Mica:	$M^- K^+ M^- K^+ M^- K^+ M^- K^+$
Intermediate:	$M^- K^+ M^- B^+ M^- K^+ M^- B^+$
Chlorite:	$M^- B^+ M^- B^+ M^- B^+ M^- B^+$

indicates how intermediates between mica and chlorite might be formed. Giving M^- and B^+ the representative compositions $[Mg_3AlSi_3O_{10}(OH)_2]^-$ and $[Mg_2Al(OH)_6]^+$, the intermediate shown would have the composition $KMg_8Al_3Si_6O_{20}(OH)_{10}$, and would have an effective interplanar distance on (001) of 24.3 Å., just the sum of those for mica and chlorite. Chlorite-brucite intermediates would probably be unstable because of the juxtaposition of two positively charged brucite layers.

A detailed account of this investigation will be published in the *Zeitschrift für Kristallographie.*

[1] Linus Pauling, These PROCEEDINGS, 16, 123–129 (1930).
[2] S. B. Hendricks, *Zeit. Kristallographie*, 71, 269 (1929).

DISCUSSION OF VARIOUS TREATMENTS WHICH HAVE BEEN GIVEN TO THE NON-STATIC LINE ELEMENT FOR THE UNIVERSE

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§1. *Introduction.*—In several previous articles,[1,2,3] I have shown the possibility of deriving a non-static cosmological line element to agree with the relatively uniform distribution of matter observationally found in the universe, and of using this line element to account both for the annihilation of matter taking place throughout the universe, and for the red-shift in the light from the extra-galactic nebulæ. After the publication of the first of these articles, I learned from a conversation with Professor H. P. Robertson that he had previously published[4] a different derivation of the same general form of line element; more recently Professor de Sitter and Sir Arthur Eddington have both been kind enough to call my attention to the previous use of a non-static line element by Lemaitre,[5] and still more recently I have discovered the early treatment of non-static line elements by Friedman.[6]

The purpose of the present article is to discuss certain features in this earlier work which overlap the treatment which I have given. Some remarks will also be made concerning an article by Eddington[7] which appeared after the publication of my first article, and one by de Sitter[8] which appeared just after the publication of my second article, both of these having come to hand while the present note was in course of preparation.

§2. *The General Form of the Line Element.*—Employing the same symbolism which I have previously used, the non-static cosmological line element can be written in the general form

$$ds^2 = -\frac{e^{g(t)}}{[1 + r^2/4R^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2, \quad (1)$$

where R is a constant, and the dependence on the time arises from the function $g(t)$. There is an entire equivalence between this line element and those of Robertson, Lemaitre and Friedman. The translation of results between the different systems of symbols can be made with the help of the following equations in which the symbols on the left hand side correspond to equation (1) and those on the right hand side to the author indicated.

$$\left.\begin{aligned} R &= R \\ g(t) &= 2f(t) \\ \frac{r}{1 + r^2/4R^2} &= r \end{aligned}\right\} \begin{array}{l}\text{Robertson}^9 \\ \\ \text{Equation 10}\end{array} \quad (2)$$

$$\left.\begin{aligned} Re^{g/2} &= R \\ \frac{1}{[1 + r^2/4R^2]^2}\left(\frac{dr^2}{R^2} + \frac{r^2}{R^2}d\theta^2 + \frac{r^2}{R^2}\sin^2\theta d\phi^2\right) &= d\sigma^2 \end{aligned}\right\} \begin{array}{l}\text{Lemaitre} \\ \text{Equation 1}\end{array} \quad (3)$$

$$\left.\begin{aligned} Re^{g/2} &= R(x_4)/c \\ \frac{r/R}{1 + r^2/4R^2} &= \sin x_1 \end{aligned}\right\} \begin{array}{l}\text{Friedman} \\ \text{Equation } D_3\end{array} \quad (4)$$

The derivation which I gave to this line element was based on five assumptions. Four[10] of these assumptions could be regarded as direct consequences of the approximately uniform distribution of material in the universe observationally found to those depths to which the Mount Wilson instruments have been able to penetrate, and the fifth[11] was the stability requirement that nebulæ which are stationary *in the system of coordinates used* should not be subject to acceleration but should remain permanently stationary.

