THE STABILITY OF ARCHES

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THE STABILITY OF ARCHES

BY

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PREFACE

THE present book is an attempt to put before the reader the principles upon which the stability of an arch is determined. Much ingenuity has been displayed in devising satisfactory methods of investigation either for assuring the stability of, or for estimating the actual maximum stresses in, an arch-ring. Most of these depend ultimately on the elastic theory. In using indirect methods for the purpose of simplification there is always the danger of losing sight of the degree of approximation attained, and there is the added difficulty always inseparable from graphical methods of assuring the necessary accuracy. This is particularly the case in the graphical investigation of the stresses in an arch-ring, and therefore particular attention has been given to the elastic theory, which is our ultimate standard of reference.

The general deductions of this theory have been confirmed by all recent experience, particularly for arches which are moderately flat; and as modern arches in masonry and concrete are usually of this type, the elastic theory affords the most satisfactory basis of investigation.

The author desires to acknowledge particularly his indebtedness to Prof. Melan's "Theorie des Gewölbes und des Eisenbetongewölbes im besonderen," Handbuch für Eisenbetonbau, Vol. I, and to "Leitfaden für das Entwerfen und die Berechnung gewölbter Brücken," by G. Tolkmitt.

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CHAPTER I.

INTRODUCTORY.

1. WHETHER we regard the arch from an architectural or from an engineering standpoint, that is to say, from its artistic or its scientific side, the elegance of its form and its combined lightness and strength make it an object of the first interest to constructors. The scientific aspect of the subject dates no further back than about 200 years, but during that period it has received its full share of consideration by mathematicians and engineers. On the architectural and practical side the use of the arch goes back to a remote antiquity, although not so remote but that indications of its origin are traceable. In the English translation of the Bible the arch is only once mentioned (Ezek. XL. 16), and then only on account of a mistranslation, we are informed. In Chaldea and Egypt in early times we find only the simplest forms of arch, and in many ruins of ancient cities, such as Persepolis, no trace of it is found. In all probability it had its origin and reached its highest development in China; for scattered over this vast country from south to north are fine examples of arched bridges of great antiquity. In the mountains which divide Manchuria from China, fine arches exist in the Great Wall built some 300 years B.C., which are still in excellent preservation, and the bridge of arches which

Marco Polo speaks of crossing in the thirteenth century serves its purpose equally well to-day; whilst all over China the arch has figured as a favourite feature in architecture and landscape gardening, as may be seen in their pictures and pottery.

2. Owing to the tendency of the arch under normal conditions to sink at the crown and thereby to cause spreading at the abutments, it requires strong lateral support, in consequence of which the earliest known examples at Nippur in Chaldea, about 4000 B.C., and at Dendera and other places in Egypt about 3500 to 3000 B.C., were used below ground level. These were of unburnt brick, and not more

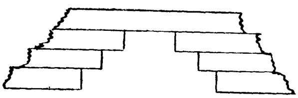


FIG. 1.

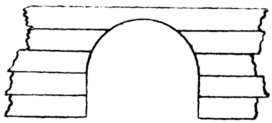


FIG. 2.

than 6 ft. span. The favourite construction in the East at this early period was to build the arch, if we may so call it, of horizontal slabs projecting one over the other, as shown in Figs. 1 and 2, a fine example of which is the Treasury of Atreus at Mycenæ. Examples of this method of construction are found in Egypt, Greece, Assyria, Italy, Mexico, India, China, and elsewhere. Such structures, however, cannot be regarded as true arches, because there is no horizontal thrust, and this must be regarded as the essential feature of the arch. Horizontal thrust, in fact, appears at first to have been regarded with disfavour, for, as the Hindus say, "the arch never sleeps," meaning, of course, that it is always exerting a thrust tending to its destruction.

Primitive arches of the form shown in Fig. 3 are found at Deir-el-Bahri and in the round towers of Ireland. Fig. 4 is an example from the Great Pyra-

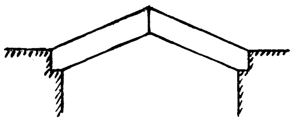


FIG. 3.

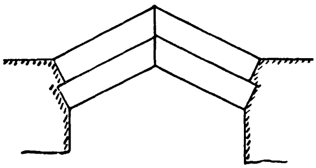


FIG. 4.

mid; whilst in Fig. 5, which occurs also in the Pyramids, is shown a combination of both types,

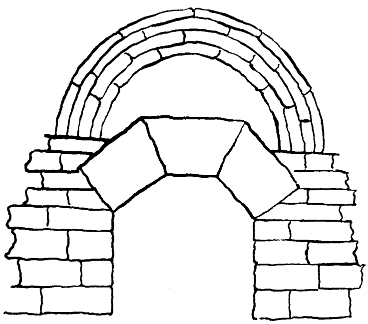


FIG. 5.

whilst Fig. 6 shows another arrangement in the tomb of the pyramid at Gizeh. In this we have an early example of the semicircular arch, a favourite form in ancient China, which appears to have been introduced into Europe by the Etruscans, who were of eastern origin. This people transmitted it to the Romans, who excelled in this class of construction, of which they have left numerous examples, the

Pont du Gard and the Aqueduct at Tarragona being particularly fine specimens.

3. The scientific treatment of the subject began about 1712 with Lahire. Previous to this we have no evidence, and we can scarcely suppose that the theory of construction in general had received much attention, because the general condition of mechanical science was not such as to permit it. Most of the bridges of ancient date, outside of military bridges, were constructed by builders who be-

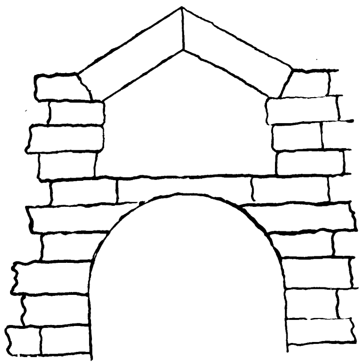


FIG. 6.

longed to a close corporation, and these expert constructors moved from place to place constructing bridges from their practical knowledge of the art, without much consideration, we may suppose, of the mechanical principles involved. Lahire (1712), however, considered the equilibrium of the voussoirs, which form the arch-ring, by treating them as a system of frictionless blocks acted on by the load and by the mutual reactions between their faces. Eytelwein advanced the discussion by introducing the frictional resistance of the surfaces; and Couplet (1730), Coulomb (1773), Boistard (1822), Navier, and others are associated with the early history of the theory. Méry first introduced the idea of the line of pressure, whilst the names of Lamé, Clapeyron, Rankine, Moseley, and many others of first-class reputation are associated with the later development of the subject.

4. From an architectural standpoint arches may be classified as :—

(*a*) Semicircular or Norman arches.
(*b*) Segmental, that is, less than semicircular.
(*c*) Pointed or Gothic.
(*d*) Horseshoe or Etruscan.
(*e*) Elliptic or pseudo-elliptic.

5. The masonry arch consists of the *arch-ring* and

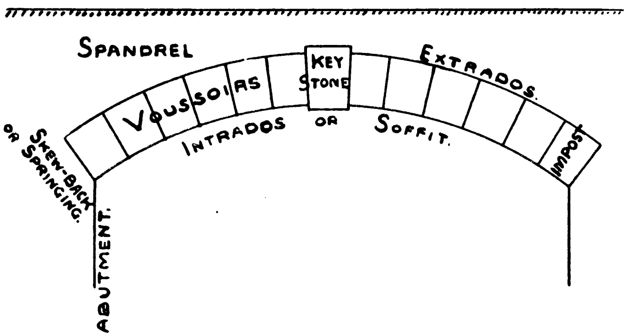


FIG. 7.

its *abutments*, which take the end thrust (Fig. 7), and intermediate *piers* perform the same function in the case of a series of arches. The arch-ring itself consists of wedge-shaped stones or *voussoirs* carefully fitted, or of bricks bonded together. The central voussoir at the *crown* of the arch is known as the *keystone*, and those adjacent to the *springings* as the *imposts*. The bearing surfaces are known as the *bed-joints*, and these are perpendicular or nearly so to the under surface or *soffit* of the arch, which surface is also spoken of as the *intrados*; whilst the outer surface is called the *extrados*. The *spandrels* or *haunches* are the lower portions of the extrados, and the spandrel *filling* or *backing* is the material between the extrados and the roadway.

6. Modern arches may be classified as :—

(*a*) Metal arches.

(*b*) Masonry and brickwork arches.

(*c*) Concrete and reinforced concrete arches.

The main difference in theory consists in the fact that metal arches may be subject to considerable tensile stress, whereas masonry and concrete arches are designed with a view to completely avoiding any tension in the arch-ring under all conditions of loading, whilst reinforced concrete arches, though often designed in the same way, may be permitted to suffer tensile stress if properly reinforced, with a consequent saving of material.

Increased knowledge of theory has naturally tended to lighter construction and increased length of spans. Few of the ancient bridges had long spans. Those in China and the East are always quite short. In England the longest span masonry arch is over the Dee at Chester, the clear span being 203 ft., whilst in America the longest span is at Washington, of 220 ft. The masonry arch near

Trezzo in Italy, built in 1380 and destroyed in 1416, is therefore remarkable in having a span of 251 ft. In 1906 a three-hinged arch of plain concrete was completed at Ulm in Germany of 215 ft. span, but for long span bridges of this type armoured concrete is generally preferred. Such an arch was constructed over the Tiber in Italy, having a span of 328 ft. with a rise of 32·8 ft. (See "Engineer," 16 May, 1913.)

7. For very large spans it is necessary to use steel and iron. The table on next page, giving particulars of various typical arches arranged in accordance with their spans, will be found useful for the sake of comparison.

8. The strength of masonry and concrete arches loaded under similar conditions varies directly as the strength of the material and the depth of the keystone, and inversely as the radius of curvature at the crown. These considerations indicate the possibilities at our disposal for increasing the span. The following figures are approximately the crushing strengths of the materials in common use.

Crushing Strengths of Arch Materials.

Good brickwork in Portland cement .	150	tons/ft.^{2}
Cement concrete	100 to 180	,,
Granite	1600	,,
Bessemer steel	3000	,,
Cast steel	3600	,,
Cast iron	5000	,,

From these figures we see that, under proper conditions, cast iron should be a very suitable material for arches, though it has in recent years been customary to regard it with suspicion on account of a few

TABLE I.

Name.	No. of Hinges.	Span in Feet.	Rise in Ft.	Material.	Thickness in Ft.		Rad. at Crown Ft.	Max. Stress, lb/in.².	Description.
					Crown.	Springings.			
1. Niagara Bridge . . .	2	854	172	Metal					Parallel chords with latticed web.
2. Rhine Bridge at Bonn .	2	626·4	75·3	,,					Spandrel-braced.
3. ,, ,, at Düsseldorf	2	604·2	92·3	,,					,,
4. Douro Bridge . . .	0	576	148·4	,,					
5. Niagara	2	559	116	,,					,,
6. Douro Bridge, Oporto .	2	533	194	,,					Crescent arch.
7. St. Louis, U.S. . . .	0	520	47	,,					Arch rib, parallel chords.
8. Zambesi River . . .	2	500	90	,,					Spandrel-braced.
9. Minneapolis . . .	3	456	90	,,					,,
10. Rochester Bridge . .	2	416	67	,,					,,
11. Austerlitz Bridge, Paris .	3	357	39	,,					Each half crescent shaped.
12. Tiber, Italy . . .	0	328	32·8	R.C.					Thickness of decking at crown 8½ [ins.
13. Syra Valley, U.S. . .	0	300	60	Masonry	6	13·3	220	620	
14. Plauen, Saxony . .	0	295·2	59	Slate	4·9				Depth of ribs, 6 ft.

15. Luxembourg . . .	0	277·75	101·5		4·7				
16. Southwark Bridge, Thames	0	240	24	Cast Iron					
17. Adda, Italy . . .	3	233	33·3	Masonry	5	7·3	250	700	
18. Cabin John, Washington .	0	224	58	,,	9·7	20·3	140		Granite. Circular.
19. Pruth	0	217	58·6	Sandstone	7	10·3	128	344	
20. Prince Regent, Munich .	3	213	21·3	Limestone	3·3	4·2		563	
21. Gutoch Bridge . . .	0	213	53·7	Masonry	6·7	9·3	133	440	
22. Grosvenor Bridge, Dee .	0	203	42·7	Stone	4·1	6·1	142		Lead joints.
23. Schwändenholz . .	0	190	47·5	Sandstone	6	8·7	119	432	
24. Donau, Munderkingen .	3	167	16·7	Concrete	3·3	3·7	233	475	
25. Nogent Valley, Marne .	0	167	83·3	Masonry	6	15	83		
26. London Bridge . . .	0	152	37·7	Granite	4·8	10	162		Ellipse.
27. Turin Bridge . . .		148	18	,,	4·9		160		Circular arc.
28. Kansas Avenue . . .		125	18·9	R.C.	1·8	6·0			1·58 °/o steel at crown.
29. Oder, Frankfort . .	0	100	12·5	Brick	2·7	4·3			Circular arc.
30. Westminster Bridge . .	0	76	38		7·6	14			
31. Kaiser Wilhelm, Berlin .	0	74	13·3	Granite	2·7	5		750	
32. Mich. C. Ry. . . .		50·3	9·5	R.C.	1·5	7·5			Circular.

Particulars of about 500 arched bridges will be found in Howe's "Symmetrical Masonry Arches," from which many of the above figures are taken.

isolated failures. There seems no reason, however, why cast iron should not be used for structures in compression, or subject to moderate tension, provided it is used in such a way as to avoid unknown initial strains.

If we compare the strengths of different arches as suggested above, with the help of these figures, we find a great variation in their relative strengths. Thus, for example, the figures in the last column below represent the relative strengths of the bridges named :—

Name.	Material.	Crushing Strength = s.	Depth of Keystone = d.	Rad. of Curv. at Crown = R.	$\frac{sd}{R}$.
Maidenhead Bridge	Brick in Cement	150 tons/ft.^{2}	5·25	169	4·66
Chester Bridge	Sandstone	400 ,,	4·0	140	11·4
Plauen ,,	Hard Slate	1670 ,,	4·11	214	38
London ,,	Granite	1600 ,,	4·75	162	47

CHAPTER II.

DISCUSSION OF THE PROBLEM.

THREE-PINNED ARCH.

9. THE problem to be solved in connection with the arch may present itself in various ways. Firstly, the form of the arch may be given and the material of which it is constructed, the nature of the supports and the load liable to come upon it, and it may be required to determine the degree of stability it possesses and the maximum stresses that may arise; secondly, the distribution of the load may be given, and it is required to determine the form of arch best adapted to the circumstances; thirdly, the form of the arch may be fixed, and the load distribution may be arranged to suit it.

Two modes of procedure are in common use in carrying out the investigation of the stability of masonry and concrete arches. The simple and older method which is best adapted to the treatment of small spans is based upon the determination of the critical line of pressure for the arch and the given load upon it; whilst the more modern and accurate method aims at finding the true line of pressure by means of the elastic theory. This method is universally applied to metal arches which are statically indeterminate; and though less reliable in the case of masonry and concrete the results have been shown experimentally to be trustworthy within the limits of accuracy required.

10. **The Linear Arch.**—Let ABCD (Fig. 8) be one of the voussoirs of a masonry arch-ring, and let EFAD be the load upon it. Then if W_1 be the resultant of their weights, and P, Q be the resultant reactions of the adjacent voussoirs, then these three forces P, Q, and W_1 being in equilibrium, must pass through the

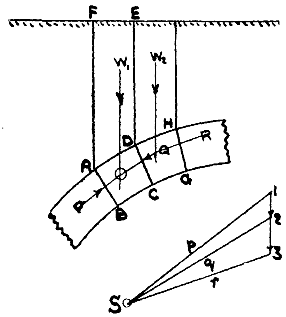


FIG. 8.

same point O and be proportional to the sides of a triangle S, 1, 2 having its sides parallel to the three forces, by the well-known theorem of the triangle of forces. Thus the side S, 2 = q represents the pressure Q on the face CD; and this force reversed then becomes the reaction on the adjacent block CDHG; it intersects the resultant load W_2 on the second voussoir and so determines the direction of the thrust R on the next one, and so on. In this way, proceeding from voussoir to voussoir, we obtain a link-polygon for the given loads whose links represent the lines of action of the forces acting at the voussoir joints. This link-polygon is the line of pressure for the arch, and since a polygon of rigid jointed rods coinciding with it would be in unstable equilibrium, this polygon is often spoken of as the "linear arch," this term being applied to the *ideal* arch for the given load system, which has no bending moment at any point along it. It is clear, however, that the position of the linear arch is indeterminate unless the magnitude direction and position of one of the forces is known; in other words, although any number of polygons may be drawn for the given loads, three independent conditions are necessary to fix that particular one which is their true line of pressure, and unless three such conditions are known the problem of finding it is statically indeterminate, that is to say, is impossible of solution by means of the simple laws governing the statics of rigid bodies. If, for example, we know three points through which the line of pressure must pass, as in the case of a three-pinned arch, when the friction of the pins is neglected, then the problem can be easily solved; but if one of the ends is fixed, by doing away with the pin-joint, then some other condition becomes necessary, such as a knowledge of the direction of the thrust at the fixed end; and if both ends are fixed, by doing away with the two end joints, then some other condition must be given or assumed before the true line of pressure can be found. In the case of a masonry or concrete arch therefore, which is usually without joints, no points at all on the line of pressure being known, three other conditions are necessary for its determination, and these are derived from the elastic properties of the arch-ring.

11. For the above reasons, and because the exact

determination of the linear arch is both troublesome and uncertain in such cases, it has often been preferred to make metal arches pin-jointed at the crown and springings; and even in the case of masonry and concrete arches semi-rigid joints are sometimes introduced at the crown and springings to fix more definitely the position of the line of pressure. Thus in the case of a bridge over the Enz, near Hofen, plates of lead $\frac{7}{8}$ in. thick were placed over the centre third of the joints, thereby ensuring that the line of pressure lies within this space; and in other cases granite rockers or hinges of cast iron or steel have been used, as shown in Fig. 9; whereby, if we neglect the effect of friction, the line of pressure is constrained to pass through the axis of the joint, and in any case cannot pass far from it unless the joint seizes through age and rust or excess of pressure.

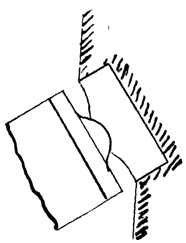


FIG. 9.

12. **The Three-pinned Arch.**—If we assume that the line of pressure passes through the axis of the three pins, then its exact position becomes easy of determination from simple statical considerations alone; and for this reason, the results being more assured, arches of this type have been preferred, where theory alone would recommend an arch-ring without joints. Moreover, the effect of temperature changes has no appreciable influence on the stresses in this type of arch.

Method I. Determination of the Line of Pressure.—Suppose A, B, C (Fig. 10) to be the three pins of an arch, and W_1, W_2, W_3, W_4 to be any system of vertical loads upon it. Set down the loads to scale along a vector line 0, 1, 2, 3, 4, and with any pole O construct a link-polygon A*abcde*, which may begin at A, and which ends on the vertical through C at *e*. Then this will be an equilibrium polygon for the loads, but will not be the linear arch unless it passes through A, B, and C. Join A*e*, and draw O5 parallel to it. This deter-

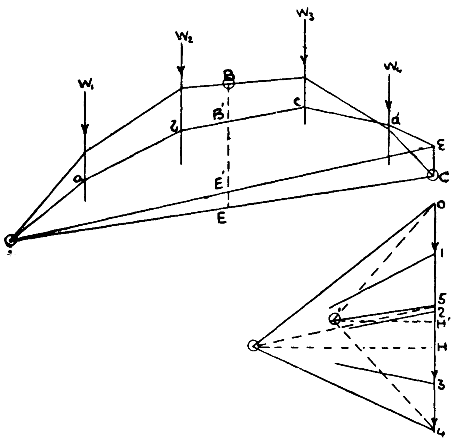


FIG. 10.

mines the vertical forces at A and C, which will be the same wherever the pole is taken. But if the link-polygon is to pass through A and C, its closing line must be parallel to AC, and therefore the pole must lie on a line parallel to AC drawn through 5. In order to make the polygon also pass through B it must have a vertical depth at B equal to BE ;

hence its vertical depth B′E′ must be increased so as to make it equal to BE, and this may be effected by decreasing the polar distance in the ratio B′E′ : BE. If then a new pole O′ be taken on the line through 5 parallel to AC and so that O′H′ : OH = B′E′ : BE, then a new polygon starting from A should pass through B and C, and this will be the linear arch required; that is to say, it will be the true line of pressure.

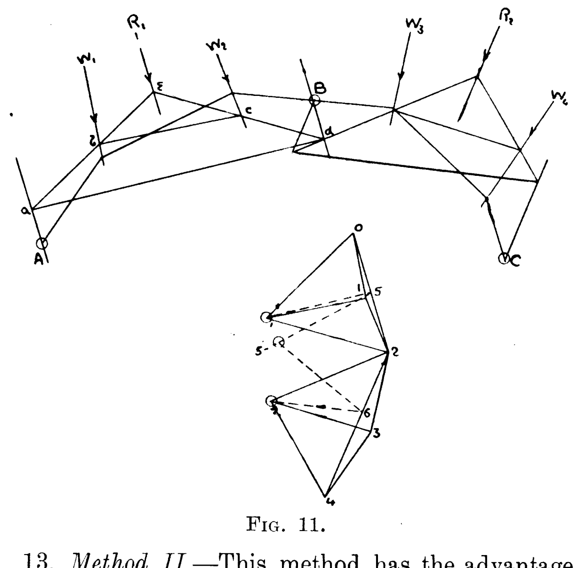


FIG. 11.

13. *Method II.*—This method has the advantage that it may be applied as well to forces which are not vertical. For example, suppose A, B, C (Fig. 11) are the pins and W_1, W_2, W_3, W_4 are any loads. Construct their vector figure 0, 1, 2, 3, 4 and with any convenient pole O_1 construct a link-polygon

a, *b*, *c*, *d* for the loads between A and B, the first and last links of which meet in *e*. Then a line R_1 through *e* parallel to 0, 2 represents the line of action of the resultant, and if we draw through A and B parallels to it, to cut the polygon in *a* and *d*, and join *ad*, a line through O_1 parallel to *ad* divides the resultant 0, 2 at the point 5, into two parts which represent the pressures on the pins; but this polygon will not be part of the linear arch unless it also passes through A and B, and in order to do this it must have its closing line *ad* to pass through A and B. Join AB and through 5 draw 55′ parallel to it; then if a new pole be taken anywhere on this line, the new polygon starting through A will pass through B also. Similarly, if we draw any link-polygon for the loads between B and C with any pole O_2, and find the pressures at B and C as before, viz. 2, 6 and 6, 4, a line through 6 parallel to BC determines the line on which the final pole must be taken to make the polygon pass also through B and C. Hence if we take the point O as pole and start a polygon from A, it should pass also through B and C and be therefore the required linear arch.

14. *Method III. The Reaction Locus.*—This method consists in finding the reactions due to each individual load and from them determining the resultant reactions by compounding the individual components. One of these resultants being known in magnitude, direction, and position, we have the three necessary conditions for fixing the line of pressure. Let A, B, C (Fig. 12) be the three pins of the arch and let W_1, W_2, etc., be any loads on it. Then if we first suppose all the loads removed

except W_1, the line of pressure for this load will pass directly through B and C, because there is no load between these points to deflect it. Produce BC therefore to cut the line of action of the load W_1 in D. Then the reaction at A must also pass through D, and AD is therefore its direction. A triangle of forces 0, 1, 2 being now drawn, the reactions 1, 2 at B and 2, 0 at A due to W_1 are found. Similarly, if we suppose all the loads except W_2

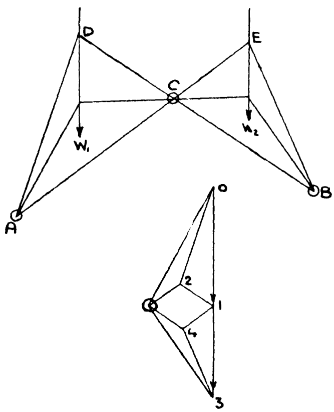


FIG. 12.

removed, the line of pressure will pass directly through A and C. Hence if we produce the line joining A to C to cut W_2 in E, the reaction at B must pass through E, and if 1·3 represent W_2, a triangle of forces 1, 3, 4 will give the values of the reactions at A and B due to W_2.

If these are the only two loads we may complete the parallelogram 2, 1, 4, O, and it will be seen that 3, O is the resultant reaction at B, and O*o* is the resultant reaction at A. If therefore we draw a link-polygon with O as pole, passing through A, it should pass also through C and B and will be the linear arch required.

In this way the reactions due to any number of loads may be found for each in turn, and these may then be compounded by means of a vector polygon to give the resultant reactions at A and B. Either of these being known the line of pressure can be drawn without further difficulty.

If we suppose the loads W_1 and W_2 to continually change their position, the points D and E will move along two straight lines which are the locus of the points in which the reactions intersect as a load moves across the arch. For this reason they are known as the *reaction locus* for this particular case, and it will be seen that as soon as the reaction locus is known, it will be no more difficult to find the reactions in the case of a two-pinned arch than it is in the present case, though the locus itself is not so simply determined.

15. **Determination of the Thrust, Shearing Force, and Bending Moment at any Point by means of the Line of Pressure. Eddy's Theorem.** —The effect on any section which is produced by the external load is usually analysed into (1) a thrust, (2) a shearing force, and (3) a bending moment. When these are known we have all that is necessary for the calculation of the stresses. Now all of these are at once readily found at any section so soon as the line of pressure is known. For suppose *abcd* (Fig. 13) to be a portion of the line of pressure for an arch and AB to be any section of the arch-ring,

whose axis passes through the centroid of the section C, and let the resultant thrust R acting along *cb* cut the plane of the section in the load-point L. Then the magnitude of R will be known from the corresponding ray O, 1 in the vector figure, and if

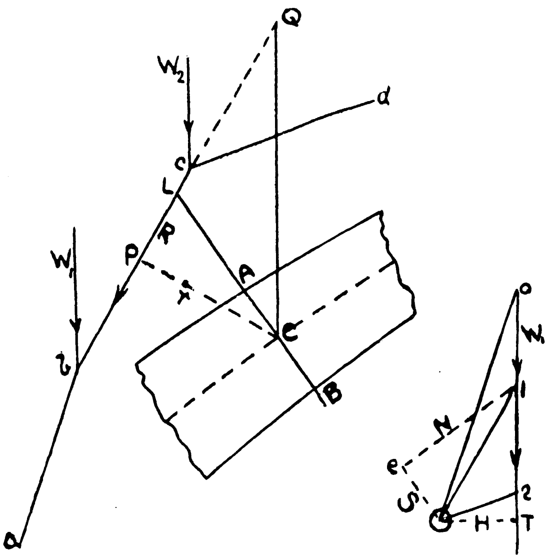


FIG. 13.

we resolve it into components 1*e* and *e*O normal and parallel to the section AB, these components will be the values of the normal thrust N on the section and the shearing force S. Also because the resultant passes through L at a distance CL from the axis of the arch it causes a bending moment whose value is R × *r*, where *r* = CP is the perpendicular distance of the resultant from the point C. Draw CQ vertical to cut R in Q. Then the triangles CPQ, TO1 are similar, and therefore

$$CQ : CP = O1 : H$$
$$\text{or } H . CQ = O1 . CP$$

But O1 measures the resultant R, and CP is its lever-arm about C. Therefore O1 . CP is the bending moment at the section and is equal to H . CQ ; that is to say, the bending moment at any section is measured by the product of the horizontal thrust H measured in the scale of loads, into the vertical intercept between the axis of the arch and the line of pressure, measured in the scale of lengths.

This important theorem is known as Eddy's Theorem, and is very convenient in the graphical treatment of the arch because the intercepts between the line of pressure and the arch-axis give a visual perception of the relative values of the bending moments at various points, similar to that which the bending moment diagram does in the case of beams under vertical loads.

