ON THE DISTANCES AND RADIAL VELOCITIES OF EXTRA-GALACTIC NEBULAE, AND THE EXPLANATION OF THE LATTER BY THE RELATIVITY THEORY OF INERTIA

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OBSERVATORY AT LEIDEN

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1. The large radial velocities of extragalactic nebulae measured by the observers at Mount Wilson during the last year have enhanced the importance of the problem of the determination of the distances of these objects. Evidently both the apparent diameter and the integrated apparent magnitude must be correlated with the distance. In trying to derive a mathematical expression for this correlation it is necessary to start with a certain number of known distances as a basis. For a few of the largest and nearest nebulae the distances have been determined directly by Hubble, Lundmark, Shapley and others from cepheids or novae in them. For others the brightest stars have been used as a criterion of the distance. This criterion, however, is less reliable than the other two, because, as Shapley has pointed out, we cannot be sure that what is assumed to be a star may not in reality be a cluster of stars. Some of the distances determined by this method have, however, been used, as the number of distances depending on cepheids or novae alone would be too small as a basis for the discussion. Other methods have also been employed, based on rotation, or the distance of condensations in spiral arms, or other hypothetical considerations; the distances so determined have not been used in my discussion.

2. The first step in the discussion was an investigation of the correlation between the apparent diameters and integrated magnitudes. The discussion is being published in full in *B. A. N.* 185, to which publication the reader must be referred for details. If we assume: 1st, that both the true diameters and the true total magnitudes are distributed according to a Gaussian (or other symmetrical) frequency function round constant average values (constant meaning independent of the distance from us), and 2nd, that the measured (or estimated) values of the apparent diameters and magnitudes are referred to a consistent scale throughout, then we should have, for all objects of one class

$$\log d + \frac{m}{5} = \text{const.},$$

where the constant may vary from one class to another. Hubble, in his great paper on extragalactic nebulae,[1] has assumed this relation. I have