The derivation of Robertson was based on two assumptions, first that space-time should be separable into space and time, and second that it should be spatially homogeneous and isotropic. The latter of these assumptions is closely related to my four assumptions that flow from the observational approximately uniform distribution of material in the universe, and the former has the same effect on the form of the line element as my stability requirement. There is thus a close relation between the starting points of Robertson and myself. I should wish to emphasize, however, that it would seem to me premature if we should assert at present that the whole of space-time is necessarily even approximately spatially homogeneous. I prefer to regard the line element (1) as an approximate expression in agreement with the observational uniform distribution of matter out to the range of the 100-inch telescope, but one which could be reasonably modified to take account of different conditions if they should be found at greater distances.

Lemaitre presents no derivation of the line element used, and the derivation given by Friedman appears less satisfying than those given by Robertson and myself, especially as he introduces at the very start an assumption (C) which would rule out the presence of radiation in the universe.

§3. *Treatment of the Red Shift by Lemaitre.*—The non-static line element under discussion was not applied to observational data by Robertson or Friedman, but was used by Lemaitre to account for the red shift in the light from extra-galactic nebulæ. In doing this he introduced two assumptions—first that the proper pressure in the universe should be permanently zero, and second that the cosmological constant occurring in Einstein's equations should have a particular value that would make the minimum proper volume of the universe occur at the time minus infinity. In the present section we shall discuss the results of these assumptions, and then in the following section compare the treatment of Lemaitre with treatments which I have tried.

a. Effect of Zero Pressure on Presence of Radiation.—The assumption of zero proper pressure at once excludes the presence of radiation in the universe, for which the pressure is one-third the energy density. Hence this assumption presumably does not agree precisely with the facts, since the universe appears to have had a long past history in which radiation has been emitted from the heavenly bodies. Moreover, it is interesting to note in this connection that the temperature of intergalactic space would only have to be of the order of one degree absolute to give a density of radiation equal to Hubble's value for the averaged out density of matter in the universe, 1.5×10^{-31} gm./cc. Nevertheless, we have no easy means of estimating the actual pressure, and can make no precise quantitative judgment as to the degree of correctness of Lemaitre's assumption on this point.

b. Effect of Zero Pressure on the Annihilation of Matter.—The assumption that the pressure in the universe is zero and hence that radiation is permanently excluded would also rule out the annihilation of matter, that is, its transformation into radiation. This can be seen from my general equation[12] for the fractional rate of annihilation of matter

$$-\frac{1}{M}\frac{dM}{dt} = 6\frac{p_0}{\rho_0}\dot{g} + \frac{3}{\rho_0}\frac{dp_0}{dt}, \tag{5}$$

where p_0 is the proper pressure in the universe, ρ_0 the proper density of matter, and $\dot{g} = dg/dt$. With the pressure p_0 permanently zero, both terms on the right hand side of the equation (5) are zero and the amount of matter in the universe would be a constant.

This result also does not appear to be in precise agreement with the facts, which seem to indicate a continuous annihilation of matter in order to maintain the radiation from the stars. In addition, we have some indications that an annihilation of matter in intergalactic dust may be associated with the production of the cosmic rays.

Moreover, as I have previously pointed out the annihilation of matter has the effect of making the line element of the universe necessarily non-static and hence necessarily leading to a shift in the wave length of light from distant objects, and, as I also showed, a red rather than a violet shift if we make use of the additional assumption that the proper motions of the nebulæ are tending to decrease rather than increase with the time.[13]

There are thus certain further arguments against Lemaitre's assumption of zero pressure since it would lead to a zero rate of annihilation of matter. Nevertheless, since we have no satisfactory theory of the enormous variation with stellar mass in the rates of annihilation actually observed in the stars, and have no satisfactory knowledge as to the possible rate of annihilation of matter in intergalactic dust, it is not possible to decide exactly what the actual figure for the average rate of annihilation should be, or to make any precise quantitative judgment of Lemaitre's hypothesis from this point of view.

