Aerodynamic Theory

A General Review of Progress

Under a Grant of the Guggenheim Fund
for the Promotion of Aeronautics

William Frederick Durand
Editor-in-Chief

Volume II

Division E
General Aerodynamic Theory—Perfect Fluids
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With 113 Figures
and 4 Plates

Berlin · Julius Springer · 1935

ISBN-13: 978-3-642-89628-6 e-ISBN-13: 978-3-642-91485-0
DOI: 10.1007/978-3-642-91485-0

GENERAL PREFACE

During the active life of the Guggenheim Fund for the Promotion of Aeronautics, provision was made for the preparation of a series of monographs on the general subject of Aerodynamic Theory. It was recognized that in its highly specialized form, as developed during the past twenty-five years, there was nowhere to be found a fairly comprehensive exposition of this theory, both general and in its more important applications to the problems of aeronautic design. The preparation and publication of a series of monographs on the various phases of this subject seemed, therefore, a timely undertaking, representing, as it is intended to do, a general review of progress during the past quarter century, and thus covering substantially the period since flight in heavier than air machines became an assured fact.

Such a present taking of stock should also be of value and of interest as furnishing a point of departure from which progress during coming decades may be measured.

But the chief purpose held in view in this project has been to provide for the student and for the aeronautic designer a reasonably adequate presentation of background theory. No attempt has been made to cover the domains of design itself or of construction. Important as these are, they lie quite aside from the purpose of the present work.

In order the better to suit the work to this main purpose, the first volume is largely taken up with material dealing with special mathematical topics and with fluid mechanics. The purpose of this material is to furnish, close at hand, brief treatments of special mathematical topics which, as a rule, are not usually included in the curricula of engineering and technical courses and thus to furnish to the reader, at least some elementary notions of various mathematical methods and resources, of which much use is made in the development of aerodynamic theory. The same material should also be acceptable to many who from long disuse may have lost facility in such methods and who may thus, close at hand, find the means of refreshing the memory regarding these various matters.

The treatment of the subject of Fluid Mechanics has been developed in relatively extended form since the texts usually available to the technical student are lacking in the developments more especially of interest to the student of aerodynamic theory. The more elementary treatment by the General Editor is intended to be read easily by the average technical graduate with some help from the topics comprised in Division A. The more advanced treatment by Dr. Munk will call for some familiarity with space vector analysis and with more advanced mathematical methods, but will commend itself to more advanced students by the elegance of such methods and by the generality and importance of the results reached through this generalized three-dimensional treatment.

In order to place in its proper setting this entire development during the past quarter century, a historical sketch has been prepared by Professor Giacomelli whose careful and extended researches have resulted in a historical document which will especially interest and commend itself to the study of all those who are interested in the story of the gradual evolution of the ideas which have finally culminated in the developments which furnish the main material for the present work.

The remaining volumes of the work are intended to include the general subjects of: The aerodynamics of perfect fluids; The modifications due to viscosity and compressibility; Experiment and research, equipment and methods; Applied airfoil theory with analysis and discussion of the most important experimental results; The non-lifting system of the airplane; The air propeller; Influence of the propeller on the remainder of the structure; The dynamics of the airplane; Performance, prediction and analysis; General view of airplane as comprising four interacting and related systems; Airships, aerodynamics and performance; Hydrodynamics of boats and floats; and the Aerodynamics of cooling.

Individual reference will be made to these various divisions of the work, each in its place, and they need not, therefore, be referred to in detail at this point.

Certain general features of the work editorially may be noted as follows:

1. Symbols. No attempt has been made to maintain, in the treatment of the various Divisions and topics, an absolutely uniform system of notation. This was found to be quite impracticable.

Notation, to a large extent, is peculiar to the special subject under treatment and must be adjusted thereto. Furthermore, beyond a few symbols, there is no generally accepted system of notation even in any one country. For the few important items covered by the recommendations of the National Advisory Committee for Aeronautics, symbols have been employed accordingly. Otherwise, each author has developed his system of symbols in accordance with his peculiar needs.

At the head of each Division, however, will be found a table giving the most frequently employed symbols with their meaning. Symbols in general are explained or defined when first introduced.

2. General Plan of Construction. The work as a whole is made up of *Divisions*, each one dealing with a special topic or phase of the general subject. These are designated by letters of the alphabet in accordance with the table on a following page.

The Divisions are then divided into chapters and the chapters into sections and occasionally subsections. The Chapters are designated by Roman numerals and the Sections by numbers in bold face.

The Chapter is made the unit for the numbering of sections and the section for the numbering of equations. The latter are given a double number in parenthesis, thus (13.6) of which the number at the left of the point designates the section and that on the right the serial number of the equation in that section.

Each page carries at the top, the chapter and section numbers.

W. F. Durand

Stanford University, California
January, 1934.

GENERAL LIST OF DIVISIONS WITH AUTHORS

Volume I.

A. Mathematical Aids

W. F. Durand — Professor (Emeritus) of Mechanical Engineering, Stanford University, Calif., Member of the National Advisory Committee for Aeronautics.

B. Fluid Mechanics, Part I

W. F. Durand

C. Fluid Mechanics, Part II

Max M. Munk — Lecturer in Aerodynamics at the Catholic University of America, Washington, D. C., and Technical Editor of the "Aero Digest".

D. Historical Sketch

R. Giacomelli — Lecturer in History of Mechanics at the University of Rome, Italy, and Editor of "L'Aerotecnica".

with the collaboration of

E. Pistolesi — Professor of Mechanics at the Royal School of Engineering at Pisa, Italy, and Editor-in-Chief of "L'Aerotecnica".

Volume II.

E. General Aerodynamic Theory—Perfect Fluids

Th. von Kármán — Director of the Guggenheim Aeronautics Laboratory, California Institute of Technology, Pasadena, Calif., and formerly Director of the Aerodynamic Institute, Aachen, Germany.

J. M. Burgers — Professor of Aero- and Hydrodynamics at the Technische Hoogeschool at Delft, Holland.

Volume III.

F. The Theory of Single Burbling

C. Witoszyński — Professor of Aerodynamics at the Warsaw Polytechnical School and Director of the Warsaw Aerodynamic Institute, Poland.

M. J. Thompson — Assistant Professor of Aeronautical Engineering at the University of Michigan, Ann Arbor, Mich.

G. The Mechanics of Viscous Fluids

L. Prandtl — Professor in Applied Mechanics at the University of Göttingen, Germany, and Director of the Kaiser Wilhelm Institute for Fluid Research.

H. The Mechanics of Compressible Fluids

G. I. Taylor — Yarrow Research Professor of the Royal Society, Fellow of Trinity College, Cambridge, England.

J. W. Maccoll — Research Officer, Department of External Ballistics, Ordnance Committee, Woolwich, England.

I. Experimental Methods—Wind Tunnels

A. Toussaint — Director of the Aerodynamic Laboratory, Saint-Cyr-l'École, France.

E. Jacobs — Associate Aeronautical Engineer, in charge of the National Advisory Committee for Aeronautics' variable-density wind tunnel, Langley Field, Virginia.

Volume IV.

J. Applied Airfoil Theory
A. Betz — Professor at the University and Director of the Aerodynamic Research Institute at Göttingen, Germany.

