CRESCENT-SHAPED MINIMAL SURFACES

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1. *The Problem of Plateau for Crescent-Shaped Minimal Surfaces.*—The problem of Plateau is to prove the existence of a minimal surface with a given boundary. A minimal surface is one definable by the Weierstrass formulas

$$x_i = \Re F_i(w), \sum_{i=1}^{n} F_i'^2 (w) = 0 \quad (i = 1, 2, \ldots, n). \tag{1.1}$$

The author has given a solution of the Plateau problem for two contours,[1] by methods which are a natural generalization of those previously used by him to give the first complete solution of the Plateau problem for a single contour.[2]

It was assumed, in the treatment of the two-contour case, that the contours did not intersect one another. If, on the contrary, the contours have a single point in common, then we may obtain a minimal surface of the topological type of the region between two internally tangent circles; indeed, the minimal surface will be representable conformally on this region. Such a region we call a *crescent*, and the minimal surface representable conformally upon it *crescent-shaped*.

The purpose of the present note is to adapt the methods of the previous papers of the writer, cited above, to solve the Plateau problem for crescent-shaped minimal surfaces: *given two Jordan curves* Γ_1 *and* Γ_2, *with one and only one point P in common, to prove—under appropriate sufficient conditions—the existence of a crescent-shaped minimal surface bounded by* Γ_1 *and* Γ_2.

That some sort of restriction on the contours is necessary is evident from considering, for instance, two externally tangent circles in the same plane; obviously they determine no proper crescent-shaped minimal surface, giving rather the sum of the two circular discs. The sufficient condition established in this paper is that if $m(\Gamma_1, \Gamma_2)$ denote the minimum of crescent-shaped areas bounded by Γ_1 and Γ_2, and $m(\Gamma_1)$, $m(\Gamma_2)$ the minimum of simply connected areas bounded by Γ_1, Γ_2 separately, then

$$m(\Gamma_1, \Gamma_2) < m(\Gamma_1) + m(\Gamma_2) \tag{1.2}$$

(the relation $\leqq$ holds in any case). We also suppose that all three quantities m are *finite*, an assumption certainly verified if the contours are rectifiable.

2. *The Functional* $A(g_1, g_2)$.—If the crescent is subjected to an in-

version with the point of tangency of its bounding circles as pole, we obtain a strip bounded by two parallel straight lines; without loss of generality, we can assume this strip to be that bounded by the lines

$$(L_1) \quad z = \text{real, and } (L_2) \quad z = \text{real} + \pi i,$$

in the complex plane of z. The inversion, being a conformal transformation, will convert a conformal representation of the required minimal surface on the crescent into a conformal representation on the strip, and vice versa.

The next step is to adapt to crescent-shaped surfaces the functional employed in the author's treatment of the one- and two-contour cases of the problem of Plateau.

Let

$$x_i = g_{1i}(z), \quad x_i = g_{2i}(z) \quad (i = 1, 2, \ldots, n) \tag{2.1}$$

denote arbitrary parametric representations of Γ_1 and Γ_2, where z describes L_1 and L_2, respectively, the common point P of the two contours being always required to correspond to the point at infinity on each line. If

$$x_i = \Re F_i(w) \quad (w = u + iv)$$

denote the harmonic functions determined in the parallel strip S by the boundary values (2.1), these functions define a crescent-shaped harmonic surface bounded by Γ_1, Γ_2. Let the first fundamental form of this surface be

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

then the fundamental functional is

$$A(g_1, g_2) = \int\int_S \tfrac{1}{2}(E + G)dudv. \tag{2.2}$$

This integral must be defined accurately as a limit; we construct the strip S_ϵ by removing a band of width ϵ from each edge of S, and define

$$A_\epsilon(g_1, g_2) = \int\int_{S_\epsilon} \tfrac{1}{2}(E + G)dudv, \tag{2.2'}$$

then

$$A(g_1, g_2) = \lim_{\epsilon \to 0} A_\epsilon(g_1, g_2).$$

It is not hard to prove that the integrand of A_ϵ stays uniformly bounded for all g_1, g_2 if ϵ is fixed > 0.

