CONCERNING THE SEPARATION OF POINT SETS BY CURVES

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It has been shown by Zoretti[1] that if K is a bounded maximal connected subset of a continuum M and e is a positive number then there exists a simple closed curve J which encloses K and contains no point of M and which has the further property that every one of its points is at a distance less than e from some point of K. It is to be observed that this does not imply that every point *within* J is at a distance less than e from some point of K. Indeed if K is, for example, a circle and e is less than its radius then there exists no J having this property. I have, however, at various times, had occasion to use the following theorem.

THEOREM 1. *If, in a plane S, M is a closed point set and K is a bounded maximal connected subset of M which does not separate S, then, for every positive number e, there exists a simple closed curve which encloses K and contains no point of M and which is such that every point within it is at a distance less than e from some point of K.*

Proof.—Let C denote some definite circle which encloses K, let r denote its radius and let d denote the shortest distance from C to K. For every positive integer n let T_n denote the set of all points $[X]$ such that X can be joined to some point of C by a simple continuous arc every point of which is at a distance equal to or greater than $d/2n$ from every point of K and at a distance less than or equal to r from the center of C. It is clear that, for every n, T_n is a bounded and connected point set and that T_n is a subset of T_{n+1}. It can easily be shown that for each n there exists a finite set of circles G_n all of radius equal to $d/3n$ and such that each point of T_n is within some one of them. Let T_n^* denote the point set obtained by adding together all the circles of the set G_n together with their interiors. Let J_n denote the boundary of that complementary domain of T_n^* which contains K. Clearly T_n^* is a continuum and J_n is a simple closed curve enclosing K. If e is any positive number there exists a positive number n such that every point within J_n is at a distance less than e from some point of K. For suppose there exists a positive number e for which this is not the case. Then for each n there exists, within J_n, a point P_n which is at a distance greater than or equal to e from every point of K. There exists a point P which is a sequential limit point of some subsequence of the sequence of points P_1, P_2, P_3, ... The point P is at a distance greater than or equal to e from every point of K. Since, by hypothesis, the closed point set K does not separate S therefore there exists a simple continuous arc which contains no point of K but which contains both P and a point of

C. Let h denote the least distance from this arc to K. Let k denote the smallest positive integer which is greater than $d/2h$. Then clearly P lies without J_k. Therefore, since P is a sequential limit point of some subsequence of $P_1, P_2, P_3, \ldots$, there exists an integer g greater than k such that P_g is without J_k. But P_g is within J_g and, since g is greater than k, the interior of J_g is a subset of the interior of J_k. Hence P_g is within J_k. Thus a contradiction has been obtained. Therefore it is true that for every e there exists a positive integer n such that every point of I_n, the interior of J_n, is at a distance less than e from some point of K. There exists a continuous one to one transformation T, with single valued inverse, which throws the point set I_n into the point set S. For any point set N which is a subset of I_n let $T(N)$ denote the image of N under this transformation. Let W denote the set of all those points of M which belong to I_n. Clearly $T(K)$ is a bounded maximal connected subset of $T(W)$. It follows, by the above mentioned theorem of Zoretti's, that there exists a simple closed curve Q which encloses $T(K)$ but contains no point of $T(W)$. There exists, within J_n, a simple closed curve J such that $T(J) = Q$. The curve J satisfies all the requirements indicated in the statement of Theorem 1.

THEOREM 2. *Suppose that, in a plane S, K and H are two closed connected and bounded point sets such that (a) neither K nor H separates the plane, (b) the set T of all points common to K and H is totally disconnected, (c) K-T is connected. Then there exists a simple closed curve which encloses K-T but encloses no point of H-T and contains T but no point of $(K + H) - T$.*