K. Airplane Body (Non-Lifting System) Drag and Influence on Lifting System
C. Wieselsberger — Professor of Aerodynamics and Director of the Aerodynamic Institute, Technische Hochschule, Aachen, Germany.

L. Airplane Propellers
H. Glauert † — Past Fellow of Trinity College, Cambridge, England; Principal Scientific Officer at the Royal Aircraft Establishment, Farnborough.

M. Influence of the Propeller on other Parts of the Airplane Structure
C. Koning — Rijks-Studiedienst voor de Luchtvaart, Amsterdam, Holland.

Volume V.

N. Dynamics of the Airplane
B. Melvill Jones — Professor of Aeronautical Engineering in the University of Cambridge, England, Member of the Aeronautical Research Committee of Great Britain.

O. Airplane Performance
L. V. Kerber — Former Chief Aerodynamics Branch Materiel Division, U. S. Army Air Corps, and former Chief, Engineering Section Aeronautics Branch, Department of Commerce.

Volume VI.

P. Airplane as a Whole—General View of Mutual Interactions Among Constituent Systems
M. Panetti — Professor of Applied Mechanics and Director of the School of Aeronautics in the R. Politecnico di Torino, Italy.

Q. Aerodynamic Theory of Airships
Max M. Munk — Lecturer in Aerodynamics at the Catholic University of America, Washington, D. C., and Technical Editor of the "Aero Digest".

R. Performance of Airships
K. Arnstein — Chief Engineer of the Goodyear Zeppelin Company, Akron, Ohio.
W. Klemperer — Research Engineer of the Goodyear Zeppelin Company, Akron, Ohio.

S. Hydrodynamics of Boats and Floats
E. G. Barrillon — Director of the Naval Experimental Tank, Paris, France.

T. Aerodynamics of Cooling
H. L. Dryden — Physicist in the United States Bureau of Standards, Chief of the Aerodynamics Section, Washington, D. C.

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DIVISION E

GENERAL AERODYNAMIC THEORY—PERFECT FLUIDS

By **Th. von Kármán,**

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NOTATION

The following table comprises a list of the principal notations employed in the present Division, with their more usual meanings. Notations not listed are either so well understood as to render mention unnecessary, or are only rarely employed and are explained as introduced. Where occasionally a symbol is employed with more than one meaning, the local context will make the significance clear.

X, Y, Z	Space axes
X, Y	Two-dimensional axes, sometimes forces along X and Y
x, y, z	Space coordinates
ξ, η, ζ	Supplementary space coordinates
z	Used for the complex $(x + iy)$
ζ	Used for the complex $(\xi + i\eta)$
x, ω, θ	Cylindrical coordinates, IV **42**
b	Half span
c	Chord
S	Area of airfoil, or area in general
α	Geometrical angle of incidence
i	Effective angle of incidence
θ	Angle of inclination, usually to axis of X
β	Angle between X and 1st axis, II **13, 14**
τ, γ	Special angles in conformal transformation, II **13, 14**
$\varkappa$	Exponent used in conformal transformation, II **19**
v_x, v_y, v_z	Component velocities along axes of x, y, z, III **1**
w_x, w_y, w_z	Component "added" velocities along axes of x, y, z, III **6**
V	Velocity in general; often initial or undisturbed velocity
w	Added velocity, usually downward
w	Used for complex velocity $u - iv$
Γ	Circulation; strength of vortex
γ	Vorticity
$\bar{\gamma}$	Strength of vortex sheet
$\gamma_x, \gamma_y, \gamma_z$	Components of vorticity, III **1**
φ, Φ	Usually potential
ψ, Ψ	Usually stream function
χ, f, F	Usually complex function $\varphi + i\psi$
f_x, f_y, f_z	Component real forces, III **6**
g_x, g_y, g_z	Component induced forces, III **6**
k_x, k_y, k_z	Component generalized forces, III **6**
K	Resultant of generalized forces
Q_x, Q_y, Q_z / $\bar{Q}_x, \bar{Q}_y, \bar{Q}_z$	Integrals of generalized forces with respect to x or ξ, III **9**
D	Drag
D_i	Induced drag
L	Lift
l	Lift per unit span
Λ	Generalized force per unit span
M	Moment

p	Pressure
ϖ	Impulsive pressure, III 3
R	Resultant air force
R	Complex force resultant $(X - iY)$, II (4.8)
E, T	Kinetic energy
W	Work
C_L	Lift coefficient
C_D	Drag coefficient
C_{Di}	Coefficient of induced drag
C_{D0}	Profile drag coefficient
C_{M0}	Moment coefficient for zero lift
λ	Aspect ratio
σ	Special coefficient, IV (22.6), (22.10), (25.10)
δ, τ	Special coefficients, IV (4.11), (6.6), (6.10)
E, K, F	Elliptic integrals
Res.	Residue (Cauchy)
Re.	Real part
Im.	Imaginary part
i	$\sqrt{-1}$ or sometimes the i^{th} term of a series
ϱ	Density
t	Time

EDITOR'S PREFACE (DIVISION E)

The present Division of this work is intended to present the general mathematical foundation of the mechanics of perfect fluids—incompressible and non-viscous—with such specialized developments as are of particular importance for application to the problems of aerodynamics.

Chapter I is in considerable degree descriptive of the phenomena involved in the flow of a fluid about a solid body and in particular in the flow of air about an airplane wing. This descriptive material is supplemented by certain general statements regarding the laws connecting these phenomena together, the whole intended to serve as a form of introduction to the more formal and mathematical treatments of the following chapters.

This more formal treatment begins with Chapter II, in which, however, the paths of relative fluid motion are assumed to be limited to planes, thus constituting so-called two-dimensional flow, and permitting of the application of certain specialized agencies—conformal transformation in particular—for their investigation.

Chapters III and IV then follow, the first dealing in more fundamental mathematical fashion with the laws of three-dimensional fluid flow and the second with several special and important applications to the problems of aerodynamics as furnishing the ground work for all rational aeronautic design.

Chapter V follows with a development of the more important aspects of the mechanics of non-uniform and of curvilinear motion. This is followed in Chapters VI and VII by a discussion of certain phenomena occurring in the wake formed downstream from a body around which a fluid is flowing. The two chapters deal with somewhat different aspects of this general problem and together serve to indicate the more important lines of development relating to this phase of the subject.

Chapters V, VI and VII are intended to present the important aspects of the more recent advances in aerodynamic theory and to give also indications of the lines along which further work on these subjects should be directed.

The notation required to present adequately the somewhat extended mathematical developments of this Division is, of necessity, of considerable complexity and multiple duty has been required of several symbols. As in other Divisions of the work, the more important symbols are listed together with their meanings and where needful, a reference to where introduced or defined. In cases of multiple use of the same symbol, the local context will usually serve to indicate which particular meaning is intended. The Editor trusts that the occasional references to the more elementary material presented in Division A and B together with the notes and appendices will serve to make somewhat easier the reading of this Division by those who may be less familiar with the more formal and mathematical methods here of necessity employed.

In the preparation of this volume, the general plan of the work having been agreed upon through discussion between both authors, Dr. von Kármán has primarily contributed Chapters II and VII, and Dr. Burgers the main part of Chapters I, III, IV, V and VI. Each author, however, has had the advantage of the reading of the other's manuscript and of suggestions and criticism based thereon, so that the volume represents their joint work.

