THE POLYTOPES WITH REGULAR-PRISMATIC VERTEX FIGURES

PART 2*.

By H. S. M. COXETER.

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Contents.

Preface.

The idea of the fundamental region of a group is familiar. (A list of references will be found at the end of this paper.) The *orthoscheme*, which Schläfli† associates with a regular polytope, is really a fundamental region for the group of symmetries of the polytope. Dr. J. A. Todd‡ has recently used this fact in order to obtain *abstract definitions* for the groups of symmetries of the regular polytopes.

* Part 1 of this paper appears in the *Phil. Trans. Royal Soc.* (A), 229 (1930), 329-425. The paragraphing of this part follows on that of Part 1.

† "Theorie der vielfachen Kontinuität", *N. Denkschr. Schweiz. Ges. Natw.*, 38 (1901).

‡ "The groups of symmetries of the regular polytopes", *Proc. Camb. Phil. Soc.*, 27 (1931), 212-231.

On 24 November, 1930, Dr. G. de B. Robinson urged me to seek a fundamental region for the "pure Archimedean" polytope n_{21}. By fitting together three special orthoschemes, I found the required fundamental region, namely a simplex all of whose dihedral angles were either $\frac{1}{2}\pi$ or $\frac{1}{3}\pi$. This result led to an empirical generalization, and to an extension of the Schläfli symbol $\{k_1, k_2, ..., k_{m-1}\}$. Afterwards I proved the general result, which can be stated as follows :

Every group of real orthogonal substitutions on m variables, having as fundamental region a simplex all of whose dihedral angles are submultiples of π, *is either the whole group of symmetries of some m-dimensional uniform polytope, or a sub-group thereof.*

An abstract definition for such a group, and in particular for the group of automorphisms of the twenty-seven lines on a cubic surface or of the twenty-eight bitangents of a plane quartic, can be written down at once.

A preliminary account of this work appears in the *Journal London Math. Soc.*, 6 (1931), 132–136. In the last line but eight of page 134, the words "central projections of" should be inserted after the word "vertices".

I should like to express here my thanks to Dr. Todd and Dr. Robinson for their inspiration and encouragement.

14. *An extension of the Schläfli symbol.*

14.1. At the end of § 5.3 we observed that the vertex figure of the polytope $t_n\{k_1, k_2, ..., k_{m-1}\}$ is a generalized prism whose two constituents are the vertex figures of

$$\{k_n, k_{n-1}, ..., k_1\} \quad \text{and} \quad \{k_{n+1}, k_{n+2}, ..., k_{m-1}\}.$$

This fact suggests the new notation

$$t_n\{k_1, k_2, ..., k_{m-1}\} = \begin{Bmatrix} k_n, k_{n-1}, ..., k_1 \\ k_{n+1}, k_{n+2}, ..., k_{m-1} \end{Bmatrix},$$

which is justified by the identities

$$t_0\{k_1, k_2, ..., k_{m-1}\} = \{k_1, k_2, ..., k_{m-1}\}$$

and $$t_{m-1}\{k_1, k_2, ..., k_{m-1}\} = \{k_{m-1}, k_{m-2}, ..., k_1\}.$$

We thus define

$$\begin{Bmatrix} i_0, i_1, ..., i_{p-1} \\ j_0, j_1, ..., j_{q-1} \end{Bmatrix} = \left[\{i_1, ..., i_{p-1}\}\, 2\cos\frac{\pi}{i_0}, \quad \{j_1, ..., j_{q-1}\}\, 2\cos\frac{\pi}{j_0}\right]^{+1}$$

(in the notation of § 7.1).

It follows from this definition that $\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$ has in general two kinds of bounding figure:

$$\begin{Bmatrix} i_0, i_1, \ldots, i_{p-2} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} \text{ and } \begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-2} \end{Bmatrix}.$$

Thence it follows that the general element (apart from vertices) is

$$\begin{Bmatrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q).$$

This result is perfectly analogous to the fact that the general element of $\{k_1, k_2, \ldots, k_{m-1}\}$ is

$$\{k_1, k_2, \ldots, k_{m'-1}\} \quad (1 \leqslant m' \leqslant m),$$

and it is easily seen to agree with (5.23).

14.2. Since

$$\{k_1, k_2, \ldots, k_{m-1}\} = \left(\{k_2, \ldots, k_{m-1}\}\, 2\cos\frac{\pi}{k_1}\right)^{+1},$$

it is natural to define

$$\begin{Bmatrix} k_1, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} = \left(\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} 2\cos\frac{\pi}{k_1}\right)^{+1}.$$

The general element (apart from vertices and edges) is now

$$\begin{Bmatrix} k_1, & i_0, i_1, \ldots, i_{p'-1} \\ & j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q).$$

Similarly, we can define inductively

$$\begin{Bmatrix} k_1, k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} = \left(\begin{Bmatrix} k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix} 2\cos\frac{\pi}{k_1}\right)^{+1}.$$

The general element of not more than n dimensions is simply the general element of $\{k_1, k_2, \ldots, k_n\}$, while the general element of more than n dimensions is

$$\begin{Bmatrix} k_1, k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p'-1} \\ & j_0, j_1, \ldots, j_{q'-1} \end{Bmatrix} \quad (0 \leqslant p' \leqslant p,\ 0 \leqslant q' \leqslant q).$$

Since $\{i_1, \ldots, i_{p-1}\}$ is p-dimensional, $\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$ is $(p+q+1)$-dimensional, whence $\begin{Bmatrix} k_1, k_2, \ldots, k_n, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$ is $(n+p+q+1)$-dimensional. Thus the number of dimensions is one more than the number of digits involved in the symbol.

Note that

$$\begin{Bmatrix} k_1, k_2, \ldots, k_n, & \begin{matrix} j_0, j_1, \ldots, j_{q-1} \\ i_0, i_1, \ldots, i_{p-1} \end{matrix} \end{Bmatrix} = \begin{Bmatrix} k_1, k_2, \ldots, k_n, & \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \end{Bmatrix}.$$

14.3. Let us now illustrate this notation by giving numerical values to the i's, j's, and k's.

$\begin{Bmatrix}3\\3\end{Bmatrix}=t_1\alpha_3=\beta_3=\{3, 4\}$, the octahedron, whose vertex figure is $[\alpha_1, \alpha_1]=\beta_2$;

$\begin{Bmatrix}3\\4\end{Bmatrix}=t_1\beta_3$, the cuboctahedron ,, ,, ,, $[\alpha_1, \beta_1]$;

$\begin{Bmatrix}3\\5\end{Bmatrix}=t_1\{3, 5\}$, the icosidodecahedron ,, ,, ,, $[\alpha_1, \alpha_1\tau]$;

$\begin{Bmatrix}3\\6\end{Bmatrix}=t_1\{3, 6\}$,, ,, ,, $[\alpha_1, \alpha_1\sqrt{3}]$;

$\begin{Bmatrix}4\\4\end{Bmatrix}=t_1\delta_3=\delta_3=\{4, 4\}$, "squared paper" ,, ,, ,, $[\beta_1, \beta_1]=\beta_2\sqrt{2}$.

$\begin{Bmatrix}3, & 3\\ & 3\end{Bmatrix}=\{3, 3, 4\}=\beta_4$,, ,, ,, $\begin{Bmatrix}3\\3\end{Bmatrix}=\{3, 4\}=\beta_3$;

$\begin{Bmatrix}4, & 3\\ & 3\end{Bmatrix}=\{4, 3, 4\}=\delta_4$,, ,, ,, $\begin{Bmatrix}3\\3\end{Bmatrix}\sqrt{2}=\beta_3\sqrt{2}$;

$\begin{Bmatrix}3, & 3\\ & 4\end{Bmatrix}=h\delta_4=\alpha_3 h$,, ,, ,, $\begin{Bmatrix}3\\4\end{Bmatrix}=t_1\beta_3=e\alpha_3$;

$\begin{Bmatrix}3 & \\ 3, & 3\end{Bmatrix}=t_1\alpha_4$,, ,, ,, $[\alpha_1, \alpha_2]$;

$\begin{Bmatrix}3 & \\ 3, & 4\end{Bmatrix}=t_1\beta_4=\{3, 4, 3\}$,, ,, ,, $[\alpha_1, \beta_2]=\gamma_3$;

$\begin{Bmatrix}4 & \\ 3, & 3\end{Bmatrix}=t_1\gamma_4$,, ,, ,, $[\beta_1, \alpha_2]$;

$\begin{Bmatrix}4 & \\ 3, & 4\end{Bmatrix}=t_1\delta_4$,, ,, ,, $[\beta_1, \beta_2]$;

$\begin{Bmatrix}3 & \\ 4, & 3\end{Bmatrix}=t_1\{3, 4, 3\}$,, ,, ,, $[\alpha_1, \alpha_2\sqrt{2}]$;

$\begin{Bmatrix}3 & \\ 3, & 5\end{Bmatrix}=t_1\{3, 3, 5\}$,, ,, ,, $[\alpha_1, \{5\}]$;

$\begin{Bmatrix}5 & \\ 3, & 3\end{Bmatrix}=t_1\{5, 3, 3\}$,, ,, ,, $[\alpha_1\tau, \alpha_2]$.

$$\left\{3,\ 3,\ \begin{matrix}3\\3\end{matrix}\right\} = \{3,\ 3,\ 3,\ 4\} = \beta_5;$$

$$\left\{4,\ 3,\ \begin{matrix}3\\3\end{matrix}\right\} = \{4,\ 3,\ 3,\ 4\} = \delta_5;$$

$$\left\{3,\ \begin{matrix}3 & \\ 3, & 3\end{matrix}\right\} = h\gamma_5;$$

$$\left\{3,\ \begin{matrix}3 & \\ 3, & 4\end{matrix}\right\} = \{3,\ 3,\ 4,\ 3\} = h\delta_5;$$

$$\left\{\begin{matrix}3 & & \\ 3, & 3, & 3\end{matrix}\right\} = t_1\alpha_5; \qquad \left\{\begin{matrix}3, & 3\\ 3, & 3\end{matrix}\right\} = t_2\alpha_5;$$

$$\left\{\begin{matrix}3 & & \\ 3, & 3, & 4\end{matrix}\right\} = t_1\beta_5; \qquad \left\{\begin{matrix}3, & 3\\ 3, & 4\end{matrix}\right\} = t_2\beta_5;$$

$$\left\{\begin{matrix}4 & & \\ 3, & 3, & 3\end{matrix}\right\} = t_1\gamma_5; \qquad \left\{\begin{matrix}4 & & \\ 3, & 3, & 4\end{matrix}\right\} = t_1\delta_5;$$

$$\left\{\begin{matrix}3 & & \\ 3, & 4, & 3\end{matrix}\right\} = t_1\{3,\ 3,\ 4,\ 3\} = \{3,\ 4,\ 3,\ 3\} = t_2\delta_5 = \left\{\begin{matrix}3, & 4\\ 3, & 4\end{matrix}\right\};$$

$$\left\{\begin{matrix}3 & & \\ 4, & 3, & 3\end{matrix}\right\} = t_1\{3,\ 4,\ 3,\ 3\}; \qquad \left\{\begin{matrix}3, & 3\\ 4, & 3\end{matrix}\right\} = t_2\{3,\ 3,\ 4,\ 3\}.$$

$$\left\{3^{m-3},\ \begin{matrix}3\\3\end{matrix}\right\}^{*} = \beta_m; \quad \left\{3,\ \begin{matrix}3 & \\ 3^{m-3} & \end{matrix}\right\} = h\gamma_m; \quad \left\{4,\ 3^{m-4},\ \begin{matrix}3\\3\end{matrix}\right\} = \delta_m; \quad \left\{3,\ \begin{matrix}3 & \\ 3^{m-4}, & 4\end{matrix}\right\} = h\delta_m.$$

$$\left\{\begin{matrix}3^n \\ 3^{m-n-1}\end{matrix}\right\} = t_n\alpha_m; \quad \left\{\begin{matrix}3^n & \\ 3^{m-n-2}, & 4\end{matrix}\right\} = t_n\beta_m; \quad \left\{\begin{matrix}3^{n-1}, & 4\\ 3^{m-n-1} & \end{matrix}\right\} = t_n\gamma_m; \quad \left\{\begin{matrix}3^{n-1}, 4 & \\ 3^{m-n-2}, & 4\end{matrix}\right\} = t_n\delta_m.$$

$$\left\{3^n,\ \begin{matrix}3^p\\3^q\end{matrix}\right\} = n_{pq}.$$

14.4. The notation can be further extended. By § 5.8 with l for n, if Π_m has an $(l+1)$-th vertex figure, then

$$t_l\,\Pi_m = [\Pi_l',\ \Pi_{m-l-1,\,l+1}]^{+1}.$$

Now let
$$\Pi_m = \left\{k_1,\ k_2,\ \ldots,\ k_n,\ \begin{matrix}i_0,\ i_1,\ \ldots,\ i_{p-1}\\ j_0,\ j_1,\ \ldots,\ j_{q-1}\end{matrix}\right\}.$$

In this case, if $l \leqslant n$,

$$\Pi_l' \text{ is the vertex figure of } \{k_l,\ k_{l-1},\ \ldots,\ k_1\}$$

* 3^n stands for 3, 3, .. , 3, with n 3's.

and

$$\Pi_{m-l-1,\,l+1} \text{ is the vertex figure of } \left\{ k_{l+1}, k_{l+2}, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}.$$

Accordingly, it is perfectly analogous to write

$$(14.41)\quad t_l \left\{ k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} k_l, k_{l-1}, \ldots, k_1 & \\ & i_0, i_1, \ldots, i_{p-1} \\ k_{l+1}, k_{l+2}, \ldots, k_n, & \\ & j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \qquad (l \leqslant n).$$

The three kinds of bounding figure are obtained by omitting the k_1 or the i_{p-1} or the j_{q-1}, respectively.

In particular,

$$t_{n-1} \left\{ k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} k_{n-1}, k_{n-2}, \ldots, k_1 & \\ & i_0, i_1, \ldots, i_{p-1} \\ k_n, & \\ & j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\},$$

and, finally,

$$t_n \left\{ k_1, k_2, \ldots, k_n, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} k_n, k_{n-1}, \ldots, k_1 \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}.$$

This last symbol means

$$\left[\{k_{n-1}, \ldots, k_1\}\, 2\cos\frac{\pi}{k_n},\ \left[\{i_1, \ldots, i_{p-1}\}\, 2\cos\frac{\pi}{i_0},\ \ \{j_1, \ldots, j_{q-1}\}\, 2\cos\frac{\pi}{j_0} \right] \right]^{+1}.$$

By (4.22), the inner square brackets can be removed, and the rows of our three-rowed symbol are permutable. Writing

$$h_r = k_{n-r},$$

we now have

$$(14.42)\qquad t_n \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} = \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$$

$$= t_p \left\{ i_{p-1}, i_{p-2}, \ldots, i_0, \begin{matrix} j_0, j_1, \ldots, j_{q-1} \\ h_0, h_1, \ldots, h_{n-1} \end{matrix} \right\}$$

$$= t_q \left\{ j_{q-1}, j_{q-2}, \ldots, j_0, \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \end{matrix} \right\}.$$

When every h, i, j is equal to 3, these results reduce to (7.35) and (7.36). Thus, in the notation of (12.11),

$$(14.43) \qquad \left\{\begin{matrix} 3^n \\ 3^p \\ 3^q \end{matrix}\right\} = 0_{npq}.$$

14.5. Since the vertex figure and general element of

$$(14.51) \qquad \left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$$

are respectively

(14.52)

$$\left[\{h_1, \ldots, h_{n-1}\}\, 2\cos\frac{\pi}{h_0},\ \{i_1, \ldots, i_{p-1}\}\, 2\cos\frac{\pi}{i_0},\ \{j_1, \ldots, j_{q-1}\}\, 2\cos\frac{\pi}{j_0}\right]$$

and

$$\left\{\begin{matrix} h_0, h_1, \ldots, h_{n'-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix}\right\} \quad (0 \leqslant n' \leqslant n,\ \ 0 \leqslant p' \leqslant p,\ \ 0 \leqslant q' \leqslant q),$$

the following existence conditions are necessary:

$$\left\{\begin{matrix} h_0, h_1, \ldots, h_{n-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}, \quad \left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-2} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}, \quad \left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-2} \end{matrix}\right\}$$

must all be finite, and the sum of the squared circumradii of the three constituents of the prism (14.52) must not exceed unity. By (2.93), the latter condition is equivalent to

$$\frac{\Delta_{n-1}(h_2, \ldots, h_{n-1})}{\Delta_n(h_1, \ldots, h_{n-1})}\cos^2\frac{\pi}{h_0} + \frac{\Delta_{p-1}(i_2, \ldots, i_{p-1})}{\Delta_p(i_1, \ldots, i_{p-1})}\cos^2\frac{\pi}{i_0} + \frac{\Delta_{q-1}(j_2, \ldots, j_{q-1})}{\Delta_q(j_1, \ldots, j_{q-1})}\cos^2\frac{\pi}{j_0} \leqslant 1,$$

or (subtracting both sides from 3)

$$(14.53) \qquad \frac{\Delta_{n+1}(h_0, h_1, \ldots, h_{n-1})}{\Delta_n(h_1, \ldots, h_{n-1})} + \frac{\Delta_{p+1}(i_0, i_1, \ldots, i_{p-1})}{\Delta_p(i_1, \ldots, i_{p-1})} + \frac{\Delta_{q+1}(j_0, j_1, \ldots, j_{q-1})}{\Delta_q(j_1, \ldots, j_{q-1})} \geqslant 2,$$

in virtue of (2.89). As usual, equality indicates degeneracy.

