$$g = B - \alpha + 2k^2\left[\alpha\frac{y^2 - z^2}{2} + \beta yz + \gamma y + \delta z\right]$$

$$h = -(\beta + A) + 2k^2\left[A\frac{y^2 - z^2}{2} + Byz + Cy + Dz\right].$$

Although a, b, c, f, g, h are not force components it may be interesting to compare the above expressions with the forces obtained from an entirely different point of view in the preceding paper. There is a certain analogy but it is not clear how the two kinds of expressions might be identified. The comparison suggests, however, that we might have to consider a, b, c, f, g, h as variable with $t - x$ and that the expressions (6) would be asymptotically approached by them as $t - x$ approaches zero.

We may note that there is a certain anisotropy which might suggest an interpretation in connection with polarization; but this polarization belongs to the field of stresses rather than to the field of forces and it is of the elliptic type.

[1] Presented before the Chicago meeting of the American Mathematical Society, Christmas, 1926; *Bull. Amer. Math. Soc.*, **33**, 157 (1927).

[2] These PROCEEDINGS, **14**, June, 1928, pp. 484–88.

[3] *Ibid.*, **10**, 294–8 (July, 1924). Also *Trans. Amer. Math. Soc.*, **27**, 106–36 (1925).

CONCERNING PLANE CLOSED POINT SETS WHICH ARE ACCESSIBLE FROM CERTAIN SUBSETS OF THEIR COMPLEMENTS[1]

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A point set K is said to be accessible from a point set R provided that if A and B are any two points of K and R, respectively, then there exists a simple continuous arc AB from A to B such that $AB - B$ is a subset of R. Schoenflies[2] has shown that if K is a closed and bounded point set which separates the plane into just two domains D_1 and D_2 and is accessible from each of these domains, then K is a simple closed curve. In this paper a closely related theorem will be established, i.e., it will be shown that if K is any closed and bounded point set in a plane S such that there exist in $S - K$, two mutually exclusive connected point sets R_1 and R_2 such that K is accessible from each of these sets, then either K is a simple closed curve or there exists a simple continuous arc which contains K. With the aid of this proposition it will be shown that if K is any *irreducible cutting*[3] of a plane bounded continuous curve M between two points A and B of M such that K is accessible from each of the components[4] R_a and R_b of $M-K$ containing A and B, respectively, then either K is a simple closed curve or there exists a simple continuous arc which contains K but which contains no other point of M. A number of other theorems will be given which either lead up to these results or follow readily from them.

A subset K of a connected point set M is said to be a *cutting*[3] of M or is said to cut M provided the set of points $M-K$ is not connected; a cutting K of M is said to be an *irreducible cutting*[3] *of* M provided no proper subset of K is a cutting of M. A subset K of a connected set M is said to be a *cutting of* M *between the points* A *and* B *of* M,[3] or to cut M between A and B, provided that $M-K$ is the sum of two mutually separated point sets M_a and M_b containing A and B, respectively; K is said to be an *irreducible cutting of* M *between* A *and* B[3] provided that K cuts M between A and B but no proper subset of K cuts M between A and B.

The points, or point sets, A and B of a connected point set M are said to be separated in M by a subset N of M provided $M-N$ is the sum of two mutually separated point sets containing A and B, respectively. Whereever it is simply stated that "A and B are separated by a point set N," it is understood that A and B are separated in the plane by the set N. If R is an open subset of a continuum M, then by the M-boundary of R is meant the set of all those points of $M-R$ which are limit points of R. If M and N are point sets, the notation $M \cdot N$ will be used to denote the set of points common to M and N.

