29.

All prime numbers either of the form  $24m + 1$  or  $24m + 7$  are contained in the expression  $aa + 6bb$ ; whereas the prime numbers of the form  $24m + 5$  and  $24m + 11$  are contained in the expression  $2aa + 3bb$ .

### Theorem 30.

No number  $aa + 6bb$  or  $2aa + 3bb$  is able to be divided by any number which is contained in any of the following forms

$$\begin{array}{ll} 24m - 1 & 24m - 5 \\ 24m - 7 & 24m - 11 \end{array}$$

### Theorem 31.

All the prime divisors of numbers in the form  $aa + 10bb$  or in the form  $2aa + 5bb$  are either 2 or 5 or are contained in one of the following forms

$$\begin{array}{ll} 40m + 1 & 40m + 7 \\ 40m + 9 & 40m + 23 \\ 40m + 11 & 40m + 37 \\ 40m + 19 & 40m + 13 \end{array}$$

### Theorem 32.

Prime numbers contained in the forms of the first column above are at once numbers of the form  $aa + 10bb$  and prime numbers contained in the second column are numbers of the form  $2aa + 5bb$ .

### Theorem 33.

No number either of the form  $aa + 10bb$  or of the form  $2aa + 5$  is able to be divided by any number which is contained in the following forms

$$\begin{array}{ll} 40m - 1 & 40m - 7 \\ 40m - 9 & 40m - 23 \\ 40m - 11 & 40m - 37 \\ 40m - 19 & 40m - 13 \end{array}$$

### Theorem 34.

All the prime divisors of numbers contained in the form  $aa + 14bb$  or in the form  $2aa + 7bb$  are either 2 or 7 or are contained in one of the following formulas

$56m + 1$	$56m + 3$
$56m + 9$	$56m + 27$
$56m + 25$	$56m + 19$
$56m + 15$	$25m + 5$
$56m + 23$	$56m + 45$
$56m + 39$	$56m + 13$

### Theorem 35.

Prime numbers of the formulas contained in the first column above are at once numbers of either the form  $aa + 14bb$  or  $2aa + 7bb$ , while on the other hand those three times those contained in the second column is contained in the formulas in the first column.

### Theorem 36.

If in the above examples the sign  $+$  is switched to  $-$ , then no number contained in this form can be a divisor of either the form  $aa + 14bb$  or the form  $2aa + 7bb$ .

### Theorem 37.

All the prime divisors of numbers contained in the form  $aa + 15bb$  or in the form  $3aa + 5bb$  are either 2, or 3, or 5 or are contained in one of the following formulas

$60m + 1$	$60m + 31$	or of these 4
$60m + 17$	$60m + 47$	$30m + 1$
$60m + 19$	$60m + 49$	$30m + 17$
$60m + 23$	$60m + 53$	$30m + 19$
		$30m + 23$

### Theorem 38.

All prime divisors of numbers contained in the form  $aa + 21bb$  or in the form  $3aa + 7bb$  are either 2, or 3, or 7 or are contained in one of the following formulas

$84m + 1$	$84m + 5$
$84m + 25$	$84m + 41$
$84m + 37$	$84m + 17$
$84m + 55$	$84m + 11$
$84m + 31$	$84m + 13$
$84m + 19$	$84m + 71$

### Theorem 39.

All the prime divisors of numbers contained in the form  $aa + 35bb$  or in the form  $5aa + 7bb$  are either 2, or 5, or 7 or are contained in one of the following formulas

$140m + 1$	$140m + 3$	or of these
$140m + 9$	$140m + 27$	$70m + 1$
$140m + 81$	$140m + 103$	$70m + 3$
$140m + 29$	$140m + 87$	$70m + 9$
$140m + 121$	$140m + 83$	$70m + 11$
$140m + 109$	$140m + 47$	$70m + 13$
$140m + 11$	$140m + 33$	$70m + 17$
$140m + 99$	$140m + 17$	$70m + 27$
$140m + 51$	$140m + 13$	$70m + 29$
$140m + 39$	$140m + 117$	$70m + 33$
$140m + 71$	$140m + 73$	$70m + 39$
$140m + 79$	$140m + 97$	$70m + 47$
		$70m + 51$

### Theorem 40.

All the prime divisors of numbers contained in any of the forms

$$\begin{array}{ll} aa + 30bb; & 2aa + 15bb; \\ 3aa + 10bb; & 5aa + 6bb \end{array}$$

are either 2, or 3, or 5, or are contained in one of the following formulas.

$$\begin{array}{ll} 120m + 1; & 120m + 11 \\ 120m + 13; & 120m + 23 \\ 120m + 49; & 120m + 59 \\ 120m + 37; & 120m + 47 \\ 120m + 17; & 120m + 67 \\ 120m + 101; & 120m + 31 \\ 120m + 113; & 120m + 43 \\ 120m + 29; & 120m + 79 \end{array}$$

These theorems will suffice to formulate the following notes, from which the nature of the divisors of formulas of the type  $paa + qbb$  will be examined thoroughly.

