PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 9 JUNE 15, 1923 Number 6

ANOTHER INTERPRETATION OF THE FUNDAMENTAL GAUGE-VECTOR OF WEYL'S THEORY OF RELATIVITY

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Read before the Academy, April 24, 1923

1. In the geometry of paths as developed by Professor Veblen and myself in a number of papers in volumes eight and nine of these PROCEEDINGS, the idea is that the paths are a generalization of straight lines in euclidean space. We take the equations of the paths in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{1}$$

where s is a parameter peculiar to each path and $\Gamma^i_{\alpha\beta}$ $(=\Gamma^i_{\beta\alpha})$ are functions of the x's. Now I make the assumption that *physical phenomena manifest themselves in paths in a space-time continuum of four dimensions and that the functions $\Gamma^i_{\alpha\beta}$ are determined by the character of the phenomena.* In this note I apply this idea to the case of electro-magnetic phenomena as developed in the general theory of relativity, and the results raise the question whether Weyl, and later Eddington, are justified in the assumption that the fundamental vector introduced by Weyl in his gauging system is the electro-magnetic potential of the field.

2. Suppose that the fundamental quadratic form of the space-time continuum is written in the general form

$$ds^2 = g_{ij}dx^i dx^j \qquad (g_{ij} = g_{ji}) \tag{2}$$

Let K_α and K^β denote the covariant and contravariant components of the electro-magnetic potential in a general system of coördinates, x^i. As usual we put

$$F_{\mu\nu} = \frac{\partial K_\mu}{\partial x^\nu} - \frac{\partial K_\nu}{\partial x^\mu}, \tag{3}$$

and

$$F^{\mu}_{\nu} = g^{\mu\alpha}F_{\alpha\nu}, \qquad F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}, \tag{4}$$

where $g^{\mu\alpha}$ is the cofactor of $g_{\mu\alpha}$ in the determinant divided by

$$g = |g_{\mu\nu}| \tag{5}$$

From (3) follows the second set of Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = 0, \tag{6}$$

and the first set in tensor form are

$$F^{\mu\nu}_{\nu} = J^{\mu} \tag{7}$$

where the left-hand member is the sum for ν of the covariant derivatives of $F^{\mu\nu}$, and J^{μ} are the contravariant components of the charge-current vector in general coördinates; thus

$$J^{\mu} = \rho_0 \frac{dx^{\mu}}{ds}, \tag{8}$$

where ρ_0 is the proper-density of the charge.[1]

The equations of motion, in tensor form, of an electric charge are

$$\mu\left(\frac{d^2x^i}{ds^2} + \begin{Bmatrix} \alpha\beta \\ i \end{Bmatrix} \frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}\right) = -F^{i}_{\nu}J^{\nu}, \tag{9}$$

where μ is the mass-density and $\begin{Bmatrix} \alpha\beta \\ i \end{Bmatrix}$ denotes the Christoffel symbols of the second kind with respect to (2).[2]

3. If we put

$$\varphi^i = -\frac{1}{\mu}F^{i}_{\nu}J^{\nu} = -\frac{1}{T}F^{i}_{\nu}J^{\nu}, \tag{10}$$

where, in consequence of (2),

$$T = \mu g_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = \mu, \tag{11}$$

equations (9) may be written

$$\frac{d^2x^i}{ds^2} + \left(\begin{Bmatrix} \alpha\beta \\ i \end{Bmatrix} - g_{\alpha\beta}\varphi^i\right)\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 0. \tag{12}$$

These equations are of the form (1), where

$$\Gamma^i_{\alpha\beta} = \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\} - g_{\alpha\beta}\varphi^i. \tag{13}$$

Elsewhere,[3] I have shown that the same paths are defined by

$$\frac{d^2x^i}{ds'^2} + \Gamma'^i_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} = 0, \tag{14}$$

if we take

$$\Gamma'^i_{\alpha\beta} = \delta^i_\alpha\varphi_\beta + \delta^i_\beta\varphi_\alpha, \qquad \frac{d^2s}{ds'^2} = -2\varphi_\alpha\frac{dx^\alpha}{ds}\left(\frac{ds}{ds'}\right)^2. \tag{15}$$

