PRINCIPAL DIRECTIONS IN THE EINSTEIN SOLAR FIELD

By J. L. Synge

Department of Mathematics, University of Toronto

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In the Newtonian manifold of space-time there is at any point one principal direction, namely that for which the three space coördinates are stationary. In the hyperbolic space-time of the Special Relativity Theory, there are no principal directions, the only directions intrinsically definable being those forming the cone

$$- dx^2 - dy^2 - dz^2 + c^2dt^2 = 0.$$

In the previous paper (*Proc. N. A. S.*, p. 198) several types of principal directions have been defined for a Riemannian N-space, but the field equations

$$G_{mn} = 0$$

render indeterminate the principal directions of Types I, II and III as there defined. Eisenhart (*Proc. N. A. S.*, Vol. 8, No. 2, p. 24) has shown that those of Type III are indeterminate for all three forms of Einstein space free from matter. However, it would appear that those of Type IV might exist in the Solar Field; they correspond to stationary values of the invariant function of direction

$$\theta = g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3s} G_{t_1t_2,t_3t} \frac{dx_s}{ds} \frac{dx_t}{ds}.$$

But, as will be seen, the value of θ at any point proves to be independent of direction and therefore the principal directions of Type IV are indeterminate. However, since θ is an invariant function of position varying from point to point, there will exist principal directions corresponding to stationary values of $d^2\theta/ds^2$ for geodesics drawn in all possible directions. These principal directions are given by

$$\left[\frac{\partial^2\theta}{\partial x_s \partial x_t} - \left\{ {st \atop m} \right\} \frac{\partial\theta}{\partial x_m}\right] dx_t = \phi g_{st}\, dx_t \quad (s = 1, 2, 3, 4) \qquad (1)$$

and are a generalization of Type II.

The manifold under consideration is of four dimensions, and, in accordance with the conventions employed in the foregoing paper small Roman indices imply a range or summation from 1 to 4, small Greek indices from 1 to 3. The line element is given by

$$ds^2 = g_{mn}\, dx_m\, dx_n$$

where

$$g_{11} = -(1-k/x_1)^{-1}, \quad g_{22} = -x_1^2, \quad g_{33} = -x_1^2 sin^2 x_2,$$
$$g_{44} = 1-k/x_1, \quad g_{mn} = 0 \quad (m \neq n).$$

Hence we have

$$g^{mn} = \begin{cases} 1/g_{mn} & (m = n) \\ 0 & (m \neq n). \end{cases}$$

Observing that

$$\frac{\partial g_{st}}{\partial x_3} = \frac{\partial g_{st}}{\partial x_4} = 0,$$

we find that any three index symbol, $\begin{bmatrix} mn \\ s \end{bmatrix}$, is zero if just one of the indices is either 3 or 4. Turning to the general expression

$$G_{mn,st} = \frac{\partial}{\partial x_t}\begin{bmatrix} ms \\ n \end{bmatrix} - \frac{\partial}{\partial x_s}\begin{bmatrix} mt \\ n \end{bmatrix} + g^{ab}\left\{\begin{bmatrix} mt \\ a \end{bmatrix}\begin{bmatrix} ns \\ b \end{bmatrix} - \begin{bmatrix} ms \\ a \end{bmatrix}\begin{bmatrix} nt \\ b \end{bmatrix}\right\} \quad (2)$$

we find

$$G_{4\nu,\sigma\tau} = g^{ab}\left\{\begin{bmatrix} 4\tau \\ a \end{bmatrix}\begin{bmatrix} \nu\sigma \\ b \end{bmatrix} - \begin{bmatrix} 4\sigma \\ a \end{bmatrix}\begin{bmatrix} \nu\tau \\ b \end{bmatrix}\right\}.$$

Now $\begin{bmatrix} 4\tau \\ a \end{bmatrix}$ vanishes unless $a = 4$, $\begin{bmatrix} \nu\sigma \\ b \end{bmatrix}$ vanishes if $b = 4$, while g^{ab} vanishes unless $a = b$. Therefore

$$g^{ab}\begin{bmatrix} 4\tau \\ a \end{bmatrix}\begin{bmatrix} \nu\sigma \\ b \end{bmatrix} = 0;$$

applying similar reasoning to the second part of the expression, we find

$$G_{4\nu,\sigma\tau} = 0.$$

Thus any tensor-component with just one index equal to 4 vanishes. Similarly any tensor-component with just one index equal to 3 vanishes.

