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ARITHMETIC IN PHILO JUDAEUS

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IN SEVERAL technical fields, notably geometry, arithmetic, rhetoric, and philosophy, there were in circulation in ancient times concise, systematic handbooks of the elements of the subject which went under several names, "introductions," "elements," "arts," and the like. Euclid's *Elements of Geometry*, of all of this class of writings, is the best known today; certain others have survived, but a much greater number have been lost. Were it possible for the modern student of Greek science to assemble a chronological series of the *elementa*, or *artes*, of a given subject, he would thus best be able to trace the development of the science in question, as successive generations added new theorems or discoveries. This, however, he cannot do, so many of the connecting links have perished; he must take as fixed points the few surviving works and fill in the intervals with fragmentary notices and testimonies extracted from authors not directly in the line of descent. Readers of Sir Thomas Heath's *Manual of Greek Mathematics*[1] will note that he has done something of the sort for geometry, noting when, and by whom, successive additions were made to the elements of that science before Euclid perfected its codification. The present writer had a similar purpose in writing a chapter on "The Development of the Greek Arithmetic before Nicomachus" in a volume devoted to that author,[2] and in the course of

[1] Oxford: Clarendon Press, 1931.

[2] D'Ooge, Robbins, and Karpinski, *Nicomachus of Gerasa* (University of Michigan Studies, "Humanistic Series," Vol. XVI). New York: Macmillan, 1926. Cited as *Nicomachus of Gerasa*.

collecting and stating the evidence it became clear that Philo Judaeus could be profitably used to show the nature and extent of elementary arithmetical knowledge in his time.[1] The purpose of the present paper is to examine Philo's arithmetic somewhat more carefully and to find out, if possible, something about a lost Philonic book which its author perhaps called Περὶ ἀριθμῶν.

Philo was no mathematician, in the professional sense, and made no contributions, important or trivial, to the mathematical sciences. He displays, however, extensive knowledge of the current arithmetic and a great fondness for using arithmetical motifs in his allegorical exegesis of the Old Testament. It is also evident that his arithmetic was of the Pythagorean variety, very similar to that of Nicomachus. In examining his arithmetical statements it will therefore be convenient to compare him with Nicomachus, whose *Introduction to Arithmetic* is the best systematic treatise on the subject, in spite of its faults, that has come down from antiquity. There will be found to be a residue of material, however, that is more properly classed with what is now called "arithmology," which is wholly Pythagorean and hardly scientific in any sense. In the analysis that follows, accordingly, a distinction will be made between Philo's arithmetic and his arithmology.

A. PHILO'S ARITHMETIC

1. *Basic philosophical conceptions* (Nicomachus *Introduction to Arithmetic* i. 1–6).[2]—The philosophical conceptions basic to mathematics appear, of course, in Philo, but in non-mathematical contexts for the most part. The mathematical sciences are mentioned together in *De vita Mosis* i. 5,[3] but nothing is said of the Platonic doctrine, quoted by Nicomachus, that they are all ultimately one.[4] To Philo, however, as to Nicomachus, number is a part of the framework of the world, associated by him with the divine logos as Nicomachus asso-

[1] *Ibid.*, p. 31.

[2] Hereafter cited simply as Nicomachus. Hoche's edition (Leipzig, 1866) is used in all page references to Nicomachus.

[3] The Cohn-Wendland text (Berlin: Reimer, 1896–1915) is cited, and page references are to that edition, except that for Philonic material derived from the Armenian version the text of Aucher as reprinted in the stereotyped Tauchnitz edition (Leipzig, 1898) is used.

[4] Nicomachus i. 3.

ciates it with God's mind and plan. The most striking passage is as follows: "Gomor vero mensura . . . . uti est verbum divinum, quo omnia mensurata sunt ac mensurantur, quae in terra sunt, ratione numero et collatione in harmoniam et consonantiam comprehensa, ex quibus species et mensurae entium cernuntur."[1] In another place he remarks τάξει δὲ ἀριθμὸς οἰκεῖον.[2]

Under this head, also, may be briefly mentioned two Platonic doctrines shared by Philo. The first is that man's recognition of number comes from observing days, months, years, etc.; *De opif. mundi* 19 is certainly based on *Timaeus* 47A ff. The second is found in *De congr. erud. grat.* 26: ἰσόπλευρα γὰρ καὶ σκαληνὰ κύκλους τε καὶ πολυγώνια καὶ τὰ ἄλλα σχήματα γεωμετρία προσεξεῦρε, σημείου δὲ καὶ γραμμῆς καὶ ἐπιφανείας καὶ στερεοῦ φύσιν, ἃ δὴ ῥίζαι καὶ θεμέλιοι τῶν λεχθέντων εἰσίν, οὐκέτι γεωμετρία . . . . ταῦτα γὰρ ἀνακεῖται φιλοσοφίᾳ καὶ ἡ περὶ ὅρων πραγματεία πᾶσα τῷ φιλοσόφῳ. This is strongly reminiscent of *Republic* 510B ff. and 533BC, and of the point there insisted upon by Plato, that mathematics does not prove its hypotheses.

2. *Definitions of number, and of odd and even* (Nicomachus i. 7).—Philo's writings contain no definitions of these things. I would, however, call attention to one sentence: κατὰ δὲ τὰ δύο γραμμή, διότι ῥύσει μὲν ἑνὸς δυάς, ῥύσει δὲ σημείου συνίσταται γραμμή.[3] One of the definitions of number given by Nicomachus is "flow of quantity made up of units," ποσότητος χύμα ἐκ μονάδων συγκείμενον, undoubtedly a Pythagorean definition, and the same idea seems to be present in the Philonic passage, which is one of several in which 1, 2, 3, and 4 are equated with point, line, surface, and solid in geometry.[4] The notion that the line, surface, and solid are derived from the "flow" of the simpler geometrical elements is clearly expressed in *De decalogo* 7.

Though Philo only hints at a definition of number, he has several interesting things to say about numbers and the "elements" of number. For example, *Quaest. et sol. in Gen.* iv. 110:

Porro distincta est unitas ab uno eo modo quo excellit distinguiturque originale exemplar a forma; indicium est enim unitas et similitudo unitatis

[1] *Quaest. et sol. in Gen.* iv. 23; cf. with Nicomachus i. 6.