16. **Application of the Preceding Methods to the Case of a Three-Pinned Arch with Distributed Loading.**—Let ACB (Fig. 14) be a circular arch of 150 ft. span and 15 ft. rise, with pin joints at A, B, and C, and suppose the dead load to be 26 cwt. per foot run over the whole span, with an additional live load of 18 cwt. per foot run on one-half the span. It is required to find the thrust, shear, and bending moment at any point.

Method I. By Means of the Line of Pressure.—In a case of this kind the line of pressure may be expected to lie so near the axis of the arch itself under ordinary conditions that the vertical intercepts between the two curves will be too small to ensure reliable measurement. It will therefore be an advantage to distort the vertical scale of the drawing

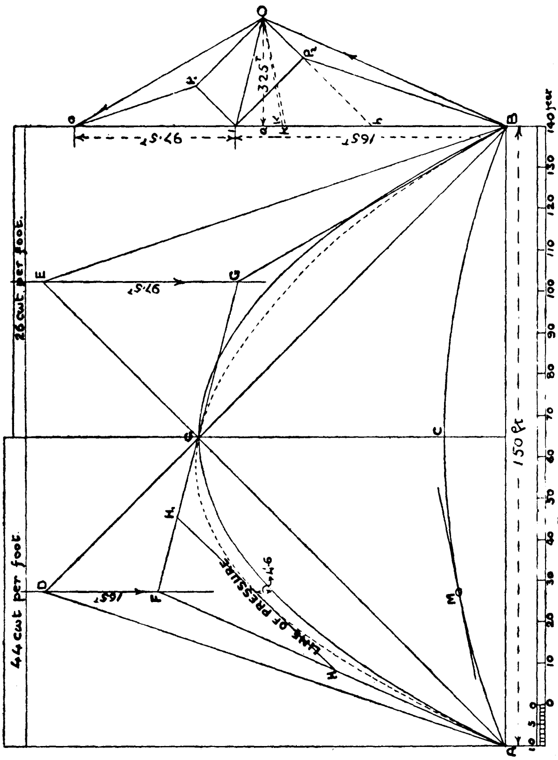


FIG. 14.

by magnifying its vertical dimensions, say five times. This should not be done graphically, because the errors will be multiplied to the same extent, but the ordinates should be calculated by the formulæ or by the Table on page 135. In this way we get the elliptical curve AC_1B which is a parallel projection of the circular arc ACB. BC_1 and AC_1 are then joined and produced to cut the resultants of the two loads on the ribs AC_1, BC_1 in D and E. This determines the directions of the reactions at A and B due to each of these loads taken separately, and if we now set down O . 1 = 97·5 tons and 1 . B = 165 tons the magnitudes of the reactions due to these loads individually will be given by the triangles P_1 . O . 1 and P_2 . 1 . B whose sides are drawn parallel to the directions of the forces, and if we complete the parallelogram $OP_1 1P_2$, Oo, OB will be the resultant reactions at B and A. The polar distance scales 65 tons, but the actual horizontal thrust of the arch will be five times greater than this, on account of the vertical distortion, and will therefore be 65 × 5 = 325 tons.

Next draw FG parallel to O . 1 and join AF, GB. These lines will be tangents to the line of pressure, and since this curve will, for uniform loading, consist of parabolic arcs, these arcs may be drawn in to touch the tangents AF, FC_1 and C_1G, GB by the method described on page 138.

This determines the line of pressure, and the bending moment at any point may now be found by measuring the vertical ordinate between the line of pressure and the arch-axis in feet and multiplying it by the polar distance, viz. 65 tons. For example, at $\frac{1}{4}$ span from the left-hand abutment the ordinate scales 4·6 ft. and the bending moment there is therefore 4·6 ft. × 65 tons = 299 tons ft.

The thrust in the arch at this point may be found by drawing the tangent to the curve, viz. HH_1 and Oh parallel to it. Its actual direction can be found by making $ak = \frac{1}{5}\,ah$, when Ok will be the direction required, and if we resolve Ok into components Ok', $k'k$ parallel respectively to the tangent and radius of the arc at M, these will give the values of the normal thrust and of the shearing force at M, and in the same way the values of the thrust, shear, and bending moment at any other point can be found, and curves of thrust and shear constructed therefrom.

17. *Method II. By Calculation.*—Let V_A, H (Fig. 15) be the vertical and horizontal components of the

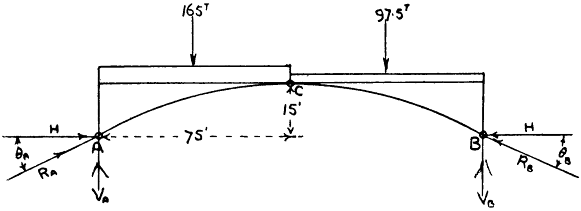


FIG. 15.

reaction at A, and V_B, H be those at B.

$$\text{Then } V_B = 97{\cdot}5 + \tfrac{1}{4} \times 67{\cdot}5 = 114\tfrac{3}{8} \text{ tons,}$$

$$\text{and } V_A = 262\tfrac{1}{2} - 114\tfrac{3}{8} = 148\tfrac{1}{8} \text{ tons.}$$

Taking moments about C, we have

$$114\tfrac{3}{8} \times 75 = H \times 15 + 97{\cdot}5 \times 37{\cdot}5 =$$

$$\underline{328\tfrac{1}{8} \text{ tons}} = \text{the hor. thrust.}$$

$$\text{Further } \tan\theta_A = \frac{V_A}{H} = \frac{148\frac{1}{8}}{328\frac{1}{8}} = {\cdot}4514. \quad \therefore \theta_A = 24^\circ\,18';$$

$$\tan\theta_B = \frac{V_B}{H} = \frac{114\frac{3}{8}}{328\frac{1}{8}} = {\cdot}3486 \quad \therefore \theta_B = 19^\circ\,13'.$$

$$R_A = H \sec\theta_A = 328\tfrac{1}{8} \times 1{\cdot}0972 = 359{\cdot}9 \text{ tons.}$$

$$R_B = H \sec\theta_B = 328\tfrac{1}{8} \times 1{\cdot}0590 = 347{\cdot}4 \text{ tons.}$$

To find the bending moment at the point M (Fig. 16).

The bending moment at M $= V_A \times 37{\cdot}5 - H \times y$
$- \frac{1}{2} \times 165 \times 18{\cdot}75.$

Now $y = R \cos \phi - R \cos \phi_0 = R (\cos \phi - \cos \phi_0)$
and $75^2 = R^2 - (R - 15)^2 = 30\,R - 225$
whence R = 195 ft. ∴ OC = 180 ft.

$\sin \phi = \frac{37{\cdot}5}{195} = {\cdot}1923.$ ∴ $\phi = 11^\circ\ 5'$ and $\cos \phi = {\cdot}9812.$

Also $\cos \phi_0 = \frac{180}{195} = {\cdot}9231.$

∴ $y = 195\ ({\cdot}9812 - {\cdot}9231) = 195 \times {\cdot}0581 = 11{\cdot}33$ ft.

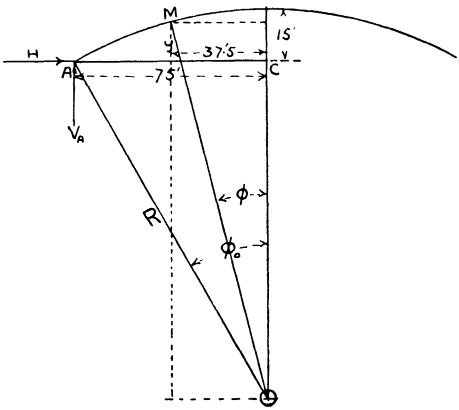


FIG. 16.

∴ the bending moment at
$M = 148\frac{1}{8} \times 37{\cdot}5 - 328\frac{1}{8} \times 11{\cdot}33 - 82{\cdot}5 \times 18{\cdot}75$
$= 5555 - 3718 - 1547 = \underline{290 \text{ tons ft.}}$

18. *Method III. By Influence Lines.*—In this procedure the bending moment at any section due to a unit load in any given position is found by draw-

ing an influence diagram for the particular section considered. (See "Moving Loads by Influence Lines" in this Series.) Thus, if we require to find the bending moment at any section D (Fig. 17) an influence line must be drawn for this particular section to show the bending moment there for a unit

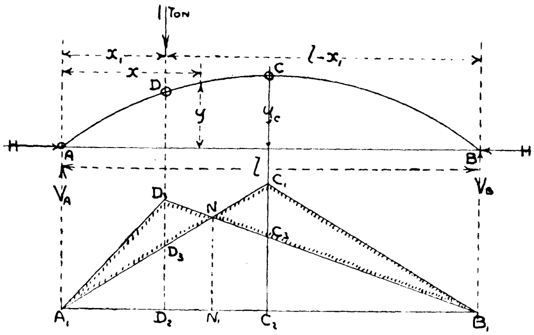


FIG. 17.

load placed anywhere on the span. Now the actual bending moment at any section is the difference between the positive bending moment due to the vertical forces, less the negative bending moment due to the horizontal thrust. But when a unit load acts at any point distant x_1 from the end A, the bending moment at D due to the vertical forces is represented by a triangle A_1D_1B, for which

$$D_1D_2 = \frac{1 \times x_1 (l - x_1)}{l} \text{ tons ft.}$$

$$\text{Also since } V_B \times l = 1 \times x_1,\ V_B = \frac{x_1}{l},$$

and taking moments about C

$$H \cdot y_c = \frac{x_1}{l} \cdot \frac{l}{2}. \quad \therefore H = \frac{x_1}{2y_c},$$ from which it appears

that H varies as x_1, and that the influence diagram for H is therefore a triangle whose maximum ordinate is C_1C_2. But when the load is at the centre $x_1 = \frac{l}{2}$ and therefore the maximum value of H is $\frac{l}{4y_c}$, whilst the bending moment at any point x due to the force H will be Hy. The ordinates of the triangle $A_1C_1B_1$ therefore when multiplied by y will give the bending moment at the point D due to the horizontal thrust, and the actual bending moment at D will therefore be given by the vertical intercept between the two triangles $A_1D_1B_1$ and $A_1C_1B_1$.

Thus in the present case we construct an influence diagram for $\frac{1}{4}$ span by setting up

$$C_1C_2 = \frac{ly_1}{4y_c} = \frac{150 \times 11 \cdot 33}{4 \times 15} = 28 \cdot 325 \text{ tons},$$

the ratio being actually multiplied by the unit load of 1 ton say. If we suppose C_1C_2 drawn 2·8325 ins. long the scale will be 1 in. = 10 tons. Join A_1C_1, B_1C_1. Then the triangle $A_1C_1B_1$ is the influence diagram for the horizontal thrust H, or the "H-line" as it is called. Next set up

$$D_1D_2 = \frac{x_1(l - x_1)}{l} = \tfrac{1}{4} \times 112 \cdot 5 = 28\tfrac{1}{8} \text{ tons},$$

and join D_1A_1, D_1B_1. Then $A_1D_1B_1$ is the influence diagram for the vertical forces.

The ordinate at any point of the shaded area will now give the bending moment at D due to the unit load at the given point. The bending moment at D due to any given system of loads in any position may now be readily found by multiplying the ordinate under each load by the magnitude of the load and summing the results. Also by a well-known property of influence lines, if the load is uniformly distributed the bending moment will be given by the area of the influence diagram under the load multiplied by the load per unit length.

The maximum and minimum effects may therefore be easily investigated in this way.

In the present case $A_1N_1 = 59{\cdot}75$ ft., $D_1D_3 = 13{\cdot}96$ ft., $C_1C_3 = 9{\cdot}58$ ft.

Hence the area $A_1D_1N = \frac{1}{2} \times 59{\cdot}75 \times 13{\cdot}96 = 417{\cdot}2$ tons ft.;

the area $NC_1C_3 = \frac{1}{2} \times 9{\cdot}58 \times 15{\cdot}25 = 73{\cdot}05$ tons ft.;

and the area $B_1C_1C_3 = \frac{1}{2} \times 9{\cdot}58 \times 75 = 359{\cdot}3$ tons ft.

$\therefore$ the positive bending moment due to
$2{\cdot}2$ tons p. ft. $= 2{\cdot}2\,(417{\cdot}2 - 73{\cdot}05) = 757{\cdot}1$ tons ft.,
and the negative bending moment due to
$1{\cdot}3$ tons p. ft. $= 1{\cdot}3 \times 359{\cdot}3 \qquad = 467{\cdot}1$ „ „

$\therefore$ the resultant bending moment $= 290{\cdot}0$ tons ft.

thus checking exactly with the calculated result.

This will not, however, be the maximum bending moment as is often assumed, because it will be seen at once that the maximum value will occur when the live load covers the left-hand portion of the arch up to the point N only, in which case the bending moment at $\frac{1}{4}$ span will be greater than before by the amount represented by the triangle NC_1C_3, viz. $73{\cdot}05$ tons ft. The maximum bending moment at $\frac{1}{4}$ span is therefore $290 + 73 = 363$ tons ft.

19. **Example of a Three-pinned Spandrel-braced Arch.**—This type of arch is very economical

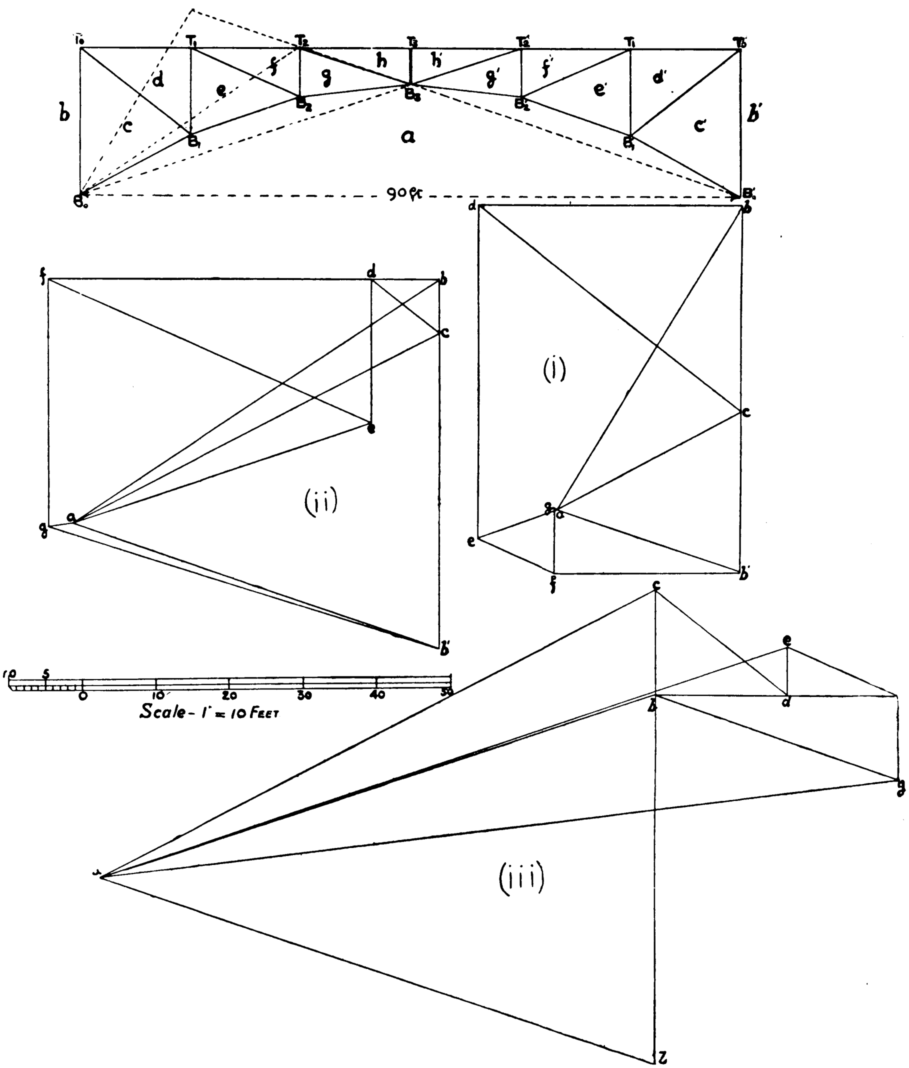


Fig. 18.

and convenient of construction in positions to which it is adapted, and is consequently commonly used.

The maximum and minimum forces occurring in each member are best found in one or other of the following ways. First, a unit load is placed at any one of the top joints, e.g. at T_2 (Fig. 18), and the reactions at B_0, $B_0{}^1$ due to this load are found by constructing the triangle of forces bb^1a, fig. 1. A reciprocal figure or force diagram is then drawn in the usual way for the whole structure, from which the forces in each member may be scaled off. These forces are tabulated, and when each is multiplied by the actual dead or live load which occurs at any joint, the force in any member which that load produces will be known; and it will therefore be easy to select that combination of loads which produces either the maximum or the minimum force in any particular member. By the second method, an influence line is drawn for the moment of resistance of each bar in the structure, from which the maximum or minimum force which can occur in it may be deduced in the usual way, by loading exclusively either the positive or negative areas.

In order to compare the two methods, which may be used mutually to check each other, we will consider the case of the three-pinned arch shown in Fig. 18, which we will suppose to carry a uniform dead load of $1\frac{1}{2}$ tons per foot and a live load of 1 ton per foot, distributed along the upper chord by means of a platform. The lower panel-points lie on a parabolic arc of 90 ft. span and 15 ft. rise.

Method I.—Suppose a 1 ton load placed in succession at each of the joints T_0, T_1, T_2, T_3 of the top chord. Force diagrams are drawn for each case in the usual way as shown in (i), (ii), and (iii) and the forces in each member are scaled off and tabulated as follows :—

1	2	3	4	5	6	7
Member.	Force due to Unit Load at			Force due to Actual Dead Load of $22\frac{1}{2}$ Tons at		
	T_1	T_2	T_3	T_1	T_2	T_3
T_0T_1 .	+ ·73	+ ·18	− ·41	+ 16·4	+ 4·05	− 9·25
T_1T_2 .	+ ·525	+ 1·07	− ·703	+ 11·8	+ 24·05	− 15·8
T_2T_3 .	0	0	0	0	0	0
$T_2^1T_3^1$.	0	0	0	0	0	0
$T_1^1T_2^1$.	− ·26	− ·505	− ·703	− 5·85	− 11·35	− 15·8
$T_0^1T_1^1$.	− ·14	− ·265	− ·41	− 3·16	− 5·95	− 9·25
B_0B_1 .	+ ·575	+ 1·135	+ 1·72	+ 12·9	+ 25·3	+ 38·7
B_1B_2 .	− ·24	+ ·87	+ 2·03	− 5·4	− 19·55	+ 45·7
B_2B_3 .	− ·025	− ·07	+ 2·23	− ·563	− 1·57	+ 50·2
$B_2^1B_3^1$.	+ ·77	+ 1·52	+ 2·23	+ 17·35	+ 34·2	+ 50·2
$B_1^1B_2^1$.	+ ·68	+ 1·355	+ 2·03	+ 15·3	+ 30·5	+ 45·7
$B_0^1B_1^1$.	+ ·57	+ 1·135	+ 1·72	+ 12·8	+ 25·55	+ 38·7
T_0B_1 .	− ·92	− ·22	+ ·52	− 20·7	− 4·95	+ 11·7
T_1B_2 .	+ ·22	− ·98	+ ·32	+ 4·95	− 22·0	+ 7·2
T_2B_3 .	+ ·55	+ 1·12	− ·74	+ 12·4	+ 25·2	− 16·65
$T_2^1B_3$.	− ·275	− ·53	− ·74	− 6·2	− 11·9	− 16·65
$T_1^1B_2^1$.	+ ·13	+ ·258	+ ·32	+ 2·93	+ 5·8	+ 7·2
$T_0^1B_1^1$.	− ·17	− ·335	+ ·52	− 3·83	− 7·53	+ 11·7
B_0T_0 .	+ ·56	+ ·135	− ·315	+ 12·6	+ 3·04	− 7·1
B_1T_1 .	+ ·91	− ·41	− ·14	+ 20·5	+ 9·2	− 3·15
B_2T_2 .	− ·155	− ·675	+ ·24	− 3·49	− 15·6	+ 5·4
B_3T_3 .	0	0	+ ·5	0	0	+ 11·25
$B_2^1T_2^1$.	+ ·088	+ ·153	+ ·242	+ 1·98	+ 3·54	+ 5·4
$B_1^1T_1^1$.	− ·050	− ·10	− ·14	− 1·125	− 2·25	− 3·15
$B_0^1T_0^1$.	− ·10	− ·205	− ·32	− 2·25	− 4·62	− 7·1

8	9	10	11	12	13	14	15
Force due to Live Load of 15 Tons at			Total Force due to Dead Load.	Total Force due to Live Load.		Max.	Min.
T_1	T_2	T_3	Tons.	Tons.	Tons.	Tons.	Tons.
+10·95	+ 2·7	− 6·15	− 7·16	+ 13·65	−18·38	+ 6·49	− 25·5
+ 7·88	+16·05	−10·55	− 12·95	+ 23·93	−32·60	+ 10·98	− 45·5
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
− 3·9	− 7·6	−10·55	− 12·95	+ 23·93	−32·60	+ 10·98	− 45·5
− 2·1	− 3·98	− 6·15	− 7·16	+ 13·65	−18·38	+ 6·49	− 25·5
+ 8·62	+17·05	+25·8	+153·95	+102·9	0	+256·87	+156·4
− 3·6	+13·05	+30·45	+112·25	+104·5	− 3·6	+216·78	+108·6
− ·38	− 1·05	+33·45	+149·82	+101·2	− 1·43	+251·1	+148·4
+11·55	+22·8	+33·45	+149·82	+101·2	− 1·43	+251·1	+148·4
+10·2	+20·38	+30·45	+112·25	+104·5	− 3·6	+216·8	+108·6
+ 8·45	+17·2	+25·8	+154·04	+102·9	0	+256·9	+154·0
−13·8	− 3·3	+ 7·8	− 13·61	+ 15·6	−24·67	+ 2·0	− 38·3
+ 3·3	−14·7	+ 4·8	+ 6·08	+ 18·7	−14·7	+ 24·8	− 8·6
+ 8·25	+16·8	+11·1	− 13·80	+ 47·2	−12·07	+ 23·4	− 25·9
− 4·125	− 7·95	+11·1	− 13·80	+ 47·2	−12·07	+ 23·8	− 25·9
+ 1·95	+ 3·87	+ 4·8	+ 6·08	+ 18·7	−14·7	+ 24·8	− 8·6
− 2·55	− 5·02	+ 7·8	− 13·61	+ 15·6	−24·67	+ 2 0	− 38·3
+ 8·4	+ 2·02	− 4·73	− 5·43	+ 10·4	−12·68	+ 5·0	− 18·1
+13·65	+ 6·15	− 2·1	+ 20·03	+ 19·8	− 6·45	+ 39·8	+ 13·6
− 2·33	−10·1	+ 3·6	− 2·86	+ 10·8	−12·43	+ 7·9	− 15·3
0	0	+ 7·5	+ 22·50	+ 15·0	0	+ 37·5	+ 22·5
+ 1·32	+ 2·32	+ 3·6	− 2·86	+ 10·8	−12·43	+ 7·9	− 15·3
− ·75	− 1·5	− 2·1	+ 20·03	+ 19·8	− 6·45	+ 39·8	+ 13·6
− ·15	− 3·07	− 4·73	− 5·43	+ 10·4	−12 68	+ 5·0	− 18·1

In the above Table, Col. 1 gives the name of each member of the arch. Cols. 2, 3, and 4 show the force induced in each of these members by a 1-ton load situate at joints T_1, T_2, T_3 respectively, as measured on the corresponding force diagram. Cols. 5, 6, and 7 show the forces actually caused in the members by the actual dead load of 22½ tons at the points T_1, T_2, and T_3. Cols. 8, 9, and 10 show the same thing for the temporary load of 15 tons at each joint. Col. 11 shows the total force in each member due to the whole dead load. These values are found by adding the forces produced by the loads at T_1, T_2, and T_3 in each bar to those produced by the loads acting on the opposite half of the arch, which is the same thing as adding them to the forces produced by T_1, T_2, and T_3 in the corresponding members of the right-hand half. Cols. 12 and 13 give the maximum and minimum forces which occur in the members, which are found by adding all the positive forces for a maximum and all the negative forces for a minimum in each case. Cols. 14 and 15 give the maximum and minimum forces due to the dead and live loads combined, taking the worst combinations that can occur.

20. *Method II. By Influence Lines.*—In applying this method to find the maximum force in any bar of a framed structure, as in the present case, an influence line must be drawn for the bending moment about the moment centre, from which the maximum bending moment can be found and the maximum force in the bar deduced.

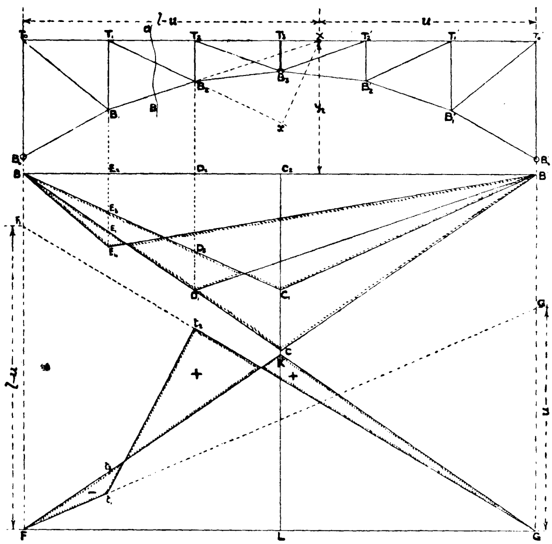


FIG. 19.

For example, if we require to find the force in the bar T_1T_2 say, of the preceding example, an influence line is drawn for the bending moment about the point B_2, which is the moment centre for that bar, because if an imaginary section be taken through the three members T_1T_2, T_1B_2, and B_1B_2, by taking moments about the point B_2, the unknown forces in the bars

T_1B_2, B_1B_2 are eliminated from the resulting equation, and the moment of the force in the bar T_1T_2 about the point B_2, will be in equilibrium with the bending moment round that point due to the external forces on the one side of the section; and since the force in $T_1T_2 \times T_2B_2$ is equal to the bending moment, the value of T_1T_2 is found by dividing this bending moment by the length of the lever arm T_2B_2.

It is therefore required, first of all, to construct an influence diagram for the moment about B_2, showing how this moment changes as a unit load moves over the arch. Now the actual moment at any point is the difference between the positive moments due to the vertical forces and the negative moments due to the horizontal thrust of the arch, and we have shown already, in § 18, how an influence diagram may be constructed to show this.

Thus, in the present case, to find the force in the bar T_1T_2, when a unit load is at T_1 say, 15 ft. from the L.H. end, we first construct the influence diagram for the point B_2. Make

$$C_1C_2 = \frac{ly_1}{4y_c} = \frac{90 \times 13{\cdot}2}{4 \times 15} = 19{\cdot}8 \text{ tons ft.}$$

Using a scale of 1 in. = 10 tons, $C_1C_2 = 1{\cdot}98$ in. Next make

$$D_1D_2 = \frac{x_1(l - x_1)}{l} = \frac{30 \times 60}{90} = 20 \text{ tons ft.} = 2 \text{ ins.},$$

and complete the diagram. The intercept E_1E at 15 ft. from the L.H. end then scales 3·5 tons ft., and the force in T_1T_2 is therefore

$$\frac{3{\cdot}5}{T_2B_2} = \frac{3{\cdot}5}{6{\cdot}75 \text{ ft.}} = {\cdot}52 \text{ tons.}$$

Similarly, when the unit load is at T_2, 30 ft. from the L.H. end, the intercept on the influence diagram scales 7·0 tons ft., and the force in T_1T_2 is then $\frac{7 \cdot 0}{6 \cdot 75} = 1 \cdot 04$ tons, and so on.

