at its middle point A_n, and having a common radius

$$R_n = Kn\varphi(n).OA_n,$$

K being a conveniently chosen number. We assume that all the zeros of $f(z)$ are outside these circles.

E. There are no zeros of $f(z)$ in the circle (C_n).

Then there is one, and only one zero of $f'(z)$ in the circle (C_n) for $n > n_0$, n_0 being a sufficiently large number.

The proof follows again from theorem I, by showing that in the circle (C_n), $|Y| < 1/\lambda_n$ when $n > n_0$.

It must be noted in this case that there is a necessary geometrical relation between α_n and R_n; it is easy to see that

$$\sin |\alpha_n| < \frac{1}{Kn\varphi(n)}\,.$$

For the functions of order zero such that $a < x^{1/2}\varphi(x) < b$, where a and b are fixed numbers, neither of the above methods applies; it is then necessary to make further hypotheses on the zeros.

DUALITY RELATIONS IN TOPOLOGY

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In some recent papers[1] I have introduced the relative cycles for point sets and the associated relative boundary relations and homologies which may be of four types: absolute, modular, relative, relative modular. Various considerations lead one also to introduce a couple of invariants analogous to the Betti-numbers and for all these I have given loc. cit. proofs of some very general relations, in particular of duality which include all those previously known.

In going over the whole question I have recently had occasion to revise the proofs and extend the results somewhat. The extensions are along the line of much information concerning the relative torsion coefficients which occur, however, only when the subset G of the carrying complex is polyhedral. I do not wish to dwell on these here. The modified proofs are noteworthy and the changes shall now be indicated in outline. Their object was to extend as far as possible Poincaré's own proof for the duality relations of an M_n without boundary and to avoid wherever possible

Kronecker indices as in the original proofs. The two basic elements in Poincaré's proof are the incidence matrices of the cells and the construction of a dual complex.

Let C_n be a complex of Veblen's type which defines an M_n and let C_n^* be its dual. The cells of C_n^* can be so oriented with respect to those of C_n, that the incidence matrix of the h and $(h - 1)$-cells of C_n is the transverse of the similar matrix for the dimensions $n - h + 1$ and $n - h$ of C_n^*. From this Poincaré's duality theorems follow and likewise Veblen's and Alexander's extensions to the modular cases. This is manifestly as straightforward and direct a procedure as could be desired, and now for its generalization.

Let us call *regular* a cell of C_n which fulfills the same requirements as if C_n defined an M_n without boundary. Let G be a subset of C_n. If every cell of C_n not on G is regular we shall say that C_n is a *manifold relatively to* G, or more briefly, that $C_n - G$ is a *manifold.*

Take first the case where G is a subcomplex of C_n. Let C_n' be the first derived complex of C_n (regular subdivision of C_n). The sum of the cells of C_n' that have a vertex on the regular cell E_h of C_n but do not meet E_h, is an $(n - h)$-cell E^*_{n-h}, the *transverse* of E_h. The sum of these transverses is a complex C^*, the dual of C_n relative G. Among its properties the following are of particular interest: (*a*) $C - C^* = N$ neighborhood of G on C sum of the cells of C_n' with a vertex on G. (*b*) The cells of C^* whose sum is the boundary of N are the transverses of those of $C_n - G$ with a vertex on G.

The comparison between the incidence matrices of $C_n - G$ and C^* yields all the duality relations corresponding to G, a subcomplex of C_n. More generally of course G may be assumed merely to be a polyhedral complex on C_n.

Suppose now that G *is an arbitrary closed set on* C_n, with $C_n - G$ a manifold. Consider a subdivision C_n' of C_n and let N be the sum of all its cells whose closure meets G. It is a neighborhood of G of the same type as above. Subdivide further the cells of N alone as far as desired and let N' be the analogous neighborhood constructed by means of the new subdivision, and so on. The totality of all the cells of $C_n - N$, $N - N'$, etc., is a denumerable set of regular cells which constitutes what may be described as an *infinite* complex K_n, defining an *infinite* manifold. The transverses of the cells of K_n constitute another infinite complex K^*, the *dual* of the first. The incidence matrices for these complexes are exactly as for ordinary (finite) complexes with a finite number of non-zero elements in each row or column, but the number of rows or columns is infinite (denumerable).

The machinery is thus at hand for extending Poincaré's scheme of things with few modifications relatively. In particular we can avoid the seemingly involved and unnatural Vietoris cycles whose place is now taken by the infinite cycles (eventually fractionary) composed of cells of K or K^*.

Infinite complexes are susceptible of other applications. They are notably convenient in proving the invariance of the relative Betti and torsion numbers. They have already been considered for $n = 2$ by Kerekjarto,[2] but there is no hint of the above applications in his work.

[1] These PROCEEDINGS, **13** (1927), 614–622, 805–807; *Ann. of Math.*, (2) **29**, (1928), 232–254.

[2] *Vorlesungen über Topologie.*

GROUPS WHICH ADMIT THREE-FOURTHS AUTOMORPHISMS

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A group G is said to admit three-fourths automorphisms if it is possible to establish at least one (1,1) correspondence between its operators in which exactly three-fourths of these operators correspond to their inverses. This is the only condition imposed on G in the present article. It is easy to prove that whenever G admits at least one such automorphism it must admit exactly three distinct ones, and it must contain exactly three abelian subgroups of index 2. In each of these three automorphisms all the operators of two of these subgroups correspond to their inverses while the remaining operators correspond to their inverses multiplied by the commutator of order 2 contained in G. The continued product of these three automorphisms is an automorphism of G in which each of the operators of the central corresponds to its inverse while every other operator corresponds to its inverse multiplied by the commutator of order 2 contained in G. Hence G must also admit such an automorphism whenever it admits a three-fourths automorphism, and this automorphism is invariant under the group of automorphisms of G while a three-fourths automorphism is not necessarily invariant under this group.

A necessary and sufficient condition that a group admits a three-fourths automorphism is that its central is of index 4, and hence its order must be divisible by 8. If its order is not a power of 2 then G must be the direct product of an abelian group of odd order and a non-abelian group of order 2^m, and every such direct product admits three-fourths automorphisms whenever its Sylow subgroup of order 2^m has this property. At most three-fourths of the operators of a non-abelian group can correspond to their inverses in an automorphism of the group, and hence the