[2] *De opif. mundi* 3.

[3] *Ibid.* 16.

[4] Cf. *ibid.* 32; *De vita Mosis* iii. 11; *De decalogo* 7; *Quaest. et sol. in Exod.* ii. 93, 94. But he calls 3 the "image of solid body," as in *Leg. all.* i. 2, from the analogy of the three dimensions of solids.

unum. quare? quoniam unum potest et multorum perfectionem recipere, ut armentum, chorus, tribus, gens, exercitus, civitas, quorum unumquodque unum est; unitas autem non est ex multitudine, quia immunis est et expers participationis, et inconiuncta est propter solitudinem sicut declarat vel nomen ipsum.

That is, unity is a Platonic idea which earthly unities may imitate; and if Philo said this of unity he probably said the same thing of other numbers. So, too, Nicomachus distinguishes between the eternal numbers, which serve as patterns, and the numbers used in science; Theon of Smyrna discusses at length the difference between "unity" and "one," terms which he says Archytus and Philolaus used indifferently.[1] The post-Platonic Pythagoreans evidently found this a fertile topic.

Of unity Philo says that it is the beginning, element, and measure[2] of number, in which he agrees with Nicomachus. Nicomachus says further that the dyad is one of the elements of number, the embodiment and cause of otherness, while unity causes sameness;[3] on the other hand, he allows that the dyad arises out of the monad.[4] Of this, in Philo, there are suggestions rather than explicit statements. *De opif. mundi* 16, already quoted, states that the flow of, or from, the monad generates the dyad, and *De praem. et poen.* 7 also is to the effect that the monad is elementary and the dyad derived. In *Quaest. et sol. in Gen.* ii. 12, of the dyad we read: "Necnon inaequalitate laborat ob ceteros longos [numeros]. Nam qui a duobus in duplicem augentur omnes alii longi sunt."[5] There is certainly the implication here that inequality, or "otherness," inheres in the heteromecic numbers because of their derivation from the evens, which start with 2; and this is precisely the doctrine of Nicomachus.

In another closely allied topic there is agreement between Philo and Nicomachus, the latter being represented in this instance by the *Theologumena arithmeticae*. The verbal agreement indeed is so close

[1] Nicomachus i. 6; Theon, p. 18, 5 ff. (Hiller) (and Moderatus of Gades cited by Hiller in his critical notes); Lydus *De mens.* ii. 5: διαφέρει δὲ μονὰς ἑνὸς ᾗ διαφέρει ἀρχέτυπον εἰκόνος· παράδειγμα μὲν γὰρ ἡ μονάς, μίμημα δὲ μονάδος ἕν.

[2] *Quaest. et sol. in Gen.* i. 77.

[3] *Nicomachus of Gerasa*, pp. 99 ff.

[4] *Ibid.*, pp. 116 f.

[5] Aucher's version, and mathematically incorrect. The heteromecic series was derived from the even numbers by addition of the successive terms, and this is probably what Philo said.

that the two must have built on the same sources. Briefly, Nicomachus says that the monad and the dyad are elementary and the beginnings of number, but not yet actual numbers, just as a point, the beginning of a line, is not a line; the triad is the first actual number because it has form, or, as it is frequently put, beginning, middle, and end. The passages to be compared are the following:

PHILO *Quaest. et sol in Gen.* iv. 8

Porro nimis naturale est illud de tribus mensuris dictum verbum; vere enim reapse tribus mensurantur omnia, principio ducto et medio et fine, quorum utrumque inane comperitur absque tertia parte, carens exsistentia. quare Homerus non frustra dixit, omnia tripliciter divisa esse; et Pythagorici triadem in numeris et in figuris rectangulum et triangulum[1] supponunt pro elemento scientiae universorum.

Theol. arith. 15 [AST]

. . . . *σύστημα δὲ μονάδος καὶ δυάδος ἡ τριὰς πρώτη. ἀλλὰ καὶ τέλους καὶ μέσου καὶ ἀρχῆς πρωτίστη ἐπιδεκτική, δι' ὧν τελειότης περαίνεται πᾶσα . . . . καὶ τὸ παρ' Ὁμήρῳ δὲ ἁρμόσοι τις ἂν τούτοις, τρίχθα καὶ πάντα δέδασται.* P. 8 [Ast]: *διόπερ ἡ πρώτη σύνοδος αὐτῶν πρῶτον ὡρισμένον πλῆθος ἀπετέλεσε, στοιχεῖον τῶν ὄντων, ὅ ἂν εἴη τρίγωνον μεγεθῶν τε καὶ ἀριθμῶν, σωματικῶν τε καὶ ἀσωμάτων. . . .*

In the light of the foregoing it is easy to understand how Philo can call 2 "empty" and 3 "full," as he does in other passages.[2] There is enough evidence to show, I think, that Philo and Nicomachus agreed very well in their ideas of these "elementary numbers," 1, 2, and 3. As a matter of fact, they both reflect the current Pythagorean theories, and many other parallels to these statements could be given.

Philo is also at one with the Pythagorizing arithmeticians in certain things which he says about the decimal system. *De plant. Noe* 18 is typical.[3] The number 10,000, he says, "is the most important and perfect bound of the numbers increasing from unity. Thus unity is the beginning of numbers, and 10,000 the end for those in the first

[1] Aucher's version. The original Greek was probably *ὀρθογώνιον τρίγωνον*.

[2] *Quaest. et sol. in Gen.* ii. 12, "numerus enim binus non mundus; primum quia vacuus est, non densus; quod autem non est plenum neque mundum est"; *Quaest. et sol. in Exod.* ii. 100, "ternio est condensus plenusque numerus, nullam habens vacuitatem, sed quicquid in dualitate discerptum erat adimplens."

