SPHERE GEOMETRY AND THE CONFORMAL GROUP IN FUNCTION SPACE

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1. *Sphere Geometry in Function Space.*—By analogy with the definition of a sphere in Euclidean n-dimensional space, a sphere in the Euclidean function-space R_x is represented by the equation[1]

$$\sigma \int f^2(y)dy - 2\int \varphi(y)f(y)dy + \omega = 0, \tag{1}$$

where $f(x)$ and $\varphi(x)$ are continuous functions on the interval $0 \leqq x \leqq 1$, and σ and ω are real numbers on the interval $0 \leqq \sigma, \omega \leqq 1$. Here, $\varphi(x)$, σ, ω are considered fixed for the moment, while $f(x)$ denotes the variable point in R_x. Writing (1) in the form

$$\int \left[f(y) - \frac{\varphi(y)}{\sigma}\right]^2 dy = \frac{\int \varphi^2(y)dy - \sigma\omega}{\sigma^2},$$

we observe that the distance from the fixed point $\varphi(x)/\sigma$ to the variable point $f(x)$ is constant. We shall call $\varphi(x)/\sigma$ the center of the sphere, while the radius r is given by $r^2 = \dfrac{\int \varphi^2(y)dy - \sigma\omega}{\sigma^2}$.

The angle θ between two spheres

$$\begin{aligned} \sigma_1\int f^2(y)dy - 2\int \varphi_1(y)f(y)dy + \omega_1 &= 0, \\ \sigma_2\int f^2(y)dy - 2\int \varphi_2(y)f(y)dy + \omega_2 &= 0, \end{aligned} \tag{2}$$

will be defined by the expression

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2},$$

where r_1 and r_2 are the radii, and d the distance between the centers. Two spheres are said to intersect orthogonally when and only when $\cos\theta = 0$. It follows at once that a necessary and sufficient condition that two spheres (2) in R_x be orthogonal, is

$$2\int \varphi_1(y)\varphi_2(y)dy - \sigma_1\omega_2 - \sigma_2\omega_1 = 0. \tag{3}$$

We shall call the left member of (3) the *polar of the quadratic functional*

$$\int \varphi^2(y)dy - \sigma\omega.$$

A sphere (1) is completely fixed when $\varphi(x)$, σ and ω are given, so that we may take $\varphi(x)$, σ, ω as the homogeneous sphere coördinates of the function-space R''_x. If then we take as element the sphere in R_x, we shall obtain a geometry which we shall call the sphere geometry of R_x since it is the analogue of the elementary sphere geometry of n-space. A second type of sphere geometry in R_x, analogous to Lie's sphere geometry in n-space, arises when use is made of the coördinates $\varphi(x)$, σ, ω, ρ connected by the relation

$$\rho^2 = \int \varphi^2(y)dy - \sigma\omega.$$

This type of geometry will be considered in a subsequent paper.

2. *The Group of Conformal Transformations.*—In order to proceed with the study of the sphere geometry in R_x we consider the linear, homogeneous functional transformations

$$\begin{cases} \tau\varphi'(x) = A(x)\varphi(x) + \int B(x, y)\varphi(y)dy + C(x)\sigma + D(x)\omega, \\ \tau\sigma' = \int E(y)\varphi(y)dy + F\sigma + G\omega, \\ \tau\omega' = \int H(y)\varphi(y)dy + K\sigma + L\omega, \end{cases} \tag{4}$$

which take a point $\{\varphi(x), \sigma, \omega\}$ of the function space R''_x into another point $[\varphi'(x), \sigma', \omega']$. Here the functions $A(x)$, $B(x, y)$, $C(x)$, $D(x)$, $E(x)$, $H(x)$ are continuous on their respective ranges, while F, G, K and L are constants; τ is a factor of proportionality, so that $\{\varphi(x), \sigma, \omega\}$ and $\{\tau\varphi(x), \tau\sigma, \tau\omega\}$ are the same point in the function space R''_x.

Let us consider in particular the special transformations (4) which leave unchanged the quadratic functional equation

$$\int \varphi^2(y)dy - \sigma\omega = 0 \tag{5}$$

so that

$$\int \varphi'^2(y)dy - \sigma'\omega' = M^2[\int \varphi^2(y)dy - \sigma\omega],$$

where M is an arbitrary constant. The following relations are shown to exist between the coefficients of a transformation of this kind:

$$\begin{cases} \int C^2(y)dy - KF = 0, \int D^2(y)dy - GL = 0, A^2(x) - M^2 = 0 \\ 2\int C(y)D(y)dy - FL + M^2 - GK = 0, \\ 2A(x)C(x) - FH(x) - KE(x) + 2\int B(y, x)C(y)dy = 0, \\ 2A(x)D(x) - GH(x) - LE(x) + 2\int B(y, x)D(y)dy = 0, \\ 2A(x)B(x, y) - E(x)H(y) + 2\int B(z, x)B(z, y)dz + 2A(y)B(y, x) - \\ \quad E(y)H(x) = 0. \end{cases} \tag{4'}$$

Thus the transformations defined by (4) and (4') take a sphere of zero radius in into another sphere of the same kind. In other words, the most general linear point transformations in the space R''_x which leave (5) invariant will be the analogue of the conformal transformations in n-space, expressed in homogeneous sphere coördinates. We shall therefore call the transformations (4) and (4') *conformal transformations in* R_x. It may be readily shown that such transformations are further characterized by the property that they transform orthogonal spheres in R_x into orthogonal spheres.