c. Effect of Zero Pressure on the Form of the Line Element.—The assumption of zero pressure prescribes a definite dependence of the line element on the time. In accordance with the principles of relativistic mechanics the proper pressure corresponding to the line element (1) is given by the equation[14]

$$8\pi p_0 = -\frac{1}{R^2}e^{-g} - \ddot{g} - \frac{3}{4}\dot{g}^2 + \Lambda, \tag{6}$$

where Λ is the cosmological constant and the dots indicate differentiation with respect to the time. Setting this expression equal to zero we can obtain as a first integral

$$\dot{g} = \sqrt{\frac{4\Lambda}{3} - \frac{4}{R^2}e^{-g} + Ae^{\frac{-3g}{2}}} \qquad (7)$$

where A is a constant of integration, and can then write the complete integral in the general form

$$t = \int \frac{dg}{\sqrt{\frac{4\Lambda}{3} - \frac{4}{R^2}e^{-g} + Ae^{-\frac{3g}{2}}}}. \qquad (8)$$

d. Effect of Lemaitre's Second Assumption on the Form of the Line Element.—If we now introduce Lemaitre's second assumption, as to the magnitude of Λ, the indicated integral in equation (8) can be explicitly evaluated. This second assumption is given by the equation

$$A = \frac{8}{3R^3\,\Lambda^{1/2}}. \qquad (9)$$

Substituting in (8), and for simplicity of expression, writing in agreement with the notation of Lemaitre

$$\Lambda = \frac{1}{R_0{}^2} \qquad (10)$$

we can obtain[15]

$$t = R_0\sqrt{3}\log\frac{\sqrt{Re^{g/2} + 2R_0} + \sqrt{Re^{g/2}}}{\sqrt{Re^{g/2} + 2R_0} - \sqrt{Re^{g/2}}} + R_0\log\frac{\sqrt{3Re^{g/2}} - \sqrt{Re^{g/2} + 2R_0}}{\sqrt{3Re^{g/2}} + \sqrt{Re^{g/2} + 2R_0}} + \text{const.} \qquad (11)$$

as an explicit expression for the dependence of g on t.

In accordance with this result the value of $Re^{g/2}$ could be regarded as increasing continuously from a minimum value R_0 with which it started at a time infinitely remote in the past, thus providing the long period for past stellar evolution desired by Lemaitre.[16]

e. Effect of Form of Line Element on Relation between Red Shift and Distance.—As shown in my previous work a linear relation between red shift and distance is to be obtained by taking a linear dependence of g on t. Hence the question naturally arises whether such a complicated relation between g and t as that given by Equation (11) would give a dependence of red shift on distance nearly enough linear to agree with the actual observations of Hubble and Humason on the red shift in the light from the nebulæ. This, however, proves to be the case.

As an exact expression for the dependence of fractional red-shift on the coordinate r we have[17]

$$\frac{d}{dr}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{1}{1 + r^2/4R^2}\,\frac{\dot{g}}{2}, \tag{12}$$

where $\dot{g}$ is the value of dg/dt at the time the light was emitted. With sufficient approximation for our present purposes this can be written as

$$\frac{d}{dL}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{\dot{g}}{2}, \tag{13}$$

where L is to be taken as agreeing with the observational distances of Hubble, and with g linear in t this gives us the desired linear dependence of red shift on distance.

As the observational value for the relation between red shift and distance Lemaitre (Equation 24) takes

$$\frac{d}{dL}\left(\frac{\delta\lambda}{\lambda}\right) = \frac{\dot{g}}{2} = 6.4 \times 10^{-10}\ (\text{years})^{-1}. \tag{14}$$

To estimate the deviation from linearity involved in the treatment of Lemaitre, we may return to the expression for $\dot{g}$ given by equation (7) and by differentiating with respect to the time and substituting equations (9) and (10) obtain

$$\frac{\ddot{g}}{2} = \frac{1}{R^2 e^g} - \frac{R_0}{R^3 e^{3g/2}}. \tag{15}$$

With the values of $Re^{g/2}$ and R_0 of Lemaitre this gives us

$$\frac{\ddot{g}}{2} \approx 3 \times 10^{-21}\ (\text{years})^{-2}. \tag{16}$$

On the other hand the most remote nebula whose red shift has so far been determined is at a distance of approximately 10^8 light years, so that the change in $\dot{g}/2$ or in the slope of $\delta\lambda/\lambda$ as a function of L in going out to the most distant nebula observed would be of the order

$$\Delta\left[\frac{d}{dL}\left(\frac{\delta\lambda}{\lambda}\right)\right] \approx 3 \times 10^{-13}\ (\text{years})^{-1}, \tag{17}$$

which for the present is negligible compared with the observational value 6.4×10^{-10} (years)$^{-1}$. This is in agreement with the fact specially pointed out in my second article that it is possible to assign a small positive value to $\ddot{g}$ which will be large enough to reduce the rate of annihilation of matter to any desired extent below that calculated from a linear dependence of g on t, and yet small enough not to affect seriously the linear dependence of red shift on distance.