[3] *ἐστὶ δὲ ὅρος οὗτος* (sc. *μυριάς*) *τῶν ἀπὸ μονάδος παραυξηθέντων ὁ μέγιστος καὶ τελειότατος, ὥστε ἀρχὴν μὲν ἀριθμῶν εἶναι μονάδα, τέλος δὲ ἐν τοῖς κατὰ τὴν πρώτην σύνθεσιν μυριάδα. παρ' ὃ καί τινες οὐκ ἀπὸ σκοποῦ βαλβῖδι μὲν μονάδα, καμπτῆρι δὲ εἴκασαν μυριάδα, τοὺς δὲ μεθορίους πάντας ἀριθμοὺς τοῖς δρόμον ἀγωνιζομένοις· ἀρχόμενοι γὰρ ὥσπερ ἀπὸ βαλβῖδος φέρεσθαι μονάδος παρὰ μυριάδα τὸ τέλος ἵστανται.*

grouping. So some have not unreasonably likened unity to the starting post and 10,000 to the turning point, and all the intervening numbers to the runners in the race; for they begin to move from unity, as it were, as from a starting post, and halt finally at 10,000." In other passages he denominates 10, 100, and 1,000 also as "turning posts" (καμπτῆρες) in this race course of the numbers,[1] with the corollary that, as each leg of the course is a repetition of the first, so too the tens, hundreds, and thousands are repetitions of the units, or new orders of units. Thus 30 is in the series of tens what 3 is among the units; 60, among the tens, and 600, among the hundreds, correspond to 6.[2] The decad is a "boundary of the infinity of the numbers, which they round, like a turning-post, and turn back."[3] The decad may be said to set its own form upon the numbers, making the tens, hundreds, thousands, etc., images of the first decade.

This conception of the decimal system is to be found even in the *Introduction to Arithmetic* of Nicomachus, where (i. 17) 10 is called the "unit of the second course" and 100 the "unit of the third course"; it also appears in the *Theologumena arithmeticae*, and Iamblichus explains it in his commentary on the Nicomachean passage.[4] The decimal system is, of course, not the exclusive property of the Pythagoreans, but the imagery and personification of the race-track simile is entirely typical of them.

3. *Classification of absolute number.*—(*a*) Even and odd; (*b*) prime and composite; (*c*) perfect, superabundant, and deficient (Nicomachus i. 7–16).

a) Even and odd: Philo of course uses these terms constantly. He does not, however, have occasion to define them, though in *De opif. mundi* 3 he says that 3 is the first odd number, 2 the first even (περιττῶν . . . . ἀρχὴ τριάς, δυὰς δὲ ἀρτίων). This, of course, is based on the theory that 1 is the beginning of number, not an actual number at

[1] *De plant. Noe* 29; *De opif. mundi* 15; *Quaest. et sol. in Gen.* ii. 32; cf. also *Quaest. et sol. in Gen.* iii. 56.

[2] *Quaest. et sol. in Gen.* ii. 5; ii. 17; iv. 164.

[3] *De opif. mundi* 15: ὅρος τῆς ἀπειρίας τῶν ἀριθμῶν . . . . περὶ ὃν ὡς καμπτῆρα εἰλοῦνται καὶ ἀνακάμπτουσι. Cf. Anatolius *Ap. Theol. arith.* 63 (Ast): ὅτι ὅρος ἐστὶ τῆς ἀπειρίας τῶν ἀριθμῶν (*sc.* δεκάς); Lydus *op. cit.* iii. 2.

[4] Nicomachus i. 19. 17; see the citations in *Nicomachus of Gerasa*, p. 219, n. 1, especially *Theol. arith.*, p. 59 (Ast).

all, but only potentially such, which was, as we have just seen, Nicomachean and Pythagorean. Nevertheless both Philo and Nicomachus, when they write out the series of odds and evens, begin the former with 1 and the latter with 2.[1]

In typical Pythagorean fashion Philo calls the odd male and the even female, an identification common enough in the *Theologumena arithmeticae* but avoided in the *Introduction to Arithmetic*.[2]

Nicomachus divided the even into even-times even, odd-times even, and even-times odd. Of these, Philo has specifically only the *ἀρτιοπέριττος*, and gives no definite evidence as to whether or not he recognized the other Nicomachean classes. It is hard to think that he did not, since his mentor Plato has something at least very similar; but, on the other hand, in *De decalogo* 6, when he is demonstrating the perfection of 10, he points out that it contains *ἄρτιοι, περιττοί,* and *ἀρτιοπέριττοι* as though this were a full classification. It is in fact the classification attributed to Philolaus, and possibly Philo was conservative in this particular.[3] He often speaks of 6 as the first *ἀρτιοπέριττος*.[4]

b) Prime and composite numbers: Philo happens to mention by name but one of this group of varieties, the primes, and gives the orthodox definition of them in the Euclidean form.[5] He does not say definitely that the prime is a classification of the odd, as Nicomachus does, but the examples which he gives are all odd numbers. He does not mention "Eratosthenes' sieve," the method of discovering primes reported by Nicomachus.

Composite numbers (*σύνθετοι*) are not specifically mentioned. The distinction between primes and composites, however, is clearly made in the following passage (*De opif. mundi* 33): *ἐκείνων* [the numbers

[1] E.g., in Philo *Quaest. et sol. in Gen.* ii. 5; in Nicomachus ii. 9. 3.

[2] *De opif. mundi* 3; *Quaest. et sol. in Gen.* i. 83; *Theol. arith.* 24 (Ast), 31, 33 (Anatolius) 35, 37 (Anatolius); Alexander *In Met.* 985*b* 26; Lydus *op. cit.* ii. 10; Iamblichus *In Nicom.* 34, 15 ff. (Pistelli).

[3] Heath (*op. cit.*, pp. 39 f.) summarizes the classifications of odd and even made by the ancients.

[4] *De opif. mundi* 3; *De spec. leg.* ii. 6; *Quaest. et sol. in Gen.* iii. 38, 49.