THEOREM 1. *If, in a plane* S, R_1, R_2 *and* R_3 *are mutually exclusive connected point sets, then* $S-(R_1 + R_2 + R_3)$ *contains not more than two points, each of which is accessible from each of these sets.*

Proof.—Suppose, on the contrary, that there exist three points X, Y, and Z in $S-(R_1 + R_2 + R_3)$ each of which is accessible from each of the sets R_1, R_2 and R_3. Then it readily follows that there exist points A and B in R_1 and R_2, respectively, and arcs AXB, AYB and AZB from A to B, no two of which have in common any point except A and B and such that the set of points $AX + AY + AZ-(X + Y + Z)$ belongs to R_1 and the set $BX + BY + BZ-(X + Y + Z)$ belongs to R_2. Of the three arcs AXB, AYB and AZB, one must lie except for the points A and B within the simple closed curve formed by the sum of the other two. The various cases are alike, so let us suppose $AZB-(A + B)$ lies within the curve $AXBYA$. Let D_1 denote the exterior of this curve, and let D_2 and D_3 be the interiors of the simple closed curves $AXBZA$ and $AYBZA$, respectively. Since R_3 is connected, it is clear that it must lie wholly in one of the domains D_1, D_2, D_3. But this is impossible, for Z is not a limit point of D_1, Y is not a limit point of D_2, and X is not a limit point of D_3, and by hypothesis each of the points X, Y and Z must be a limit point of R_3. Thus the supposition that theorem 1 is not true leads to a contradiction.

The hypothesis of theorem 1 can be weakened somewhat, since in the proof we use only the fact that all of the points X, Y and Z are accessible from R_1 and R_2 and are limit points of R_3.

THEOREM 2. *If R_1, R_2 and R_3 are mutually exclusive arcwise connected subsets of a continuous curve M every subcontinuum of which is a continuous curve, then not more than two points of $M-(R_1 + R_2 + R_3)$ can be limit points of each of the sets R_1, R_2 and R_3.*

Proof.—Suppose, on the contrary, that there exist three points of $M-(R_1 + R_2 + R_3)$ each of which is a limit point of each of the sets R_1, R_2 and R_3. But by a theorem of the author's[5] every limit point of R_i ($i = 1, 2, 3$) is accessible from R_i, and by theorem 1, not more than two points of $M-(R_1 + R_2 + R_3)$ can be accessible from each of the sets R_1, R_2, and R_3. This contradiction proves theorem 2.

COROLLARY. *If R_1, R_2 and R_3 are mutually exclusive connected open subsets of a continuous curve M every subcontinuum of which is a continuous curve having M-boundaries B_1, B_2 and B_3, respectively, then $B_1 \cdot B_2 \cdot B_3$ contains not more than two points.*

THEOREM 3. *If every point of the closed and bounded point set K in a plane S is accessible from each of two mutually exclusive connected subsets R_1 and R_2 of $S-K$, then either K is a simple closed curve or there exists a simple continuous arc which contains K.*

Proof.—Suppose K is not a simple closed curve. Let t denote any component of K which contains more than one point. Then t is not a simple closed curve. For suppose it is. Let I and E, respectively, denote the interior and exterior of t. Then neither I nor E can contain both of the sets R_1 and R_2. For suppose E contains both R_1 and R_2; then every point of t is accessible from each of the three mutually exclusive connected subsets R_1, R_2, and I of $S-t$; but this contradicts theorem 1. A like contradiction is obtained if we suppose I contains both R_1 and R_2. Hence it follows that R_1 lies wholly in one of the sets I and E, and R_2 lies wholly in the other. And since every point of K is a limit point of each of the sets R_1 and R_2, then K must be a subset of t, and hence must be identical with t. But t is, by supposition, a simple closed curve, and by our original hypothesis K is not a simple closed curve. Thus the supposition that t is a simple closed curve leads to a contradiction.

Since t is not a simple closed curve, then[6] it contains two points X and Y such that $t-(X + Y)$ is connected. Let Z be any other point of t. Then since, by hypothesis, every point of K is accessible from each of the sets R_1 and R_2, it readily follows that there exist points A and B of R_1 and R_2, respectively, and simple continuous arcs AXB, AYB and AZB such that the point set $AX + AY + AZ - (X + Y + Z)$ is a subset of R_1 and the set $BX + BY + BZ - (X + Y + Z)$ is a subset of R_2. The set of points $t - (X + Y)$, since it is connected, must lie wholly either within or without the simple closed curve $AXBYA$. The two cases are practically alike, so let us suppose $t - (X + Y)$ lies in the interior I of the curve $AXBYA$. Then the arc AZB divides I into two domains D_1 and D_2 whose boundaries contain X and Y, respectively, and each of which, it is easily seen, must contain points of $t - (X + Y)$. Then if S_1 denotes the set of points $D_1 \cdot t + X$ and S_2 the set $D_2 \cdot t + Y$, it is easily seen that S_1 and S_2 are mutually separated sets and that $S_1 + S_2 = t - Z$. Hence t is disconnected by the omission of any one of its points except X and Y, and therefore[6] t is a simple continuous arc from X to Y.