### Note 1.

The form  $paa+qbb$  has no divisors but that which is at once a divisor of  $aa+pqbb$ . The rationale for this is clear; for were a number a divisor of the form  $paa+qbb$ , the same then divides the form  $ppaa+pqbb$ , which is  $aa+pqbb$ , by putting  $a$  in place of  $pa$ . Thus it suffices to consider the single form  $aa+Nbb$ ; from this rationale the divisors for  $paa+qbb$  are completed.

### Note 2.

Among the prime numbers which divide any of the numbers contained in the form  $aa+Nbb$ , the prime 2 occurs. For if  $N$  were an odd number with  $a$  and  $b$  taken as odd numbers, the form  $aa+Nbb$  will be divisible by 2; and if  $N$  were an even number, with  $a$  taken to be even, the form again will be divisible by 2. Then indeed this number  $N$  will be able to be a divisor of the form  $aa+Nbb$ , which by taking  $a=N$  is clear.

### Note 3.

All the remaining prime divisors of the form  $aa+Nbb$  are thus able to be expressed as  $4Nm+\alpha$ ; moreover in turn, all prime numbers contained in the form  $4Nm+\alpha$  are at once divisors of the form  $aa+Nbb$ . In addition, if the expression  $4Nm+\alpha$  permits divisors of the form  $aa+Nbb$ , then no number of the form  $4Nm-\alpha$  will be able to be a divisor of any number contained in the form  $aa+Nbb$ .

### Note 4.

It will be moreover that the particular values of  $\alpha$  depend on the character of the number  $N$ ; and indeed always unity will be among the values for  $\alpha$ . Then also, because prime numbers are being sought for in the formula  $4Nm+\alpha$ , it is clear that no even number nor any number which has a common divisor with  $N$  can constitute a value of  $\alpha$ .

### Note 5.

As well, all the values of  $\alpha$  will be less than  $4N$ , for if they were greater, by decreasing the number  $m$ , ones less than  $4N$  can be obtained. Hence the values of  $\alpha$  will be odd numbers less than  $4N$ , and also prime to  $N$ . But indeed not all of these odd numbers prime to  $N$  will furnish suitable values for  $\alpha$ , but half of them are excluded by the rule that if  $x$  were a value of  $\alpha$ , then  $-x$ , that is  $4N-x$ , may not be a value of it; and in turn if  $x$  were not a value of  $\alpha$ , then  $4N-x$  will be certain to be a value of it.

## Note 6.

So that  $4Nm + \alpha$  will contain all the prime divisors of the formula  $aa + Nbb$ , a value of this  $\alpha$  will be defined in the following way. Were  $p, q, r, s$ , etc. distinct prime numbers, excepting 2, which will be considered separately, then

if it were	a value of $\alpha$ will be
$N = 1$	1
$N = 2$	2
$N = p$	$p - 1$
$N = 2p$	$p - 1$
$N = 2p$	$2(p - 1)$
$N = pq$	$(p - 1)(q - 1)$
$N = 2pq$	$2(p - 1)(q - 1)$
$N = pqr$	$(p - 1)(q - 1)(r - 1)$
$N = 2pqr$	$2(p - 1)(q - 1)(r - 1)$
etc.	

## Note 7.

Moreover, in the same way that unity always appears among the values of  $\alpha$ , thus too any number which is an odd square and relatively prime to  $N$  shall have a place among the values of  $\alpha$ . For by putting  $b$  as the even number  $2c$ , the formula would be  $aa + 4Ncc$ , which if it were a prime number must be contained in the expression  $4Nm + \alpha$ . Therefore  $\alpha$  will be a residue of  $aa$ , because  $aa$  remains when this is divided by  $4N$ . In a similar way, among the values of  $\alpha$  all numbers of the form  $aa + N$  should appear, that is which remain from division by  $4N$ ; for by putting  $b = 2c + 1$  it would be  $aa + Nbb = aa + N + 4N(cc + c)$ , which, if it were a prime number, gives that  $aa + N$  is a value of  $\alpha$ .