W. F. Durand

Stanford University, California
July, 1934

DIVISION E

GENERAL AERODYNAMIC THEORY PERFECT FLUIDS

By

Th. von Kármán and **J. M. Burgers**
Pasadena, California Delft, Holland

CHAPTER I

BASIC IDEAS OF WING THEORY: FLOW AROUND AN AIRFOIL

1. Introductory Remarks. The basic principle underlying the construction of all flying machines is the property of an inclined plane, when it is moved horizontally through the air, of producing a supporting or lifting force. This property has long been known and the most remarkable feature connected with it is that the lifting force may be very great compared with the resistance which the plane experiences in the direction opposite to its motion. The ratio of the supporting force, or *lift*, as it is usually called, to the resistance, or *drag*, may be especially high when the angle of inclination of the plane to its path (the angle of attack) is small, and when the plane is of special form, resembling the wings of a bird.

Modern researches have shown that the resistance itself may be decomposed into two constituent parts, one of which is due to the friction of the air along the sides of the plane, while the other finds its origin in the circumstance that the appearance of the lift is accompanied by the creation of a definite flow pattern in the neighborhood and in the wake of the plane, which demands a continuous supply of energy. While the former part of the resistance, the frictional resistance (or the so-called *profile drag*), can be reduced by making the surface of the plane very smooth and by fairing its profile (or transverse section), the other part of the resistance depends directly on the magnitude of the lift and on the plan form of the plane. Any study of the former part requires consideration of the viscosity of the fluid medium, a subject treated in Division G of this work. Investigation of the latter part, on the other hand, may be realized through the application of the fundamental principles of fluid mechanics applied to the ideal fluid as defined in Division B, and since the mathematical difficulties of dealing with viscous fluids tend to obscure somewhat the character of treatment desirable at the start, it will be convenient in the present division to limit the treatment broadly to the case of viscosity assumed small or vanishing, and thus to include consideration of the induced part of the drag only.

It is the object of the present division to give a general account of the mathematical theory of the motion of the air about airplane wings. Such a theory must furnish formulae by means of which the lifting force can be calculated, or at least estimated, and it must furnish data for determining the amount of energy to be expended in producing the flow system. Before entering upon this subject it seems desirable, however, to give first a general description of the motion of the air around an airfoil, and of the nature of the reaction experienced by the airfoil in consequence of this flow. This will afford an opportunity, moreover, of explaining the process by which those features of the flow system that are responsible for the appearance of the lift, are originated.

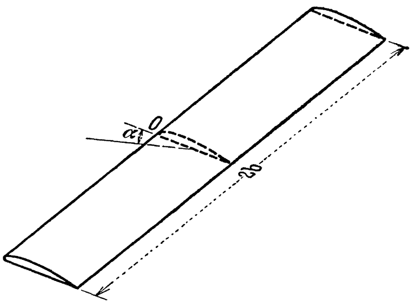

Fig. 1.

2. Principle Data Characterizing an Airfoil. As is well known the ordinary type of airfoil can be described as having the form of a flat or slightly cambered body, usually symmetrical with respect to a median plane (see Fig. 1). The dimension perpendicular to the median plane is called the *span* (to be denoted by $2b$); in all practical cases it is much greater than the dimension parallel to that plane. The section made by the median plane is called the (principal) *profile* of the airfoil (Fig. 2). Other profiles are obtained by making sections parallel to the median plane at other points of the span. These profiles may be all identical in shape and position, as is the case for instance with most of the model airfoils used in experimental investigations, or they may vary from the median plane to the ends of the airfoil, as is the case commonly met with in practice.

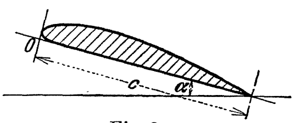

Fig. 2.

In order to determine the form of a profile two lines are introduced, one of which is called the *chord*, usually defined as the tangent touching the lower boundary of the profile at two points; or when such a tangent cannot be conveniently drawn, another line may be assigned more or less arbitrarily as the chord. The second reference line is drawn perpendicular to the chord, in such a way that it just touches the "nose" of the profile. The positions of points on both sides of the profile can then be given with respect to these lines as coordinate axes.

The principal characteristics of the shape of a profile are the *length of the chord* (c), the *camber* and the *thickness*; further characteristics are the position of maximum camber, of maximum thickness, etc. As to the position of the various profiles, the chords may be all parallel, or there may be a certain *twist* in the airfoil.

The position of the airfoil with respect to the direction of motion (or, as in the case of windtunnel experiments, with respect to the direction of the airflow) is specified through two angles: (1) the angle between this direction and the median plane, called the *angle of yaw* and (2) the angle which the projection of this direction upon the median plane makes with the chord. We shall be concerned for the present only with the case in which the first angle is zero, so that the direction of the airflow is contained in the median plane. The second angle is called the *geometrical angle of incidence*; it will be denoted by the letter α.

A characteristic of great importance in determining the amount of energy to be expended in producing the flow around an airfoil is the so-called *aspect ratio*—the ratio of the span to the chord. In the case of an airfoil of varying profile the definition adopted is that of the ratio of span to the mean chord. If we introduce the area S of the airfoil, defined as the area of its projection upon a plane surface parallel to the span and to the chord of the median section, the mean chord will be found by dividing S by the span $2\,b$. Hence the aspect ratio (denoted by the letter λ) is given by the formula:

$$\lambda = \frac{(2\,b)^2}{S} \qquad (2.1)$$

3. Reaction of the Air upon an Airfoil. In the ordinary case of a symmetrical airfoil held in such a position that the direction of the original motion of the air is contained in the plane of symmetry, the resultant reaction R likewise is contained in this plane. It is customary to resolve this reaction into two components, the *lift* L, perpendicular to the direction of the original flow, and the *drag* D, parallel to this direction (see Fig. 3). Corresponding to these components, two non-dimensional coefficients are introduced, the lift coefficient C_L and the drag coefficient C_D, which are defined by means of the equations:

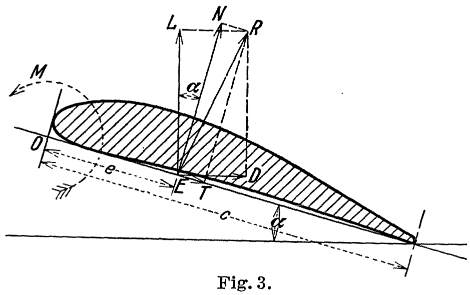

Fig. 3.

$$L = \frac{1}{2}\varrho\, V^2\, S\, C_L \qquad (3.1)$$

$$D = \frac{1}{2}\varrho\, V^2\, S\, C_D \qquad (3.2)$$

Here V is the original velocity of the air relative to the airfoil, ϱ is the density of the air, S (as before) is the area of the airfoil. The introduction of the factor 1/2, which is done to make apparent the relation of the forces to the impact pressure $(1/2)\,\varrho\,V^2$, is not adopted by all laboratories. When it is left out, a different set of coefficients is obtained, having values just one half of those defined by (3.1) and (3.2).

In most cases C_L and C_D can be treated as approximately independent of the velocity and (for geometrically similar airfoils) of the dimensions of the airfoil, though actually they depend to a certain degree upon the Reynolds' number connected with the flow around it. This matter, however, falls outside the scope of the present Division and the reader is referred to Divisions I and J for its treatment.