Here we have tacitly assumed that $npq > 0$. The existence of

$$\begin{Bmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$$

depends solely on the existence of the regular polytope

$$\{i_{p-1}, i_{p-2}, \ldots, i_0, \; j_0, j_1, \ldots, j_{q-1}\},$$

since the former polytope is the t_p truncation of the latter.

14.6. In Chapter 17 we shall see that the vertices of

(14.61) $$\begin{Bmatrix} h_{n-1}, h_{n-2}, \ldots, h_0, & i_0, i_1, \ldots, i_{p-1} \\ & j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$$

are the centres of the bounding $\begin{Bmatrix} h_0, h_1, \ldots, h_{n-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{Bmatrix}$'s of (14.51), so that the existence of (14.61) will follow from the existence of (14.51). But this fact need not be used here; we shall simply find all possible h's, i's, and j's for which (14.51) exists, and then observe that the corresponding polytopes (14.61) are familiar. By (14.42), there cannot be further values of the h's, i's, j's for which (14.61) exists.

When $h_0 = \ldots = h_{n-1} = i_0 = \ldots = i_{p-1} = j_0 = \ldots = j_{q-1} = 3$, (14.53) becomes

$$\tfrac{1}{2}\left(\frac{n+2}{n+1} + \frac{p+2}{p+1} + \frac{q+2}{q+1}\right) \geqslant 2,$$

or

(14.62) $$npq \leqslant n+p+q+2,$$

which is the same as the existence condition (7.32) for O_{npq}.

Assuming that $npq > 0$ (as we may, by the remark at the end of §14.5), the only digit in the symbol $\begin{Bmatrix} 3^n \\ 3^p \\ 3^q \end{Bmatrix}$ that can be increased is the last in a row. For, by (14.42),

$$\begin{Bmatrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{Bmatrix} = t_n \begin{Bmatrix} 4, 3^{n-1}, & 3 \\ & 3 \end{Bmatrix} = t_n \, \delta_{n+3} = t_2 \, \delta_{n+3}.$$

This being degenerate, we cannot introduce further digits into any row.

$\begin{Bmatrix} 3^{n-1}, h \\ i \\ j \end{Bmatrix}$ is impossible if $h > 4$ or $i > 3$, since

$$\frac{\Delta_{n+1}(3, 3, \ldots, 3, h)}{\Delta_n(3, \ldots, 3, h)} + \frac{\Delta_2(i)}{\Delta_1} + \frac{\Delta_2(j)}{\Delta_1}$$

$$= \frac{\{(n+1)/2^n\} - (n/2^{n-1}) \cos^2 \pi/h}{(n/2^{n-1}) - \{(n-1)/2^{n-2}\} \cos^2 \pi/h} + \frac{\sin^2 \pi/i}{1} + \frac{\sin^2 \pi/j}{1}$$

$$= \frac{1}{2(n-1)} \left(n - \frac{1}{1-(n-1)\cos 2\pi/h}\right) + \sin^2 \frac{\pi}{i} + \sin^2 \frac{\pi}{j},$$

which decreases when h or i increases.

Thus, if $npq > 0$, there are only two families of polytopes of the kind we are investigating:

(i) The three semi-reciprocals

$$\left\{3^n, {3^p \atop 3^q}\right\} = n_{pq}, \quad \left\{3^p, {3^q \atop 3^n}\right\} = p_{qn}, \quad \left\{3^q, {3^n \atop 3^p}\right\} = q_{np}$$

and their common truncation

$$\begin{Bmatrix} 3^n \\ 3^p \\ 3^q \end{Bmatrix} = 0_{npq},$$

with the existence condition $npq \leqslant n+p+q+2$.

(ii) The three semi-reciprocals

$$\left\{4, 3^{n-1}, {3 \atop 3}\right\} = \delta_{n+3}, \quad \left\{3, {3 \atop 3^{n-1}, 4}\right\} = h\delta_{n+3} = \left\{3, {3^{n-1}, 4 \atop 3}\right\}$$

and their common truncation

$$\begin{Bmatrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{Bmatrix} = t_2 \delta_{n+3}.$$

14.7. The simplest examples of three-rowed symbols are

$$\begin{Bmatrix} 3 \\ 3 \\ 3 \end{Bmatrix} = \{3, 4, 3\}, \quad \begin{Bmatrix} 3 \\ 3 \\ 4 \end{Bmatrix} = t_1 \delta_4,$$

$$\begin{Bmatrix} 3 \\ 3 \\ 3, 3 \end{Bmatrix} = t_2 \beta_5, \quad \begin{Bmatrix} 3 \\ 3 \\ 3, 4 \end{Bmatrix} = \{3, 4, 3, 3\}.$$

14.8. Since $\left\{\begin{matrix}3\\3\end{matrix}\right\} = \{3, 4\}$, whenever a row of an extended Schläfli symbol ends with "3, 4", this combination can be replaced by a pair of 3's, one over the other. In this manner, the identities

$$\left\{3, \begin{matrix}3\\3, 4\end{matrix}\right\} = \{3, 3, 4, 3\} \quad \text{and} \quad \left\{\begin{matrix}3\\3\\3, 4\end{matrix}\right\} = \{3, 4, 3, 3\}$$

lead respectively to

$$\left\{\begin{matrix} & 3\\ 3, & 3\\ & 3\end{matrix}\right\} = \{3, 3, 4, 3\} \quad \text{and} \quad \left\{\begin{matrix}3\\3\\3\\3\end{matrix}\right\} = \{3, 4, 3, 3\}.$$

The symbols

$$\left\{\begin{matrix} & j & \\ h, & & \\ & 3^{m-5}, & \begin{matrix} i\\ k\end{matrix}\end{matrix}\right\}, \quad \left\{\begin{matrix} & k & \\ i, & & \\ & 3^{m-5}, & \begin{matrix} j\\ h\end{matrix}\end{matrix}\right\}, \quad \left\{\begin{matrix} & h & \\ j, & & \\ & 3^{m-5}, & \begin{matrix} k\\ i\end{matrix}\end{matrix}\right\}, \quad \left\{\begin{matrix} & i & \\ k, & & \\ & 3^{m-5}, & \begin{matrix} h\\ j\end{matrix}\end{matrix}\right\},$$

wherein $h = i = j = k = 3$, can be associated with Du Val's cycle of four semi-reciprocal $h\delta_m$'s. (These are semi-reciprocal in the sense that the reciprocal of each possesses the vertices of the other three together.) In each member of this cycle, the centres of the β_{m-1}'s are the vertices of the opposite member, while the centres of the $h\gamma_{m-1}$'s are the vertices of the two adjacent members.

14.9. It may have seemed pedantic to consider general values for the numbers involved in the symbols (14.51) and (14.61), when never more than two of these numbers can actually exceed 3. It is therefore worth while to remark that, in a Minkowskian or hyper-Minkowskian space (with a certain number of "time-like" dimensions), the h's, i's, and j's can be as great as we please, the restriction (14.53) being withdrawn.

In particular, Du Val has investigated the "pure Archimedean" polytopes n_{21} with $n > 5$.

Such considerations, however, are outside the scope of the present work.

15. *Spherical simplexes whose dihedral angles are submultiples of π.*

15.1. An ordinary spherical triangle can be regarded as the intersection of a sphere with three independent planes through its centre. The angles of the spherical triangle are just the angles between pairs of the planes. If the sphere is of unit radius, these three angles suffice to determine the spherical triangle in both shape and size. But they must not be too small. In fact, if the angles are (2 3), (3 1), (1 2), the area of the spherical triangle is

$$(2\ 3)+(3\ 1)+(1\ 2)-\pi.$$

This function of the angles must consequently be positive. The limit of a sequence of spherical triangles of diminishing angles on spheres of suitably increasing radii is a *plane* triangle, for which

$$(2\ 3)+(3\ 1)+(1\ 2)-\pi=0.$$

These notions can easily be extended to m dimensions. A *spherical simplex* is defined as one of the 2^m parts into which an $(m-1)$-sphere is divided by m independent primes through its centre. Of the two supplementary angles between a pair of the primes, that one which is inside the spherical simplex is called a *dihedral angle*. If the $(m-1)$-sphere is of unit radius, the spherical simplex is completely determined by its $\frac{1}{2}m(m-1)$ dihedral angles. As in three dimensions, these dihedral angles must not be too small. But when $m>3$, the content of a spherical simplex is no longer a simple function of the angles. We accordingly seek a more tractable criterion.

Let the m primes be called $1, 2, \ldots, m$; and let* $(r\ s)$ denote the dihedral angle between the primes r and s, so that $(s\ r)=(r\ s)$. Using Cartesian coordinates, with the origin at the centre of the $(m-1)$-sphere, let the prime r have the equation

$$\sum_{i=1}^{m} a_{ri}x_i=0,$$

where

$$\sum_{i=1}^{m} a_{ri}^2=1 \quad (r=1, 2, \ldots, m).$$

We can suppose the signs of the a's adjusted so that the spherical simplex is just the aggregate of points satisfying

$$\sum_{i=1}^{m} x_i{}^2=1, \qquad \sum_{i=1}^{m} a_{ri}x_i \geqslant 0.$$

It follows that

$$\sum_{i=1}^{m} a_{ri}a_{is}=-\cos(r\ s).$$

* Not to be confused with the (ij) of § 9.2.

The m primes being independent,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \cdots & \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{vmatrix} \neq 0.$$

Squaring this inequality we have*

$$(15.11) \quad \begin{vmatrix} 1 & -\cos(1\ 2) & -\cos(1\ 3) & \dots & -\cos(1\ m) \\ -\cos(2\ 1) & 1 & -\cos(2\ 3) & \dots & -\cos(2\ m) \\ -\cos(3\ 1) & -\cos(3\ 2) & 1 & \dots & -\cos(3\ m) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\cos(m\ 1) & -\cos(m\ 2) & -\cos(m\ 3) & \dots & 1 \end{vmatrix} > 0.$$

15.2. The section of our spherical simplex by the prime through the origin perpendicular to all the primes $1, 2, \ldots, i-1, i+1, \ldots, m$ is a spherical simplex of one fewer dimensions, whose dihedral angles are

$$(r\ s) \quad (r \neq i \neq s).$$

Hence the above determinant must remain positive when any number of rows are removed, along with the corresponding columns. The inequalities obtained in this manner provide the required criteria for the existence of a spherical simplex of given dihedral angles.

Note that, provided that no obtuse angles are admitted, the value of our determinant diminishes when any one of the angles is diminished. The limit of a sequence of spherical simplexes of diminishing dihedral angles on $(m-1)$-spheres of suitably increasing radii is a *Euclidean* simplex, for which the determinant vanishes. But the simpler determinants, derived by omitting corresponding rows and columns, remain definitely positive.

Let us now enumerate the spherical and Euclidean simplexes all of whose dihedral angles are submultiples of π. (This restriction implies that no dihedral angle shall be obtuse.)

15.3. If the primes $1, 2, \ldots, m$ fall into two sets, say $1, 2, \ldots, i$ and $i+1, i+2, \ldots, m$, such that every prime of the former set is perpendicular to every prime of the latter, then our determinant breaks up into two

* This result is due to Schläfli (*loc. cit.* in Preface).

factors, whence the existence of the spherical simplex depends on the existence of two simpler simplexes, viz., that whose dihedral angles are $(r\,s)$ with $r \leqslant i$ and $s \leqslant i$, and that whose dihedral angles are $(r\,s)$ with $r > i$ and $s > i$.

In particular, there is a spherical simplex *all* of whose dihedral angles are right, namely the spherical simplex bounded by

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_m = 0.$$

15.4. Apart from one trivial case, it is impossible to have a *closed chain* of acute dihedral angles, such as

$$(1\,2), \quad (2\,3), \quad \ldots, \quad (i-1\,i), \quad (i\,1).$$

For, to take the most favourable possibility, suppose that

$$m = i, \quad (1\,2) = (2\,3) = \ldots = (i-1\,i) = (i\,1) = \tfrac{1}{3}\pi,$$

and
$$(r\,s) = \tfrac{1}{2}\pi \ (1 < |r-s| < i-1).$$

Our determinant becomes

$$\begin{vmatrix} 1 & -\frac{1}{2} & 0 & 0 & \ldots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ldots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{1}{2} & 0 & 0 & 0 & \ldots & -\frac{1}{2} & 1 \end{vmatrix} = 0 \quad (i > 2).$$

Therefore the simplex in this case is Euclidean; and any further diminution of the angles, or insertion of extra angles, will render it non-existent.

15.5. Apart from one trivial case, it is impossible to have more than three acute $(r\,s)$'s with a common r (or s). For, to take the most favourable possibility, suppose that

$$m = 5, \quad (1\,2) = (1\,3) = (1\,4) = (1\,5) = \tfrac{1}{3}\pi \text{ and } (r\,s) = \tfrac{1}{2}\pi \quad (r > 1, \ s > 1).$$

Our determinant becomes

$$\begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & & 0 & 1 \end{vmatrix} = 0.$$

Therefore the simplex in this case is Euclidean; and any further diminution of the angles, or insertion of new angles, will render it non-existent.

15.6. Apart from one trivial case, it is impossible to have a double occurrence of three acute $(r\,s)$'s with a common r. For, to take the most favourable possibility, suppose that

$$(1\ 3) = (2\ 3) = (3\ 4) = (4\ 5) = \ldots = (m-4\ m-3) = (m-3\ m-2)$$
$$= (m-2\ m-1) = (m-2\ m) = \tfrac{1}{3}\pi,$$

with right angles for all the rest. In this case, too, the determinant vanishes. Therefore the simplex is again Euclidean; and any further diminution of angles, or insertion of extra angles, will render it non-existent.

15.7. The only type of simplex which remains to be considered is that whose acute dihedral angles form three open chains all emanating from one bounding prime, thus:

$$\begin{array}{llll} (1\ 2), & (2\ 3), & \ldots, & (n\ n+1); \\ (1\ n+2), & (n+2\ n+3), & \ldots, & (n+p\ n+p+1); \\ (1\ n+p+2), & (n+p+2\ n+p+3), & \ldots, & (n+p+q\ n+p+q+1). \end{array}$$
$$(m = n+p+q+1.)$$

Let us rename these angles as follows:

$$\begin{array}{llll} \pi/h_0, & \pi/h_1, & \ldots, & \pi/h_{n-1}; \\ \pi/i_0, & \pi/i_1, & \ldots, & \pi/i_{p-1}; \\ \pi/j_0, & \pi/j_1, & \ldots, & \pi/j_{q-1}. \end{array}$$

Then our simplex can conveniently be denoted by the symbol

$$(15.71) \qquad \begin{pmatrix} h_0, & h_1, & \ldots, & h_{n-1} \\ i_0, & i_1, & \ldots, & i_{p-1} \\ j_0, & j_1, & \ldots, & j_{q-1} \end{pmatrix}.$$

15.8. In order to clarify this notation, we may rename the m bounding primes:

$$(15.81) \qquad 0,\ 1,\ 2,\ \ldots,\ n,\ 1',\ 2',\ \ldots,\ p',\ 1'',\ 2'',\ \ldots,\ q''.$$

Then π/h_0, π/i_0, π/j_0 are the angles between 0 and 1, 1′, 1″ respectively,

π/h_r is the angle between r and $r+1$, π/i_r between r' and $(r+1)'$, and π/j_r between r'' and $(r+1)''$. Every other angle is a right angle.

We might have regarded our acute dihedral angles as forming only two chains, one emanating from the middle of the other. But this aspect destroys the symmetry which exists between the h's, i's, and j's, obscuring the fact that the three rows of the symbol (15.71) can be permuted bodily.

If $n = 0$, so that there are no h's, we are left with a single chain of acute dihedral angles; thus

$$(15.82) \qquad \begin{pmatrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix} = (i_{p-1}, i_{p-2}, \ldots, i_0, \; j_0, j_1, \ldots, j_{q-1}).$$

A simplex of this type, say*

$$(k_1, k_2, \ldots, k_{m-1}),$$

is what Schläfli calls an *orthoscheme*. In this case, returning to the notation of § 15.1, we have

$$(r\; r+1) = \pi/k_r,$$

but all $(r\, s)$'s for which r and s differ by more than 1 are right angles. So (15.11) becomes

$$\Delta_m(k_1, k_2, \ldots, k_{m-1}) > 0,$$

in the notation of § 3.5.

The simplest orthoscheme is (k): an arc of length π/k. The next simplest is (k_1, k_2): a right-angled spherical triangle, of angles π/k_1 and π/k_2. The simplest other simplex of type (15.71) is $\begin{pmatrix} h \\ i \\ j \end{pmatrix}$: a hyperspherical tetrahedron with three right dihedral angles at one vertex, the remaining three dihedral angles being π/h, π/i, π/j.

Analogous symbols for the special simplexes discussed in § 15.5 and § 15.6 are respectively

$$\begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3, & 3, \ldots, & 3, & 3 \\ 3, & & & 3 \end{pmatrix}.$$

The special kind of simplex discussed in § 15.4 is unique, in that the number

* Not to be confused with the $(x_1, x_2, \ldots, x_n)$ of § 3.6.

of acute dihedral angles is equal to (instead of less than) the number of bounding primes. In this case the brackets must close up to form a complete circle; so the appropriate symbol is

$$\textcircled{3^m} \quad (m = i).$$

The enumeration of orthoschemes (with integral k's) involves exactly the same work as the enumeration of regular polytopes, as undertaken in § 3.5; so we need not repeat it here. The connection will appear later. Since obviously

$$(k_{m-1}, k_{m-2}, \ldots, k_1) = (k_1, k_2, \ldots, k_{m-1}),$$

the result is as follows:

m	Spherical orthoschemes	Euclidean orthoschemes
2	(k)	
3	(3, 3), (3, 4), (3, 5)	(4, 4), (3, 6)
4	(3, 3, 3), (3, 3, 4), (3, 3, 5), (3, 4, 3)	(4, 3, 4)
5	(3, 3, 3, 3), (3, 3, 3, 4)	(4, 3, 3, 4), (3, 3, 4, 3)
>5	(3^{m-1}), $(3^{m-2}, 4)$	$(4, 3^{m-3}, 4)$

For the sake of completeness, we might add: for $m = 1$, the very simple simplex

$$(\ )$$

which has, and is, a single vertex; and, for $m = 2$ (Euclidean), the straight segment

$$(\infty),$$

so called because it subtends a zero angle at infinity.