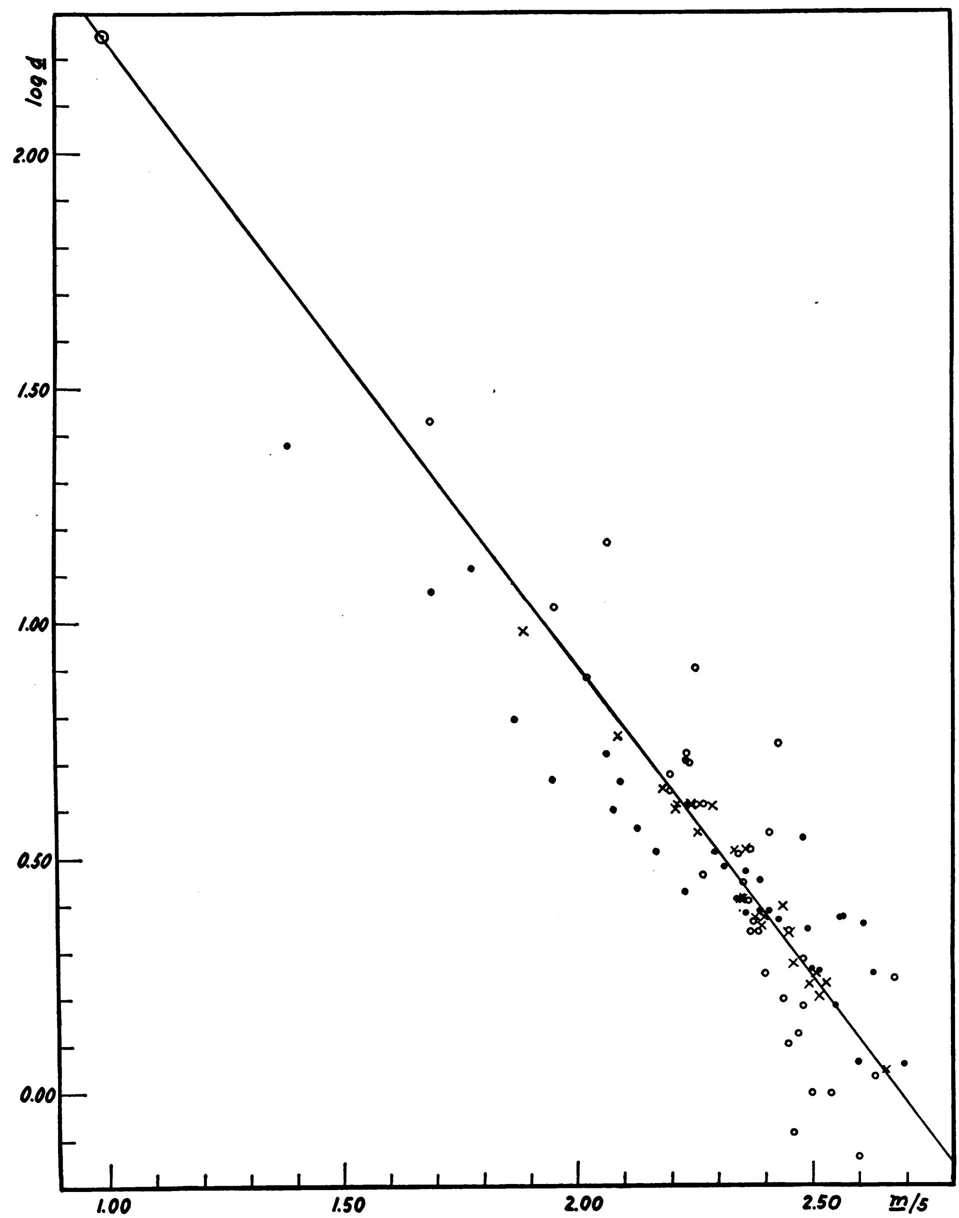

FIGURE 1

Correlation between log *d* and *m*/5 for spiral nebulae. The horizontal coördinate is *m*/5, the vertical coördinate is log *d*. Dots are means of from five to fifteen nebulae taken in order of increasing *m*/5, circles are similar means taken in order of decreasing log *d*. The crosses are means of these two kinds of means, and represent the points through which the straight line was drawn by least squares. The large circle at the top is the Andromeda nebula N. G. C. 224.

based my investigation on the data given in that paper which refer to 291 spirals, 85 elliptical, and 11 irregular nebulae, not counting the "unclassified" or "peculiar" objects. To the irregular nebulae I have added the two Magellanic Clouds and N. G. C. 6822, making the total number of this class 14. The following formulae were derived.

$$\text{Spirals:} \qquad \frac{m}{5} = 2.31 - (0.75 \pm 0.03)\,(\log d - 0.50)$$

$$\text{Elliptical nebulae:} \quad \frac{m}{5} = 2.30 - (0.87 \pm 0.04)\,(\log b + 0.26) \qquad (1)$$

$$\text{Irregular nebulae:} \quad \frac{m}{5} = 1.39 - (0.93 \pm 0.15)\,(\log d - 1.27)$$

For the spirals small corrections have been applied to the diameters and magnitudes to reduce them all to the class *Sb*. For the elliptical nebulae the *minor* diameter $b = d\,(1 - e)$ is used instead of the *major* diameter d.

Although the probable errors in the formulae (1) cannot be assumed to be a true measure of the uncertainty, still the deviation from the theoretical value 1 is undoubtedly real in the case of the spirals and the elliptical nebulae. The data for the spirals are represented in figure 1. It will be seen that a line drawn at an angle of 45° would not represent the observations satisfactorily.

The only admissible explanation of the difference of the coefficients from unity is by systematic errors in the scale of the diameters, or the magnitudes, the small and faint nebulae being estimated too small, or too bright, or both. It should be noted that the effect of an absorption of light in space would be to make the coefficient larger than unity instead of smaller. It is, indeed, hardly to be wondered at that the existing measures, or one should rather say estimates, of the diameters and the magnitudes of objects of so widely different aspect as the large and the small spirals should not conform to a consistent scale. The problem of establishing a standard scale of diameters and of magnitudes for the extragalactic nebulae is a very difficult one, but it is of paramount importance, and it is to be hoped that practical astronomers will give it all the attention that it deserves.

3. As has been already stated, the determination of formulas expressing the correlation between distance and apparent diameter, and between distance and apparent total magnitude, was based on distances determined by Hubble, Lundmark or Shapley from cepheids, novae, or the brightest stars. As a convenient unit of distance I use 10^{24} cm., or approximately a million light years, which is denoted by A. The numbers of distances used were: 21 spirals (11 *Sb* and 10 *Sc*), 6 elliptical nebulae (four of which belong to the Virgo cluster, of which the distance is about

6A, the two others being N. G. C. 205 and 221 at distance 1A, which are, however, not typical for their class) and 6 irregular nebulae. The distances of the spirals range from 1A to 6A, those of the irregular nebulae from 0.1A to 4.5A. For the spirals and for the irregular nebulae linear formulas were derived from these data for the relation between log r and log d and between log r and $m/5$, but the uncertainty of the coefficients is naturally very large, owing to the small range of distances. For the elliptical nebulae no coefficient could be determined. The two formulas for log r_d and log r_m were so adjusted as to satisfy exactly the correlations (1) found between log d and $m/5$. The finally adopted formulas are:

$$\begin{aligned} &\text{Spirals:} && \log r_d = 1.28 - 0.75 \log d, && \log r_m = -1.40 + \frac{m}{5} \\ &\text{Elliptical:} && \log r_d = 0.77 - 0.87 \log b, && \log r_m = -1.30 + \frac{m}{5} \qquad (2) \\ &\text{Irregular:} && \log r_d = 1.04 - 0.85 \log b, && \log r_m = -1.30 + 0.91\frac{m}{5} \end{aligned}$$