It is desirable to have a direct expression for A in terms of g_1 and g_2; this can be obtained by means of formulas established in Two Contours (§ 2). Let $G(x, y; u, v)$ denote the Green's function for the region S; then G can always be given the form

$$G(x, y;\ u, v) = \Re \log \frac{S(z, w)}{S_1(z, w_0)}, \tag{2.3}$$

where

$$z = x + iy, \quad w = u + iv, \quad w_0 = u - iv.$$

If

$$Z(z, w) = \frac{\partial}{\partial z} \log S(z, w), \tag{2.4}$$

and

$$P(z, w) = \frac{\partial}{\partial w} Z(z, w) = \frac{\partial^2}{\partial z \partial w} \log S(z, w), \tag{2.5}$$

then, with the help of Green's theorem, the explicit form for A is:

$$A(g_1, g_2) = \frac{1}{4\pi} \sum_{\alpha\beta} \int_{L_\alpha} \int_{L_\beta} \sum_{i=1}^{n} [g_{\alpha i}(z) - g_{\beta i}(\zeta)]^2\ P(z, \zeta) dz\, d\zeta. \tag{2.6}$$

Here and elsewhere in this paper, integration on L_1 is to the right, on L_2 to the left.

It is to be observed that, although complex elements enter into this formula, the value of $A(g_1, g_2)$, according to (2.2), is always positive real.

The next step is to obtain explicitly the Green's function for the strip S. We state the result immediately:

$$G(x, y;\ u, v) = \Re \log \frac{\sinh \frac{1}{2}(z - w)}{\sinh \frac{1}{2}(z - w_0)}. \tag{2.7}$$

It will be an easy exercise for the reader to verify that this has all the requisite properties: regular harmonic in the strip S except at the one point w where there is a logarithmic singularity, reducing to zero on L_1 and L_2.

By comparison with (2.3),

$$S(z, w) = \sinh \tfrac{1}{2}(z - w), \tag{2.8}$$

and therefore, by (2.4) and (2.5),

$$Z(z, w) = \tfrac{1}{2} \coth \tfrac{1}{2}(z - w), \tag{2.9}$$

$$P(z, w) = \frac{1}{4 \sinh^2 \frac{1}{2}(z - w)}. \tag{2.10}$$

Hence, by substitution in (2.6),

$$A(g_1, g_2) = \frac{1}{4\pi} \sum_{\alpha\beta} \int_{L_\alpha} \int_{L_\beta} \sum_{i=1}^{n} [g_{\alpha i}(z) - g_{\beta i}(\zeta)]^2 \frac{dz d\zeta}{4 \sinh^2 \frac{1}{2}(z - \zeta)} \tag{2.11}$$

The integral is improper, on account of the singularity of the integrand for $z = \zeta$, but always exists either as a finite positive quantity or $+\infty$; its value is to be found by integrating first with $|z - \zeta| \geqq \epsilon$, furnishing $\overline{A}_\epsilon(g_1, g_2)$, and then defining

$$A(g_1, g_2) = \lim_{\epsilon \to 0} \overline{A}_\epsilon(g_1, g_2).$$

By a formula of Two Contours [2.13, loc. cit.] and (2.9) above, the harmonic surface determined by g_1, g_2 is

$$x_i = \Re F_i(w)$$

with

$$F_i(w) = \frac{1}{2\pi i} \sum_\alpha \int_{L_\alpha} g_{\alpha i}(z) \coth \tfrac{1}{2} (z - w) dz + c_i, \qquad (2.12)$$

where c_i is a constant whose exact value is unnecessary.

We shall need too the formula, easily derived from the last one [cf. Two Contours, 2.16]:

$$\sum_{i=1}^{n} F_i'^2 (w) = \frac{1}{2\pi^2} \sum_{\alpha\beta} \int_{L_\alpha} \int_{L_\beta} \sum_{i=1}^{n} [g_{\alpha i}(z) - g_{\beta i}(\zeta)]^2 \frac{dz d\zeta}{16 \sinh^2 \frac{1}{2} (z - w) \sinh^2 \frac{1}{2} (\zeta - w)}. \qquad (2.13)$$

3. *Attainment of the Minimum of* $A(g_1, g_2)$.—Each of the sets $[g_1]$, $[g_2]$ of parametric representations of Γ_1, Γ_2, respectively, is compact. Therefore the composite elements $[g_1, g_2]$ form a compact set.