From (2) we find

$$G_{4\nu,\sigma 4} = \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_\nu \partial x_\sigma} - \frac{1}{2} g^{11}\begin{bmatrix} \nu\sigma \\ 1 \end{bmatrix}\frac{\partial g_{44}}{\partial x_1} - \frac{1}{4} g^{44}\frac{\partial g_{44}}{\partial x_\nu}\frac{\partial g_{44}}{\partial x_\sigma}, \quad (3)$$

and hence

$$G_{4\nu,\sigma 4} = 0 \quad (\nu \neq \sigma).$$

The surviving members of the class given in (3) are

$$\left.\begin{aligned} G_{41,14} &= \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_1^2} - \frac{1}{4} g^{11}\frac{\partial g_{11}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} - \frac{1}{4} g^{44}\left(\frac{\partial g_{44}}{\partial x_1}\right)^2 = -\frac{k}{x_1^3} \\ G_{42,24} &= \frac{1}{4} g^{11}\frac{\partial g_{22}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} = \frac{1}{2}\frac{k}{x_1}\left(1 - \frac{k}{x_1}\right) \\ G_{43,34} &= \frac{1}{4} g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} = \frac{1}{2}\frac{k}{x_1} \sin^2 x_2\left(1 - \frac{k}{x_1}\right) \end{aligned}\right\} \quad (4).$$

From (2) we find

$$G_{\mu\nu,\sigma\tau} = \frac{\partial}{\partial x_\tau}\begin{bmatrix}\mu\sigma\\ \nu\end{bmatrix} - \frac{\partial}{\partial x_\sigma}\begin{bmatrix}\mu\tau\\ \nu\end{bmatrix} + g^{\alpha\beta}\left\{\begin{bmatrix}\mu\tau\\ \alpha\end{bmatrix}\begin{bmatrix}\nu\sigma\\ \beta\end{bmatrix} - \begin{bmatrix}\mu\sigma\\ \alpha\end{bmatrix}\begin{bmatrix}\nu\tau\\ \beta\end{bmatrix}\right\}. \quad (5)$$

Since the presence of just one 3 among the indices makes the tensor-component vanish, the surviving independent members of this class are

$$G_{31,13}, \quad G_{31,23}, \quad G_{32,23}, \quad G_{21,12};$$

we calculate them from (5):—

$$G_{31,13} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{11}}{\partial x_1} - \frac{1}{4}g^{33}\left(\frac{\partial g_{33}}{\partial x_1}\right)^2 = -\frac{1}{2}\sin^2 x_2\,\frac{k/x_1}{1-k/x_1} \quad (6)$$

$$G_{31,23} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_1 \partial x_2} - \frac{1}{4}g^{22}\frac{\partial g_{33}}{\partial x_2}\frac{\partial g_{22}}{\partial x_1} - \frac{1}{4}g^{33}\frac{\partial g_{33}}{\partial x_2}\frac{\partial g_{33}}{\partial x_1} = 0$$

$$\left.\begin{aligned} G_{32,23} &= \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_2^2} + \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{22}}{\partial x_1} - \frac{1}{4}g^{33}\left(\frac{\partial g_{33}}{\partial x_2}\right)^2 = kx_1\sin^2 x_2 \\ G_{21,12} &= \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{22}}{\partial x_1}\frac{\partial g_{11}}{\partial x_1} - \frac{1}{4}g^{22}\left(\frac{\partial g_{22}}{\partial x_1}\right)^2 = -\frac{1}{2}\frac{k/x_1}{1-k/x_1} \end{aligned}\right\} \quad (7)$$

The complete list of surviving components, derivable from (4), (6) and (7), is as follows:—

of type $G_{s_1s_2,s_31}$ $\begin{cases} G_{21,21} & G_{12,21} \\ G_{31,31} & G_{13,31} \\ G_{41,41} & G_{14,41}\end{cases}$; of type $G_{s_1s_2,s_32}$ $\begin{cases} G_{12,12} & G_{21,12} \\ G_{32,32} & G_{23,32} \\ G_{42,42} & G_{24,42}\end{cases}$;

of type $G_{s_1s_2,s_33}$ $\begin{cases} G_{13,13} & G_{31,13} \\ G_{23,23} & G_{32,23} \\ G_{43,43} & G_{34,43}\end{cases}$; of type $G_{s_1s_2,s_34}$ $\begin{cases} G_{14,14} & G_{41,14} \\ G_{24,24} & G_{42,24} \\ G_{34,34} & G_{43,34}\end{cases}$.

