If $[z(0)]$ were null it would follow that $c_{s+1}, \ldots, c_n$ would be zero since the rank of the last n-s columns of $E(0)$ is n-s.

We would, moreover, have

$$u_i'(0) = u_i\,(0) = 0 \qquad (i = 1, \ldots, n)$$

so that $[u(x)]$ would be identically null contrary to the assumption that the solutions of the base are linearly independent. Thus $[z(0)] \neq [0]$.

Now the solutions $u_i(x)$ and $z_i(x)$ are conjugate. If we use the fact that $[u(0)] = [0]$ the condition that $u_i(x)$ and $z_i(x)$ be conjugate solutions takes the form,[3]

$$R_{ik}(0)z_i(0)u_k'(0) = R_{ik}(0)z_i(0)z_k(0) = 0. \quad (i, k = 1, \ldots, n)$$

But this is contrary to the assumption that the problem is positively regular.

Thus $E(0) \neq 0$ and the order of $D(x)$ at $x = 0$ equals the nullity s of $D(0)$.

[1] "A Generalization of the Sturm Separation and Comparison Theorems in n-Space," *Math. Ann.*, **103**, 52–69 (1930).

[2] "The Foundations of the Calculus of Variations in the Large in m-Space. (First paper.) *Trans. Am. Math. Soc.*, **31** (1930). Lemma § 7, p. 385. Lemma 2 of p. 387 should read, "Let D (x) be a determinant whose n columns are n conjugate, etc."

The error in the proof in § 7 was first called to my attention by G. A. Bliss. The proof given in § 7 can be modified so as to be correct but the present proof seems simpler.

[3] See Bolza, *Variationsrechnung*, p. 626.

AUTOMORPHISMS OF ORDER 2 OF AN ABELIAN GROUP

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1. Invariant Automorphisms of Order 2. It is known that an invariant operator of the group of isomorphisms of an abelian group G must transform every operator of G into the same power of itself. Hence it results that when the order of G is of the form p^m, where p is an odd prime number, its group of isomorphisms involves one and only one invariant operator of order 2. When $p = 2$ this group of isomorphisms contains no invariant operator of order 2, one invariant operator of this order, or three such operators as G involves no operator of order 4, operators of order 4 but none of order 8, or operators of order 8. If we combine these facts with the known theorem that the group of isomorphisms of every abelian

group is the direct product of the groups of isomorphisms of its Sylow subgroups there results the following theorem: *The number of the invariant operators of order* 2 *under the group of isomorphisms of an abelian group is* $2^k - 1$, *where* k *is the number of the distinct prime numbers which divide the order of the group when this order is either odd or when the group involves operators of order* 4 *but none of order* 8. *When this group involves operators of order* 8 k *is one larger than the numbers of these primes and when it involves operators of order* 2 *but none of order* 4, k *is one less than this number.*

2. NON-INVARIANT AUTOMORPHISMS OF ORDER 2. In every automorphism of order 2 of an abelian group G of odd order it is possible to find two subgroups such that every operator of one of them corresponds to itself while every operator of the other corresponds to its inverse and that G is the direct product of these two subgroups. It may be noted that when an abelian group is the direct product of two subgroups than either both of these subgroups are characteristic or neither of them has this property. This results directly from the fact that every characteristic proper subgroup of an abelian prime power group must include the fundamental characteristic subgroup thereof. From the same theorem it follows that an abelian prime power group cannot be the direct product of two or more of its characteristic proper subgroups. Hence it results that if an abelian group is the direct product of characteristic proper subgroups, then all of these subgroups are either Sylow subgroups or the direct products of Sylow subgroups of the entire group.

A necessary and sufficient condition that an automorphism of order 2 of G is invariant under the group of isomorphisms of G is that the two related subgroups are characteristic. Since both of these subgroups must be non-characteristic in a non-invariant automorphism of order 2 there must be some odd prime number p which divides both of their orders. Each of the sets of reduced independent generators of these two subgroups must therefore involve at least one operator whose order is divisible by p. If this operator is multiplied successively by the first p different powers of the operator of order p generated by an independent generator of the other set while the remaining independent generators of these two sets remain unchanged there result p different sets of independent generators of G. The generators of one of these two subgroups remain unchanged and hence this subgroup remains unchanged. In the case of the other subgroup the generators are changed and also the p subgroups are different as otherwise they would involve an operator of order p from the unchanged subgroup. It therefore results that *every complete set of conjugate operators of order* 2 *of the group of isomorphisms of an abelian group of odd order involves at least* p^2 *distinct operators if it involves more than one, where* p *is a prime divisor of the order of the group.*

When the order of G is of the form 2^m, there may be complete sets of conjugate operators of order 2 under the group of isomorphisms of G such that each of the sets involves only two elements. It is easy to prove that there are always at least two such sets whenever G has two and only two largest invariants, and these exceed 4. When this condition is satisfied G contains a characteristic subgroup H of index 4 composed of all its operators which are not of highest order. There are two automorphisms of order 2 of G in which H is composed of all the operators which correspond to themselves and in which no cyclic subgroup of highest order corresponds to itself. These therefore constitute a complete set of conjugates under the group of isomorphisms of G. A second such complete set results when H is composed of all the operators of G which correspond to their inverses under an automorphism of order 2 and hence the inverse commutator subgroup of G is its characteristic subgroup of order 4.

