THEOREM 16. *The multiplication algebra of a pure Riemann matrix of the first kind is either a field $R(\theta)$ or a generalized quaternion algebra over such a field.*

[1] "On the Wedderburn Norm Condition for Cyclic Algebras," *Bull. Am. Math. Soc.*, **37**, pp. 301–312 (1931).

[2] "Quadratic Formen im algebraischen Zahlkorpern," *Jour. reine angewandete Math.*, **153**, pp. 113–130 (1923).

[3] *Cf.* the author's "New Results in the Theory of Normal Division Algebras," *Trans. Am. Math. Soc.*, **32**, pp. 171–195 (1930).

[4] See the author's paper, "The Structure of Pure Riemann Matrices with Non-Commutative Multiplication Algebras," *Rend. Circ. Mat. Pal.*, **55**, pp. 57–115 (1931), for the definitions and properties of Riemann matrices of the two kinds.

ELECTRIC CONDUCTIVITY AND OPTICAL ABSORPTION IN METALS, ONCE MORE

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The question how many free electrons there are per cu. cm. of a metal is an important, if not indeed a vital, one in theories of electric conduction. Optical evidence has for twenty years been generally regarded as showing the number of free electrons to be about the same as the number of atoms. This belief, however, always assumes, so far as I have observed, that the free electrons are the only conduction electrons. But what happens to the optical argument if we once admit that electrons which are not "free," in the ordinary sense of the word, may have a part in conduction, by going directly from atoms to adjacent ions? I shall try in this paper to go somewhat more deeply into this question than I have done previously. I shall pay especial attention to the investigations, theoretical and experimental, of Meier[1] and of Hagen and Rubens.[2]

Meier uses the formulas of Voigt, but these are consistent with those of Drude, and I shall begin with Drude's general equation of motion for an electron, whether "free" or "bound," within a metal. It is equation (1) of Chapter 5 in his "Theory of Optics."

$$m \frac{\partial^2 \xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta} \xi - re^2 \frac{\partial \xi}{\partial t}. \qquad (1)$$

The term eX is the impressed electric force corresponding to the equation

$$X = X_0 \cos nt. \qquad (2)$$

The next term represents the elastic force tending to carry the electron back to its zero position. The final term represents the energy-absorbing, heat-producing, resistance encountered by the electron in its motion.

This resistance term is of a highly artificial character, for it represents the moving electron as encountering a continuous resistance proportional to the velocity. This conception of resistance, though fictitious, is convenient and permissible for the motion of electrons through steady fields, for it accords with the rate of production of joulean heat in such motions. But to use the same conception for the motion of electrons in the rapidly alternating fields of light-waves is a hazardous procedure which must affect with some measure of doubt all conclusions to which it leads.

Meier infers from his experiments and calculations that a very appreciable amount of energy is absorbed by the "bound" electrons, saying that in many cases they are of "prevailing influence." I do not undertake to criticize any of his conclusions regarding these electrons. But I raise the question whether the conduction electrons of which his experiments give evidence are necessarily, or probably, free electrons. I shall undertake in this paper to indicate the way in which the dual theory of conduction can deal with the matter before us.

Drude adapts equation (1) to the case of a free electron by making the second term, the elasticity term, in the second member zero. He thus gets

$$m \frac{\partial^2 \xi}{\partial t^2} = eX - re^2 \frac{\partial \xi}{\partial t}. \qquad (3)$$

Now the dual theory rejects this equation as applied to a free electron. In accordance with the usual conception of free-electron conduction it assumes that a free electron gains velocity, from an assisting electric field, during its free path, thus storing up increased kinetic energy to be turned into heat by collision at the end of the path. Moreover, this theory supposes the free electrons to be few, compared with the atoms, and their paths very long in duration compared with the period of a light-wave. Accordingly, many alternations of impulse will be applied to an electron during its path and these impulses will neutralize each other in their effect upon the velocity with which the electron impinges on an atom or an ion at the end of its path. The final term in equation (3) is meaningless for such a case. According to the dual theory the effect of the "free" electrons in optical absorption is negligible. Equation (3) should be interpreted as applying to the other class of conduction electrons, the "associated" electrons.

