numerable elements, and conversely any arithmetic determines such a class, giving connections with the algebra of logic. These connections are destroyed if we assume "multiplication" is not commutative.

For the case when multiplication is not a commutative, we replace "division" by "left division" and "right division," with analogous changes for other relations such as "equivalence." We then develop the theory of the greatest common divisor, least common multiple, equivalence with respect to unit factors and so on. It is shown that for an arithmetic, we must assume the units of the system are commutative with all the other elements of the system. Any "arithmetic" may be converted into a group by adjunction, and the theory of congruence and the fundamentals of Kronecker's theory of forms carry over almost unchanged. The complete development will be published shortly in a mathematical journal.

ON THE STRUCTURE OF A PLANE CONTINUOUS CURVE[1]

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Communicated October 11, 1927

If x and y are distinct points of a continuous curve M, the set of all points $[z]$, such that z lies on some arc of M with end-points x and y, is called the *arc-curve* xy and is denoted by $M(x + y)$. This was defined in a previous paper, "Concerning the Arc-Curves and Basic Sets of a Continuous Curve."[2] Among other results this paper contained the following theorems for the case where M lies in a plane:

A.—The arc-curve xy is itself a continuous curve.

B.—If K is a maximal connected subset of $M - M(x + y)$, then K has only one limit point in $M(x + y)$.

C.—If P is a point of a set K such that $K - P$ is the sum of two non-vacuous mutually separated sets K_1 and K_2, and A and B are distinct points of $K_1 + P$, then no point of K_2 is a point of any arc whose end-points are A and B and which lies wholly in K.

It is the purpose of this paper to characterize types of points of a continuous curve and types of continuous curves by a set which is the limit of the arc-curve xy as y approaches x. Throughout the paper the letter M is used to denote a plane continuous curve and all point sets mentioned are considered as subsets of M. The theorems listed above will be referred to as "Theorem A," etc.

THEOREM 1.—*If the point x lies on no simple closed curve of M and α is any arc of M whose end-points are x and any other point z of M, then x is a limit point of the points of M which lie on α and separate x and z in M.*[3]

Proof. Suppose there exists an arc α of M whose end-points are x and z and such that x is not a limit point of the points of α which separate x and z in M. Then there exists a point y of α distinct from x and z such that no point of the subarc xy of α separates x and z in M. Let $[d]$ be the set of maximal connected subsets of $M-xy$, and for any maximal connected subset d let R_{xd} and R_{yd} be the first and last limit points of d on the subarc xy of α in the order x to y. If $R_{xd} \neq R_{yd}$, there exists an arc A_d, whose end-points are R_{xd} and R_{yd} and which lies in d except for these points.[4] Since the point x lies on no simple closed curve of M, $R_{xd} \neq x$ unless $R_{yd} = x$. If w is any point of $xy - x$, there exists a maximal connected subset d_w, such that w is between R_{xd_w} and R_{yd_w} on the arc xy if $w \neq y$, and if $w = y$, $R_{yd_w} = w \neq R_{xd_w}$.

Since only a finite number of the sets of $[d]$ have a diameter greater than any given positive number,[5] there exists a set d_1 of $[d]$, such that (1) $R_{yd_1} = y$ and (2) if d' is any set such that $R_{yd'} = y$, then $R_{xd'}$ lies on the subarc yR_{xd_1} of xy. In general, there exists a set d_n $(n > 1)$ of $[d]$, such that (1) $R_{xd_{n-1}}$ lies between R_{yd_n} and R_{xd_n} on xy and (2) if d' is any set of $[d]$ such that $R_{xd_{n-1}}$ lies between $R_{xd'}$ and $R_{yd'}$ on xy, then $R_{xd'}$ lies on the subarc yR_{xd_n} of xy.

We have the order $y, R_{yd_2}, R_{xd_1}, R_{yd_3}, R_{xd_2}, R_{yd_4}, R_{xd_3}, \ldots, x$ on the arc xy. All of the points of this series are distinct except that for any i we may have $R_{xd_i} = R_{yd_{i+2}}$. Using the fact that each time d_n is chosen to satisfy condition (2), it is not difficult to see that x is the sequential limit point of $R_{xd_1}, R_{xd_2}, \ldots$. If $R_{xd_i} \neq R_{yd_{i+2}}$, let $R_{xd_i}R_{yd_{i+2}}$ denote the subarc of xy which has these two points as end-points, and if $R_{xd_i} = R_{yd_{i+2}}$, let $R_{xd_i}R_{yd_{i+2}}$ denote this single point. Let yR_{yd_2} denote the subarc of xy whose end-points are y and R_{yd_2}. Let