3. CYCLIC COMMUTATOR SUBGROUPS OF A GROUP OF ORDER 2^m. A necessary and sufficient condition that an automorphism is of order 2 is that every commutator which arises from this automorphism corresponds to its inverse thereunder. Hence it results that if G is any abelian group of order 2^m and if H is any subgroup of G which gives rise to a cyclic quotient group, then a necessary and sufficient condition that G admits an automorphism of order 2 in which H is composed of all its operators which correspond to themselves is that the co-set of G which corresponds to the square of an independent generator of this quotient group involves operators whose order is equal to that of this generator. The number of these operators is equal to the number of the automorphisms of order 2 of G in which H is composed of all the operators of G which correspond to themselves thereunder, since this number is equal to twice the number of the cyclic subgroups of this order generated by these operators and each such subgroup gives rise to two such automorphisms of order 2. The number of the automorphisms of order 2 of G which are characterized by the fact that H is composed of all the operators of G which correspond to themselves under each of them may also be found as follows: If the co-set which corresponds to the square of an independent generator of the quotient group involves no operator whose order is less than the order of this generator, then this number is equal to the number of the operators of H whose orders do not exceed the order of this quotient group. On the other hand, if this co-set involves operators whose order is less than the order of this quotient group, then the number of these automorphisms of order 2 is equal to the number of the operators of H whose order is equal to the order of the quotient group in question. These results follow directly from the fact that in the latter case an operator of lowest order in this co-set can be taken as an independent generator of the subgroup of G corresponding to a subgroup of index 2 of this quotient group, while in the former case any operator of this co-set whose order is equal to the order of this quotient

group can be used as such a generator. *It is always possible to choose a set of independent generators of G so that all except at most three of its elements appear in H.*

By means of the results of the two preceding paragraphs it is easy to determine the number of the automorphisms of order 2 of G which correspond to cyclic commutator subgroups. For this purpose it is only necessary to find successively all the subgroups of G which give rise to such cyclic quotient groups that the co-set corresponding to the square of an independent generator of this quotient group involves an operator whose order is equal to the order of this generator, and then to count the number of the corresponding automorphisms of order 2 by either of the two methods noted above. It also results from these paragraphs that an independent generator of the commutator subgroup corresponding to an automorphism of order 2 cannot be a fourth power of an operator of G when the order of this generator exceeds 2, and that when such a generator is a square of an operator of G then this operator must be transformed under this automorphism into a power of itself. It must therefore be then transformed either into its inverse or into its semi-inverse, the semi-inverse of an operator of order 2^{α} being its $2^{\alpha-1}-1$ power. *The independent generators of the commutator subgroup corresponding to an automorphism of order 2 of G must be either non-squares or squares of non-squares of G whenever their orders exceed 2.*

4. COMMUTATOR SUBGROUPS OF TYPE (1, 1, 1...) OF A GROUP OF ORDER 2^m. When the subgroup H composed of all the operators of an abelian group G of order 2^m which correspond to themselves under an automorphism of order 2 gives rise to an abelian quotient group of type (1, 1, 1...) then H must involve at least one subgroup which is simply isomorphic with this quotient group. The number of such automorphisms for a given H is equal to the product of the number of such subgroups contained in H and the order of the group of isomorphisms of one of these subgroups. By considering successively all the subgroups of G which give rise to such quotient groups we can obtain the total number of the possible automorphisms of order 2 of G which give rise only to commutators whose orders divide 2. It is, however, often more convenient to determine this number by another method which we proceed to explain.