## Note 8.

It can be understood as well that if  $x$  were a value of  $\alpha$ , then too  $xx$  (which indeed is clear from the preceding) and all higher powers of  $x$ , so that  $x^\mu$  itself should have a place among the values of  $\alpha$ . Then, if aside from  $x$  also  $y$  were a value of  $\alpha$ , then too  $xy$  and in general  $x^\mu y^\nu$  gives a value of  $\alpha$ . Certainly if  $x^\mu y^\nu$  were greater than  $4N$ , by dividing this the remainder will be a value of  $\alpha$ . In a similar way, if in addition  $z$  were a value of  $\alpha$ , then further  $x^\mu y^\nu z^\xi$  will be a value of  $\alpha$ . And then from this inquiry, from one or several values of  $\alpha$  all the other values can be found by easy work.

## Note 9.

Were  $x$  some number prime to  $4N$  and less than it, then either  $+x$  or  $-x$  will be a value of  $\alpha$ . If therefore  $x$  were a prime number, from the following table it can be understood whether the case  $+x$  or  $-x$  for the value of  $\alpha$  obtains.

If	it will be
$N = 3n - 1$	$\alpha = +3$
$N = 3n + 1$	$\alpha = -3$
$N = \begin{cases} 5n + 1 \\ 5n + 4 \end{cases}$	$\alpha = +5$
$N = \begin{cases} 5n + 2 \\ 5n + 3 \end{cases}$	$\alpha = -5$
$N = \begin{cases} 7n + 3 \\ 7n + 5 \\ 7n + 6 \end{cases}$	$\alpha = +7$
$N = \begin{cases} 7n + 1 \\ 7n + 2 \\ 7n + 4 \end{cases}$	$\alpha = -7$
$N = \begin{cases} 11n + 2 \\ 11n + 6 \\ 11n + 7 \\ 11n + 8 \\ 11n + 10 \end{cases}$	$\alpha = +11$
$N = \begin{cases} 11 + 1 \\ 11n + 3 \\ 11n + 4 \\ 11n + 5 \\ 11n + 9 \end{cases}$	$\alpha = -11$

If a prime number is given, whether the sign  $+$  or  $-$  obtains for the value of  $\alpha$  will thus be investigated. Both cases shall be pursued, the one in which the given prime number is of the form  $4u + 1$ , the other in which it is of the form  $4u - 1$ . In the first case it will be  $\alpha = +(4u + 1)$  if it were  $N = (4u + 1)n + tt$ , or  $\alpha = -(4u + 1)$  if it were  $N \neq (4u + 1)n + tt$ . In the latter case however it will be  $\alpha = +(4u - 1)$  if it were  $N \neq (4u - 1)n + tt$  or  $\alpha = -(4u - 1)$  if  $N = (4u - 1)n + tt$ . Here it should be noted that in the way as the sign  $=$  denotes equality, so the sign  $\neq$  denotes the impossibility of equality. But if it were moreover in both cases  $N = (4u \pm 1)n + s$ , it will also be  $N = (4u \pm 1)n + s^\nu$ , where  $\nu$  denotes a certain integral number, from which such a table for any prime number may be constructed without effort.

### Note 10.

Since  $4Nm + 1$  is among the forms of the prime divisors of  $aa + Nbb$ , the expression  $aa + Nbb$  will not be able to be divided by any number which is contained in the form  $4Nm - 1$ . In a similar way should  $4Nm + tt$  exhibit the form of divisors of the expression  $aa + Nbb$ , it follows that no number of the type  $4Nm - tt$  will be able to be a divisor of any number contained in the form  $aa + Nbb$ , whenever  $a$  and  $b$  are relatively prime numbers. Then from this impossibility will follow this equation  $(4Nm - tt)u = aa + Nbb$ , and thus it will be  $4Nmu - ttu - Nbb = aa$ , if indeed the numbers  $4Nmu - ttu$  and  $Nbb$  were relatively prime, because with certainty it follows, if  $b = 1$  and  $t = 1$ , that this obtains.

### Corollary.

No number contained in the form  $4abc - b - c$  is ever able to be a square.