[5] *De decalogo* 7: *τόν τε πρῶτον κόσμον ὅς μονάδι μόνῃ μετρεῖται, οὗ παράδειγμα ὁ τρεῖς, ὁ πέντε, ὁ ἑπτά.* Cf. Euclid vii, def. 12: *πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.* The definition of Nicomachus is similar but states that the primes are always odd: *ὅταν ἀριθμὸς περισσὸς μόριον μηδὲν ἕτερον ἐπιδέχηται εἰ μὴ τὸ παρώνυμον ἑαυτῷ, ὃ καὶ ἐξ ἀνάγκης μονὰς ἔσται* (i. 11. 2).

of the first decade] γὰρ οἱ μὲν γεννῶσιν οὐ γεννώμενοι, οἱ δὲ γεννῶνται μέν, οὐ γεννῶσι δέ, οἱ δὲ ἀμφότερα καὶ γεννῶσι καὶ γεννῶνται· μόνη δ' ἑβδομὰς ἐν οὐδενὶ μέρει θεωρεῖται.[1]

c) Perfect, superabundant, and deficient numbers: Again, Philo mentions only one of these classes—the perfect number—but defines it precisely as the mathematicians do.[2] He mentions but two of them, 6 and 28,[3] whereas Nicomachus lists four; and says nothing about superabundant and deficient numbers.[4]

4. *Relative number.*—(*a*) Equality and inequality; (*b*) the ratios.

a) Equality and inequality: To this subject Nicomachus devotes the chapter which introduces his treatment of relative number (i. 17). Equality and inequality are the highest generic divisions of relative number, he states; things are equal when in comparison one neither exceeds nor falls short of the other, and the relation equality admits of no difference or degree. Such doctrines, I think, were familiar to Philo, although we can judge only by a few incidental phrases, such as ἀνισότης, ἐν ᾧ τό τε ὑπερέχον καὶ τὸ ὑπερεχόμενον (*De iustitia* 14), and the reference to τὸ ἴσον in *Quis rer. div. heres* 28. Again, in *De opif. mundi* 32, he says that all right angles are equally "right," which is curiously like Nicomachus' statement about the absolute nature of equality.

b) Ratios: We have no definition of ratio (λόγος) or of the various specific ratios in Philo, but ample evidence that he knew and used all the terms found in Nicomachus. For instance, in *De decalogo* 6 he says that the "perfect" number 10 contains all the διαφορὰς λόγων τῶν ἐν ἀριθμοῖς, πολυπλασίων καὶ ἐπιμερῶν καὶ ὑποεπιμερῶν[5]—the multiples, superpartients, and subsuperpartients—which leaves unaccounted for only the superparticulars, multiple superpartients, multiple superparticulars, and the reciprocal ratios out of the classes

[1] Cf. *Leg. all.* i. 5.

[2] *De opif. mundi* 34: The number 28 is τέλειον καὶ τοῖς αὑτοῦ μέρεσιν ἰσούμενον, which is almost word for word Euclid's definition (*op. cit.* 23), ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν, as well as that of Nicomachus (i. 16. 1) and Theon of Smyrna (p. 45, 10 [Hiller]).

[3] **6,** *De opif. mundi* 3; *Leg. all.* i. 2; *De decalogo* 7, *Quaest. et sol. in Gen.* iii. 38; *In Exod.* ii. 87; **28,** *De opif. mundi* 34; *De vita Mosis* iii. 5; *Quaest. et sol. in Exod.* ii. 87. He calls other numbers—e.g., 10—perfect, but in a different sense.

[4] Nevertheless in *Quaest. et sol. in Gen.* iii. 49 it is pointed out that the sum of the factors of 8 is 7, and (*ibid.* i. 91) that the factors of 120 add up to 240.

[5] ἐπιμορίων is read by most MSS for ὑποεπιμερῶν in *De decalogo* 6 (C.-W., IV, 273, 4).

enumerated by Nicomachus. He constantly mentions specific ratios[1] and, like both Nicomachus and Theon, especially the ones which represent numerically the fundamental musical concords.[2]

5. *Figurate numbers.*—This subject, together with proportions, makes up the second book of Nicomachus' *Introduction.* The doctrine of figurate numbers, based on the Pythagorean conception of number as capable of assuming spatial form in one, two, or three dimensions, in so far as it relates to squares and cubes is something with which we are still familiar, though triangular, hexagonal, or pyramidal numbers, and the like are strange to us. It will be seen, however, that all this was perfectly natural to Philo and that he employed most of the expressions found in this part of Nicomachus' treatise.

a) Dimensions: Nicomachus (ii. 6) begins his discussion of figurate numbers with definitions of and statements concerning the dimensions, intervals, point, line, surface, and solid—matters which are fundamental to both geometry and this highly geometrical division of ancient arithmetic. The extent to which Philo cites the same material will perhaps best be illustrated by comparing portions of *De decalogo* 7 with excerpts from Nicomachus.

PHILO, *De decalogo* 7

τὴν μέντοι δεκάδα . . . . καὶ διὰ ταῦτα ἄν τις θαυμάσαι περιέχουσαν τήν τε ἀδιάστατον φύσιν καὶ τὴν διαστηματικήν· ἡ μὲν οὖν ἀδιάστατος τάττεται κατὰ σημεῖον μόνον, ἡ δὲ διαστηματικὴ κατὰ τρεῖς ἰδέας γραμμῆς καὶ ἐπιφανείας καὶ στερεοῦ· τὸ μὲν γὰρ δυσὶ σημείοις περατούμενόν ἐστι γραμμή, τὸ δ' ἐπὶ δυὸ διαστατὸν ἐπιφάνεια, ῥυείσης ἐπὶ πλάτος γραμμῆς, τὸ δ' ἐπὶ τρία στερεόν, μήκους καὶ πλάτους βάθος προσλαβόντων, ἐφ' ὧν ἵσταται ἡ φύσις· πλείους γὰρ τριῶν διαστήσεις οὐκ ἐγέννησεν. ἀρχέ-

NICOMACHUS [ed. HOCHE]

P. 94, 8: *ἔσται οὖν ἡ μονὰς σημείου τόπον ἐπέχουσα καὶ τρόπον ἀρχὴ μὲν διαστημάτων καὶ ἀριθμῶν.* . . . . 24: *ἀδιάστατος ἄρα ἡ μονάς.*