I shall now show that Z is not a limit point of $K - t$. Suppose the contrary is true. Now the segment $t - (X + Y)$ lies either within or without the curve $AXBYA$. Suppose it lies in the interior I of $AXBYA$. The arc t divides I into two domains G_1 and G_2 whose boundaries contain A and B, respectively. Then either G_1 or G_2 must contain a point P of K. But if P lies in G_1 it is not accessible from R_2, and if P lies in G_2 it is not accessible from R_1; and by hypothesis every point of K is accessible from each of the sets R_1 and R_2. Thus, in this case, the supposition that any point Z of t except X and Y is a limit point of K leads to a contradiction. A similar contradiction is obtained if it is supposed that $t - (X + Y)$ lies without the simple closed curve $AXBYA$. Thus it follows that every component of K is either a point or a simple continuous arc t such that no point of t, save possibly its end-points, is a limit point of $K - t$. Therefore, by a theorem due to R. L. Moore and J. R. Kline,[7] there exists a simple continuous arc which contains K.

Since every simple continuous arc in the plane is a subset of some simple closed curve, we have the following immediate corollary.

COROLLARY. *Under the hypothesis of theorem 3, there exists a simple closed curve which contains K.*

Definitions.—A continuous curve M is said to be cyclicly connected[8] provided every two points of M lie together on some simple closed curve in M. If the cyclicly connected continuous curve C is a subset of a continuous curve M, then C is called a maximal cyclic curve of M if and only if C is not a proper subset of any other cyclicly connected continuous curve which is a subset of M.

THEOREM 4. *Let K be an irreducible cutting of a bounded continuous curve M between two points A and B of M, let R_a and R_b denote the components of $M-K$ containing A and B, respectively, let N be the boundary of any complementary domain D of M, and suppose that K contains more than one point. Then* (1) *K lies wholly in some maximal cyclic curve C of M,* (2) *$K \cdot N$ is either vacuous or it lies on some simple closed curve in N;* (3) *if every point of $K \cdot N$ is accessible from each of the sets R_a and R_b, then $K \cdot N$ contains not more than two points; and* (4) *if K is a subset of N, then K consists of exactly two points.*

Proof.—(1) If X and Y are any two points of K, then since $R_a + X + Y$ and $R_b + X + Y$ are connected[9] subsets of M having in common just the points X and Y, it is clear that X and Y are not separated in M by any single point of M. But if two points of a continuous curve M do not lie together in the same maximal cyclic curve of M, there exists[8] a point of M which separates these two points in M. Thus it follows that there exists a maximal cyclic curve C of M which contains every point of K.

(2) Suppose $K \cdot N$ is not vacuous. By (1) K is a subset of some maximal cyclic curve C of M. Let G denote the complementary domain of C which contains D. By a theorem of the author's[8] the boundary J of G is a simple closed curve. Since K is a subset of C, it is clear that J must contain $K \cdot N$ and must itself be a subset of N.

(3) Every point of $K \cdot N$ is accessible from D. Hence every point of $K \cdot N$ is accessible from each of the three mutually exclusive connected sets R_a, R_b and D, no one of which contains a point of $K \cdot N$. Therefore, by theorem 1, $K \cdot N$ contains not more than two points.

It is to be noted that, in consideration of the proof of theorem 1 and the fact that every point of K is a limit point of each of the sets R_a and R_b, the conclusion of (3) holds if it is assumed merely that there exists one of the sets R_a and R_b such that every point of $K \cdot N$ is accessible from this set.

