troduce an insuperable difficulty regarding specific heat. Even with the assumption that I have made as to the length of l, the influence of the free electrons, and still more the influence of ionization with increase of temperature, on the specific heat, presents a rather serious question. Thus, to take what is probably the most unfavorable case set forth in the preceding tables, if in Mg $q = 1.5$ and $s = 6.5$ and $\lambda'_0 = 460\ R$ and $\gamma = 12\%$, we have, as the heat absorbed by the free electrons in the rise of 1 gm. of Mg from 0 °C. to 1° C., 0.025 cal., while the heat required by the accompanying ionization is 0.124 cal., a total of 0.149 cal., which is rather more than half the total specific heat of Mg at 0° C. It would be easy, however, to choose values of q, s and λ'_0 for Mg which would serve the purpose of this paper while affecting the specific heat less. Moreover, it is possible that l may be greater than I have taken it to be.

Table 17 of my paper in these *Proceedings* for March, 1920, now requires revision, Al, Fe, Mo and Tl going into Section (A); but the general testimony of the table remains unchanged, the *mean* of the last column being now -1.50% for Section (A) and $+2.47\%$ for Section (B).

It remains to be seen whether the theory I am developing can deal successfully with the Peltier effect. A preliminary examination, already made, of this matter is encouraging.

[1] Equipartition of energy, as holding for electrons *within* an atom, cannot be regarded as an established principle.

[2] *Physic. Rev., Ithaca,* April, 1918 (329).

[3] In my previous paper δ_0 had a different meaning.

[4] *London, Phil. Mag.*, **28,** 1914 (692–702).

[5] *Physic Rev., Ithaca,* May 1919 (374–391).

MOTION ON A SURFACE FOR ANY POSITIONAL FIELD OF FORCE

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1. The first part of the paper presents a study of the geometric properties of the system of trajectories generated by the motion of a particle on any constraining surface (spread of two dimensions) under any positional field of force. The complete characteristic properties are derived.[1] Starting at any point on the surface in a given direction, and with a given speed, a unique trajectory is generated. The complete system of trajectories forms a triply infinite system of curves corresponding uniquely to a given field of force. The five geometric properties stated in section 2 are characteristic of the system of trajectories, and any triply infinite system of curves on a surface possessing these five properties may be considered as generated by the motion of a particle in a unique field of force.

An additional property serves to characterize the motion when the field of force is conservative.

Another part of the paper presents briefly an analogous study for certain other classes of triply infinite systems of curves on a surface, in particular, brachistochrones and catenaries in a conservative field of force. For all such systems five characteristic properties (sec. 3) differing but slightly from those for trajectories are derived.

2. The motion of a particle on a surface

$$x = x(u,v),\ y = y(u,v),\ z = z(u,v)$$

may be most simply expressed by the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \phi,\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \psi,$$

where T is the kinetic energy, and ϕ, ψ are the components of the force given as functions of the coördinates[2] u,v. The differential equation of such a system found by eliminating the time has the form

$$v''' = A + Bv'' + Cv''^2,$$

where A,B,C are special functions of u,v,v'.

The following five characteristic properties of dynamical trajectories under any positional field of force are derived by studying the differential equation of such a system.

I. The ∞^1 curves passing through a given point O in a given direction ξ have associated with them their orthogonal projections in the tangent plane to the surface at O. The locus of the foci of the osculating parabolas of the associate system is a bicircular quartic with O as node, and the straight line in the direction ξ both as tangent line to the quartic and also as one of the asymptotes to the hyperbola which is the inverse of the quartic with respect to the given point O.

II. The focal locus or bicircular quartic associated by Property I with each direction ξ through a point O, is such that the tangent line to the quartic in the direction ξ bisects the angle between the force vector through O and the second tangent to the quartic.

III. Through every point O on the surface and in every direction ξ through that point, there passes *one* curve of the system which hyperosculates its corresponding geodesic circle of curvature. The locus of the centers of geodesic curvature of the ∞^1 hyperosculating trajectories which pass through O is a conic passing through O in the direction of the force vector.

IV. The points of the surface and the directions of the force vectors through these form a set of differential elements defining a simple system of ∞^1 curves on the surface, viz., the lines of force. At any point O, the geodesic curvature of the line of force through O is equal to three times the geodesic curvature of the hyperosculating trajectory of Property III which passes through O in the same direction.