§4. *Comparison of Treatments by Lemaitre and the Author.*—It will now be interesting to compare the treatment of the non-static line element by Lemaitre with the treatments given in my first and second papers and certain other possibilities. This can be done with the help of the following table which presents some of the significant features of these different treatments.

CASE	PRESSURE	FORM OF g	ANNIHILATION	RED SHIFT
L	$p_0 = 0$	Equation 11	$-\frac{1}{M}\frac{dM}{dt} = 0$	$\frac{\delta\lambda}{\lambda} \backsimeq kL$
T_1	$p_0 \ll \rho_0$	$g = 2kt$	$-\frac{1}{M}\frac{dM}{dt} = 3k$	$\frac{\delta\lambda}{\lambda} = kL$
T_2	$p_0 \ll \rho_0$	$g = 2(kt + lt^2 + mt^3\ldots)$	$-\frac{1}{M}\frac{dM}{dt} < 3k$	$\frac{\delta\lambda}{\lambda} \backsimeq kL$
T_3	$p_0 \gg \rho_0$	$g = 2kt$	$-\frac{1}{M}\frac{dM}{dt} \gg 3k$	$\frac{\delta\lambda}{\lambda} = kL$
T_4	$p_0 \ll \rho_0$	$e^g = (1 + kt)^2$	$-\frac{1}{M}\frac{dM}{dt} = 3k$	$\frac{\delta\lambda}{\lambda} \backsimeq kL$
T_5	$p_0 =$ const.	Equation 11	$-\frac{1}{M}\frac{dM}{dt} = \frac{12p_0}{\rho_0}k$	$\frac{\delta\lambda}{\lambda} \backsimeq kL$

The first column in this table gives a symbol by which the case can be designated. The second and third columns give the assumptions which are made as to the pressure in the universe and the form of dependence of the line element on the time, some assumption on these points being necessary in order to permit any application at all. The last two columns give the resulting value for the calculated rate of annihilation of matter, and the resulting dependence of the red shift on distance.

Case L is that of Lemaitre. It has the disadvantage of the very complicated dependence of g on t, given by equation (11), which still remains complicated in the symbolism of Lemaitre. It also suffers from the assumptions of absolutely zero pressure in the universe and absolutely zero rate of annihilation of matter as discussed above. Nevertheless our actual knowledge as to pressure and rate of annihilation is insufficient to rule it out as possibly being a good approximation.

Case T_1 is the one presented in my first paper, and the first of those discussed in my second paper. It has the advantage of the simplest possible dependence of g on t which leads immediately to a linear dependence of red shift on distance. It also has the possible advantage that it makes the logarithmic rate of annihilation of matter a constant so long as $p_0 \ll \rho_0$. Such a logarithmic rate of decay would seem theoretically reasonable except for the actual complications found in the rate of decay of matter in stars of different masses. The case in question gives a rate of annihilation of matter as high as that observationally found in the most massive stars, far higher than that in a typical star such as the sun. If, however, it should be found that the cosmic rays arise from an exceedingly high rate of annihilation of intergalactic dust, this case might be an approximately correct description of the present state of the universe.

Case T_2 is the second of those discussed in my second paper, and gives perhaps the most satisfactory treatment of our present knowledge of the state of the universe now possible. The pressure is still taken small compared with the density of matter, but the dependence of g on t is modified by the possibility of terms in t of higher order than the first. In particular it was shown in my second paper that it was possible to assign a small positive value to the coefficient l in the expression $g = 2(kt + lt^2)$, which would be big enough to reduce the calculated value for the present rate of annihilation of matter as far as desired, and yet at the same time small enough not to affect appreciably the linear relation between red shift and distance out to a depth of 10^8 light years, which is as far as the observations of Hubble and Humason have yet been pushed.