15.9. It is easily proved by induction that the proper determinant for

$$\begin{pmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix}$$

is equal to

$$\left(\frac{\Delta_{n+1}(h_0, h_1, \ldots, h_{n-1})}{\Delta_n(h_1, \ldots, h_{n-1})} + \frac{\Delta_{p+1}(i_0, i_1, \ldots, i_{p-1})}{\Delta_p(i_1, \ldots, i_{p-1})} + \frac{\Delta_{q+1}(j_0, j_1, \ldots, j_{q-1})}{\Delta_q(j_1, \ldots, j_{q-1})} - 2\right)$$
$$\times \Delta_n(h_1, \ldots, h_{n-1})\, \Delta_p(i_1, \ldots, i_{p-1})\, \Delta_q(j_1, \ldots, j_{q-1}).$$

We saw in § 15.2 that the existence of a spherical simplex depends, not only on (15.11), but also on the existence of a spherical simplex of one fewer dimensions, derived by suppressing any one of the original bounding primes. If we suppress the prime 0 (in the notation of § 15.8), we obtain the orthoscheme

$$(h_1, \ldots, h_{n-1}, 2, i_1, \ldots, i_{p-1}, 2, j_1, \ldots, j_{q-1}),$$

whose existence depends solely on the joint existence of the three orthoschemes

$$(h_1, \ldots, h_{n-1}), \quad (i_1, \ldots, i_{p-1}), \quad (j_1, \ldots, j_{q-1}).$$

But the suppression of the prime r $(r > 0)$ leads to the simplex

$$\begin{pmatrix} h_0, h_1, \ldots, h_{r-2}, 2, h_{r+1}, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix},$$

whose existence depends solely on the joint existence of

$$\begin{pmatrix} h_0, h_1, \ldots, h_{r-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix} \quad \text{and} \quad (h_{r+1}, \ldots, h_{n-1}).$$

Similarly for the suppression of r' or r''.

Hence the simplex (15.71) certainly exists if

$$\begin{pmatrix} h_0, h_1, \ldots, h_{n-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix}, \quad \begin{pmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-2} \\ j_0, j_1, \ldots, j_{q-1} \end{pmatrix}, \quad \begin{pmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-2} \end{pmatrix}$$

all exist and are definitely spherical (not Euclidean), and if further the inequality (14.53) is satisfied. In this inequality we have " $>$ " for a definitely spherical simplex, and " $=$ " for a Euclidean one.

When all the h's, i's, and j's are equal to 3, (14.53) becomes (14.62). Thus the simplex

$$\begin{pmatrix} 3^n \\ 3^p \\ 3^q \end{pmatrix}$$

is spherical when $$npq < n+p+q+2,$$

and Euclidean when

$$npq = n+p+q+2.$$

The particular cases are as follows:

$m = n+p+q+1$	Spherical	Euclidean
$p+q+1$	$\begin{pmatrix} 3^p \\ 3^q \end{pmatrix} = (3^{p+q})$	
$n+3$	$\begin{pmatrix} 3^n \\ 3 \\ 3 \end{pmatrix}$	
6	$\begin{pmatrix} 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$	
7	$\begin{pmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3, 3 \\ 3, 3 \\ 3, 3 \end{pmatrix}$
8	$\begin{pmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3, 3, 3 \\ 3, 3, 3 \\ 3 \end{pmatrix}$
9		$\begin{pmatrix} 3, 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{pmatrix}$

The next simplest possibility is

$$\begin{pmatrix} 3^{n-1}, 4 \\ 3^p \\ 3^q \end{pmatrix}.$$

In this case, since $\Delta_{n+1}(3^{n-1}, 4) = 1/2^n$ (as we saw in §3.5), we have $\frac{1}{2}\left(1+\frac{p+2}{p+1}+\frac{q+2}{q+1}\right) \geqslant 2$, or $pq \leqslant 1$. Since the orthoschemes have already been considered, we can assume that $pq > 0$. Therefore we must have

$$p = q = 1.$$

The simplex $$\begin{pmatrix} 3^{n-1}, 4 \\ 3 \\ 3 \end{pmatrix},$$

being Euclidean, is the last possibility.

Recapitulating, the only possible simplexes whose dihedral angles are submultiples of π are the following:

$(\;)$ $(m = 1,$ spherical or Euclidean),

(k) $(m = 2,$ spherical),

(∞) $(m = 2,$ Euclidean),

$\begin{pmatrix} 3^n \\ 3^p \\ 3^q \end{pmatrix}$ $(npq \leqslant n+p+q+2, \quad m = n+p+q+1,$ spherical or Euclidean),

$(3^{m-2}, 4)$ (spherical),

$(4, 3^{m-3}, 4)$ (Euclidean),

$(3, 5)$ $(m = 3,$ spherical),

$(3, 6)$ $(m = 3,$ Euclidean),

$(3, 3, 5), \quad (3, 4, 3)$ $(m = 4,$ spherical),

$(3, 3, 4, 3), \quad \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$ $(m = 5,$ Euclidean),

$\begin{pmatrix} 3^{m-4}, 4 \\ 3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 3, & & 3 \\ & 3^{m-5}, & \\ 3, & & 3 \end{pmatrix}$ (Euclidean),

$\circledcirc 3^m$ (Euclidean),

and an endless variety of new spherical simplexes derivable from pairs of known spherical simplexes in the manner described in § 15.3.

16. *Groups whose fundamental regions are simplexes.*

16.1. Let
$$R_1, \; R_2, \; \ldots, \; R_m$$
denote the reflections in the m primes of § 15.1. These operations clearly generate a *group* of congruent transformations or orthogonal substitutions. Since a reflection is a *negative* operation (*i.e.* a transformation whose matrix has a negative determinant), the group contains both positive and negative operations; accordingly it is said to be *extended*.

It is well known that the product $R_r R_s$ is a rotation through angle $2\,(r\ s)$ about the secundum of intersection of the two primes r and s. Since

$$R_s^2 = 1 \quad \text{and} \quad (R_r R_s)(R_s R_t) = R_r R_t,$$

all these products are expressible in terms of $m-1$ of them, provided that these $m-1$ involve all the R's. A suitable set of products is*

$$S_r = R_{r+1} R_1 \quad (r = 1, 2, \ldots, m-1).$$

These rotations also generate a group. Since a rotation is a positive operation, every operation of this group must be *positive*; accordingly the group is said to be *unextended.*

Since the operation R_1, of period 2, belongs to the former group but not to the latter, and since

$$R_s = S_{s-1} R_1 \quad (s = 2, 3, \ldots, m),$$

it follows that the unextended group is a sub-group of index 2 in the extended group.

The operation R_s transforms the simplex of § 15.1 into a new simplex, having the prime s in common with the original one. The operations

$$R_r R_s, \quad (R_r R_s)^2, \quad (R_r R_s)^3, \quad \ldots$$

transform the original simplex into a cycle of new ones, all meeting in the secundum of intersection of r and s. *If every dihedral angle of the simplex is a sub-multiple of* π, these simplexes will not overlap. In fact we shall have

$$(R_r R_s)^{\pi/(r\,s)} = 1.$$

In this case, the simplex is called a *fundamental region* for either of the groups.

16.2. We shall let g_m denote the order of the extended group, so that $\frac{1}{2}g_m$ is the order of the unextended group. g_m may be finite or infinite; we shall soon see that it is finite or infinite according as the fundamental region is spherical or Euclidean.

The operations of the extended group satisfy the relations

$$(16.21) \qquad \begin{cases} R_s^2 = 1 & (s = 1, 2, \ldots, m), \\ (R_r R_s)^{\pi/(r\,s)} = 1 & (r, s = 1, 2, \ldots, m;\ r \neq s), \end{cases}$$

* This S_r is not to be confused with the S_m of § 8.9.

which are equivalent to

(16.22) $$R_1^2 = 1, \quad (S_r R_1)^2 = 1 \quad (r = 1, 2, \ldots, m-1)$$

and

(16.23) $$\begin{cases} S_{r-1}^{\pi/(r\,1)} = 1 & (r = 2, 3, \ldots, m), \\ (S_{r-1} S_{s-1}^{-1})^{\pi/(r\,s)} = 1 & (r, s = 2, 3, \ldots, m;\ r \neq s). \end{cases}$$

The g_m operations of the extended group transform the fundamental region into a net of g_m simplexes, fitting together so as to fill the whole $(m-1)$-space (spherical or Euclidean) at least once. An obvious extension of an argument used by Burnside* proves that the net fills the space exactly once, and that the equations (16.21) constitute an *abstract definition* for the extended group.

Since the extended group can be derived from the unextended group by the insertion of R_1, which is related to the S's by (16.22), it follows that the equations (16.23) constitute an abstract definition for the unextended group.

Since the g_m simplexes, each of finite content, fit together to fill the $(m-1)$-space just once, it follows that g_m is finite or infinite according as the space is spherical or Euclidean.

16.3. The $\frac{1}{2}g_m$ operations of the unextended group transform the fundamental region into one half of the net of simplexes, namely into a set of $\frac{1}{2}g_m$ simplexes of which no two have a common bounding prime. Any negative operation of the extended group (*e.g.* R_1) transforms this half of the net of simplexes into the other half. It is useful to regard every simplex as being "shaded" or "non-shaded" according to the half-net to which it belongs. A beautiful account of the case when $m = 3$ is given by F. Klein in Chapter 1 of his *Lectures on the icosahedron*†.

16.4. Let us now consider the particular simplexes that can serve as fundamental regions. Take first the kind of simplex discussed in § 15.3, viz. that for which

$$(r\,s) = \tfrac{1}{2}\pi \quad \text{whenever} \quad r \leqslant i < s.$$

The relation $(R_r R_s)^2 = 1$ simply means that R_r and R_s commute. Hence, substituting in (16.21), we see that the extended group is in this case the

* W. Burnside, *Theory of groups of finite order* (2nd ed., 1911), 399 (§ 291).

† Second edition in English, 1913.

direct product of the groups generated by

$$R_1,\ R_2,\ \ldots,\ R_i,$$

and by $$R_{i+1},\ R_{i+2},\ \ldots,\ R_m.$$

In particular, the group corresponding to the simplex all of whose dihedral angles are right angles is the direct product of m groups each of order 2.

If $$(r\,1) = \tfrac{1}{2}\pi \quad (r = 2, 3, \ldots, m),$$

(16.23) becomes

$$\begin{cases} S_{r-1}^2 = 1 & (r = 2, 3, \ldots, m), \\ (S_{r-1} S_{s-1})^{\pi/(r\,s)} = 1 & (r, s = 2, 3, \ldots, m,\ r \neq s). \end{cases}$$

These equations being of the same form as (16.21), it follows that the unextended group corresponding to a simplex for which one bounding prime is perpendicular to all the others, is simply isomorphic with the extended group corresponding to the simplex (of one fewer dimensions) which these other primes cut out on the special one.

16.5. Corresponding to the simplex $\textcircled{3^m}$ discussed in § 15.4, we have the extended group defined by

$$(16.51)\qquad \begin{cases} R_s^{\,2} = 1 & (s = 1, 2, \ldots, m), \\ (R_r R_s)^2 = 1 & (1 < |r-s| < m-1), \\ (R_r R_s)^3 = 1 & (r-s = -1 \text{ or } m-1), \end{cases}$$

and the unextended group defined by

$$(16.52)\qquad \begin{cases} S_1^{\,3} = 1, \quad S_{m-1}^3 = 1, & \\ S_u^{\,2} = 1 & (u = 2, 3, \ldots, m-2), \\ (S_u S_v^{-1})^2 = 1 & (|u-v| > 1), \\ (S_u S_{u+1}^{-1})^3 = 1 & (u = 1, 2, \ldots, m-2). \end{cases}$$

Since the simplex is Euclidean, both these groups are of infinite order. It is convenient to give them the respective symbols

$$\boxed{3^m} \quad \text{and} \quad \boxed{3^m}'.$$

E.g., the infinite group discussed in Burnside's § 299 is our

$$\boxed{3^3}',$$

its fundamental region being the plane equilateral triangle $\textcircled{3^3}$.

16.6. Corresponding to the simplex $\begin{pmatrix}3\\3\\3\\3\end{pmatrix}$ discussed in § 15.5, we have the extended group defined by

$$(16.61)\quad \begin{cases} R_1^2 = R_2^2 = R_3^2 = R_4^2 = R_5^2 = 1, \\ (R_2R_3)^2 = (R_2R_4)^2 = (R_2R_5)^2 = (R_3R_4)^2 = (R_3R_5)^2 = (R_4R_5)^2 = 1, \\ (R_1R_2)^3 = (R_1R_3)^3 = (R_1R_4)^3 = (R_1R_5)^3 = 1, \end{cases}$$

and the unextended group defined by

$$(16.62)\quad \begin{cases} S_1^3 = S_2^3 = S_3^3 = S_4^3 = 1, \\ (S_1S_2^{-1})^2 = (S_1S_3^{-1})^2 = (S_1S_4^{-1})^2 \\ \qquad = (S_2S_3^{-1})^2 = (S_2S_4^{-1})^2 = (S_3S_4^{-1})^2 = 1. \end{cases}$$

Since the simplex is Euclidean, both these groups are of infinite order. It is convenient to give them the respective symbols

$$\begin{bmatrix}3\\3\\3\\3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}3\\3\\3\\3\end{bmatrix}'.$$

Similarly, we obtain two infinite groups

$$(16.63)\quad \begin{bmatrix}3, & & 3\\ & 3^{m-5}, & \\ 3, & & 3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}3, & & 3\\ & 3^{m-5}, & \\ 3, & & 3\end{bmatrix}'$$

from the Euclidean simplex discussed in § 15.6.

16.7. Having mentioned all the "trivial" cases, let us turn our attention to the two groups which have the fundamental region (15.71). We shall call the extended and unextended groups

$$(16.71)\quad \begin{bmatrix}h_0, & h_1, & \ldots, & h_{n-1}\\ i_0, & i_1, & \ldots, & i_{p-1}\\ j_0, & j_1, & \ldots, & j_{q-1}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}h_0, & h_1, & \ldots, & h_{n-1}\\ i_0, & i_1, & \ldots, & i_{p-1}\\ j_0, & j_1, & \ldots, & j_{q-1}\end{bmatrix}'$$

respectively.

The following change of notation is convenient:

$$(16.72)\quad \begin{cases} O = R_1, & \\ N_r = R_{r+2} & (r = 0, 1, \ldots, n-1), \\ P_r = R_{r+n+2} & (r = 0, 1, \ldots, p-1)^*, \\ Q_r = R_{r+n+p+2} & (r = 0, 1, \ldots, q-1); \end{cases}$$

$$(16.73)\quad \begin{cases} N_r' = N_r O = S_{r+1} & (r = 0, 1, \ldots, n-1), \\ P_r' = P_r O = S_{r+n+1} & (r = 0, 1, \ldots, p-1), \\ Q_r' = Q_r O = S_{r+n+p+1} & (r = 0, 1, \ldots, q-1). \end{cases}$$

The extended group is defined by

$$(16.74)\quad \begin{cases} O^2 = 1, & \\ N_r^2 = 1 & (r = 0, 1, \ldots, n-1), \\ P_r^2 = 1 & (r = 0, 1, \ldots, p-1), \\ Q_r^2 = 1 & (r = 0, 1, \ldots, q-1), \\ (ON_0)^{h_0} = (N_{r-1} N_r)^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ (OP_0)^{i_0} = (P_{r-1} P_r)^{i_r} = 1 & (r = 1, 2, \ldots, p-1), \\ (OQ_0)^{j_0} = (Q_{r-1} Q_r)^{j_r} = 1 & (r = 1, 2, \ldots, q-1), \\ (ON_r)^2 = (OP_r)^2 = (OQ_r)^2 = 1 & (r > 0), \\ (N_r N_s)^2 = (P_r P_s)^2 = (Q_r Q_s)^2 = 1 & (|r-s| > 1), \\ (P_r Q_s)^2 = (Q_r N_s)^2 = (N_r P_s)^2 = 1; & \end{cases}$$

and the unextended group by

$$(16.75)\quad \begin{cases} N_0'^{h_0} = N_r'^2 = (N_{r-1}' N_r')^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ P_0'^{i_0} = P_r'^2 = (P_{r-1}' P_r')^{i_r} = 1 & (r = 1, 2, \ldots, p-1), \\ Q_0'^{j_0} = Q_r'^2 = (Q_{r-1}' Q_r')^{j_r} = 1 & (r = 1, 2, \ldots, q-1), \\ (N_r' N_s')^2 = (P_r' P_s')^2 = (Q_r' Q_s')^2 = 1 & (|r-s| > 1), \\ (P_r' Q_s')^2 = (Q_r' N_s')^2 = (N_r' P_s')^2 = 1 & (r+s > 0), \\ (P_0' Q_0'^{-1})^2 = (Q_0' N_0'^{-1})^2 = (N_0' P_0'^{-1})^2 = 1. & \end{cases}$$

* This P_r is not to be confused with the P_{m} of § 8.3.