Next to find the force in a lower chord member, say B_1B_2. In this case the point T_1 will be the moment centre, and the influence line for H will have its maximum ordinate $= \frac{ly_2}{4y}$ by the same reasoning as before, where y_2 is the height of the point T_1 above the springing-joint = 20 ft. The maximum value of the bending moment due to H will therefore be $CC = \frac{90 \times 20}{60} = 30$ tons ft. Also the ordinate for the bending moment due to the vertical forces for the position T_1 will be

$$E_2E_4 = \frac{15 \times 75}{90} = 12 \cdot 5 \text{ tons ft.}$$

If we plot these values and complete the diagram we find from it that for a unit load at T_1 the bending moment about T_1 is $E_1E_4 = 2 \cdot 5$ tons ft., and the force in $B_1B_2 = \frac{2 \cdot 5}{\text{lever arm of } B_1B_2} = -0 \cdot 23$ tons ft., the negative sign indicating tension, because the moment about T_1 is clockwise.

To find the force in a diagonal member. Suppose we consider, for example, the diagonal T_1B_2. The moment centre for this member will be the point X in which the bars T_1T_2 and B_1B_2 cut by the section $\alpha\beta$ intersect.

Now if the unit load be on the left T_1,

$$M = V_B' \cdot u - Hy_2 \quad . \quad . \quad (1)$$

and when it is on the right of T_2

$$M_x = V_B(l - u) - Hy_2 \quad . \quad . \quad (2)$$

but if the unit load be distant v from T_o, $V_B = \dfrac{l - v}{l}$

and $V_B' = \dfrac{v}{l}$.

$$\therefore M_x = \frac{v}{l} \cdot u - Hy_2 \quad . \quad . \quad (1')$$

$$\text{and } M_x = \frac{l - v}{l} \cdot (l - u) - Hy_2 \quad . \quad (2')$$

The first terms in these two equations are both represented by straight lines, and if we put $v = l$ in the expression $\dfrac{v}{l} \cdot u$, we get for the ordinate at the extreme right the value u; and if we put $v = o$ in the expression $\dfrac{l - v}{l}(l - u)$ we get $l - u$ as the value of the ordinate on the extreme left. If we now draw in the lines GF_1, FG_1, and drop verticals from T_1 and T_2 to meet them in t_1 and t_2, the broken line Ft_1, t_2, G represents the influence diagram for the bending moment at X due to the vertical forces alone. If then we set up $KL = \dfrac{ly_2}{4y_c}$ as before, the triangle FKG will be the influence line for the bending moment due to the horizontal thrust H. The first of these bending moments will cause tension in the bar T_1B_2 and the second compression. The effect of any load on the moment at X can now be found from the influence diagram, and the force which it causes in the bar T_1B_2 can be deduced by dividing this moment by the lever arm. Thus, for example, in the present case, t_1t_3 scales 3·4 tons, whilst the lever arm Xx is 15·4 ft., which makes the tensile force in the bar T_1B_2 = 0·22 ton when the unit load is at T_1. Similarly, the force may be found for any other position of the unit load, and, therefore, for any system of loads in any position on the span.

Having thus found for any particular bar the force caused in it when a load is placed at any point of the span, the maximum force which will occur in it under any arrangement of loading can easily be derived, because when calculating the maximum tensile force in any member, all the temporary loads which cause compression, and therefore diminish the tensile force, must be left out of consideration; and similarly when calculating the maximum compressive force all those temporary loads which cause tension must be omitted.

By comparing the results obtained above by this method, it will be seen that they agree closely with those obtained by Method I.

CHAPTER III.

THE ELASTIC THEORY OF THE ARCH.

21. WHEN an arch has less than three joints, the three conditions necessary to fix the position of the line of pressure must be made up by a consideration of the elastic properties of the arch-ring. It has been objected to the elastic theory that the arch-ring not being homogeneous throughout, especially when joints exist, the theory cannot be properly applied ; but this objection has little force in fact, because the arch-ring in such cases is usually composed of distinct parts, like voussoirs, which are bounded by joints sensibly normal to the axis, and there is no theoretical objection to neighbouring parts having different coefficients of elasticity, so long as the parts themselves have uniform elastic properties.

The theory is, however, somewhat complex, and involves assumptions and simplifications more or less approximate, so that the results derived from its application have until comparatively recent years been rightly regarded with a certain amount of suspicion. This distrust has been largely removed by the careful and complete series of experiments carried out by the Institution of Austrian Engineers and Architects, the results of which were published about the year 1900.

These experiments were made on arches of masonry and brickwork and of concrete, plain and armoured, having spans of from 1·35 to 23 m. (75·4 ft.) span, and 4·6 m. (15·1 ft.) rise. The large brick and stone arches of 23 m. span were built in Portland cement mortar, 1 cement to 2·6 sand. The crown thickness was 0·6 m. (1 ft. 11¾ ins.) and the thickness at the springings was 1·14 m. (3 ft. 7½ ins.). The concrete arch was 0·7 m. (2 ft. 3½ ins.) thick throughout, whilst the Monier arch was 0·35 m. (1 ft. 1¾ ins.) at the crown and 0·6 m. (1 ft. 11⅝ ins.) at the springings. The longitudinal rods were 0·55 in. diameter, and the transverse rods 0·276 in. diameter, the mesh being 2½ ins. The width was 2 m. (6·56 ft.). The loading was applied gradually from one abutment to the crown, and the distortion of the axis was carefully measured until the arches showed signs of collapse. From these measurements the values of E were calculated and were found to be as follows :—

For masonry . .	60,400 kg./cm.^{2}	=	858,000 lb./in.^{2}
,, brickwork . .	27,800 ,,	=	393,000 ,,
,, plain concrete .	246,000 ,,	=	3,475,000 ,,
,, reinforced concrete	333,800 ,,	=	4,700,000 ,,

At a certain critical load cracks began to appear in the arch-ring between ¼ and ⅓ or between ⅔ and ¾ of the span, and at the supports, agreeing in general with the position of the joints of rupture as deduced from theory.

The breaking loads were as follows :—

Brickwork arch . . .	67·5 tons.
Masonry	74·02 ,,
Concrete	83·27 ,,
Monier concrete . . .	146·12 ,,

The breaking load for the masonry arches exceeded the critical load by about 30 per cent, for the brickwork arches by about 59 per cent, for the plain concrete by 31 per cent, and for the reinforced concrete by 86 per cent.

The general result of the experiments was to show that the elastic theory might be applied with confidence to arches of masonry, brickwork, and concrete of such proportions as are now common, and as a direct outcome of them a stone arch of 213 ft. span and 58·6 ft. rise was built over the Pruth in Austria. A full account of these experiments is given in Volume I, "Handbuch für Eisenbetonbau".

The general accuracy of the elastic theory of the arch is also confirmed by experiments on models under polarized light. Dr. E. G. Coker, who has made many experiments on this subject, has kindly directed my attention to a memoir of M. Mesnager ("Détermination complète sur un modèle réduit des tensions qui se produiront dans un ouvrage. Annales des Ponts et Chaussées") in which the author describes a glass model of a bridge over the Rhone which was constructed for the purpose of checking the results as calculated in the usual way. This model, examined under polarized light, gives an ocular demonstration of the change of position of the neutral axis of the arch-ring when the load is applied, which is quite in accordance with the deductions of theory, whilst the stresses obtained by the optical method did not differ more than 14·7 per cent, as a maximum, from the results of the elastic method.

22. **The Development of the Elastic Theory of**

the Arch. Angular Distortion of a Rib due to Bending.—Let a be the area of a horizontal strip of the cross-section of an arch-ring (Fig. 20), distant u from the axis CC through the centroid of the section, and let A_1B_1, A_2B_2 be neighbouring sections, δs apart. Then if the arch-ring bend through an angle

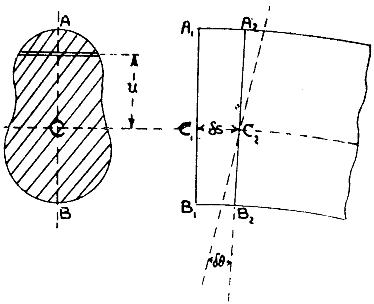


FIG. 20.

$\delta\theta$ on the length δs when the load is applied, and if δx is the change of length produced thereby on the length δs at distance u from the axis CC, then the stress induced is $s = \mathrm{E} \cdot \frac{\delta x}{\delta s}$, and the moment of resistance of the strip is therefore

$$sau = \mathrm{E} \cdot \frac{\delta x}{\delta s} \cdot a \cdot u.$$

The total moment of resistance of the cross-section is therefore

$$\mathrm{R} = \Sigma\left[\mathrm{E} \cdot \frac{\delta x}{\delta s} \cdot a \cdot u\right],$$

or since $\delta x = u \cdot \delta\theta$

$$\mathrm{R} = \Sigma\left[\mathrm{E} \cdot \frac{u \cdot \delta\theta}{\delta s} \cdot au\right];$$

and since for any given cross-section of an arch-ring, E, δs and $\delta\theta$ may be regarded as constants without sensible error, we get

$$R = E \cdot \frac{\delta\theta}{\delta s} \Sigma [au^2],$$

or since $\Sigma [au^2] = I$

$$R = EI \cdot \frac{\delta\theta}{\delta s},$$

and since for equilibrium the moment of resistance R of the cross-section must be equal and opposite to the bending moment M due to the external forces, we get

$$M = EI \cdot \frac{\delta\theta}{\delta s} \text{ or } \delta\theta = \frac{M \cdot \delta s}{EI};$$

that is to say, the angle of flexure due to the bending moment is proportional to $\frac{M}{EI}$.

Hence, if a rib be fixed at one end and free at the other, and a known system of loads be applied to it, the bending moment at every point will be known and the total angular displacement between its extremities will be

$$\Delta\theta = \Sigma \left[\frac{M \cdot \delta s}{EI}\right] \quad . \quad . \quad . \quad (1)$$

23. **Horizontal and Vertical Displacements due to Bending.**—Let C (Fig. 21) be the centre of gravity of a small segment of rib, and suppose this section to turn through an angle $\delta\theta$ on the length δs, when the load is applied, due to the elasticity of the material, and let y be the vertical distance of the free end B below C, and x the horizontal distance between them. Due to the flexure of the element δs, suppose B to move to B′, and let δx, δy be the horizontal and vertical displacements of B due to

the change in angle $\delta\theta$. Then since these displacements are very small, BB′ is sensibly perpendicular to CB and δx is perpendicular to y. Hence in the similar triangles CDB, BD′B′

$$\frac{\delta x}{y} = \frac{BB'}{CB'} = \delta\theta$$

and similarly $$\frac{\delta y}{x} = \frac{BB'}{CB'} = \delta\theta.$$

Hence the horizontal displacement is $\delta x = y \cdot \delta\theta$ (2)

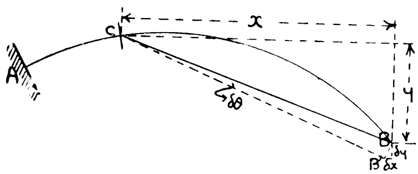


FIG. 21.

and the vertical displacement is $\delta y = x \cdot \delta\theta$. (3)

Introducing the value of

$$\delta\theta = \frac{M \cdot \delta s}{EI} \text{ we get}$$

$$\delta x = y \cdot \frac{M \cdot \delta s}{EI} \quad . \quad . \quad . \quad (4)$$

and $$\delta y = x \cdot \frac{M \cdot \delta s}{EI} \quad . \quad . \quad . \quad (5)$$

These are the small horizontal and vertical components of the actual displacement due to flexure on a small length δs of the rib, and the total horizontal and vertical displacements between the extremities will be the sum of all these

or $$\Delta x = \Sigma\left[y \cdot \frac{M \cdot \delta s}{EI}\right] \quad . \quad . \quad . \quad (6)$$

and $$\Delta y = \Sigma\left[x \cdot \frac{M \cdot \delta s}{EI}\right] \quad . \quad . \quad . \quad (7)$$

24. **Horizontal and Vertical Displacements due to Axial Thrust.**—Let N be the normal thrust on any cross-section of the rib, whose area is A. Then if the resultant act along the axis, the stress on the section will be uniform, and if we denote it by s_0, then $s = \frac{N}{A}$. The compressive strain on a small length δs will therefore be $\frac{s_0 \cdot \delta s}{E} = \frac{N}{AE} \cdot \delta s$, and if we denote compression by the negative sign, the total compression along the whole length will be

$$\Delta s = -\Sigma\left[\frac{N}{AE} \cdot \delta s\right]$$

and the corresponding horizontal and vertical displacements due to the thrust are therefore

$$\Delta x = -\Sigma\left[\frac{N}{AE} \cdot \delta x\right] \quad . \quad . \quad (8)$$

and

$$\Delta y = -\Sigma\left[\frac{N}{AE} \cdot \delta y\right] \quad . \quad . \quad (9)$$

25. **Horizontal and Vertical Displacements due to Change of Temperature.**—Let a be the coefficient of expansion of the material, and t the number of degrees rise in temperature. Then the expansion due to this rise on a length δs will be $at \cdot \delta s$ and the corresponding horizontal and vertical displacements will be $at \cdot \delta x$ and $at \cdot \delta y$, whilst the total displacements between the ends will be

$$\Delta x = \Sigma\left[at \cdot \delta x\right] \quad . \quad . \quad (10)$$

and

$$\Delta y = \Sigma\left[at \cdot \delta y\right] \quad . \quad . \quad (11)$$

26. **Total Horizontal and Vertical Displacements.**—Combining the values of Δx from equations (6), (8), and (10) and of Δy from equations (7), (9), and (11) we get

total horizontal displacement

$$= \Sigma\left[y \cdot \frac{\mathrm{M} \cdot \delta s}{\mathrm{EI}}\right] - \Sigma\left[\frac{\mathrm{N}}{\mathrm{AE}} \cdot \delta x\right] + \Sigma\left[at \cdot \delta x\right] \quad . \quad (12)$$

and total vertical displacement

$$= \quad \left[x \cdot \frac{\mathrm{M} \cdot \delta s}{\mathrm{EI}}\right] - \Sigma\left[\frac{\mathrm{N}}{\mathrm{AE}} \cdot \delta y\right] + \Sigma\left[at \cdot \delta y\right] \quad . \quad (13)$$

It is only in very flat arches that the effect of the thrust is appreciable, and as its inclusion complicates the formulæ it will be desirable in the first case to neglect its effect, and to investigate later what allowance should be made for it. Likewise the effect of change of temperature may be left for future consideration, whilst we discuss first of all the stresses due to bending which form by far the principal contribution to the actual result.

27. **Primary Equations of Condition for a Hingeless Arch.**—As we have already seen, the problem of the arch requires three independent conditions for its solution, and these conditions are, for an arch without joints, derived necessarily from its elastic properties.

If we assume that the abutments are rigid, so that when the load is applied the span remains unaltered, then the total horizontal displacement as expressed by equation (12) must be zero. Further, if the level of the supports remains the same, then the total vertical displacement as expressed by equation (13) must also be zero. Finally, if the ends of the arch-ring are assumed rigidly fixed in direction, so that however the arch itself may bend under the load, the slope of its axis at these ends is unaltered, then $\Delta\theta$ in equation (1) will also be zero, and therefore if we consider only the bending moment due to the load, and consider E as constant, we get from the above conditions

$$\Sigma \frac{My \,.\, \delta s}{I} = 0 \quad . \quad . \quad . \quad (14)$$

$$\Sigma \frac{Mx \,.\, \delta s}{I} = 0 \quad . \quad . \quad . \quad (15)$$

$$\text{and } \Sigma \frac{M \,.\, \delta s}{I} = 0 \quad . \quad . \quad . \quad (16)$$

These are the fundamental equations of condition upon which the solution of the arch problem depends, whatever variations may be adopted in applying them.

In the special case when the moment of inertia of the arch-ring is constant, I will also disappear from the equations, and if, as is often done, the axis of the arch is divided into *equal* lengths δs, δs being then also constant will vanish too, and the above equations will then reduce to the simpler forms:—

$\Sigma\,[My] = 0$; (14′); $\Sigma\,[Mx] = 0$; (15)′; and $\Sigma\,[M] = 0$; (16)′.

Sometimes also, when the moment of inertia of the arch-ring is variable, with the object of simplifying the equations as much as possible, the lengths δs are so chosen as to keep the ratio $\frac{\delta s}{I}$ constant, in which case the equations take the same simple forms as those last derived.

These, however, are merely convenient variations in the method of applying the fundamental equations.

28. **Remarks on the Assumptions.**—As regards the first assumption that the span does not alter when the load is applied, this is not exactly true, because when the centering of an arch is removed the

material necessarily yields, if only on account of its elasticity, and if no allowance were made for this the alteration of form might seriously affect the stresses. In arches up to about 150 ft. span, however, the centering is usually built so that the rise is greater than the final rise by about $\frac{1}{800}$ of the span, or by the amount of its probable deflection. The abutments may also yield to a certain extent, and any increase in the span will cause an increase in the horizontal thrust amounting to about 17 tons for every $\frac{1}{10}$ ft., according to Balet ("Analysis of Elastic Arches"), whilst a rise in temperature of 15° F. will cause an increase of about 3 tons in the thrust per foot of width. In masonry and concrete fluctuation of temperature is less than for metal arches and may be taken as not more than 20° F. in this country, and less when the average depth of filling exceeds about 30 ins. As regards the third assumption, the fixure of the terminals is never absolutely rigid, because the elasticity of the abutments will permit a slight rotation. This effect must, however, be very slight in most cases and may safely be disregarded, especially when the depth of the arch-ring increases towards the abutments.

29. **Allowance for the Thrust and Change of Temperature.**—It has been shown in sect. 26 (eq. 12), that the total horizontal displacement of the free end of an arch rib is given by the expression

$$\Delta x = \Sigma y \,.\, \frac{M \,.\, \delta s}{EI} - \Sigma \frac{N}{AE} \,.\, \delta x + \Sigma at \,.\, \delta x$$

and if we assume the span to remain unaltered, then

$$\Sigma y \,.\, \frac{M \,.\, \delta s}{EI} - \Sigma \frac{N}{AE} \,.\, \delta x + \Sigma at \,.\, \delta x = 0 \text{ (equation 13).}$$

The value of a for stone and concrete varies from ·0000044 to ·0000078 per degree Fahrenheit, and t need not be taken higher than about 20° F. for highway bridges. The effect of the axial thrust N is to shorten the span, and its effect may therefore be regarded as equivalent to a fall in temperature, and included as such in the temperature effect. The terms involving N are, however, always very small in comparison with those involving the bending moment, and a small correction may be applied to allow for them. Thus in the formula for H, sect. 31, viz.,

$$H = \frac{\Sigma \frac{\mu y}{I} \cdot \delta s}{\Sigma \frac{y^2}{I} \cdot \delta s}$$

the effect of the thrust may be sufficiently allowed for by adding to the denominator the term $\frac{v}{R} \cdot \frac{l}{n \cdot A_o}$ where $\frac{l}{n}$ is the mean length of the horizontal segments into which the span is divided, A is the mean area of the cross-sections, v is the distance of the centre of curvature at the crown from the X axis, and R is the radius of curvature at the crown, the value $\frac{v}{R}$ being usually nearly equal to unity for flat arches.

The maximum stresses due to change of temperature usually arise at the crown and springings, and their values may be estimated with sufficient accuracy by means of the following formulæ due to Melan :—

For the horizontal thrust due to change of temperature

$$H_t = \frac{45}{4} \cdot \frac{E\alpha t I_o}{hh'}$$

where I is a mean value of the moments of inertia of the cross-sections and h, h' are the rise of the arch and the rise of the line of resistance respectively, and $h' = h\left(1 + \frac{45}{4} \cdot \frac{I_o}{AH^2}\right)$ approximately.

The bending moment at the crown due to change of temperature then becomes

$$M_t = \frac{15}{4} \cdot \frac{E\alpha t \,.\, I_o}{h}$$

and the extreme stresses due to change of temperature are

(*a*) at the crown $\qquad s_t = \frac{15}{16} E\alpha t \frac{t_o^2}{hh'}\left(1 \pm 2\frac{h}{t_o}\right)$;

(*b*) at the springings $s_t = \frac{15}{16} E\alpha t \frac{t_o^2}{hh'}\left(1 \quad 4\frac{h}{t_o}\right)$.

CHAPTER IV.

THE TWO-HINGED ARCH.

30. THE two-pinned arch having joints at the abutments, but continuous at the crown, is chiefly used for steel bridges. No particular advantage appears to be gained by employing pin joints at the abutments only, because the determination of the forces is a statically indeterminate problem, whilst the construction is complicated by introducing pin joints; and for any except small arches the extra rigidity gained by dispensing with the crown pin does not appear to be sufficient to compensate for the other disadvantages.

The actual calculations are, however, less complex than in the case of the hingeless arch, because only one elastic condition is necessary to replace the loss of one pin. This condition may be derived by assuming the span to remain unaltered by the loading. We then have by equation 14 of last chapter

$$\Sigma \frac{My \cdot \delta s}{I} = 0.$$

Now by Eddy's Theorem (sect. 15) if ACB (fig. 22) be the axis of the arch, hinged at A and B, and ADB be the line of pressure, whose polar distance is H, the bending moment M at any point is H $\times$ CD, where CD is the vertical intercept between the axis of the arch and the line of pressure, measured in the linear scale of the drawing. If we introduce this value into equation (14) above, we get

$$\Sigma\left[\frac{\text{H . CD . } y \text{ . } \delta s}{\text{I}}\right] = 0 \quad . \quad . \quad (17)$$

and if we suppose the *axis* divided into small equal

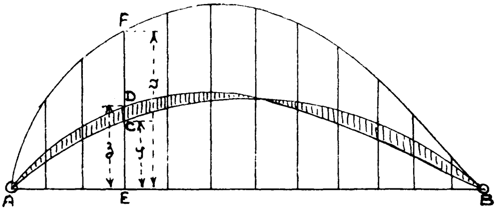


FIG. 22.

lengths δs, δs being constant and the horizontal thrust H being also constant, this reduces to

$$\Sigma\left[\frac{\text{CD . } y}{\text{I}}\right] = 0.$$

But CD at any point = ED − EC = $z - y$ where z is the ordinate of the line of pressure and y that of the arch-axis. Hence we get $\Sigma\left[\frac{(z - y)y}{\text{I}}\right] = 0$

$$\text{or } \Sigma\left[\frac{y^2}{\text{I}}\right] = \Sigma\left[\frac{y \text{ . } z}{\text{I}}\right] \quad . \quad . \quad (18)$$

Now if any link-polygon be drawn through A and B, such as AFB, for the given loads upon the arch, its ordinates u will have a constant ratio to those of the true linear arch ADB, which is merely an equilibrium polygon for the same load system with a different polar distance. If then we write the ratio $\frac{z}{u} = r$, $z = r \text{ . } u$, and substituting this value of z in equation (18) we get

$$\Sigma\left[\frac{y^2}{\mathrm{I}}\right] = \Sigma\left[\frac{ru \,.\, y}{\mathrm{I}}\right] = r\Sigma\left[\frac{u \,.\, y}{\mathrm{I}}\right]$$

$$\text{whence } r = \frac{\Sigma\left[\dfrac{y^2}{\mathrm{I}}\right]}{\Sigma\left[\dfrac{u \,.\, y}{\mathrm{I}}\right]} \quad . \quad . \quad . \quad (19)$$

or when the cross-section of the arch-ring is rectangular, since $\mathrm{I} = \frac{bt^3}{12}$ where t is the thickness of the ring, we get

$$r = \frac{\Sigma\left[\dfrac{y^2}{t^3}\right]}{\Sigma\left[\dfrac{u \,.\, y}{t^3}\right]} \quad . \quad . \quad . \quad (20)$$

The values of the numerator and denominator of these expressions may obviously be found by measurement from the drawing and subsequent calculation and summation, and if the polar distance corresponding to the trial-polygon AFB be changed in the inverse ratio $\frac{1}{r}$, the equilibrium-polygon drawn with the new pole will be the true line of pressure; or

$$\frac{\text{true polar distance}}{\text{assumed polar distance}} = \frac{1}{r} = \frac{\Sigma\left[\dfrac{u \,.\, y}{\mathrm{I}}\right]}{\Sigma\left[\dfrac{y^2}{\mathrm{I}}\right]} \quad . \quad (21)$$

or in the special case when the value of I is constant

$$\frac{1}{r} = \frac{\Sigma\,[u \,.\, y]}{\Sigma\,[y^2]} \quad . \quad . \quad . \quad (21')$$

It may be remarked that the determination of the value of r fixes the value of the polar distance and therefore of the true horizontal thrust of the arch, and when this is known there is no difficulty in drawing the line of pressure or in calculating the bending moments, etc.

31. *Example.*—It will be of interest to consider, by way of comparison, the same arch and the same loading as in the example of chapter II, except that the arch-ring will now be made continuous at the crown. This arch was 150 ft. span, with a rise of 15 ft., loaded with 26 cwt. per foot, with 18 cwt. per foot additional load covering the left-hand half.

Method I. Calculation of the Horizontal Thrust.—If an arch of span l carry a uniform load of w per foot run, we have by the fundamental equation (14), sect. 27,

$$\Sigma \frac{My \, . \, \delta s}{I} = 0.$$

Now M at any section is the difference between the bending moment on a simply supported beam of the same span and loaded in the same manner and the bending moment due to the horizontal thrust H. Hence if we denote by μ the bending moment on the simply supported beam, $M = \mu - Hy$.

Consequently the above equation becomes

$$\Sigma \frac{(\mu - Hy)y \, . \, \delta s}{I} = 0$$

$$\text{or } \Sigma \frac{\mu y \, . \, \delta s}{I} = H \Sigma \frac{y^2 \, . \, \delta s}{I}$$

$$\text{whence } H = \frac{\Sigma \dfrac{\mu y \, . \, \delta s}{I}}{\Sigma \dfrac{y^2 \, . \, \delta s}{I}} \text{ approximately,}$$

$$\text{or exactly } H = \frac{\int \frac{\mu y \, . \, ds}{I}}{\int \frac{y^2 \, . \, ds}{I}}$$

or, if we assume the arch-ring to be of uniform cross-section,

$$H = \frac{\int \mu y \, . \, ds}{\int y^2 \, . \, ds}.$$

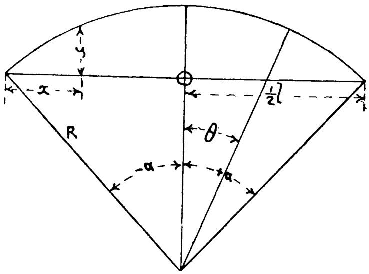


FIG. 23.

$$\text{Now } \mu = \frac{w}{2} \, . \, x(l - x) \quad \text{(Fig. 23)}$$

$$\text{and } x = \frac{l}{2} - R \sin \theta.$$

$$\therefore \mu = \frac{w}{2}\left(\frac{l}{2} - R \sin \theta\right)\left(l - \frac{l}{2} + R \sin \theta\right)$$

$$\doteq \frac{w}{2}\left(\frac{l^2}{4} - R^2 \sin^2 \theta\right).$$

Also $y = R(\cos \theta - \cos \alpha)$, $\sin \alpha = \frac{l}{2R}$ and $ds = R \, . \, d\theta$.

$$\therefore H = \frac{2\int_0^\alpha \frac{w}{2}\left(\frac{l^2}{4} - R^2 \sin^2 \theta\right) R(\cos \theta - \cos \alpha) \; R \, . \, d\theta}{2\int_0^\alpha R^2(\cos \theta - \cos \alpha)^2 \, . \, R \, . \, d\theta}$$

which reduces to

$$H = \frac{w}{2R} \cdot \frac{\frac{l^2}{4}(\sin \alpha - \alpha \cos \alpha) - \frac{R^2}{3} \sin^3 \alpha + \frac{R^2}{2} \cos \alpha(\alpha - \frac{1}{2} \sin 2\alpha)}{\frac{1}{2}(\alpha + \frac{1}{2} \sin 2\alpha) - \sin 2\alpha + \alpha \cos^2 \alpha}.$$

Now in our case $\sin \alpha = \cdot 3846176$, $\alpha = 22° \; 37' \; 12'' = \cdot 39476$ radians, and $R = 195$ ft.