Case T_3, which was also considered in my second paper, is interesting in illustrating the effect of assuming that the density of radiation in the universe is large compared with the averaged out density of matter. It gives an exceedingly high rate of annihilation of matter because of the great effect (with $p_0 \gg \rho_0$) of the first term on the right-hand side of the equation previously cited

$$-\frac{1}{M}\frac{dM}{dt} = \frac{6p_0}{\rho_0}\dot{g} + \frac{3}{\rho_0}\frac{dp_0}{dt}. \tag{18}$$

In general, the assumption of a pressure in the universe large compared with the density of matter would seem difficult.

Case T_4, which has not previously been published, is interesting in providing another very simple dependence on the time which gives approximately the same results as Case T_1.

Case T_5, which has also not been previously published, is of interest in taking the pressure in the universe as constant. This might have an approximate validity since there are some processes (emission of radiation) which tend to increase the pressure and others (expansion of the universe) which tend to decrease it. With p_0 constant we can take the relation between g and t as that presented by equation (11), given above for the Lemaitre theory, with the change that corresponding to equations (9) and (10) we should have instead

$$A = \frac{8}{3R^3\sqrt{\Lambda - 8\pi p_0}} \quad \text{and} \quad \Lambda - 8\pi p_0 = \frac{1}{R_0^2} \tag{19}$$

§5. *Remarks Concerning the Articles of Eddington and de Sitter.*—After the publication of my first paper, an article appeared by Eddington (loc. cit.) making use of the non-static line element to discuss the stability of the Einstein universe. In this article he has incidentally made two criticisms of my first paper with which I cannot entirely agree.

Referring perhaps to the title of my paper, "The Effect of the Annihilation of Matter on the Wave-Length of Light from the Nebulæ," Eddington says that the conversion of matter into radiation cannot be regarded as an explanation of the recession of the nebulæ, since it tends to retard the expansion of the universe.

I am quite willing to agree that the title of my paper was in some ways unfortunately chosen, since in the case of a number of mathematically interrelated phenomena, it is dangerous to label one of them cause and the rest effects, and since as shown by Lemaitre the non-static line element can also be applied to an expanding universe without introducing any annihilation of matter.

I should wish to reëmphasize, however, that the annihilation of matter necessarily leads to a non-static line element, and to a universe in which there is a red rather than a violet shift if we are willing to exclude the possibility that the proper motions of the nebulæ tend to increase with the time. I should also wish to emphasize that the annihilation of matter must certainly be accompanied either by an expansion of the universe or by a rise in pressure. This is very clearly shown by the equation already cited

$$-\frac{1}{M}\frac{dM}{dt} = \frac{6p_0}{\rho_0}\dot{g} + \frac{3}{\rho_0}\frac{dp_0}{dt}, \tag{20}$$

which makes it evident that a positive value for the fractional rate of annihilation $(-1/M)(dM/dt)$ must be accompanied either by an expanding universe, i.e., $\dot{g}$ positive, or by a rising pressure, i.e., dp_0/dt positive.

The second criticism which Eddington made was based on the fact that in my first paper I assumed a simple linear formula $g = 2kt$ to express the dependence of the line element on the time. It was definitely stated in the article, however, that this was to be regarded as a first approximation, and since then I have discussed the introduction of higher terms in my second paper. It may be remarked, moreover, that as far as the motions of the nebulæ are concerned the simple linear formula appears to be a good approximation out to 10^8 light years. It cannot be stated, however, that it is necessarily a good approximation for the calculation of rates of annihilation.

Shortly after the publication of my second paper, an interesting treatment of the non-static line element for the universe was published by de Sitter (loc. cit.). In this treatment he first considers the forms of the line element that would correspond to a universe with no annihilation of matter, but not necessarily with zero pressure. The different forms considered correspond to different choices of the parameters. The forms are very complicated, that of Lemaitre as given by equation (11) being a specially simple case. De Sitter then considers the possibility of allowing for the annihilation of matter by permitting one of the parameters in his previous equations to vary with the time in such a way as to give a rate of annihilation of the order of that found in a dwarf star of a somewhat later type than the sun or even less. In this connection he remarks in a footnote on the high rate of annihilation given by the treatment in my first paper, but by this time my second paper had already been published showing the possibility of obtaining as low a rate of annihilation as desired by introducing terms of higher order than the first in the expression for the dependence of g on t.