Proof.—Since K does not separate S it can easily be proved that there exists a ray r (of an open[2] curve) which has as its origin some point of H and which has no point in common with K. Since T is closed and bounded there exists a sequence of bounded point sets $D_1, D_2, D_3, \ldots$ such that (a) for every n, the set $S - D_n$ is closed and D'_{n+1} is a subset of D_n, (b) T is the set of points common to the point sets $D_1, D_2, D_3, \ldots$, (c) not all points of K belong to D'_1.[3] For each n let K_n denote the set of all those points of K which do not belong to D'_n. There exists a finite set G_1 of circular domains, all of radius less than 1, such that (a) every point of K'_1 belongs to some domain of the set G_1, (b) no point of $D'_2 + r + H$ is in, or on the boundary of, any domain of G_1. There exists a finite set G_2 of circular domains, all of radius less than $^1/_2$, such that (a) every point of $K'_2 - K_1$ is in some domain of G_2, (b) no point of $D'_3 + r + H$ is in, or on the boundary of, any domain of G_2. There exists a set G_3 of circular domains, all of radius less than $^1/_3$, such that (a) every point of $K'_3 - K_2$ is in some domain of G_3, (b) no point of $D'_4 + r + H$ or of $S - D_1$, is in or on the boundary of, any domain of G_3. This process may be continued. Thus there exists an infinite sequence $G_1, G_2, G_3, \ldots$ such that (a) for every n, G_n is a finite set of circular domains, all of radius less than $1/n$, (b) for every n, G_{n+2} covers $K'_{n+2}-K_{n+1}$ but does not cover, or have on the boundary of any one of its domains, any point of $D'_{n+3} + r + H$ or of $S-D_n$.

Let D denote the point set obtained by adding together the domains of all the sets $G_1, G_2, G_3, \ldots$ The point set D' is a continuous curve. For suppose first that P is a point of D' not belonging to T. There exists a circle C, with center at P, which does not contain or enclose any point of T. There exists a positive integer m such that if $n > m$ then no domain of the set G_n contains a point within C. Let Q_m denote the point set obtained by adding together the circular regions of the sets $G_1, G_2, G_3, \ldots G_m$ together with the circles bounding these regions. The point set Q_m is the sum of a finite number of closed point sets each of which is composed of a circle together with its interior. But a circle plus its interior is connected im kleinen[4] and if each of a finite number of point sets is connected im kleinen so is their sum. Hence Q_m is connected im kleinen at the point P. Since Q_m is a subset of D' and contains every point of D' that lies within C therefore D' is connected im kleinen at P.