The elasticity term of equation (1) disappears for an associated electron, as it does for a free electron, but for a different reason. It disappears because the work done by elastic forces when an electron goes out from an atom to unite with an adjacent ion is zero, on the whole. As to the

final term in equation (3), the resistance term, it doubtless still gives an imperfect picture of the nature of resistance, but this picture is not so fatally false in this case as is that of a free electron. For the time of transit of an electron from an atom to a neighboring ion is probably of the same order of magnitude as the vibration-periods of "bound" electrons, which, according to the conclusions of Meier, are comparable with the periods of the light-waves used in his experiments. That is, the direction of the impressed electric field will not reverse many times, and may not reverse at all, during the period of an associated-electron "transit." Accordingly, the associated electrons can absorb energy from the field during their transits and generate heat at the expense of this energy when they make their impacts at the end of these transits.

Meier, studying the reflection of polarized light from various metals, found for each metal and for each wave-length of light used a value of n, the refractive index, and of κ, the coefficient of absorption. Using the nomenclature of Voigt, though with the fundamental ideas of Drude, he writes

$$2n^2\kappa = \frac{\rho_1 \nu_1'/\nu}{\nu^2 + \nu_1'^2} + \sum \frac{\rho\nu'\nu}{(\nu_0^2 - \nu^2)^2 + \nu'^2\nu^2}, \qquad (4)^3$$

and

$$n^2(1 - \kappa^2) = 1 - \frac{\rho_1}{\nu^2 + \nu_1'^2} + \sum \frac{\rho(\nu_0^2 - \nu^2)}{(\nu_0^2 - \nu^2)^2 + \nu'^2\nu^2}. \qquad (5)$$

In each of these equations the summation term has reference to the bound electrons. The term containing factors with the subscript 1 has to do with the conduction electrons.

Curves were plotted, from the observational data, showing $n^2\kappa$ and $n^2(1 - \kappa^2)$ as functions of the wave-length. Then came a laborious process of finding, by trial, values of ρ_1, ν_1', ρ, ν' and ν_0 which, when used in equations (4) and (5), would give values of $n^2\kappa$ and $n^2(1 - \kappa^2)$, according well with the observational curves already plotted for these quantities.

The quantity which especially interests us is ρ_1, defined as $4\pi N_1 e^2 \div m$, N_1 being the number of conduction electrons per cu. cm. of the metal. The process of evaluating N_1 numerically involved an estimate of $e \div m$ somewhat different from the one now accepted, but this difference is not important for our present purpose. Working by a method which he regards as unquestionably sound, Meier gets for each of the ten metals examined a value, p_1, for the number of conduction electrons per atom. This number varies from about 0.46, in copper, to about 2.22, in bismuth, the mean for all the metals examined being not far from 1.

Now in one aspect this average is in excellent agreement with the requirements of the dual theory of conduction; for it is natural to suppose that each atom can allow one of its electrons, and only one, to leave it

for transit to an adjacent ion. On the other hand, if Meier's results are to be interpreted as meaning that about one electron per atom, as a rule, is actually traveling at every instant in the direction of the current, this conception is not in accord with the picture of associated-electron conduction given by the dual theory. For this picture presents any one associated electron as quiescent, so far as conduction is concerned, for the greater part of the time, but sharing now and then, doubtless a vast number of times a second, in a sort of train movement through the metal. If there is any inconsistency between this view of the matter and Meier's conclusions regarding the number of conducting electrons, this inconsistency may perhaps reasonably be attributed to the highly artificial conception of resistance which is embodied in the equations used, a matter which has already been commented on in this paper.

But Meier estimated p by a second method, by taking, for each metal, the conductivity, per conduction electron, indicated by his investigation and finding how many electrons of this conductivity would be needed to account for the specific conductivity of the metal as measured with steady currents. He thus found values, which we will call p_2 values, that were notably larger in most cases than the p_1 values. He was surprised and puzzled by this fact but presently hit, somewhat doubtfully, on what is probably the correct explanation, for he remarks that "possibly a difference of damping constant—in conduction and oscillations plays a part here."