Thus, for the spirals and the elliptical nebulae, the systematic errors of scale, which cause the deviation of the coefficient in (1) from unity,

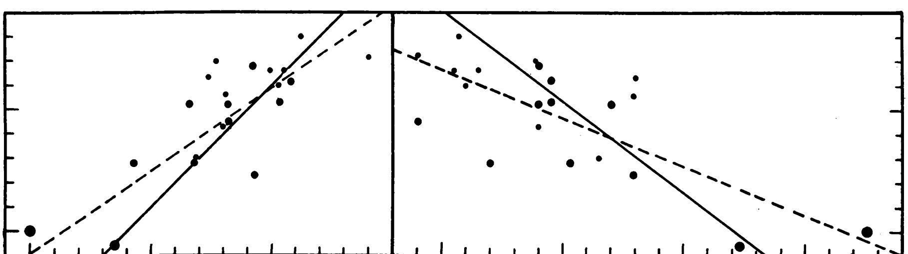

FIGURE 2

Correlation between distance and magnitude and between distance and diameter. The vertical coördinate is log r. The horizontal coördinate in the left hand half of the figure is $m/5$, and in the right hand half log d. The broken lines are those derived directly from the plotted data. The full lines are the adjusted and finally adopted lines, corresponding to the formulae (2). The diameters of the dots are proportional to the square roots of the weights.

have been thrown entirely on the diameter, the magnitude scale being assumed to be correct. This, of course, is rather arbitrary, and probably not exact, but no more definite conclusion can be derived from the material at present available. The values of log r_d and log r_m derived from the formulas (2) form a consistent system, of which the scale, however, is still very uncertain. The data for the spirals have been represented in figure 2.

The average absolute magnitude according to the formulas (2) is -15.56

for the spirals and -16.06 for the elliptical nebulae. The former agrees well with Hubble's adopted value. The effect of the spreading in absolute magnitude cannot be separated from that of the errors of observation. The combination of the two effects corresponds, according to a rough determination, to a probable error of $\pm 0^{m}.5$ for the spirals and $\pm 0^{m}.3$ for the elliptical nebulae.

4. Comparing the distances thus determined with the observed radial velocities, the strong correlation is at once evident. The adopted linear formula is:

$$\log r_v = 0.30 \pm .02 + \log (1000\ V/c), \tag{3}$$

where V is the radial velocity corrected for the rotation of the galaxy, and c the velocity of light. For the determination of the coefficient only those nebulae were used of which the corrected radial velocity exceeds +300 km./sec. Some nebulae, of which the distances or the radial velocities were for some reason not reliable, were also not used. Thus practically the straight line was drawn through the origin of coördinates and the center of gravity of a group of nebulae at an average distance of about $5A$. The data are represented in figure 3. The three nebulae N. G. C. 4853, 4860, 4865 have been entered at the mean distance corresponding to the mean of the diameters, the magnitudes being unknown. Hubble's distance for the Coma cluster would bring 4853 and 4860 exactly on the curve, while for 4865 the distance derived from the diameter agrees with the curve. These three nebulae, of course, have not been used in the determination of the coefficient. Also N. G. C. 7619 has not been used.

For the small and faint nebulae there can be no doubt, as has already been pointed out by Hubble,[2] that the formula (3) gives a better determination of the distance than (2). As it depends on (2), it involves the same scale, and carries this scale on to large distances, with an error probably not exceeding 10%, apart from the errors of observation of the radial velocity (which are, however, generally much smaller).

The meaning of the formula (3) is that the radial velocity is proportional to the distance, according to the formula

$$\frac{V}{c} = \frac{r}{2000}. \tag{3'}$$

The individual velocities are distributed round the value (3′) with a spreading corresponding to a probable error of about ± 140 km./sec.