P. 85, 4: . . . . *γραμμὴ γάρ ἐστι τὸ ἐφ' ἓν διαστατόν* . . . . *ἐπιφάνεια γάρ ἐστι τὸ διχῇ διαστατόν· τρία δὲ διαστήματα στερεόν, στερεὸν γάρ ἐστι τὸ τριχῇ διαστατὸν καὶ οὐκ ἔστιν οὐδαμῶς ἐπινοεῖν στερεόν, ὃ πλεόνων τέτευχε διαστημάτων ἢ τριῶν, βάθους, πλάτους, μήκους.*

P. 86, 15: *οὕτως δὴ* . . . . *ἡ μὲν μονὰς ἀρχὴ παντὸς ἀριθμοῦ ἐφ' ἓν*

[1] E.g., doubles, *διπλάσιοι, De opif. mundi* 15, 30, 37; *Quaest. et sol. in Gen.* iii. 49 and iv. 71; triples, *τριπλάσιοι, De opif. mundi* 15, 30; quadruples, *ibid.* 15; sextuples and decuples, *Quaest. et sol. in Gen.* ii. 5; *ἐπίτριτος* and *ἡμιόλιος, De opif. mundi* 15, 37; *superbipartiens tertias,* 50:30, *Quaest. et sol. in Gen.* ii. 5; *ἥμισυς, Quod det. pot. insid.* 19; *διπλασίων λόγος, τριπλασίων λόγος, De opif. mundi* 30; *ἑξαπλάσιος λόγος, ibid.* 31.

[2] *De opif. mundi* 15, 31; *De vita Mosis* iii. 11; *Quaest. et sol. in Gen.* iv. 27.

τυποι δὲ τούτων ἀριθμοὶ τοῦ μὲν ἀδιαστάτου σημείου τὸ ἕν, τῆς δὲ γραμμῆς τὰ δύο, καὶ ἐπιφανείας μὲν τρία, στερεοῦ δὲ τέσσαρα, ὧν ἡ σύνθεσις . . . . ἀποτελεῖ δεκάδα, κ.τ.λ.

διάστημα κατὰ μονάδα προβιβαζομένου . . . . 22: *γραμμικοὶ μὲν εἰσιν ἀριθμοὶ ἁπλῶς ἅπαντες οἱ ἀπὸ δυάδος ἀρχόμενοι . . . . ἐπίπεδοι δὲ οἱ ἀπὸ τριάδος ἀρχόμενοι.*

This is sufficient to show that Philo knew the sort of arithmetic that Nicomachus taught, in practically the same terms. He has also another definition of the point, "that which has no parts,"[1] and of the line, "length without breadth,"[2] and just as Nicomachus does, and as Plato did before either of them, in connection with the three dimensions he speaks of the six categories of motion—up, down, forward, back, right, and left.[3]

b) Plane numbers: The ancients sometimes defined and regarded these as numbers which are the product of two factors,[4] but also as those which, analyzed into their component units, could be arranged in geometrical forms in a plane.[5] Philo is certainly familiar with the second conception, which appears in several enumerations of triangular, square, pentagonal, hexagonal, and heptagonal numbers[6] in the course of his allegorical interpretations. Even a Pythagorean arithmetician like Nicomachus, however, was likely to think of the "rectangular" numbers, the squares, heteromecics and promecics, as the product of two factors,[7] and Philo probably did so as well. At least he uses the common expression *ἰσάκις ἴσοι* to designate squares, im-

[1] *De congr. erud. grat.* 26, *οὗ μέρος οὐδέν.*

[2] *μῆκος ἀπλατές, De opif. mundi* 16; *De congr. erud. grat.* 26. On the dimensions cf. also *Quaest. et sol. in Exod.* ii. 61.

[3] *De somn.* i. 5; *De opif. mundi* 41; *Leg. all.* i. 2, 4; *De decalogo* 7. Cf. Nicomachus, p. 85, 9 ff. (Hoche); Plato *Tim.* 43B, 34A and *Laws* 894C; *Theol. arith.* 36 (Ast); (Plut.) *Epit.* iii. 15. 10; Martianus Capella vii. 736; Anatolius *Ap. Theol. arith.* 42 (Ast); Lydus *op. cit.* ii. 11; Macrobius *In somn. Scip.* i. 6. 81. In some of these passages (following Plato) circular motion is added as a seventh; in the *Laws* Plato speaks of ten varieties, not all spatial.

[4] Theon of Smyrna, p. 31, 9 (Hiller). This was the Euclidean definition (*Elem.* vii, def. 16).

[5] See, e.g., Nicomachus' definition of triangular numbers, ii. 8. 1.

[6] Especially *Quaest. et sol. in Gen.* i, 83; cf. *ibid.* 91 (the fifteenth triangular number is 120, etc.); *ibid.* ii. 5 and iii. 56. In several cases he points out that certain numbers are summations of the natural series up to a certain point; these are of course triangular, by definition; thus, **10,** *De decalogo* 7 and various other passages; **28,** *Quaest. et sol. in Exod.* ii. 87, etc.; **36,** *Quaest. et sol. in Gen.* iii. 49; **55,** *De vita Mosis* iii. 4; **300,** *Quaest. et sol. in Gen.* ii. 5.

[7] E.g., ii. 18. 2.

plying multiplication, and says that 6 is *ἀπὸ ἑτερομήκους . . . . τοῦ δὶς τρία.*[1]

Philo makes several references to three theorems concerning squares, cubes, and heteromecic numbers. The first two are that the addition of the successive odd numbers produces the squares, of the even numbers the heteromecics.[2] These theorems are also to be found in Nicomachus and Theon, and were known to the early Pythagoreans.[3] In modern notation they are

$$1+3+5+\ldots\ldots+(2n-1)=n^2$$

and

$$2+4+6+\ldots\ldots+2n \quad =n(n+1)\,.$$

The third theorem is stated thus by Philo: That in any analogous series, beginning with unity—for example, with the ratio 2 or 3—every other term will be a square, every third term a cube, and every sixth term both a cube and a square.[4] As such a series would be expressed thus,

$$1,\ a,\ a^2,\ a^3,\ a^4,\ a^5,\ a^6,\ a^7,\ a^8\ \ldots\ldots\ a^n,$$

the truth of the observation is evident. Philo's statement of the theorem is fuller than that of Nicomachus, who confines himself to saying that every other term is a square,[5] and the same as that of Theon;[6] the latter, however, adds at this point that every square is divisible by 3 or becomes so when diminished by unity, and is similarly divisible by 4.

c) Solid numbers: Philo mentions, among these, only pyramids with a triangular base and cubes. In *De opif. mundi* 16 he picturesque-

[1] *De opif. mundi* 16; *De decalogo* 7; *Leg. all.* i. 2; cf. Nicomachus i. 19. 19, Theon p. 26, 14 (Hiller), etc.