Inserting these values we get $H = 187 \cdot 55 \, w$ as the value of the horizontal thrust. Now the horizontal thrust due to a uniform load on one-half the span is one-half of the horizontal thrust when the whole span is covered.

Hence for a load of $w = 2 \cdot 2$ tons per ft. on the left-hand half of the arch we get

$$H_1 = \frac{2 \cdot 2 \times 187 \cdot 55}{2} = 206 \cdot 3 \text{ tons},$$

and for a load of $1 \cdot 3$ tons per ft. on the right-hand half of the arch we get

$$H_2 = \frac{1 \cdot 3 \times 187 \cdot 55}{2} = 121 \cdot 9 \text{ tons}.$$

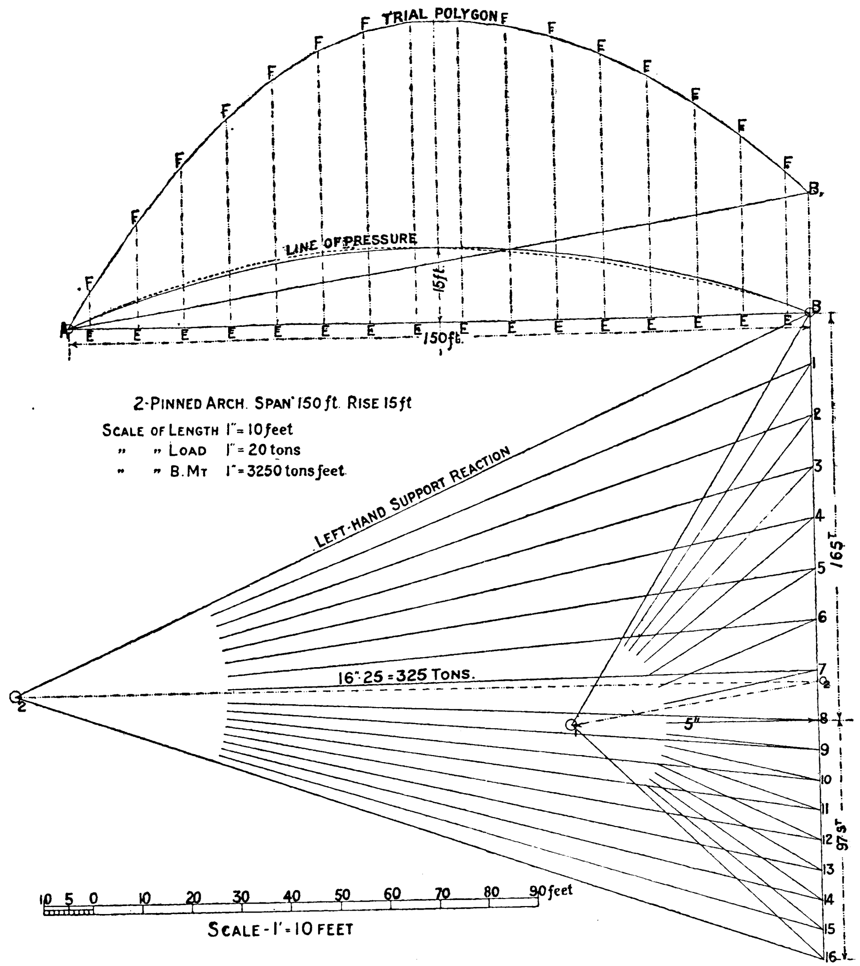
Hence the horizontal thrust due to both is

$$H = 206 \cdot 3 + 121 \cdot 9 = \underline{328 \cdot 2 \text{ tons}},$$

which is slightly greater than the horizontal thrust when the arch was 3-pinned.

The bending moment at any point can now be readily found from the equation $M = \mu - Hy$.

32. *Method II. Graphically.*—The arch was drawn to a scale of 1 in. = 10 ft., and the span having been divided into sixteen equal parts, the loads on these were set off along a vector line as shown in Plate I, to a scale of 1 in. = 20 tons, and a pole O_1 having been chosen in a convenient position, a link-polygon, AFF . . . B_1, was constructed,



Stability of Arches.] PLATE I. [*To face page* 55.

and the closing line 0·17 put in. A line O_1a was then drawn through O_1 parallel to AB_1, dividing the vector line into two parts at a which represent the vertical loads on the pins. The position of a will, of course, be independent of the position of the pole O_1, but if the link-polygon is to pass through B as well as A, the pole must lie on the horizontal through a, for it is only so that the closing line can be horizontal. The true pole must therefore lie on the horizontal through a, and the polar distance O_2a must at the same time be such as to satisfy the condition (21′) sect. 30. The axis of the arch was therefore divided into equal lengths δs, and verticals EF were drawn through the mid-points of these. The lengths of these verticals were then measured, for convenience, in $\frac{1}{2}$ in. units and the corresponding ordinates of the arch with the same $\frac{1}{2}$ in. scale. These lengths were found to be as follows :—

Ordinate.	1	2	3	4	5	6	7	8
y	0·38	1·02	1·58	2·05	2·42	2·70	2·88	2·98
y^2	0·144	1·040	2·496	4·202	5·856	7·290	8·294	8·880
EF	1·28	3·62	5·57	7·18	8·48	9·32	9·82	9·93
y . EF	0·46	3·69	8·80	14·72	20·52	25·16	28·28	29·59

Ordinate.	9	10	11	12	13	14	15	16
y	2·98	2·88	2·70	2·42	2·05	1·58	1·02	0·38
y^2	$\Sigma[y^2] = 2 \times 38{\cdot}2 = 76{\cdot}4$							
EF	9·60	9·05	8·25	7·22	5·98	4·52	2·88	1·05
y . EF	28·61	26·06	22·27	17·47	12·26	7·14	2·94	0·38
	$\Sigma[y\,.\,\mathrm{EF}] = 248{\cdot}35$							

$$\therefore \frac{1}{r} = \frac{248{\cdot}35}{76{\cdot}4} = 3{\cdot}25,$$

Now the assumed polar distance was 5 ins.; therefore the true polar distance is

$$5 \text{ ins.} \times 3{\cdot}25 = 16{\cdot}25 \text{ ins.} = \underline{325 \text{ tons.}}$$

If we draw a line through the pole O_1 parallel to the closing line AB_1 of the link-polygon, this determines the vertical reactions at the supports, and if we take the true pole on the horizontal through the point *a* so obtained, a new link-polygon starting from one pin should pass through the other, and will be the true line of pressure.

The values of the bending moments cannot as a rule be found accurately from the bending moment diagram, because the line of pressure lies very near the axis of the arch; but when the horizontal thrust has been found, the bending moment at any point of the arch-ring may be readily calculated by subtracting from the bending moment which the load would cause on a simply supported beam of the same span the bending moment due to the horizontal thrust. Thus the bending moment at $\frac{1}{4}$ span in the present case will be

$$Hy + \tfrac{1}{2}\, wx^2 - V_A \,.\, x = 325 \times 11{\cdot}3 + \tfrac{1}{2} \times 2{\cdot}2 \times 37{\cdot}5^2 - 148\tfrac{1}{8} \times 37{\cdot}5 = 335{\cdot}5 \text{ tons/ft.}$$

The graphical determination of the thrust will be found to be quite as reliable as when determined by calculation, for on account of the fact that difference terms are involved, the arithmetic must be more than usually accurate to obtain a reliable result.

33. *Method III. By means of the Reaction Locus.*—When a single load W rests at any point on the arch, the reactions at the supports A and B must intersect on the line of action of the load in some

point C (Fig. 24) whose position may be found if the horizontal thrust of the arch is known. For if nl be the horizontal distance of W from the left-hand support, from the similar triangles CAD, *cad*

$$\frac{z}{nl} = \frac{cd}{ad} = \frac{V_A}{H} = \frac{W}{H}(1 - n)$$

$$\text{whence } z = \frac{Wln}{H}(1 - n).$$

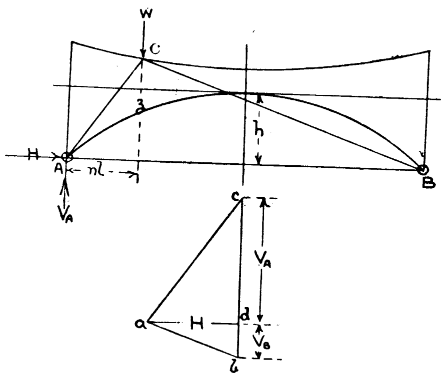


FIG. 24.

Parabolic Arch.—If we suppose the axis of the arch to be a parabola, and the moment of inertia of the ring at any point to be proportional to the secant of the inclination, then denoting the moment of inertia of the arch-ring at the crown by I_o, and its value at any other section by I, we have $I = I_o . \frac{ds}{dx}$

$$\text{and } H = \int \frac{\mu y \,.\, ds}{I} \div \int \frac{y^2 \,.\, ds}{I} = \int \mu y \,.\, dx \div \int y^2 \,.\, dx.$$

Now when x is less than nl,

$$\mu = V_A \,.\, x = W(1 - n)x,$$

and when x is greater than nl,

$$\mu = V_A \cdot x - W(x - nl) = Wn(l - x).$$

Therefore

$$\int_0^l \mu y \cdot dx = \int_0^{nl} W(1-n)\, x \cdot y \cdot dx + \int_{nl}^l Wn(l - x)y \cdot dx$$

$$\text{where } y = \frac{4hx}{l^2}(l - x)$$

$$= W(1-n)\frac{4h}{l^2}\int_0^l x^2(l - x)dx + \frac{4Wnh}{l^2}\int_{nl}^l x(l-x)^2 \cdot dx$$

$$= \frac{Whl^2n(1 - n)}{3}(1 + n - n^2).$$

$$\text{Also} \int_0^l y^2 \cdot dx = \tfrac{8}{15} h^2 l.$$

$$\therefore H = \tfrac{5}{8} W\frac{l}{h} n(1 - n)(1 + n - n^2)$$

$$\text{and therefore } z = h \cdot \frac{8}{5(1 + n - n^2)}.$$

This is the equation to the reaction locus, and to facilitate the plotting of it values of $\frac{z}{h}$ are given below for different values of n up the centre of the span :—

n	0	·05	·10	·15	·20	·25	·30	·35	·40	·45	·50
z/h	1·600	1·527	1·468	1·415	1·379	1·347	1·322	1·304	1·290	1·283	1·280

Circular Arch.—The equation to the reaction locus for a circular arch is $y = \frac{1 + Bk}{1 - Ak} \cdot y_o$ which becomes a very complex expression when the values of A, B, and k are introduced in terms of the known quantities. It may, however, be readily plotted with the aid of the following tables (see Dubois'

"Graphical Statics") where α and β have the meanings shown in Fig. 25 and $k = \frac{\mathrm{I}}{\mathrm{A}r^2}$, I being the

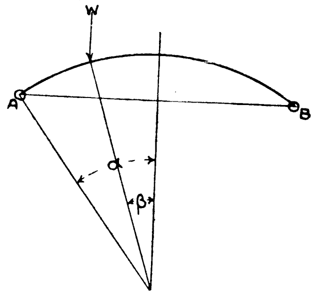


Fig. 25.

moment of inertia of the arch-ring assumed constant, A its area, and r the radius of the circular arc.

Table I, *giving values of* A *and* B *for given values of* α *and* β.

	β	$\alpha=0$	$\alpha=10^\circ$	$\alpha=20^\circ$	$\alpha=30^\circ$	$\alpha=40^\circ$	$\alpha=50^\circ$	$\alpha=60^\circ$	$\alpha=90^\circ$	
A	0	1·20	1·19	1·17	1·14	1·08	1·00	0·88	0	$\frac{a^2}{h^2}$
	0·2	1·21	1·20	1·18	1·15	1·10	1·01	0·90	0	
	0·4	1·24	1·24	1·21	1·18	1·13	1·05	0·94	0	
	0·6	1·29	1·29	1·27	1·24	1·20	1·13	1·02	0	
	0·8	1·38	1·38	1·36	1·34	1·30	1·24	1·18	0	
	1·0	1·50	1·50	1·49	1·47	1·45	1·41	1·36	0	
B	a	0·234	0·233	0·221	0·203	0·178	0·146	0·107	0	$\frac{a^4}{h^4}$

Example.—Suppose $\alpha = 40^\circ$. Then by Table I $\mathrm{B} = 0{\cdot}178\,\frac{a^4}{h^4}$, where a is the half-span and h is the rise.

TABLE II, *giving values of* y_o.

β	$\alpha=0$	$\alpha=10°$	$\alpha=20°$	$\alpha=30°$	$\alpha=40°$	$\alpha=50°$	$\alpha=60°$	$\alpha=90°$
0	1·280	1·282	1·288	1·300	1·316	1·340	1·375	1·571
0·2	1·290	1·292	1·298	1·309	1·327	1·348	1·380	1·571
0·4	1·322	1·324	1·329	1·340	1·354	1·374	1·403	1·571
0·6	1·379	1·380	1·385	1·393	1·405	1·421	1·443	1·571
0·8	1·468	1·469	1·471	1·476	1·483	1·490	1·504	1·571
1·0	1·600	1·600	1·599	1·587	1·594	1·591	1·588	1·571
$\times\ a$	$\times\ h$							

$A = 1{\cdot}08\,\frac{a^2}{h^2},\ \ 1{\cdot}10\,\frac{a^2}{h^2}$, etc., for values of $\beta = 0, 0{\cdot}2\,a$, etc., and by Table II.

$y_o = 1{\cdot}316h,\ 1{\cdot}327h$, etc., for corresponding values of β.

Thus when $\beta = {\cdot}4 \times 40°$, for example

$$y = \frac{1 + 0{\cdot}178\,\frac{a^4}{h^4}\,.\,k}{1 - 1{\cdot}13\,\frac{a^2}{h^2}\,.\,k} \times 1{\cdot}354h.$$

Hence for an arch of 150 ft. span and 15 ft. rise,

$$\frac{a}{h} = 5$$

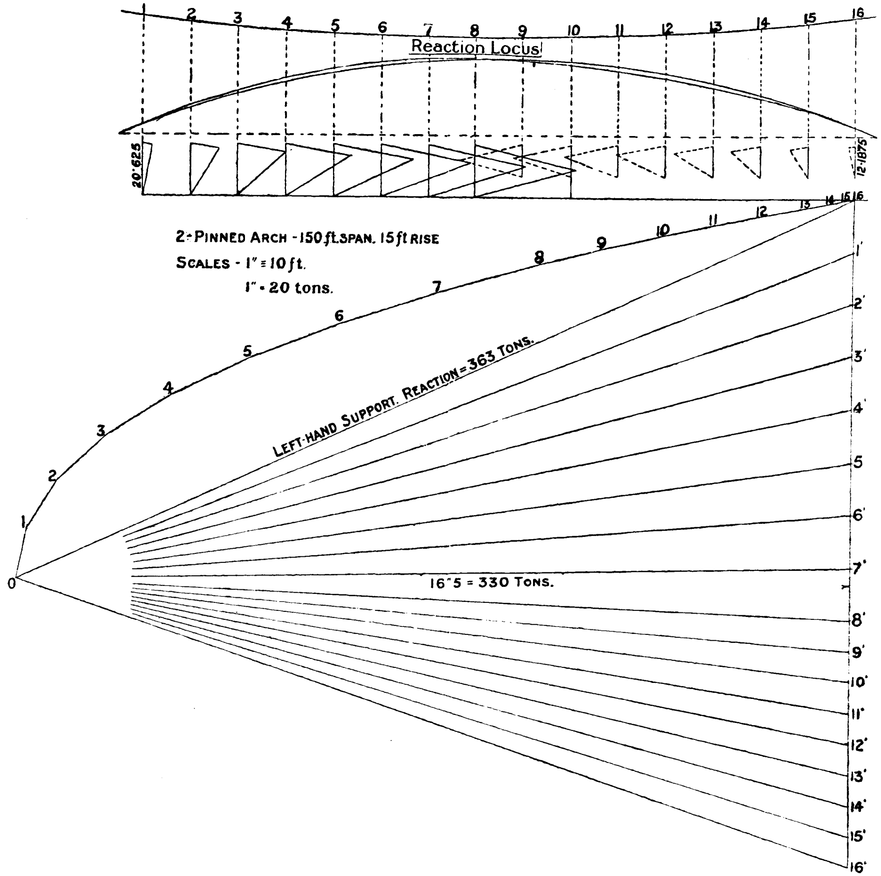
$$\text{and } y = \frac{1 + 0{\cdot}178 \times 5^4k}{1 - 1{\cdot}13 \times 5^2k} \times 1{\cdot}354 \times 15,$$

and if the depth of the arch-ring is 18 ins. and the radius of the arch-axis 195 ft.,

$$k = \frac{1{\cdot}5^2}{12 \times 195^2} = \frac{1}{202800}.$$

$$\therefore y = \frac{202911}{202772} \times 20{\cdot}31 = 20{\cdot}37 \text{ ft.}$$

In the case of flat arches, such as the one under



Stability of Arches.] PLATE II. [*To face page* 61.

consideration, no sensible difference will be caused by regarding the curve as parabolic. This has been done in the present instance (Plate II). The reaction locus was calculated from the tabular values above. The span was divided into sixteen equal parts, and verticals were drawn through the mid-points of each of these to represent the lines of action of the loads, as in the former solution. The directions of the reactions due to each load in turn were then determined by joining the pins A and B to the points in which each load intersects the reaction locus. A triangle of forces was then drawn for each load, as shown, from which the magnitudes of the reactions were found. Their values at A were then plotted from 0 to 16, giving 0·16 as the resultant reaction at A. The vertical loads 16·1′, 1′·2′ etc., were next compounded with this reaction, and a link-polygon was drawn starting from A. This polygon passed through B, as it should do, thereby checking the accuracy of the drawing. It is therefore the true line of pressure. The polar distance found in this way scaled 330 tons, as against 328·2 tons by calculation.

It will therefore be seen that either of the graphical methods may be relied upon to give a result sufficiently near the truth for all practical purposes. An absolute check should not in any case be expected, because the assumptions are not exactly the same in all three cases.

34. **Construction of an Influence Line for the Bending Moment at any Section of a Two-pinned Parabolic Arch.**—It has been shown above that the horizontal thrust for a parabolic arch is

$$H = \tfrac{5}{8} W \frac{l}{h} n(1 - n)(1 + n - n^2)$$

where $n = \frac{x}{l}$ and x is the distance of the load W from one end. Hence for a unit load

$$H = \tfrac{5}{8}\frac{l}{h} \cdot n(1 - n)(1 + n - n^2).$$

If the values of H for different values of n be calculated from this formula and the results plotted at the corresponding points, a fair curve drawn through the points so obtained will give the horizontal thrust when the load occupies any position on the span. Now the actual bending moment M at any point is equal to the bending moment μ due to the same load on a simply supported beam less the bending moment Hy due the horizontal thrust; therefore

$$M = \mu - Hy \text{ (Fig. 26)}$$

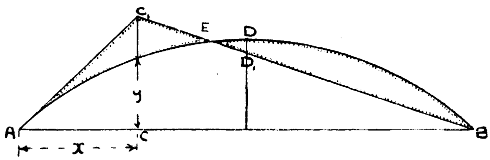


FIG. 26.

$$\text{or } \frac{M}{y} = \frac{\mu}{y} - H \qquad \text{where } \mu = \frac{x(l - x)}{l}$$

for a unit load.

If therefore we require to construct an influence diagram for the bending moment at a particular section C, we set up $CC_1 = \frac{x(l - x)}{ly}$ and join AC_1, BC_1. An influence line ADB for the horizontal thrust, or the H-line for the section, is then plotted from the equation for the thrust given above, when the shaded area between the two figures will be an influence diagram for values of $\frac{M}{y}$, from which therefore the values of M can be found directly.

By way of example and for the sake of comparison with the 3-pinned arch already considered, an influence diagram will be drawn for a 2-pinned arch of the same span and rise for a section C at $\frac{1}{4}$ span.

The following table will facilitate the calculation of the ordinates of the H line :—

Values of $\frac{5}{8} . n (1 - n) (1 + n - n^2)$ for Different Values of n.

$n=$	·05	·10	·15	·20	·25	·30	·35	·40	·45	·50
$\frac{5}{8} n (1-n) (1+n-n^2)$	·031	·061	·09(	·116	·139	·159	·174	·188	·193	·195

If we multiply each of these values by $\frac{l}{h} = \frac{150}{15} = 10$, in the present case we get the values of H due to a unit load in the corresponding positions. These values when plotted will therefore give the H line, and the ordinate between this and the triangle AC_1B will give the values of $\frac{M}{y}$ for a unit load in any position on the span. The values of M at once follow. The bending moment at the section for any system of concentrated loads in any given position may therefore be easily found in the usual way by multiplying each load into its ordinate on the diagram and summing the results with due regard to sign. At the same time the influence area will give the value of M for a distributed load. Now in the present instance

$$CC_1 \text{ at } \tfrac{1}{4} \text{ span} = \frac{x(l - x)}{ly} = \frac{37{\cdot}5 \times 112{\cdot}5}{150 \times 11{\cdot}3} = 2{\cdot}49.$$

The influence diagram will therefore be as shown in Fig. 26, and the bending moment at $\frac{1}{4}$ span will be found by multiplying the area AC_1E by 2·2 tons per ft., the negative area EDD_1 also by 2·2 tons per ft., and the negative area BD_1D by 1·3 tons per ft.

CHAPTER V.

THE HINGELESS ARCH.

35. An arch without hinges is not only less flexible than an arch with joints, but the resistance which the fixed ends offer to bending assist in relieving the stresses in other portions of the arch, when other conditions remain the same. Its main disadvantages are that the calculation of the stresses requires great accuracy, and the results obtained by graphical methods are not to be trusted except as a check on the calculations. At the same time the stresses due to change of temperature may appreciably increase the stresses due to the loading. If, however, the arch can be erected in cold weather, the effect of any rise in temperature will be to cause a reduction in the maximum stresses, and the fixure of the ends will therefore have a beneficial effect in this respect. As the cost of construction is in general not greater than in other kinds of arch, the advantages appear to favour this type, so that the theoretical difficulties appear to be the sole reason why the hingeless arch has not been universally adopted for long spans, where the difficulties of erection are the same in any case.

36. **General Expression for the Bending Moment at any Section of a Symmetrical Hingeless**

Arch under Vertical Loading.—Let ACB (fig. 27) be the axis of the arch, and $A_1C_1B_1$ be the line of pressure (at present unknown) for the load upon it. Then by Eddy's Theorem, sect. 15, the bending moment M at any point is represented by the product of the horizontal thrust H and the vertical intercept between the arch-axis and the line of pressure at that point.

Thus at the point C, $M = H \times CC_1$.

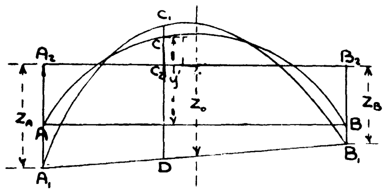


FIG. 27.

Let a horizontal line A_2B_2 be taken as the X-axis, and its mid-point O as origin, and let the vertical through C cut it in C_2 and the line A_1B_1 in D.

Then $CC_1 = C_1D - C_2D - CC_2$.

$\therefore M = H(C_1D - C_2D - y) = H \cdot C_1D - H(C_2D + y)$.

But $H \times C_1D = \mu =$ the bending moment due to the same system of loads on a simply supported beam.

$$\therefore M = \mu - H(C_2D + y).$$

But $C_2D = z_o + \frac{z_A - z_B}{l} \cdot x$ where z_o is the mean ordinate of the trapezium $A_1A_2B_2B_1$.

$$\therefore M = \mu - Hy - Hz_o - H\frac{z_A - z_B}{l} \cdot x.$$

$$\text{Let } X_1 = H\frac{z_A - z_B}{l} \text{ and } X_2 = H \cdot z_o.$$

Then $M = \mu - Hy - X_1x - X_2$.

Introducing this value of M into equations (14), (15), and (16) of sect. 27, we get

$$\Sigma\frac{\mu y}{I}.\delta s - H\Sigma\frac{y^2}{I}\delta s - X_1\Sigma\frac{xy}{I}.\delta s - X_2\Sigma\frac{y}{I}.\delta s = 0 \quad . \quad (14)'$$

$$\Sigma\frac{\mu x}{I}.\delta s - H\Sigma\frac{xy}{I}.\delta s - X_1\Sigma\frac{x^2}{I}.\delta s - X_2\Sigma\frac{x}{I}.\delta s = 0 \quad . \quad (15)''$$

$$\Sigma\frac{\mu}{I}.\delta s - H\Sigma\frac{y}{I}.\delta s - X_1\Sigma\frac{x}{I}.\delta s - X_2\Sigma\frac{1}{I}.\delta s = 0 \quad . \quad (16)''$$

Now for an arch which is symmetrical about a vertical axis $\Sigma\frac{xy}{I}.\delta s = 0$ for symmetrical values of x, since the terms have equal and opposite values on each side of the centre. Also $\Sigma\frac{x}{I}.\delta s = 0$ for the same reason; and if we choose our origin so that $\Sigma\frac{y}{I}.\delta s = 0$, the above equations reduce to

$$\Sigma\frac{\mu y}{I}.\delta s - H\Sigma\frac{y^2}{I}.\delta s = 0 \quad . \quad . \quad (14)'''$$

$$\Sigma\frac{\mu x}{I}.\delta s - X_1\Sigma\frac{x^2}{I}.\delta s = 0 \quad . \quad . \quad (15)'''$$

$$\Sigma\frac{\mu}{I}.\delta s - X_2\Sigma\frac{1}{I}.\delta s = 0 \quad . \quad . \quad (16)'''$$

Further, if we divide up the arch-axis so that the ratio $\frac{\delta s}{I}$ is kept constant, we can reduce these to the form

$$\Sigma\mu y - H\Sigma y^2 = 0 \quad . \quad . \quad (14)^{IV}$$

$$\Sigma\mu x - X_1\Sigma x^2 = 0 \quad . \quad . \quad (15)^{IV}$$

$$\Sigma\mu - X_2 . n = 0 \quad . \quad . \quad (16)^{IV}$$

where n is the number of divisions into which the arch-ring is divided.

From these last equations we get

$$H = \frac{\Sigma \mu y}{\Sigma y^2} \quad . \quad . \quad . \quad (22)$$

$$X_1 = \frac{\Sigma \mu x}{\Sigma x^2} \quad . \quad . \quad . \quad (23)$$

$$X_2 = \frac{\Sigma \mu}{n} \quad . \quad . \quad . \quad (24)$$

37. In order to illustrate the application of these formulæ, we will apply them to the same example as before, except that in the present case the ends will be assumed as rigidly fixed, and to make the example more generally useful, the arch-ring will be supposed of variable depth. The material will be taken as concrete weighing 150 lb./ft.^{3}, reinforcement will be provided to the extent of 2 per cent of the crown area, and this will be embedded symmetrically about the axis at 2 ins. from the outer surfaces of the arch-ring. The thickness at the crown will be taken as 2 ft. and at the springings as 3 ft., the extrados and intrados being circular arcs.