The integrand of A_ϵ in (2.2′) is uniformly bounded, and, under this circumstance, a theorem of Lebesgue permits us to pass to the limit under the integral sign. This is to say that A_ϵ is a continuous functional of g_1, g_2: if $\bar{g}_1 \to g_1$, $\bar{g}_2 \to g_2$, then $A_\epsilon(\bar{g}_1, \bar{g}_2) \to A_\epsilon(g_1, g_2)$.

The integrand $\frac{1}{2}(E + G)$ being positive, the approach of A_ϵ to A when $\epsilon \to 0$ is monotonic increasing; under this condition, the following lemma[3] enables us to affirm the lower semi-continuity of $A(g_1, g_2)$: *if a functional A on a Fréchet L-set can be expressed as the limit of a continuous functional A_ϵ which tends to A in increasing, then A is lower semi-continuous.*

From the compactness of the range $[g_1, g_2]$ and the lower semi-continuity of $A(g_1, g_2)$, it follows, as in the author's previous papers, that the minimum of $A(g_1, g_2)$ is attained for a certain (g_1^*, g_2^*); the proof is essentially the same as the Weierstrass proof of the attainment of the minimum of a (lower semi-) continuous function of a real variable on a closed interval, depending on the Bolzano-Weierstrass theorem expressing compactness: every infinite sequence of values of the argument (g_1, g_2)

contains a convergent sub-sequence, and on the definition of lower semi-continuity:

$$A(g_1, g_2) \leqq \liminf A(\bar{g}_1, \bar{g}_2) \text{ when } \bar{g}_1 \longrightarrow g_1, \bar{g}_2 \longrightarrow g_2.$$

4. *Exclusion of Improper Representations.*—We have next to show that the minimizing representation (g_1^*, g_2^*) is proper.

In the author's previous papers, the improper representations of a contour Γ have been classified as follows:

(*a*) A partial arc of Γ (Γ_1 or Γ_2) corresponds to a single point of L (L_1 or L_2).

(*b*) A partial segment of L corresponds to a single point of Γ.

(*c*) All of Γ corresponds to a single point of L and all of L to a single point of Γ.

Type (*a*) is excluded by the fact that then:

$$A(g_1, g_2) = +\infty,$$

for the integrand of (2.11) becomes infinite to the second order in the vicinity of the point of discontinuity, being asymptotically

$$\frac{l^2}{(z - \zeta)^2},$$

where l is the length of the chord of the arc of discontinuity of Γ. $l > 0$ because this arc is only a partial arc of Γ, and, by hypothesis, Γ has no double points.

Type (*b*) can be excluded by the mode of reasoning used in One Contour, § 18, based on a certain theorem of Fatou and Schwarzian symmetry—after it has been proved, as will be done in the next section, that

$$\sum_{i=1}^{n} F_i'^2(w) = 0.$$

It is in order to exclude type (*c*) that we need the condition (4.6) below, restricting the two contours.

Besides the functional $A(g_1, g_2)$, let us consider

$$A(g_1) = \frac{1}{4\pi} \int_{L_1} \int_{L_1} \sum_{i=1}^{n} [g_{1i}(z) - g_{1i}(\zeta)]^2 \frac{dzd\zeta}{(z - \zeta)^2},$$

$$A(g_2) = \frac{1}{4\pi} \int_{L_2} \int_{L_2} \sum_{i=1}^{n} [g_{2i}(z) - g_{2i}(\zeta)]^2 \frac{dzd\zeta}{(z - \zeta)^2};$$

these are the functionals appropriate to the construction of a simply connected minimal surface bounded by Γ_1 or Γ_2, respectively. Define the three positive numbers

$$m(\Gamma_1, \Gamma_2) = \min A(g_1, g_2), \tag{4.1}$$

$$m(\Gamma_1) = \min A(g_1), \quad m(\Gamma_2) = \min A(g_2),$$

the minima being relative to all possible parametric representations of Γ_1, Γ_2. Concretely, $m(\Gamma_1, \Gamma_2)$ is the lower bound of the areas of all crescent-shaped surfaces bounded by Γ_1, Γ_2, and $m(\Gamma_1)$, $m(\Gamma)$ the lower bounds of the areas of all simply connected surfaces bounded, respectively, by Γ_1, Γ_2; but we shall be concerned only with the definitions (4.1).