(4) If K is a subset of N, then by (2), K lies on some simple closed curve in N. Hence K has no continuum of condensation, and by a theorem of R. L. Wilder's,[10] every point of K is accessible from each of the sets R_a and R_b. Therefore, by (3), K contains not more than two points; and since by hypothesis K contains more than one point, it follows that K consists of exactly two points. This completes the proof of theorem 4.

THEOREM 5. *Let the sets M, K, A, B, R_a, and R_b be defined exactly as in the statement of theorem 4. Suppose every point of K is accessible from each of the sets R_a and R_b. Then either K is a simple closed curve or there exists a simple continuous arc t which contains K but which contains no point of M not belonging to K, i.e., $t \cdot M \equiv K$.*

Proof.—(1) Let us suppose that K is not a simple closed curve and that it contains more than one point, since the case where K contains only one point is trivial. Then by theorem 3 there exists some simple continuous arc which contains K. Hence, every component of K is either a point or an arc, and no component of K separates the plane. Furthermore, the components of K form an upper semi-continuous[11] collection of continua. Therefore, as established by R. L. Moore,[11] if G denotes the collection whose elements are the components of K and all the *points* in the plane not belonging to K, then the sum of all the elements of G constitutes a space S' which is homeomorphic with the ordinary Euclidean point plane.

Then since K is a closed totally disconnected set of elements of G, and since, by hypothesis, $M - K$ is not connected, it follows by a theorem of

R. G. Lubben's[12] that there exists a simple closed curve J of elements of G which contains K but contains no point of $M - K$, and which encloses one of the sets R_a and R_b, say R_a, but does not enclose the other.

Now let H denote the *point set* consisting of all points X such that X belongs to some element of J. It is clear that H is a continuum. It must be a continuous curve. For if not, then[13] it contains a countable infinity of mutually exclusive continua $T, H_1, H_2, H_3, \ldots$, all of diameter > some positive number given in advance, and such that T is the sequential limiting set of the sequence of continua $H_1, H_2, \ldots$. Now since K is closed, $H-K$ is the sum of a countable number of mutually exclusive segments $S_1, S_2, \ldots$ of J. Since no one of these segments contains a point of K, then for each i, the elements of S_i are all points, and hence S_i is a point set. Furthermore S_i belongs to some complementary domain D_i of M, and since, by theorem 4, part (3), the boundary N_i of D_i contains not more than two points of K, it follows that there exist two points A_i and B_i of K on N_i such that $S_i + A_i + B_i$ is closed and, in fact, is a simple continuous arc from A_i to B_i. Therefore, the sets $S_1, S_2, \ldots$ are ordinary arc segments. It readily follows that every point of T must belong to K, and since every component of K is either a point or an arc, T must be an arc. And since, by theorem 3, no interior point of T can be a limit point of $K-T$, it follows that there exist a subarc T^* of T and an infinite sequence $H_1^*, H_2^*, \ldots$ of continua all of diameter > some positive number d given in advance, and such that, for every integer $i > 0$, H_i^* is a subset of H_i and contains no point whatever of K. Hence, for each i, the continuum H_1^* must belong to one of the segments $S_1, S_2, \ldots$. Since the segments $S_1, S_2, \ldots$ are ordinary arc segments, and since each of the continua $H_1^*, H_2^*, \ldots$ is of diameter $> d$, then no one of the segments $[S_i]$ can contain an infinite number of the continua $[H_i^*]$. It follows that infinitely many of the segments $[S_i]$ must be of diameter $> d$. But each of these segments lies in some complementary domain of M, no two of them can lie together in a single complementary domain of M, and only a finite number of the complementary domains of M are of diameter $> d$. Hence, not more than a finite number of the segments $[S_i]$ can be of diameter $> d$. Thus the supposition that H is not a continuous curve leads to a contradiction.