### Note 11.

If  $N$  were a number of the form  $4n - 1$ , then the forms of the divisors are reduced to less than twice the number, so that they may be comprised in the form  $2Nm + \alpha$ . Indeed if  $4Nm + \alpha$  were a form of divisors, then too  $4Nm + 2N + \alpha$  will be a form of divisors. In this way were  $2Nm + tt$  a form of divisors, it follows that no number  $4Nm - tt$  can be a divisor of the form  $aa + Nbb$ . Thus it will be  $(2Nm - tt)u = aa + Nbb$ ; from this development  $N = 4n - 1$  arises.

### Corollary.

No number of the form  $2abc - b - c$ , if either  $b$  or  $c$  are odd numbers of the form  $4n - 1$ , is ever able to be a square.

### Note 12.

If  $N$  we an odd number of the form  $4n + 1$  or also an oddly even number, then the forms of the divisors will not be able to be reduced to less than twice the number. Indeed if  $4Nm + \alpha$  were a divisor of the form  $aa + Nbb$  then  $4Nm + 2N + \alpha$  will not be able to be a divisor of the same form. Thus  $2(2m + 1)N + tt$  will not be a divisor of the form  $aa - Nbb$ , and thus the equation  $(2(2m + 1)N + tt)u = aa - Nbb$  will be impossible if indeed  $a$  and  $b$  were relatively prime numbers: and  $N$  were an odd number of the form  $4n + 1$  or an oddly even number. From which follows this

## Corollary.

No number of the form  $2abc - b + c$ , with  $a$  arising as an odd number and  $b$  either oddly even or odd of the form  $4n + 1$ , is ever able to be a square.

## Scholion 1.

These show that the character of the divisors of the formulas  $aa + Nbb$  has been developed sufficiently, and at once take care of all the forms of divisors which are to be found without burden, by which having noted the forms of the numbers become known which are never able to present divisors for the formula  $aa + Nbb$ . Therefore by this all the values of  $N$  may be known, either prime or composite; it remains however that we pursue the case in which  $N$  denotes a negative number, either prime or composite; indeed it is clear for the formula  $paa - qbb$  to have no divisor which is not a divisor of  $aa - pqbb$ , that is  $pqaa - bb$ , from which it suffices to pursue just the forms  $aa - Nbb$ .

## Theorem 41.

All the prime divisors of numbers contained in the form  $aa - bb$  are either 2 or  $4m \pm 1$ , namely no number is permitted which is not a divisor of two different squares. In turn moreover all numbers except those which are oddly even are themselves the difference of two squares.

## Theorem 42.

All the prime divisors of numbers contained in the form  $aa - 2bb$  are either 2 or of the form  $8m \pm 1$ . And all prime numbers of the form  $8m \pm 1$  are contained infinitely many ways in the form  $aa - 2bb$ .

## Theorem 43.

All the prime divisors of numbers contained in the form  $aa - 3bb$  are either 2 or 3 or are of the form  $12m \pm 1$ . And also in turn all prime numbers of this type are contained at once in the form  $aa - 3bb$  or in the form  $3aa - bb$  infinitely many ways.

## Theorem 44.

All the prime divisors of the form  $aa - 5bb$  are either 2 or 5 or are contained

$$\begin{array}{l|l} \text{in one or the other of these forms} & \text{or in such a one} \\ 20m \pm 1 & 20m \pm 9 \quad | \quad 10m \pm 1. \end{array}$$

And all prime numbers contained in these forms are at once divisors of the form  $aa - 5bb$ .

### Theorem 45.

All the prime divisors of the form  $aa - 7bb$  are either 2 or 7 or are contained in one of the following forms:

$$28m \pm 1; \quad 28m \pm 3; \quad 28m \pm 9$$

and also in turn all prime numbers contained in these forms are divisors of the form  $aa - 7bb$ .

### Theorem 46.

All the prime divisors of the form  $aa - 11bb$  are either 2 or 11 or are contained in one of the following forms

$$44m \pm 1; \quad 44 \pm 5; \quad 44m \pm 7; \quad 44m \pm 9; \quad 44m \pm 19$$

and also in turn all prime numbers contained in these forms are at once divisors of the form  $aa - 11bb$ ; this reciprocity holds in all the following theorems.