The equations of the principal directions of Type IV are

$$\theta g_{st}\,dx_t = g^{s_1t_1}g^{s_2t_2}g^{s_3t_3}G_{s_1s_2,s_3s}G_{t_1t_2,t_3t}\,dx_t \quad (s = 1, 2, 3, 4); \quad (8)$$

these become

for $s = 1$, $\theta g_{11}dx_1 = 2g^{11}\left[(g^{22}G_{21,12})^2 + (g^{33}G_{31,13})^2 + (g^{44}G_{41,14})^2\right]dx_1$;

for $s = 2$, $\theta g_{22}dx_2 = 2g^{22}\left[(g^{11}G_{12,21})^2 + (g^{33}G_{32,23})^2 + (g^{44}G_{42,24})^2\right]dx_2$;

for $s = 3$, $\theta g_{33}dx_3 = 2g^{33}\left[(g^{11}G_{13,31})^2 + (g^{22}G_{23,32})^2 + (g^{44}G_{43,34})^2\right]dx_3$;

for $s = 4$, $\theta g_{44}dx_4 = 2g^{44}\left[(g^{11}G_{14,41})^2 + (g^{22}G_{24,42})^2 + (g^{33}G_{34,43})^2\right]dx_4$.

On substitution we obtain

$$\theta dx_1 = 3\frac{k^2}{x_1^6}dx_1, \qquad \theta dx_2 = 3\frac{k^2}{x_1^6}dx_2,$$

$$\theta dx_3 = 3\frac{k^2}{x_1^6}dx_3, \qquad \theta dx_4 = 3\frac{k^2}{x_1^6}dx_4.$$

Thus the principal directions of Type IV are indeterminate, and (8) define an invariant function of position

$$\theta = g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3s} G_{t_1t_2,t_3t} \frac{dx_s}{ds}\frac{dx_t}{ds} = 3\frac{k^2}{x_1^6}.$$

Substituting this value for θ in (1), we obtain, after reduction, the following equations for principal directions:—

$$\phi dx_1 = 9\frac{k^2}{x_1^8}\left(15\frac{k}{x_1} - 14\right) dx_1, \qquad \phi dx_2 = 18\frac{k^2}{x_1^8}\left(1 - \frac{k}{x_1}\right) dx_2,$$

$$\phi dx_3 = 18\frac{k^2}{x_1^8}\left(1 - \frac{k}{x_1}\right) dx_3, \qquad \phi dx_4 = 9\frac{k^3}{x_1^9} dx_4.$$

These equations determine the following directions:—

(i) the parametric lines of x_1, $(dx_2 = dx_3 = dx_4 = 0)$;
(ii) any direction making $dx_1 = dx_4 = 0$;
(iii) the parametric lines of x_4, $(dx_1 = dx_2 = dx_3 = 0)$.

It might be said that these principal directions illustrate both the radial and the stationary characters of the field.

FIELDS OF PARALLEL VECTORS IN THE GEOMETRY OF PATHS

BY L. P. EISENHART

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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1. In a former-paper (these PROCEEDINGS, Feb. 1922) Professor Veblen and the writer considered the geometry of a general space from the point of view of the paths in such a space—the paths being a generalization of straight lines in euclidean space. From this point of view it is natural to think of the tangents to a path as being parallel to one another. In this way our ideas may be coördinated with those of Weyl and Eddington who have considered parallelism to be fundamental rather than the paths which we so consider. It is the purpose of this note to determine the geometries which possess one or more fields of parallel vectors, which accordingly define a significant direction, or directions, at each point of the space.

2. The equations of the paths are taken in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{2.1}$$

where x^i $(i = 1, \ldots n)$ are the coördinates of a point of a path expressed as functions of a parameter s; $\Gamma^i_{\alpha\beta}$ are functions of the x's such that $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$.