5. We call "statical" solutions of the field equations of the general theory of relativity

$$G_{\mu\nu} - {}^1/_2\, g_{\mu\nu}\, G + \lambda g_{\mu\nu} + \kappa T_{\mu\nu} = 0, \tag{4}$$

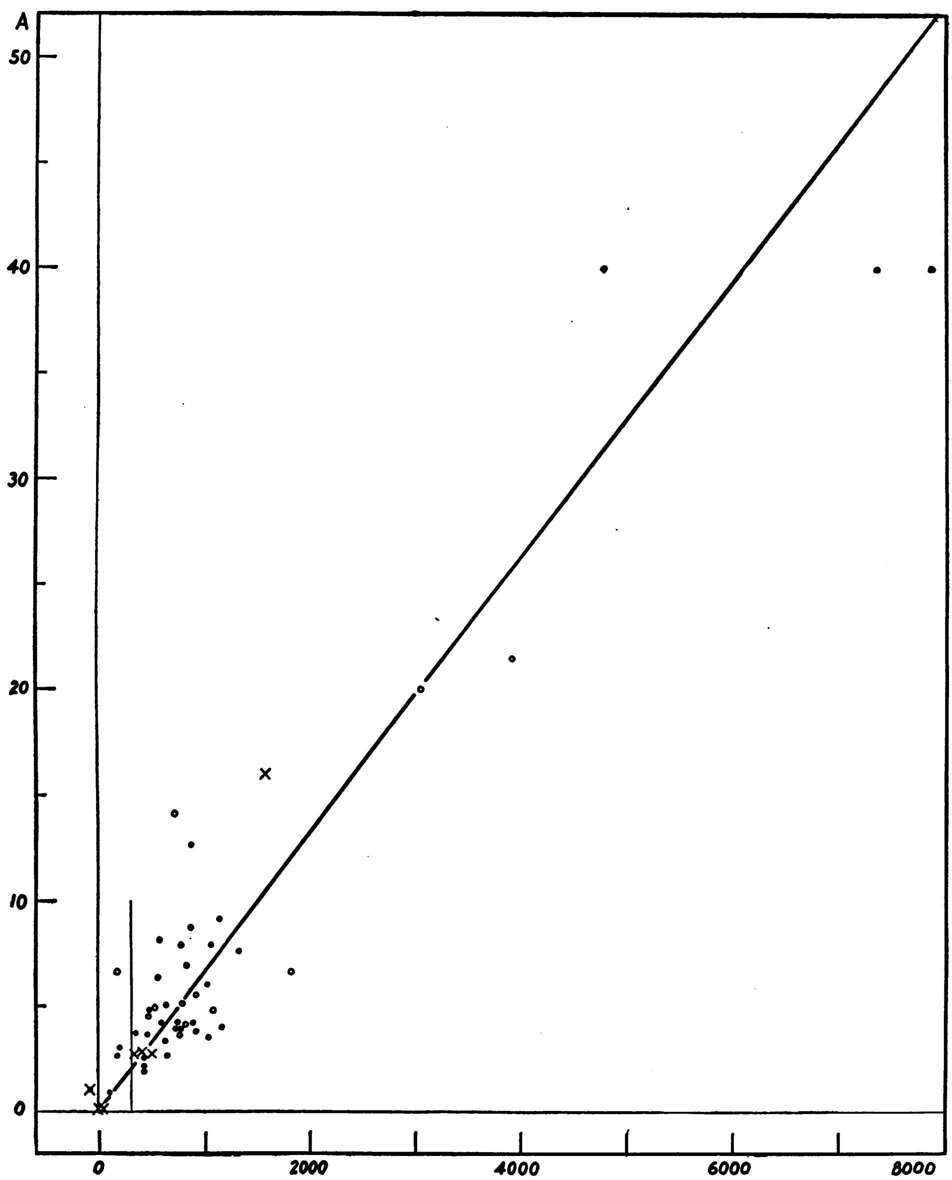

FIGURE 3

Correlation between distance and radial velocity. The vertical coördinate is the distance, the horizontal coördinate the radial velocity. Dots are spirals, circles elliptical and crosses irregular nebulae. The three dots at distance 40*A* represent N. G. C. 4865, 4853, 4860. The two circles near the middle of the figures are N. G. C. 7619 and 6359, and the cross at distance 16*A* is N. G. C. 4824. None of these have been used in determining the straight line. N. G. C. 3227, 4051 and 6359 are represented in the figure, though they were published after the completion of the investigation.

those solutions in which the $g_{\mu\nu}$ are functions of the three space coördinates x_1, x_2, x_3 only, and "dynamical" solutions those in which they depend also on the fourth coördinate $x_4 = t$. We restrict ourselves to those solutions in which the three-dimensional space has complete spherical symmetry, and is filled homogeneously with matter. Under this restriction the three-dimensional line-element can be written

$$R^2 d\sigma^2 = R^2[d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi d\theta^2)], \tag{5}$$

where

$$\chi = \frac{r}{R} \tag{5'}$$

is the radius vector expressed in the radius of the universe as a unit, and the material energy tensor is

$$T_{ij} = -g_{ij}\,p, \quad T_{44} = g_{44}(\rho_0 + 3p), \quad T = \rho_0, \tag{6}$$

