(Gibb's canonical distribution) depends obviously upon properties in the small, i.e., upon the behavior of the flow in the shells between S_x and $S_{x+\delta}$, $a \leqq x \leqq b$, δ being arbitrarily small. The true condition for that tendency must therefore be an infinitesimal one. It is evidently of great interest to find such a condition.

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[1] Cf. B. O. Koopman, these PROCEEDINGS, 17, 315–318 (1931).

[2] Cf. J. v. Neumann, these PROCEEDINGS, 18, 70–82 (1932).

[3] Cf. J. W. Gibbs, *Statistical Mechanics*, Chapter VII. We refer in particular to the sections 23 and 27 of P. and T. Ehrenfest's article, "Statistische Mechanik," in the *Encyclopaedie der math. Wiss.*, IV 2.

[4] The actual existence of the limit except in a point set of zero measure has been proved by Birkhoff, these PROCEEDINGS, 17, 650–660 (1931). This fact is, however, not needed for our purposes.

[5] An elementary proof has been given by the author, these PROCEEDINGS, 18, 93–100 (1932).

[6] E. Hopf, these PROCEEDINGS, 18, 204–209 (1932). The same subject has been simultaneously and independently treated by Koopman and v. Neumann, these PROCEEDINGS, March, 1932. These authors have found another interesting form of the equation (3). Furthermore, they give an example of a unitary group U_t, for which (3) holds, whereas (2) is not fulfilled in general.

*REGULAR FAMILIES OF CURVES. II**

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The purpose of this note is three-fold: (*a*) A new definition of the function μ is given.[1] (*b*) A condition for the regularity of a family of curves is stated. (*c*) If a closed set of "invariant points" be added to a regular family of curves, an "extended family of curves" is formed; theorems are stated on the covering of the curves by a set of "tubes," and on the introduction of a function continuous throughout the extended family.

1. *The New Function* μ.—Let R be any metric separable space, and let $a_1, a_2, \ldots,$ be a sequence of points dense in R. For any x in R, define $f_n(x)$ as follows:

$$f_n(x) = \frac{1}{1 + \rho(x, a_n)}. \tag{1}$$

Let S be any subset of R; we put

$$\mu_n(S) = \max_{x \subset S} f_n(x) - \min_{x \subset S} f_n(x), \tag{2}$$

and
$$\mu(S) = \sum_{n=1}^{\infty} \frac{\mu_n(S)}{2^n}. \tag{3}$$

If we restrict ourselves to closed subsets of R, and note that $|f_n(x) - f_n(y)| \leqq \rho(x, y)$, the two fundamental properties of the function μ stated in Paper I[2] are easily seen to hold. We have thus dropped the restriction that our sets be totally bounded.

We note in passing that Brouwer's Reduction theorem (see for instance, Kerekjarto, "Topologie" I, p. 45) can be proved easily with the aid of the function μ. If the set M is not irreducible, we remove from it nearly as much as possible, i.e., we take a subset with a small μ; if the resulting set M_1 is not irreducible, we take a subset of it with a small μ, etc. The limiting set M^* is easily seen to be irreducible.

2. *The Regularity of a Family of Curves.*—A necessary and sufficient condition that a family of curves be regular is that for each arc pq of a curve C and any $\epsilon > 0$ there is a $\delta > 0$ such that if $p' \subset V_\delta(p)$, then there is a point q' on the curve C' through p' such that (1) each point of one of the arcs pq, $p'q'$ is within ϵ of some point of the other, and (2) if r' and s' are on $p'q'$ and within δ of each other, then the arc $r's'$ is of diameter $< \epsilon$.

It follows that the families of curves in the examples given in Paper I are regular. We note that the curves in a regular family need not be equicontinuous, as is shown by the following example (using polar coördinates). One curve is $r = 1$; the others are $r = 1 - 1/(\Theta + \gamma)$, $(\Theta > 1 - \gamma)$, one curve corresponding to each γ, $0 \leqq \gamma < 2\pi$.

3. *Extended Families of Curves.*—We assume here that the regular family of curves forms an open subset R' of a compact metric space R. The remaining points $R_1 = R - R'$ we call invariant points. For instance, R might be a closed and bounded part of Euclidean n-space, and R_1, a closed subset of R. Moreover, we now restrict the curves forming R' to be images of an open segment or of a circle.

We assume also that the family of curves is orientable, that is, we can assign a direction to each curve so that neighboring curves are similarly oriented. By this we mean that the regularity condition in Paper I is satisfied, with q' lying in the positive or negative direction from p' according as q lies in the positive or negative direction from p. We shall say the curves of R' with the points of R_1 form an extended family of curves.

4. *The Tubes.*—Take a point p of a curve C, and a cross-section S_0 through p. Transforming S_0 by means of the function μ a small amount in each direction gives cross-sections S_1 and S_2. The arcs $p_1p_0p_2$, $p_i \subset S_i$, $i = 0, 1, 2$, are the arcs of the tube T containing p. We call S_1 and S_2 its ends.

In an extended family of curves there exists a sequence of tubes T_1,

$T_2, \ldots$, all lying in R', and covering R', and such that any closed set in R' has points in but a finite number of the tubes. We can give the sequence of tubes either one of the following properties: (1) Any tube contains points of at most a single end of any previous tube; (2) if the point p within the arc p_1p_2 of one of the tubes is contained in a preceding tube, then all of p_1p_2 is contained in preceding tubes.

5. *The Function f.*—In an extended family of curves a function f can be introduced with the following properties.

(1) With each point p of R and any t, $-\infty < t < +\infty$, there is associated a single point $q = f(p, t)$, which lies on the curve C through p, or coincides with p if p is an invariant point. Further, for each point q of C there is a t such that $q = f(p, t)$.

(2) $f(p, t)$ is continuous in both variables; that is, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho[f(p, t), f(p', t')] < \epsilon \text{ if } \rho(p, p') < \delta \text{ and } |t - t'| < \delta.$$

(3) For any point p and any two numbers t_1 and t_2,

$$f[f(p, t_1), t_2] = f(p, t_1 + t_2);$$

also

$$f(p, 0) = p.$$

This theorem is proved by defining f successively in the tubes $T_1, T_2, \ldots$, with the second property in §4.

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[1] The definition of μ here given was discovered during a conversation with S. Lefschetz.

[2] Equations (3) and (4).