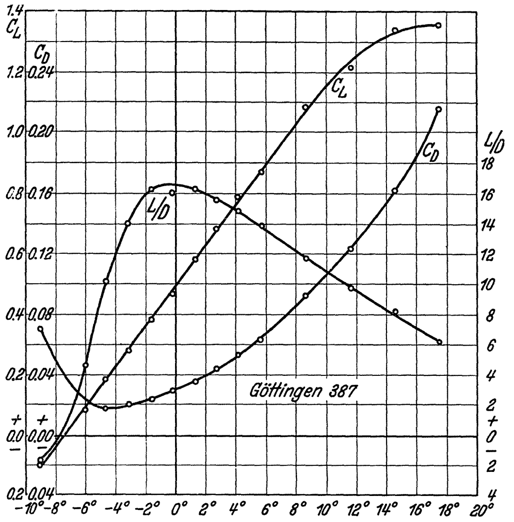

Fig. 4.

Leaving aside this question, the principal point to be considered is that the coefficients C_L and C_D are functions of the angle of incidence α. The coefficient C_L increases with increasing values of α up to a certain maximum value, corresponding to the so-called *critical angle of incidence*; when this angle is surpassed, C_L decreases again, in some cases very rapidly. When the angle of incidence is greater than the critical value, it is said that the airfoil is in a *stalled* position.

For values of α well below the critical value, C_L is approximately a linear function of α, becoming zero in most cases for a certain negative value, called the *angle of zero lift.* Upon further decrease of α below the latter value, C_L assumes negative values.

The drag coefficient C_D is always positive and as has been stated already in the introductory remarks, it is usually much smaller than C_L. It has some minimum value for an angle of incidence not much different from that of zero lift.

The coefficients C_L and C_D can be represented graphically as functions of α, as indicated in Fig. 4. Often the value of C_L/C_D, which is equal to L/D and is called the *lift/drag-ratio*, is represented together with them. Another mode of representation is obtained by taking C_D and C_L as coordinates with the values of α inscribed along the curve. In this way the so-called *polar curve* is found (see Fig. 5, curve marked C_D), a curve of great importance in both practical and theoretical applications. For convenience many laboratories choose a greater scale for C_D than for C_L in making the polar diagram.

Instead of resolving the total reaction R into lift and drag, another method, useful for a number of purposes may be employed; *viz.* into *normal force* N, perpendicular to the chord of the median section, and *tangential force* T, parallel to this chord (see Fig. 3). The corresponding coefficients are denoted by C_N and C_T. These coefficients can be calculated from C_L and C_D by means of the equations:

$$\left.\begin{aligned} C_N &= C_L \cos\alpha + C_D \sin\alpha \\ C_T &= -C_L \sin\alpha + C_D \cos\alpha \end{aligned}\right\} \qquad (3.3)$$

When the airfoil is not symmetric, or when the direction of the airflow makes an angle with the plane of symmetry, the total reaction has a further component (usually of small magnitude) in the direction of the span, called the *lateral force*. Corresponding to this component a coefficient of lateral force could be introduced, depending in general upon both the angle of incidence and upon the angle of yaw.

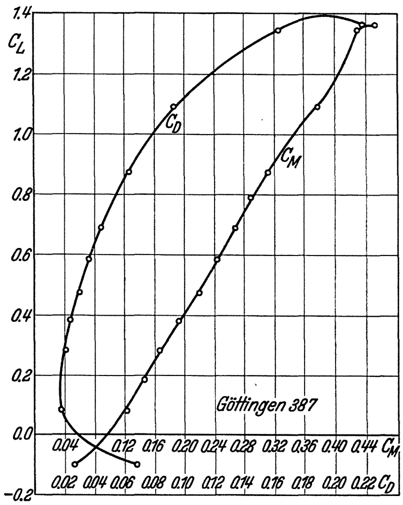

Fig. 5.

4. Moment of the Reaction of the Air upon an Airfoil. For the investigation of the stability of airplanes and for various other problems it is necessary to know also the *moment* of the forces experienced by the airfoil. Returning to the symmetrical case, this moment will be contained in the median plane, and is defined with respect to the point of intersection O[1] of the reference lines introduced in this plane (see Fig. 3). Its magnitude is denoted by M; it is measured in the sense as indicated in Fig. 3. Corresponding to it a moment coefficient is introduced by means of the equation:

$$M = \frac{1}{2}\varrho\, V^2 S\, c\, C_M \qquad (4.1)$$

where c is the mean chord of the airfoil.

The moment coefficient again is a function of the angle α. In the polar diagram it can be represented as a function of the lift coefficient C_L

[1] Other points of reference may be and sometimes are used, such as the point lying at a distance of one-quarter of the chord from the point O, or again a point near or at the center of the airfoil profile. In the present discussion, unless stated otherwise, the moment will always be taken with reference to the point O.

(see Fig. 5, curve marked C_M). It has been found that the relation between these two coefficients is approximately expressed by the equation:

$$C_M = C_{M\,0} + n\,C_L \tag{4.2}$$

where $C_{M\,0}$ is the moment coefficient for zero lift. The value of the factor n can be deduced from the theory of the flow around an airfoil, and thus appears to be equal to 0.25 [see II (8.21), (8.22)[1]]; the values experimentally obtained come very near to the theoretical result. The coefficient $C_{M\,0}$ has usually a positive value; it vanishes for a flat plate and for an airfoil which has a profile symmetrical with respect to the line chosen as chord.

Instead of the moment coefficient, the position of the point of intersection E of the resultant reaction R with the chord of the median section may be given. This point is called the *center of pressure* of the airfoil (or of the profile, as the case may be). The relation between the distance e of this point from the point O and the coefficient C_M can be obtained as follows (see again Fig. 3): We resolve the reaction R at the point E into its normal and tangential components; as the latter component does not contribute to the moment about O, we have

$$M = N\,e,$$

from which we deduce:

$$C_M = C_N \frac{e}{c} \tag{4.3}$$

In most cases the difference between normal force and lift is rather small; hence for an approximate calculation this equation may be replaced by:

$$C_M \cong C_L \frac{e}{c} \tag{4.4}$$

5. The Circulatory Flow around an Airfoil. The force experienced by the airfoil is due mainly to the difference in the pressures acting over the lower and the upper surfaces. Now in the case of steady motion of air along a body at rest, the pressure can be calculated from the velocity with the aid of Bernoulli's equation. Denoting the pressure by p, the absolute value of the velocity at any point by v, and using p_o and V to indicate respectively the pressure and velocity at infinity, we have:

$$p + \frac{1}{2}\varrho\,v^2 = p_0 + \frac{1}{2}\varrho\,V^2 \tag{5.1}$$

[1] For reasons connected with the deduction of the mathematical formulae the notation used in II 8 is slightly different from that used here, in as much as in II 8 the moment M and the coefficient C_M are taken with respect to the center of the chord, whereas the coefficient for the moment with reference to the leading edge is denoted by C_m. Moreover the quantities with the subscript 0 in II (8.19) refer to the case in which the angle of incidence is zero. The quantity corresponding to $C_{M\,0}$ above is denoted by $C_\mu = C_{m\,0} - C_{L\,0}/4$ in II (8.21) and (8.22). — See also II (14.14).

It must be remarked that the discussion of Chapter II is restricted to the case of two-dimensional motion around an airfoil of infinite span. As is shown in IV 4, formula (4.2) remains valid for a rectangular airfoil of finite span. In the case of an airfoil having a considerable angle of sweep-back complications may arise, as has been pointed out by Glauert (see IV 10).