Now let I and E, respectively, denote the interior and exterior of J. Since neither I nor E contains a point of K, then the elements of I and of E are all points, and hence I and E are ordinary domains. And since I contains R_a and E contains R_b, it follows[9] that the point set H is the common boundary of the two domains I and E. But it was shown above that H is a continuous curve. Hence, by a theorem of R. L. Moore's,[14] H is a simple closed curve. Now let A_1 and B_1 denote the end-points of S_1, and let t denote that arc of H from A_1 to B_1 which does not contain S_1.

Then the arc t contains K but contains no point of $M-K$. This completes the proof of the theorem.

Note.—The above proof for theorem 5 is by no means an elementary one. The following proof, obtained by Mrs. G. T. Whyburn, is of interest both because of its more elementary character and because of the interesting way in which it utilizes theorem 3.

Proof.—(2) Let us suppose K is not a simple closed curve, and that it contains more than one point. By theorem 4, part (3), the boundary of no complementary domain of M contains more than two points of K. Let $D_1, D_2, \ldots$ be those complementary domains of M such that, for each integer $i > 0$, the boundary of D_i contains two distinct points A_i and B_i of K. For each $i > 0$, the domain D_i contains a segment S_i of an arc whose end-points are A_i and B_i.

Let N denote the set of points $R_a + R_b + K$. Clearly[9] N is a continuum. And since, by theorem 3, every component of K is either a point or an arc t no interior point of which is a limit point of $K - t$, it is readily shown that N must be a continuous curve. Now let $I_1, I_2, I_3, \ldots$ be the complementary domains of N such that, for each integer $i > 0$, the boundary of I_i contains two distinct points X_i and Y_i of K. Then there must exist at least one of the segments $[S_i]$ having X_i and Y_i as its end-points. For by theorem 4, part (2), X_i and Y_i lie together on some simple closed curve J_i in the boundary N_i of I_i. And of the two arcs t_1 and t_2 of J_i from X_i to Y_i, one of them must lie, save for the points X_i and Y_i, wholly in R_a and the other, save for its end-points, wholly in R_b; for if, on the contrary, one of the sets R_a and R_b, say R_a, contains points P_1 and P_2 belonging to t_1 and t_2, respectively, then $t_1 - (X_i + Y_i)$ and $t_2 - (X_i + Y_i)$ must each belong to R_a. Hence R_b must lie wholly either within or without J_i, say without J_i; then I_i must lie within J_i and if P_1P_2 denotes an arc[14] in R_a from P_1 to P_2, then P_1P_2 separates the exterior of J_i, and hence not both of the points X_i and Y_i can be accessible from R_b, contrary to hypothesis. Thus it follows that $t_1 - (X + Y)$ belongs to one of the sets R_a and R_b, say to R_a, and $t_2 - (X_i + Y_i)$ belongs to R_b. Then there must exist a complementary domain of M whose boundary contains both X_i and Y_i; for if not then M would contain[15] a simple closed curve C which separates X_i from Y_i, and clearly this is not possible, for C would contain an arc joining a point P of t_1 and a point Q of t_2 and lying otherwise wholly in I_i, which is absurd since P belongs to R_a and Q to R_b. Hence there exists a complementary domain D of M whose boundary contains both X_i and Y_i; accordingly there exists at least one of the segments $[S_i]$ whose end-points are X_i and Y_i. For each positive integer i, let us select just one segment S_i^* of the collection $[S_i]$ whose end-points are X_i and Y_i. Let E denote the sum of all such segments $[S_i^*]$ thus selected, and let T denote the point set $K + E$.

Since K is closed and bounded, and since not more than a finite number of the segments $[S_i^*]$ are of diameter $>$ any preassigned positive number, it readily follows that T is closed and bounded. And since T contains an arc segment S_i^* which joins every two points X_i and Y_i of K which lie together on the boundary of a complementary domain of N and which lies wholly in some complementary domain of M, it follows with the help of theorems by C. M. Cleveland[15] and R. L. Moore[16] that T is connected. Hence, T is a bounded continuum.