$$\alpha_1 = x + A_{d_1} + R_{xd_1}R_{yd_3} + A_{d_3} + \ldots + A_{d_{2i-1}} + R_{xd_{2i-1}}R_{yd_{2i+1}} + \ldots,$$
$$\alpha_2 = x + yR_{yd_2} + A_{d_2} + R_{xd_2}R_{yd_4} + A_{d_4} + \ldots + A_{d_{2i}} + R_{xd_{2i}}R_{yd_{2i+2}} + \ldots.$$

Since α_j $(j = 1, 2)$ is connected, x is the sequential limit point of $[R_{xd_i}]$ and the diameter of A_{d_i} approaches zero as i increases indefinitely,[5] the set α_j is an arc of M whose end-points are x and y. The arcs α_1 and α_2 have only x and y in common; therefore, $\alpha_1 + \alpha_2$ is a simple closed curve of M containing x, which is contrary to hypothesis.

Since every point of M is either a cut-point or a non-cut point, and non-cut points which lie on no simple closed curve are end-points,[6] we have the following.

COROLLARY. *If P is a cut-point of M which lies on no simple closed curve of M or if P is an end-point of M, then P is a limit point of the cut-points of M.*

THEOREM 2.—*If x is a point of M and C_x denotes the set consisting of x and all points $[y]$ such that M contains a simple closed curve containing both x and y, then C_x is a continuous curve.*

Proof. Obviously C_x is connected, and by an argument similar to that used in proving theorem 1 we may show that C_x is closed. It remains to see that C_x is connected *im kleinen*. If P is a point of C_x and ϵ is a positive number, there exists a number $\delta_\epsilon > 0$ such that every point of M within a distance δ_ϵ of P can be joined to P by an arc of M, every point of which is within a distance ϵ of P. Let Q be any point of C_x within a distance δ_ϵ of P and let PQ denote an arc of M with end-points P and Q and every point of which is within a distance ϵ of P. We shall show that every point of PQ belongs to C_x, which proves the theorem.

Suppose PQ contains a point Z which does not belong to C_x. Then $P \neq Z \neq Q$. Since C_x is closed, there are first points P_1 and P_2 of C_x on the subarcs ZP and ZQ of PQ in the order Z to P and Z to Q. If either P_1 or P_2, say P_1, is the point x, the arc P_1ZP_2 of PQ plus either of the arcs from x to P_2 of a simple closed curve of M containing both x and P_2, is a simple closed curve of M containing both x and Z, and thus Z belongs to C_x. If neither P_1 nor P_2 is x, let J_1 and J_2 denote simple closed curves of M which contain P_1 and x and P_2 and x, respectively. If J_1 contains P_2, then the arc of J_1 from P_1 to P_2 which contains x together with the arc P_1ZP_2 of PQ is a simple closed curve of M containing x and Z. We proceed similarly if J_2 contains P_1. If J_1 does not contain P_2 and J_2 does not contain P_1, let α denote one of the arcs of J_2 from P_2 to x and let P_3 be the first point of J_1 on α in the order P_2 to x. If $P_3 = x$, then α plus the arc P_1ZP_2 of PQ plus either of the arcs of J_1 from P_1 to x is a simple closed curve of M containing x and z. If $P_3 \neq x$, then the subarc P_3P_2 of α plus the arc P_1ZP_2 of PQ plus the arc of J_1 from P_1 to P_3 which contains x is a simple closed curve of M containing x and Z. Thus in any case the point Z belongs to the set C_x

THEOREM 3.—*If K is a maximal connected subset of $M - C_x$, K has only one limit point in C_x.*

With the use of a theorem due to Wilder,[4] the proof of theorem 3 follows that of theorem 2.

THEOREM 4.—*If K is a maximal connected subset of $C_x - x$, then $K + x$ is a cyclicly connected[7] continuous curve.*

Proof. From theorem 2 we see that $K + x$ is a continuous curve. Suppose $K + x$ contains a cut-point P, that is, $K + x - P$ is the sum of two non-vacuous mutually separated sets K_1 and K_2. Since K is connected, P is not the point x. Then x lies in one of the two sets K_1 or K_2 and let y be a point of the other. Since y belongs to C_x there exist two arcs of M from y to x which have only their end-points in common. But this is impossible, since every arc from y to x must contain P. Then $K + x$

contains no cut-point and by a theorem due to G. T. Whyburn,[8] $K + x$ is a cyclicly connected continuous curve.