Hence the higher pressure along the underside of the airfoil must be accompanied by relatively lower velocities, whereas the lower pressure on the upperside must be accompanied by relatively higher velocities. It thus becomes apparent that there is a certain asymmetry in the distribution of the motion along both sides.

In order to obtain a measure of this asymmetry we make use of the conception of *circulation*, which has been introduced in Division A VI 4. Assume a closed curve, encircling a section of the airfoil, as indicated in Fig. 6. Then the integral: $\Gamma = \int v_s\, d\, s$ (5.2)
where $d\, s$ is an element of the curve, and v_s is the component of the velocity in the direction of $d\, s$, taken in the counterclockwise sense along this curve, has a negative value. Usually its absolute value is given and this is called the circulation around the section considered. The actual direction of the circulation in the case of Fig. 6 is clockwise (as indicated by the arrows marked on s).

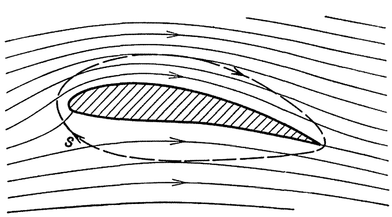

Fig. 6.

As has been demonstrated in Divisions A VI 4 and B III 1, the value of the circulation is independent of the shape of the curve provided (1) that one shape can be derived from another by a continuous deformation, in which the curve is not cut by any body present in the field, and (2) that the motion of the air is irrotational. The latter condition is not satisfied in any case of actual motion. In the case of the airfoil we have vortex motion (*a*) in the boundary layers along the surfaces of the airfoil (both upper and under sides) (*b*) in the wake behind it. The vortices in the wake are, on the one hand, vortices with their axes approximately parallel to the span; they are the vortices which have originated in the boundary layer, and have been "washed out" of it by the flow of the air. The other part of the vortices present in the wake are the so-called *trailing* vortices, with axes approximately parallel to the direction of the motion of the air. As will be seen presently these latter vortices are of great importance in the theory of the airfoil, and it will be necessary to give much attention to them. In the ordinary case of uniform motion of an airfoil the vortices with axes parallel to the span are of relatively small importance; their theory falls outside the scope of the present division, and belongs to the subject of viscous flow[1].

Returning now to the definition of the circulation around a particular section of an airfoil, the following statement must be made. We consider a series of closed curves, all lying in the plane of the section, so

[1] It must be remarked, however, that in the case of non-uniform motion attention must be given to vortices with axes parallel to the span arising in connection with changes in the magnitude of the circulation, as will be seen in Chapter V.

that in changing from one form to another we do not cut any of the trailing vortices. The curves must include the boundary layer, so that they lie in the region of irrotational flow, with the exception of that part of the curve which cuts the wake. When now this latter part is always given such a direction that it cuts the wake at right angles, then the value of the circulation is the same for all curves under consideration[1].

6. The Kutta-Joukowski Theorem. It is to be expected that a mathematical relation will exist between the reaction experienced by the airfoil and the circulation around it. In the case of plane motion (two-dimensional motion), such as could be obtained in the case of an airfoil of infinite span having constant profile and constant angle of incidence for all sections, where consequently also Γ has the same value for all sections, such a relation is given by the so-called Kutta-Joukowski theorem, which states that:

$$l = \varrho V \Gamma \tag{6.1}$$

l being the component of the force experienced in the direction normal to the velocity V, and referred to unit of span. In the above equation the absolute value is taken for Γ and the sign of the force is determined by the rule that the direction of the vector l is deduced from the direction of the vector V by a rotation through 90^0 in the sense opposite to the circulation[2].

This theorem is of great importance in the theory of airfoils. It can be demonstrated in various ways, and it will be necessary to return to it on many occasions. One demonstration can be developed through a direct calculation of the resultant fluid pressure acting on the various points of the section. For the case of an infinite cylinder with circular cross section this demonstration has been given in Division B V 5; a general deduction for sections of any form can be made by the use of certain theorems relating to momentum, as has been shown in Division B V 7 and as further developed in Chapter II.

Usually the theorem is extended without further consideration to the case of three-dimensional motion around an airfoil of finite span,

[1] If the condition that the line must cut the wake at right angles were not stipulated, the magnitude of the circulation would become indefinite, as an arbitrary number of vortices, "washed out" from the boundary layers along the surfaces of the airfoil, and rotating either in one sense or in the other, could be included. The necessity of this condition was pointed out by G. I. TAYLOR, Phil. Trans. London A 225, p. 238, 1925.

In the considerations of the present division the condition is of no further consequence. It is of importance, however, in the theory of viscous flow. As pointed out by BETZ, it has an interesting application in the theory of periodic wing systems (see BETZ, A., Handb. d. Phys., Bd. VII, p. 234).

[2] The theorem was given by W. M. KUTTA in 1902 and independently by N. E. JOUKOWSKI in 1905; it was also known to C. A. BJERKNES in 1897. The general connection between circulation and lift seems to have been observed for the first time by F. W. LANCHESTER in 1894.

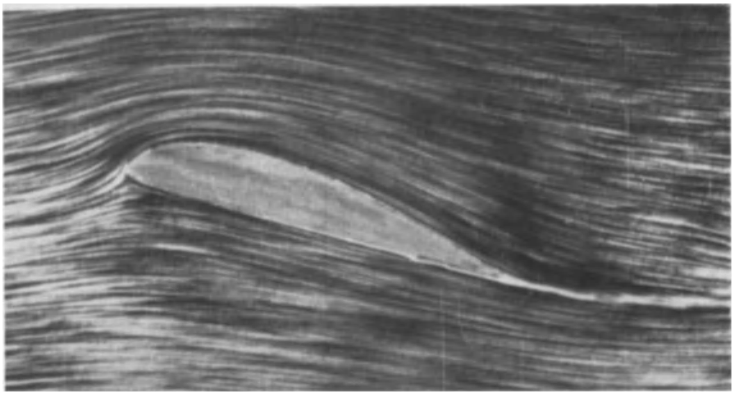

Fig. 7.

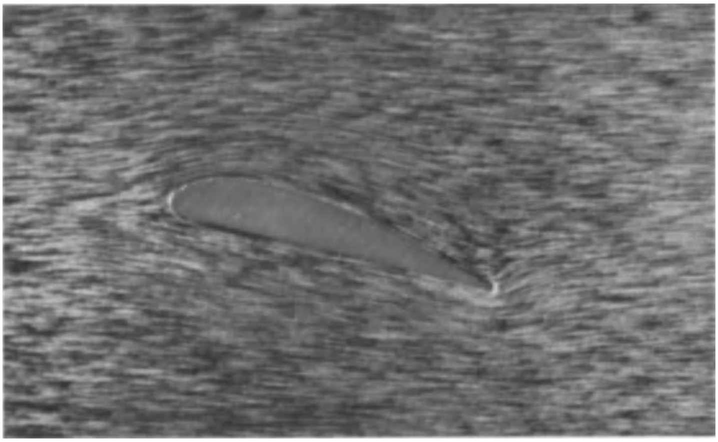

Fig. 11.