From a practical point of view the treatment of de Sitter suffers from the great complexity in the forms of line element obtained, and from a theoretical point of view in the fact that they correspond to a universe in which no annihilation of matter takes place, and the allowance for annihilation is made as a later correction without inclusion in the explicit form of the line element. Furthermore, although his forms of line element apparently give the whole history of the expansion of the universe, he himself emphasizes that he does not wish to lay stress on the implications for the distant past or future, and determines the parameters for his different forms from considerations of the present density and rate of expansion of the universe. Hence in some ways, I prefer the treatment which had already been published in my second article, Case T_2 of the table, in which we assume a simple expansion in powers of t, which only attempts to represent the dependence of the line element on the time in the neighborhood of the present, and in which as many parameters as possible are to be determined from our present knowledge of density and pressure, rate of expansion, *and* rate of annihilation.

NOTE: Another article by de Sitter (*Proc. Nat. Acad.*, **16**, 474, 1930) has arrived just as this paper was to be sent to the Editor. It contains, however, no further material which needs comment from the present point of view.

§6. *Conclusion.*—In conclusion it would seem as if our present knowledge of the universe could best be treated by the methods of my second article in accordance with the Case T_2 of the table, by taking the cosmological line element in the form

$$ds^2 = -\frac{e^{2(kt+lt^2)}}{[1 + r^2/4R^2]^2}(dx^2 + dy^2 + dz^2) + dt^2, \qquad (21)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and R, k and l are constant parameters.

As I have shown this line element corresponds to a uniform distribution of material in the universe, and nebulæ stationary in the coördinates x, y, z will remain so. Furthermore, in the case of an expanding universe, with $(kt + lt^2)$ positive and the accompanying red shift from distant objects, the proper motions of the nebulæ would tend to decrease with the time.

At the time $t = 0$ (i.e., the present) the density of matter ρ_0 and the pressure p_0 would be related to the parameters by[18]

$$4\pi(\rho_0 + 4p_0) = \frac{1}{R^2} - 2l. \tag{22}$$

The dependence of the red shift on distance would be approximately linear provided l is sufficiently small[19]

$$\frac{\delta\lambda}{\lambda} \approx kL. \tag{23}$$

And the rate of annihilation of matter would be given by[20]

$$-\frac{1}{M}\frac{dM}{dt} = 12\frac{p_0}{\rho_0}k + \left(1 + \frac{4p_0}{\rho_0}\right)\frac{\dfrac{3}{R_2} - 18l}{\dfrac{1}{R^2} - 2l}k. \tag{24}$$

We thus have three equations to determine the three parameters R, k and l in terms of physically measurable quantities. Taking the pressure p_0 small compared with the density of matter ρ_0, and using Hubble's estimate for the average density of matter in the universe we have

$$\frac{1}{R^2} - 2l \approx 10^{-22}\,(\text{years})^{-2}; \tag{25}$$

from the red shift we have

$$k = 5.1 \times 10^{-10}\,(\text{years})^{-1}, \tag{26}$$

which at present is the most accurately determined of the parameters, and finally with p_0 small compared with ρ_0, l can be chosen to give a rate of annihilation as small as that for example from the late dwarf stars, without appreciably affecting the linear relation between red shift and distance. Any actual choice for l, however, should perhaps wait until we have more knowledge as to the rate of annihilation of intergalactic dust. At the present moment, the use of any more complicated expression for the dependence of the line element on the time, or the introduction of higher temporal terms than lt^2 would perhaps be premature.

Further progress in the use of the non-static line element for describing the behavior of matter in the universe would seem possible both along theoretical and observational lines.

On the theoretical side we should be greatly helped by some acceptable interpretation and determination of the cosmological constant Λ, and this might result from a successful unified field theory, or from the proper fusion of general relativity and quantum mechanics. A satisfactory theory which would account for the enormous range in the rates of annihilation of matter found in stars of different mass would also be extremely helpful.

