is usually the case when close lines are photometered. The point is being studied by a new photometer, which has been constructed in our new physical research laboratory. It may be that the curves indicate a number of very close and very weak satellites.

Plate 6 represents the photometer curve kindly made for me at the California Institute of Technology. It corresponds almost exactly to the one obtained in the Jefferson Laboratory. The x-ray lines have been very kindly photometered for me in half a dozen laboratories throughout the United States. None of the curves, however, are superior to those represented in Plates 5 and 6.

A new δ line (O $\longrightarrow$ K electrons) has appeared on some photographs of the K series of tungsten made some time ago in my laboratory by Miss Armstrong, Assistant Professor of Physics at Wellesley College, when she was acting as one of my assistants. The spectra that she photographed also show two β lines as well as the γ and the two α lines of W.

Accurate details of these K series spectra of Mo and W will be published as soon as the photographs can be examined by means of the new photometer I have designed and am having set up in our new physical research laboratory.

It gives me great pleasure to sincerely thank my assistant, Mr. Lanza, who has spent so much time and care in setting up my apparatus and taking photographs.

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*NOTE ON PERRON'S SOLUTION OF THE DIRICHLET PROBLEM**

BY HASSLER WHITNEY†

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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In 1923 Perron[1] gave a method of attacking the Dirichlet Problem, the only real difficulty of which lay in the proof of the fact that his function u was harmonic. Simplifications of this proof were later given by several authors.[2] We give here a short proof involving only Harnack's first convergence theorem and inequality, both of which follow directly from the Poisson integral, and two lemmas from Perron's paper (Hilfssätze I and II). The convergence theorem is: *Let $u_1, u_2 \ldots,$ be a sequence of functions harmonic in the closed region S, and converging uniformly in S to a limit u. Then u is harmonic in S.* The inequality we state in the form: *Let χ be harmonic and $\geqq 0$ within the circle K of radius a and with center P. Then if L is a circle about P of half the radius,*

$$\chi(Q) \leqq 3\chi(\text{P})$$

for any Q in L.

Following Wiener[3] and Petrowsky,[3] we say a function ψ is an upper function for $G + R$ and boundary values f on R if it is continuous and superharmonic ($M_K\psi \leqq \psi$ for any circle K) in G, and $\Psi \geqq \bar{f}$ on R. The solution of the Dirichlet problem is given by the function u, the lower limit of all upper functions. u is easily seen to exist (see Perron's paper).

To show that u is harmonic in G, let P be any point of G, let K be a circle about P lying in G, and let L be a circle about P of half the radius. It is sufficient to show that to any $\epsilon > 0$ there corresponds a harmonic function U such that

$$\left| U - u \right| \leqq \epsilon \text{ in } L.$$

For then to a sequence of ϵ's converging to 0 corresponds a sequence of harmonic functions approaching u uniformly. Harnack's first convergence theorem now applies, and u is harmonic in L. But as P was any point of G, u is harmonic in G.

Let ψ be an upper function such that $\psi(P) \leqq u(P) + \epsilon/3$. As $M_K\psi \leqq \psi$ and $M_K\psi$ is an upper function, we have

$$u(P) \leqq M_K\psi(P) \leqq u(P) + \frac{\epsilon}{3}. \tag{1}$$

$M_K\psi$ is harmonic in K. I state that it is our required function; that is, for any P' in L,

$$\left| M_K\psi(P') - u(P') \right| \leqq \epsilon. \tag{A}$$

If (A) is false, then there is a point Q in L for which

$$M_K\psi(Q) = u(Q) + \eta,\ \eta > \epsilon. \tag{B}$$

Let ϕ be an upper function for which $\phi(Q) < u(Q) + \eta - \epsilon$ Put $\phi' = \min. (\phi, M_K\psi)$. Then $M_K\phi' \leqq \phi' \leqq \phi$, and hence

$$M_K\phi'(Q) < u(Q) + \eta - \epsilon.$$

This with (B) gives

$$M_K\phi'(Q) < M_K\psi(Q) - \epsilon.$$

But as $M_K\phi'$ is an upper function and $M_K\phi' \leqq \phi' \leqq M_k\psi$, $u \leqq M_k\, \phi' \leqq M_K\, \psi$ and (1) gives

$$M_K\psi(P) - M_K\phi'(P) \leq \frac{\epsilon}{3}.$$

Thus the function $\chi = M_K\psi - M_K\phi'$ is harmonic within K, is $\geqq 0$ there, is $\leqq \epsilon/3$ at P, and is $> \epsilon$ at Q in L. But this contradicts Harnack's inequality, so (A) holds, and u is harmonic in G.

The proof holds in any number of dimensions, if the constant in Harnack's inequality is changed properly.

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† NATIONAL RESEARCH FELLOW.

[1] O. Perron, *Math. Zeit.*, **18,** 42–54 (1923); we follow the notations in this paper.

[2] R. Remak, *Math. Zeit.*, **20,** 126–130 (1924); T. Radó and F. Riesz, *Ibid.* **22,** 41–44 (1925); R. Remak, *J. I. Math.*, **156,** 227–230 (1926). The proof here given is in essence like that of the last-named paper.

[3] I. Petrowsky, *Rec. Math. Moscou*, **35,** 105–110 (1928); N. Wiener, *J. Math. Phys. Mass. Inst. Techn.*, **4,** 21–32 (1925).

PROOF OF THE QUASI-ERGODIC HYPOTHESIS

BY J. V. NEUMANN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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1. The purpose of this note is to prove and to generalize the quasi-ergodic hypothesis of classical Hamiltonian dynamics[1] (or "ergodic hypothesis," as we shall say for brevity) with the aid of the reduction, recently discovered by Koopman,[2] of Hamiltonian systems to Hilbert space, and with the use of certain methods of ours closely connected with recent investigations of our own of the algebra of linear transformations in this space.[3] A precise statement of our results appears on page 79.

We shall employ the notation of Koopman's paper, with which we assume the reader to be familiar. The Hamiltonian system of k degrees of freedom corresponding with the Hamiltonian function $H(q_1, \ldots, q_k, p_1, \ldots, p_k)$ defines a steady incompressible flow $P \longrightarrow P_t = S_tP$ in the space Φ of the variables $(q_1, \ldots, q_k, p_1, \ldots, p_k)$ or "phase-space," and a corresponding steady conservative flow of positive density ρ in any invariant sub-space $\Omega \subset \Phi$ (Ω being, e.g., the set of points in Φ of equal energy). The Hilbert space $\mathfrak{H}$ consists of the class of measurable functions $f(P)$ having the finite Lebesgue integral $\int_\Omega |f|^2\rho d\omega$, the "inner product"[4] of any two of them (f, g) and "length" $\|f\|$ being defined by the equations

$$(f, g) = \int_\Omega f\bar{g}\rho d\omega; \; \|f\| = \sqrt{(f,f)}. \tag{1}$$

The transformation U_t is defined as follows:

$$U_tf(P) = f(S_tP) = f(P_t); \tag{2}$$

obviously it has the group property