Now there does not exist a complementary domain D of the continuum $N + T$ which has a boundary point P_1 in R_a and also a boundary point P_2 in R_b. For suppose there does exist such a domain D. Then D lies wholly in some complementary domain G of N. (1) In case D is identical with G, then G can contain no point of E, and therefore the boundary U of G can contain not more than one point P of K. Now P must separate P_1 and P_2 in U; this being so, it readily follows that P also separates P_1 and P_2 in M. But since K, by hypothesis, is an irreducible cutting of M between A and B, then K must be an irreducible cutting of M between P_1 and P_2. Hence K must consist of a single point P. But we have supposed that K contains more than one point. Hence, in this case, the supposition that the domain D exists leads to a contradiction. (2) In case D is not identical with G, then G must contain one segment S^* of the collection $[S_i^*]$, and therefore the boundary F of G must contain the end-points X^* and Y^* of S^* which belong to K. Then since the sum of the points X^* and Y^* separates P_1 and P_2 in U, it readily follows that the segment S^* separates G into two domains the boundary of one of which contains P_1 but not P_2 and the boundary of the other contains P_2 but not P_1. But the boundary of D contains both P_1 and P_2. Hence, in any case, the supposition that the domain D exists leads to a contradiction.

Let G_a denote the sum of all those complementary domains of $N + T$ whose boundary contains a point of R_a, and G_b the sum of all those whose boundary contains a point of R_b. Then from what we have just shown it follows that $R_a + G_a$ and $R_b + G_b$ are mutually exclusive connected point sets. Now if X is any point of T which belongs to K, then X is accessible from each of the sets $R_a + G_a$ and $R_b + G_b$, for by hypothesis it is accessible from R_a and also from R_b and clearly some point of R_a (R_b) is accessible from each component of G_a (G_b). And if P is any point of T not belonging to K, then P belongs to some segment S^* of $[S_i^*]$. The segment S^* separates some complementary domain G of N into two domains G_1 and G_2 one of which, it is clear from the second paragraph of this proof, must belong to G_a and the other to G_b. Hence, P is accessible from each of the sets $R_a + G_a$ and $R_b + G_b$. Therefore, since T is connected, it follows by theorem 3 that T must be either an arc or a simple closed curve. If T is a simple closed curve, then since K is not a simple closed curve, there exists a component S of $T-K$. And $T-S$ is, then, a simple continuous arc which contains K but no point of $M-K$. This completes the proof.

In proof (1) of theorem 5 we have established the following theorem.

THEOREM 6. *Under the hypothesis of theorem 5, there exists a simple closed curve J which contains K but no point of $M-K$ and which separates R_a and R_b.*

THEOREM 7. *If, in addition to the hypothesis in theorem 5, it is assumed that the boundaries of no two of the complementary domains of M have a common point, then there exists a simple closed curve which contains K and is a subset of M.*

Theorem 7 is readily established with the aid of theorem 6.

THEOREM 8.—*Let the sets M, A, B, K, R_a and R_b be defined exactly as in the statement of theorem 4. Then if every component T of K is a continuous curve and there exist two non-cut points of T each of which is accessible from both of the sets R_a and R_b, there exists a simple closed curve which contains K but no point of $M-K$ and which separates R_a and R_b.*

Proof.—It follows by a theorem of W. L. Ayres[17] that either K is itself a simple closed curve or each component T of K is either a point or an arc. If K is itself a simple closed curve, our theorem is an immediate consequence of theorem 1. So let us suppose K is not a simple closed curve. Then if T is any component of K which consists of more than one point, T must be an arc whose end-points are accessible from each of the sets R_a and R_b. Then by almost identically the same argument as used in the third paragraph of the proof of theorem 3 it is shown that no interior point of T is a limit point of $K-T$. Hence it follows that K can have no continuum of condensation; and since K is the M-boundary of each of the sets R_a and R_b, then by a theorem of R. L. Wilder's[10] *every point* of K must be accessible from each of the sets R_a and R_b. Therefore, by theorem 6, there exists a simple closed curve which contains K but no point of $M-K$ and which separates R_a and R_b.

In conclusion I will point out that if M is a continuous curve every subcontinuum of which is a continuous curve, then[5] the accessibility hypotheses in theorems 4, 5, 6 and 8 are always satisfied.