### Theorem 47.

All the prime divisors of the form  $aa - 13bb$  are either 2 or 13 or are contained in one of the following forms:

$$\begin{array}{ll|l} 52m \pm 1 & 52m \pm 3 & \text{which are reduced to these} \\ 52m \pm 9 & 52m \pm 25 & 26m \pm 1 \\ 52m \pm 23 & 52m \pm 17 & 26m \pm 3 \\ & & 26 \pm 9. \end{array}$$

### Theorem 48.

All the prime divisors of numbers of the form  $aa - 17bb$  are either 2 or 17, or are contained in one of the following forms:

$$\begin{array}{ll|l} 68m \pm 1 & 68m \pm 9 & \text{which are reduced to these} \\ 68m \pm 13 & 68m \pm 19 & 34m \pm 1 \\ 68m \pm 33 & 68m \pm 25 & 34m \pm 9 \\ 68m \pm 21 & 68m \pm 15 & 34m \pm 13 \\ & & 34m \pm 15. \end{array}$$

### Theorem 49.

All the prime divisors of numbers of the form  $aa - 19bb$  are either 2 or 19 or are contained in one of the following forms

$$\begin{array}{lll} 76m \pm 1 & 76m \pm 3 & 76m \pm 9 \\ 76m \pm 27 & 76m \pm 5 & 76m \pm 15 \\ 76m \pm 31 & 76m \pm 17 & 76m \pm 25. \end{array}$$

### Theorem 50.

All the prime divisors of numbers of the form  $aa - 6bb$  are either 2 or 3 or are contained in one of these forms:

$$24m \pm 1; \quad 24 \pm 5.$$

### Theorem 51.

All the prime divisors of numbers of the form  $aa - 10bb$  are either 2 or 3 or are contained in these forms:

$$\begin{array}{ll} 40m \pm 1 & 40m \pm 3 \\ 40m \pm 9 & 40m \pm 13. \end{array}$$

### Theorem 52.

All the prime divisors of numbers of the form  $aa - 14bb$  are either 2 or 7 or are contained in these forms

$$\begin{array}{lll} 56m \pm 1 & 56m \pm 5 & 56m \pm 25 \\ 56m \pm 13 & 56m \pm 9 & 56m \pm 11 \end{array}$$

### Theorem 53.

All the prime divisors of numbers of the form  $aa - 22bb$  are either 2 or 11 or are contained in these forms

$$\begin{array}{lll} 88m \pm 1 & 88m \pm 3 & 88m \pm 9 \\ 88m \pm 27 & 88m \pm 7 & 88m \pm 21 \\ 88m \pm 25 & 88m \pm 13 & 88m \pm 39 \\ & & 88m \pm 29. \end{array}$$

### Theorem 54.

All the prime divisors of numbers of the form  $aa - 15bb$  are either 2 or 3 or 5 or are contained in these forms:

$$60m \pm 1; \quad 60m \pm 7; \quad 60m \pm 11; \quad 60m \pm 17.$$

### Theorem 55.

All the prime divisors of numbers of the form  $aa - 21bb$  are either 2 or 3 or 7 or are contained in these forms:

$$\begin{array}{cc|l} 84m \pm 1 & 84m \pm 5 & \text{which are reduced to these} \\ 84m \pm 25 & 84m \pm 41 & 42m \pm 1 \\ 84m \pm 37 & 84m \pm 17 & 42m \pm 5 \\ & & 42m \pm 17 \end{array}$$

### Theorem 56.

All the prime divisors of numbers of the form  $aa - 33bb$  are either 2 or 3, or 11 or are contained in these forms

$$\begin{array}{cc|l} 132m \pm 1 & 132m \pm 17 & \text{which are reduced to these} \\ 132m \pm 25 & 132m \pm 29 & 66m \pm 1 \\ 132m \pm 35 & 132m \pm 65 & 66m \pm 17 \\ 132m \pm 49 & 132m \pm 41 & 66m \pm 25 \\ 132m \pm 37 & 132m \pm 31 & 66m \pm 29 \\ & & 66m \pm 31 \end{array}$$

### Theorem 57.

All the prime divisors of numbers of the form  $aa - 35bb$  are either 2 or 5 or 7 or are contained in these forms:

$$\begin{array}{ccc} 140m \pm 1 & 140m \pm 9 & 140m \pm 59 \\ 140m \pm 29 & 140m \pm 19 & 140m \pm 31 \\ 140m \pm 13 & 140m \pm 23 & 140m \pm 67 \\ 140m \pm 43 & 140m \pm 33 & 140m \pm 17 \end{array}$$