16.8. When $p = q = 0$, we have two groups whose fundamental region is the orthoscheme $(h_0, h_1, \ldots, h_{n-1})$, namely:

$$[h_0, h_1, \ldots, h_{n-1}],$$

defined by

$$(16.81)\quad \begin{cases} O^2 = N_r^2 = 1 & (r = 0, 1, \ldots, n-1), \\ (ON_0)^{h_0} = (N_{r-1}N_r)^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ (ON_r)^2 = 1 & (r > 0), \\ (N_r N_s)^2 = 1 & (|r-s| > 1); \end{cases}$$

and

$$[h_0, h_1, \ldots, h_{n-1}]',$$

defined by

$$(16.82)\quad \begin{cases} N_0'^{h_0} = N_r'^2 = (N_{r-1}' N_r')^{h_r} = 1 & (r = 1, 2, \ldots, n-1), \\ (N_r' N_s')^2 = 1 & (|r-s| > 1). \end{cases}$$

(16.81) and (16.82) are due to Todd*, who obtained them as abstract definitions for the extended and unextended groups of the regular polytope $\{h_0, h_1, \ldots, h_{n-1}\}$. The groups given by Burnside in his § 296† are respectively:

I, $[n]'$;

II, $[2, n]'$, which is the same as $[n]$;

III, $[3, 3]'$;

IV, $[3, 4]'$;

V, $[3, 5]'$.

Further, the infinite groups discussed in his §§ 300, 301 are respectively

$$[4, 4]' \quad \text{and} \quad [3, 6]'.$$

The extended groups

$$[3, 3], \quad [3, 4], \quad [3, 5]$$

are mentioned by Klein‡. Todd has considered the groups

$$[3^{m-1}], \quad [3^{m-2}, 4], \quad [3, 3, 5], \quad [3, 4, 3],$$
$$[3^{m-1}]', \quad [3^{m-2}, 4]', \quad [3, 3, 5]', \quad [3, 4, 3]'$$

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* *Loc. cit.* in Preface.

† *Theory of groups*, 408.

‡ *Lectures on the icosahedron*, 24.

in detail, generating each group, save the last of all, by means of *two* operations. $[3^{m-1}]$ and $[3^{m-1}]'$ are easily recognizable as the symmetric and alternating groups of degree $m+1$. $[3^{m-2}, 4]$ and $[3^{m-2}, 4]'$ are discussed by Dr. A. Young in the fifth of his papers on *Substitutional analysis**; his "(*AB*) subgroup" can be identified with our $\begin{bmatrix} 3^{m-3} \\ 3 \\ 3 \end{bmatrix}$.

L. E. Dickson† gives an abstract definition for $\begin{bmatrix} 3, 3 \\ 3 \\ 3 \end{bmatrix}'$ closely resembling ours.

16.9. The rest of our finite groups, namely

$$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}', \quad \begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}', \quad \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}',$$

are less familiar. We shall find the orders of all of them, and identify the first and fifth with important geometrical groups, viz., the group of automorphisms of the twenty-seven lines on the general cubic surface, and the group of automorphisms of the twenty-eight bitangents of the general plane quartic curve.

17. *How each of the groups is related to a uniform polytope.*

17.1. When, as in §5.1, $t_n \Pi_m$ is regarded as the part of space which is inside both Π_m and Π_m', the elements $t_n \Pi_s$ $(s > n)$ arise as actual truncations of the s-dimensional elements of Π_m, and the elements $t_{n-m+s} \Pi_{s, m-s}$ $(s \geqslant m-n)$ as actual truncations of the s-dimensional elements of Π_m'. But, if $r+s = m-1$, the s-dimensional elements of Π_m' *correspond* to the r-dimensional elements of Π_m, in the sense that their centres are collinear

* *Proc. London Math. Soc.* (2), 31 (1930), 273.

† *Linear groups* (Leipzig, 1901), 293.

with the common centre of Π_m and Π_m'. Also the vertices of $t_n \Pi_m$ correspond to the Π_n's of Π_m. Thus, for every element Π_r of Π_m, there is a corresponding element, say Π^r, of $t_n \Pi_m$, viz.:

$$\Pi^r = t_{n-r-1} \Pi_{m-r-1, r+1} \qquad (r < n),$$

$$\Pi^n = a_0,$$

$$\Pi^s = t_n \Pi_s \qquad (s > n).$$

If $\Pi_m = \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$, we have

$$\Pi^r = t_{n-r-1} \left\{ h_{n-r-2}, h_{n-r-3}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$$

(17.11)
$$= \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-r-2} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \qquad (r < n)$$

and

$$\Pi^s = t_n \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix} \right\}$$

(17.12)
$$= \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix} \right\} \qquad (s = n+p'+q'+1; \quad p' \geqslant 0, \quad q' \geqslant 0).$$

Putting $q' = q$, we see that the elements

$$\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \quad \text{of} \quad \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$$

correspond to the elements

$$\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\} \quad \text{of} \quad \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\},$$

and [by (17.11) with the h's and i's interchanged] to the $(p-p'-1)$-

dimensional elements

$$\{i_{p-1}, i_{p-2}, \ldots, i_{p'+2}\} \quad \text{of} \quad \left\{ i_{p-1}, i_{p-2}, \ldots, i_0, \begin{matrix} j_0, j_1, \ldots, j_{q-1} \\ h_0, h_1, \ldots, h_{n-1} \end{matrix} \right\},$$

and [by (17.12) with the h's and j's interchanged] to the elements

$$\left\{ j_{q-1}, j_{q-2}, \ldots, j_0, \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \end{matrix} \right\} \quad \text{of} \quad \left\{ j_{q-1}, j_{q-2}, \ldots, j_0, \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \end{matrix} \right\}.$$

When every h, i, j is equal to 3, this means that the elements $O_{np'q}$ of O_{npq} correspond to the elements $n_{p'q}$ of n_{pq}, and to the elements $\alpha_{p-p'-1}$ of p_{qn}, and to the elements $q_{np'}$ of q_{np}. We have thus incidentally proved the generalized "semi-reciprocation theorem" enunciated at the end of § 7.8.

Clearly, also, the *vertices* of $\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$ correspond to the elements

$$\{h_{n-1}, h_{n-2}, \ldots, h_1\}, \quad \{i_{p-1}, i_{p-2}, \ldots, i_1\}, \quad \{j_{q-1}, j_{q-2}, \ldots, j_1\}$$

of the other three polytopes, respectively.

17.2. Let W_0 be a vertex of $\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}$, and let $W_{n'pq}$, $W_{np'q}$, $W_{npq'}$ be the centres of "adjacent" elements

$$\left\{ \begin{matrix} h_0, h_1, \ldots, h_{n'-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}, \quad \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p'-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}, \quad \left\{ \begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix} \right\}$$

respectively. By saying that these elements are to be "adjacent", we mean that W_0 must be a common vertex of the elements whose centres are W_{0pq}, W_{n0q}, W_{np0}, and that the elements whose centres are $W_{n'pq}$, $W_{np'q}$, $W_{npq'}$ must respectively belong to those whose centres are $W_{(n'+1)pq}$, $W_{n(p'+1)q}$, $W_{np(q'+1)}$.

Having defined

$$W_0, \quad W_{n'pq}, \quad W_{np'q}, \quad W_{npq'}$$

$$(0 \leqslant n' < n, \quad 0 \leqslant p' < p, \quad 0 \leqslant q' < q),$$

let $\quad X_n, \quad X_{n-n'-1}, \quad X_{p'q}, \quad X_{pq'}$

denote the centres of the corresponding elements of

$$\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ \cdot \quad \cdot \qquad \cdot \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}.$$

The conditions of adjacency now state that the element

$$\{h_{n-1}, h_{n-2}, \ldots, h_{n-r+1}\}$$

whose centre is X_r (or the vertex X_0, if $r = 0$) belongs to the element $\{h_{n-1}, h_{n-2}, \ldots, h_{n-r}\}$ whose centre is X_{r+1} $(r < n)$, that the element $\{h_{n-1}, h_{n-2}, \ldots, h_1\}$ whose centre is X_n belongs both to the element $\{h_{n-1}, h_{n-2}, \ldots, h_0, j_0, j_1, \ldots, j_{q-1}\}$ whose centre is X_{0q} and to the element $\{h_{n-1}, h_{n-2}, \ldots, h_0, i_0, i_1, \ldots, i_{p-1}\}$ whose centre is X_{p0}, and that the elements

$$\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p'-1} \\ \cdot \quad \cdot \qquad \cdot \\ j_0, j_1, \ldots, j_{q-1} \end{matrix} \right\}, \quad \left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ \cdot \quad \cdot \qquad \cdot \\ j_0, j_1, \ldots, j_{q'-1} \end{matrix} \right\},$$

whose centres are $X_{p'q}$, $X_{pq'}$, belong respectively to the analogous elements whose centres are $X_{(p'+1)q}$, $X_{p(q'+1)}$.

We next have to define certain primes passing through the common centre (W_{npq} or X_{pq}) of our $(n+p+q+1)$-dimensional polytopes. Each such prime is determined by $n+p+q$ further points lying on it, which points may be taken equally well as W's or as X's:

Prime	Determining W's	Determining X's
0	All save W_0	All save X_n
r_h $(0 \leqslant r < n)$	All save W_{rpq}	All save X_{n-r-1}
r_i $(0 \leqslant r < p)$	All save W_{nrq}	All save X_{rq}
r_j $(0 \leqslant r < q)$	All save W_{npr}	All save X_{pr}

The $n+p+q+1$ primes

(17.21) $\quad 0, 0_h, 1_h, \ldots, (n-1)_h, \quad 0_i, 1_i, \ldots, (p-1)_i, \quad 0_j, 1_j, \ldots, (q-1)_j$

bound a spherical simplex, whose vertices are central projections of the W's (or X's) on an $(n+p+q)$-sphere concentric with the polytopes. This simplex is called a *fundamental simplex* for each of our four polytopes (viz. the three semi-reciprocals and their common truncation).

When the polytopes are degenerate, the definition is simpler, since the W's actually coincide with the X's, thus determining a *Euclidean* fundamental simplex. In this case we can regard 0, r_h, r_i, r_j as primes of the $(n+p+q)$-space filled by the polytopes.

17.3. In order to identify the fundamental simplex with (15.71), we must calculate its dihedral angles. These are simply the angles between pairs of our primes, and may conveniently be called

$$(0\, r_h),\quad (0\, r_i),\quad (0\, r_j),\quad (r_h\, s_h),\quad (r_i\, s_i),\quad (r_j\, s_j),\quad (r_i\, s_j),\quad (r_j\, s_h),\quad (r_h\, s_i).$$

Let us first suppose that $\qquad q = 0.$

In this case the W's are the centres of certain elements of

$$\begin{Bmatrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \end{Bmatrix},$$

while the X's are the centres of elements, one of every kind, of the regular polytope

$$\{h_{n-1}, h_{n-2}, \ldots, h_0, i_0, i_1, \ldots, i_{p-1}\}.$$

In fact, X_0 is a vertex, X_1 is the centre of an edge through X_0, X_2 is the centre of a plane face through this edge, and so on. The process continues as far as X_{n+p}, the centre of a bounding figure, provided that we write $X_{n+p'+1}$ for $X_{p'0}$.

If we put

(17.31) $$h_r = k_{n-r}, \quad i_r = k_{n+r+1}, \quad n+p+1 = m,$$

then the points $X_0, X_1, \ldots, X_{m-1}$ are the centres of elements of

$$\{k_1,\ k_2,\ \ldots,\ k_{m-1}\},$$

and it is natural to complete the sequence by letting X_m denote the centre of the whole polytope. Since the circumscribing sphere-analogue of any element is a section of the circumscribing sphere-analogue of any higher element containing that element, all the lines $X_0 X_1$, $X_1 X_2$, ..., $X_{m-1} X_m$ are mutually perpendicular. Hence, if r denotes the prime determined by all the X's save X_{r-1}; and $(r\, s)$ denotes the angle between the primes r and s, it follows that $(r\, s)$ is a right angle whenever r and s differ by more than 1. It remains to prove that

(17.32) $$(r\; r+1) = \pi/k_r \quad (r = 1, 2, \ldots, m-1).$$

[The remaining angle, $(m\;\; m+1)$, is irrelevant to our present purpose.]

When $m = 2$, we have a regular k_1-gon, centre X_2. X_0 is one end of the side whose centre is X_1. The angle $X_1 X_2 X_0$ is clearly

$$(1\,2) = \pi/k_1.$$

We can therefore use induction, and assume the corresponding result for all regular $(m-1)$-dimensional polytopes. Since the analogous X's for the bounding figure $\{k_1, k_2, \ldots, k_{m-2}\}$ whose centre is X_{m-1} are precisely

$$X_0,\ X_1,\ \ldots,\ X_{m-1},$$

we have

$$(17.33) \qquad (r\; r+1) = \pi/k_r \quad (r = 1, 2, \ldots, m-2).$$

Further, if $X_0', X_1', \ldots, X_{m-1}'$ are the analogous X's for the vertex figure $\{k_2, k_3, \ldots, k_{m-1}\}$ at the vertex X_0, then X_{r-1}' and X_r are collinear with X_0 $(r = 1, 2, \ldots, m)$. Therefore

$$(17.34) \qquad (r\; r+1) = \pi/k_r \quad (r = 2, 3, \ldots, m-1).$$

(17.33) and (17.34) together give (17.32).

Reverting to the other notation, we have

$$(0\, r_h) = (n+1\;\, n-r), \quad (0\, r_i) = (n+1\;\, n+r+2),$$

$(r_h\, s_h) = (n-r\;\, n-s)$, $(r_i\, s_i) = (n+r+2\;\, n+s+2)$, $(r_h\, s_i) = (n-r\;\, n+s+2)$, and therefore, if $q = 0$,

$$(17.35) \qquad \begin{cases} (0\, r_h) = (0\, r_i) = \tfrac{1}{2}\pi & (r > 0), \\ (r_h s_h) = (r_i s_i) = \tfrac{1}{2}\pi & (|r-s| > 1), \\ (r_h s_i) = \tfrac{1}{2}\pi, & \\ (0\, 0_h) = \pi/h_0, \quad (0\, 0_i) = \pi/i_0, & \\ \big((r-1)_h r_h\big) = \pi/h_r, \quad \big((r-1)_i r_i\big) = \pi/i_r & (r > 0). \end{cases}$$

Interchanging q and n, and j and h; if $n = 0$,

$$(17.36) \qquad \begin{cases} (0\, r_i) = (0\, r_j) = \tfrac{1}{2}\pi & (r > 0), \\ (r_i s_i) = (r_j s_j) = \tfrac{1}{2}\pi & (|r-s| > 1), \\ (r_i s_j) = \tfrac{1}{2}\pi, & \\ (0\, 0_i) = \pi/i_0, \quad (0\, 0_j) = \pi/j_0, & \\ \big((r-1)_i r_i\big) = \pi/i_r, \quad \big((r-1)_j r_j\big) = \pi/j_r & (r > 0). \end{cases}$$

Similarly, if $p=0$,

$$(17.37)\quad \begin{cases} (0\,r_j) = (0\,r_h) = \tfrac{1}{2}\pi & (r>0), \\ (r_j\,s_j) = (r_h\,s_h) = \tfrac{1}{2}\pi & (|r-s|>1), \\ (r_j\,s_h) = \tfrac{1}{2}\pi, & \\ (0\,0_j) = \pi/j_0, \quad (0\,0_h) = \pi/h_0, & \\ \big((r-1)_j\,r_j\big) = \pi/j_r, \quad \big((r-1)_h\,r_h\big) = \pi/h_r & (r>0). \end{cases}$$

Let X'_{n-1}, $X'_{n-n'-2}$, $X'_{p'q}$, $X'_{pq'}$ be the points associated with

$$\left\{ h_{n-2},\, h_{n-3},\, \ldots,\, h_0,\, \begin{matrix} \dot{i}_0,\, \dot{i}_1,\, \ldots,\, \dot{i}_{p-1} \\ \dot{j}_0,\, \dot{j}_1,\, \ldots,\, \dot{j}_{q-1} \end{matrix} \right\}$$

in the same way as the points X_n, $X_{n-n'-1}$, $X_{p'q}$, $X_{pq'}$ are associated with

$$\left\{ h_{n-1},\, h_{n-2},\, \ldots,\, h_0,\, \begin{matrix} \dot{i}_0,\, \dot{i}_1,\, \ldots,\, \dot{i}_{p-1} \\ \dot{j}_0,\, \dot{j}_1,\, \ldots,\, \dot{j}_{q-1} \end{matrix} \right\}.$$

If the former polytope is chosen to be the actual vertex figure of the latter at the vertex X_0, then X'_{r-1} and X_r are collinear with X_0 ($r=1, 2, \ldots, n$). So also are $X'_{p'q}$ and $X_{p'q}$, and $X'_{pq'}$ and $X_{pq'}$. Therefore the angles involved in (17.36) are independent of n. Thus the restriction "$n=0$" can be removed. Similarly, the restrictions "$p=0$" and "$q=0$" can be removed from (17.37) and (17.35) respectively. (17.35), (17.36) and (17.37) together give *all* the dihedral angles of the fundamental simplex, viz.:

$$(17.38)\quad \begin{cases} (0\,r_h) = (0\,r_i) = (0\,r_j) = \tfrac{1}{2}\pi \quad (r>0), \\ (r_h\,s_h) = (r_i\,s_i) = (r_j\,s_j) = \tfrac{1}{2}\pi \quad (|r-s|>1), \\ (r_i\,s_j) = (r_j\,s_h) = (r_h\,s_i) = \tfrac{1}{2}\pi, \\ (0\,0_h) = \pi/h_0, \quad (0\,0_i) = \pi/i_0, \quad (0\,0_j) = \pi/j_0, \\ \big((r-1)_h\,r_h\big) = \pi/h_r, \quad \big((r-1)_i\,r_i\big) = \pi/i_r, \quad \big((r-1)_j\,r_j\big) = \pi/j_r \quad (r>0). \end{cases}$$

Thus the primes (17.21) can be identified with the primes (15.81), and the fundamental simplex with the simplex (15.71).