In order to divide the arch-ring into segments so that the ratio $\frac{\delta s}{I}$ may be constant, in accordance with the condition assumed above, the values of the moment of inertia were calculated by dividing the half axis into five equal parts. The moment of inertia of the concrete is $I_c = \frac{bt^3}{12}$ where b is taken as 1 ft. and t is the thickness of the arch-ring at the section. The moment of inertia of the steel is $I_s = A_s \times$ (distance from axis)^2, where A_s is the total area of the steel at the section. The total moment of inertia of the cross-section is then $I = I_c + 15\ I_s$, taking the ratio of the elastic moduli as 15. The values of I at six equidistant sections were estimated to be as follows :—

Section.	1	2	3	4	5	6	
I	1·08	1·23	1·42	1·78	2·46	3·32	ft.^4

A horizontal base-line 0·7 (Fig. 28) was then drawn equal in length to 1·6, and the values $\frac{1}{2}$I were plotted above and below it, to any convenient scale so as to obtain a diagram representing the moments of inertia

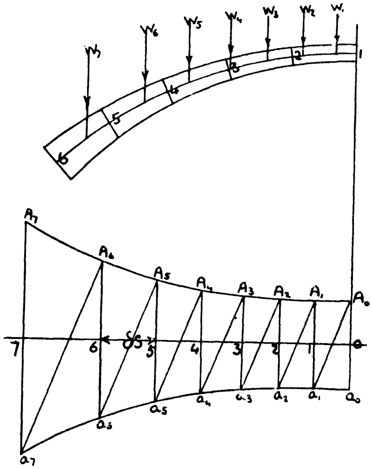


FIG. 28.

at any point. Then if parallel lines be drawn at any slope as shown in Fig. 29 the ratio $\frac{\delta s}{I}$ will be approximately constant throughout, and by trial and error it is easy to find a slope such as will divide the arch-ring into a convenient number of parts, and which will close up at the point 7. The values of δs so found are then set off along the axis of the arch, and the loads W_1, W_2, etc., acting on these lengths are assumed to act at their mid-points.

The corresponding values of δx measured along the horizontal are given in the following table along with other data required for solving the equations (22), (23), and (24). Column 1 gives the section considered; column 2 gives the vertical ordinate y' of the arch-axis at the section. By addition we get $\frac{1}{2}\Sigma\ [y'] = 79{\cdot}40$ ft., and as $n = 14$ we have $\frac{79{\cdot}40}{7} = 11{\cdot}34$ ft. as the mean value. Now in order that the condition $\Sigma\ \frac{y}{I} \cdot \delta s = 0$, or in our case $\Sigma y = 0$, may be satisfied, the axis A_2B_2, from which the values of y are measured, must be drawn at this distance above the line AB. Column 3 gives the values of $y = y' - 11{\cdot}34$. The algebraic total of this column should be zero, but due to unavoidable small errors we find it to be $-\ 0{\cdot}04$. This error is then distributed equally in column 4, thereby reducing the error of the total to $+\ 0{\cdot}002$. Column 5 gives the values of x' corresponding to the values of y', measured from the left-hand end; column 6 gives the horizontal lengths of the segments δx into which the arch is divided, and column 7 gives the loads on these segments, viz. for 2·2 tons per foot on the left half and 1·3 tons per foot on the right half of the span. These loads were next plotted on a vector line, and a

TABLE I, *for the Calculation of the Horizontal Thrust* H.

1	2	3	4	5	6	7	8	9	10
Section.	y' in Ft.	y in Ft.	Corrected y in Ft.	x' in Ft.	δx in Ft.	w in Tons.	μ in Tons/Ft.	μy Tons/Ft.^{2}	y^2 in Ft.^{2}
1	3·40	− 7·94	− 7·934	8·60	17·50	38·50	1,740	− 13,800	62·95
2	8·35	− 2·99	− 2·984	24·10	13·05	28·71	2,940	− 8,774	8·90
3	11·10	− 0·24	− 0·234	35·85	10·45	22·99	3,900	− 913	0·05
4	12·90	+ 1·56	+ 1·566	45·90	9·50	20·90	4,270	+ 6,687	2·45
5	14·00	+ 2·66	+ 2·666	54·85	8·40	18·48	4,430	+ 11,810	7·11
6	14·65	+ 3·31	+ 3·316	63·20	8·25	18·15	4,970	+ 16,480	11·00
7	15·00	+ 3·60	+ 3·606	71·10	7·85	17·27	4,980	+ 17,960	13·00
7′	79·40	− 11·17	− 11·152		75·00	10·20	4,840	+ 17,450	105·46
6′	$\frac{1}{7}$ of 79·40	+ 11·13	+ 11·154			10·73	4,620	+ 15,320	2
5′	= 11·34	− 0·04	+ 0·002			10·92	4,330	+ 11,540	210·92
4′						12·35	3,880	+ 6,076	
3′						13·58	3,280	− 768	
2′						16·97	2,400	− 7,160	
1′						22·75	1,000	− 7,934	
						$\Sigma w = 262·50$		$\Sigma \mu y = 63,974$	

bending moment diagram was drawn for them in the usual way, as if they were on a simply-supported beam, and the values of the bending moments μ at each section so found are given in column 8. Column 9 gives the values of μy as obtained from columns 4 and 8; whilst column 10 gives the squares of column 4.

Inserting the values of $\Sigma \mu y$ and Σy^2 so found in equation (22), we get

$$H = \frac{\Sigma \mu y}{\Sigma y^2} = \frac{63974 \text{ tons/ft.}^2}{210{\cdot}92 \text{ ft.}^2} = \underline{303 \text{ tons.}}$$

If we include the effect of the normal thrust on the arch the denominator will be increased by an amount which is approximately equal to

$$\frac{I}{\delta s} \cdot \frac{v}{R} \cdot \frac{l}{n \, . \, A_0}$$

where v is the distance of the centre of curvature at the crown from the horizontal axis of co-ordinates = about 191 ft. in our case, and since $\frac{I}{\delta s} = \frac{3}{2}$ and $A_0 = 2{\cdot}5$ sq. ft. say the value of the above term is approximately 6 tons. If we take this into account we get $\underline{H = 295 \text{ tons.}}$

In order to find the values of x_1 and x_2, and thereby fix the position of the line of pressure and the bending moments, the values of μx and x^2 required in equation (23) are found and tabulated below.

TABLE II, *for the Calculations of* X_1 *and* X_2.

1	2	3	4	5
Section.	μ in Tons/Ft.	x in Ft.	μx in Tons/Ft.^{2}	x^2 in Ft.^{2}
1	1,740	+66·40	+115,500	4,409·0
2	2,940	+50·90	+149,700	2,590·8
3	3,900	+39·15	+152,700	1,532·7
4	4,270	+29·10	+124,200	846·8
5	4,430	+20·15	+ 89,280	406·0
6	4,970	+11·80	+ 58,650	139·2
7	4,980	+ 3·90	+ 19,420	15·2
7′	4,840	− 3·90	− 18,880	9,939·7
6′	4,620	−11·80	− 54,520	2
5′	4,330	−20·15	− 87,260	19,879·4
4′	3,880	−29·10	−112,900	
3′	3,280	−39·15	−128,400	
2′	2,400	−50·90	−122,200	
1′	1,000	−66·40	− 66,400	
	$\Sigma\mu=51,580$		−709,450	
			+590,560	
			$\Sigma\mu x=-$ 18,890	

From the above values we get by equation (23)

$$X_1=\frac{\Sigma\mu x}{\Sigma x^2}=\frac{-18890}{19879}=\cdot950 \text{ ft.}=H\frac{z_A-z_B}{l}$$

and by equation (24)

$$X_2=\frac{\Sigma\mu}{n}=\frac{51580}{14}=3684 \text{ tons/ft.}^2=H\,.\,z_o=H\frac{z_A+z_B}{2}.$$

$$\therefore\ z_A+z_B=\frac{2\times3684}{295}=24\cdot98$$

$$\text{and } z_A-z_B=\frac{\cdot95\times150}{295}=0\cdot48.$$

$$\text{Whence } z_A=12\cdot73 \text{ ft.}$$
$$\text{and } z_B=12\cdot25 \text{ ft.}$$

The fixing couples at the supports can now be found for

$$AA_1 = z_A - AA_2 = 12{\cdot}73 - 11{\cdot}34 = 1{\cdot}39 \text{ ft.}$$
$$\text{and } BB_1 = z_B - BB_2 = 12{\cdot}25 - 11{\cdot}34 = 0{\cdot}91 \text{ ft.}$$
$$\text{and } M_A = H \times AA_1 = 295 \times 1{\cdot}39 = 410 \quad \text{tons/ft.}$$
$$\text{and } M_B = H \times BB_1 = 295 \times 0{\cdot}91 = 268{\cdot}5 \quad ,,$$

The value of the reaction at A can be found by taking moments about B. Thus

$$V_A \times 150 + M_B - M_A - (165 \times 112\tfrac{1}{2} + 97{\cdot}5 \times 37\tfrac{1}{2}) = 0$$
$$\text{whence } V_A = \underline{149 \text{ tons.}}$$

The bending moment at any section can then be found from the equation

$$M = \mu - Hy - Hz_o - H\frac{z_A - z_B}{l} . x$$
$$= \mu - H\left(y + z_o \frac{z_A - z_B}{l} . x\right).$$

Thus at section 1 we get

$$M = 1740 - 295\,(-7{\cdot}934 + 12{\cdot}49 + {\cdot}0032 \times 66{\cdot}4)$$
$$= 1740 - 1406 = 334 \text{ tons/ft.},$$

and at section 7, near the centre,

$$M = 4980 - 295\,(3{\cdot}606 + 12{\cdot}49 + {\cdot}0032 \times 3{\cdot}90)$$
$$= 4980 - 4750 = 230 \text{ tons/ft.}$$

38. **Treatment of the Hingeless Arch by means of Influence Lines.**—In order to investigate the worst conditions of loading at any given section of the arch-ring, the most convenient method is to construct an influence line for the thrust, bending moment, or other effect which it is desired to investigate.

We have seen in chapter V, sect. 34, that under suitable conditions $\Sigma\frac{y}{I} . \delta s = 0$ and that if the arch-axis be so divided as to make $\frac{\delta s}{I}$ constant, then $\Sigma[y] = 0$, and in equations 22, 23, and 24 of chapter v we have

$$H = \frac{\Sigma \mu y}{\Sigma y^2} \ (22), \quad X_1 = \frac{\Sigma \mu x}{\Sigma x^2} \ (23), \text{ and } X_2 = \frac{\Sigma \mu}{n} \ (24)$$

where $X_1 = H \frac{z_A - z_B}{l}$ and $X_2 = H \cdot z_o$ (see sect. 36).

Having then divided up the axis of the arch-ring so as to make $\frac{\delta s}{I}$ constant in the way already explained (sect. 37), the position of the x axis is drawn in as before, and this determines the values of the intercepts y between the x axis and the arch-axis.

Graphical Evaluation of the Terms $\Sigma \mu y$, Σy^2, etc.—Let AD_1B (Fig. 29) represent the influence line

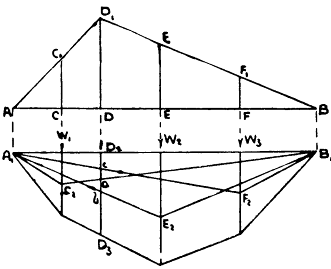


FIG. 29.

for bending moment for a unit load on a simply supported beam AB. Then when a unit load W_1 acts in any position C, $W_1 . CC_1$ is the bending moment μ_1 which it produces at a given section D. Similarly if a load W_2 act at E, $W_2 . EE_1$ is the bending moment μ_2 which it produces at D, and so on for other loads. Hence the total bending moment at D is $W_1 . CC_1 + W_2 . EE_1 + \text{etc.} = \Sigma W . \mu$.

Let $A_1C_2B_1$, $A_1E_2B_1$, $A_1F_2B_1$ be the bending moment diagrams for loads W_1, W_2, W_3.

Then the bending moment diagram for the total load is made up of the ordinates of these triangles; and therefore the ordinate D_2D_3 is the sum of D_2a, D_2b, and D_2c. But D_2a represents the bending moment at D due to the load W_1 acting at $C = W_1 . CC_1$; similarly D_2b represents the bending moment at D due to W_2 acting at $E = W_2 . EE_1$, and so on. Hence the total ordinate D_2D_3 represents the value of $\Sigma W\mu$, and therefore if the loads W are the ordinates y, $\Sigma\mu y$ is represented by the ordinate $z = D_2D_3$ of the bending moment diagram drawn for these loads, at the section D under consideration. We therefore set off the values of y vertically along a vector line (ii), Plate III, and with any pole O_2 we construct a link-polygon A_2C_2. Also setting off the values of y horizontally as at (i), with a pole O_1 at the same polar distance as before, we draw a link-polygon for these so-called loads. Then the intercept mm_1 cut off on the X axis by the first and last links gives the total moment of the forces y about this axis $= \Sigma y^2$. In the present case $mm_1 = 2 \times 5{\cdot}8$ ft. $= 11{\cdot}6$ ft. Similarly if we suppose the same points as before loaded with the values of x instead of those of y, the ordinates of the link-polygon will give the value of $\Sigma\mu x$. The so-called loads x are therefore plotted along a vector line in (iii), and any pole O_3 being taken, another link-polygon A_3C_3 is drawn, and its closing line A_3C_3. Then the ordinates of this figure represent the values of $\Sigma\mu x$ for any section, whilst the intercept $C_{3''}$ on the vertical axis of the arch, between the first and last links produced to cut it, represents the sum of the moments of the x loads about that axis. Hence $2C_{3''}$ represents the value of Σx^2 for the whole arch, and therefore

$$X_1 = \frac{\Sigma\mu x}{\Sigma x^2} = \frac{z_3}{2C_3 n} \text{ tons.}$$

If therefore we suppose $2C_3 n$ to represent the unit load to scale, the value of the ordinate z_3 at any point gives the value of X_1 for a unit load at that point, and is therefore the influence diagram for X_1.

Similarly since

$$H = \frac{\Sigma\mu y}{\mu y^2} = \frac{z_2}{2mm_1},$$

if we suppose $2mm_1$ to represent the unit load to scale, the value of the ordinate Z_2 at any point will be the horizontal thrust for the unit load at that point, and consequently the link-polygon, or more accurately the curve which envelops it, is an influence line for the value of the horizontal thrust due to the unit load in any position. It is therefore spoken of as the H polygon.

Finally, if we set down the values of unity to any scale along a vector line at (iv) and with any pole O_4 construct another link-polygon $A_4C_4B_4$, the ordinates Z_4 of this polygon will give the values of $\Sigma 1 \times \mu$ for any position of the load. If we make the polar distance $= \frac{1}{2}n = 07$ where n is the number of the unit loads, then $X_2 = \frac{\Sigma\mu}{n} = \frac{1}{2}z_4$, and the ordinates of the polygon measured to twice the linear scale are the influence values for X_2.

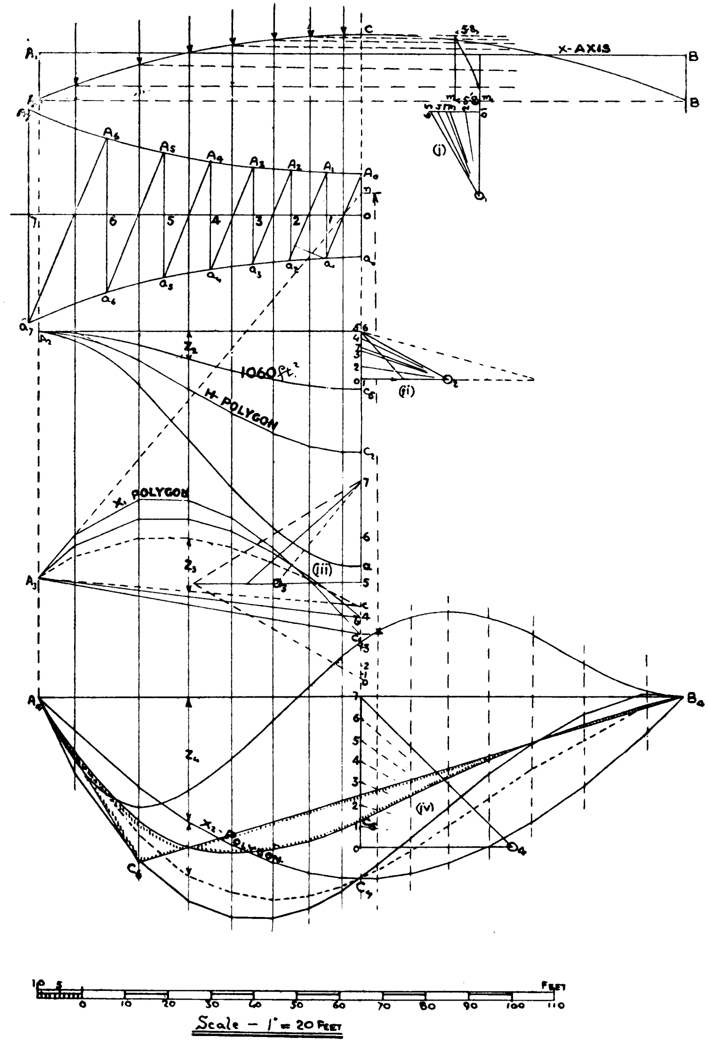
In this way we have influence diagrams representing the values of H, X_1, X_2. Now the bending moment at any section of the arch is

$$M = \mu - (Hy + X_1x + X_2).$$

If therefore we adhere to the scale of the X_2 diagram, that is to say, twice the linear scale, and alter the ordinates of the H diagram in the ratio $\frac{y}{mm_1}$ and the ordinates of the X_1 diagram in the ratio $\frac{x}{C_{3n}}$, where y and x are the values taken at the point for which the value of M is required, the algebraic sum of the corrected ordinates will determine the influence line for the values of $Hy + X_1x + X_2$ at the section considered.

In order to obtain an influence diagram for M, therefore, we construct the influence line for the values of μ at the section considered. This diagram is a triangle, and using the same vertical scale, viz. twice the linear scale, the ordinate of this triangle on the vertical axis of the arch, should measure $x' = \frac{l}{2} - x$. The triangle being drawn in, the influence line for M will be the diagram between it and the curve already found for the values of $Hy + X_1x + X_2$.

In Plate III the above procedure is illustrated. Having drawn the curves A_2C_2, A_3C_3, and $A_4C_4B_4$, the ordinates of the first two polygons are altered in the ratio $\frac{y}{mm_1}$ and $\frac{x}{C_3n}$ by changing the polar distances in the inverse ratios. This has been done for two sections, viz. the section A at the extreme left, and the section 6, for which the corresponding values of x and y were measured. In this way the new polygons A_2a, A_2C_6, A_3b, A_3c were obtained. The



Stability of Arches. PLATE III. [*To face page* 79.

ordinates of the X_1 and X_2 polygons were then added and those of the H polygon subtracted, the result being the curves $A_4C_7B_4$, $A_4C_6B_4$. In the first case, since there is no bending moment at A for a freely supported beam, the curve obtained is the influence diagram for the bending moment at A, and its area, as measured by planimeter was found to be $+ 2{\cdot}22$ sq. ins. for the left-hand half of the girder, and $- 2{\cdot}17$ sq. ins. for the right-hand half. Therefore since the left-hand half carries a load of 2·2 tons per foot and the right-hand half carries a load of 1·3 tons per foot, and remembering that the scale of the drawing was 1 in. = 20 ft. and that the vertical scale is twice the linear scale, the total bending moment at A due to the whole load is $(2{\cdot}22 \times 2{\cdot}2 - 2{\cdot}17 \times 1{\cdot}3) \times \frac{400}{2}$ tons/ft. = 412·6 tons/ft., which checks closely with the result hitherto obtained by calculation (see p. 74).

In the second case, dealing with the bending moment at section 6, having obtained the resultant curve for the values of $Hy + X_1x + X_2$, the triangle $A_4C_6B_4$ is drawn having its intercept on the vertical axis = the distance of the section from the left-hand end. The shaded area then represents the influence diagram for M, and the algebraic sum of its areas, which is here negative and represents $- 154$ tons/ft., is the bending moment at section 5.

In this way influence lines for the bending moment at any section may be drawn, and it is then easy to see what distribution of loading will produce the most severe moment, and to obtain the value of it.

Now the maximum stresses due to the bending moment are always much in excess of the stresses due to the axial thrust, and therefore the distribution of load which determines the maximum bending moment at any section may be taken as that which produces maximum stress at the section. Also the normal thrust N for this loading may be found approximately from the formula $N = H \sec \phi$, where ϕ is the angle which the section makes with the vertical.

39. By way of illustration suppose we require the worst conditions of stress for the section 5. Having constructed the influence diagram for the bending moment at this section, we find that the areas of the positive and negative parts are + 132, − 308, and + 36 sq. ft., or + 66, − 154, and + 18 tons/ft. per ton of load covering these segments. Hence the resultant bending moment due to the dead load of 1·3 tons per ft. $= (66 + 18 - 154) \times 1·3 = -91$ tons/ft., and the maximum positive moment due to the live load of ·9 ton per foot covering the two extreme segments is $·9\ (66 + 18) = 75·6$ tons/ft., whilst the maximum negative bending moment due to the live load is $·9 \times 154 = 138·6$ tons/ft. The actual greatest and least values of the bending moment at the section are therefore $-91 + 75·6 = -15·4$ tons/ft., and $-91 - 138·6 = -229·6$ tons/ft.

The horizontal thrust H for these conditions of loading are found from the H line to be 260 tons and 376 tons respectively, whilst $\phi = 15\frac{1}{2}°$.

$$\therefore N = 260 \sec 15\tfrac{1}{2}° \text{ and } 376 \sec 15\tfrac{1}{2}° \text{ resp.}$$
$$= 270 \text{ and } 390 \text{ tons.}$$

M. Mesnager's Method.—This method is described in "Engineering," 17 March, 1916. It affords an easy means of calculating the stresses in a concrete arch which satisfies the conditions that its axis is parabolic and that its moment of inertia increases according to the law $I \cos \theta = I_o$, where I_o is the moment of inertia at the crown, and I is the moment of inertia at any other section inclined θ to the vertical. Then if the thickness at the crown is t, the thickness at any other point $\frac{t}{\sqrt[3]{\cos \theta}}$. If the arch is built in this way the bending moments at any point can easily

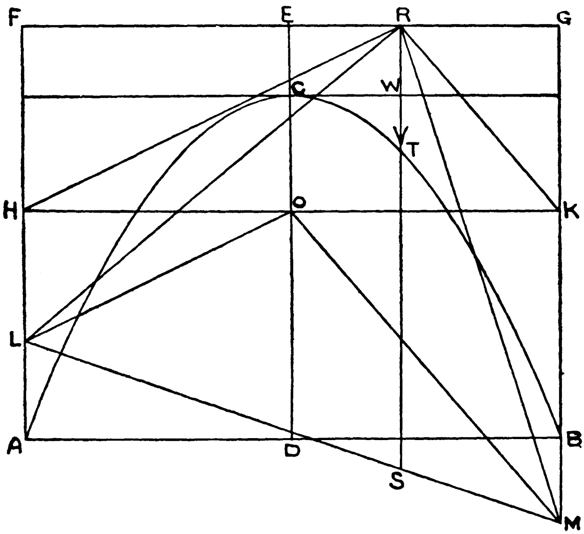


FIG. 30.

be found graphically, by the following construction, a proof of which is given in the issue of "Engineering" referred to above. Let ACB be the arch axis, and let W be a load in any position. Make CE = $\frac{1}{5}$ CD and through E draw a horizontal FG. Make

$CO = \frac{1}{3}$ CD and through O draw HK horizontal. Also let R be the point in which the line of action of W cuts FG. Join RH, RK, and from O draw OL, OM parallel to RH, RK. Join RM, RL. Then the intercept between these two lines and the arch axis gives at every point the bending moment for the arch produced by the load W on the same scale that the triangle LRM represents the bending moment produced by the same load on a simply supported girder. Thus RT represents the bending moment on the arch-ring to the same scale that RS represents the bending moment on a simply supported girder.

Since the ratio of all the vertical lines in the figure will be the same whatever the scale of vertical distortion, a diagram may be drawn to a large scale, and this diagram can then be used for finding the bending moments on an arch of the above type, whatever its rise and span, the result being rigidly accurate so long as the work done in compressing the arch-ring is negligible in comparison with that done in bending it, an assumption usually regarded as sufficiently accurate for practical purposes.

CHAPTER VI.

MASONRY AND CONCRETE ARCHES.

40. PREVIOUS to the development of the elastic theory, the investigation of the stability of masonry and brickwork arches was always carried out by testing whether an equilibrium polygon could be drawn to lie within the middle-third of the arch-ring; because if this were possible it was argued that the arch would be stable. This method of solution, however, gives no clue to the actual stresses set up, because the actual line of pressure is undetermined; and therefore where the greatest economy of material is desirable and necessary, as in the case of very large arches, the determination of the actual line of pressure is required, and this is only possible by means of the elastic theory of the arch; but in the case of small bridges, in which the thickness of the arch-ring is of secondary importance, the older and less accurate method may be used and is useful in checking the results obtained by calculation.

41. It has already been explained that, treated as a statical problem, no solution for a hingeless arch is possible, because the three conditions necessary to fix the position of the line of pressure are absent. It was, however, realized by the early engineers that an arch must necessarily be stable provided that:—

(1) There is no tension at any joint of the arch-ring.

(2) That the crushing strength of the material is not exceeded.

(3) That the line of thrust nowhere makes an angle with the normal at any bed-joint greater than the least value of the friction angle.

These are the three fundamental conditions of stability for an arch-ring. Now in order that the first of these conditions may be satisfied, it may easily be shown that, assuming a linear distribution of stress, it is only necessary for the line of thrust to lie within certain critical limits, which for a rectangular cross-section are the middle-third points of the arch-ring; or if we allow for frictional resistance, it may be taken, according to Scheffler, that provided the line of pressure lies within the middle half the primary condition of stability is satisfied.

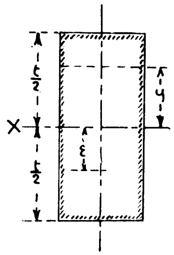


FIG. 31.

For when the resultant thrust on any section of a structure acts out of centre in the plane of the principal axis, the stress at any point distant y (Fig. 31) from the axis XX through its centroid is given by the formula

$$s = \frac{N}{A}\left(1 + \frac{ey}{k_0^2}\right)$$

where e is the eccentricity of the resultant thrust, N is the component of this thrust normal to the section, and y is to be taken positive or negative according as it is on the same side of the axis XX as the resultant or not. Now for a rectangular section

$$k_0^2 = \frac{t^2}{12}$$

in which case the formula becomes

$$s = \frac{N}{A}\left(1 + \frac{12ey}{t^2}\right).$$

Therefore if the stress becomes zero at the extreme edge when $y = -\frac{t}{2}$ we get

$$O = 1 - \frac{6e}{t}.$$

$$\text{or } e = \frac{t}{6};$$

that is to say, when the resultant departs from the axis of the section by one-sixth of its depth in either direction the stress on the remote edge is zero, and if more than this it becomes tensile. This is the well-known "Rule of the Middle-Third," viz. that so long as the resultant thrust lies within the middle-third of the depth, there will be no tensile stress in the arch-ring.

42. The question, however, recurs as to where the resultant does actually lie when the shape of the arch is known and the loading and other conditions are given. In the absence of the elastic theory it is necessary to rely upon the somewhat metaphysical argument known as the "Principle of Least Work". This important principle, which all experience tends to justify, has been stated very clearly by Rankine as follows:—

"If a body be in equilibrium under the action of any system of forces, this system will consist of forces of two kinds, which may be described as *active* and *passive* forces, standing to one another in the relation of *cause* and *effect*, the passive forces being called into play by the active forces; and since these passive forces will not increase after they have once produced equilibrium with the active forces, we conclude that the passive forces will be the least which are required to produce equilibrium."

In other words, we assume that Nature effects her purpose with the least expenditure of mechanical energy. It follows from this "Principle of Least

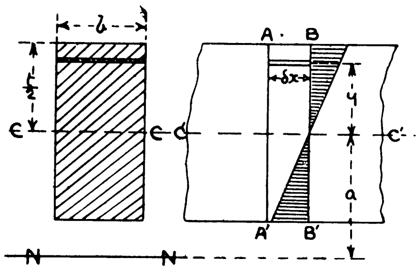


FIG. 32.

