are 1,2,5,5 or a set (1). In the latter case the remaining coefficients are arbitrary. The only interesting case is, therefore, $f = q + ev^2 + \ldots$ such that no quaternary form, obtained by deleting all but four terms of f, represents all integers. Then $e < 16$, since otherwise $f = 15$ would require $q = 15$, which was seen to be impossible. The values 6,7,8,9,10 of e are excluded, since the abridged form $x^2 + 2y^2 + 5z^2 + ev^2$ was seen to represent all positive integers. Hence $e = 5,11,12,13,14$ or 15. In these respective cases, we see that f represents 15 when $z = u = v = 1$; $x = 2, v = 1$; $x = y = v = 1$; $y = v = 1$; $x = v = 1$; $v = 1$; with all further variables zero. We may, therefore, state the

THEOREM. *If, for $n \geqq 5$, $f = a_1x_1^2 + \ldots + a_nx_n^2$ represents all positive integers, while no sum of fewer than n terms of f represents all positive integers, then $n = 5$ and $f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2$ ($e = 5,11,12,13,14,15$), and these six forms f actually have the property stated.*

[1] Ramanujan, *Proc. Cambridge Phil. Soc.*, **19**, 11 (1916–9).

[2] Dickson, *Bull. Amer. Math. Soc.*, **33** (1927).

CONGRUENCES OF PARALLELISM OF A FIELD OF VECTORS

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1. In a geometry of paths the equations of the paths may be written in the form

$$\frac{dx^i}{dt}\left(\frac{d^2x^j}{dt^2} + \Gamma^j_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) - \frac{dx^j}{dt}\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) = 0, \quad (1.1)$$

t being a general parameter and Γ^i_{kl} functions of the k's which are symmetric in k and l; a repeated Latin index indicates the same from 1 to n of that index. These functions serve to define infinitesimal parallelism of vectors and accordingly are called the coefficients of the affine connection. We may go further and say that if λ^i are the components of a field of contravariant vectors and C is any curve of the space (at points of which the x's are given as functions of a parameter t), the vectors of the field at points of C are parallel with respect to C, when, and only when,

$$(\lambda^h\lambda^i_{,j} - \lambda^i\lambda^h_{,j})\frac{dx^j}{dt} = 0, \quad (1.2)$$

where

$$\lambda^i_{,j} = \frac{\partial\lambda^i}{\partial x^j} + \lambda^h\Gamma^i_{hj}. \quad (1.3)$$

These equations are of such a form that if vectors λ^i are parallel with respect to C so also are $\lambda^i\varphi$, where φ is any function of the x's. This is a generalization of the parallelism of Levi-Civita for a Riemannian space.

2. If for a given vector-field λ^i the determinant $|\lambda^i_{,j}|$ is not zero, a necessary and sufficient condition that the determinant $|\bar{\lambda}^i_{,j}|$ for $\bar{\lambda}^i = \varphi\lambda^i$ be zero, that is, that the determinant

$$\left| \varphi\lambda^i_{,j} + \lambda^i \frac{\partial\varphi}{\partial x^j} \right| \tag{2.1}$$

be zero, is that φ be a solution of the equation

$$\begin{vmatrix} \varphi & -\dfrac{\partial\varphi}{\partial x^1} & \cdots\cdots & -\dfrac{\partial\varphi}{\partial x^n} \\ \lambda^1 & \lambda^1_{,1} & \cdots\cdots & \lambda^1_{,n} \\ \lambda^2 & \lambda^2_{,1} & \cdots\cdots & \lambda^2_{,n} \\ \lambda^n & \lambda^n_{,1} & \cdots\cdots & \lambda^n_{,n} \end{vmatrix} = 0. \tag{2.2}$$

Moreover, the rank of (2.2) is $n - 1$ for each solution. Hence, in considering a vector-field λ^i we assume that the components are changed by a factor φ, if necessary, so that the rank of $|\lambda^i_{,j}|$ is at most $n - 1$. We say that then the field is *normal*. This is a generalization of a unit, or null, vector-field in a Riemannian space. For, in this case we have $\lambda_i\lambda^i_{,j} = 0$ and consequently $|\lambda^i_{,j}| = 0$.

If the rank of $|\lambda^i_{,j}|$ is $n - r$, there are r independent vector-fields μ^i_α $(\alpha = 1, \ldots, r)$ which satisfy

$$\mu^j\lambda^i_{,j} = 0, \tag{2.3}$$

and the general solution is given by

$$\mu^i = \sum_\alpha^{1,\ldots,r} \psi^\alpha \mu^i_\alpha, \tag{2.4}$$

where the ψ's are arbitrary functions of the x's.

When μ^i satisfies (2.3), along each curve of the congruence defined by

$$\frac{dx^1}{\mu^1} = \cdots\cdots = \frac{dx^n}{\mu^n},$$

the vectors λ^i are parallel, as follows from (1.2). Moreover, it follows that the vectors $\varphi\lambda^i$ are parallel also, whatever be φ. Accordingly we say that each solution μ^i of (2.3) defines a *congruence of parallelism* of the field λ^i.