17.4. As in §16.7, let O, N_r, P_r, Q_r denote the reflections in the primes 0, r_h, r_i, r_j, respectively. With the help of the assumption made in §2.4, we can prove by induction that these reflections are symmetries of $\left\{ h_{n-1},\, h_{n-2},\, \ldots,\, h_0,\, \begin{matrix} \dot{i}_0,\, \dot{i}_1,\, \ldots,\, \dot{i}_{p-1} \\ \dot{j}_0,\, \dot{j}_1,\, \ldots,\, \dot{j}_{q-1} \end{matrix} \right\}$, and so also of the related polytopes.

To do this we first prove, by induction with respect to m, the corresponding result for $\{k_1, k_2, \ldots, k_{m-1}\}$, then observe that every symmetry of $\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\}$ is also a symmetry of $\left\{\begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$, and finally use induction with respect to n.

Apart from two exceptional cases, the same method serves to prove that these reflections actually generate the whole group of symmetries of $\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$, thus identifying this group of symmetries with the group

$$(17.41) \qquad \left[\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right]$$

whose abstract definition is (16.74).

The first exceptional case is when

$$\{i_0, i_1, \ldots, i_{p-1}\} = \{j_0, j_1, \ldots, j_{q-1}\},$$

since then $\left\{\begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$ has twice as many symmetries as

$$\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\}.$$

(See § 5.5.) The second exceptional case is when

$$i_0 = j_0,\ i_1 = 4 = j_1 \quad \text{and} \quad i_2 = i_3 = \ldots = 3 = j_2 = j_3 = \ldots,$$

since then $\left\{\begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$ has $\binom{p+q}{p}$ times as many symmetries as $\{i_{p-1}, i_{p-2}, \ldots, i_0, j_0, j_1, \ldots, j_{q-1}\}$*.

Thus, whenever the group of symmetries of

$$\left\{ h_{n-1}, h_{n-2}, \ldots, h_0, \begin{matrix} i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$$

---

* The only actual examples of this second exception are:

$$p = 1,\ q = 2,\ i_0 = 3,\ n = 0 \text{ or } 1,\ h_0 = 3;$$

$$p = 2,\ q = 2,\quad = 3,\ n = 0.$$

is not identical with (17.41), it contains the latter as a sub-group. The same can, of course, be said for the group of symmetries of $\left\{\begin{matrix} h_0, h_1, \ldots, h_{n-1} \\ i_0, i_1, \ldots, i_{p-1} \\ j_0, j_1, \ldots, j_{q-1} \end{matrix}\right\}$, since this polytope is a truncation of the other.

17.5. When every h, i, j is equal to 3, the results are as follows. If n, p, q are all different, or if $p = q = 0$, the groups of symmetries of n_{pq}, of p_{qn}, of q_{np}, and of 0_{npq}, are one and the same group, viz. $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$. If $n \neq p = q \neq 0$, the group of symmetries of p_{qn}, *i.e.* of p_{pn}, is $\begin{bmatrix} 3^n \\ 3^p \\ 3^p \end{bmatrix}$, and is a sub-group of index 2 in the group of symmetries of n_{pp} or of 0_{npp}. Finally, if $n = p = q \neq 0$, the group of symmetries of n_{pq}, *i.e.* of n_{nn}, contains $\begin{bmatrix} 3^n \\ 3^n \\ 3^n \end{bmatrix}$ as a sub-group of index 2, and is itself a sub-group of index 3 in the group of symmetries of 0_{nnn}.

By (12.23), it follows that the order of the group $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$ is always

(17.51) $$(n+1)!\ (p+1)!\ (q+1)!\ [npq].$$

For the values of $[npq]$, see §12.6. The order of the unextended group $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}'$ is just half as great. Tabulating results in the finite cases, we have:

n	p	q	Order of $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}$	Order of $\begin{bmatrix} 3^n \\ 3^p \\ 3^q \end{bmatrix}'$
0	p	q	$(p+q+2)!$	$\frac{1}{2}(p+q+2)!$
n	1	1	$2^{n+2}(n+3)!$	$2^{n+1}(n+3)!$
2	2	1	51840	25920
3	2	1	2903040	1451520
4	2	1	696729600	348364800

In particular, as we saw in § 13.1, $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3, \end{bmatrix}$ is the group of automorphisms of the 27 lines on a general cubic surface. $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3, \end{bmatrix}'$, being a self-conjugate sub-group of index 2, must therefore be identical with the simple group* $A(4, 3)$.

17.6. For a one-rowed symbol there are no "exceptional cases": $[k_1, k_2, \ldots, k_{m-1}]$, which is the same as $[k_{m-1}, k_{m-2}, \ldots, k_1]$, is precisely the group of symmetries of $\{k_1, k_2, \ldots, k_{m-1}\}$; while $[k_1, k_2, \ldots, k_{m-1}]'$ is the group of *positive* symmetries. The finite cases are as follows:

$[3^{m-1}]$, of order $(m+1)!$ (the symmetric group);

$[3^{m-1}]'$, of order $\frac{1}{2}(m+1)!$ (the alternating group);

$[3^{m-2}, 4]$, of order $2^m m!$; $[3^{m-2}, 4]'$, of order $2^{m-1} m!$;

$[k]$, of order $2k$; $[k]'$, of order k (the cyclic group);

$[3, 5]$, of order 120; $[3, 5]'$, of order 60 (the icosahedral group);

$[3, 3, 5]$, of order 14400; $[3, 3, 5]'$, of order 7200;

$[3, 4, 3]$, of order 1152; $[3, 4, 3]'$, of order 576.

These groups are not all distinct. In fact, the following two simple isomorphisms are well known:

$$[3, 3] \sim [3, 4]', \quad [3, 3, 3]' \sim [3, 5]'.$$

17.7. If a certain finite polytope $\Pi^{(1)}_{m_1}$ has a fundamental simplex determined by the centres of m_1 particular elements $\Pi^{(1)}_{r_1}$, and similarly for $\Pi^{(2)}_{m_2}$, then the fundamental simplex of the generalized prism

$$[\Pi^{(1)}_{m_1}, \ \Pi^{(2)}_{m_2}]$$

is defined as being determined by the centres of the m_1+m_2 elements

$$[\Pi^{(1)}_{r_1}, \ \Pi^{(2)}_{m_2}] \quad \text{and} \quad [\Pi^{(1)}_{m_1}, \ \Pi^{(2)}_{r_2}].$$

It is easily seen that the essential features of a fundamental simplex are maintained, since the reflections in its bounding primes are symmetries of the whole prism.

* L. E. Dickson, *Linear groups* (1901), 306–307.

If the vertices of the prism are given by coordinates of the form

$$(x_1, \ldots, x_p;\ x_{p+1}, \ldots, x_q)$$

as in § 4.1, then the prime determined by the centres of all but one of the elements $[\Pi_{r_1}^{(1)}, \Pi_{m_2}^{(2)}]$ and of all the elements $[\Pi_{m_1}^{(1)}, \Pi_{r_2}^{(2)}]$ has an equation of the form

$$F(x_1, \ldots, x_p) = 0.$$

Likewise, the prime determined by the centres of all the elements $[\Pi_{r_1}^{(1)}, \Pi_{m_2}^{(2)}]$ and of all but one of the elements $[\Pi_{m_1}^{(1)}, \Pi_{r_2}^{(2)}]$ has an equation of the form

$$G(x_{p+1}, \ldots, x_q) = 0.$$

Since these two primes are perpendicular, the fundamental simplex is of the type considered in § 15.3, the "two simpler simplexes" being the fundamental simplexes of $\Pi_{m_1}^{(1)}$ and $\Pi_{m_2}^{(2)}$.

In this connection it only remains to observe that the extended group which has this simplex for a fundamental region is the direct product of the groups similarly related to the simpler simplexes; and that the group of symmetries of the prism either is, or contains as a sub-group, the direct product of the groups of symmetries of the constituents.

17.8. We saw in § 14.8 that, if $\dot{j}_{q-2} = 3$ and $\dot{j}_{q-1} = 4$, the row

$$\dot{j}_0, \dot{j}_1, \ldots, \dot{j}_{q-1}$$

of an extended Schläfli symbol can be replaced by

$$\dot{j}_0, \dot{j}_1, \ldots, \dot{j}_{q-3}, \begin{matrix} 3 \\ 3 \end{matrix}$$

without altering the polytope represented. Let us see how this transformation affects the corresponding simplex.

The simplex represented by the transformed symbol is bounded by primes r_h, r_i, 0 and $0_j, 1_j, \ldots, (q-3)_j$, along with two extra primes, say $(q-2)_j$ and $(q-2)_j'$, each inclined at an angle $\frac{1}{3}\pi$ to $(q-3)_j$, but perpendicular to one another and to all the other primes. Let $(q-1)_j$ denote the bisector of the angle $\big((q-2)_j (q-2)_j'\big)$. Then, since $(q-2)_j$ and $(q-2)_j'$ are similarly situated with respect to the rest of the primes, the new prime $(q-1)_j$ divides the simplex into two equal halves, simplexes whose bounding primes are obtained from those of the whole simplex by replacing $(q-2)_j'$ or $(q-2)_j$ by $(q-1)_j$. Of the half which involves $(q-2)_j$, the bounding prime $(q-1)_j$ is perpendicular to all the others except $(q-2)_j$.

The angle $\big((q-2)_j\,(q-1)_j\big) = \frac{1}{2}\big((q-2)_j\,(q-2)_j'\big) = \frac{1}{4}\pi$. Thus the effect of changing $\begin{smallmatrix}3\\3\end{smallmatrix}$ into 3, 4 is to halve the simplex, and so to double the order of the group for which the simplex is a fundamental region. In fact, a new operation of period 2, viz. the reflection in $(q-1)_j$, is introduced into the group.

The group of symmetries of δ_m or $\{4, 3^{m-3}, 4\}$ is $[4, 3^{m-3}, 4]$ or $\begin{bmatrix}3, 4 \\ 3^{m-4}, 4\end{bmatrix}$. Therefore

$$\begin{bmatrix}3 & \\ 3 & \\ 3^{m-4}, & 4\end{bmatrix}$$

is a sub-group of index 2 in this group, and

$$\begin{bmatrix}3 & \\ 3 & \\ 3^{m-5}, & \begin{smallmatrix}3\\3\end{smallmatrix}\end{bmatrix}$$

is a sub-group of index 4. This last symbol is given in a more symmetrical form in (16.63). When $m = 5$, it reduces to

$$\begin{bmatrix}3\\3\\3\\3\end{bmatrix}.$$

17.9. Consider the point whose coordinates consist of r repetitions of $1-(r/m)$ followed by $m-r$ repetitions of $-(r/m)$. For all values of r, this point lies in the $(m-1)$-space

(17.91) $$x_1+x_2+\ldots+x_m = 0.$$

The point is the origin both when $r = 0$ and when $r = m$. The m points obtained by giving r the values $0, 1, \ldots, m-1$ are the vertices of a Euclidean simplex whose bounding primes are

(17.92) $$x_1-x_2 = 0, \quad x_2-x_3 = 0, \quad \ldots, \quad x_{m-1}-x_m = 0, \quad x_1-x_m = 1.$$

Since these, regarded as primes in m dimensions, are all perpendicular to (17.91), we can obtain the angles between them by the usual rule. In fact, calling them $1, 2, \ldots, m$, we have

$$(1\ 2) = (2\ 3) = \ldots = (m-1\ m) = (m\ 1) = \tfrac{1}{3}\pi \quad \text{and} \quad (r\ s) = \tfrac{1}{2}\pi$$
$$(1 < |r-s| < m-1).$$

Thus this simplex is of the kind considered in § 15.4.

The reflections in the first $m-1$ of the primes (17.92) are simply the transpositions of consecutive pairs of coordinates. These generate the symmetric group on $x_1, x_2, \ldots, x_m$, which is the group of symmetries of α_{m-1}. Since this is a sub-group of index 2 in the group of symmetries of $e\alpha_{m-1}$ (see § 6.9), and since the remaining reflection, viz. that in $x_1 - x_m = 1$, changes the origin into the point $(1; 0, 0, \ldots, 0; -1)$, it follows that $\boxed{3^m}$ (defined in § 16.5) is a sub-group of index 2 in the group of symmetries of $\alpha_{m-1} h$.

Having now considered every possibility, we can assert that *every group of real orthogonal substitutions on m variables, having as fundamental region a simplex all of whose dihedral angles are submultiples of π, is either the whole group of symmetries of some m-dimensional uniform polytope, or a sub-group thereof.*

18. *The twenty-seven lines and the twenty-eight bitangents.*

18.1. The most important of our *extended* groups is

$$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix},$$

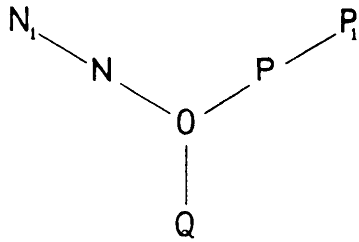
since this, being the group of symmetries of 2_{21} or $(PA)_6$, is also the group of automorphisms of the lines on a general cubic surface.

By (16.74) it has the *abstract definition*

$$(18.11)\quad \begin{cases} O^2 = 1, \\ N^2 = N_1^2 = 1, \\ P^2 = P_1^2 = 1, \\ Q^2 = 1, \\ (ON)^3 = (NN_1)^3 = 1, \\ (OP)^3 = (PP_1)^3 = 1, \\ (OQ)^3 = 1, \\ (ON_1)^2 = (OP_1)^2 = 1, \\ (PQ)^2 = (P_1 Q)^2 = (QN)^2 = (QN_1)^2 \\ \quad = (NP)^2 = (N_1 P)^2 = (NP_1)^2 = (N_1 P_1)^2 = 1. \end{cases}$$

For simplicity we have written N for N_0, P for P_0, Q for Q_0.

This abstract definition is conveniently represented by the following diagram:



The six generating operations are each of period 2; all pairs of them not directly linked in the diagram are permutable, and the products of linked pairs are of period 3. Analogous diagrams can be made for all our abstract definitions of extended groups (each link, in the general case, being marked with the period of the corresponding product).

18.2. In the notation of § 9.4, the α_4

$$b_3 b_4 b_5 b_6 c_{12}$$

belongs to both the β_5's

$$b_2 b_3 b_4 b_5 b_6 c_{12} c_{13} c_{14} c_{15} c_{16} \quad \text{and} \quad b_1 b_3 b_4 b_5 b_6 c_{12} c_{32} c_{42} c_{52} c_{62},$$

and is therefore of *type* 2_{01} (see § 7.5). On the other hand, the α_4

$$b_2 b_3 b_4 b_5 b_6$$

belongs to the former of these β_5's and also to the α_5 $(= 2_{20})$

$$b_1 b_2 b_3 b_4 b_5 b_6,$$

so that it is of type 2_{10}.

It follows that the requirements of § 17.2 are satisfied if we take

X_0	to be the vertex	b_6,
X_1	,, ,, centre of the edge	$b_5 b_6$,
X_2	,, ,, ,, ,, α_2	$b_4 b_5 b_6$,
X_{01}	,, ,, ,, ,, α_4	$b_3 b_4 b_5 b_6 c_{12}$,
X_{11}	,, ,, ,, ,, β_5	$b_2 b_3 b_4 b_5 b_6 c_{12} c_{13} c_{14} c_{15} c_{16}$,
X_{20}	,, ,, ,, ,, α_5	$b_1 b_2 b_3 b_4 b_5 b_6$.

These six points determine the six primes

$$1_h, \ 0_h, \ 0, \ 0_i, \ 1_i, \ 0_j$$

(by omitting them in turn, and joining the remaining five to the centre X_{21}). By § 17 . 4, the generating operations

$$N_1,\ N,\ O,\ P,\ P_1,\ Q$$

are the reflections in these primes respectively.

18 . 3. Consider the transpositions (12)*, (23), (34), (45), (56), and the bifid substitution [456 . 123], defined in (9 . 44).

(56)	alters	X_0	but leaves the other X's invariant,
(45)	,,	X_1	,, ,, ,, ,, ,
(34)	,,	X_2	,, ,, ,, ,, ,
(23)	,,	X_{01}	,, ,, ,, ,, ,
(12)	,,	X_{11}	,, ,, ,, ,, ,
[456 . 123]	,,	X_{20}	,, ,, ,, ,, .