As stated in § 1, we assume that all three quantities (4.1) are finite.

If g_1 degenerates into a point p_1 on Γ_1, and g_2 is not degenerate, we can prove by reasoning similar to that of Two Contours, p. 342 (which we shall not repeat here), that

$$A(g_1, g_2) > m(\Gamma_1) + A(g_2),$$

whence

$$A(g_1, g_2) > m(\Gamma_1) + m(\Gamma_2). \tag{4.2}$$

If both g_1 and g_2 are degenerate, in points p_1 and p_2, respectively, not both coincident with the unique point P common to Γ_1 and Γ_2, then

$$A(g_1, g_2) = +\infty. \tag{4.3}$$

If p_1 and p_2 are both coincident with the point P, then

$$A(g_1, g_2) = m(\Gamma_1) + m(\Gamma_2). \tag{4.4}$$

This formula shows that always

$$m(\Gamma_1, \Gamma_2) \leqq m(\Gamma_1) + m(\Gamma_2). \tag{4.5}$$

Let us now introduce the assumption that we have the strict inequality:

$$m(\Gamma_1, \Gamma_2) < m(\Gamma_1) + m(\Gamma_2), \tag{4.6}$$

then it will be seen by the formulas (4.2, 3, 4) that the possibility of a minimizing representation of type (c) is excluded.

5. *Vanishing of the First Variation of $A(g_1, g_2)$.*—The functional $A(g_1, g_2)$ attaining its minimum for (g_1^*, g_2^*), its first variation must then vanish—a condition which will lead to

$$\sum_{i=1}^{n} F_i'^2(w) = 0, \tag{5.1}$$

proving that the harmonic surface determined by (g_1^*, g_2^*) according to the formulas (2.12) is minimal.

We employ the following variation of the independent variables:

$$z' = z + \tfrac{1}{2}\lambda \coth \tfrac{1}{2}(z - w), \tag{5.2}$$

and the same in ζ, on L_1 and L_2, with w an arbitrary point of the strip S, and λ a real parameter. Although the only admissible variations are those which convert L_1 and L_2 into themselves in a monotonic continuous way, with preservation of the point at infinity, and the transformation (5.2) does not do this; still this variation is allowable, for $|\lambda|$ sufficiently small, as a combination of admissible variations [cf. Two Contours, pp. 332–334].

It is readily calculated that by (5.2) and the analogue in ζ, we have

$$\frac{dz'd\zeta'}{4\sinh^2\frac{1}{2}(z'-\zeta')} = \frac{dzd\zeta}{4\sinh^2\frac{1}{2}(z-\zeta)} - \lambda\frac{dzd\zeta}{16\sinh^2\frac{1}{2}(z-w)\sinh^2\frac{1}{2}(\zeta-w)} + \ldots, \qquad (5.3)$$

the dots denoting terms in higher powers of λ.

The principal feature of the calculation is the use of the identity

$$\sinh^2\tfrac{1}{2}(z-w) + \sinh^2\tfrac{1}{2}(\zeta-w) - 2\sinh\tfrac{1}{2}(z-w)\sinh\tfrac{1}{2}(\zeta-w)\cosh\tfrac{1}{2}(z-\zeta) = \sinh^2\tfrac{1}{2}(z-\zeta), \qquad (5.4)$$

which can be obtained by squaring the subtraction formula for sinh: $\sinh\frac{1}{2}(z-w)\cosh\frac{1}{2}(\zeta-w) - \cosh\frac{1}{2}(z-w)\sinh\frac{1}{2}(\zeta-w) = \sinh\frac{1}{2}(z-\zeta)$, and making certain simple transformations, including the subtraction formula for cosh:

$$\cosh\tfrac{1}{2}(z-w)\cosh\tfrac{1}{2}(\zeta-w) - \sinh\tfrac{1}{2}(z-w)\sinh\tfrac{1}{2}(\zeta-w) = \cosh\tfrac{1}{2}(z-\zeta).$$