[2] See especially *Quaest. et sol. in Gen.* ii. 14; also *ibid.* 5 and 12, and iii. 56. In the latter passage the statement that $100=1+3+5+\ldots\ldots+19$ is based on this principle. In *ibid.* i. 91 we have $1+3+5+7+\ldots\ldots+15=64$ (a square) and $2+4+6+8+\ldots\ldots+14=56$ (heteromecic).

[3] Nicomachus ii. 9. 3 (squares), ii. 17. 2, 18.2, 20.3 (heteromecics); Theon, p. 26, 14; p. 28, 3; p. 34, 1; p. 39, 10 (squares); p. 27, 8; p. 31, 14 (heteromecics); Heath, *op. cit.*, pp. 44, 48.

[4] *De opif. mundi* 36: *τὸν μὲν τρίτον ἀπὸ μονάδος, εἰ διπλασιάζοι τις, εὑρήσει τετράγωνον, τὸν δὲ τέταρτον κύβον, τὸν δ' ἐξ ἀμφοῖν ἕβδομον κύβον ὁμοῦ καὶ τετράγωνον, κτλ.* Cf. *ibid.* 30. Philo counts the terms in Greek fashion.

[5] ii. 20. 5.

[6] P. 34, 16 ff. (Hiller); see *Nicomachus of Gerasa*, p. 58.

ly illustrates the formation of the pyramid by referring to the children's game, like jackstones, played with nuts or acorns, in which three of the objects are laid in the form of a plane triangle, with a fourth above to complete the pyramid; thus 4 is the first pyramidal number.[1]

Cubes he calls, like Nicomachus, ἰσάκις ἴσοι ἰσάκις, the product of a number taken three times as a factor.[2]

Before leaving the subject of figurate numbers it should be observed that Philo shares with Nicomachus a peculiarly Pythagorean view of the square and heteromecic numbers. Unity and the dyad embody, respectively, sameness and otherness, as has been stated above; sameness and otherness therefore dwell in the odd and even series, based on 1 and 2, and, further, in the squares and heteromecics derived from the addition of these series; and, still further, as sameness and otherness are the one good and the other bad, so there may be virtue or vice in these numbers. So in *Quaest. et sol. in Gen.* ii. 5 we read: "Plena enim et perfecta natura paritatis est factrix iuxta quadranguli[3] naturam, par autem et infinitum inaequalitatis iuxta alterius longi compositionem"; and again, of the number 2 (*ibid.* 12): "Necnon inaequalitate laborat ob ceteros longos; nam qui a duobus in duplicem augentur omnes alii longi sunt.[4] atqui inaequale non est mundum, sicut neque materiale, sed quod ab illo est fallibile est et incomptum," etc. Similarly (*ibid.* iv. 110), Philo speaks of the *prava natura dualitatis* and the *probitas unitatis*. This, of course, is not mathematics, but it is additional evidence that Philo had absorbed thoroughly one of the types of arithmetic current in his time.

6. *Proportions.*—Nicomachus knew of ten kinds of proportions,[5] but only the three original varieties, the arithmetic, geometric, and harmonic, appear in Philo's extant writings. He uses the term ἀνα-

[1] Anatolius, p. 32, 5 (Heiberg) uses this same illustration of the game καρυατίζειν, perhaps deriving it from Philo. Nicomachus ii. 13. 2 ff. discusses pyramidal numbers and their formation.

[2] *De decalogo* 7. He frequently cites 8 as the first cube; cf. *Quaest. et sol. in Gen.* i. 91; ii. 5; iii. 49. In the latter passage he points out that 64, factored as 8×8, is a square, and as $4\times4\times4$ a cube (cf. *ibid.* i. 91). In *op. cit.* iii. 56 he states that $100=1^3+2^3+3^3+4^3$.

[3] Aucher's version has *trianguli*, which is evidently wrong.

[4] Again Aucher seems to be wrong, for what Philo actually said was undoubtedly that the *sum* of the "doubles" (i.e., even numbers) gives heteromecics.

[5] Heath, *op. cit.*, pp. 51–53, describes them and their discovery.

λογία indiscriminately of them all.[1] We can even quote his definitions of these three. The arithmetic progression, or proportion, is most simply defined in *De decalogo* 6, *ᾗ τῷ ἰσαρίθμῳ ὑπερέχει καὶ ὑπερέχεται*, which is very like a Nicomachean expression (ii. 27. 1), *ἴσῳ ὑπερέχουσαν καὶ ὑπερεχομένην*.[2] In *De decalogo* 6, further, there is this description of the geometric proportion, *καθ' ἣν οἷος ὁ λόγος πρὸς τὸν πρῶτον τοῦ δευτέρου, τοιοῦτος καὶ πρὸς τὸν δεύτερον τοῦ τρίτου*, and of the harmonic, *καθ' ἣν ὁ μέσος τῶν ἄκρων τῷ ἴσῳ μορίῳ ὑπερέχει τε καὶ ὑπερέχεται*. Both of these are closely in agreement with Nicomachus.[3] A longer and more complete definition of the harmonic proportion occurs in *De opif. mundi* 37: *ἁρμονικῆς δὲ ἀναλογίας διττὴ κρίσις· μία μὲν ὅταν ὃν λόγον ἔχει ὁ ἔσχατος πρὸς τὸν πρῶτον, τοῦτον ἔχῃ ἡ ὑπεροχή, ᾗ ὑπερέχει ὁ ἔσχατος τοῦ μέσου, πρὸς τὴν ὑπεροχὴν ᾗ ὑπερέχεται ὑπὸ τοῦ μέσου ὁ πρῶτος . . . . ἑτέρα δὲ βάσανος τῆς ἁρμονικῆς ἀναλογίας, ὅταν ὁ μέσος τῶν ἄκρων ἴσῳ μορίω καὶ ὑπερέχῃ καὶ ὑπερέχηται.* This is paralleled in Nicomachus, page 131, 19–21, and page 132, 22–133, 2. The first definition, in modern terminology, is, if $a > b > c$,