Therefore

$$(18.31)\quad \begin{cases} N_1 = (56), \quad N = (45), \quad O = (34), \quad P = (23), \quad P_1 = (12), \\ \text{and} \qquad\qquad Q = [456 . 123]. \end{cases}$$

In terms of the lines on the cubic surface (Schläfli's notation), the operation (12) consists in interchanging the two halves of the double-six

$$\begin{pmatrix} a_1\, b_1\, c_{23}\, c_{24}\, c_{25}\, c_{26} \\ a_2\, b_2\, c_{13}\, c_{14}\, c_{15}\, c_{16} \end{pmatrix},$$

and the operation [456 . 123] consists in interchanging the two halves of the double-six

$$\begin{pmatrix} a_4\ a_5\ a_6\ c_{23}\, c_{31}\, c_{12} \\ c_{56}\, c_{64}\, c_{45}\, b_1\ b_2\ b_3 \end{pmatrix}.$$

18 . 4. Clearly the generating operations

$$N_1,\ N,\ O,\ P,\ P_1$$

can be taken to be any open chain of transpositions, and then Q can be either of the two bifid substitutions which separate the numbers involved in N and N_1 from the numbers involved in P and P_1. In order to employ

* Not to be confused with the (1 2) of § 15 . 1.

the notation of § 13.5, it is natural to choose the chain

$$(23),\quad (36),\quad (61),\quad (14),\quad (45).$$

Q must now be either [236.145] or [145.236]. Taking the latter value, we have

$$Q = [145.236] = (13)[345.126](13).$$

But, as we observed in § 13.9,

$$[345.126] = H_4.$$

Moreover, $$(13) = (61)(36)(61).$$

Hence, by § 13.5,

$$N_1 = H_6 H_8 H_6,\quad N = H_2,\quad O = H_0,\quad P = H_7,\quad P_1 = H_3 H_1 H_3,$$

and $$Q = H_0 H_2 H_0 H_4 H_0 H_2 H_0.$$

On substituting in (18.11) we obtain an abstract definition for the group in terms of the H's, and so ultimately in terms of the *two* operations ω and H_0. But the new abstract definition is excessively complicated; in fact, the definition in terms of *six* operations is altogether preferable.

18.5. The most important of our *unextended* groups is

$$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}',$$

since this, being (by § 17.5) the group of *positive* symmetries of 3_{21} or $(PA)_7$, is also (by § 11.5) the group of automorphisms of the bitangents of a plane quartic of genus 3.

By (16.75) it has the *abstract definition*

$$(18.51)\quad \begin{cases} N'^3 = N_1'^2 = N_2'^2 = (N' N_1')^3 = (N_1' N_2')^3 = 1, \\ P'^3 = P_1'^2 = (P' P_1')^3 = 1, \\ Q'^3 = 1, \\ (N' N_2')^2 = 1, \\ (P_1' Q')^2 = (Q' N_1')^2 = (Q' N_2')^2 = (N_1' P')^2 \\ \quad = (N_2' P')^2 = (N' P_1')^2 = (N_1' P_1')^2 = (N_2' P_1')^2 = 1, \\ (P' Q'^{-1})^2 = (Q' N')^2 = (N' P')^2 = 1. \end{cases}$$

For simplicity we have written N' for $N_0'^{-1}$, P' for P_0', Q' for Q_0'.

18.6. In the notation of § 9.3, the α_5

$$c_{12} C_{38} C_{48} C_{58} C_{68} C_{78}$$

belongs to both the β_6's

$$c_{12} c_{13} c_{14} c_{15} c_{16} c_{17} C_{28} C_{38} C_{48} C_{58} C_{68} C_{78}$$

and

$$c_{12} c_{32} c_{42} c_{52} c_{62} c_{72} C_{18} C_{38} C_{48} C_{58} C_{68} C_{78},$$

and is therefore of type 3_{01}. On the other hand, the α_5

$$C_{28} C_{38} C_{48} C_{58} C_{68} C_{78}$$

belongs to the former of the β_6's and also to the α_6 $(= 3_{20})$

$$C_{18} C_{28} C_{38} C_{48} C_{58} C_{68} C_{78},$$

so that it is of type 3_{10}.

It follows that the requirements of § 17.2 are satisfied if we take

X_0	to be the vertex		C_{78},
X_1	,, ,, centre of the edge		$C_{68} C_{78}$,
X_2	,, ,, ,, ,,	α_2	$C_{58} C_{68} C_{78}$,
X_3	,, ,, ,, ,,	α_3	$C_{48} C_{58} C_{68} C_{78}$,
X_{01}	,, ,, ,, ,,	α_5	$c_{12} C_{38} C_{48} C_{58} C_{68} C_{78}$,
X_{11}	,, ,, ,, ,,	β_6	$c_{12} c_{13} c_{14} c_{15} c_{16} c_{17} C_{28} C_{38} C_{48} C_{58} C_{68} C_{78}$,
X_{20}	,, ,, ,, ,,	α_6	$C_{18} C_{28} C_{38} C_{48} C_{58} C_{68} C_{78}$.

These seven points determine the seven primes

$$2_h,\ 1_h,\ 0_h,\ 0,\ 0_i,\ 1_i,\ 0_j$$

(by omitting them in turn, and joining the remaining six to the centre X_{21}). By § 17.4, the generating operations of the corresponding extended group, viz.

$$N_2,\ N_1,\ N_0,\ O,\ P_0,\ P_1,\ Q_0,$$

are the reflections in these primes respectively. Finally, by (16.73),

$$N_2' = N_2 O = O N_2, \quad N_1' = N_1 O = O N_1, \quad N' = (N_0 O)^{-1} = O N_0,$$

$$P' = P_0 O, \quad P_1' = P_1 O, \quad Q' = Q_0 O.$$

18.7. Consider the transpositions (12), (23), (34), (45), (56), (67), and the bifid reflection [4567 . 1238], defined in § 9.3.

(67)	alters X_0	but leaves	the	other	X's	invariant,	
(56)	,, X_1	,,	,,	,,	,,	,	
(45)	,, X_2	,,	,,	,,	,,	,	
(34)	,, X_3	,,	,,	,,	,,	,	
(23)	,, X_{01}	,,	,,	,,	,,	,	
(12)	,, X_{11}	,,	,,	,,	,,	,	
[4567 . 1238]	,, X_{20}	,,	,,	,,	,,	.	

Therefore

$$N_2 = (67),\quad N_1 = (56),\quad N_0 = (45),\quad O = (34),\quad P_0 = (23),$$

$$P_1 = (12),\quad \text{and}\quad Q_0 = [4567 . 1238].$$

Finally,

$$(18.71)\quad \begin{cases} N_2' = (34)(67),\quad N_1' = (34)(56),\quad N' = (345),\quad P' = (234), \\ P_1' = (12)(34),\quad \text{and}\quad Q' = [4567 . 1238](34). \end{cases}$$

18.8. Let us now express the generating operations

$$N_2',\ N_1',\ N',\ P',\ P_1',\ Q'$$

in terms of the cyclic permutation and bifid substitution of § 11.5. We shall call the latter K_0, so that*

$$K_0 = [1357 . 2468]\,\mathrm{ST}.$$

It is convenient also to let

$$K_n = (1234567)^{-n} K_0 (1234567)^n,$$

so that $K_1 = [2461 . 3578]\,\mathrm{ST}$, and so on. (ST is permutable with every operation.)

From (9.35) we derive fourteen relations such as

$$K_0 K_1 K_0 = (18)\,\mathrm{ST} = K_1 K_0 K_1.$$

---

* The expression at the end of § 11.5 lacks the S, in error.

Six of these lead to expressions for the N''s and P''s. In order to obtain Q', we observe that

$$[4567\,.\,1238] = (26)\,K_4\,\mathrm{ST}(26),$$

and

$$(26) = (28)(68)(28).$$

The results are as follows:

$$N_2' = K_2 K_4 K_2 K_0 K_5 K_0, \quad N_1' = K_4 K_2 K_6 K_4, \quad N' = K_2 K_4 K_2 K_3 K_5 K_3,$$

$$P' = K_1 K_3 K_1 K_2 K_4 K_2, \quad P_1' = K_2 K_0 K_4 K_2,$$

and

$$Q' = K_1 K_2 K_1 K_5 K_6 K_5 K_1 K_2 K_1 K_4 K_1 K_2 K_1 K_5 K_6 K_5 K_2 K_1 K_4 K_2.$$

On substituting in (18.51), we obtain an abstract definition for the group in terms of the K's, and so ultimately in terms of the *two* operations (1234567) and K_0. It is possible that this new abstract definition could be simplified by the exercise of some ingenuity.

19. *The hundred and twenty tritangent planes**.

19.1. If $m < 9$, a set of m points of general position in a plane determines a finite number of rational curves which have the property of being completely specified by their multiplicities at these points. It is shown by Du Val, in a paper which will shortly appear, that these rational curves are in correspondence with the vertices of $(PA)_m$. These curves have one variable intersection with any cubic passing simply through the m points. When $m < 8$, such cubics represent the prime sections of the Del Pezzo surface† F_2^{9-m} (of order $9-m$, in $9-m$ dimensions), and the rational curves represent the lines on these surfaces. Thus *there is a perfect correspondence between the lines on the Del Pezzo surface $F_2{}^n$ and the vertices of $(PA)_{9-n}$.*

Since $F_2{}^3$ is the cubic surface in ordinary space, while $F_2{}^2$ is the double plane branching along a quartic curve of genus three, these are the cases considered in Chapter 18.

Since plane cubics through eight points all pass through a ninth, there is no corresponding surface when $m = 8$. If, however, we consider sextic curves passing doubly through the eight points, we obtain a surface having on it 240 conics which correspond to the vertices of $(PA)_8$. This surface

* *Chapter* 19 *was added* 11 *June*, 1932.

† *Rend. di Palermo*, 1 (1887), 241.

is, in fact, a double quadric cone in ordinary space, having for branch curve the sextic of genus four in which it is cut by a general cubic surface. The section of the double cone by one of the 120 tritangent planes of this curve breaks up into a pair of conics which coincide in space but lie on different sheets of the double cone. Hence *the* 120 *tritangent planes of this special quadri-cubic curve correspond to the* 120 *pairs of opposite vertices of* $(PA)_8$. This theorem is due to Todd.

Let i, j, k denote any three numbers from among 1, 2, ..., 8. The rational curves determined by eight base points are actually* as follows:

8 points c_i, which can be regarded as curves of order zero;

28 lines a_{ij}, of which a_{78} joins c_7, c_8;

56 conics b_{ijk}, of which b_{678} is determined by the five points $c_1, c_2, \ldots, c_5$;

56 cubics c_{ij}, of which c_{18} goes once through $c_2, c_3, \ldots, c_7$, and twice through c_8;

56 quartics a_{ijk}, of which a_{678} goes once through $c_1, c_2, \ldots, c_5$, and twice through c_6, c_7, c_8;

28 quintics b_{ij}, of which b_{78} goes twice through $c_1, c_2, \ldots, c_6$, and once through c_7, c_8;

8 sextics c_i', of which c_8' goes twice through $c_1, c_2, \ldots, c_7$, and thrice through c_8.

(The order of the suffixes of a or b is immaterial, but c_{ij} and c_{ji} are distinct.)

The corresponding vertices of $(PA)_8$ are respectively

$$c_{i9},\ a_{ij9},\ b_{ijk},\ c_{ij},\ a_{ijk},\ b_{ij9},\ c_{9i},$$

where, using the coordinates (10.21) for $(PA)_8\ 3\sqrt{2}$,

$$\begin{array}{lll} a_{789} & \text{is} & (1, 1, 1, 1, 1, 1;\ -2, -2, -2), \\ b_{123} & \text{is} & (2, 2, 2;\ -1, -1, -1, -1, -1, -1), \\ c_{19} & \text{is} & (3;\ 0, 0, 0, 0, 0, 0, 0;\ -3), \end{array}$$

and so on. The relation is such that two curves having $r-1$ free intersections correspond to two vertices whose mutual distance is $\sqrt{r}$ times the edge $(3\sqrt{2})$.

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* Noether, *Math. Annalen*, 33 (1888), 534. Coble, *Algebraic geometry and theta-functions* (New York, 1929), 209.

19.2. The group of automorphisms of the rational curves, being the group of symmetries of $(PA)_8$, is

(19.21) $$\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}.$$

By (16.74), this group is generated, and in fact abstractly defined, by eight involutory operations

$$O,\ N,\ N_1,\ N_2,\ N_3,\ P,\ P_1,\ Q$$

which satisfy the relations

$$(ON)^3 = (NN_1)^3 = (N_1N_2)^3 = (N_2N_3)^3 = 1,$$
$$(OP)^3 = (PP_1)^3 = 1,$$
$$(OQ)^3 = 1,$$

all other pairs being permutable.

The α_6 $\quad c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}\, a_{129}$

belongs to both the β_7's

$$c_{29}\, c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}\, a_{129}\, a_{139}\, a_{149}\, a_{159}\, a_{169}\, a_{179}\, a_{189}$$

and $\quad c_{19}\, c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}\, a_{129}\, a_{239}\, a_{249}\, a_{259}\, a_{269}\, a_{279}\, a_{289},$

and is therefore of type 4_{01}. On the other hand, the α_6

$$c_{29}\, c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}$$

belongs to the former of these β_7's and also to the $\alpha_7 (= 4_{20})$

$$c_{19}\, c_{29}\, c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89},$$

so that it is of type 4_{10}.

It follows that the requirements of § 17.2 are satisfied if we take

X_0	to be the vertex					c_{89},
X_1	,,	,,	centre of the edge			$c_{79}\, c_{89}$,
X_2	,,	,,	,,	,,	α_2	$c_{69}\, c_{79}\, c_{89}$,
X_3	,,	,,	,,	,,	α_3	$c_{59}\, c_{69}\, c_{79}\, c_{89}$,
X_4	,,	,,	,,	,,	α_4	$c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}$,
X_{01}	,,	,,	,,	,,	α_6	$c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}\, a_{129}$,
X_{11}	,,	,,	,,	,,	β_7	$c_{29}\, c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}\, a_{129}\, a_{139} \cdots a_{189}$,
X_{20}	,,	,,	,,	,,	α_7	$c_{19}\, c_{29}\, c_{39}\, c_{49}\, c_{59}\, c_{69}\, c_{79}\, c_{89}$.

These eight points determine the eight primes

$$3_h,\ 2_h,\ 1_h,\ 0_h,\ 0,\ 0_i,\ 1_i,\ 0_j$$

(by omitting them in turn, and joining the remaining seven to the centre X_{21}). By § 17.4, the generating operations

$$N_3,\ N_2,\ N_1,\ N,\ O,\ P,\ P_1,\ Q$$

are the reflections in these primes respectively.

In § 11.7 we saw that the group is generated by the two special symmetries

$$(12345678) \quad \text{and} \quad V_{123}. \tag{19.22}$$

(The former is simply a cyclic permutation of the first eight coordinates, while the latter, defined in § 10.6, interchanges a_{123} and b_{123}, c_{14} and a_{234}, c_{41} and b_{234}, a_{456} and b_{789}, and so on, but leaves c_{12}, c_{45}, a_{345} and b_{345} unaltered.) It is therefore natural to consider the involutory operations

$$(12),\ (23),\ (34),\ (45),\ (56),\ (67),\ (78),\ \text{and}\ V_{123}.$$

(78) alters X_0 but leaves the other X's invariant,
(67) ,, X_1 ,, ,, ,, ,
(56) ,, X_2 ,, ,, ,, ,
(45) ,, X_3 ,, ,, ,, ,
(34) ,, X_4 ,, ,, ,, ,
(23) ,, X_{01} ,, ,, ,, ,
(12) ,, X_{11} ,, ,, ,, ,
V_{123} ,, X_{20} ,, ,, ,, .

Therefore

$$\left\{\begin{array}{l} N_3=(78),\quad N_2=(67),\quad N_1=(56),\quad N=(45),\quad O=(34), \\ \qquad\qquad P=(23),\quad P_1=(12),\quad \text{and}\quad Q=V_{123}. \end{array}\right. \tag{19.23}$$

By § 11.7 all the transpositions can be deduced from (19.22); and by substituting the resultant expressions for O, N, N_1, etc., we could obtain an abstract definition (albeit extremely cumbrous) in terms of these two operations.

19.3. The 28 bitangents of the plane quartic can be denoted by unordered pairs of the numbers 1, 2, ..., 8. Pascal* has devised an analogous notation for the 120 tritangent planes of the quadri-cubic curve, by unordered triads of the numbers 0, 1, ..., 9. In the special case when the quadric through the curve is a cone, the relation with $(PA)_8$ is as follows.

If i, j, k are any three of the numbers 1, 2, ..., 9, the opposite vertices a_{ijk} and b_{ijk} correspond to the plane ijk, and the opposite vertices c_{ij} and c_{ji} correspond to the plane $ij0$.

Pascal's notation does not immediately exhibit the full symmetry of the configuration of tritangent planes. Bath† has discovered a *bifid substitution* analogous to that used for permuting the bitangents of the plane quartic. The substitution

$$0123/456789,$$

for example, interchanges 014 and 234, 456 and 789, but leaves 012 and 345 unaltered. Thus V_{123}, *when regarded as permuting the joins of opposite vertices* ("diameters", say) *instead of the vertices themselves, is identical with* 0123/456789; and similarly for any V_{ijk}.

But the interchanges involved when the digit 0 occurs *after* the stroke in Bath's symbol show that a substitution such as 1234/567890 is *not* a symmetry of $(PA)_8$. In fact, the diameters 123 and 345 are perpendicular, whereas their transforms 123 and 125 are inclined at $\frac{1}{3}\pi$. Hence, *when the canonical curve lies on a cone, the only bifid substitutions that remain valid are those in which the digit* 0 *occurs before the stroke.*

19.4. It is well known that the canonical curve of genus p is of order $2p-2$, in $p-1$ dimensions, and has $2^{p-1}(2^p-1)$ $(p-1)$-tangent primes. These correspond to the odd theta-characteristics of genus p, and so their group of automorphisms is the special Abelian linear group‡ $A(2p, 2)$, of order

$$\prod_{r=1}^{p} 2^{2r-1}(2^{2r}-1). \qquad (19.41)$$

When $p = 3$, this order is

$$2.3.8.15.32.63 = 1\ 451\ 520.$$

In fact, since the bitangents of the plane quartic correspond to the pairs of opposite vertices of 3_{21}, and since in seven dimensions reflection in a

* *Annali di Mat.* (2), 20 (1892), 198. Actually, Pascal uses "10" instead of "0".