where ρ_0 is the material (invariant) density, which must be a constant (i.e., independent of x_1, x_2, x_3), and p is the pressure, which is the sum of the radiation pressure of the radiant energy and the dynamical pressure representing the random motions of the material particles. Although in Einstein's theory the distinction between gravitation and inertia is irrelevant, and purely a matter of taste, we can say that, by postulating complete homogeneity, we neglect gravitation and consider the field of inertia alone. The four-dimensional line-element then is

$$ds^2 = -R^2 d\sigma^2 + f dt^2. \tag{7}$$

It is well known that under these restrictions there are only two possible statical solutions of the field equations (4) (apart from the limiting case $R = \infty$ corresponding to classical Newtonian mechanics), which I have called (A) and (B),[3] viz:

$$\begin{aligned} &(A) \quad R = \text{const} = \sqrt{\frac{1}{\lambda}}, \quad f = \text{const.} = c^2, \\ &(B) \quad R = \text{const} = \sqrt{\frac{3}{\lambda}}, \quad f = c^2 \cos^2\chi. \end{aligned} \tag{8}$$

In the system (A) the universe is filled with matter, the density depending on the radius by the formula

$$\kappa\rho_0 = \frac{2}{R^2}, \tag{9}$$

where κ is the constant of gravitation. The matter has no systematic motion:[4] the path of a test-body (material particle) is a straight line (geodesic) described with constant velocity.

In the system (B) the density ρ_0 and the pressure p are both zero:[4] the universe is empty. The path of a test-body is in this case a hyperbola, of which the asymptotes pass through the origin of coördinates, and the radial component of the velocity is given by

$$\frac{V^2}{c^2} = \frac{r^2}{R^2}\left(1 - \frac{r_0^2}{r^2}\right)\left(1 + \frac{v_0^2}{r^2}\right), \tag{10}$$

r_0 being the minimum distance, and $v_0 = r_0(d\theta/cdt)_0$ the transverse velocity at this distance.[5] Since the time spent by the body in the neighborhood of its minimum distance is only a short fraction of its whole course, we can in all but exceptional cases take as a good approximation

$$\text{(B)} \qquad \frac{V}{c} = \pm \frac{r}{R}. \tag{10'}$$

We have thus two determinations of the radius of the three-dimensional world, viz.:

$$\begin{aligned} &\text{(A)} \qquad R_A^2 = \frac{2}{\kappa\rho_0}, \\ &\text{(B)} \qquad R_B = \frac{r}{V/c}. \end{aligned} \tag{11}$$

The system (A) is only admissible if there are no systematic motions, and is thus excluded by the observed systematic radial velocities of the extragalactic nebulae. These velocities conform to the law (10′) of the system (B), it remaining unexplained, however, why all velocities are positive and none negative. This system can, however, only be admitted if the density of matter in the universe is so small that $\rho_0 = 0$, or emptiness, can be taken to be a good approximation to the truth.

Now we know that the part of the universe that we can reach with our telescopes is filled with spirals and other extragalactic nebulae. These appear to be often condensed in clusters, but as an approximation we may take a uniform density of one nebula in a cube of $1A$ side. Taking the average mass of one nebula as unit of mass, the density will be $\rho_0 = 1$. If for this average mass we take 10^{11} ⊙ (estimate by Dr. Oort), and for the units of length and of time we take $1A$, and the corresponding time so as to make $c = 1$, the constant of gravitation in these units becomes

$$\kappa = 3{\cdot}7.10^{-7}.$$

With this value of κ, and the value (3′) of V/cr, we find from (11) the following values of R_A and R_B:

$$R_A = 2300\,A, \qquad R_B = 2000\,A. \tag{12}$$

The fact that these two values are practically the same, or at least of the same order of magnitude, shows that emptiness can *not* be taken as a good approximation in the system (B). The world is, in fact, practically full, according to the criterion of system (A).

We thus come to the conclusion that *both the solutions* (A) *and* (B) *must be rejected*, and as these are the only statical solutions of the equations (4), the true solution represented in nature must be a *dynamical* solution.

6. A dynamical solution of the equations (4), with the line-element (5), (7) and the material energy tensor (6) is given by Dr. G. Lemaître in a paper[6] published in 1927, which had unfortunately escaped my notice until my attention was called to it by Professor Eddington a few weeks ago. In this solution f and R in (7) are assumed to be functions of t alone. We can put $\sqrt{f}\,dt = d\tau$, and use τ as a new independent variable. In other words we can take f = constant $= c.^2$ Evidently, if the coördinate χ has no systematic motion, which can easily be shown to be the case, then the systematic motion of r will be, by (5′), $dr/dt = rdR/Rdt$, or, if by a dot we denote a differential quotient d/cdt, by (11)

$$\frac{V}{c.r} = \frac{\dot{R}}{R} = \frac{1}{R_B}, \tag{13}$$

will be a pure function of t, independent of x_1, x_2, x_3, of which the value at the present moment is, by (12), $0\cdot5.10^{-3}\,A^{-1}$.