The present paper may be regarded as an endeavor to explain this difference which Meier, on the experimental evidence, surmised, but which he, with his conception of conductivity, had not anticipated and could not understand.

TABLE 1

METAL	p_1	p_2	$p_1 \div p_2$	λ_0 IN $\mu\mu$	MAX. $\frac{\lambda_0}{c}$	MIN. $\frac{\lambda_0}{c}$
Hg	2.13	2.19	0.97			
Bi	2.22	3.85	0.58	150		5×10^{-16}
Ni	0.66	3.85	0.17	305 to 150	1.02×10^{-15}	5×10^{-16}
Au	0.74	4.61	0.16	370 to 150	1.23×10^{-15}	5×10^{-16}
Ag	0.82	7.28	0.11	270, 245, 200	0.9×10^{-15}	6.7×10^{-16}
Co	1.08	10.8	0.10	525 to 150	1.75×10^{-15}	5×10^{-16}
Cu	0.46	5.9	0.078	500 to 200	1.67×10^{-15}	6.7×10^{-16}
Pt	1.33	27.5	0.048	600	2.00×10^{-15}	
Zn	0.89	18.9	0.047	750 to 150	2.50×10^{-15}	5.0×10^{-16}
Steel	1.32	32.6	0.040	750 to 200	2.50×10^{-15}	6.7×10^{-16}

Meier's material enables us to make table 1. The wave-lengths λ_0 are those corresponding to the frequency of the "chief characteristic vibrations" of the bound electrons. Accordingly, $\lambda_0 \div c$ is the vibration-period of a bound electron. The order of arrangement of the metals in this table is that given by Meier, who remarks that the less $p_1 \div p_2$ is for any metal, the greater, as a rule, is the value of maximum λ_0 for that

metal—that is, the greater the maximum vibration-period among the bound electrons of the metal. This observation seems to me important.

The fact that p_2 is greater than p_1 means that the conduction electrons are less effective for the alternating fields of the light-waves used than they are for steady fields. Can it be shown that such a difference should appear, with sufficiently short wave-lengths, if conduction is effected mainly by electron "transits" from an atom to an adjacent ion? An affirmative answer to this question seems to be indicated by the fact, noted by Meier, that the degree of failure, the amount of falling off, in the conductivity per electron in the alternating field of light-waves appears to have a pretty close relation to the natural vibration-period of the more loosely "bound" electrons. For it seems probable that the time occupied by an electron in "transit" is not very different from the vibration-period of these loosely bound electrons, since an outward swing of an outer electron would seem to be the natural beginning of a transit.

More than one authority has dealt with the ratio between steady-current conductivity and light-wave conductivity of metals. Thus Richardson, on page 432 of his "Electron Theory of Matter," gives a formula relating to this matter, crediting it to H. A. Wilson elaborating "a method originally given by Jeans;" but this formula is not in convenient shape for my use, as it does not explicitly contain the duration time of a "path." Thomson, on page 84 of his "Corpuscular Theory of Matter," says, "We can easily show that if k is the conductivity under steady forces, then when forces vary as sin nt the conductivity will be proportional to $k\,\frac{\sin^2 nT}{n^2T^2}$, where $2T$ is the interval between two collisions" [of a free electron with the atoms]. This formula I can readily adapt to my use by taking k to be the same as my "associated-electron conductivity," κ_a, and substituting duration of a "transit" for duration of a "path." Moreover, I let τ, instead of Thomson's $2T$, represent the duration of a transit. Thus I get for my present use $\kappa_a\,\frac{\sin^2 (1/2 n\tau)}{(1/2 n\tau)^2}$ as the adapted Thomson formula.

I must admit that I have never succeeded in my attempts to deduce Thomson's expression. Not being willing to use it blindly, I have invented a crude method of dealing with the problem in hand, a method which involves simplifying assumptions and can hardly be regarded as strictly accurate. This method has not given me a brief formula, and an exposition of it cannot well be attempted here, though it may be published in some future number of these PROCEEDINGS. Fortunately it gives numerical results agreeing, in all the cases where I have used it, to the third place with those derived from the adapted Thomson formula. In the following tabulation τ means the duration of a "transit," t_1 the com-

plete period of a light-wave; $\kappa_a' \div \kappa_a$ is the ratio of the reduced transit-conductivity to the normal, steady-current, transit conductivity.