Multiplying (5.3) by

$$\sum_{i=1}^{n}['g_{\alpha i}(z') - 'g_{\beta i}(\zeta')]^2 = \sum_{s=1}^{n}[g_{\alpha i}(z) - g_{\beta i}(\zeta)]^2,$$

and integrating, we get

$$A('g_1, 'g_2) = A(g_1, g_2) - \lambda\cdot\frac{1}{4\pi}\sum_{\alpha\beta}\int_{L_\alpha}\int_{L_\beta}\sum_{i=1}^{n}[g_{\alpha i}(z) - g_{\beta i}(\zeta)]^2 \frac{dzd\zeta}{16\sinh^2\frac{1}{2}(z-w)\sinh^2\frac{1}{2}(\zeta-w)} + \ldots. \qquad (5.5)$$

If $(g_1, g_2) = (g_1^*, g_2^*)$, this function of λ has a minimum for $\lambda = 0$, and therefore the coefficient of the first power of λ vanishes:

$$\sum_{\alpha\beta}\int_{L_\alpha}\int_{L_\beta}\sum_{i=1}^{n}[g_{\alpha i}(z) - g_{\beta i}(\zeta)]^2\frac{dzd\zeta}{16\sinh^2\frac{1}{2}(z-w)\sinh^2\frac{1}{2}(\zeta-w)} = 0. \qquad (5.6)$$

But, according to the formula (2.13), this is the same as

$$\sum_{i=1}^{n}F_i'^2(w) = 0. \qquad (5.7)$$

6. *Conformal Mapping.*—Analogous to the author's previous work, the preceding theory gives for $n = 2$ the conformal mapping of any crescent-shaped plane region—i.e., one bounded by two Jordan curves with a unique point in common, one curve enclosing the other—on a circular crescent, including the continuous attachment of the conformal map to a topological correspondence between the boundaries.

[1] "The Problem of Plateau for Two Contours," *Jour. Math. Phys.*, **10**, 315–359 (1931). This paper will be cited as "Two Contours."

[2] "Solution of the Problem of Plateau," *Trans. Amer. Math. Soc.*, **33**, 263–321 (1931). This paper will be cited as "One Contour."

[3] One Contour, p. 282.

GROUPS GENERATED BY TWO OPERATORS OF ORDER 3 WHOSE COMMUTATOR IS OF ORDER 2

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Let s and t represent two operators of order 3 such that their commutator s^2t^2st is of order 2. It is known that then each of the four commutators whose elements are powers of s and t

$$s^2t^2st \qquad s^2tst^2 \qquad st^2s^2t \qquad sts^2t^2$$

is of order 2 and hence there result the following identities

$$s^2t^2st = t^2s^2ts,\ s^2tst^2 = ts^2t^2s,\ st^2s^2t = t^2sts^2,\ sts^2t^2 = tst^2s^2.$$

Since $t^2s^2ts.s^2tst^2 = t^2s^2t^2st^2$ and $t^2s^2t^2st^2.t^2s^2t^2st^2 = t^2s^2t^2sts^2t^2st^2 = t^2s^2t^2.tst^2s^2.st^2 = 1$, it follows that the first and the second of these four commutators are commutative. Similarly it follows from the identities $s^2t^2st.t^2sts^2 = s^2t^2s^2ts^2$ and $s^2t^2s^2ts^2.s^2t^2s^2ts^2 = s^2t^2s^2tst^2s^2ts^2 = s^2t^2s^2.sts^2t^2.ts^2 = 1$, that the first of these four commutators is also commutative with the third.

It seems desirable to employ a different method to prove that the first and fourth of these commutators are commutative but a similar method will be employed to prove that the fourth of these commutators is also commutative with the second and third, as follows: From the identities $s^2tst^2.tst^2s^2 = s^2ts^2t^2s^2$, and $s^2ts^2t^2s^2.s^2ts^2t^2s^2 = s^2ts^2t^2sts^2t^2s^2 = s^2t.t^2s^2ts.s^2t^2s^2 = 1$, it results that the second and the fourth are commutative. Similarly, the fact that the third and the fourth are commutative results from the identities $t^2sts^2.sts^2t^2 = t^2st^2s^2t^2$ and $t^2st^2s^2t^2.t^2st^2s^2t^2 = t^2st^2s^2tst^2s^2t^2 = t^2s.s^2t^2st.t^2s^2t^2 = 1$.

It has not yet been proved that the first and last of these four commu-