### Theorem 58.

All the prime divisors of numbers of the form  $aa - 30bb$  are either 2 or 3 or 5 or are contained in these forms

$$\begin{array}{lll} 120m \pm 1 & 120m \pm 13 & 120m \pm 49 \\ 120m \pm 37 & 120m \pm 7 & 120m \pm 29 \\ 120m \pm 17 & 120m \pm 19 & \end{array}$$

### Theorem 59.

All the prime divisors of numbers of the form  $aa - 105bb$  are either 2 or 3 or 5 or 7 or are contained in these forms

$420m \pm 1$	$420m \pm 13$	which are reduced to these
$420m \pm 169$	$420m \pm 97$	$210m \pm 1$
$420m \pm 23$	$420m \pm 121$	$210m \pm 13$
$420m \pm 107$	$420m \pm 131$	$210m \pm 23$
$420m \pm 109$	$420m \pm 157$	$210m \pm 41$
$420m \pm 59$	$420m \pm 73$	$210m \pm 53$
$420m \pm 101$	$420m \pm 53$	$210m \pm 59$
$420m \pm 151$	$420m \pm 137$	$210m \pm 73$
$420m \pm 89$	$420m \pm 103$	$210m \pm 79$
$420m \pm 79$	$420m \pm 187$	$210m \pm 89$
$420m \pm 41$	$420m \pm 113$	$210m \pm 97$
$420m \pm 209$	$420m \pm 197$	$210m \pm 101$
		$210m \pm 103$

### Note 13.

Therefore all the prime divisors of numbers contained in the form  $aa - Nbb$  are either 2, are divisors of the number  $N$ , or are comprised in the form  $4Nm \pm \alpha$ . And if  $4Nm + \alpha$  were a form of divisors, then also  $4Nm - \alpha$  would be a form of divisors: on the other hand, for the forms  $aa + Nbb$ , if  $4Nm + \alpha$  were the form of a divisor then this form does not permit a divisor  $4Nm - \alpha$ .

### Note 14.

Therefore by taking  $4Nm \pm \alpha$  as the general form of divisors of numbers contained in the expression  $aa - Nbb$ , the letter  $\alpha$  signifies many different numbers; unity is always contained among these, then indeed from the discussion on prime divisors, among the values for  $\alpha$  there will be no even number and no divisor of the number  $N$ . Then moreover it is clear that all the values of  $\alpha$  can be arranged such that each is less than  $2N$ . For if  $4Nm + 2N + b$  were a divisor, then by putting  $m - 1$  in place of  $m$ ,  $4Nm - (2N - b)$  will be a divisor. Therefore

the values of  $\alpha$  will be odd numbers prime to  $N$ , less than  $2N$ ; of all the odd numbers prime to  $N$  and less than  $2N$ , exactly half will give suitable values for  $\alpha$ , while the remaining exhibit formulas in which no divisor whatsoever is contained. Indeed always just as many formulas of divisors will be obtained as those which are not permissible, except for the single case in which  $N = 1$ .

### Note 15.

For the number of values of  $\alpha$  attained for the formula of the divisors of  $4Nm \pm \alpha$ , since from the varying of the sign each formula is doubled, the same rule prevails which I gave above in note 6. For in the last theorems, in which it was  $N = 105, 3, 5, 7$ , the number of values of  $\alpha$  will be equal to 2, 4, 6 = 48, or as each formula would be a pair, the number of formulas would be 24, just as many indeed as we have exhibited.

### Note 16.

Moreover just as unity always appears among the values of  $\alpha$ , so to any square numbers which is prime with  $4N$  provides a suitable value for  $\alpha$ . For by putting  $b = 2c$ , the formulae  $aa - Nbb$  becomes  $aa - 4Ncc$  or  $4Ncc - aa$ , from which it is clear for any square number  $aa$ , which is prime to  $4N$ , to exhibit a suitable value for  $\alpha$ , namely by taking the residue which remains from the division of  $aa$  by  $4N$ . In a similar way by putting  $b = 2c + 1$ , the formula  $Nbb - aa$  becomes  $4N(cc + c) + N - aa$ , from which too all the numbers  $N - aa$  or  $aa - N$ , which are prime to  $4N$ , provide suitable values for  $\alpha$ . Then too it is to be noted that if  $x, y, z$  are values of  $\alpha$ , then too for  $x^\mu y^\nu z^\xi$ , and any such products which result from numbers  $x, y, z$  and any powers, to be able to exhibit values of  $\alpha$ ; from which knowing one or several values of  $\alpha$ , all can be obtained with easy work.