Thus D' is connected im kleinen at every point of $D'-T$. But if it contained any irregular points then it would necessarily contain a continuum of irregular points and this continuum would be a subset of T contrary to the hypothesis that T is totally disconnected. With the aid of the fact that $K-T$ is connected it is easy to see that D is connected and that it is a domain. Let E denote the unbounded complementary domain of D'. Let β denote the boundary of E. Since D' is a continuous curve so is[5] β. Hence the outer boundary[5] of E with respect to D is a simple closed curve J. The curve J is a subset of D' and it encloses D. The point set H contains no point within J. For suppose it does. Then clearly $H-T$ contains a point U within J. It is clear that every point of J belongs either to T or to the circular boundary of a domain of one of the sets $G_1, G_2, G_3, \ldots$ But no one of these circles contains any point of $r + H$. It follows that J contains no point of $r + H-T$. But r contains points without J and it contains no point of T. Therefore, since it is connected, r is a subset of the exterior of J. But r contains a point of $H-T$. Thus there exists a point V which belongs to $H-T$ and lies in the exterior of J. Since the closed point set K does not separate S there exists a simple continuous arc UV which contains no point of K. This arc contains as a subset an arc ACB which has no point in common with K and which has in common with H only its end-points A and B which lie respectively within J and without J. Neither of the bounded continua H and ACB separates the plane and they have in common only the two points A and B. It follows by a theorem of Miss Mullikin's[6] that $S-(ACB + H)$ is the sum of two mutually separated domains. It is clear that one of these domains (call it I) is bounded and the other one (E) is unbounded. Let M_1 denote the set of all points $[X]$ such that X lies in some segment of ACB whose extremities are the extremities of a segment of J which lies wholly in I. If X is a point of M_1 then J contains[7] two points A_X and B_X lying in the order AA_XXB_XB on ACB and such that (*a*) there is a segment t_X of J which has A_X and B_X as extremities and which lies wholly in I, (*b*) no other segment of J lies wholly in I and has its end-points on the intervals AA_X and BB_X of the arc ACB. Thus with each point X of M_1 we have associated a definite segment t_X. Let M_2 denote the point set obtained by adding together all the segments t_X for all points X of M_1. Let t denote the point set $M_2 + (ACB - M_1)$. It can be seen that t is a simple continuous arc with extremities at A and B. The point set $t + H$ has just two complementary domains and one of these domains (call it R) is bounded. The domain R is a subset of I and no segment of J lies in R and has both of its end-points on t. Clearly A and B belong to the boundary on R. But A is within J and B is without J. It follows that R contains points within J and points without J. Hence, since it is connected, R contains at least one point of J. Let W denote such a point. Since J encloses the point A of the boundary of R and the boundary of R is connected therefore J contains points without R. Therefore W belongs to some interval q of the curve J which lies wholly in R, except for its end-points which belong to the boundary of R. Since the end-points of q do not both belong to t, one of them belongs to H. Hence q contains a sub-interval WP which lies wholly in R except for the point P which belongs to H. Let U denote the collection whose elements are the boundaries of the domains of the collections $G_1, G_2, G_3, \ldots$ Since every point of WP except P belongs to some circle of the set U, but the point P does not, therefore P is the sequential limit point of some sequence of distinct points $P_1, P_2, P_3, \ldots$ which lie on WP and no two of which lie on the same circle of the set U. But if e is a positive number there do not exist more than a finite number of circles of the set U of radius more than e. It follows that if, for each n, O_n denotes the center of a circle of the set U which contains P_n then the distance from O_n to P_n approaches zero as a limit as n approaches infinity. Since P does not belong to t there exists a circle C with center at P which encloses no point of t. There exists an integer n such that O_n and P_n are both within C. Since O_n and P_n are, respectively, within and on some common circle of the set U there is no point of H between them. Since they are within C there is no point of t between them. Thus there is no point of the boundary of R between them. But P_n belongs to R. Therefore so does O_n. But O_n belongs to $K - T$. Thus $K - T$ contains a point of R. By a similar argument, with the assistance of an inversion of the plane about a circle with center at some point of R, it may be shown that $K - T$ also contains a point of the unbounded complementary domain of $t + H$. But, by hypothesis, $K - T$ is connected. Hence it contains a point of $t + H$. But this is impossible. Thus the supposition that J encloses a point of H has led to a contradiction. In particular there is no point of T within J. Since $K - T$ is within J and every point of T is a limit point of $K - T$ therefore there is no point of T without J. Hence T is a subset of J. The truth of Theorem 2 is therefore established.

THEOREM 3. *If T is a totally disconnected closed subset of the boundary of a simply connected domain D and there exists a continuum K containing T and such that $K - T$ is a subset of D then there exists a simple closed curve J containing T and enclosing $K - T$ and such that $J - T$ is a subset of D.*

Theorem 3 may be established with the aid of Theorem 2.

THEOREM 4. *If P is a point of the boundary of the simply connected domain D and there exists a continuum K which contains P but lies, except for P, wholly in D then P is accessible from D.*

Theorem 4 is a corollary of Theorem 3.

THEOREM 5. *If H is a countable collection of (two or more) mutually exclusive unbounded continua and no one of them separates the plane and their sum N is a closed point set and α and β are two continua of the collection H and there exists a simple continuous arc which contains a point of α but no other point of N and there also exists an arc which contains a point of β but no other point of N, then there exists an open curve which separates α from β and contains no point of N.*

Proof.—Since N is not[8] connected there exists a point O which does not belong to it. Let T denote an inversion of the plane S about some circle with center at O. Let G denote the set of all continua g such that, for some continuum h of the set H, $g = T(h) + O$. Let M denote the sum of all the continua of the set G. Let C denote a circle enclosing M. An element x of G will be said to be of class 1 if there exists a simple continuous arc AB whose end-points A and B belong to $x - O$ and C respectively and which has only the point A in common with M. If a and c are two G-elements[9] of class 1 and b and d are two other G-elements (whether of class 1 or not) then a and c are said to ordinally separate b and d under certain conditions described on Pages 194 and 195 of S.C. Let a and b denote the point sets $T(\alpha) + O$ and $T(\beta) + O$ respectively. By a theorem of Miss Mullikin's,[6] N does not separate S. Hence M does not separate S. From this and the hypothesis it follows that a and b are of class 1. If there are only a finite number of continua in the set G the truth of Theorem 5 can be easily established with the aid of Theorem 2. Let us suppose that there are infinitely many continua in G. Then it is easy to see that there exists an element c of class 1 and distinct from a and from b. There are two possible cases.