* The Editor regrets to state that this article was lost in transmission from author to editorial office—which accounts for the delay in publication.—E. B. W.

[1] Presented to the American Mathematical Society, December 28, 1927.

[2] Schoenflies, A., *Gottingen Nachrichten*, **1902**, p. 185. Cf. also, Moore, R. L., *Trans. Amer. Math. Soc.*, **17**, 158, 159 (1916); Swingle, P. M., *Bull. Amer. Math. Soc.*, **32**, 110–11 (1926) (abstract), and a remark by C. Kuratowski in *Fund. Math.*, **6**, 143 (1924).

[3] Cf. Whyburn, G. T., *Bull. Amer. Math. Soc.*, **33**, 408 (1927) (abstract). Paper to appear in full in *Fund. Math.*, **13**.

[4] A component of a point set K is a maximal connected subset of K; cf. Hausdorff, *Mengenlehre*, **1927**.

[5] Cf. Whyburn, G. T., "Concerning the Complementary Domains of Continua," forthcoming in the *Annals of Mathematics*, theorem 15. For the theorem for the case where the sets R_1, R_2 and R_3 are open subsets of M, see Whyburn, G. T., these PROCEEDINGS, **13**, 650–657 (1927), theorem 3.

[6] Moore, R. L., *Trans. Amer. Math. Soc.*, **21**, 333–347 (1920).

[7] Moore, R. L., and Kline, J. R., *Annals of Mathematics*, **20**, 182–223 (1919).

[8] Cf. Whyburn, G. T., these PROCEEDINGS, **13**, 31–38 (1927) and *Bull. Amer. Math. Soc.*, **33**, 305–308 (1927).

[9] This follows from the fact that K is the M-boundary of each of the sets R_a and R_b. Cf. the abstract of my paper "On Irreducible Cuttings of Continua," presented to the American Mathematical Society, December 28, 1927.

[10] Wilder, R. L., *Fund. Math.*, **7**, 340–377 (1925), theorem 1.

[11] Moore, R. L., *Trans. Amer. Math. Soc.*, **27**, 416–428 (1925).

[12] Lubben, R. G., *Bull. Amer. Math. Soc.*, **32**, 114 (1926) (abstract).

[13] Moore, R. L., *Bull. Amer. Math. Soc.*, **29**, 291–297 (1923).

[14] Moore, R. L., *Math. Zeit.*, **15**, 254–260 (1922).

[15] Cleveland, C. M., these PROCEEDINGS, **13**, 275–276 (1927).

[16] Moore, R. L., *Ibid.*, **13**, 711–716 (1927), theorem 2.

[17] Ayres, W. L., *Bull. Amer. Math. Soc.*, **33**, 565–571 (1927), theorems 4 and 5. For an additional hypothesis needed in these theorems, see a note in the *Bulletin of the American Mathematical Society*, **34**, 107–108 by W. L. Ayres. It is of interest to note that while, as shown by theorem 8 of the present paper, Ayres' condition (4) in his note makes the point set K such that it is a subset of a simple closed curve, his condition (4′) does not do this. His condition (4) implies (4′) but (4′) does not imply (4).

INCOMPLETE SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS[1]

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The theory of a system of total differential equations

$$\frac{\partial u_\alpha}{\partial x_i} = a_{\alpha i} \qquad \alpha = 1, 2, \ldots, r;\ i = 1, 2, \ldots, n \tag{1}$$

which are completely integrable, that is, for which the conditions

$$\frac{\partial a_{\alpha i}}{\partial x_j} + \sum_{\beta=1}^{r} \frac{\partial a_{\alpha i}}{\partial u_\beta} a_{\beta j} = \frac{\partial a_{\alpha j}}{\partial x_i} + \sum_{\beta=1}^{r} \frac{\partial a_{\alpha j}}{\partial u_\beta} a_{\beta i} \tag{2}$$

are satisfied identically in u and x, is known to correspond exactly to the theory of a jacobian system of simultaneous linear homogeneous equations,[2] but the usual treatment of the incompletely integrable case for the two