TABLE 2

$t_1 \div \tau$	=	18	10	6	2	4/3	1
$\kappa_a' \div \kappa_a$	=	0.990	0.968	0.912	0.405	0.090	0

When τ is greater than t_1, the ratio $\kappa_a' \div \kappa_a$ will not necessarily be zero, but it will be small, at the most about 0.04, according to the Thomson formula.

Going back now to table 1 and remembering that, according to my conception, the $p_1 \div p_2$ of that table should be the same as the $\kappa_a' \div \kappa_a$ of table 2, we see that in order to account for the value $p_1 \div p_2 = 0.97$, the one given for mercury, the ratio $t_1 \div \tau$ should be about 10. Now the range of wave-length used with mercury was from $325\mu\mu$ to $630\mu\mu$, and accordingly the value of $t_1 \div \tau$, τ being constant and t_1 proportional to the wave-length, must have nearly doubled between the low and the high limits of t_1 for this range. At only one point could the value $t_1 \div \tau = 10$ have held. My theory would therefore lead me to expect a smaller value of $\kappa_a' \div \kappa_a$ near the short wave-length end than near the long wave-length end of the range used, though the difference might be too small for certain detection. Meier, however, made no allowance for any dependence of conductivity on wave-length. Moreover, he attributed none of the absorption in mercury to action of the bound electrons. In the case of this one metal, the conduction electrons, taken with their normal conductive power, were sufficient, he believed, to account for the observed facts, within the errors of observation and calculation. Nevertheless, an examination of the two curves which he gives for $n^2(1 - \kappa^2)$, as a function of λ, in the case of mercury (see his figure 40) shows that from wave-length $400\mu\mu$ to wave-length $325\mu\mu$ the two diverge in a way that may indicate a progressive failure of conductive power in the electrons.

As to the other metals dealt with, in all of which a reduced conductive power was indicated by the absorptive power attributed to the conduction electrons, it may be doubted whether the agreement between the experimental and the theoretical curves is close enough to prove the correctness of Meier's assumption that the absorptive power of these electrons is independent of the wave-length within the range of light employed. In many cases the two curves cross each other, sometimes more than once, a fact which at least suggests the possibility that closer agreement might have been obtained by assuming a variation of conductive absorptive power with variation of wave-length.

The smallest value of $p_1 \div p_2$ in table 1 is 0.04, for steel. This, according to table 2, corresponds to a value of $t_1 \div \tau$ lying between 4/3 and 1, but nearer the former. Let us call it x. The value of $t_1 \div \tau$ must

have been less than x for $\lambda = 225\mu\mu$ and greater than x for $\lambda = 600\mu\mu$. the limits of wave-length used with steel. It seems probable, then, that with a wave-length of 4μ, which is about 7 times $600\mu\mu$, the ratio $t_1 \div \tau$ would be near 10 and the ratio $\kappa_a' \div \kappa_a$ near 1. We may, then, conclude that all the metals listed in table 1 would have $\kappa_a' \div \kappa_a$ near 1 when $\lambda = 4\mu$.

Especial attention is directed to $\lambda = 4\mu$ because this was the short wave-length limit in the famous work of Hagen and Rubens[4] on absorption in metals. These investigators, using seven metals and several alloys, studied especially the quantity $(100 - R)$, in which 100 stands for the intensity of radiation falling at perpendicular incidence on a mirror of the metal and R represents the intensity of the radiation reflected. This quantity $(100 - R)$ is called the *penetrating radiation*. It has been shown by various writers[5] that Maxwell's original theory of radiation, when molecules and their motions are disregarded, leads to the general formula

$$(100 - R) = \frac{200}{\sqrt{\Lambda t_1}} \qquad (6)$$

where Λ is the electric conductivity of the metal in absolute electrostatic measure, and t_1 is the full period of the radiation wave in seconds. Letting