Work" that of all the equilibrium-polygons that can be constructed for a system of loads on an arch, that one is the true line of pressure which lies nearest to the axis of the arch-ring. For if AA′, BB′ (Fig. 32) be two neighbouring sections, unit distance apart, and s is the stress at any distance y from the axis C′C′, then $\frac{s}{s^o} = \frac{y + a}{a}$ where s is the stress at unit distance from the neutral axis NN, and since the strain-energy in tension or compression per unit length is $\frac{1}{2}$ volume $\times \frac{s^2}{E}$ the total strain-energy developed between the two sections will be given by the expressions

$$\sum_{y_1}^{y} \left[\frac{\frac{1}{2} b \,.\, \delta y \,.\, \delta x s^{o2} \left(1 + \frac{y}{a}\right)^2}{\mathrm{E}} \right]$$

$$= \frac{\delta x \,.\, s^{\,2}}{2\mathrm{E}} \sum_{y_1}^{y_2} \left[b \,.\, \delta y + \frac{2y}{a} b \,.\, \delta y + \frac{y^2}{a^2} \,.\, b \,.\, \delta y \right]$$

$$= \frac{\delta x \,.\, s_o^{\,2}}{2\mathrm{E}} \left[\mathrm{A} + \mathrm{O} + \tfrac{1}{2} \frac{1}{a^2} \mathrm{I}_o \right] = \frac{\mathrm{A} s_o^{\,2} \,.\, \delta x}{2\mathrm{E}} \left[1 + \frac{k_o^{\,2}}{a^2} \right]$$

and this is least when a is greatest. But the position of the load-point L and the position of the neutral axis N are connected by the relation CN . CL $= k_o^2$, so that when a is greatest, the load-point approaches nearest to the axis. Hence we derive the following important conclusion, *that the resultant line of thrust will always endeavour to set itself as near the axis of the arch-ring as possible, placing itself in such a position that whilst remaining an equilibrium-polygon for the given loads it will develop the minimum amount of strain-energy in the arch-ring.*

We conclude therefore that if an equilibrium-polygon can be drawn for the given load system that will lie altogether within the middle-third of the arch-ring, the primary condition of stability will be satisfied, because the true line of pressure will lie nearer to the axis than the one found, and therefore the arch will be more stable than if this one were the true line of thrust.

Now if we consider the manner in which an ordinary segmental arch usually yields under normal conditions of loading, we find that cracks begin to form along the soffit at the crown, and along the extrados towards the springings as shown at A, B, and C. This indicates that the line of pressure has passed outside the middle-third limits. In the *critical position* when the line of pressure just touches the middle-third lines as at A, C, and B (Fig. 33), the sections at which contact first occurs are known as the *joints of rupture,* because the arch will first yield

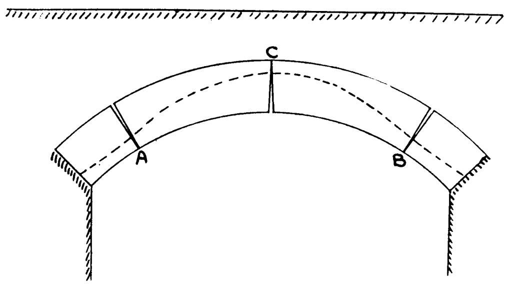


FIG. 33.

at these points, and the equilibrium-polygon is then known as the *critical line of pressure.*

Should the arch yield in the manner indicated above, the critical line of pressure is known as the curve of *minimum thrust,* because the horizontal thrust of the arch is then the least that can occur consistent with the first condition of stability. On the other hand, if the arch is so shaped or so loaded that it yields as shown in Fig. 34, the critical line of pressure is known as the *curve of maximum thrust,* because the horizontal thrust is then the greatest that can occur within the same limitations.

43. **Construction of the Critical Line of Pressure. Reduced Load Curve.**—In dealing with the load upon an arch it is usual to consider a portion

of the arch contained between two longitudinal sections 1 ft. or 1 m. apart, as the case may be, and to assume that the weights of the arch-ring and its

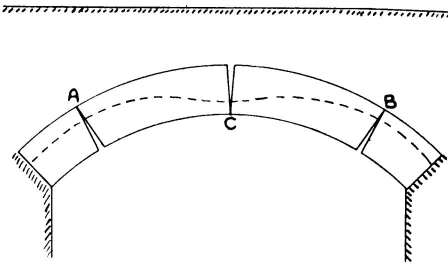


FIG. 34.

load act vertically; and this assumption is on the safe side, because if the arch-ring have a rise equal to or less than half the span, any horizontal forces due to earth pressure or other causes will increase its stability. Usually an arch has a concrete filling over the haunches, as shown in Fig. 35, upon which

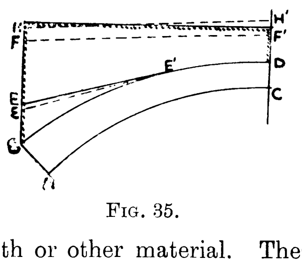


FIG. 35.

rests the earth or other material. These materials may be of much less specific weight than that of the arch-ring. In such a case it is convenient to construct a *reduced load curve* by diminishing the ordinates of each material so as to obtain a diagram which represents the amount of masonry having the same weight. Thus, e.g. if the weight of the arch-ring be 160 lb./ft.^{3}, that of the concrete backing 140 lb./ft.^{3}, of the earth filling 100 lb./ft.^{3}, and if there is also a live load of 200 lb./ft.^{2}, the ordinates between the line EE′ and the extrados are reduced in the ratio $\frac{140}{160}$ giving the curve *e*E′; the ordinates above EE′D are reduced in the ratio $\frac{100}{160}$ and added to the last, whilst the live load will be represented by an additional height of masonry of $\frac{200}{160}$ ft. on the scale of the drawing. In this way we get the line HH′ which is the reduced load curve, such that the area HH′CAB below it represents a quantity of masonry whose weight is equivalent to that of the combined weights of the other materials.

44. **Method of Fictitious Joints.**—A further simplification in the construction is introduced by disregarding the actual voussoirs and having constructed the reduced load curve, dividing it into vertical strips of equal width. If these widths are not large, their resultant weights may without sensible error be taken as acting along their centre lines. For if we take the trouble to consider the load resting on each voussoir, as is often done, and compound it with the weight of the voussoir itself, the difference in the results obtained will be quite inappreciable, because the weight of the triangular prism *ceb* (Fig. 36) is so small compared with the pressures on its faces that the line of pressure is not sensibly deflected by it in passing from *cb* to *ce*, and therefore no appreciable error is made in regarding the

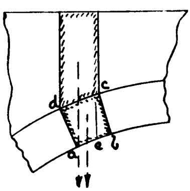


FIG. 36.

load as consisting of segments bounded by vertical planes, whose weights are assumed to act along the mid-ordinates, and if these planes are equidistant, the weights contained by them are proportional to the mid-ordinates under the reduced load curve, and these may therefore be used as vectors to represent them.

45. Construction of the Critical Line of Pressure for Minimum Thrust at the Crown, and for

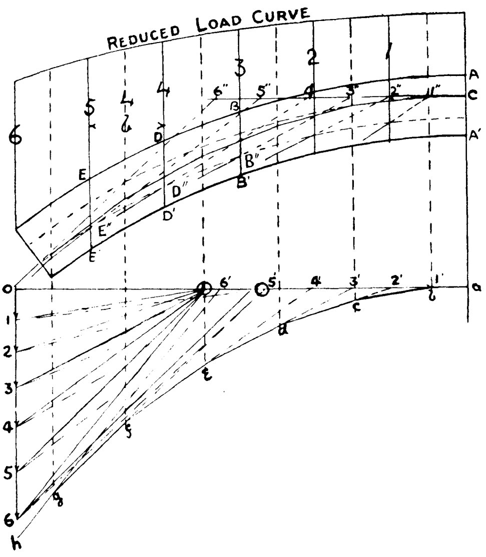
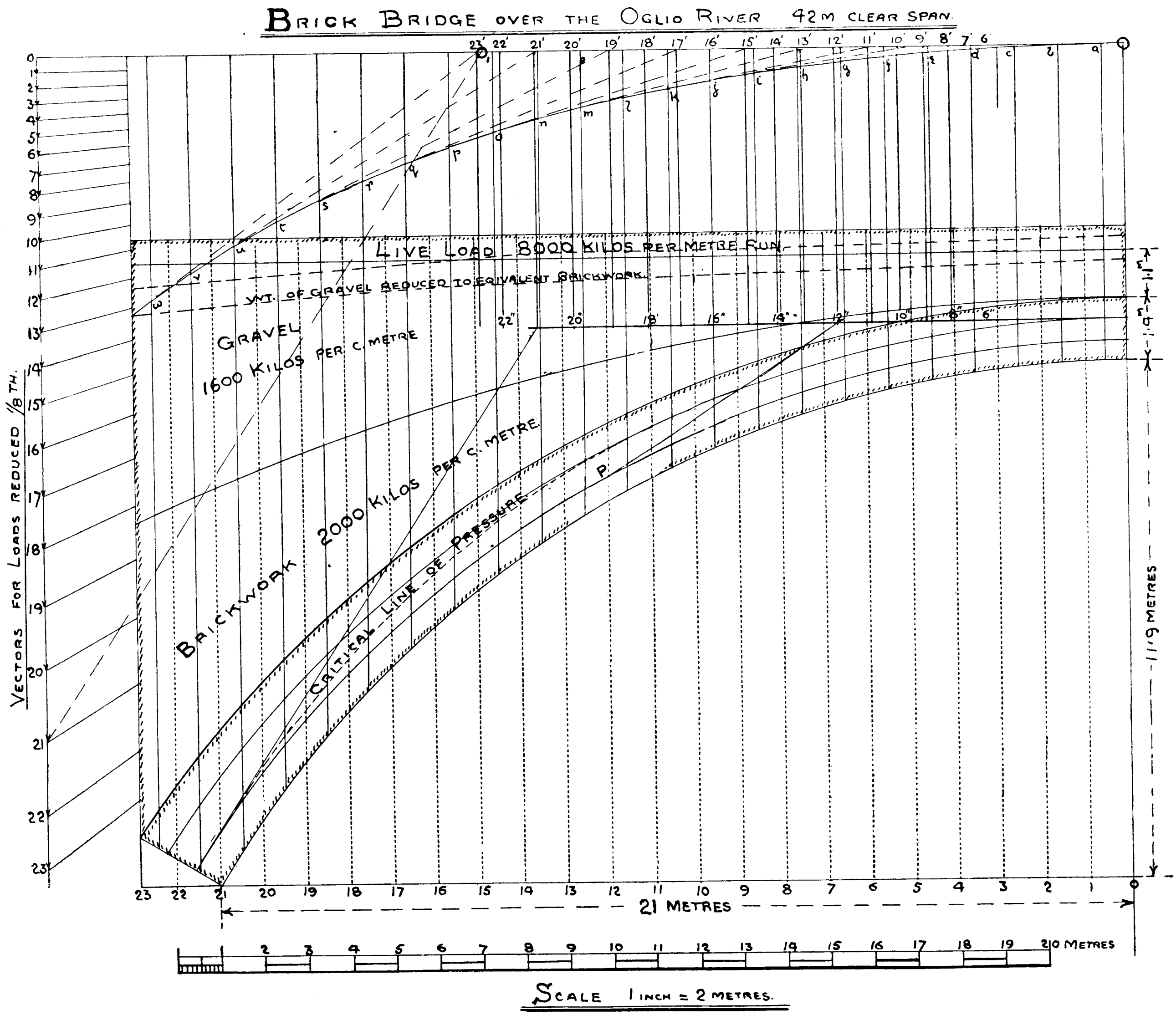


FIG. 37.

Symmetrical Loading.—Having constructed the reduced load diagram and divided it into vertical strips of equal width, vectors 0·1, 1·2, etc. (Fig. 37), are set off proportional to the mid-ordinates, and a pole O being taken on the horizontal through *o*, a link-polygon *ab* to *h* is drawn, whose sides are produced to cut the first link *ab* produced in the points 1′, 2′, etc. Now if we consider the equilibrium of any portion of the arch-ring, such as AA′B′B, and assume the line of pressure to act at the upper middle-third point C at the crown, then if B″ were by chance the point of rupture, the resultant thrust at this point will just be tangential to the lower middle-third line at B″, and its direction must be such as to pass through the point 3″ in which the resultant of the load intersects the horizontal thrust through C, according to the principle that when a body is in equilibrium under the action of three forces, the directions of those forces must pass through the same point. B″3″ will therefore be the direction of the resultant thrust at B″. But if we find that B″3″ produced *cuts* the lower middle-third line, B″ cannot be the point of rupture, because this occurs at the point where the line of pressure *touches* the middle-third line. We therefore try the point D″ in the same way by joining D″4″ and seeing if it cuts the lower middle-third line, and so on until we arrive at a point, say E″, where the line E″5″ first touches. This will be the joint of rupture, and a line through the point on the vector figure parallel to E″5″ will determine the pole O′ which corresponds to the critical line of pressure. If the link-polygon is now redrawn with this pole, the result will be the critical line of pressure for minimum thrust at the crown. If, however, no such tangent can be found, it does not necessarily follow that an equilibrium-polygon cannot be drawn lying within the middle-third



Brick Bridge over the Oglio River 42m clear span.
Live Load 8000 Kilos per metre run.
Wt. of gravel reduced to equivalent brickwork.
Gravel
1600 Kilos per c. metre
Brickwork 2000 Kilos per c. metre
Critical line of pressure
P
Vectors for loads reduced 1/8th.
1.19 metres
21 Metres
10 Metres
Scale 1 inch = 2 metres.

limits, for this may sometimes be effected by starting at a point below C at the crown.

46. In order to make clear the above description the line of pressure is drawn in Plate IV for a bridge over the Oglio, in Italy. This bridge is of brickwork and has a clear span of 21 m. and a rise of 11·9 m. The arch is a circular arc, the radius of the intrados being 24·4 m. and the radius of the extrados 27·5 m. with a crown thickness of 1·4 m. The width is 7·5 m. The weight of the brickwork is taken as 2000 kilo. per cub. metre, the load up to DE being also brickwork of approximately the same weight.

Radius of DE = 54·5 m. From DE to the horizontal FG the backing is gravel weighing 1600 kg. per c. m. GE = 1·1 m. The greatest live load to be carried is 8000 kg. per metre run.

The half arch was drawn to a scale of 1 in. = 2 m. and was then divided into twenty-two strips each 1 m. wide, with another one of less width at the end. The mid-ordinates were drawn in, as shown. These ordinates are proportional to the weights of the corresponding strips and represent those weights to a scale of 1 in. = $2^m \times 7{\cdot}5^m \times 2000$ kg./cm.$^3 = 30{,}000$ kg. The live load being 8000 kg. per metre will be represented by $\frac{8000}{30000} = {\cdot}267$ in. The ordinates of the gravel are reduced in the ratio $\frac{1600}{2000} = \frac{4}{5}$, and these together with the live load ordinates are added to the brickwork, so that we get the reduced load curve representing the equivalent amount of brick. The mid-ordinates of the reduced load diagram were then reduced $\frac{1}{8}$ and plotted along a vector line as shown, the scale being now 1 in. =

240,000 kg. A pole O was arbitrarily chosen on the horizontal through *o*, and a link-polygon *abc* . . . *w* was drawn whose links were produced to cut the first link in the points 1′, 2′, 6′, 7′, etc. These points were then projected upon a horizontal drawn through the upper middle-third point at the crown, giving 1″, 2″, 6″, 7″, etc. These represent the points through which the resultant pressures on each section of the arch-ring must go on the assumption that these resultants pass through the lower middle-third point. Thus if we join the point 12″ say, to the lower middle-third point of section 12, viz. P, the trial line 12″ P will be the direction of the resultant thrust on this assumption. But when produced it passes outside the lower middle-third line, and therefore the joint of rupture cannot be at this section. We find that as we approach the abutment the trial lines become more nearly tangential to the lower middle-third line, and in this example we find that it is only when we reach the abutment that the trial line becomes actually a tangent. We therefore draw through 23 a line having this direction, and this determines the pole O_1 for the critical line of pressure. Starting from the upper middle-third point at the crown this line of pressure is drawn in as shown.

The polar distance thus found measures the horizontal thrust corresponding to the critical line of pressure to the vector scale of 1 in. = 240,000 kg., and would in this case give the actual thrust of the arch because no other link-polygon could be drawn within the middle-third lines which would lie nearer to the axis than the one drawn; in other words, the critical line of pressure will in this case be the true line of pressure or approximately so.

47. Simplified Treatment of Small Segmental Arches.—In the case of small segmental arches it will usually be found, as in the preceding example, that the joints of rupture lie so near the springings that no error worth considering will be introduced if we assume that the line of pressure is tangential to the lower middle-third line at the springings

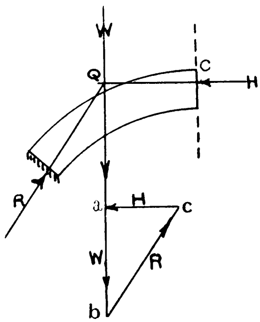


FIG. 38.

themselves. This greatly simplifies the investigation, because if we find the centre of gravity of the reduced load-area, the horizontal thrust H acting through the upper middle-third point C at the crown (Fig. 38) will meet the vertical through the centre of gravity in Q, and therefore the resultant reaction R acting through the lower middle-third point at the springing joint must have the direction RQ. A triangle of forces *abc* will then determine the magnitudes of H and R when W is known.

Further, experience shows that an arch in process of construction will support itself when extending beyond the abutments to some distance, and it may safely be taken that it may be trusted to do so in any case as far as a joint AA′ making an angle of about 30° (Fig. 39) with the horizontal as shown in Fig. 39. This part of the arch may therefore be regarded as forming part of the abutment itself, and it is only in this way that we can explain the stability of some arches which must otherwise be regarded as violating the primary condition of stability. In this case the springing joint may be regarded as at AA′.

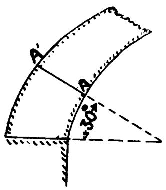


FIG. 39.

48. Construction of the Critical Line of Pressure for an Arch, when the Load is Asymmetric.—When the load upon an arch is asymmetric, the difficulty of constructing the critical line of pressure is greatly increased, because the highest point of the curve will no longer be at the centre of

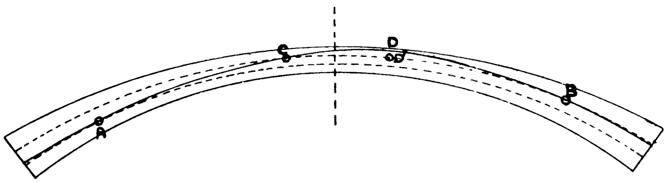


FIG. 40.

the arch as it must be for symmetrical loading. It will, however, lie somewhere near the centre on the side on which the load is greatest. A good way is therefore to select a point C (Fig. 40) in a reasonable position and two other points A and B on the lower and upper middle-third lines respectively at points near the springings where we may suppose the joints of rupture are likely to occur. A link-polygon for the given loads is then drawn to pass through the points A, C, and B as described in Chapter II, and if this lies within the middle-third lines, the primary condition of stability is satisfied. But if it is found to cut these lines, then the point D where the polygon recedes furthest from the axis should be noted, and a point D′ being taken on the middle-third line at this section another link-polygon is drawn to pass through A, D′, and B.

In the same way it may be found after drawing the first trial-polygon that the points A and B may require to be chosen in a different position to enable the link-polygon to lie within the middle-third, and the trial-polygon will enable us to select the most suitable positions; but if no such polygon can be drawn, then we must either (i) thicken the arch-ring; (ii) alter the form of the arch; or (iii) alter the distribution of the load.

49. **Adjustment of the Arch to Suit the Load.**—Let Fig. 41 represent an arch which has been designed to approximate to the form desired and which is proportioned, as far as one can estimate roughly, to carry the dead load, the proportions being those commonly adopted. For the sake of convenience in drawing suppose the vertical scale to be magnified four times. The lower figure represents the arch and its load so magnified, *ac* being the reduced load curve for the dead load. The area under this curve divided into five strips of equal width, and the loads, as represented by the mid-ordinates of these strips reduced to $\frac{1}{4}$ size were set down along a vector line 0·1 . . . 5, as shown. A pole O_1 was taken as the horizontal

through 5 at any convenient distance and a link-polygon *defgh* was drawn, its first and last links intersecting in *l*. A vertical through *l* then represents the line of action of the resultant load on the semi-arch.

Draw through the mid-point of the crown a horizontal CL to represent the horizontal thrust at the

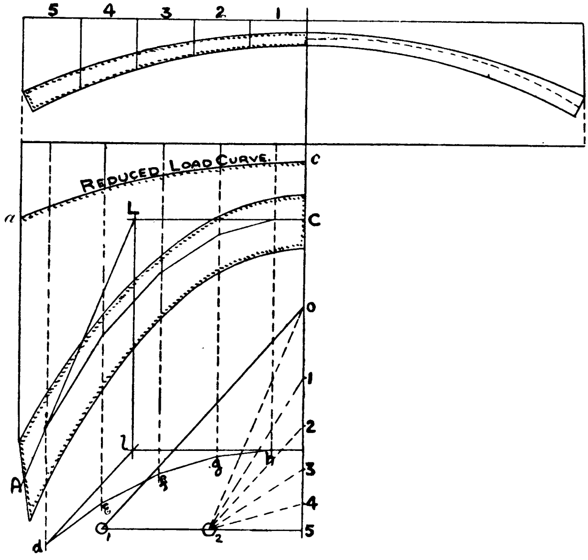


FIG. 41.

crown, meet the resultant load in L and join the centre A of the springing joint to L. Then AL represents the reaction at the springing when the line of pressure acts at its centre. Draw oO_2 parallel to AL, and with O_2 as a new pole, construct another link-polygon starting at A, which should pass through C. This will represent the line of pressure if we neglect any small alteration in the load due to the modified form of the arch-ring. We now alter the axis of the arch so as to make it conform to this curve, and the new form of the arch-axis so obtained is shown by a dotted line in the figure.

50. **Adjustment of the Load to Suit the Arch.** —The third method of design consists in altering the

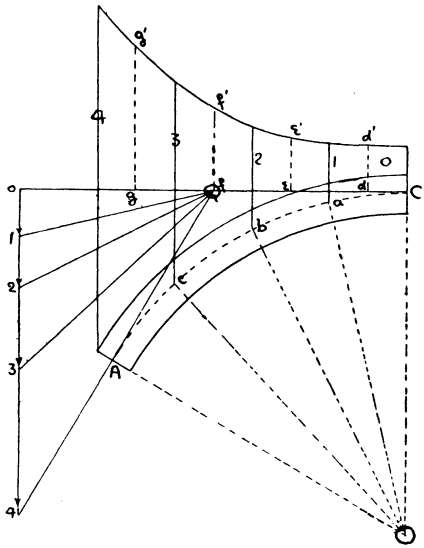


FIG. 42.

distribution of the load so as to make the line of pressure coincide approximately with the assumed form of the arch. This may be effected by altering the specific weights of the backing materials or by building openings in the spandrels when the load over the haunches is too great or by means of transverse arches, etc. In order to determine the distribution of load required to bring about the desired result, the half-span (Fig. 42) is divided into a convenient number of equal parts. Verticals drawn through the points so obtained intersect the arch-axis in the points a, b, c. At A, c, b, a, C draw radial lines OA, Oc, etc. Perpendiculars to OA and OC at A and C respectively intersect in Q, and therefore CQ, AQ are the directions of the line of pressure at C and A. If therefore we take Q as a pole, the lines Qo and Q_4 will be the directions of the rays of the vector figure corresponding to the first and last links of the line of pressure, and if we draw lines through Q parallel to the tangents at a, b, and c, these lines will cut off lengths 01, 1·2, 2·3, 3·4 on any vertical which are proportional to the loads on the corresponding segments of the arch.

If, therefore, we set up ordinates dd', ee', etc., equal to these lengths, at the mid-points of the horizontal segments into which the half-span was divided, and draw a fair curve $d'e'f'g'$ through the point so obtained, this curve will represent the distribution of load which will make the line of pressure coincide with the axis of the arch. If we estimate the probable weight of the load and the arch-ring for the segment adjacent to the crown, and note that this weight is represented by the length 0·1 in the vector figure, the scale of the diagram will be determined with sufficient accuracy, and the magnitudes of the loads on the other segments will be represented by the lengths 1·2, 2·3, 3·4 to the same scale.

51. **Abutments of an Arch.**—As we have already seen, if an ordinary flat arch yields under its load,

the line of pressure rises at the crown and sinks at the springings; but if the earth yield because the horizontal pressure due to earth or otherwise forces the abutment inward, the line of pressure sinks at the crown and rises at the springings as indicated in the sketch. In the first case therefore the line of minimum thrust should be used, and in the second case the line of maximum thrust.

In considering the stability of an abutment therefore, if we disregard the earth-pressure the line of

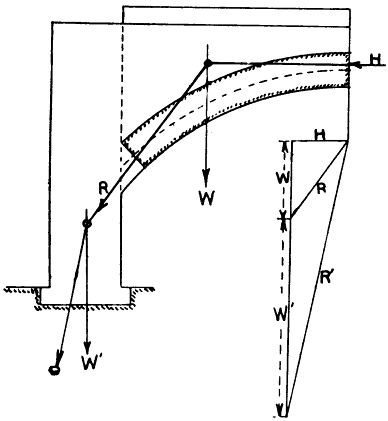


FIG. 43.

minimum thrust must be assumed as our critical line of pressure, and the arch must be considered as fully loaded, because the horizontal thrust is then greatest. The resultant W (Fig. 43) of the load on the half arch is then found, and a △ of forces determines the values of H and R. The weight W of the abutment is next calculated and compounded with the previous resultant R as shown. We thus get R′ which must lie within the middle-third of the base, and satisfy the other conditions of stability.

If this does not occur the width or slope of the abutments must be modified until the required conditions are fulfilled.

52. **Intermediate Piers.**—In the case of intermediate piers the most unfavourable condition of the load occurs when one span is fully loaded and the adjacent span unloaded. In this case, the line

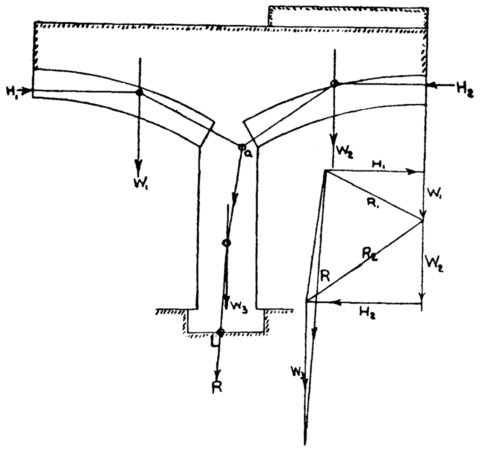


FIG. 44.

of maximum thrust is assumed as the critical line of pressure for the unloaded span, and the minimum line of thrust for the loaded span.

The resultant thrusts R_1 and R_2 at the springings (Fig. 44) intersect in *a*. The resultant of these is compounded with the weight of the pier W_3, and the final resultant cuts the base in L.

CHAPTER VII.

DESIGN OF MASONRY AND CONCRETE ARCHES.

53. WHEREAS in the construction of metal arches it is a matter of small importance what the exact form of the arch may be, in the case of masonry and concrete it is desirable that the line of pressure, in the worst case of loading, shall remain within or not far outside the middle-third of the arch-ring, and in order that this may be so with a minimum thickness of arch-ring the axis of the arch should deviate as little as possible from the line of pressure for the load it carries. In this case the arch-ring will be under pure thrust and the stresses will therefore have their least value. Any deviation from the true form necessitates a heavier arch, and, moreover, the deformations due to the load are increased, the theory becomes less correct, and the weight and cost of material are greater than need be, together with the expense of erection. Hence the problem is, to give the arch such a form that its axis for a definite mean load shall coincide with the line of pressure for that load. The direct solution of this problem is difficult, though it may be approximated to graphically by first designing the arch for its assumed load by any of the standard formulæ given in the text-books, then

determining the line of pressure for this arch, and afterwards modifying the form of the arch so as to make it conform to the line of pressure; or again the load on the arch may be modified by openings or arches in the spandrels or by changing the specific weight of the filling material, as explained in the last chapter.