THEOREM 5.—*If the continuous curve M contains a cut-point and a simple closed curve, it contains a cut-point which lies on a simple closed curve of M.*

Theorem 5 may be established by the methods of theorem 1.

THEOREM 6.—*If u and v are distinct points of $M(x + y)$, then $M(u + v)$ is a subset of $M(x + y)$.*

This theorem follows easily from theorem B.

Definition. Let x be a point of M and $y_1, y_2, y_3, \ldots.$ be a sequence of points of M distinct from x, such that the distance from x to y_i approaches zero as i increases indefinitely. Let K denote the set of all points which are common to all the sets $M(x + y_i)$. The set K exists and is non-vacuous. If the set K is independent of the choice of the sequence of points $y_1, y_2, y_3, \ldots.$ approaching x as a limit, then K will be called the *limiting arc-curve of M at the point x* and will be denoted by Lim $M(x)$.

THEOREM 7.—*If x is a point of M which lies on no simple closed curve of M, the limiting arc-curve at x exists and is the point x.*

Proof. If the theorem is not true, there exists a sequence of points $y_1, y_2, y_3, \ldots.$ such that the sets $M(x + y_i)$ have in common a point $z \neq x$. By theorem 1, if α is an arc of M whose end-points are x and z then α contains a point p distinct from x and z such that p separates x and z in M. Then $M - p = M_x + M_z$, where M_x and M_z are non-vacuous mutually separated sets containing x and z, respectively. As x is not a limit point of M_z, there exists a number $\epsilon > 0$ so that no point of M_z is within a distance ϵ of x. There exists an integer n_ϵ such that if $i > n_\epsilon$, the point y_i lies within a distance ϵ of x. Then $M_x + p$ contains y_i for $i > n_\epsilon$, and by theorem C, $M(x + y_i)$ cannot contain z. But we assumed that $M(x + y_i)$ contains z for every i. Hence, for every sequence $[y_i]$ approaching x, the point x is the only point common to all the sets $M(x + y_i)$.

THEOREM 8.—*If x is a cut-point of M and lies on some simple closed curve of M, the limiting arc-curve at x does not exist.*

Proof. Suppose $M - x$ is the sum of two non-vacuous mutually separated sets M_1 and M_2 and that the limiting arc-curve at x exists. From theorem C it follows that if all points of the sequence $y_1, y_2, y_3, \ldots.$ belong to M_1, the limiting arc-curve at x is a subset of $M_1 + x$. Similarly, if all points of the sequence belong to M_2, Lim $M(x)$ is a subset of $M_2 + x$. But the Lim $M(x)$ is independent of the choice of the sequence and thus it must be common to the sets $M_1 + x$ and $M_2 + x$; that is, if the limiting arc-curve exists at a cut-point x, it must consist of the single point x. But if x lies on some simple closed curve J of M and $[y_i]$ is a sequence, every point of which belongs to J, the arc-curve $M(x + y_i)$ contains J for every i. Thus J is a subset of the set of common points for this sequence. Then the set of common points depends on the choice of the sequence. Hence Lim $M(x)$ does not exist.

THEOREM 9.—*If x is a non-cut-point of M which lies on a simple closed curve of M, the limiting arc-curve at x exists and is the set C_x.*

Proof. As x is not a cut-point of M, $C_x - x$ is connected; then from theorem 4, C_x is a cyclicly connected continuous curve. It is not difficult to see that if y and z are any points of C_x distinct from x and from each other, there exists an arc of C_x which has end-points x and y and contains z. With this and theorem 3, we see that $M(x + y) = C_x$ for every point $y \neq x$ of C_x. Also if y is a point of a maximal connected subset D of $M - C_x$, then $M(x + y)$ is contained in $D + C_x$ and contains C_x. Then C_x is a subset of $M(x + y)$, if y is any point of M distinct from x. Now suppose there exists a sequence of points $[y_i]$ of M approaching x such that the arc-curves $M(x + y_i)$ have in common a point w which is not a point of C_x. Let D_w denote the maximal connected subset of $M - C_x$ containing the point w. By theorem 3, D_w has only one limit point in C_x, and since x is not a cut-point of M, this limit point is not x. There exists a number $\epsilon > 0$ such that no point of D_w is within a distance ϵ of x. There exists a number $n_\epsilon > 0$ such that for $i > n_\epsilon$, the distance from x to y_i is less than ϵ. For $i > n_\epsilon$, y_i must lie in $M - D_w$, and by theorem C, for $i > n_\epsilon$, $M(x + y_i)$ is a subset of $M - D_w$. Then the point w is not common to all the sets $M(x + y_i)$. Therefore, $C_x = $ Lim $M(x)$.