TABLE 3

	$\lambda = 4\mu$	$\lambda = 8\mu$	$\lambda = 12\mu$
Mean value of C for 4, 8, 12μ,	$C_4 = 19.4$	$C_8 = 13.0$	$C_{12} = 11.0$
Mean deviation from mean value of C	$\delta_4 = 21.0\%$	$\delta_8 = 14.5\%$	$\delta_{12} = 9.6\%$
Theoretical value of C' from eq. (7)	$C_4' = 18.25$	$C_8' = 12.9$	$C_{12}' = 10.54$

κ stand for the conductivity in the units employed by Jäger and Dissel-horst,[6] and putting λ for the wave-length in μ, Hagen and Rubens get from equation (6)

$$(100 - R)\sqrt{\kappa} = \frac{36.5}{\sqrt{\lambda}} = C, \qquad (7)$$

C being a quantity which, for a given value of λ, should be the same for all metals.

Working with wave-lengths 4μ, 8μ and 12μ, H. and R. obtained with the metals (except bismuth) which they studied, and five alloys, results agreeing fairly well with equation (7). They made the test in the following way: Multiplying the observed $(100 - R)$ of each material by $\kappa^{1/2}$, κ being the steady-current conductivity of the metal, and taking the mean of all the products, except that for bismuth, for each of the three wave-lengths, they called this mean C. From the second member, $36.5 \div \lambda^{1/2}$,

of equation (7) they got for each λ a value which they called C'. Table 3 is taken, with slight modifications of form, from their table on page 884.

I have certain observations to make relative to the numbers in this table. For the 4μ case $C \div C' = 1.063$. This ratio could have been made 1.000 by using in finding C a value k about 12 per cent less than the normal steady-current value. A natural interpretation of this fact would be that, in spite of the inference above drawn from the behavior of the metals studied by Meier, 4μ is a somewhat too short wave-length to show the full absorptive power to be expected from the conduction electrons. Confirmation of this view might seem to be found in the 8μ case, where $C \div C'$ is 1.008, and could have been made 1.000 by using in the calculation of C a value of κ only 1.6 per cent below the steady current value. But when we go on to the 12μ case we have $C \div C' = 1.044$, and to make it 1.000 a value of k nearly 9 per cent below the normal value would have been required.

It appears, then, that we do not find an unquestionable approach to the condition $C \div C' = 1.000$ as we increase the wave-length. The mean value of $C \div C'$, for the three wave-lengths, is 1.038, corresponding to a value of 1.077 for $\kappa \div \kappa'$, and there is a possibility that some such ratio as this would continue to much longer wave-lengths. If this should prove to be the case, the discrepancy might be explained in either of two ways. First, one might suppose that imperfection of the mirror surfaces of the metals made R too small and $(100 - R)$ too large. Second, some consideration might be given to the fact that the dual theory of conduction indicates a discrepancy in the direction noted and of something like the amount noted.

For it is to be remembered that, according to this theory, the κ that is effective in light absorption is not, even with long wave-lengths, the full steady-current value of the conductivity. It is merely what I call κ_a, and so it is κ_a, not the full κ, that should be used in calculating C. Excluding bismuth, because H. and R. exclude it in taking their mean values, I, from my own study of the electrical and thermal properties of 17 metals, including 2 alloys, find 1.096 as the mean value of $\kappa \div \kappa_a$ at 0° C. According to this, use of κ, instead of κ_a, as a factor in the value of C would make the ratio $C \div C'$ between 4 and 5 per cent greater than it should be. This is a little more than enough to account for the value, 1.038, found by H. and R. for the mean value of this ratio.

The metals and alloys of my list are not just the same as those used by H. and R., and I wish to guard against attaching too much importance to the numerical relations here noted. The main point which I hope to make in this connection is that the experimental evidence is not unfavorable to my conception of conduction.

A like remark is justified when we consider the peculiar behavior of

bismuth. The mean value of $C \div C'$ for bismuth, according to the work of H. and R., is excpetionally large, being 1.37. The value of $\kappa \div \kappa_a$ in bismuth is, according to my investigation, exceptionally large, being 1.51 at 0° C. Use of κ instead of κ_a would account for 1.23 as the value of $C \div C'$ in bismuth, to be compared with the mean value, 1.038, found by H. and R. for metals and alloys in general. H. Murmann (*Zeits. Z. Physik*, **54** (1929)) using wave-lengths ranging from 25μ to 110μ measured the amount of radiation actually transmitted by thin films of various metals, including bismuth. But his results have little bearing on the question here at issue, for neither the thickness nor the conductivity of a film was known, but only the product of these two quantities.