54. **Empirical Formulæ for the Thickness of the Arch-ring at the Crown.**—The thickness of the arch-ring will depend to some extent upon the load which the arch is likely to carry, and in view of this it is convenient to classify arches under four groups:—

(1) Light arches which carry only their own weight and a light load, such as arches which form a ceiling without floors.

(2) Mean arches, used in buildings, which carry a floor, as in the case of cellar vaults.

(3) Heavy arches for road bridges, tunnels, etc., subjected to heavy loads but only slight impact and vibration.

(4) Very heavy arches for railway bridges, subject to heavy loading and strong vibration.

The thickness at the crown may then be calculated approximately by the formula

$$t_o = \left(\frac{n}{10} + \frac{1}{100s}\right)\left(3{\cdot}28 + \frac{l}{10}\right)$$

where n is the number of the group to which the arch belongs in the above classification, s is the ratio of rise to span, and l is the span. Thus for an arch carrying a floor and having a span of 3 ft. and with $s = \frac{1}{5}$ we have $n = 2$ and

$$t_o = \left(\frac{2}{10} + \frac{1}{20}\right) \times 3{\cdot}58 = {\cdot}895 \text{ ft.},$$

a result rather too large for such small arches.

For $n = 3$, $l = 100$ ft., $s = \frac{1}{5}$ we get

$$t_o = \left({\cdot}3 + \frac{1}{20}\right)(3{\cdot}28 + 10) = 4{\cdot}64 \text{ ft.}$$

Trautwine's formulæ for circular and elliptical arches are as follows:—

for first-class stone: $t_o = 0{\cdot}25\sqrt{\text{R} + 0{\cdot}5l} + 0{\cdot}2$;

for second-class work:—

$$t_o = 0{\cdot}281\sqrt{\text{R} + 0{\cdot}5l} + 0{\cdot}225;$$

for brickwork or fair rubble:—

$$t = 0{\cdot}333\sqrt{\text{R} + 0{\cdot}5l} + 0{\cdot}267,$$

where R is the radius of curvature at the intrados at the crown, l is the clear span, and the units are in feet.

Low's formula: $t_o = 0{\cdot}125\sqrt{10(l - h) + 2\text{H}}$ where h is the clear rise, and H is the height of the surcharge above the extrados at the crown.

Rankine's formulæ: $t_o = \sqrt{0{\cdot}12\text{R}}$ for a single arch

$t_o = \sqrt{0{\cdot}17\text{R}}$ for a series of arches.

Perronet's formula for circular or elliptical arches:

$$t_o = 1 + 0{\cdot}035l.$$

55. **Form of the Arch-Ring.**—Let CC_1 (Fig. 45) be the line of resistance for a distributed load of w per foot on the arch. At a distance x from the crown C the downward load is $\int_o^x w\,dx$ and the horizontal thrust is H. Hence the slope of the curve at this point is

$$\tan \alpha = \frac{dy}{dx} = \frac{1}{\text{H}}\int_o^x w\,dx.$$

Consequently $\dfrac{d^2y}{dx^2} = \dfrac{w}{H}$.

If therefore R is the radius of curvature at the point

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{w}{H}.$$

But $\dfrac{d^2y}{dx^2} = \dfrac{1}{R}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \dfrac{1}{R}(1 + \tan^2 a)^{\frac{3}{2}} = \dfrac{1}{R}\sec^3 a.$

Hence $H = wR \cos^3 a$

Now the normal pressure at C_1 is $N = H \sec a$, and if the compressive stress is to be the same for all

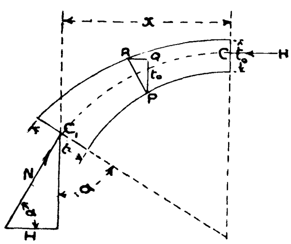


FIG. 45.

sections the thickness of the arch-ring t must be such that

$$\frac{\text{the pressure at the crown}}{t_0} = \frac{\text{pressure at } C_1}{t}$$

$$\text{or } \frac{H}{t_0} = \frac{H \sec a}{t}.$$

$\therefore\ t = t_0 \sec a$, or in other words, the projection of any section on the vertical must be equal to the crown thickness.

Further it may be shown that at the crown $R = r + t_0$ where r is the radius of curvature of the intrados there.

According to the above investigation the proper thickness of the arch-ring at any point may easily be found geometrically. For if P be any point of the intrados, and we set up PQ $= t$ and draw QR horizontal to cut the radius through P in R, PR will be thickness required. It is usually sufficient to determine the thickness at one other point P′ and to draw a circular arch passing through both. The position of this second point is at the joint of rupture and this depends on the form of the arch. For semi-

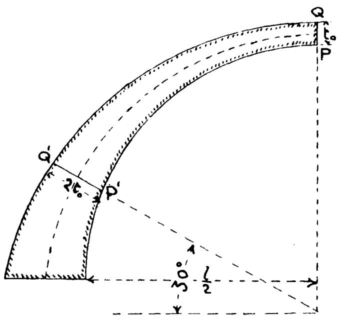


FIG. 46.

circular arches it may be taken at the section which makes an angle of 30° with the horizontal (Fig. 46) and corresponds at the intrados with the half-rise $= \frac{l}{4}$.

The thickness is then $t = t_o \sec 60° = 2t_o$.

For elliptic or pseudo-elliptic arches the joint of rupture at the haunches may be taken at the section which is inclined at 45° or thereabout (Fig. 47), and the thickness $t = 1{\cdot}4 t_o$; or more conveniently, we may take $t = k \,.\, t_o$ where k is a co-efficient which depends upon the ratio of rise to span, and whose value may be taken as follows :—

For a ratio $\frac{1}{3}$ $k = 1{\cdot}80$

" " $\frac{1}{4}$ $k = 1{\cdot}60$

" " $\frac{1}{5}$ $k = 1{\cdot}40$

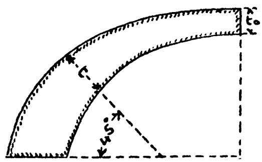


FIG. 47.

Croizette-Desnoyer's formulæ for thickness at the springings :—

S.	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$	
$\frac{t_s}{t_o}$	1·40	1·24	1·15	1·10	for segmental arches.

S.	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	
$\frac{t_s}{t_o}$	1·80	1·60	1·40	for elliptic and pseudo-elliptic arches.

56. **Thickness of the Abutments.**—The thickness of the abutments may be found from the formula

$$T = \left(\cdot 3 + \frac{1}{100s}\right)\ (l - h + H + 2t_o)$$

(Fig. 48), where H is the height of the springings above the foundation.

Thus for an arch of 100 ft. span, with $s = \frac{1}{5}$, $H = 15$ ft., $t_o = 4{\cdot}5$ ft.,

$$T = \left(\cdot 3 + \frac{1}{20}\right)\ (100 - 20 + 15 + 9) = 36{\cdot}4 \text{ ft.}$$

Trautwine's formula is $a = \cdot 2\text{R} + 0\cdot 1\text{H} + 2$

$$\text{T} \geqq \tfrac{2}{3}h.$$

Rankine states that a varies from $\frac{1}{3}$ to $\frac{1}{5}$ radius of the intrados at the crown in existing structures.

German practice $a = 1 + 0\cdot 04\ (sl + 4\text{H})$.

Piers.—These are usually from $2\frac{1}{2}$ to 3 times the thickness of the arch-ring at the crown, except in the case of abutment piers, which are constructed when a series of arches are in line; these are made sufficiently strong to take the centre thrust from either side in case of the collapse of one of the spans.

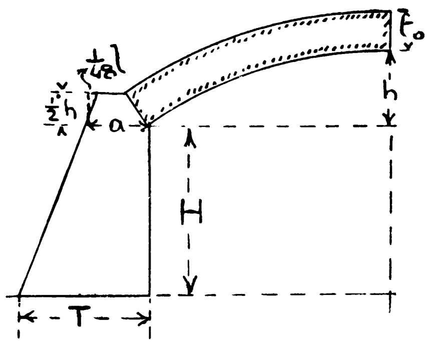


FIG. 48.

The results of these formulæ often differ widely from one another; they can therefore only be regarded as useful for the purpose of preliminary calculations and the design should be verified by the methods already described.

57. Tolkmitt's Investigation of the Form of the Arch-Ring.—Tolkmitt ("Leitfaden für das

Entwerfen und die Berechnung gewölbter Brücken," 1912) has dealt with the form of the arch-ring in a more scientific way, and his method of procedure has found considerable favour on the Continent. He begins by assuming that the axis of the arch coincides with the line of resistance for the dead load on it, and then investigates the curve which the axis must assume in order to satisfy this condition for a given system of loading. Taking the centre of the crown at the intrados as origin of co-ordinates, and ξ, η as the co-ordinates of any point on the axis, and calling A the area under the reduced load-curve = NMOS,

$$A = \int_0^\xi z \, . \, d\xi \text{ and } \tan \alpha = \frac{A}{H} = \frac{d\eta}{d\xi}$$

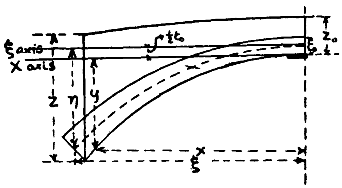


FIG. 49.

where α is the angle of inclination PQS of the normal to the vertical.

$$\text{Whence } \frac{d^2\eta}{d\xi^2} = \frac{1}{H} \cdot \frac{dA}{d\xi} = \frac{z}{H} \quad . \qquad . \quad (1)$$

When z is a function of the co-ordinates this equation can be solved, and its solution determines the form of the arch-ring.

In the first place applying the result to a load diagram whose upper boundary is horizontal, and taking the intrados at the crown as origin of co-ordinates, he shows that the form of the intrados may be expressed approximately by the equation (Fig. 49)

$$y = \frac{z_o x^2}{2(1 - e)\,\mathrm{H} - (\frac{1}{8} + e)\,x^2} \quad . \quad (2)$$

where
$$e = \frac{t_o z_o}{\mathrm{H}} . \quad . \quad . \quad . \quad (3)$$

and z_o is the height of the reduced load surface above the intrados. The radius of curvature of this curve at the crown when $x = 0$ is

$$\mathrm{R}_o = \frac{\mathrm{H}}{z_o}(1 - e) \quad . \quad . \quad . \quad (4)$$

and substituting the value of H from this equation in (2) we get

$$y = \frac{z_o x^2}{2\mathrm{R}z_o - (\frac{1}{8} + e)\,x^2} \quad . \quad . \quad (5)$$

With the help of (3) and (4) we get

$$\mathrm{H} = (\mathrm{R} + t_o)\,z_o \quad . \quad . \quad . \quad (6)$$

and then by (3)

$$e = \frac{t_o}{\mathrm{R} + t_o} \quad . \quad . \quad . \quad (7)$$

Now let $m = \dfrac{z_o}{\frac{1}{8} + e} = \dfrac{z_o}{\frac{1}{8} + \dfrac{t_o}{\mathrm{R} + t_o}} \quad . \quad . \quad (8)$

The equation (5) may be written

$$y = \frac{mx^2}{2m\mathrm{R} - x^2} \quad . \quad . \quad . \quad (9)$$

The compressive stress at the crown for uniform distribution of stress is

$$s_o = \frac{\mathrm{H}}{t_o} \quad . \quad . \quad . \quad (16)$$

In designing an arch, assuming the thickness at the crown, and the load there, then for a given span we can calculate the proper form of the arch. Thus, for example, suppose $t_o = 2$ ft., $z_o = 4$ ft., $l = 60$ ft. as above, and $s_o = 20{,}000$ lb./ft.^{2}, the weight of the material of the arch-ring being 120 lb./ft.^{3}

Then s_o in masonry units will be 166,

and by (4) $R_o = \dfrac{H}{z_o}(1 - e)$ where $H = s_o t_o$ by (16)

$$\text{and } e = \frac{t_o z_o}{H} \text{ by (3).}$$

$$\therefore\ R_o = \frac{s_o t}{z_o}\left(1 - \frac{t_o z_o}{s_o t_o}\right) = t_o\left(\frac{s}{z_o} - 1\right)$$

$$= 2\left(\frac{166}{4} - 1\right) = 81 \text{ ft.,}$$

$$\text{and by (8)}\quad m = \frac{4}{\frac{1}{8} + \frac{4}{166}} = 54{\cdot}2.$$

$$\therefore \text{ by (9)}\quad y = \frac{54{\cdot}2x^2}{2 \times 54{\cdot}2 \times 81 - x^2} = \frac{54{\cdot}2x^2}{8780 - x^2},$$

and when $x = 30$ ft., $y = h = \dfrac{54{\cdot}2 \times 900}{8780 - 900} = 6{\cdot}2$ ft.

For given values of l, h, and t_o, the radius of curvature R at the crown decreases or increases according as z_o becomes greater or smaller; that is to say, as the load increases the arch rises between the crown and springings, and as the load decreases it falls. It follows therefore that in designing an arch, neither the fully-loaded nor the unloaded condition should be used as the basis of calculation, but rather a mean condition, viz. one in which half the moving load is distributed uniformly over the arch; for the axis of the arch can, of course, be an equilibrium-polygon for one condition of loading only, and the deviations of this equilibrium-polygon from the extreme cases above and below it will be least if the axis of the arch is made an equilibrium-polygon for the *mean load*, where by the mean load is to be understood

$$z_o = t_o + e + \frac{w}{2}$$

where w is the height of the moving load expressed in masonry units, and e is the height of the dead load over the extrados in masonry units.

Also at the supports $y = h$, when $x = \frac{l}{2}$.

Substituting these values in (5) and (9) we get

$$R = \frac{l^2}{8z_o}\left(\frac{z_o}{h} + \frac{1}{8} + \frac{t_o}{R + t}\right) \quad . \quad (10)$$

$$\text{and } R = \frac{l^2}{8} \cdot \frac{h + m}{hm} = \frac{l^2}{8}\left(\frac{1}{h} + \frac{1}{m}\right) \quad . \quad (11)$$

and putting this last value of R into (9)

$$y = \frac{mx^2}{\frac{l^2}{4} \cdot \frac{h + m}{h} - x^2} \quad . \quad . \quad (12)$$

Formula (10) is a quadratic in R which on solution gives

$$R = -t_o + \frac{l^2}{16z}\left[\frac{z_o}{h} + \frac{1}{8} + \frac{8t\,z}{l^2} + \sqrt{\left(\frac{z}{t} + \frac{1}{8} + \frac{8t\,z_o}{l^2}\right)^2 + \frac{32t_oz_o}{l^2}}\right. \quad . \quad (13)$$

This combined with (6) gives the horizontal thrust

$$H = \frac{l^2}{16}\left[\frac{z}{h} + \frac{1}{8} + \frac{8t_oz}{l^2} + \sqrt{\left(\frac{z_o}{t_o} + \frac{1}{8} + \frac{8t\,z}{l^2}\right)^2 + \frac{32t\,z_o}{l^2}}\right] \quad . \quad (14)$$

and since by (8) $m = \dfrac{z}{\frac{1}{8} + e}$ and by (3) $e = \dfrac{t_o z_o}{\text{H}}$

$$\text{we get } m = \frac{8\,\text{H}z}{\text{H} + 8t_o z_o}. \quad . \quad . \quad (15)$$

58. *Example.*—Given an arch having a clear span of 60 ft. with a rise of 15 ft. and a crown thickness of 2 ft. with a reduced load of 2 ft. over the crown, it is required to find the horizontal thrust and the proper form of the arch-ring.

$$\text{Here } z_o = 2\text{ ft.} + 2\text{ ft.} = 4\text{ ft.}\,;\ \frac{z_o}{h} = \frac{4}{15} = \cdot 267$$

$$\frac{8t_o z_o}{l^2} = \frac{8 \times 2 \times 4}{60^2} = \cdot 018.$$

$\therefore$ by (14)

$$\text{H} = \frac{60^2}{16}\left[\cdot 267 + \cdot 125 + \cdot 018 + \sqrt{(\cdot 410)^2 + 4 \times \cdot 018}\right]$$

$$= 225\,[\cdot 410 + \cdot 490] = 202 \cdot 5 \text{ ft.}^3\,;$$

that is to say, the horizontal thrust is equal to the weight of 202·5 cub. ft. of the masonry of the arch-ring.

$$\text{Further by (15) } m = \frac{8 \times 202 \cdot 5 \times 4}{202 \cdot 5 + 8 \times 2 \times 4} = 24 \cdot 3 \text{ ft.}$$

$$\text{and by (12) } y = \frac{24 \cdot 3x^2}{\dfrac{60^2}{4} \cdot \dfrac{15 + 24 \cdot 3}{15} - x^2} = \frac{24 \cdot 3x^2}{2358 - x^2}.$$

for $x =$	5	10	15	20	25	30 ft.
$y =$	0·26	1·08	2·56	4·96	12·62	15 ft.

The following values are the ordinates for a parabolic arc of the same rise and span :—

$y =$	0·42	1·67	3·75	6·67	10·42	15 ft.

and by (11) $R = \frac{3600}{8} \cdot \left(\frac{1}{15} + \frac{1}{24 \cdot 3}\right) = 215$ ft.

59. **Determination of the Thickness of the Arch-Ring.**—The thickness of the arch-ring must satisfy the two following conditions: first, the permissible stress must not be exceeded, and second, it must be possible to draw an equilibrium-polygon for the given load which shall lie within the middle-third of the arch-ring when the arch is loaded on one-half.

Since the actual position of the line of pressure is not known, it is not advisable to base the first condition upon the maximum stress at the edge of the arch-ring, but rather on the value $s_o = \frac{H}{t_o}$, which is the mean stress at the crown. This stress then must not be exceeded. Also in order that the second condition may be fulfilled $t_o \geqq \cdot 06 \frac{wl^2}{H}$

$$\text{and since } H = s_o t_o \text{ we get } t_o \geqq \tfrac{1}{4} l \sqrt{\frac{w}{s_o}}.$$

Example 1.—An arch of 60 ft. span and 15 ft. rise is built of material whose maximum permissible stress is 2000 lb./ft^2. It carries a load of 200 lb./ft.^{2} Find the least thickness of the arch-ring if its weight is 150 lb./ft.^{3}

Here $l = 60$, $w = \frac{200}{150}$ ft. $= \frac{4}{3}$, $s_o = \frac{20000}{150} = 133$ ft.

$$\therefore t_o = \tfrac{1}{4} \times 60 \sqrt{\frac{4}{3 \times 133}}$$

$$= \frac{30}{20} = 1 \cdot 5 \text{ ft.}$$

Example 2.—Given that the load per foot run on an arch is $w = 600$ lb. per sq. ft. and the maximum permissible stress $s = 20{,}000$ lb./ft.^{2} If the span is 50 ft., find the proper rise and thickness of arch-ring. Height of filling over crown = 3 ft. Weight of filling material 120 lb./ft.^{3}, weight of arch-ring 150 lb./ft.^{3}

Here $s_o = \dfrac{20000}{150} = 133$ ft. expressed in masonry units

and $e = 3 \times \dfrac{120}{150} = 2{\cdot}4$ ft., $w = \dfrac{600}{150} = 4$ ft., $l = 50$ ft.

$$\therefore\ t_o = \tfrac{1}{4} \times 50\sqrt{\frac{4}{133}} = 2{\cdot}1 \text{ ft.}$$

Then $z = 2{\cdot}1 + 2{\cdot}4 + \frac{4}{2} = 6{\cdot}5$ ft.

$$m = \frac{z_o}{\frac{1}{8} + \frac{z_o}{t_o}} = \frac{6{\cdot}5}{\frac{1}{8} + \frac{6{\cdot}5}{133}} = 37{\cdot}4.$$

$$r = t_o\left(\frac{s}{z_o} - 1\right) = 2{\cdot}1\left(\frac{133}{6{\cdot}5} - 1\right) = 41 \text{ ft.}$$

$$y = \frac{mx^2}{2mr - l^2}.$$

$$\therefore \text{ when } x = \tfrac{1}{2}l = 25 \text{ ft.}$$

$$y = h = \frac{37{\cdot}4 \times 25^2}{2 \times 37{\cdot}4 \times 41 - 625} = 9{\cdot}55 \text{ ft.}$$

60. **Determination of the Best Form of Arch when the Load-Curve has any Form.**—On account of openings in the spandrels, or the use of filling materials having different specific weights, or other causes, the upper boundary of the reduced load-area is often not horizontal, as we have been assuming. In the case of ordinary bridges for roads or railways the deviation from a horizontal line is, however, never very great, and therefore it is permissible to

assume as a first approximation that the upper boundary is a horizontal line at a height z_0 above the intrados and to determine in this way the form of the arch. The curve so found is not, however, the correct curve, but serves as a basis for further calculations.

Thus suppose ABCDE (Fig. 50) to be the actual

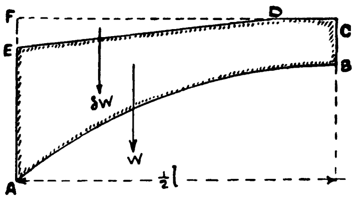


FIG. 50.

reduced load-area in a given case. Then this area may be regarded as arising from the case in which the surface is horizontal as indicated by CF, by supposing the part DEF subtracted, for the corrections due to the change of form of the intrados may be regarded as negligible. Let W be the weight of the material represented by the area ABCF, δW be the weight represented by DEF, and δM be the moment of δW about A. Then if δH and δR are the corresponding changes in the horizontal thrust H and in the radius of curvature R, and if the actual values be denoted by affixing a dash to the symbols used above, we get for the actual load-area

$$W' = W - \delta W, \quad R' = R - \delta R, \quad M' = M - \delta M,$$
$$H' = H - \delta H.$$

Then since the moment of the load-area may be regarded as equal to the product of the horizontal thrust into the rise of the line of pressure, and since the points in which the line of pressure cuts the vertical support reactions in either case may be taken as coincident, we have the relation

$$\frac{\mathrm{H} - \delta\mathrm{H}}{\mathrm{H}} = \frac{\mathrm{M} - \delta\mathrm{M}}{\mathrm{M}}.$$

Also $(\mathrm{R}' + t_o)z_o = \mathrm{H}'$ and $(\mathrm{R} + t_o)z_o = \mathrm{H}$ (eq. 6),

whence $$\delta\mathrm{H} = \frac{\mathrm{H}}{\mathrm{M}} \cdot \delta\mathrm{M} \quad . \quad . \quad (17)$$

and $$\delta\mathrm{R} = \frac{\delta\mathrm{H}}{z_o} = \frac{\mathrm{H}}{\mathrm{M}} \cdot \frac{\delta\mathrm{M}}{z_o} \quad . \quad . \quad (18)$$

The values of δH, δM and δR are positive if the load, area rises towards the supports, and negative when it falls.

The procedure therefore is as follows: H and R are first calculated on the supposition that the surface of the reduced load-area is horizontal. M is then found by the approximate formula

$$\mathrm{M} = \frac{l^2}{8}\left[z_o + \frac{h}{h + m}\left(\frac{m}{6} + \frac{h}{15}\right)\right] \quad . \quad (19)$$

δH and δR are next calculated by means of formulæ (17) and (18).

Thence we deduce the correct horizontal thrust H − δH and the correct radius of curvature at the crown R − δR. The form of the arch may now be improved by inserting for R its corrected value in the equation $y = \dfrac{mx^2}{2m\mathrm{R} - x^2}$ and changing the value of m so that the curve may pass through the supports as before; that is to say, make $y = h$ when $x = \frac{1}{2}l$.

61. *Example.*—In Fig. 50 suppose $\frac{1}{2}l = 30$ ft., $h = 15$ ft., $t_o = 2$ ft., $z_o = 4$ ft., and the slope of

DE to be 1 in 10. CD = 6 ft., DF = 24 ft., EF = 2·5 ft.

For a horizontal surface CF, by formula (14),

$$H = \frac{l^2}{16}\left[\frac{z_o}{h} + \frac{1}{8} + \frac{8\,t_o\,z_o}{l^2} + \sqrt{\left(\frac{z}{h} + \frac{1}{8} + \frac{8\,t\;z_o}{l^2}\right)^2 + \frac{32\,t\;z_o}{l^2}}\right.$$

$$= \frac{3600}{16}\left[\frac{4}{15} + \frac{1}{8} + \frac{8 \times 2 \times 4}{3600} + \sqrt{(\cdot 3935)^2 + \frac{32 \times 2 \times 4}{3600}}\right.$$

$$= 225\,[\cdot 3935 + \cdot 475] = \underline{195 \text{ cub. ft.}}$$

Also by formula (15)

$$m = \frac{8Hz_o}{H + 8\,t_o\,z_o} = \frac{8 \times 195 \times 4}{195 + 8 \times 2 \times 4} = \underline{24{\cdot}1 \text{ ft.}};$$

and by (11)

$$R = \frac{l^2}{8}\left(\frac{1}{h} + \frac{1}{m}\right) = \frac{3600}{8}\left(\frac{1}{15} + \frac{1}{24{\cdot}1}\right) = \underline{48{\cdot}6 \text{ ft.}}$$

Then by the figure

$$\delta W = \frac{24 \times 2{\cdot}5}{2} = 30 \text{ ft.}^3,$$

$$\text{and } \delta M = \delta W \times \frac{24}{3} = 240 \text{ ft.}^4$$

$$\therefore \text{ by (19)} \quad M = \frac{l^2}{8}\left[z_o + \frac{h}{h+m}\left(\frac{m}{6} + \frac{h}{15}\right)\right]$$

$$= \frac{3600}{8}\left[4 + \frac{15}{15 + 24{\cdot}1}\left(\frac{24{\cdot}1}{6} + \frac{15}{15}\right)\right] = \underline{2666 \text{ ft.}^4}$$

Hence by (17)

$$\delta H = \frac{H}{M}\,.\,\delta M = \frac{195}{2666} \times 240 = \underline{17{\cdot}5 \text{ ft.}^3},$$

which is negative, because δM is negative,

$$\text{and by (18)} \quad \delta R = \frac{H}{M}\cdot\frac{\delta M}{z_o} = \frac{17{\cdot}5}{4} = \underline{4{\cdot}4 \text{ ft.}}$$

The actual horizontal thrust of the arch is therefore

$$H' = H - \delta H = 195 - 17{\cdot}5 = \underline{177{\cdot}5 \text{ ft.}^3}$$

and the radius of curvature at the crown is

$$R' = R - \delta R = 48{\cdot}6 - 4{\cdot}4 = \underline{44{\cdot}2 \text{ ft.}}$$

Further, if in the equation $y = \dfrac{mx^2}{2mR - x^2}$, we put $y = h = 15$ ft. and $x = \frac{1}{2}l = 30$ ft., and the corrected value of $R = 44{\cdot}2$ ft.,

then $15 = \dfrac{m' \times 900}{2m' \times 44{\cdot}2 - 900}$ where m' is the corrected value of m, from which we get $m' = 31{\cdot}5$ ft.

The corrected form of the arch is therefore

$$y = \frac{m'x^2}{2m'R' - x^2} = \frac{31{\cdot}5x^2}{2 \times 31{\cdot}5 \times 44{\cdot}2 - x^2} = \underline{\frac{31{\cdot}5x^2}{2780 - x^2}}.$$

CHAPTER VIII.

LOADS AND STRESSES.

62. THE weight of masonry and concrete arches and the dead load they carry is so much greater than in the case of metal arches that the effect of impact and vibration is not of much importance. Moreover, the live load is relatively so small in bridges of any size that if the line of pressure for the fully-loaded arch coincide with the axis of the arch, it may safely be assumed that under the worst conditions of loading it will not pass beyond the critical limits. Also concentrated loads are so much better distributed by the filling, that they may be dealt with by reducing them to an equivalent distributed load, and it is further convenient to express the live load as well as the filling in terms of an equivalent weight of masonry whose density is that of the arch-ring.