Case 1. Suppose there exists no G-element which is separated from c by a and b. If x and y are two G-elements, distinct from each other and from a and b, and x is of class 1 then y is separated either from a by x and

b or from b by x and a. In the first case y will be said to *follow* x. In the second case it will be said to *precede* x. If x, y and z are G-elements the statement that z is *between* x and y means that (a) x and y are both of class 1, (b) z either follows x and precedes y or follows y and precedes x. If x and y are two distinct G-elements of class 1 then by the segment xy is meant the collection of all those G-elements which are between x and y while by the *interval* xy is meant the collection consisting of x and y and all the G-elements of the segment xy. With the aid of results established in the course of the proof of Theorem 10 of S.C., it may be seen that either (a) there exist two distinct G-elements e and f, of class 1, such that e precedes f and such that there is no G-element between them, or (b) there exist two infinite sequences $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$ of G-elements, of class 1, such that, for every m and n, (1) y_m precedes z_n, (2) y_n precedes, or is identical with, y_{n+1} and z_{n+1} either precedes, or is identical with, z_n, (3) there exists no G-element which is common to all the segments y_1z_1, $y_2z_2, \ldots$ In Case 1(a) let K denote the point set obtained by adding together a and e and all those point sets of G (if there are any) which precede e and let H denote the point set obtained by adding together f and b and all those point sets of G (if there are any) which follow f. In Case 1(b) let K denote the point set obtained by adding together all the point sets of the sequence $a, y_1, y_2, y_3 \ldots$ together with all the point sets g of the collection G such that g precedes some point set of this sequence and let H denote the point set obtained by adding together all the point sets of the sequence $b, z_1, z_2, z_3, \ldots$ together with all point sets g of the collection G such that g follows some point set of this sequence. In either case H and K are bounded continua with only the point O in common and $H + K = M$. Clearly there exists a ray r (of an open curve) which has, as its origin, some point of $b - O$ and which does not have any other point in common with $H + K$. The argument given in the first paragraph of the preceding proof of Theorem 2 may now be applied without modification. As in the beginning of the second paragraph of that proof, let D denote the point set obtained by adding together the domains of all the sets $G_1, G_2, G_3, \ldots$ Now let L denote the greatest connected subset of D which contains the connected point set $a - O$. The point set L is a domain. By an argument similar to that employed, in the proof of Theorem 2, to show that D' was a continuous curve, it may be shown that L' is a continuous curve. Let E denote the unbounded complementary domain of L'. The outer boundary of E with respect to L is a simple closed curve J. The point set $a - O$ lies within J. But O is a limit point of $a - O$. Hence O is either on or within J. But, since it is unbounded, the ray r contains points without J, and $r + b - O$ is a connected point set which contains no point of J. Hence $r + b - O$ is wholly without J. But O is a limit point of $r + b - O$. Hence O is not within J. Thus the simple closed curve J contains O and encloses $a-O$ but encloses no point of b and contains no point of $M-O$. The point set $T^{-1}(J)$ is an open curve which separates α from β and contains no point of N.

In Case 2 there exists a G-element which is separated from c by a and b. This case may be treated with the help of methods used in Case 1.

[1] Zoretti, L., "Sur les fonctions analytiques uniformes," *J. Math. pures appl.*, **1,** 1905 (9–11).

[2] An open curve is a point set which is in one to one reciprocal continuous correspondence with a straight line. For a point set theoretic definition, see my paper "On the foundations of plane analysis situs," *Trans. Amer. Math. Soc.*, **17,** 1916 (131–164).

[3] If M is a set of points, M' denotes the set of all those points which are limit points of M.

[4] The point set M is said to be connected im kleinen at the point P if, for every positive number e, there exists a positive number d such that every point of M at a distance from P less than d lies in a connected subset of M which contains P and is of diameter less than e. An *irregular point* of M is a point of M at which M is not connected im kleinen. Cf. Hahn, H., *Jahresber. D. Math. Ver.*, **23,** 1914 (318–322). Also Mazurkiewicz, S., *Fund. Math.*, **1,** 1920 (166–209) and, possibly, earlier papers, in Polish, referred to therein. Also Nalli, P., *Rend. Circ. Mat., Palermo*, **32,** 1911 (391–401).