When $|\lambda^i_{,j}|$ is of rank $n - r$, we say that the field λ^i is *general* or *special*, according as the rank of the matrix of the last n rows of (2.2) is $n - r + 1$ or $n - r$. When the field is special, and also when it is general and

$r > 1$, equation (2.2) is satisfied by every function φ. When $r = 1$ and the field is general, equation (2.2) reduces to

$$\mu^j \frac{\partial \varphi}{\partial x^j} = 0. \tag{2.5}$$

Suppose that the field is general and that φ is a solution of (2.5) when $r = 1$, or any function whatever when $r > 1$. The equations

$$\mu^j \left(\varphi \lambda^i_{,j} + \lambda^i \frac{\partial \varphi}{\partial x^j} \right) = 0 \tag{2.6}$$

are satisfied by all vectors μ^i defined by (2.4) for which the functions ψ^α satisfy the equation

$$\sum_\alpha^{1,\ldots,r} \psi^\alpha \mu^j_\alpha \frac{\partial \varphi}{\partial x^j} = 0.$$

If there were a solution of (2.6) not expressible in the form (2.4), then from (2.6) we have equations of the form $\lambda^i = \varphi^j \lambda^i_{,j}$, in which case the rank of the matrix of the last n rows of (2.2) is $n - r$. Hence when the field is general, all the solutions of (2.6) are expressible in the form (2.4), that is, on replacing λ^i by $\varphi\lambda^i$ no new congruences of parallelism are obtained.

When the field is special, the determinant (2.1) is of rank $n - r$ at most. Consequently if φ is such that not all of the equations

$$\mu^j_\alpha \frac{\partial \varphi}{\partial x^j} = 0 \qquad (\alpha = 1, \ldots, r)$$

are satisfied, there is another solution, say $\mu^i_{\alpha+1}$, of equations (2.6). Evidently it is such that $\mu^j_{\alpha+i} \dfrac{\partial \varphi}{\partial x^j} \neq 0$. If μ^i is any other solution of (2.6) not of the form (2.4), on eliminating λ^i from (2.6) and from the similar equations when μ^j is replaced by $\mu^j_{\alpha+1}$, we have

$$\left(\mu^j_{\alpha+1} \mu^k \frac{\partial \log \varphi}{\partial x^k} - \mu^j \mu^k_{\alpha+1} \frac{\partial \log \varphi}{\partial x^k} \right) \lambda^i_{,j} = 0, \tag{2.7}$$

and consequently μ^i is expressible linearly in terms of μ^i_β ($\beta = 1, \ldots, r + 1$). Hence for the given function φ all solutions of (2.6) are expressible linearly in terms of these $r + 1$ vectors. For another function, say φ_1, there is at most one field $\mu^i_{\alpha+2}$ other than $\mu^i_\alpha (\alpha = 1, \ldots, r)$. But in this case we have the equations obtained from (2.7) on replacing φ in the first term of the left-hand member by φ_1 and μ^j throughout by $\mu^j_{\alpha+2}$. Consequently the change of the function φ does not yield new congruences of parallelism.

Gathering these results together, we have:

When a vector-field λ^i is normal and the rank of $|\lambda^i_{,j}|$ is $n - r$, there are r

independent congruences of parallelism, unless the rank of the matrix of the last n rows of (2.2) *is* $n - r$; *in the latter case there are* $r + 1$ *independent congruences of parallelism; moreover, in either case any linear combination of the vectors defining a congruence of parallelism also defines such a congruence.*

When, in particular, $\lambda^i_{,j} = \lambda^i \sigma_j$, where σ_j is any covariant vector, the vectors λ^i are parallel with respect to any curve in the space, but only in case σ_j is a gradient is parallelism independent of the curve.

3. We consider the converse problem: Given a vector field μ^i to determine the vector fields λ^i of which the former is a congruence of parallelism. We assume that the coördinate system x^i is that for which $\mu^\sigma = 0$ $(\sigma = 2, \ldots, n)$.[1] In this coördinate system the equations (2.3) for the determination of the λ's is

$$\frac{\partial \lambda^i}{\partial x^1} + \lambda^j \Gamma^i_{j1} = 0. \tag{3.1}$$

Any set of functions λ^i satisfying these equations are the components in the x's of a vector field with respect to which the congruence μ^i is the congruence of parallelism. A set of solutions is determined by arbitrary values of λ^i for $x^1 = 0$, that is, by n arbitrary functions of $x^2, \ldots, x^n$. In particular, the n sets of solutions λ^i_α, where α for $\alpha = 1, \ldots, n$ determines the set and i the component, determined by the initial values $(\lambda^i_\alpha)_0 = \delta^i_\alpha$ are independent, since the determinant $|\lambda^i_\alpha|$ is not identically zero; here δ^i_α is zero or 1 according as $i \neq \alpha$ or $i = \alpha$. Moreover, from the form of (3.1) it follows that $\lambda^i = \sum\limits_{\alpha}^{1,\ldots,n} \varphi^\alpha \lambda^i_\alpha$ is also a solution, where the φ's are any functions of $x^2, \ldots, x^n$. Hence we have:

For any congruence μ^i *there exist* n *independent vector fields* λ^i_α *with respect to which the given congruence is the congruence of parallelism; moreover, the field*

$$\lambda^i = \sum_{\alpha}^{1,\ldots,n} \varphi^\alpha \lambda^i_\alpha \qquad (i = 1, \ldots, n)$$

possesses the same property, when the φ's *are any solutions of the equation*

$$\mu^i \frac{\partial \varphi}{\partial x^i} = 0,$$

the coördinates x^i *be any whatever.*

[1] Eisenhart, *Riemannian Geometry*, Princeton Univ. Press, 1926, p. 5.