The field equations (7) become

$$\begin{aligned} \frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} &= \lambda - \kappa p \\ \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} &= \frac{1}{3}(\lambda + \kappa\rho), \end{aligned} \tag{14}$$

where $\rho = \rho_0 + 3p$ is the density of "relative mass," while ρ_0 is the density of material, or "invariant," mass. From these equations we easily find

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = 0, \tag{15}$$

which is the equation of the conservation of energy. If we introduce the volume $\mathbf{V} = \pi^2 R^3$ of space, this can be written

$$d(\mathbf{V}\rho) + pd\mathbf{V} = 0, \tag{15'}$$

showing the analogy of the homogeneously distributed matter with a gas, from which follows the decrease of the pressure by a diabatic expansion of the universe. We put further

$$\kappa\rho_0 = \frac{\alpha}{R^3}, \qquad \kappa p = \frac{\beta}{R^4}, \tag{16}$$

α thus being proportional to the total mass in the universe ($\alpha = \kappa M/\pi^2$). The equation (15) then becomes

$$R\dot{\alpha} + 3\dot{\beta} = 0. \tag{15''}$$

This history of the universe is then described by any two of the equations (14) and (15), or (14) and (15′), or (14) and (15″), to which an assumption regarding α or β must be added. It is most convenient to take the second of (14) and (15″).

7. Lemaître takes α = constant, from which by (15″) follows β = constant. It is, however, certain that material mass is continuously converted into energy by the radiation of the stars, and $\dot{\alpha}$ must consequently have a finite negative value, however small. We can measure the rate of conversion of matter into radiant energy against the rate of expansion of the universe, and put

$$\frac{\dot{\alpha}}{\alpha} = -\gamma\frac{\dot{R}}{R},$$

γ being positive. As a convenient hypothesis we can take γ to be a constant. Then

$$\alpha = \alpha_0 R^{-\gamma},$$

and from (15″)

$$\beta = \beta_0 + \frac{\alpha_0\gamma}{3(1-\gamma)}R^{1-\gamma},$$

and consequently

$$\kappa p = \frac{\beta_0}{R^4} + \frac{\alpha_0\gamma}{3(1-\gamma)}\cdot\frac{1}{R^{3+\gamma}}. \tag{17}$$

The first term of (17) is the kinematic pressure, corresponding to the random velocities of the extragalactic nebulae, treated as molecules. We have thus for the kinematic pressure

$$\frac{p}{\rho_0} = \frac{\beta_0}{R^4}\Big/\frac{\alpha_0}{R^{3+\gamma}} = \frac{\beta_0}{\alpha_0}\cdot\frac{1}{R^{1-\gamma}}$$

Now in the kinetic theory of gases p/ρ is proportional to the square of the random velocities. Consequently by the expansion of the universe the random velocities decrease proportionally to $1/R^{1/2-\gamma/2}$, or, since in the actual universe γ is a very small quantity, to $1/\sqrt{R}$.

The second term in (17) is the radiation pressure which is proportional to the radiant energy. The total amount of radiant energy in the universe is thus proportional to $1/R^{\gamma}$, and consequently *decreases*, i.e., the increase of radiant energy by the radiation of the stars is more than balanced by the decrease by the adiabatic expansion of the universe. This can easily be verified by making up the account of loss and gain of energy, gain by the radiation of matter and loss by the degradation consequent upon the increase of wave-length corresponding by Doppler's principle to the velocity of the source. Thus this theory incidentally gives a complete answer to the old question what becomes of the energy which is continually being poured out into space from the stars. It is more than used up in the work done in expanding the universe. It would, however, not be correct to say that the universe is expanded *by* the radiation. It would expand just the same if there were no radiation ($\gamma = 0$). The expansion is due to the term with λ in the equation (4), which represents an inherent expanding tendency of space, counteracting the binding force of gravitation represented by κ.

We can make an estimate of the numerical value of γ. The total amount of energy radiated in the unit of time by a single extragalactic nebula can easily be computed from its absolute magnitude. Comparing this with the adopted mass we find approximately

$$\frac{\dot{\alpha}}{\alpha} = -10^{-19},$$

and comparing this with the observed value (13) of $\dot{R}/R$, using the value (12) of R_B, we have

$$\gamma = 2.10^{-7}.$$

We can thus safely neglect the influence of γ, and consequently treat α and β as constants.