Weight and Strength of Arch Materials.

		Weight per Cub. Ft. Lb.	Values of s_0 lb./ft.^{2}
Masonry :	Sandstone or Limestone .	135 to 150	40,000 to 60,000
	Granite	165	100,000
	Brick	120	30,000 to 40,000
	Concrete	140	50,000 to 80,000
	Reinforced Concrete .	150	—
	Gravel and Earth . .	90 to 120	—

63. **Dead Load.**—The dead load consists of the weight of the arch itself and the filling, etc., which it carries. The following are the average weights for the materials involved :—

The height of the filling over the extrados at the crown may be taken

$$\text{for railway arches as } \frac{300}{\rho} \text{ ft.}$$

$$\text{and for highway arches as } \frac{185}{\rho} \text{ ft.}$$

where ρ is the density of the material in lb. per cub. foot.

Average Permanent Load on Road Bridges.

Iron road bridges :—		
Timber platform and ballast .	100 lb./ft.^{2}	= 120 lb./ft.^{2}
Cross girders	20 ,,	
Iron bridges with brick arches :—		
Arches	48 lb.	= 210 lb./ft.^{2}
Concrete and asphalt . . .	42 ,,	
Metalling	100 ,,	
Cross girders	20 ,,	

64. **Live Load.**—The following formulæ for the live load, which may be assumed, are given in Melan's "Plain and Reinforced Concrete Arches" (translation by D. B. Steinmann) :—

$$\text{for very heavy vehls. } w = \left(100 + \frac{12000}{l}\right)\frac{3+e}{3e} \text{ lb. p. ft.}^2$$

$$\text{,, heavy vehicles } \quad w = \left(100 + \frac{6500}{l}\right)\frac{3+e}{3e} \quad \text{,,}$$

$$\text{,, light vehicles } \quad w = \left(100 + \frac{2600}{l}\right)\frac{3+e}{3e} \quad \text{,,}$$

$$\text{,, railway bridges } w = \left(1000 + \frac{20000}{l}\right)\frac{3+e}{7+4e} \quad \text{,,}$$

When half the arch is loaded we must write $\frac{1}{2}l$ instead of l.

The load to be allowed for a dense crowd of people may be taken as 100 lb. per sq. foot.

Highway Bridges.—For small spans heavy wagons give the most severe loading, but for large spans a crowd of people. The axle distances for heavy wagons are about 10 ft., so that for spans up to 20 ft. only one axle at a time comes upon the one half of the bridge. In the case of a bridge 20 ft. wide, which carries on one half an axle load of 6 tons, if we consider the axle load as spread over 10 ft. width, the average live load will be $\frac{2 \times 6}{10 \times 10} = \cdot 12$ ton/ft.^{2}

Taking the heaviest freight wagon of 24 tons, 25 feet long, and a team of six horses weighing ·35 ton each, extending over 36 ft., this gives upon a length of 61 ft. and a width of 10 ft. an average load of

$$\frac{24 + 6 \times \cdot 35}{10 \times 61} = \cdot 0428 \text{ ton.}$$

Without the horses, however, on a length of 25 ft. we get

$$\frac{24}{10 \times 25} = \cdot 096 \text{ ton/ft.}^2$$

In dealing with concentrated loads on small spans these are supposed distributed over an area drawn at 35° to the vertical from the boundary of the load.

65. **Railway Bridges.**—In the case of locos the axle spacings are as low as 5 ft., and although the pressure is better distributed by means of rails, sleepers, and ballast than in the case of roadway bridges with a thin roadway, yet even in the case of spans as small as 30 ft. the full weight of three axles must be allowed on the half span. If the maximum wheel-load for a 72-ton engine be taken

as 8 tons, this gives a total load of 48 tons distributed over a width of say 12 ft. of archway (= length of sleepers + 3 ft.), that is to say, over $15 \times 12 =$ 180 sq. ft. of surface.

Hence we get an average of $\frac{48}{180} = \cdot 267$ ton/ft.^{2}

On longer spans, however, the equivalent distributed load will diminish as the span increases up to a certain point. Thus the following table gives the equivalent distributed load for railway bridges of varying span calculated for a maximum load of 66-ton engine and 1-ton tender with an increase of $2\frac{1}{2}$ per cent for future contingencies.

Span in feet .	10	15	20	25	30	40	50	100	150	200
Equivalent load in tons per ft.	3·69	3·11	2·88	2·61	2·45	2·22	2·10	1·93	1·92	1·92

66. *Example.*—A highway bridge has an opening of 100 ft. and a rise of 12 ft. It is constructed of cement concrete weighing 135 lb./ft.^{3}, whose compressive strength under test was 4000 lb./in.^{2} The permissible mean compressive stress is taken as $\frac{1}{10}$ of this value, and the height of the surcharge at the crown is 2 ft. The densities of the concrete and filling are 140 lb./ft.^{3} and 100 lb./ft.^{3} respectively. Roadway level.

Here the height of filling at the crown reduced to masonry units is $2 \times \frac{100}{140} = 1\cdot 43$ ft., and by formula sect. 64, when the arch is loaded on one half we get for the heavy loading

$$w = \left(100 + \frac{12000}{50}\right)\frac{3 + 1\cdot 43}{3 \times 1\cdot 43} = 350 \text{ lb./ft.}^2$$

Then by sect. 59 $\qquad t = \frac{1}{4}l \sqrt{\frac{w}{s_0}}$

where $\qquad s_o = 400 \times 144 = 57600$ lb./ft.^{2}

$$= \tfrac{1}{4} \times 100\sqrt{\frac{350}{57600}} = 1{\cdot}95 \text{ ft., or say } \underline{2 \text{ ft.}}$$

The form of the arch, etc., may then be deduced as in previous examples.

If we now put the value $e = {\cdot}01\dfrac{wl^2}{\mathrm{H}}$ into the

formula $\qquad s = s_o\left(1 + \dfrac{6e}{t_o}\right)$

we get $\qquad s = s_o\left(1 + {\cdot}06\dfrac{wl^2}{\mathrm{H}t_o}\right)$

$$= s_o + {\cdot}06\, w\left(\frac{l}{t_o}\right)^2,$$

or the maximum stress is greater than the mean stress by ${\cdot}06\, w\left(\dfrac{l}{t_o}\right)^2.$

67. **Stresses in the Arch-Ring.**—If the pressure upon the cross-section of an arch be uniformly distributed over it, then the magnitude of this stress is everywhere $s_o = \dfrac{\mathrm{N}}{\mathrm{A}}$ where N is the normal thrust and A is the area ; but if the resultant pressure does not act along the axis, then regarding the arch as elastic and capable of resisting tension, the stress intensity at any distance y from the axis is given by the formula

$$s = \frac{\mathrm{N}}{\mathrm{A}}\left(1 + \frac{ey}{k_o^2}\right) = s_o\left(1 + \frac{ey}{k_o^2}\right)$$

where e is the eccentricity of the resultant thrust and y is positive or negative according as it lies on the same side of the axis as the resultant, or on the side remote from it, and for a rectangular cross-section $k_o^2 = \frac{t^2}{12}$, so that the stresses at the edges are

$$s = s_o\left(1 + \frac{6e}{t}\right).$$

If, however, tensile stresses are not permissible, then e must not be greater than $\frac{t}{6}$ for rectangular cross-sections, where t is the thickness of the arch-ring (see chap. vi., p. 85), that is to say, the resultant thrust must lie within the middle-third of the depth. So long as this is so, the value of s will not exceed $2\frac{N}{A}$, or the maximum stress will not exceed twice the stress when the distribution is uniform.

Now in the case of masonry and plain concrete arches it is not usual to allow for any tensile stress, so that if e is greater than $\frac{1}{6}t$, the above formula can no longer be used, but that part of the section in which, according to the formula tensile stress would occur, must be regarded as ineffective. In this case the effective thickness AC of the section (Fig. 51) will be less than its full thickness AB and will in fact be

$$AC = 3AL = 3(\tfrac{1}{2}t - e).$$

Hence the maximum stress will be

$$s_{\text{max.}} = 2\frac{N}{3(\frac{1}{2}t - e)} = \frac{4N}{3(t - 2e)}.$$

The maximum stress therefore increases very quickly with e and becomes infinitely great when $e = \frac{1}{2}t$, whereas when tensile stress is permissible, the maximum stress in this case will be only four times the mean stress. It is therefore very important that the resultant thrust shall not have a large eccentricity.

The probable position of the critical sections at which the maximum stresses occur can only be found by means of the elastic theory of the arch, because, as we have shown, it is only in this way that the actual position of the resultant thrust can be fixed, although we can assert it will lie within

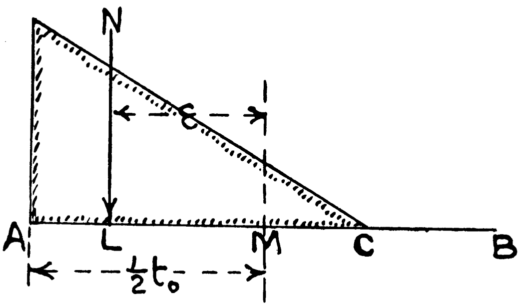


FIG. 51.

the critical limits provided that an equilibrium can be drawn for the given loading within those limits (see chap. vi.). Hence a limiting value for the maximum stress may be derived which is tolerably reliable. Tolkmitt in his treatise already referred to shows that for the half-span loaded with w tons per foot, no line of pressure can be drawn which is closer to the axis than

$$\tfrac{1}{2}\delta' = 0{\cdot}01 \frac{wl^2}{H} \text{ at the crown}$$

$$\text{and } \tfrac{1}{2}\delta'' = 0{\cdot}0125 \frac{wl^2}{H} \text{ at the springings,}$$

where δ' and δ'' are the total limits of deviation from the axis. The maximum value of $\frac{1}{2}\delta'$ lies at $\frac{1}{5}l$ from the crown, and since the thickness of the arch-ring does not vary much within this distance from the crown, the thickness at the crown may be made

$$t_o = 6 \times \cdot 01 \frac{wl^2}{H}.$$

68. Stresses in the Arch-Ring by the Elastic Theory.—When the steel reinforcement is asymmetric with respect to the axis of the arch-ring, the centroid of the cross-section is displaced towards the side on which the steel is in excess; or, where there is only single reinforcement, towards the side on which it occurs. It is only when the resultant thrust passes through this point that the stress over the section can be uniform. When this does not occur, that is to say, when the resultant of the load is eccentric, the stress will vary over the section, and its value may be found without difficulty by the ordinary formula

$$s = \frac{N}{A}\left(1 + \frac{ey}{k_o^2}\right)$$

so long as there is no tensile stress, the moment of inertia being that of the section reduced to concrete.

When, however, the resultant acts outside the critical limits, so that tension occurs, the calculation becomes more troublesome, for although the same reasoning applies as in the case of beams, it is no longer true that the neutral axis passes through the centroid of the cross-section, its position depending now on the value of the bending moment. Referring to Fig. 52, and with the usual assumptions, we obtain the following relations :—

$$s_s = m \,.\, c \, \frac{h' - n}{n} \quad . \quad . \quad . \quad (1)$$

$$s_s' = m \,.\, c \, \frac{n - a'}{n} \quad . \quad . \quad (2)$$

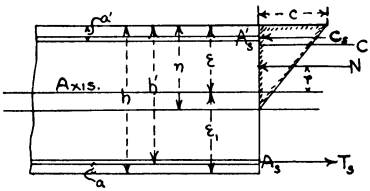


FIG. 52.

from (1) $mcn + s_s n = mch'$ or $n(mc + s_s) = mch'$.

$$\therefore n = \frac{mch'}{mc + s_s} = \frac{m}{m + \frac{s_s}{c}} \,.\, h' \quad . \quad . \quad (3)$$

$$C = \frac{cn}{2} \,.\, b \quad . \quad . \quad . \quad (4)$$

$$C_s = s_s'A_s' = mcA_s' \,.\, \frac{n - a'}{n} \quad . \quad . \quad (5)$$

$$T_s = s_s \,.\, A_s = mcA_s \frac{h' - n}{n} \quad . \quad . \quad (6)$$

In the case of beams with no axial thrust, the resultant axial force being zero, we have

$$C + C_s - T_s = 0,$$

but in the present case this becomes

$$C + C_s - T_s = P \quad . \quad . \quad (7)$$

$$\therefore \frac{cn}{2} \,.\, b + mcA_s' \frac{n - a'}{n} - mcA_s \frac{h' - n}{n} = N$$

$$c \left[\frac{nb}{2} + mA_s' \frac{n - a'}{n} - mA_s \,.\, \frac{h' - n}{n} \right] = N$$

$$c\,[n^2 . b + 2mA_s'(n - a') - 2mA_s(h' - n)] = 2Nx_o.$$

$$\therefore c = \frac{2Nx_o}{n^2b + 2mA_s'(n - a') - 2mA_s(h' - n)} \quad . \quad (8)$$

If the reinforcement is symmetrical about the axis then $A_s = A_s'$ and

$$c = \frac{2Nn}{n^2\,b + 2mA_s(2n - h)} \quad . \quad . \quad (8')$$

In order to obtain the value of n, taking moments about a point in the neutral axis, we get

$$N(n - e + r) = c . \frac{nb}{2} \cdot \frac{2}{3}n + A_s's_s'(n - a') + A_ss_s(h' - n)$$

$$= c . \frac{n^2b}{3} + mA_s' . c\frac{(n - a')^2}{n} + mA_sc\frac{(h' - n)^2}{n}.$$

$$\therefore 3N(x_o - e + r) = c[n^3b + 3mA_s'(n - a')^2 + 3mA_s(h' - n)^2],$$

$$\text{or } c = \frac{3Nn(n - e + r)}{n^3b + 3mA_s'(n - a')^2 + 3mA_s(h' - n)^2}. \quad (9)$$

and therefore equating (8) and (9) we get

$$\frac{2}{n^2b + 2mA_s'(n - a') - 2mA_s(h' - n)}$$

$$= \frac{3(x_o - e + r)}{n^3b + 3mA_s'(n - a')^2 + 3mA_s(h' - n)^2}. \quad (10)$$

Writing $v = e - r$ and transforming, this becomes

$$n^3 - 3v . n^2 - \frac{6m}{b} . n[A_s'(v - a') - A_s(h' - v)]$$

$$+ \frac{6m}{b}[A'a'(v - a') - A_sh'(h' - v)] = 0. \quad (11)$$

and if $A_s = A_s'$, this becomes

$$n_o^3 - 3v . n^2 - \frac{6mA_s}{b} . n(2v - h) + \frac{6mA_s}{b} \times$$

$$[a'(v - a') - h'(h' - v)] = 0 . \quad . \quad (11')$$

This equation determines the value of x_o, and then from (8) or (9) the value of c may be found and the stresses in the steel from (1) and (2).

Example (compare Melan's " Plain and Reinforced Concrete Arches ").—In an arch rib of 1 ft. depth, the extreme moments per foot of width at the most severely stressed section are $M_1 = 14880$ lb./ft. and $M_2 = -9720$ lb./ft. The axial thrusts corresponding to the two cases of loading amount to 9500 lb. and

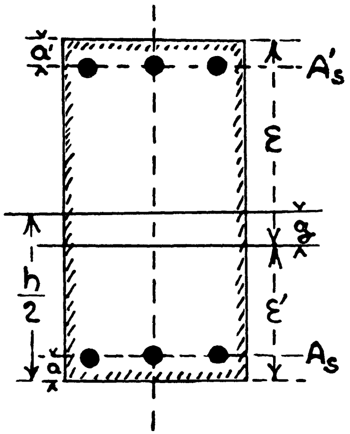


FIG. 53.

8100 lb. respectively. If we take the amount of reinforcement at $a' = 2{\cdot}11$ per cent and $a'' = 0{\cdot}623$ per cent as obtained from Prof. Melan's diagrams where $a' = \dfrac{A_s}{h}$ and $a'' = \dfrac{A_s'}{h}$, and assuming the steel is placed at $0{\cdot}1h$ from the upper and lower faces of the arch-ring (fig. 53), we have for the first case, viz. $M_1 =$ 14880 lb./ft. by equation (8)

$$c = \frac{2Pn}{n^2b + 2mA_s'(n - a') - 2mA_s(h' - n)}$$

where P = 9500 pounds.

$$b = 1 \text{ ft.}\ ;\ m = 15\ ;\ A_s' = \frac{\cdot 623}{100} \times 1\ ;\ a' = \cdot 1\ ;$$

$$A_s = \frac{2\cdot 11}{100} \times 1\ ;\ h' = \cdot 9 \text{ ft.}$$

and n is found from equation (11), viz. :—

$$n^3 - 3v \,.\, n^2 - \frac{6m}{b} \,.\, n[A_s'(v - a') - A_s(h' - v)]$$

$$+ \frac{6m}{b}[A_s'a'(v - a') - A_sh'(h' - v)] = 0$$

where $v = e - r$ and $r = \dfrac{M_1}{P_1} = \dfrac{14880}{9500} = 1\cdot 566.$

$$\text{Now } g = \frac{mA_s\left(\frac{h}{2} - a\right) - mA_s'\left(\frac{h}{2} - a'\right)}{bh + m(A + A_s')}$$

$$= \frac{15 \times \cdot 0211\,(0\cdot 5 - 0\cdot 1) - 15 \times \cdot 00623\,(0\cdot 5 - 0\cdot 1)}{1 \times 1 + 15\,(\cdot 00623 + 4\cdot 0211)}$$

$$= \frac{\cdot 1266 - \cdot 03738}{1 + \cdot 410} = \frac{\cdot 08922}{1\cdot 410} = \cdot 0633 \text{ ft.}$$

$$\text{and } e = \frac{h}{2} + g = \cdot 5 + \cdot 0633 = \cdot 5633.$$

$$\therefore v = \cdot 5633 - 1\cdot 566 = -1\cdot 003.$$

$$\therefore n^3 + 3\cdot 009n^2 - 90n\,[\cdot 00623(-1\cdot 003 - \cdot 1) - \cdot 0211\,(\cdot 9 + 1\cdot 003)] + 90\,[\cdot 00623 \times \cdot 1\,(-1\cdot 103) - \cdot 0211 \times \cdot 9 \times 1\cdot 903] = 0.$$

$$\therefore n^3 + 3\cdot 009n - 90n\,(-\cdot 00687 - \cdot 04015) + 90\,(-\cdot 000687 - \cdot 03613) = 0$$

$$n^3 + 3\cdot 009n^2 + 4\cdot 2318n - 3\cdot 3135 = 0.$$

Whence $n = \cdot 533$ ft.

for n	y
$= \cdot 5$	$-0{\cdot}27$
$\cdot 52$	$-0{\cdot}16$
$\cdot 51$	$-0{\cdot}24$
$\cdot 53$	$-0{\cdot}077$
$\cdot 54$	$+0{\cdot}0066$

$$\therefore c = \frac{2 \times 9500 \times \cdot 533}{\cdot 533^2 \times 1 + 30 \times \cdot 00623\ (\cdot 533 - \cdot 1) - 30 \times \cdot 0211\ (\cdot 9 - \cdot 533)}$$

$$= \frac{10127}{\cdot 28409 + \cdot 0809 - \cdot 2323} = \frac{10127}{\cdot 1327} = 76310 \text{ lb./ft.}^2$$

$$= 530 \text{ lb./in.}^2$$

$$c' = c \times \frac{e'}{e} = 530 \times \frac{\cdot 437}{\cdot 563} = 411 \text{ lb./in.}^2$$

and by (1) $s_s = mc\,\dfrac{h' - n}{n} = 15 \times 530\,\dfrac{\cdot 9 - \cdot 533}{\cdot 533}$

$$= 7950 \times \frac{\cdot 367}{\cdot 533} = 5450 \text{ lb./in.}^2$$

and by (2) $s_s' = mc\,\dfrac{n - a'}{n} = 15 \times 530\,\dfrac{\cdot 533 - \cdot 1}{\cdot 533}$

$$= 7950 \times \frac{\cdot 433}{\cdot 533} = 6460 \text{ lb./in.}^2$$

It will be seen that this method of calculation is very troublesome, and consequently many attempts have been made to devise methods which will give approximately the same results. None of these appear to be very satisfactory so far as agreement in the results is concerned, though possibly sufficiently so, considering the general uncertainty of the data.

Prof. Melan proceeds as follows :—

The moment of inertia of the cross-section

$$I = 5[(0{\cdot}9 - x)\,(2{\cdot}7 - x)a' + (x - 0{\cdot}1)\,(x - 0{\cdot}3)a'']\,d^3 = i\,.\,d^3$$

where

$$x = -15(a'+a'') + \sqrt{15^2(a'+a'')^2 + 30(0{\cdot}9a' + 0{\cdot}1a'')}$$

$$\text{where } x = \frac{n}{d}.$$

Then $R_c = \frac{i}{x} . d^2c = m . d^2c$ = moment of resistance of the section

$$\text{or } R_s = \frac{i}{0{\cdot}9 - x} . d^2 . \frac{s}{15} = \frac{i}{0{\cdot}9 - x} . \frac{s}{15c} d^2 . c = m' . d^2c.$$

The coefficients m and m', or the resisting moments per unit stress for $d = 1$, depends only on a' and a'', and if these are plotted, the values of m and m' will be represented by curves. The chart so formed (which is given in Melan's "Plain and Reinforced Concrete Arches," translated by D. B. Steinmann) is very convenient for designing purposes and particularly when the sections are subject to reversed moments. In the example above the values of a' and a'' there assumed were determined from the chart for the assumed value of s = 13500 lb./in.^{2} = 30 c. The stresses obtained in this way were

in the upper surface of the concrete	450	lb./in.^{2}
in the lower ,, ,, ,,	295	,,
in the upper steel	13,500	,,
in the lower ,, ...	6,460	,,

It is more usual, however, to make the reinforcement the same both on the upper and lower sides.

APPENDIX.

Calculation of the Ordinates of a Circular Arc.

$$CG = \sqrt{r^2 - x^2}; \; OF = \sqrt{r^2 - (\tfrac{1}{2}l)^2}.$$

$$\therefore \; CE = \sqrt{r^2 - x^2} - \sqrt{r^2 - (\tfrac{1}{2}l)^2},$$

and if we make $\dfrac{l}{2} = 1$

$$CE = \sqrt{r^2 - x^2} - \sqrt{r^2 - 1}$$

$$\text{and } (r - h)^2 + 1^2 = r^2.$$

$$\therefore \; r = \tfrac{1}{2}h + \frac{1}{2h}$$

$$\text{and } r^2 = \tfrac{1}{4}\left(h^2 + \frac{1}{h^2}\right) + \tfrac{1}{2}.$$

$$\therefore \; y = \sqrt{\left(h^2 + \frac{1}{h^2}\right) + \tfrac{1}{2} - x^2} - \sqrt{\tfrac{1}{4}\left(h^2 + \frac{1}{h^2}\right) - \tfrac{1}{2}}.$$

Coefficients of $\dfrac{y}{l}$ for different values of $\dfrac{rise}{span}$.

$\dfrac{x}{\frac{1}{2}l} =$	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
for $\dfrac{h}{l} =$ ·1	·099	·096	·091	·084	·075	·064	·051	·036	·019	0
·12	·119	·115	·110	·102	091	·078	·063	·045	·023	0
·14	·138	·134	·128	·118	·106	·092	·074	·053	·028	0
·16	·158	·154	·146	·136	·121	·105	·085	·061	·033	0
·18	·178	·174	·165	·154	·135	·120	·098	·070	·038	0
·20	·198	·192	·183	·169	·151	·130	·104	·074	·039	0

Example.— If $l = 150$ ft. and $h = 15$ ft., then $\frac{h}{\frac{1}{2}l} = 0 \cdot 2$.

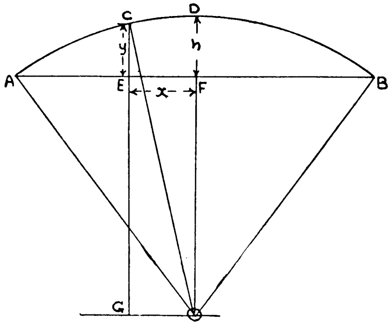


FIG. 54.

Therefore the ordinates are $198 \times 150 = 29 \cdot 7$, $\cdot 192 \times 150 = 28 \cdot 8$, and so on.

Construction for a Parabola.

Method I. By Points.—Let A·5 (Fig. 55) be the half-base and AO be the height of the parabola it is required to construct, AO being its axis. Complete the rectangle AOB5 and divide OB into any convenient number of equal parts, such as 0·1, 1·2, etc. Similarly divide B 5 into the same number of equal parts at 1′, 2′, 3′, 4′. Join 01′, 02′, etc. Then the points in which these lines cut the corresponding verticals determine points on the parabola.

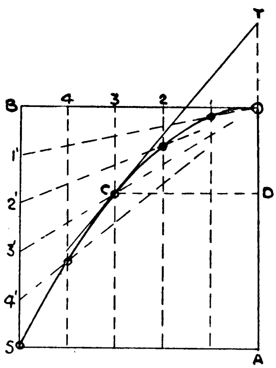


FIG. 55.

Method II.—The following method avoids the necessity of dividing the sides of the rectangle into equal parts, and is therefore quicker. Let CB (Fig.

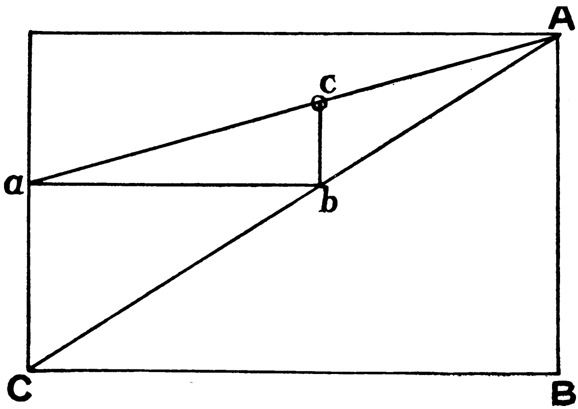


Fig. 56.

56) be the base of the semi-parabola and BA its height. Join AC. To obtain a point on the curve draw any line A*a*, and draw *ab* horizontally to cut it in *b*. Erect a vertical *bc* to cut A*a* in *c*. Thus *c* is a point on the parabola.

Method III. By Tangents.—Let AT, BT (Fig. 57) be any two tangents, and let it be required to construct a

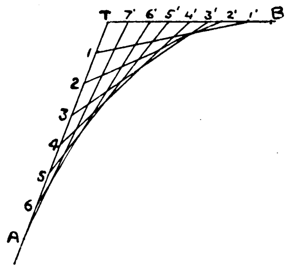


Fig. 57.

parabola to touch these tangents at A and B. Divide BT, AT into any convenient number of equal parts, as shown, and join 11′, 22′, etc. These lines will envelop the parabola, and practically define it.

To Draw a Tangent at any Point of a Parabola.—Let it be required to draw a tangent to the parabola in Fig. 55 to touch it in any given point C. Draw CD parallel to the base, and make OT = OD. Join CT. Then CT is the tangent required.

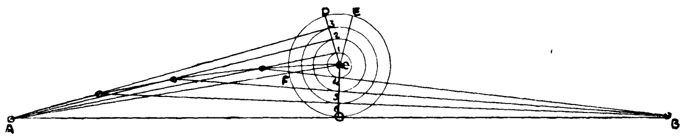


FIG. 58.

To Construct a Flat Circular Arc the Length of Chord and Rise being given.—Let AB (Fig. 58) be the chord or span and OC the rise. With O as centre and radius OC describe a circle. Join AC, cutting this circle in F, and make FD = FO. Join CD. Divide CD, CO into as many parts as the points required on the arc AC, and draw A1, A2, A3 to meet B4, B5, B6. The points of intersection are points on the arc required. Similarly for the other half.

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