Summary

1. The resistance term in Drude's fundamental equation of motion of an electron within a metal is highly artificial, for light-wave fields, and any conclusions to which it leads are affected by some measure of doubt.

2. The form which this equation takes when it is applied to a conduction electron is quite as appropriate for an electron that passes directly from an atom to an adjacent ion as it is for a "free" electron.

3. The conduction electrons which Willi Meier (in 1909) found to be about as numerous as the atoms may reasonably be taken to be the "associated" electrons, the electrons that may execute "transits" from an atom to an adjacent ion.

4. The formula $k\frac{\sin^2 nT}{n^2T^2}$, given by J. J. Thomson on page 84 of his "Corpuscular Theory of Matter," when applied to "transit" conductivity gives the same results as those obtained in a different way by the author, indicating that when the ratio *wave-period to transit-period* drops from 10 to 1, the absorptive power of the transit electrons drops from 97% of its maximum value to zero.

5. Meier's results, obtained with wave-lengths ranging from $250\mu\mu$ to $670\mu\mu$, seem to indicate that for wave-lengths of 4μ or greater the transit electrons should have almost their full, steady-current, conductive power and a corresponding absorptive power.

6. Hagen and Rubens, using wave-lengths 4μ, 8μ and 12μ, got for metals and alloys in general results which seem to indicate that the absorption-conductivity is somewhat less than the steady-current conductivity at all three wave-lengths, the difference being about 12% at 4μ, about 1.6% at 8μ and about 9% at 12μ, with no clear evidence that it would disappear at greater wave-lengths. In the case of bismuth the difference is exceptionally large.

7. This difference, which is hardly to be accounted for by the ordinary theory of the relation between conduction and absorption, may be due to some imperfection of the experimental method used by H. and R. On the other hand, it is worth noting that the dual theory of conduction requires a difference of the kind and of the order of magnitude here observed. For this theory makes the paths of the few free electrons last so long that the rapidly alternating fields of even the longest light-waves would have no net effect on them. Therefore it confines the conduction-absorption power of metals to the action of the "associated," or "transit," electrons, and for metals in general the conductivity due to these electrons is about 8 or 10% less than the total conductivity. For bismuth κ_a is exceptionally small, about 67% of the total κ.

[1] Meier, *Inaugural Dissertation*, Barth, Leipsic, 1909.

[2] Hagen and Rubens, *Ann. d. Physik*, **11**, 873–901 (1903).

[3] Meier's paper gives $\rho\nu' \div \nu$ in the numerator within the summation sign, but I think it should be $\rho\nu'\nu$.

[4] Hagen and Rubens, *Ann. d. Physik*, **11**, 873–901 (1903).

[5] Loc. cit., p. 886.

[6] Their κ is the reciprocal of the resistance, in ohms, of a piece of the metal 1 m. long and 1 sq. m. in area of cross-section.

THE SPECTRAL ERYTHEMIC REACTION OF THE HUMAN SKIN TO ULTRA-VIOLET RADIATION

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Introduction.—At present the physician has neither a unit of dosage nor a meter for accurately measuring the amount of ultra-violet radiation used for healing purposes.

In the absence of an inanimate dosage-meter the patient is used as an indicator, and the dosage is estimated by the erythema produced on the inner forearm. This method is in common use as a guide in ultra-violet radiation therapy. Furthermore, in view of the wide variation in erythemic susceptibility of pigmented and unpigmented skin (brunette and blonde), and in view of the fact that irradiation cannot be continued safely beyond skin tolerance, it is highly probable that any unit of dosage or any inanimate dosage-meter, that may be adopted, will have to take this physiological effect into consideration.

Hence, in connection with the question of the unit of dosage, and particularly in connection with methods of standardizing the dosage, an