Since a definite point of a continuous curve M must satisfy one, and not more than one, of the hypotheses of theorems 7, 8 and 9, we have

THEOREM 10.—*If x is a point of a continuous curve M, then (a) x lies on no simple closed curve of M, (b) x is a non-cut-point of M and lies on a simple closed curve of M, (c) x is a cut-point of M and lies on a simple closed curve of M, if and only if (a) Lim $M(x) = x$, (b) Lim $M(x) = C_x \neq x$, (c) Lim $M(x)$ does not exist.*

THEOREM 11.—*In order that a continuous curve M be acyclic,*[9] *it is necessary and sufficient that Lim $M(x) = x$ for every point x of M.*

Theorem 11 is a consequence of theorem 10(*a*).

THEOREM 12.—*In order that a continuous curve M be cyclicly connected, it is necessary and sufficient that M contain a point x such that Lim $M(x) = M$.*

Proof. That the condition is necessary follows from the property of cyclicly connected continuous curves stated above, namely, that if x and y are distinct points of such a curve, $M(x + y) = M$. We shall show that it is sufficient. If M is not cyclicly connected, M contains a cut-point P.[8] Then $M - P$ is the sum of two non-vacuous mutually separated sets M_1 and M_2, and we suppose that $M_1 + P$ contains the point x. Let y be any point of M_1 distinct from x. By theorem C, $M(x + y)$ is a subset of $M_1 + P$ and thus contains no point of M_2. Since $M(x + y)$ contains Lim $M(x)$, Lim $M(x) \neq M$.

It is evident that if M is cyclicly connected, then Lim $M(x) = M$ for *every* point x of M and conversely.

THEOREM 13.—*If M is a continuous curve, either* (1) *Lim* $M(x) = x$ *for every poin. x of M, or* (2) *Lim* $M(x) = M$ *for every point x of M, or* (3) *M contains a point x such that the limiting arc-curve of M at x does not exist.*

Proof. If (1) does not hold, then M contains a simple closed curve by theorem 11. If (2) is not true, M is not cyclicly connected by theorem 12. Hence M contains a cut-point.[8] By theorem 5, M contains a cut-point which lies on a simple closed curve of M. By theorem 8, the limiting arc-curve of M does not exist at this point. Thus, if neither (1) nor (2) is true, (3) must be true.

[1] Presented to the American Mathematical Society, May 7, 1927.

[2] This paper has been submitted for publication in *Trans. Amer. Math. Soc.*

[3] A point P of a set H is said to separate two points x and y in H if H-P is the sum of two mutually separated sets, one containing x and the other containing y.

[4] Wilder, R. L., *Fund. Math.*, **7**, 1925 (342).

[5] Cf. my paper, "Note on a Theorem Concerning Continuous Curves," *Annals Math.*, **28**, 1927 (501–2).

[6] Ayres, W. L., *Annals Math.*, **28**, 1927 (396–418).

[7] A continuous curve H is said to be cyclicly connected if and only if every two points of H lie together on some simple closed curve which is a subset of H. Cf. Whyburn, G. T., these PROCEEDINGS, **13**, 1927 (31–38).

[8] Loc. cit., theorem 1.

[9] A continuous curve is said to be acyclic if it contains no simple closed curve. This term has been introduced recently by H. M. Gehman.

ON THE ARITHMETIC OF ABELINA FUNCTIONS

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Communicated October 15, 1927

1. Systematic application of abelian functions to the theory of numbers discloses many new arithmetical phenomena of considerable interest. Particularly is this the case when the periods of the functions are connected by one or more singular relations in the usual sense. An example is afforded by the novel type of arithmetical invariance made precise in §2, and illustrated in §3, which, roughly, is as follows.

Consider a set of $n > 1$ quadratic forms in the same s indeterminates. The sets of rational integral values of the s indeterminates which represent an arbitrary set of n integers representable simultaneously in the n forms are separated into sets of residue classes with respect to a modulus α. If a particular set of h of these classes is such that the number of repre-