8. We are now left with the second equation (14), in which ρ_0 and p have the values (16), α and β being constants. We put

$$z = \frac{R}{R_0},$$

R_0 being a certain initial value. The equation then becomes

$$\dot{z}^2 = \frac{\lambda}{3}\,\frac{(z^2 - 2z + 1 + a)(z^2 + 2z + b)}{z^2} = \frac{\lambda}{3}\,\frac{Z^2}{z^2}, \qquad (18)$$

where

$$\begin{aligned} R_0^2\lambda &= \frac{3}{3 - a - b} \\ \frac{\alpha}{R_0} &= \frac{6(1 + a - b)}{3 - a - b} \\ \frac{\beta}{R_0^2} &= \frac{(1 + a)b}{3 - a - b}. \end{aligned} \qquad (19)$$

The relation between the radius R and the time is given by

$$\sqrt{\frac{\gamma}{3}} \cdot (t - t_0) = \int \frac{zdz}{Z}, \tag{20}$$

t_0 being a constant. Putting

$$x = z - 1,$$

$$X^2 = x^2 + 4x + B^2, \qquad B^2 = 3 + b,$$

the integral becomes

$$\int \frac{(x+1)dx}{X \cdot \sqrt{x^2 + a}}.$$

This is an elliptic integral of the third kind. In the case $a = 0$ it can be expressed by logarithms, thus:

$$\int \frac{(x+1)dx}{x.X} = lg(x + 2 + X) + \frac{1}{B} lg \frac{x + X - B}{x + X + B},$$

omitting an additive constant, which can be included in t_0. The first term becomes positively infinite for $x = \infty$, the second term becomes negatively infinite for $x = 0$, i.e., $z = 1$, $R = R_0$. The radius of the world thus increases from the value R_0 to infinity, both the initial and the final state being reached asymptotically. This is the solution of Lemaître, who, however, only considers the case $b = 0$, or $B = \sqrt{3}$.

An interesting case would be the limiting case $B = 2$, which gives $b = 1$, $\alpha = 0$, i.e., a universe devoid of matter but filled with radiation. In that case we would have

$$\int \frac{(x+1)dx}{x(x+2)} = lg\sqrt{x(x+2)} = lg \frac{\sqrt{(R - R_0)(R + R_0)}}{R_0}.$$

This universe would still expand from $R = R_0$ for $t = -\infty$ to $R = \infty$ for $t = +\infty$.

In the actual universe, however, the value of b is very small. We have seen that the random velocities are of the order of 150 km./sec., or $0.5.10^{-3}$. The ratio $p/\rho = \beta/\alpha R$ is proportional to the square of this, from which we find, using the actual value $z = 2$ which will be derived below,

$$\frac{\beta}{R_0^2} = 5.10^{-7} \frac{\alpha}{R_0},$$

or, by (19)

$$b(1 + a) = 3.10^{-6} (1 + a - b),$$

or

$$b = 3.10^{-6}$$

The condition $a = 0$ is the condition that the equation $Z^2 = 0$ shall have a double root $z_0 = 1$, which must be taken as the lower limit of the integral. If a is different from zero, this double root separates into two, $z_0 = 1 \pm \sqrt{-a}$, which are real if a is negative, and complex if a is positive. In the case that a is negative, the lower limit of the integral is the largest of these two real roots $z_0 = 1 + \sqrt{-a}$, $x_0 = \sqrt{-a}$; if a is positive, the lower limit must be taken $z_0 = 0$, $x_0 = -1$. In both cases the integral from the lower limit to an arbitrary value, say $x = 1$, is finite. I find by numerical integration, for the case $b = 0$,

$$\text{for } a = -0.01: \int_{0 \cdot 1}^{1} \frac{(x+1)dx}{\sqrt{x^2 - 0.01}\,.\,X} = 1.82$$

$$\text{for } a = +0.01: \int_{-1}^{1} \frac{(x+1)\,dx}{\sqrt{x^2 + 0.01}\,.\,X} = 3.35.$$

By means of the value of λ, which will be derived below, the corresponding values of $t - t_0$ are found to be 2.3×10^9 years and 4.4×10^9 years, respectively, periods which are short in comparison to the lifetime of a star. We must thus suppose that in the actual universe the value of a is very small. If not exactly zero, it must necessarily be negative, for it is inconceivable that the radius of the universe should have started from an initial value exactly zero, however long ago that may have been.