$$a:c = a-b:b-c\,,$$

and the second, if b is the harmonic mean between a and c, then

$$\text{if } a = b + \frac{a}{n}\,,$$

$$b = c + \frac{c}{n}\,.$$ [4]

Nicomachus climaxes his discussion of proportions with a description of the one which he calls "most perfect," to which also Iamblichus refers as the "musical" proportion, a discovery of the Babylonians introduced to Greece by Pythagoras.[5] It appears in Plato's *Timaeus* 36A, and, by Iamblichus' account, was used by such Pythagoreans

[1] So does Nicomachus, but says (ii. 24. 1) that in the strict sense of the word only the geometric proportion is properly so called. He also uses *μεσότης* to apply to all of them.

[2] Cf. also *De opif. mundi* 37.

[3] ii. 24 (geometric) and 25 (harmonic, esp. sec. 3). Euclid's definition of the geometric proportion is in a different form (*Elem.* vii, def. 21). Philo defines only the continuous geometric proportion, but he cites examples of the disjunctive as well.

[4] Heath, *op. cit.*, p. 51. This definition was given by Archytas: Heath, *loc. cit.; Nicomachus of Gerasa*, p. 21.

[5] *Nicomachus of Gerasa*, pp. 25, 64, 284–86; Heath, *loc. cit.*

as Aristaeus of Croton, Timaeus Locrius, Philolaus, and Archytas. In modern notation it is a series of the form

$$a, \frac{2ab}{a+b}, \frac{a+b}{2}, b,$$

all four of which are in geometrical progression, while the second term is the harmonic mean, and the third the arithmetic mean, between the two extremes. This form of proportion is not only known to Philo and mentioned at least four times,[1] but two examples of such a series are cited by him in contrast to the one (6, 8, 9, 12) which appears in Nicomachus' *Introduction to Arithmetic* (ii. 29). Philo calls this proportion the πλινθίον (*laterculus*), or sometimes simply διάγραμμα; the first one, that used by Nicomachus, is "that in double ratio," and the second, 6, 9, 12, 18, "that of the triples," with reference, of course, to the ratios of the respective extreme terms, 6:12 and 6:18. Although the *laterculus* of the triples does not appear in the *Introduction* of Nicomachus or in Iamblichus' commentary upon it, it nevertheless occurs in the *Theologumena arithmeticae*,[2] which is sufficient evidence that Nicomachus, Iamblichus, and the neo-Pythagoreans generally knew both sets of numbers. They appear to have used them in three ways, in commenting upon the Platonic psychogony,[3] in discussing the musical ratios,[4] and in demonstrating the prevalence of number in nature with particular reference to the length of the period of gestation.[5]

B. PHILO'S ARITHMOLOGY

To give a complete account of Philo's arithmology would unduly prolong this paper without adding anything of significance from the mathematical point of view. From Philo alone a very complete arithmological treatise could be compiled. Practically every number that is mentioned in the Scriptures which he interprets is commented upon,

[1] *De opif. mundi* 37; *Quaest. et sol. in Gen.* i. 91; iii. 38; iv. 27.

[2] *Ibid.* 35–36, 39, 47 (Ast). In the first passage the proportion is used in connection with the enumeration of the ratios of musical intervals; in the other two, with reference to the length of the period of gestation.

[3] This is suggested by Iamblichus' citation of the *Timaeus*.

[4] Nicomachus ii. 29; Philo *De opif. mundi* 37; *Theol. arith.* 35–36 (Ast); Iamblichus *Comm. in Nicom.* 118, 19 ff. (Pistelli); Martianus Capella vii. 737.

[5] Philo *Quaest. et sol. in Gen.*, in the various places cited; *Theol. arith.* 39, 47 (Ast). The use of the series is not the same in all details.

and obscure significances are brought to light. The study of these Philonic passages, too, in their relation to other expressions of the same topics, of which there are legion, is a fascinating one. To the writer there seems to be evidence that there was in ancient times a compilation of this Pythagorean material which in various forms and at various times was drawn upon by Philo, Theon, Nicomachus, Lydus, and many others.[1]

One respect in which Philo's arithmology differs very decidedly from all the other ancient texts of this character, however, should be pointed out. The others, practically without exception,[2] confine themselves to the first decade of numbers; Philo deals with any number which may present itself in the Old Testament. Thus besides the first decade he remarks in an arithmological way on 13, 14, 15, 20, 24, 25, 28, 30, 35, 36, 40, 45, 49, 50, 55, 60, 70, 75, 80, 90, 99, 100, 120, 127, 165, 175, 200, 280, and 300.[3]

C. PHILO'S BOOK ON NUMBERS

There is a handful of references in Philo's extant works to a book about numbers, presumably his own. These allusions are as follows:

1. *De opif. mundi* 16: πολλαῖς δὲ καὶ ἄλλαις κέχρηται δυνάμεσι τετράς, ἃς ἀκριβέστερον καὶ ἐν τῷ περὶ αὐτῆς ἰδίῳ λόγῳ προσυποδεικτέον.

[1] Cf. "Posidonius and the Sources of Pythagorean Arithmology," *Classical Philology*, XV (1920), 309–22, and "The Tradition of Greek Arithmology, *ibid.*, XVI (1921), 97–123. In the latter paper, p. 102, n. 2, the arithmological passages of Philo are enumerated.