### Note 17.

So that the way in which the values for the letter  $\alpha$  are perpetually obtained may be made clear, the following table is considered, similarly as that in note 9

is had.

It will be	if it were
$\alpha = 3$	$N = 3n + 1$
$\alpha \neq 3$	$N = 3n - 1$
$\alpha = 5$	$N = 5n \begin{cases} +1 \\ -1 \end{cases}$
$\alpha \neq 5$	$N = 5n \begin{cases} +2 \\ -1 \end{cases}$
$\alpha = 7$	$N = 7n \begin{cases} +1 \\ +2 \\ -3 \end{cases}$
$\alpha \neq 7$	$N = 7n \begin{cases} -1 \\ -2 \\ +3 \end{cases}$
$\alpha = 11$	$N = 11n \begin{cases} +1 \\ -2 \\ +3 \\ +4 \\ +5 \end{cases}$
$\alpha \neq 11$	$N = 11n \begin{cases} -1 \\ +2 \\ -3 \\ -4 \\ -5 \end{cases}$
$\alpha = 13$	$N = 13n \begin{cases} +1 \\ -1 \\ +3 \\ -3 \\ +4 \\ -4 \end{cases}$
$\alpha \neq 13$	$N = 13n \begin{cases} +2 \\ -1 \\ +5 \\ -5 \\ +6 \\ -6 \end{cases}$

### Note 18.

From this table therefore, prime numbers, which provide suitable values for  $\alpha$ , are easy able discerned, and inappropriate ones can be rejected. Namely for a given prime number  $p$ , all the square numbers in the form  $pn + \theta$  are able to be comprehended, which follow by taking square numbers for  $\theta$ , that is residues, which remain after the division of squares by  $p$ . Thereby if  $N$  were a number of the type  $pn + tt$ , then among the forms of the divisors of  $4Nm \pm \alpha$  are the formulas  $aa - Nbb$ , that is  $Nbb - aa$ , and it will be had that  $\alpha = p$ , but if on the other hand the number  $N$  were not contained in the form  $pn + tt$ , then no number contained in the form  $4Nm \pm p$  will be able to be a divisor of any number of the form  $aa - Nbb$ .

### Note 19.

If  $N$  is an odd number of the form  $4n + 1$  then the forms of the divisors of the expression  $aa - Nbb$  are able to be reduced to twice as small as  $4Nm \pm \alpha$ , so that they can be presented in this way:  $2Nm \pm \alpha$ . Indeed in this case, if  $4Nm \pm \alpha$  were a form of divisors, then too  $4Nm \pm (2N - \alpha)$  will be a form of divisors, in such a way as with the case  $N = 13$ , where  $52m \pm 3$  was a form of divisors of  $aa - 13bb$ , and likewise  $52m \pm 23$  will be a form of divisors.

### Note 20.

But if moreover  $N$  were an oddly even number, or an odd number of the form  $4n - 1$ , then this reduction of dividing the form into two smaller ones does not succeed. Indeed in this case of the formula  $aa - Nbb$  if  $4Nm \pm \alpha$  were the form of divisors, then  $4Nm \pm (2N - \alpha)$  will not be such, that is: no number contained in the form  $2(2m \pm 1)N \pm \alpha$  will be a divisor of any number of the form  $aa - Nbb$ . Therefore by putting  $\alpha = tt$ , it will be

$$(2(2m \pm 1)N \pm tt)u \neq aa - Nbb,$$

of which the following is a consequence.

### Corollary.

No number contained in the form  $2abc \pm c + b$  is able to be a square, if  $a$  were an odd number, and  $b$  were either oddly even number or odd of the form  $4n - 1$ .

### Scholion 2.

Innumerable special formulas are able to be deduced from the above, which are never able to be made into squares. We considered the first form  $aa + Nbb$  and it would be a formula of the type  $4Nm + A$ , so that no number contained in this form is able to be a divisor of the form  $aa + Nbb$ . It will therefore be  $aa + Nbb = (4Nm + A)u$ , where the sign  $\neq$  denotes that the equation is impossible, from which it follows that  $aa \neq 4Nmu + Au - Nbb$ . Were it  $b = Ac$  it would be  $aa \neq 4Nmu + Au - NAAcc$ . It is put in turn  $u = NAcc + d$ , and it will be  $aa \neq 4NNAmcc + 4Nmd + Ad$ . Were it  $d = 4NNn$  it will be  $aa \neq 16N^3mn + 4NNAmcc + 4NNAn$ . Were this formula divided by the square  $4NN$  and it put  $c = 1$ , it will be the formula  $4Nmn + Am + An$ , which is never able to be a square if the form  $aa + Nbb$  is not able to be divided by any number contained in the form  $4Nm + A$ . From the above theorems therefore we gather that no number which is contained in any of the following expressions is able to be made into a square.