We may show without difficulty that the product of two conformal transformations in R_x is another conformal transformation in R_x. The question of the existence of a unique inverse is determined by a method used by Hildebrandt[2] in the inversion of a projective transformation[3] in R_x. The system of linear integral equations (4) is reduced to a single Fredholm integral equation, whose Fredholm determinant

$$\begin{vmatrix} FG \\ KL \end{vmatrix} + \sum_{n=1}^{\infty} \int \cdots \int \begin{vmatrix} B(x_i, x_j)/A & C(x_i)/A & D(x_i)/A \\ E(x_j) & F & G \\ H(x_j) & K & L \end{vmatrix} dx_1 \ldots dx_2. \quad (i, j = 1 \ldots n) \tag{6}$$

will be called the *determinant of the conformal transformation.* It is seen at once that a conformal transformation (4) has a unique inverse if and only if the expression (6) is different from zero. It follows, further, that every non-singular conformal transformation of this kind, has a unique inverse and this inverse is again a non-singular conformal transformation in R_x.

Finally, since the determinant of the product of two conformal transformations in R_x is equal to the product of the separate Fredholm determinants of these transformations, it follows that the totality of non-singular transformations in R_x form a group which we shall call *the conformal group in* R_x.

3. *Infinitesimal Conformal Transformations.*—To obtain the most general infinitesimal conformal transformation in R_x we may, without loss of generality, take L in (4) to be unity, since this coefficient may be incorporated in the factor of proportionality τ. Thus we must determine the transformations of the form

$$\begin{cases} \dfrac{d\varphi(x)}{dt} = \alpha(x)\varphi(x) + \int \beta(x, y)\varphi(y)dy + \gamma(x)\sigma + \delta(x)\omega, \\ \dfrac{d\sigma}{dt} = \int \epsilon(y)\varphi(y)dy + a\sigma + b\omega, \\ \dfrac{d\omega}{dt} = \int \mu(y)\varphi(y)dy + c\sigma \end{cases}$$

which leave invariant the expression (5), and these are shown to be

$$\begin{cases} \dfrac{d\varphi(x)}{dt} = k\varphi(x) + \int \beta(x, y)\varphi(y)dy + \gamma(x)\sigma + \delta(x)\omega, \\ \dfrac{d\sigma}{dt} = 2\int \delta(y)\varphi(y)dy + 2k\sigma, \\ \dfrac{d\omega}{dt} = 2\int \gamma(y)\varphi(y)dy. \end{cases} \tag{7}$$

where $\gamma(x)$ and $\delta(x)$ are arbitrary continuous functions, k an arbitrary constant and $\beta(x, y) + \beta(y, x) = 0$. The transformations (7) are precisely the regular infinitesimal conformal transformations obtained by Kowalewski[4] as the most general infinitesimal angle-preserving transformations in R_x. This again brings out the analogy with the situation in n-space.

Kowalewski showed that the infinitesimal conformal transformations in R_x written in non-homogeneous coördinates form a group in the sense that the commutator of any two such transformations is one of the same kind. Using a more general definition of commutator, we find that the transformations (7) constitute a group in the sense of Kowalewski.

A given infinitesimal conformal transformation in R_x generates a one-parameter group of non-singular conformal transformations in R_x, and there are formulas determining the coefficients of the generated finite transformations in terms of the coefficients of the infinitesimal transformations. The method by which these results are established is that used by Barnett[5] to obtain the corresponding results for the projective group in R_x.

1 All integrations will be Riemannian and the range will be from 0 to 1.

2 T. H. Hildebrandt, *Bull. Am. Math. Soc.*, **26**, p. 400 (1920).

3 L. L. Dines, *Trans. Am. Math. Soc.*, **20**, p. 45 (1919).

4 G. Kowalewski, *Compt. Rend.*, **153**, p. 1452 (1911).

5 I. A. Barnett, *Bull. Am. Math. Soc.*, **36**, p. 273 (1930).

CONCERNING UNIORDERED SPACES

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The present paper is devoted to a solution of a problem proposed by G. T. Whyburn.[1]

In what follows the letter Z will denote a collection (or *system*) of closed point sets such that both the sum of every two elements of Z and every closed subset of an element of Z are elements of Z.

Definition.[2]—A separable metric space S is said to be *uniordered relative to a system* Z provided that for every point p of S there exists a monotone decreasing sequence of neighborhoods $U_1, U_2, U_3, \ldots$, of p whose boundaries $B_1, B_2, B_3, \ldots$ are elements of Z, and such that p is the only point common to all the sets $U_1 + B_1, U_2 + B_2, U_3 + B_3, \ldots$.