9. From (13) and (18) we have

$$\sqrt{\frac{3}{\lambda}} \cdot \frac{Z}{z^2} = \frac{1}{R_B}, \tag{21}$$

and from (9), (16) and (19)

$$\frac{\lambda(1 + a - b)}{z^3} = \frac{1}{R_A^2}. \tag{22}$$

Dividing the square of (21) by (22) we find

$$Z^2 = 3\,\frac{R_A^2}{R_B^2}\,(1 + a - b)z, \tag{23}$$

which is an equation of the fourth degree in z. For $b = 0$ it becomes of the third degree, and if we put also $a = 0$ it becomes:

$$z^3 - 3z + 2 = 3\,\frac{R_A^2}{R_B^2}. \tag{24}$$

The values (12) of R_A and R_B give

$$\frac{R_A^2}{R_B^2} = \frac{4}{3},$$

by which the equation (24) becomes

$$z^3 - 3z - 2 = 0, \tag{24'}$$

from which we find

$$z = 2. \tag{25}$$

The present value of R is thus twice the initial value R_0.

Then from (19) and (22) we have

$$\lambda = \frac{z^3}{1 + a - b} \cdot \frac{1}{R_A^2}$$

$$R_0^2 = \frac{3(1 + a - b)}{3 - a - b} \cdot \frac{R_A^2}{z^3}$$

Using the value (12) of R_A, (25) of z and $a = 0$, $b = 0$, we find

$$\lambda = 1.5 . 10^{-6} A^{-2}$$

$$R_0 = 0.8 . 10^3 A,$$

and consequently the present radius of the universe is

$$R = 1.6 . 10^3 A = 5 . 10^8 \text{ parsecs.}$$

These numerical values are, of course, still very uncertain.

10. The dynamical solution of the field equations thus is found to account for the expansion of the universe, which is observed in the radial velocities of the extragalactic nebulae. The universe is homogeneously filled with matter and has spherical symmetry throughout its history, its radius increasing from an initial value R_0 to infinity. The initial state corresponds to Einstein's solution (A). This solution, however, is unstable, and the radius must change. The change is taken to be an expansion, in accordance with the observed positive radial velocities, but it should be pointed out that the differential equation only gives the square of $\dot{R}$, and leaves the sign undetermined. This is the fundamental problem of why the time has a definite sign, which is inevitably connected with the description of nature by differential equations of the second order. It should be noted, however, that the dynamical solution requires all observed radial velocities to have *the same* sign, while in the solution (B) the sign was indeterminate for each individual body.

The date of this initial state may, according to the value of the indeterminate constant a, be either a finite or an infinite time ago. The three constants γ, a and b have in the theory as here worked out been brought into the form of pure numbers, so that their value is independent of the units used. It has been found that all three must be very small in the actual universe. These constants are of a very different character from the constants λ, κ, c, which are much more fundamental, and can be made equal to unity by an appropriate choice of units. The constant λ, which is a measure of the inherent expanding force of the universe, is still very mysterious, and it is difficult to see what its real meaning is. It might even be thought to be one constant too many, unless we may hope that it will ultimately be found to be in some way connected with Planck's constant h. Evidently the dynamical solution of Lemaître is not yet the last word, but it can hardly be doubted that it represents an important step towards the true interpretation of nature.

LEIDEN, April, 1930

[1] *Astroph. J.*, **64**, pp. 321–369; *Mt. Wilson Contributions* No. 324, 1926.

[2] *Proc. Nat. Acad. Sci.*, **15**, p. 172, 1929.

[3] On Einstein's theory of gravitation and its astronomical consequences, third paper, *Mon. Nat. R. Astron. Soc.*, **78**, pp. 3–28, 1917.

[4] It should be noted that in both (A) and (B) the pressure is not necessarily zero. In (A) there may be radiant energy and random motions of the matter. The complete equation (9) is $\kappa(4p + \rho_0) = 2/R^2$. In (B) the corresponding equation is $4p + \rho_0 = 0$, thus if ρ_0 is positive, the pressure must be negative. See "third paper," *M. N.*, **78**, footnote to p. 21.

[5] See "third paper," *M. N.*, **78**, pp. 18–19.

[6] Un Univers homogène de masse constante et de rayon croissant, rendant compte de la vitesse radiale des nébuleuses extra-galactiques, *Annales de la Société scientifique de Bruxelles*, **47**, p. 49, April, 1927. A similar solution had previously been given by A. Friedman, *Zeitschrift für Physik*, **10**, p. 377, 1922.

THE CELLULAR DIVISION AND APPROXIMATION OF REGULAR SPREADS

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The theorems and proofs outlined below are complete and will be elaborated elsewhere.

The terms *cell simplex* and *complex* are used in the sense of Veblen's definitions.[1]

Let $(y) = (y_1, \ldots, y_n)$ be rectangular coördinates in Euclidean n-space, $n \geq 2$, and let $(u) = (u_1, \ldots, u_i)$ be coördinates in an auxiliary i-space, $i < n$. A *regular i-cell* in the space of the y's will mean an i-cell definable by a correspondence with a simplex in the space of the u's in such a way that

(1) the y's as functions of the u's are continuous with their first partial derivatives, and