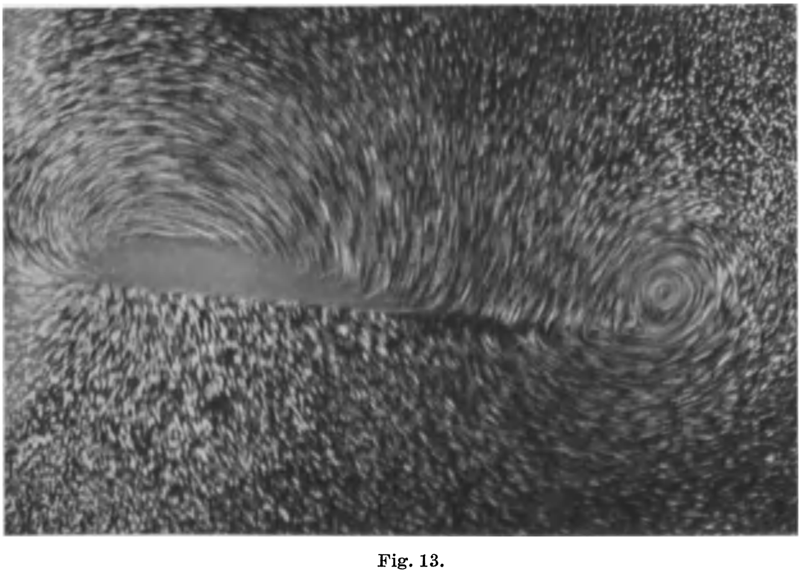

Fig. 13.

Photographic Flow Pictures (from PRANDTL-TIETJENS, Hydro- und Aeromechanik II).

where the circulation Γ varies from one section to another; then l is the lift per unit span at a particular section. If the coordinate y is measured in the direction of the span, the origin being taken in the median plane, the total lift L is given by the integral:

$$L = \int_{-b}^{+b} l\,d\,y = \varrho\,V \int_{-b}^{+b} \Gamma\,d\,y \tag{6.2}$$

It will be necessary to further investigate this point (see III **31**) as the extension is not self-evident, and as difficulties may arise in some cases, for instance with an airfoil of low aspect ratio.

A photographic picture of the flow around an airfoil is given in Fig. 7[1]. It is obtained with a model moved through water, the camera being attached to the carriage to which also the model was rigidly connected, so that the lines of flow relative to the airfoil are made visible. The presence of the circulation can be deduced from the fact that the streamlines on the upper side of the airfoil are slightly drawn together, whereas those on the under side are at a somewhat greater distance from each other than they are upstream of the airfoil. A much more convincing picture of the circulation is obtained if the camera is kept at rest with respect to the fluid at infinity, in which case the lines of flow relative to the fluid become visible. A photograph taken in this way is reproduced in Fig. 13 (Plate I); see also 8.

It is of interest to remark that (6.1) has been tested experimentally by a direct measurement of the velocity distribution around an airfoil of constant profile, stretching entirely across a windchannel of rectangular section, so that the flow could be assumed to be two-dimensional. The circulation calculated from the observed velocities for various curves satisfying the conditions mentioned, appeared to be constant with a reasonable degree of accuracy, and corresponded with the value deduced from the measured lift per unit span[2].

7. Vortex System Connected with the Circulatory Motion around the Airfoil. It is evident that the motion in the neighborhood of the airfoil, especially when the latter possesses a high aspect ratio, must show a certain resemblance to the motion around a vortex of strength equal to Γ, with its axis parallel to the direction of the span, and superposed upon a field of rectilinear motion of velocity V. In the case of two-dimensional motion the field around the airfoil and that around the vortex become more and more nearly the same with increase of the

[1] Figs. 7, 11 and 13 are given on Plate I and are taken from PRANDTL-TIETJENS, Hydro- und Aeromechanik II (Berlin 1931).

[2] See BRYANT, L. W. and WILLIAMS, D. H.: Phil. Trans. London A 225, p. 199, 1925 (Techn. Rep. Aeron. Res. Committee, Teddington, R. and M. No. 989, Febr. 1924).

distance toward infinity. The similarity can be made more accurate by dividing the circulation Γ over a definite system of vortices, the arrangement of which depends upon the details of the form of the airfoil. Though in the three-dimensional case other factors must be taken into account, the resemblance is still present. We return to this point in a later section; at the moment it is sufficient to mention that on account of this correspondence the term *bound vortex* is often used for the airfoil with the circulatory motion around it.

Now a well known theorem concerning vortex motion states that a vortex cannot end freely in the interior of a fluid. This theorem, which is a direct consequence of Stokes' theorem on circulation (see Division A IX 3), can be extended to the case of bound vortices. The theorem then asserts that ordinary (actual, or *free*) vortices must extend from the surface of the airfoil into the field of motion, of such an intensity that the circulation around these vortices is equal to the circulation around the body.

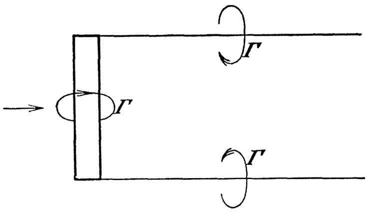

Fig. 8.

To be more precise, taking first the case of a constant value of Γ for all sections of the airfoil, it is necessary that from each end shall extend a vortex with the same circulation around it (that is, a vortex of strength Γ). As steady motion is possible only when the vortex lines coincide with the lines of flow, the picture to be expected must be as represented in Fig. 8. The vortices under consideration are the trailing vortices mentioned in 5. They are approximately parallel to the direction of the original velocity V, being slightly distorted in consequence of the superposed velocities present in the neighborhood of the airfoil.

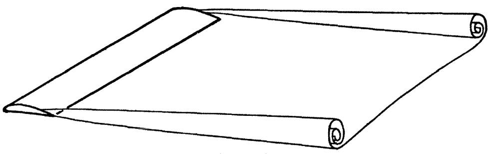
Fig. 9.

When the circulation is not constant along the span, then between any two sections I and II, having circulations Γ_I and Γ_{II}, vortices must depart from the airfoil, having a total strength equal to $\Gamma_I - \Gamma_{II}$. In the case of a gradual change of Γ along the span, as is commonly met with in practice, a whole sheet of trailing vortices extends downstream from the airfoil. In consequence of the motions these vortices impart to each other, this sheet, at a certain distance downstream from the airfoil, usually rolls up into two vortices, each of strength equal to the circulation around the median section of the airfoil, as sketched in Fig. 9 (after Prandtl)[1].

8. Origin of the Circulation around the Airfoil. Though we shall be concerned mainly with the case of steady motion, it will be useful to give some attention to the process by which the circulation is originated. It has been deduced both from experimental observation and from theoretical reasoning, that when a body starts moving in a fluid originally at rest, the motion first generated is an irrotational flow, satisfying with a high degree of approximation the ordinary condition for Dirichlet-motions[2], *viz.* that at the surface of the body the normal component of the velocity of the fluid is equal to the normal component of the velocity of the surface element, while at infinity the velocity is zero. On the other hand taking the description of the flow with reference to a system of coordinates rigidly connected with the body, we have the condition that the normal component of the velocity of the fluid at the surface of the body must be zero, whereas the velocity at infinity has the value denoted by V.

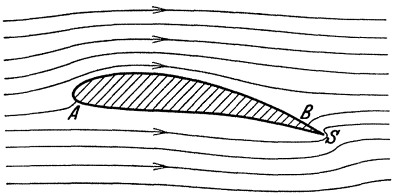

Fig. 10.