† *Journal London Math. Soc.*, 3 (1928), 84.

‡ Dickson, *Linear groups*, 89, 100.

point is a *negative* transformation, $A(6, 2)$ *is simply isomorphic with*

$$\begin{bmatrix} 3, & 3, & 3 \\ 3, & 3, & \\ 3 & & \end{bmatrix}'.$$

When $p = 4$, (19.41) becomes

$$2.3.8.15.32.63.128.255 = 47\ 377\ 612\ 800.$$

If the curve lies on a cone, one of the 136 even theta-characteristics is special*; so the group of automorphisms is a sub-group of index 136 in $A(8, 2)$, namely the corresponding *first hypoabelian group*†, of order

$$348\ 364\ 800 = 96.10!$$

Since the tritangent planes of the special quadri-cubic curve correspond to the pairs of opposite vertices of 4_{21}, their group of automorphisms is a self-conjugate sub-group of index 2 in (19.21). But this sub-group is not the same as

$$\begin{bmatrix} 3, & 3, & 3, & 3 \\ 3, & 3 & & \\ 3 & & & \end{bmatrix}',$$

since in eight dimensions reflection in a point is a *positive* transformation. These two sub-groups of the whole group of symmetries of 4_{21} have a common self-conjugate sub-group of index 2, namely the group of *positive* symmetries of the *diameters* of 4_{21}, which is the simple group $FH(8, 2)$‡, of order

$$174\ 182\ 400.$$

Since (19.21) is generated by (19.22), we can use the same symbols to represent the corresponding generators of the first hypoabelian group, provided we identify opposite vertices of 4_{21} by writing

$$T = 1$$

or

(19.42) $$V_{124} V_{235} V_{346} V_{457} V_{561} V_{672} V_{713} = (89)$$

(see §10.7). In fact, the abstract definition of the latter group is derived from that given in §19.2 by inserting one extra relation, namely (19.42) expressed in terms of O, N, N_1, etc.

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* The corresponding theta-function vanishes for zero values of the arguments. See Schottky, *Journal für Math.*, 103 (1887), 185.

† Dickson, *Linear groups*, 201.

‡ *Ibid.*, 216.

19.5. We proceed to prove that the whole group $A(8, 2)$ is generated by these same operations along with the transposition

$$(90).$$

Since the operations of the first hypoabelian group include all the permutations of 1, 2, ..., 9, it will be sufficient if we prove that $A(8, 2)$ is generated by all the permutations of 0, 1, 2, ..., 9 together with all Bath's bifid substitutions. Let G denote the group so generated. If G were not the same as $A(8, 2)$, it would have to be a sub-group; therefore we merely have to prove that G is of order 47 377 612 800.

Of the 120 tritangent planes of the general quadri-cubic curve, a pair such as

$$079, 089$$

leads by permutations and bifid substitutions to a set of 7140 pairs, namely

1260	of type	*abc*, *abd*;
3780	,,	*abc*, *ade*;
2100	,,	*abc*, *def*.

Since $7140 = \binom{120}{2}$, this shows that *all pairs of tritangent planes are equivalent.*

Again, a triad such as

$$069, \ 079, \ 089$$

leads to a set of 152320 triads, namely:

2520	of type	*abc*, *abd*, *abe*;
840	,,	*abc*, *abd*, *acd*;
7560	,,	*abc*, *abd*, *cde*;
37800	,,	*abc*, *abd*, *aef*;
25200	,,	*abc*, *abd*, *efg*;
75600	,,	*abc*, *ade*, *bfg*;
2800	,,	*abc*, *def*, *ghi*.

Defining the *sum* of the symbols of three planes as the set of digits obtained by juxtaposing the symbols and cancelling repeated digits*, we see that these particular triads are such that the sum of their symbols has one or

* Pascal, *Annali di Mat.* (2), 20 (1892), 199.

five or nine digits. (For example, $abc+abd+abe=abcde$.) Following Pascal, we call these *even* triads. The sum of the symbols of any of the remaining 128520 possible triads ("odd" triads) has either three or seven digits. The geometrical significance of this distinction is that the nine points of contact of an even triad of tritangent planes do not lie on a quadric (other than that through the whole curve).

If two planes of an even triad are fixed, there are 64 possibilities for the third. For instance, a plane making an even triad with 079 and 089 may involve both or neither of the digits 7, 8. It is easily seen that all such planes can be derived from any one of them by permutations and bifid substitutions not affecting the two fixed planes.

If an even triad is fixed, there are 36 possibilities for a fourth plane which makes an even triad with every pair of the fixed triad. For instance, a plane making an even triad with every pair of 069, 079, 089 may either be 678 or involve none of the digits 6, 7, 8.

A plane making an even triad with every pair of

059, 069, 079, 089

must not involve any of the digits 5, 6, 7, 8; so there are $\binom{6}{3}=20$ possibilities, and these can be put in correspondence with the vertices of $t_2\alpha_5$.

A plane making an even triad with every pair of

049, 059, 069, 079, 089

must not involve any of 4, 5, 6, 7, 8; so there are 10 possibilities*. But since the vertex figure of $t_2\alpha_5$ is $[\alpha_2, \alpha_2]$, which has only nine vertices, we should expect one of these ten to be special. Such is easily seen to be the case, the special plane being, of course, 123.

There is no plane that will make an even triad with every pair of

123, 049, 059, 069, 079, 089.

But there are four possibilities if we replace 123 by any other symbol formed with three of the digits 0, 1, 2, 3, 9. *E.g.*, with 039 we can have any of

012, 019, 029, 129.

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* Noether, *Math. Annalen*, 14 (1879), 270. His

$$(\alpha_\rho),\ (qq'\alpha_\rho\alpha_\sigma\alpha_\tau),\ (q\alpha_\rho\alpha_\sigma),\ (q'\alpha_\rho\alpha_\sigma)$$

are Pascal's

$$0\rho 9,\ \rho\sigma\tau,\ 0\rho\sigma,\ \rho\sigma 9,$$

respectively.

Finally, 019 is the only plane that will make an even triad with every pair of

029, 039, 049, 059, 069, 079, 089*.

Let us now translate these results into terms of groups. Those operations of G that keep one plane fixed form a sub-group G_1 of index 120. Those operations that keep two planes fixed form a sub-group G_2 of index 119 in G_1. Those that keep an even triad fixed form a sub-group G_3 of index 64 in G_2. And so on, through the sequence of numbers 36, 20, 9, 4. Finally, the only operations that will keep fixed an octad like

019, 029, 039, 049, 059, 069, 079, 089

are (90) and identity; so G_7 is of order 2.

We can now deduce the orders of all these groups. In particular, G_3 is of order

$$2.4.9.20.36 = 51\,840,$$

and G itself is of order

$$2.4.9.20.36.64.119.120 = 47\,377\,612\,800.$$

Therefore G is $A(8, 2)$, as we desired to prove.

19.6. The above procedure is closely analogous to that of Chapter 9, where we obtained successive vertex figures of certain polytopes. Since G_6, G_5, G_4 are the groups of symmetries of $(IA)_3$, $(IA)_4$, $(IA)_5$ respectively†, while G_3 has the same order as $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$, it is natural to expect some relation between G_3 and $(IA)_6$.

Let j, j' be two of the numbers 1, 2, 3; k, k' two of 4, 5, 6; l, l' two of 7, 8, 9. The 36 planes which form even triads with every pair of

123, 456, 789

are $$jj'0,\ kk'0,\ ll'0,\ jkl.$$

In the case when the quadri-cubic curve lies on a cone, these 36 planes correspond to the pairs of opposite vertices

$$c_{jj'} \text{ and } c_{j'j},\quad c_{kk'} \text{ and } c_{k'k},\quad c_{ll'} \text{ and } c_{l'l},\quad a_{jkl} \text{ and } b_{jkl}$$

of the $(IA)_6 3\sqrt{2}$ (10.32). Since these planes acquire no extra auto-

* Pascal, *Atti R. Acc. Lincei* (*Rend.*) (5), 2 (1893), 122.

† (8.16).

morphisms when the quadri-cubic curve is taken to be general, G_3 *is simply isomorphic with* $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$; or, in geometrical language, *those automorphisms of the tritangent planes which keep fixed an even triad* (*i.e.*, three planes whose points of contact do not lie on an extra quadric) *constitute a group simply isomorphic with the group of automorphisms of the lines on the cubic surface.*

This theorem is due to Pascal*, who regards the set of tritangent planes with one omitted as corresponding to a configuration of 119 planes in four dimensions. Pairs of these planes meet in a point or in a line according as the corresponding tritangent planes make an even or odd triad with the one that was omitted. Let two planes meeting only in a point be called *skew* ("gobbo"). There are 64 planes skew to any particular one; and, skew to any one among these, there are 36. Pascal shows that these 36 correspond to the double-sixes of lines on the cubic surface, which we recognize as corresponding to the diameters of $(IA)_6$. He gives a detailed account† of this configuration of 36 planes in four dimensions, from which we see that the relation with $(IA)_6$ is as follows.

The 36 planes, 360 skew pairs, 1080 skew triads (of the " second kind "), 1080 skew tetrads, and 216 skew pentads correspond to the pairs of opposite vertices, edges, α_2's, α_3's, and α_4's; while the 120 skew triads (of the " first kind ") and the 135 tetrahedra correspond to the diagonal hexagons (or pairs of diagonal $\alpha_2\sqrt{3}$'s) and diagonal $\beta_4\sqrt{2}$'s.

19.7. We shall now prove that $A(8, 2)$ *is generated by the two operations*

(19.71) $\qquad (012345678)$ *and* $0123/456789$

[*cf.* (11.71)].

It is convenient to abbreviate the latter symbol to 0123/. (10.63) and (10.64) obviously generalize to give

$$fghi/ . (ij) = (ij) . fghj/ = fghj/ . fghi/,$$

$$bcde/ . bfgh/ = bfgh/ . bijk/,$$

where *bcdefghijk* is a permutation of 0123456789. Therefore $A(8, 2)$ is generated by

(19.72) $\qquad hijk/ \quad (h, i, j, k < 9).$

* *Atti R. Acc. Lincei* (*Rend.*) (5), 2 (1893), 68.

† *Ibid.*, 71.

The operations (19.71) together lead to

$$1234/,\quad 2345/,\quad 5678/,\quad 6780/,$$

which give the transpositions

$$(15),\quad (50)$$

and

$$(01),$$

whence we can deduce all the permutations of 0, 1, ..., 8, and so also the rest of the operations (19.72).

19.8. Since $A(8, 2)$ is generated by the symmetries of the diameters of 4_{21} along with the transposition (90), it is natural to extend (19.23) by writing

(19.81) $$Q_1 = (90).$$

(We must suppose the operations O, N, N_1, etc., modified in the manner described at the end of § 19.4, so that

$$Q = 0123/456789.)$$

It is easily seen that Q_1 is permutable with

$$N_3,\ N_2,\ N_1,\ N,\ O,\ P,\ P_1,$$

whereas

$$(QQ_1)^3 = 1.$$

There are, of course, other relations; but these show that $A(8, 2)$ *may be regarded as a sub-group of the (infinite) group of symmetries of the Minkowskian polytope*

$$4_{22} \text{ or } (IA)_9.$$

This result is not surprising when we recall that G_r is precisely the group of symmetries of $(IA)_{9-r}$ when $r > 3$, and is a sub-group (of index 2) when $r = 3$.

Instead of (19.81), we might have written

$$P_2 = (01),$$

thus exhibiting $A(8, 2)$ as a sub-group of the group of symmetries of the Minkowskian polytope

$$4_{31} \text{ or } (SA)_9.$$

19.9. At the end of a paper on $(PA)_6$ and $(PA)_7$*, I have given geometrical interpretations for the elements, diagonals, etc., of $(PA)_7$ as

* *Proc. Camb. Phil. Soc.*, 24 (1928), 1–9. (The symbol Ta_4, used there, should have been $t_1 a_4$.)

special sets of bitangents of the plane quartic. We shall now attempt to do the same for $(PA)_8$ and the tritangent planes. Naturally the elements of $(PA)_8$ correspond only to sets of tritangent planes of the "special" quadri-cubic curve (lying on a cone); but there are usually analogous sets in the general case. Thus, corresponding to the 3360 pairs of opposite edges of $(PA)_8$, we have 3360 special pairs of tritangent planes of the special curve, and $119 . 120/2 = 7140$ pairs of tritangent planes of the general curve. Again, corresponding to the 30240 pairs of opposite triangular faces and the 1120 pairs of opposite diagonal triangles, we have $30240+1120$ special triads of tritangent planes of the special curve, and $64 . 7140/3 = 152320$ *even* triads of tritangent planes of the general curve.

Most of the numbers in the second column of the table at the end of this chapter are taken from § 12.8 (the last column of page 414). The number of diagonal $\alpha_2\sqrt{3}$'s of $(PA)_8$ is $28 . 240/3 = 2240$, since those at any vertex correspond to the diameters of $(PA)_7$. The number of diagonal {3, 4, 3}'s is $315 . 240/24 = 3150$, since those at any vertex correspond to the diagonal cubes of $(PA)_7$ (or to the pairs of opposite diagonal tetrahedra). In each {3, 4, 3} (*e.g.*, that lying in the 4-space

$$x_1+x_2 = x_3+x_4 = x_5+x_6, \quad x_7 = x_8 = x_9, \quad \Sigma x = 0)$$

we can inscribe three γ_4's or three $\beta_4\sqrt{2}$'s, making 9450 of each altogether.

The $\beta_4\sqrt{2}$'s inscribed in the diagonal {3, 4, 3}'s are particularly interesting since they correspond to tetrads of tritangent planes whose twelve points of contact all lie on an extra quadric. The "sum" of the symbols of such a tetrad has either no digits or all the ten.

In the case of the general quadri-cubic curve, every odd triad determines a fourth plane making with it a tetrad of this kind. *E.g.*, with the triads

235, 145, 136 and 012, 034, 056

we must associate 246 and 789

respectively. Thus the number of such tetrads is

$$128520/4 = 32\,130^*.$$

Let us take one of these tetrads, and associate with it as many more planes as possible, in such a way that every triad is odd. We find that just four more planes can be added, that this can be done in five ways, and that each new set of four is a tetrad of the same kind. We are thus led to consider octads of planes such as those which correspond to the

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* Pascal, *Atti R. Acc. Lincei* (*Rend.*) (5), 2 (1893), 204.

diagonal $\beta_8\sqrt{2}$'s of $(PA)_8$. *Each octad can be divided into two of the* 32130 *tetrads in seven ways. E.g.*, in the octad

(19.91) 124, 235, 346, 457, 561, 672, 713, 890,

the three planes whose symbols involve a particular one of the digits 1, 2, 3, 4, 5, 6, 7 form with 890 a tetrad of the proper kind. Thus the number of such octads is

$$5\,.\,32130/14 = 11\,475.$$

For the *special* quadri-cubic curve, we consider only those octads that correspond to diagonal $\beta_8\sqrt{2}$'s; then each tetrad belongs to three (instead of five) octads, and the number is

$$3\,.\,9450/14 = 2\,025.$$

TABLE OF THE PRINCIPAL SETS OF TRITANGENT PLANES.

Number of planes in a set.	Number of sets for the *special* quadri-cubic curve.	Corresponding configuration in 4_{21}.	Typical set	Number of sets for the *general* quadri-cubic curve.
1	120	α_0	089	120
2	3360	α_1	079, 089	7140
3 {	30240	α_2	069, 079, 089	} 152320
	1120	$\alpha_2\sqrt{3}$ or $\{6\}$	123, 456, 789	
4	120960	α_3	059, 069, 079, 089	1370880
5	241920	α_4	049, 059, 069, 079, 089	5483520
4	9450*	$\beta_4\sqrt{2}$	012, 034, 056, 789	32130
8	9450	γ_4	{ 235, 145, 136, 246 / 146, 236, 245, 135 }	
12	3150	$\{3, 4, 3\}$	Combination of the two above	
6 {	241920	$\alpha_5 = 4_{00}$	039, 049, 059, ..., 089	8225280
	—		123, 049, 059, ..., 089	913920
7 {	69120	$\alpha_6 = 4_{10}$	029, 039, 049, ..., 089	} 4700160
	34560	$\alpha_6 = 4_{01}$	129, 039, 049, ..., 089	
8	8640	$\alpha_7 = 4_{20}$	019, 029, 039, ..., 089	587520
14	1080	$\beta_7 = 4_{11}$	{ 029, 039, 049, ..., 089 / 129, 139, 149, ..., 189 }	
8	2025	$\beta_8\sqrt{2}$	124, 235, ..., 713, 890†	11475

* The diagonal $\beta_8\sqrt{2}$'s lead to many more $\beta_4\sqrt{2}$'s, but these 9450 are special (being inscribed in $\{3, 4, 3\}$'s).

† (19.91).

NOTES.

20.1. *Corrections to Part* 1

[*Phil. Trans. Royal Soc.*, A, 229 (1930), 329–425].

329, line 14. *For* Π^{r+u} *read* Π_r^{+u}.

335, line 27. *For* ; *read* , .

337, line 6. *For* generally *read* usually.

346, line $16\frac{1}{2}$. *Ignore the comma after* 3.

347, line 2. *Interchange* τ *and* $\sqrt{5}$.

347, line 10. *Ignore the upright stroke.*

352, line 17. *For* e *read* E.