[5] That the boundary of a complementary domain of a continuous curve is itself a continuous curve is established, with the aid of results due to Miss Torhorst and Schoenflies, on Page 259 of my paper "Concerning continuous curves in the plane," *Math. Zeitschrift*, **15,** 1922 (254–260). Mazurkiewicz establishes the same result in a similar manner in his paper "Sur les continus homogenes," *Fund. Math.*, **5,** 1924 (137). If D and E are two mutually exclusive domains and the boundary of E is a subset of the boundary of D and the point set K is a subset of E then the boundary of E will be called the *outer boundary of D relative to K*. In my paper "Concerning continuous curves in the plane," *Math. Zeitschrift*, **15,** 1922 (254–260), I showed that if the boundary of D is a continuous curve then, in case E is unbounded, the outer boundary of D with respect to E (in this case called merely the outer boundary of D) is a simple closed curve. By an inversion of the plane about a circle with center at some point of E the case where E is bounded may be reduced to the case where it is unbounded. Thus in every case the outer boundary of D with respect to E is a simple closed curve. In his paper "Sur les coupures irreductibles du plan," *Fund. Math.*, **6,** 1924 (130–145), Kuratowski calls attention to the fact that in case neither of the points A and B, mentioned in Theorem 5 of my above mentioned *Zeitschrift* paper, lies in the unbounded complementary domain of β then the outer boundary of R may not separate A from B. However, A is separated from B by the outer boundary of R relative to that complementary domain of the boundary of R which contains that one of the points A and B which does not lie in R. Thus my argument applies to the case where one of the points A and B lies in an unbounded complementary domain of the boundary of R and every other case may be reduced to this one by an inversion of the plane.

[6] *Trans. Amer. Math.*, **24,** 1922 (144–162).

[7] Cf. Theorem 37 of my paper "On the foundations of plane analysis situs," *loc. cit.*

[8] Sometime in 1923 I proved that no unbounded continuum is the sum of a countable number of (two or more) mutually exclusive continua. This result was announced at the Summer meeting of the American Mathematical Society, September 6, 1923. The same result has been established by S. Mazurkiewicz in an article in vol. 5 of *Fund. Math.* This volume bears the date 1924, but a reprint of the article left the press before the appearance of the entire volume, just how long before I do not know. I submitted

my proof for publication in that journal, the manuscript being mailed September 28, 1923. Sometime in November I received the reprint of the article by Professor Mazurkiewicz, who is one of the editors. My own paper appeared in vol. 6, 1924 (189–202). It will be referred to hereafter as S. C.

[9] By a G-element is meant a continuum which is an element of the set G.

THE SOIL POPULATION[1]

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Microörganisms causing plant and animal diseases occur, in most instances, in pure culture, or in such great abundance over contaminating forms that their rôle in a particular infection can be established with comparative ease. The same is true to a large extent of the so-called industrial fermentations, which are carried out by pure cultures and where the presence of other organisms usually has a deleterious effect. Microörganisms concerned in the dairy industry, in the preparation of silage, in bread making, preparation of beverages, oriental food products, etc., may not occur and act in pure culture, but the transformations brought about are comparatively so simple that they can readily be differentiated and the rôle of specific organisms in each can be readily established. As a matter of fact, the use of pure cultures in many of these processes presents definite advantages and has helped greatly in advancing the process in question. The contaminating organisms often destroy the desired product or produce undesirable substances.

However, the microörganisms concerned in soil processes and in sewage purification do not occur in pure culture; they bring about various complex (as well as simple) reactions, and usually act in associations, one organism rapidly utilizing the products of another, thus stimulating it to further action. A soil organism isolated in pure culture may often bring about only a semblance of a certain specific reaction; in the soil, however, the same organism will carry on the particular reaction very vigorously. This is particularly true of certain cellulose decomposing bacteria, which are many times more active in the soil or when accompanied by various other non-cellulose decomposing forms than in pure culture. This is due largely to three factors: (1) the soil is a medium highly complex in composition and no artificially prepared culture medium can approach it, in the various physical, chemical, and physico-chemical problems which it represents; (2) the organism does not act in the soil in pure culture and often the