[2] The connection of 30 with "generation" was a topic used by other arithmologists; see *ibid.*, XVI (1921), 110, where also another parallel between Philo and Lydus, involving these larger numbers, is mentioned. Of 36 Philo says "quem homologiam Pythagorici appellarunt" (see next note), which perhaps shows that this number also appeared in the arithmological texts. The fact that $55 = 1 + 2 + 3 + \ldots + 10$ is found in Anatolius, pp. 39–40, as well as in Philo.

[3] **13,** *Quaest. et sol. in Gen.* iii. 61; **14,** *ibid.* and *Quaest. et sol. in Exod.* i. 9; *De sept.* 18; **15,** *Quaest. et sol. in Gen.* i. 91; **20,** *Quaest. et sol. in Gen.* iv. 27; **24,** *ibid.* ii. 5; **25,** *ibid.* i. 91; **28,** *Quaest. et sol. in Exod.* ii. 87; **30,** *Quaest. et sol. in Gen.* ii. 5; iv. 27; **35,** *ibid.* i. 91; **36,** *ibid.* iii. 49; **40,** *ibid.* i. 25; ii. 14; iii. 56; iv. 154; **45,** *ibid.* iv. 27; **49,** *ibid.* iii. 39; **50,** *ibid.* ii. 5; iii. 39; iv. 27; *Quod det. pot. insid.* 19; *De vita Mosis* iii. 4; *De spec. leg.* ii. 21; *De vita contemp.* 8; **55,** *Quaest. et sol. in Gen.* i. 83; *De vita Mosis* iii. 4; **60,** *Quaest. et sol. in Gen.* iv. 164; **70,** *Quaest. et sol. in Gen.* i. 77; **75,** *Quaest. et sol. in Exod.* i. 9; *De migr. Abra.* 36; **80,** *Quaest. et sol. in Gen.* iii. 38; **90,** *ibid.* 36; **99,** *ibid.* 39, 61; *De mut. nom.* 1; **100,** *Quaest. et sol. in Gen.* iii. 39, 56; iv. 151; *De mut. nom* 1; **120,** *Quaest. et sol. in Gen.* i. 91; **127,** *ibid.* iv. 71; **165,** *ibid.* i. 83; **175,** *ibid.* iv. 151; **200,** *ibid.* i. 83; **280,** *Quaest. et sol. in Exod.* ii. 87; **300,** *Quaest. et sol. in Gen.* ii. 5.

2. *Ibid.* 43: ταῦτα καὶ ἔτι πλείω λέγεται καὶ φιλοσοφεῖται περὶ ἑβδομάδος. (This is at the end of his account of the number 7.)

3. *De vita Mosis* ii. 11: ἔχει δὲ καὶ τὰς ἄλλας ἀμυθήτους ἀρετὰς ἡ τετράς, ὧν τὰς πλείστας ἠκριβώσαμεν ἐν τῇ περὶ ἀριθμῶν πραγματείᾳ.

4. *Quaest. et sol. in Gen.* ii. 14: "secundo numerus XL plurimarum productor est virtutum ut alias suggestum est."

5. *Ibid.* iv. 110: "quam vero habeat naturam decas, tam secundum intellegibilem substantiam quam sensibilem, iam dictum est in libro in quo de numeris actum est."

6. *Ibid.* iii. 49: "habet ceteras quoque ampliores virtutes numerus octavus, de quo alias diximus."

7. *Ibid.* iv. 151: "haec quoque indicata sunt quum de numeris actum est." (The discussion has concerned various "perfect" numbers.)

With the possible exception of the second, and perhaps the first, these passages all seem to show that Philo had himself written a book about numbers, probably under the title Περὶ ἀριθμῶν, before he composed the commentaries on the Old Testament which have survived to us. If there are any references to this book outside of the Philonic corpus they have escaped my notice. It may have been a youthful composition which was "lost" not long after the author's own time. It is barely possible, but not, I think, probable, that Lydus had access to it.[1]

What was the character of this lost work? Philo's own allusions to it show, for one thing, that it had to do with the "powers" or "virtues" of specific numbers, among which were certainly some, and therefore probably all, of the first decade, and at least two—40 and 100—larger numbers. Philo's treatment of numbers in his extant works also gives evidence of what this lost work may have been. By all this evidence he is to be classed with the arithmologists, rather than with purely scientific writers like Euclid or even a Pythagorean like Nicomachus who, at least upon occasion, tried to deal with arithmetic systematically if not in a purely scientific spirit. The fact also that in Philo's day the compilation of arithmological treatises was a fairly popular pursuit joins to create probability. To summarize, it seems likely that Philo's Περὶ ἀριθμῶν was not an elements of arithmetic, or an introduc-

[1] *Class. Phil.*, XVI, 106.

tion to arithmetic, but an arithmology, in which were collected the Pythagorizing topics that appear in the extant works, and more. The material was probably nothing new, but taken from older sources, notably that one which seems to underlie much of the arithmology preserved to us in dozens of authors. But since this older treatise, and the arithmologies generally, did not deal with the larger numbers, we may suppose that Philo himself expanded it to include those which he found in the Old Testament and which he desired to expound by means of Pythagorean allegory. This he could easily do by analyzing them into factors, or other components, within the first decade, and deducing thus their inherent virtues; this he actually does in cases that can be cited. Finally, there is so much repetition of arithmological topics in the extant works that we may readily believe these topics also to have been found in much the same form in the Περὶ ἀριθμῶν.

This examination of Philo's mathematics is not intended to show that he was a mathematician or that he made any contribution to the science. Such a reputation was not his in antiquity, and it cannot be bestowed upon him now. His mathematical pronouncements have to be extracted from a mass of arithmological lore which has no scientific value in itself. It has been shown, however, that he was familiar with the elements of an arithmetic similar to that expounded by Nicomachus and Theon less than a century after his lifetime. Indeed, he displays a knowledge of so much that it is fair to assume that he knew much more. His testimony, in brief, is that the Greek *arithmetica*, as Nicomachus knew it and compiled it, existed also in Philo's day, and was accessible to and generally known by the well-educated man, even if he were not a professional mathematician.

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