$4mn - (m + n)$	$4mn + 3(m + n)$
$8mn - (m + n)$	$8mn + 7(m + n)$
$8mn - 3(m + n)$	$8mn + 5(m + n)$
$12mn - (m + n)$	$12mn + 11(m + n)$
$12mn - 7(m + n)$	$12mn + 5(m + n)$
$20mn - (m + n)$	$20mn + 19(m + n)$
$20mn - 3(m + n)$	$20mn + 17(m + n)$
$20mn - 7(m + n)$	$20mn + 13(m + n)$
$20mn - 9(m + n)$	$20mn + 11(m + n)$
$24mn - (m + n)$	$24mn + 23(m + n)$
$24mn - 5(m + n)$	$24mn + 19(m + n)$
$24mn - 7(m + n)$	$24mn + 17(m + n)$
$24mn - 11(m + n)$	$24mn + 13(m + n)$
$28mn - (m + n)$	$28mn + 27(m + n)$
$28mn - 9(m + n)$	$28mn + 19(m + n)$
$28mn - 11(m + n)$	$28mn + 17(m + n)$
$28mn - 15(m + n)$	$28mn + 13(m + n)$
$28mn - 23(m + n)$	$28mn + 5(m + n)$
$28mn - 25(m + n)$	$28mn + 3(m + n)$

etc.

It is to be noted also that in the formulas in the second column the numbers  $m$  and  $n$  for the coefficient of  $m + n$  must be prime. This condition requires the restriction that we established in the beginning, that in the form  $aa + Nbb$  the numbers  $a$  and  $b$  should be relatively prime numbers: for unless this condition is observed, any number whatsoever would be able to be a divisor of this form. As well, it is clear from the preceding that with this condition observed, if  $4Nmn - A(m + n)$  is not able to be a square, then likewise it is wide open that  $4Nmn - A(m + n) \pm 4Np(m + n)$  is not able to be a square.

### Scholion 3.

Now the expression  $aa - Nbb$  shall be considered when it has no divisor contained in the formula  $4Nm \pm A$ . It will therefore be  $aa - Nbb \neq 4Nmu \pm Au$  or  $aa \neq 4Nmu + NAA \pm Au$ . It may be put  $NA \pm u = d$ , or  $u = \pm d \pm NA$ , and it will be  $aa \neq \pm 4Nmd + 4NNA m + Ad$ ; it becomes  $d = \pm 4NNn$  and it will become  $16N^3mn \mp 4NNA m \pm 4NNA n \neq aa$ , from which it follows that no number contained in the formula  $4Nmn \pm A(m - n)$  can be a square. Neither therefore will any number contained in the expression  $4Nmn \pm A(m - n) \pm 4Np(m - n)$  be able to be a square, but only if the condition recalled from earlier is observed, that  $a$  and  $b$  are relatively prime numbers. Consequently from the preceding theorems the following formulas are deduced, which are never able to permit square numbers.

$8mn \pm 3(m - n)$	$8mn \pm 5(m - n)$
$12mn \pm 5(m - n)$	$12mn \pm 7(m - n)$
$20mn \pm 3(m - n)$	$20mn \pm 17(m - n)$
$20mn \pm 7(m - n)$	$20mn \pm 13(m - n)$
$24mn \pm 7(m - n)$	$24mn \pm 17(m - n)$
$24mn \pm 11(m - n)$	$24mn \pm 13(m - n)$
$28mn \pm 5(m - n)$	$28mn \pm 23(m - n)$
$28mn \pm 11(m - n)$	$28mn \pm 17(m - n)$
$28mn \pm 8(m - n)$	$28mn \pm 15(m - n)$

etc.

by attending to which moreover it is easily seen that both of the numbers  $m$  and  $n$  must be prime to the coefficient of  $(m - n)$ : for otherwise, if the letters given in the formula  $12mn \pm 5(m - n)$  were put  $m = 5p$  and  $n = 5q$ , it would follow that  $12 \cdot 25pq \pm 25(p - q)$  and thus the formula  $12pq \pm (p - q)$  ought to be a square, which however is false.