353, line 19. *Ignore the stop after* 4.81. *For* $_{,\ldots}$ *read* , ...

360, line 21. *Interchange* 2 *and* τ (*or any other pair*).

360, line 23. *Interchange* 1 *and* τ^{-1} (*or any other pair*).

361, line 15. *For* ± 3 *read* 3.

369, line 9. *For* an nth *read* a t_n.

388, line 21. *Insert* — *after* —1, . *Ignore the comma before the second semicolon.*

406, line 23. *For* $[1357.2468]\mathrm{T} = \mathrm{T}_{1357}\mathrm{ST}_{2468}$ *read*

$$\mathrm{ST}\,[1357.2468] = \mathrm{ST}_{1357}\mathrm{ST}_{2468}.$$

408, line 15. *For* automorphism *read* automorphisms.

20.2. *Miller's proof that every finite uniform polytope has a circumcentre**.

A set of points are said to be *equivalent* if, for every pair A, B of the points, there exists a congruent transformation which changes A into B, leaving the set unchanged as a whole. In § 1.8 we made the assumption that *a finite set of equivalent points necessarily lie on a sphere-analogue.* J. C. P. Miller, assisted by J. A. Todd and L. C. Young, has

* *Cf.* E. Catalan, "Mémoire sur la théorie des polyèdres", *Journal de l'Ecole Polytechnique*, 41 (1865), 33. It is interesting to note that Catalan's definition (p. 25) of "polyèdre semi-régulier du premier genre" should admit Miller's non-uniform solid (§2.1); so, too, the reciprocal of Miller's solid is really one "du second genre".

constructed a proof to justify this assumption. For simplicity he employs the terminology of three dimensions; to obtain the general statement, we have merely to read " m-dimensional sphere-analogue " or " $(m-1)$-sphere " for " sphere ", and " $(m-1)$-dimensional sphere-analogue " or " $(m-2)$-sphere " for " circle ".

LEMMA. *There is a unique smallest sphere which encloses all the points.*

Proof of lemma. Since the set of points, say Π, is supposed finite, we may take an enclosing sphere of finite radius. If the smallest of such spheres is not unique, consider two distinct smallest spheres. Since these are equal and both enclose Π, they must intersect. Describe a new sphere concentric with, and passing through, their circle of intersection. This sphere will be smaller than the others, and, containing their common part, must enclose Π; which is absurd. Thus the lemma is established.

Proof of theorem. S, the smallest sphere enclosing Π, must evidently have at least one point A of Π on it. Any other point B of Π must lie on or within S. If possible, let B be definitely within. Since the points are equivalent, there exists a congruent transformation which changes A into B but leaves Π unchanged as a whole. This transformation changes S, which passes through A, into an equal sphere S' passing through B. Since B does not lie on S, S' must be distinct from S. Since S encloses Π, and Π is transformed into itself, S' must enclose Π. But this contradicts the lemma. Hence B, which was arbitrarily chosen from the points of Π, lies on S; and so all the points lie on S.

20.3. *Uniform (degenerate) polytopes not uniquely determined by their vertex figures.*

In § 2.1 we made the assumption "that, given any uniform polytope, there is no other uniform polytope of different shape having the same vertex neighbourhood." There is no reason to doubt the validity of this assumption when the polytope is *finite*; but J. C. P. Miller has refuted it in the degenerate case, by describing two distinct uniform polytopes, say M_4 and M_4', which have the same vertex neighbourhood. Their common vertex figure is obtained if we cut a cuboctahedron of unit edge in halves along an equatorial hexagon, and replace one half by a hexagonal pyramid of unit altitude. Thus each polytope consists of a net of tetrahedra, octahedra, and triangular prisms, filling three-dimensional space. In order to avoid repetition, we shall at once describe the analogous polytopes filling m-dimensional space.

In § 6 . 8 we saw that the vertex figure of $a_m h$ is ea_m, and that the bounding figures of the latter polytope correspond to all the elements of either of two reciprocal a_m's. It follows that with every bounding figure of $a_m h$ can be associated a type symbol $0_{nn'}$ (distinct from $0_{n'n}$), where $n+n' = m-1$, in such a way that every $0_{nn'}$ at one vertex corresponds to an a_n of a definite one of the two reciprocal a_m's of which the actual vertex figure at that vertex is an "expansion" (in the sense of § 6 . 9). *E.g.*, the triangles of $a_2 h$ can be labelled alternately 0_{10} and 0_{01}. Further, there is a definite type symbol $0_{n(n'-1)}$ for the $(m-1)$-dimensional element in which a $0_{nn'}$ meets a $0_{(n+1)(n'-1)}$*. *E.g.*, every edge of $a_2 h$ is of type 0_{00}.

Now, by considering definite integer values of a particular coordinate, say x_{m+1}, in (6 . 81), we see that it is possible (in $m+1$ ways) to select a series of parallel $(m-1)$-spaces, together containing all the vertices of $a_m h$, and each filled with elements of $a_m h$ forming an $a_{m-1} h$. Every bounding figure of each $a_{m-1} h$ is already marked with a type symbol of the form $0_{n(n'-1)}$. The new polytope M_{m+1}† is constructed by cutting $a_m h$ along each one of these $(m-1)$-spaces, shifting the resultant layers apart, and inserting a layer of prisms $[0_{n(n'-1)}, a_1]$. The modification M'_{m+1} is derived by sliding every *alternate* layer of elements of $a_m h$, between its two bounding $(m-1)$-spaces, in such a way that each inserted prism, instead of joining two $0_{n(n'-1)}$'s, joins a $0_{n(n'-1)}$ to a $0_{(n'-1)n}$.

Clearly the vertex figure of either of these new polytopes is obtained if we cut an ea_m in halves along an equatorial ea_{m-1} [such as that obtained by fixing $x_{m+1} = 0$ in (6 . 82)] and replace one half by the pyramid-analogue $(ea_{m-1} \underset{\sqrt{2}}{\text{———}} a_0)$.

Exceptionally, M_3' is the same as M_3, viz. alternate strips of triangles and squares filling a plane. (Vertex figure : a cyclic pentagon, of sides $\sqrt{2}$, 1, 1, 1, $\sqrt{2}$.) But consider the degenerate prism $[M_3, \delta_2]$, consisting of alternate layers of triangular prisms and of cubes, filling three-dimensional space. It is clear that every *alternate* layer of triangular prisms can be turned bodily through a right angle, so as to give a new uniform polytope, say $[M_3, \delta_2]'$, having the same vertex figure as $[M_3, \delta_2]$.

(This vertex figure is, of course, a bipyramid of slant edge $\sqrt{2}$ on the above-mentioned vertex figure of M_3.)

20 . 4. *Coordinates for pentagonal polytopes.*

The $\{5, 3\}4\tau^{-1}$ of § 3 . 6 corresponds in position to the second $t_1\{3, 5\}2\tau^{-1}$ of § 5 . 7 ; *i.e.*, the latter, apart from size, is an *actual*

* There are, in fact, type symbols for *all* the elements of $a_m h$, except vertices.

† Not to be confused with Schoute's M_n, which is our γ_n.

truncation of the former. So, too, the $\{5, 3\}2\tau^{-1}$ of § 3.6 corresponds in position to the *first* $t_1\{3, 5\}2\tau^{-1}$. But, in order to bring the $\{3, 5\}2$ of § 3.6 into the position corresponding to the last-mentioned pair of polyhedra, it is necessary to perform a transposition among the coordinates τ, 1, 0.

Similarly, the $\{5, 3, 3\}2\tau^{-2}$ of § 3.6 and the $t_1\{3, 3, 5\}2\tau^{-1}$ of § 5.7 (both corrected as in § 20.1) correspond in position to the $t_1\{5, 3, 3\}2\tau^{-2}$ of § 5.7. But, in order to bring the $\{3, 3, 5\}2\tau^{-1}$ of § 3.6 into the corresponding position, it is necessary to perform a transposition among τ, 1, τ^{-1}, 0. After this alteration has been made, if $(x_1;\ x_2;\ x_3;\ x_4)$ is any vertex of the $\{3, 3, 5\}2\tau^{-1}$, then $(x_1+x_2;\ x_1-x_2;\ x_3+x_4;\ x_3-x_4)$* is a vertex of the $\{3, 3, 5\}2\sqrt{2}\tau^{-1}$ at the bottom of the page (346).

20.5. *Du Val's coordinates for* 5_{21}.

By applying the transformation

$$x'_{2r-1} = \tfrac{1}{2}(x_{2r-1}+x_{2r}),\quad x'_{2r} = \tfrac{1}{2}(x_{2r-1}-x_{2r}) \qquad (r = 1,\ 2,\ 3,\ 4)$$

to the coordinates (§ 9.1) for $5_{21}2\sqrt{2}$, we obtain, as the vertices of

$$5_{21}2,$$

the totality of points whose eight Cartesian coordinates are either all even or all odd or four even and four odd, with a restriction in the third case. If x_i, x_j, x_k, x_l are the four even coordinates (or the four odd ones), the suffixes i, j, k, l always form one half of one of the following bifid symbols:

[1234.5678], [1256.3478], [1278.3456],

[1357.2468],

[1368.2457], [1458.2367], [1467.2358].

A suitable permutation of the coordinates transforms these bifid symbols into those occurring in (9.36), which have the simple property that all are derivable from any one by cyclic permutation of the digits 1, 2, 3, 4, 5, 6, 7.

Of the points so defined, those distant 2 from the origin must be the vertices of

$$4_{21}2.$$

* *Cf.* Robinson, "On the orthogonal groups in four dimensions", *Proc. Camb. Phil. Soc.*, 27 (1930), 37-48. His δ_4 ($=\{3, 4, 3\}$) must not be confused with our δ_4 ($=\{4, 3, 4\}$).

These are easily seen to consist of the vertices

$$\pm(2, 0, 0, 0, 0, 0, 0, 0)$$

of a $\beta_8 2\sqrt{2}$, together with the vertices of fourteen $\gamma_4 2$'s of the form

$$x_e = x_f = x_g = x_h = 0, \quad \pm x_i = \pm x_j = \pm x_k = \pm x_l = 1,$$

where $[efgh.ijkl]$ is one of our special bifid symbols. In contrast with (9.21), this set of coordinates has the advantage that every axial prime is a prime of symmetry of the polytope.

By selecting the points of this set distant 2 from (0, 0, 0, 0, 0, 0, 0; 2), we obtain the vertices of

$$3_{21} 2$$

in the perfectly symmetrical form

$$x_e = x_f = x_g = x_h = 0, \quad \pm x_i = \pm x_j = \pm x_k = 1,$$

where $(i\,j\,k)$ is one of the seven triads

(1 2 4), (2 3 5), (3 4 6), (4 5 7), (5 6 1), (6 7 2), (7 1 3),

and e, f, g, h are the rest of the numbers 1, 2, 3, 4, 5, 6, 7. The essential properties of these triads are that every two digits determine a unique third forming a triad with them, that every two triads have a single digit in common, and that every digit belongs to just three triads. The digits of the first triad are the residues, mod 7, of the powers of 2 (or quadratic residues), and the rest of the triads are derived by cyclic permutation of the seven digits.

The vertices of 3_{21} have not hitherto been expressed by *rational* coordinates in *seven* dimensions. We observe, incidentally, that these fifty-six points are the vertices of seven cubes.

For the vertices of 5_{21}, we have now two expressions in eight dimensions and one in nine. By considering fundamental systems of theta-characteristics, Du Val has discovered an expression in *ten* dimensions. His statement is as follows.

The vertices of

$$5_{21} 5\sqrt{2}$$

are all the points in ten dimensions whose coordinates satisfy the equations

$$x_1+x_2+x_3+x_4+x_5 = x_6+x_7+x_8+x_9+x_{10} = 0$$

and the congruences

$$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv x_5 \equiv 2x_6 \equiv 2x_7 \equiv 2x_8 \equiv 2x_9 \equiv 2x_{10} \pmod 5.$$

The consequent coordinates for the vertices of

$$4_{21}\, 5\sqrt{2}$$

are

$$\begin{array}{rrrrrrrrrr}
(0, & 0, & 0, & 0, & 0; & 5, & 0, & 0, & 0, & -5), \\
(1, & 1, & 1, & 1, & -4; & 3, & 3, & -2, & -2, & -2), \\
(2, & 2, & 2, & -3, & -3; & 1, & 1, & 1, & 1, & -4), \\
(3, & 3, & -2, & -2, & -2; & 4, & -1, & -1, & -1, & -1), \\
(4, & -1, & -1, & -1, & -1; & 2, & 2, & 2, & -3, & -3), \\
(5, & 0, & 0, & 0, & -5; & 0, & 0, & 0, & 0, & 0).
\end{array}$$

20.6. *Degenerate prisms.*

In § 17.7 we defined the fundamental simplex of the finite generalized prism. It may seem unsatisfactory that nothing has been said about the degenerate prism (§ 4.8). However, the extension is easily made.

Let

$$\Sigma^{(1)}_{i-2},\ \Sigma^{(2)}_{i-2},\ \ldots,\ \Sigma^{(i)}_{i-2}$$

be the bounding simplexes of the (Euclidean) fundamental simplex Σ_{i-1} of a degenerate polytope Π_i, and let $(f\ g)$ be the angle between the spaces of $\Sigma^{(f)}_{i-2}$ and $\Sigma^{(g)}_{i-2}$. Similarly, let

$$\Sigma^{(i+1)}_{m-i-2},\ \Sigma^{(i+2)}_{m-i-2},\ \ldots,\ \Sigma^{(m)}_{m-i-2}$$

be the bounding simplexes of the fundamental simplex Σ_{m-i-1} of another degenerate polytope Π_{m-i}, and let $(k\ l)$ be the angle between $\Sigma^{(k)}_{m-i-2}$ and $\Sigma^{(l)}_{m-i-2}$. We should expect the fundamental simplex Σ_{m-1} of the degenerate prism $[\Pi_i, \Pi_{m-i}]$ to have bounding spaces corresponding to $\Sigma^{(1)}_{i-2}, \ldots, \Sigma^{(i)}_{i-2}, \Sigma^{(i+1)}_{m-i-2}, \ldots, \Sigma^{(m)}_{m-i-2}$, such that the angle between the spaces corresponding to $\Sigma^{(f)}_{i-2}$ and $\Sigma^{(g)}_{i-2}$ is $(f\ g)$, while the spaces corresponding to $\Sigma^{(g)}_{i-2}$ and $\Sigma^{(k)}_{m-i-2}$ are perpendicular. Now this is precisely the state of affairs in the (finite) prism $[\Sigma_{i-1}, \Sigma_{m-i-1}]$; the angle between the spaces of $[\Sigma^{(f)}_{i-2}, \Sigma_{m-i-1}]$ and $[\Sigma^{(g)}_{i-2}, \Sigma_{m-i-1}]$ is $(f\ g)$, while the spaces of $[\Sigma^{(g)}_{i-2}, \Sigma_{m-i-1}]$ and $[\Sigma_{i-1}, \Sigma^{(k)}_{m-i-2}]$ are perpendicular. In fact, the vertices of Σ_{m-1} all coincide with the point at infinity in the direction normal to the $(m-2)$-space of $[\Sigma_{i-1}, \Sigma_{m-i-1}]$, and its bounding $(m-2)$-spaces join this point at infinity to the bounding $(m-3)$-spaces of $[\Sigma_{i-1}, \Sigma_{m-i-1}]$. Since $[\Pi_{m-i}, \Pi_i]$ actually lies in (or rather fills) this $(m-2)$-space, we can neglect the

$(m-1)$-th dimension, and acknowledge a *fundamental prism* in $m-2$ dimensions in place of the fundamental simplex in $m-1$.

More generally, the degenerate prism

$$[\Pi_{m_1+1}, \Pi_{m_2+1}, \ldots]$$

has the fundamental prism

$$[\Sigma_{m_1}, \Sigma_{m_2}, \ldots],$$

where Σ_{m_1} is the fundamental simplex of Π_{m_1+1}, and so on. The reflections in the bounding spaces of this fundamental prism generate the direct product of the groups generated by the reflections in the bounding spaces of $\Sigma_{m_1}, \Sigma_{m_2}, \ldots$. This direct product is either the whole group of symmetries of the degenerate prism or a sub-group thereof.

The fundamental simplex of δ_2 (see §§ 3.5, 15.8) being

$$(\infty) = \alpha_1 \tfrac{1}{2},$$

the fundamental prism of

$$[\delta_2, \delta_2, \ldots] = \delta_{m+1}$$

is

$$[\alpha_1 \tfrac{1}{2}, \alpha_1 \tfrac{1}{2}, \ldots] = \gamma_m \tfrac{1}{2}.$$

Since $\gamma_m \frac{1}{2}$ can be divided (by primes of symmetry through one of its vertices) into $m!$ repetitions of the simplex $(4, 3^{m-2}, 4)$, the corresponding group (viz., the direct product of m groups of the form $S^2 = T^2 = 1$) is a sub-group of index $m!$ in the group of symmetries of δ_{m+1}.

On the other hand, if $a, b, \ldots$ are all different, the (extended) group with $[\alpha_1 \frac{1}{2} a, \alpha_1 \frac{1}{2} b, \ldots]$ for fundamental region is precisely the group of symmetries of $[\delta_2 a, \delta_2 b, \ldots]$. The unextended group corresponding to the rectangle $[\alpha_1 \frac{1}{2} a, \alpha_1 \frac{1}{2} b]$—that is, the group of rotations (positive symmetries) of $[\delta_2 a, \delta_2 b]$—is discussed by Burnside (§ 298).

20.7. *Bibliography for "Fundamental regions".*

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