This initial motion nowhere presents a circulation around the airfoil. The general course of the lines of flow around a particular section of the airfoil at this stage is given in Fig. 10, from which it will be seen that there are two stagnation points, A and B. A photographic picture of this state of flow is given in Fig. 11 on Plate I (taken again with a camera at rest relative to the model).

Returning to Fig. 10 it must be observed that in consequence of the very small radius of curvature of the profile at the trailing edge S (in theoretical investigations the radius of curvature is usually taken equal to zero here, giving a sharp trailing edge), the velocity of the fluid moving around the trailing edge will be very high; hence the pressure will be low at S.

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[1] Photographic pictures of the vortex motion behind the tips of an airfoil have been obtained by various authors; the reader may be referred for instance to L. Prandtl, C. Wieselsberger und A. Betz, Ergebnisse der Aerodynamischen Versuchsanstalt Göttingen II, p. 78 (München und Berlin 1923), and to F. W. Caldwell and E. N. Fales, Nat. Adv. Comm. Aeronautics, Washington, Rep. No. 83, 1920.

A detailed investigation of the process of rolling up of the vortex system behind the airfoil, both theoretically and experimentally, has been given by H. Kaden, Ingenieur-Archiv 2, p. 140, 1931 (see VI 3—5).

[2] For convenience the term "Dirichlet-motion" is used here to denote the irrotational motion of an ideal fluid along a body, as studied in all textbooks on hydrodynamics.

In the case of an ideal fluid, experiencing no frictional forces, the kinetic energy of the motion at the point S is just sufficient to drive the fluid to the stagnation point B, where the pressure is equal to the impact pressure. In any actual case, however, viscous forces are present; they give an additional retardation to the fluid on the way from S to B, and this retardation proves to be so effective, that in the immediate neighborhood of the surface of the airfoil the fluid does not reach B, but stops somewhere on the way. In consequence of this a backflow appears, caused by the pressure gradient acting from B towards S, which separates the original flow from the surface of the airfoil, and produces a vortex as indicated in Fig. 12.

This vortex, which is formed along the whole trailing edge of the airfoil and is called the *starting vortex*, appears to be unstable: it separates from the trailing edge and is carried along by the general motion. The process can be observed experimentally without great difficulty; a photograph of such a vortex, taken a short time after its birth, is given in Fig. 13[1] (Plate I).

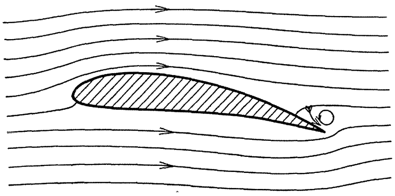

Fig. 12.

While the starting vortex is moving away, the two trailing vortices are formed from the fluid particles which are driven from the lower (high pressure) surface of the airfoil around the end edges to the upper (low pressure) surface. They make the connection between the ends of the airfoil and the ends of the starting vortex.

Simultaneously with this process the motion around the airfoil itself is changed; the line of stagnation points B is displaced, until it approximately coincides with the trailing edge. The fluid then no longer moves around the trailing edge, but flows off tangentially at both sides with a velocity differing only slightly from the original velocity V, as pictured already in Fig. 6.

In the case as indicated in Figs. 12, 13, the starting vortex possesses a circulation in the counterclockwise sense. In consequence of its sepa-

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[1] For more photographs picturing the process of the development of the circulation around an airfoil the reader may be referred for instance to: L. PRANDTL und O. TIETJENS, Hydro- und Aeromechanik II (Berlin 1931), Taf. 17—22. Cinematographic films also have been prepared in the Göttingen laboratory. Other photographs have been published by W. S. FARREN, Proc. 3rd Intern. Congress for Applied Mechanics (Stockholm 1930), I, p. 323 and by P. B. WALKER, Techn. Rep. Aeron. Res. Committee, Teddington, R. and M. No. 1402 (Jan. 1931); they are moreover especially remarkable as they have been used for an accurate experimental determination of the magnitude of the circulation in the first stages of the motion. The results obtained in this way are in very good agreement with those predicted theoretically by WAGNER (see V 5).

ration from the airfoil, a circulation of the opposite magnitude is left remaining around the airfoil. The "bound vortex" originating in this way, together with the trailing vortices and the starting vortex, form a closed vortex line of rectangular form.

The starting vortex is soon dispersed by disturbances which always arise, and after a certain time only the two trailing vortices are to be seen.

Every change in velocity and likewise every change in the angle of incidence of the airfoil is accompanied by the formation of a new starting vortex, of the same sign as the original one when the velocity or the angle of incidence is increased (supposing that the latter does not surpass its critical value) and of opposite sign in the contrary case. The theory of these vortices, which are of importance *e. g.* in the case of flapping wings, is considered in Chapter V.

In consequence of the action of viscosity, vortices with axes parallel to the direction of the span are also produced continuously in the boundary layers on both sides of the airfoil. These vortices are carried along by the general motion of the fluid, and are thus to be found likewise in the wake behind the airfoil. They are, however, of much smaller intensity, and those coming from different sides are of opposite signs and thus in the mean cancel each other. Hence they do not contribute appreciably to the circulation, provided (as noted above) that the curve along which the circulation is measured cuts the wake at right angles.

The condition that, in the final flow, the rear stagnation point B shall coincide with the trailing edge of the profile, which was originally suggested by Joukowski, is of great importance for the mathematical calculation of the magnitude of the circulation, as will appear in Chapter II.

Actually the condition is not quite rigorous; experimental investigations have shown that there remains at the upperside of the airfoil a region of irregular vortex motion which has a certain influence on the magnitude of the circulation, reducing its value somewhat as compared with that deduced from the supposition just mentioned. The extent of this region depends on various details of the form of the profile; for any given profile it extends gradually upstream farther and farther as the angle of incidence is increased, and when this angle exceeds a certain value, causes a breakdown of the whole type of motion as outlined. This value is the critical angle of incidence, already mentioned in **3**; it is usually of the order of magnitude 16^0 to 20^0. Thick profiles with a well rounded leading edge usually have higher critical angles than thinner ones; especially high critical angles are obtained by slotted airfoils. We shall, however, not enter into a consideration of these phenomena, which will be treated in Division J of this work.

In framing a method for calculating the circulation in Chapter II we shall hold to the assumption that the rear stagnation point actually coincides with the trailing edge. If necessary a correction can be introduced afterwards by multiplying the calculated value by an efficiency factor peculiar to the profile considered. This factor is usually of the order of magnitude of 85 per cent.

9. Equivalence of an Airfoil and a System of Vortices. It has already been mentioned that the presence of the circulatory motion around an airfoil can be assimilated to that around a system of vortices, kept in a fixed position (as indicated by the term "bound vortices") in a fluid moving originally with constant velocity.

In order to obtain this system of vortices in full detail we imagine the airfoil taken out of the field and replaced by fluid at rest. Then along both surfaces which originally formed the upper and under side of the airfoil, we have a discontinuity in the tangential component of the motion. There is no motion in the region between these surfaces, while outside of them the tangential component differs from zero. Hence these surfaces must be considered as *vortex sheets*. Properly speaking these sheets are identical with the boundary layers existing on both sides of the airfoil, as in any actual fluid the velocity at the surface of a fixed body is always zero[1]. The actual thickness of these layers, which of course is of importance in connection with the magnitude of the frictional resistance, is, however, of small consequence in any geometrical discussion of the vortex system as a whole.