On the observational side more information as to the cause and character of the cosmic rays would of course be desirable. Primarily, however, our observational interests must lie in the furtherance of the remarkable determinations of the relation between red shift and distance which are being made by Hubble and Humason. It is perhaps too much to hope that they could obtain an observational accuracy which would detect the small deviations from linearity of red shift with distance which would be sufficient to account for a low rate of annihilation of matter. Even with the two-hundred inch telescope under construction for this Institute such an accuracy would perhaps still be impossible. However, it is a matter of the gravest interest to make sure that no large deviation from linearity occurs. And in addition it is not premature to hope that sufficient accuracy can be obtained in the measurement of the angular extensions and the luminosities of the nebulæ to make it possible to test the interrelation between these two methods of estimating distance, which was shown in my third paper to be a necessary consequence of the non-static line element. This will furnish a significant although not very extensive test of the general idea that the red shift is to be explained by a non-static line element.

[1] Tolman, *Proc. Nat. Acad. Sci.*, **16**, 320 (April 15, 1930).

[2] Tolman, *Ibid.*, **16**, 409 (June 15, 1930).

[3] Tolman, *Ibid.*, **16**, 511 (July 15, 1930).

[4] Robertson, *Ibid.*, **15**, 822 (1929).

[5] Lemaitre, *Ann. Société Sci. Bruxelles*, **47**, Series A, 49 (1927).

[6] Friedman, *Zeit. Phys.*, **10**, 377 (1922).

[7] Eddington, *Monthly Notices R. A. S.*, **90**, 668 (May, 1930).

[8] De Sitter, *Bull. Astron. Inst. of the Netherlands*, **5**, 211 (June 24, 1930).

[9] Robertson's r is the same as my $\bar{r}$ in the form of the line element given by equation (5), Reference 3.

[10] Reference 2, Assumptions *a*, *b*, *d*, *e*, in §2.

[11] Reference 2, Assumption *c* in §2.

[12] Reference 2, Equation (14).

[13] Reference 2, §4, 5.

[14] Reference 1, Equations 34; Reference 2, Equations 2. Note the omission of a dot in the latter place.

[15] This corresponds to Lemaitre's equation (30).

[16] The article of Friedman is interesting in considering a number of possibilities as to the past history of the universe, including that of a periodic expansion and contraction, all, however, with the restriction $p_0 = 0$. De Sitter also considers a number of such possibilities, all, however, with the restriction $dM/dt = 0$.

[17] Reference 2, Equation 23. This equation was derived setting the constant $g_2 = 0$. This is also desired for our present purposes, however, to make our equation (13) agree with Lemaitre's equation (24).

[18] Reference 2, § 8 (*e*). Note that this should read $4\pi(\rho_0 + 4p_0)$ or $4(\rho_{00} + p_0)$ instead of $4\pi(\rho_0 + p_0)$.

[19] Reference 2, §8 (*g*).

[20] Reference 2, Equation 28.

ON THE SUMMABILITY OF FOURIER SERIES. THIRD NOTE

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1. In the present note we continue the discussion of the summability of Fourier series by the methods $[H, q(u)]$ of Hurwitz-Silverman-Hausdorff.[1] We give necessary and sufficient conditions for effectiveness, and show that the problem is closely connected with the theory of Fourier transforms.

We recall that the function $q(u)$ associated with the method $[H, q(u)]$ is supposed to satisfy Hausdorff's conditions: $q(u)$ is of bounded variation in (0, 1) and continuous at $u = 0$, $q(0) = 0$, $q(1) = 1$. Let $f(x)$ be any integrable function of period 2π. The n^{th} (H, q)-mean of the Fourier series of $f(x)$ is

$$H_n[f(x), q] = \int_{-\pi}^{+\pi} f(t + x)H_n(t)\, dt, \tag{1}$$

where

$$H_n(t) = \frac{1}{\pi}\Re\left\{(1 - e^{it})^{-1}\int_0^1 (1 - u + ue^{it})^n dq(u).\right\} \tag{2}$$

We put

$$\varphi(t) = f(x + t) + f(x - t) - 2S, \tag{3}$$

$$\psi(t) = \int_0^t \varphi(\tau)d\tau, \quad \Phi(t) = \int_0^t |\varphi(\tau)|d\tau. \tag{4}$$

We shall use the following terminology. A point x will be called *regular* or *pseudo-regular* with respect to $f(x)$ according as there exists an $S = S_x$,