V. Construct any isothermal net on the surface. At any point O, this net determines two orthogonal directions in which there pass two isothermal curves of the net and two hyperosculating curves of Property III. If ρ_1, ρ_2, R_1, R_2 are the radii of geodesic curvature of these four curves, s_1, s_2, the arc lengths along the isothermal curves, and ω, the tangent of the angle between the force vector and the isothermal curve with arc s_2, then, as we move along the surface from O, these quantities vary so as to satisfy the relation

$$\frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right) - \frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right) - \frac{1}{\rho_1\kappa_1} + \frac{1}{\rho_2\kappa_2} - \frac{\partial^2(\log \omega)}{\partial s_1 \partial s_2} = 0,$$

where

$$\frac{1}{\kappa_1} = \omega\left(\frac{1}{\rho_1} - \frac{3}{R_1}\right), \frac{1}{\kappa_2} = \frac{1}{\omega}\left(\frac{1}{\rho_2} - \frac{3}{R_2}\right).$$

If, in these five properties, we replace the force vector through a point by the tangent line to the conic of property III, we may state the theorem:

In order that a triply infinite system of curves (∞.[1] *in each direction through each point) on a surface may be identified with a system of dynamical trajectories under any positional field of force, the given system must possess properties I, II, III, IV, V.*

When the surface is a developable surface, the bicircular quartic of property I reduces to a circle passing through the given point.

If the field of force is conservative, the conic of property III reduces to a rectangular hyperbola.

3. If we consider the motion of a particle on a surface in a conservative field of force from one position P_o to another P_1 with the sum of the kinetic and potential energies equal to a given constant, h, certain systems of ∞^2 curves are defined by

$$\int_{(P_o)}^{(P_1)} \sqrt[n]{W + h}\, ds = \text{minimum},$$

where W is the work function (negative potential) and ds is the element of arc length. Among such systems, called "n" systems, we may mention *dynamical trajectories* ($n = 2$), *brachistochrones* ($n = -2$), and *catenaries* ($n = 1$). We may find the differential equation of the ∞^2 curves of an "n" system by the methods of the calculus of variation.[3] If, by differentiation, we eliminate the constant of energy, h, from this equation, we shall get the differential equation of the ∞^3 curves of a complete "n" system. This equation has the form

$$v''' = A + Bv'' + Cv''^2,$$

where A, B, C are special function of u, v, v', and involve n as a parameter.

The following five characteristic properties of an "n" system on a surface are derived by studying the differential equation of such a system.

I'. The "n" system possesses property I of section 2.

II′. For an "n" system, the focal locus or bicircular quartic associated by property I′ with each direction ξ through a point O, is such that the tangent of the angle which the tangent line to the quartic in the direction ξ makes with the force vector is to the tangent of the angle which the second tangent line to the quartic makes with the first tangent line as $n + 1$ is to 3.

III′. The "n" system possesses property III of section 2. The conic described in this property is a rectangular hyperbola.

IV′. For an "n" system, at any point O on the surface, the geodesic curvature of the line of force through O is equal to $(n + 1)$ times the geodesic curvature of the hyperosculating curve of property III′ which passes through O in the same direction.

V′. For an "n" system, as we move along the surface from a point O, the quantities ρ_1, ρ_2, R_1, R_2, ω, s_1, s_2, defined in property V of section 2, vary so as to satisfy the relation

$$\frac{\partial}{\partial s_2}\left(\frac{1}{\kappa_1}\right) - \frac{\partial}{\partial s_1}\left(\frac{1}{\kappa_2}\right), - \frac{1}{\rho_1\kappa_1} + \frac{1}{\rho_2\kappa_2} - \frac{\partial^2(\log \omega)}{\partial s_1 \, \partial s_2} = 0,$$

where

$$\frac{1}{\kappa_1} = \omega\left(\frac{1}{\rho_1} - \frac{n+1}{R_1}\right), \frac{1}{\kappa_2} = \frac{1}{\omega}\left(\frac{1}{\rho_2} - \frac{n+1}{R_2}\right).$$

If, in these five properties, we replace the direction of the force vector through a point by the direction of the tangent line to the rectangular hyperbola of property III′, we may state the theorem:

In order that a triply infinite system of curves on a surface may be identified with an "n" system, the given system must possess properties I′, II′, III′, IV′, V′.

[1] This is a summary of a long memoir offered for publication to the *Ann. Math., Princeton, N. J.* For a study of the corresponding problem in a plane or in ordinary 3-space, see Kasner, E., *Trans. Amer. Math. Soc., New York,* **7,** 1906 (401–424); also *Ibid.,* **8,** 1907 (135–158). For a summary of these two papers, see *Princeton Colloquium Lectures,* Chap. I.

[2] Dots refer to total derivatives with respect to the time t, and primes refer to total derivatives with respect to the parameter u.

[3] For a characterization of the ∞^2 curves of a natural system on a surface, defined by $\int F\,ds$ = *minimum,* where F is a function of the coördinates only, see Lipka, J., *Ann. Math., Princeton, N. J.,* **15,** 1913 (71–77).