Suppose that G has α invariants of which β exceed 2 and let H_0 be any subgroup of G which satisfies the following conditions: It is of type (1, 1, 1...), it includes the subgroup of order 2^{β} of G which is generated by its operators of order 2 which are squares, its order is 2^{γ}. We proceed to find the number of the automorphisms of order 2 of G giving rise to commutator subgroups of type (1, 1, 1...) and whose invariant operators of order 2 are those contained in H_0. This implies that H_0 contains at least one subgroup whose order is $2^{\alpha-\gamma}$. For each such subgroup we may establish as many automorphisms of order 2 of the subgroup of order 2^{α} generated by the operators of order 2 in G as the order of the group of isomorphisms of this subgroup. Moreover, every operator whose order exceeds 2 in a given set of independent generators of G may be transformed into itself multiplied by any one of the operators of H_0. When $\alpha > \gamma$ the number of these automorphisms of order 2 corresponding to a fixed H_0 is therefore equal to the product of the following three numbers: $2^{\beta\gamma}$, the number of the subgroups of order $2^{\alpha-\gamma}$ contained in H_0, the order of the group of isomorphisms of such a subgroup. When $\alpha = \gamma$ this product must be diminished by unity on account of the identity automorphism.

5. COMMUTATOR SUBGROUPS ARISING FROM AUTOMORPHISMS OF ORDER 2 OF AN ABELIAN GROUP OF ORDER 2^m. Let H be the subgroup composed of all the operators of an abelian group G of the order 2^m which correspond to themselves under an automorphism of order 2 of G and suppose that the operators of G are arranged in co-sets with respect to H in such a way that the commutators of highest order appear last. To a subgroup of index 2 of G there then corresponds a subgroup of index 2 of the commutator subgroup such that all the commutators which do not appear therein are of highest order. As the given subgroup of index 2 of G cannot involve all its operators of highest order it results that it is always possible to select the operators of a set of reduced independent generators of an abelian group and of the commutator subgroup arising from any automorphism of this group in such a way that an independent generator of highest order of the group gives rise to an independent generator of highest order of the related commutator subgroup.

While we *may* select any operator of highest order which does not appear in the given subgroup of index 2 of G as an operator of a set of reduced independent generators of G we *must* select such an operator s of lowest order to obtain such a set, and all of the remaining operators of the set may always be chosen from the operators of this subgroup. The operators of a set of reduced independent generators of the commutator subgroup arising from the said automorphism of order 2 of G may then be so chosen that s gives rise to an operator of highest order c in the set. The order of c can clearly not exceed the order of s and it may be equal to one-half of this order. Since c and s^{-2} appear in the same co-set of G with respect to H it results that *if s is an independent generator of lowest order of an abelian group of order 2^m which gives rise to a commutator c of highest order under an automorphism of order 2 then the order of c must be one of the following five numbers: the order of s, one half of this order, the order of a smaller independent generator of the group, twice this order, or 2.*

In every automorphism of order 2 of any abelian group G the operators which correspond to their inverses constitute a subgroup K of G which includes the commutator subgroup of G. A necessary and sufficient condition that G is the direct product of K and its subgroup H composed of the operators which correspond to themselves under this automorphism is that the order of G is odd. This is also a necessary and sufficient condition that G is the direct product of K and C. When the order of G is of the form 2^m the number of the invariants of K is the same as the number of the invariants of H and at least equal to the number of the invariants of C, and the group generated by H and C contains the square of all the operators of G and is of index 2^i under G, where i represents the number of the invariants of C. Hence the order of the cross-cut of C and H is 2^i. When either H or K is cyclic then the three subgroups H, K and C must be cyclic and their common cross-cut is of order 2. If this condition is satisfied and G is also decomposable into H and K there are $m/2 - 1$ or $(m - 1)/2$ such groups as m is even or odd.

[1] G. A. Miller, *Trans. Amer. Math. Soc.* **10**, 472, (1909).

ON THE UNIFIED FIELD THEORY. VI

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Let us suppose that the functions h^i_1 in group G_0 have the values δ^i_1 throughout a finite region R of the four dimensional continuum; these special values of the h^i_1 can obviously be imposed, at least throughout a sufficiently small region of the continuum, by a suitable coördinate transformation. A coördinate system for which the h^i_1 have the above special values will be referred to as a *cannonical coördinate system*. Interpreting the coordinate x^1 as the time t let us now assume that a disturbance is produced at a time $t = 0$ throughout a closed region R_0 of space.[1] After a time $t = \varphi(x^2, x^3, x^4)$, the effect of this disturbance will be felt at a point (x^2, x^3, x^4) outside the region R_0. For a definite value of t the equation $\varphi(x^2, x^3, x^4) = t$ will represent the surface of the space (x^2, x^3, x^4) which, at the instant t, separates the region affected from the region unaffected by the disturbance. If we assume that the values of the h^i_α and their derivatives vary in a continuous manner when we pass from one region to the other, then it follows from the result in sect. 8 of Note V under the hypothesis of canonical coördinates that the surface $t - \varphi(x^2, x^3, x^4) = 0$ must be a characteristic surface of the field equations. Wave surfaces are thus identified with the characteristic surfaces.

Now assume the expression for the element of distance in the region R in the form