From (8), (10) and the skew-symmetric character of $F_{\mu\nu}$, we have

$$\varphi_\alpha\frac{dx^\alpha}{ds} = 0, \tag{16}$$

and hence from (15) we see that s and s' can be taken as the same. From (13) and (15) it follows that the functions Γ for the paths may be given the form

$$\Gamma^i_{\alpha\beta} = \left\{ \begin{matrix} \alpha\beta \\ i \end{matrix} \right\} + \delta^i_\alpha\varphi_\beta + \delta^i_\beta\varphi_\alpha - g_{\alpha\beta}\varphi^i. \tag{17}$$

The expressions (17) are those given by Weyl[4] for the coefficients of the affine connection of a manifold whose character is determined directly by gravitation and electro-magnetism. Weyl assumes that the vector φ_α in (17) is the electro-magnetic potential. From the foregoing results it would seem that φ_α is not the electro-magnetic potential but is functionally connected with it, K_α, as given by (10), (3) and (7). There are other considerations which point to this view.

4. In accordance with the Weyl[5] geometry a new gauge system is introduced by altering the length of the standard at each point in the ratio $\lambda^{1/2}$, where λ is an arbitrary point function. If the standard is decreased in the ratio $\lambda^{1/2}$, a length will be increased in this ratio and for the new system

$$ds'^2 = \lambda ds^2, \qquad g'_{\alpha\beta} = \lambda g_{\alpha\beta}, \tag{18}$$

the coördinate system being independent of the gauge.

If the components of a tensor in the new gauge system are λ^e times those in the old system, the tensor is said to be of weight e. Thus from (18) it follows that $g_{\alpha\beta}$ is of weight one, and from (12) that φ^i is of weight minus one, and consequently φ_μ ($= g_{\mu i}\varphi^i$) is of weight zero. Furthermore it follows from (3), (4) and (7) that if K_α is taken of weight one, then φ^i is of weight minus one from (10), which is in accordance with (12).

Pauli[6] pointed out that the Einstein equations

$$R_{ik} - \frac{1}{2} g_{ik} R = - kT_{ik} \tag{19}$$

are inconsistent, when it is assumed, as Weyl has done, that the convariant components K_α of the electro-magnetic potential are of weight zero. In fact, on this hypothesis, the left-hand and right-hand members of (19) are respectively of weights zero and minus one. Jüttner[7] called attention to the same difficulties in the case of the equations

$$R_{ik} - \frac{1}{4} g_{ik}R = - kS_{ik}, \tag{20}$$

where

$$S_{ik} = \frac{1}{4} g_{ik}F_{\alpha\beta}F^{\alpha\beta} - F_{i\alpha}F_{k\beta}g^{\alpha\beta}.$$

He tried to overcome the inconsistencies by changing the form of these equations, while still holding to the hypothesis that K_α is of weight zero.

Mr. A. Bramley of the Department of Physics of Princeton University has shown, in a paper to be offered to the *Philosophical Magazine*, that equations (19) and (20) are consistent, if the weight of K_α is taken to be one, and if K_α is not supposed to be the fundamental vector φ_α of the gauging system, but functionally related to it in such a way that φ_α is of weight zero. Equations (10) are consistent with the hypothesis made by Bramley. In view of the foregoing considerations it is a question whether Weyl's assumption that the gauging vector φ_i is the electro-magnetic potential should be rejected, and the relation (10) be adopted to give the relation between φ_α and the electro-magnetic potential K_α.

1 Eddington, *The Mathematical Theory of Relativity*, Cambridge, 1923, pp. 173, 190.

2 Eddington, *l. c.*, p. 190.

3 These PROCEEDINGS, 8, 1922 (234); also Veblen, *Ibid.* (347).

4 Cf. Weyl, *Space, Time and Matter*, English translation, pp. 125, 284; also Eddington, *l. c.*, pp. 202, 203.

5 *L. c.*, p. 127; also, Eddington, *l. c.*, pp. 200, 203, Eddington shows that (17) is invariant for gauge-transformations, and for this reason (17) is used in place of (13). As shown by (14), (15) and (16) the forms (17) and (13) lead to the same paths.

6 *Encycl. Math. Wiss.*, vol. V_2, part 4, p. 767.

7 *Math. Ann.*, 87, 1922 (282).