As most airfoils are of rather small thickness, a sufficiently good approximation in many cases can be obtained by introducing the limiting assumption of an infinitely thin airfoil, both surfaces of which coincide. We then have a single vortex sheet only.

As we shall have to do frequently with vortex systems of various types in the subsequent treatment, it may be well to recall the following definitions:

Vortex motion is present whenever the distribution of the velocity is not irrotational. The intensity of vortex motion, or, the *vorticity,* is measured by the rotation of the velocity; thus denoting the components of the velocity by v_x, v_y, v_z, the components of the vorticity by γ_x, γ_y, γ_z we have the well known relations (see Division A VII 3):

$$\gamma_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \text{ etc.}$$

The strength of an *isolated vortex* surrounded by a field of irrotational motion is measured by the line integral of the velocity along a closed curve encircling the vortex. For this strength the letter Γ will be used, the same as for circulation in general. From Stokes' theorem on circulation (see Division A IX 3) it follows that the strength of a vortex can also be expressed as the integral of the normal component γ_n of the vorticity over a cross section of the vortex.

Using the concept of circulation, we can also define the component of the vorticity γ_s in any particular direction s as the quotient of the circulation around

[1] This is also the reason why the boundary layers must be included in the curve for the determination of the circulation, as stated at the end of **6.**

a small element of surface orientated normally to the direction s, by the area of this element.

In order to arrive at the concept of a *vortex sheet* we may start from the following example: consider the region between the two planes defined by $y = -\delta/2$, $y = +\delta/2$, their distance apart thus being δ, and assume in this region a vortex motion, having a component γ_z only. If then the distance δ is diminished without limit, while γ_z is increased proportionally to $1/\delta$, we shall obtain a vortex sheet. We can arrive at the same concept by considering a system of rectilinear vortices, all parallel to the z-axis and situated in the x, z-plane, provided we increase the number of vortices while decreasing their strength in such a way, that finally a continuous distribution is approached.

The *strength of a vortex sheet* is measured by the circulation around a strip of the sheet having unit length in the direction normal to the axes of the vortices, that is in the direction of x in the example given above. It is easily to be seen that in the first picture this strength is obtained as the limiting value of the integral of the vorticity γ_z over the thickness of the sheet. In the second picture the strength is obtained as the limiting value of the product of the number of vortices per unit length (measured in the direction of x) into the strength of a separate vortex, if for simplicity we suppose[1] that these quantities do not depend on x.

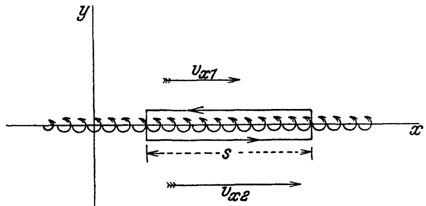

Fig. 14a.

We can relate the strength of the vortex sheet directly to the distribution of the velocity, without having recourse to the above pictures. If we take a strip of length s (see Fig. 14a) and if the velocity component v_x on the positive side of the sheet has the value v_{x1} and on the negative side has the value v_{x2}, then the line integral along the contour indicated in the diagram (which measures the circulation) has the value $(v_{x2} - v_{x1})\,s$, if taken in the counterclockwise sense (looking in the direction of $-z$). If then s becomes unity, we have for the strength of the vortex sheet, denoted by $\bar{\gamma}$, the value:

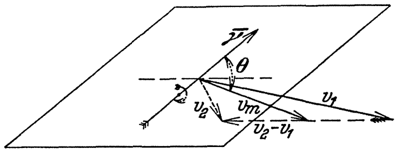

Fig. 14b.

$$\bar{\gamma} = v_{x2} - v_{x1} \tag{9.1}$$

Thus the strength of a vortex sheet is determined by the difference of the velocities on its two sides[2].

This result may be generalized if it is observed that there may be moreover a velocity component v_z. This component must have the same value on both sides of the sheet, if we keep to the supposition that the axes of all vortices are parallel to the z-axis. We might still further generalize by introducing a velocity

[1] The extension to a more general case can be made in an obvious way by introducing an integral.

[2] The dimensions of the various quantities introduced are as follows: The dimensions of the vorticity are those of angular velocity, that is $(1 \div T)$. The dimensions of Γ (circulation, or, strength of an isolated vortex) are those of a velocity multiplied by a length, thus $(L^2 \div T)$. The dimensions of the strength of a vortex sheet are the same as those of a velocity $(L \div T)$.

component v_y (which also must have the same value on both sides of the sheet, in consequence of the equation of continuity), but this introduces complications which are of no interest for the present discussion. We thus keep to the case where the resultant velocity on both sides of the sheet is tangential to the latter. Indicating now by v_1, v_2 the velocities along both sides of the sheet, and considering these quantities as vectors, we can free ourselves from the introduction of coordinate systems, and express our result by saying that the strength $\bar{\gamma}$ of the vortex sheet is given by the geometrical difference of these vectors:

$$\bar{\gamma} = v_2 - v_1 \tag{9.2}$$

The vector of the vorticity corresponding to this difference is in the plane of the sheet and directed normally to the vector $v_2 - v_1$ (see Fig. 14b).

When the vectorial mean of the two velocities

$$v_m = \frac{1}{2}(v_2 + v_1) \tag{9.3}$$

is introduced, and when the angle between the vectors v_m and $\bar{\gamma}$ is called θ, then the absolute magnitudes of v_2 and v_1 can be expressed as a function of the absolute values of v_m and $\bar{\gamma}$ by means of the equations:

$$\left.\begin{aligned} v_1^2 &= \left(v_m + \frac{1}{2}\bar{\gamma}\sin\theta\right)^2 + \left(\frac{1}{2}\bar{\gamma}\cos\theta\right)^2 \\ v_2^2 &= \left(v_m - \frac{1}{2}\bar{\gamma}\sin\theta\right)^2 + \left(\frac{1}{2}\bar{\gamma}\cos\theta\right)^2 \end{aligned}\right\} \tag{9.4}$$

These are the relations to be observed when an airfoil is replaced by a system of vortices, distributed over a single sheet, lying midway between the upper and under surfaces of an airfoil. Apart from the fact that these vortices are kept in a fixed position in space and do not move with the fluid, they satisfy the ordinary geometrical relations to which vortices are subject. As was mentioned in 7, the system of the bound vortices combined with the free or trailing vortices extending downstream satisfies the condition of continuity for vortices; moreover, if the whole system of vortices is known, the distribution of the velocity throughout the field can be calculated from them by means of Biot and Savart's formula without having further regard to the airfoil itself (see Division B III 2).

The correspondence between the airfoil and the system of bound vortices can be pursued further, as it is possible to express the force experienced by an element of the airfoil per unit of area in terms of the strength of the vortex sheet. This force is evidently equal to the difference of the pressures on both sides, and is directed normal to the element (assuming again the case of a very thin airfoil). Taking the index 1 for the upper side and 2 for the lower side, and expressing the pressure difference by means of the velocities according to Bernoulli's theorem, we obtain (see Fig. 14b):

$$p_2 - p_1 = \frac{1}{2}\varrho\,(v_1^2 - v_2^2) \tag{9.5}$$

In consequence of (9.3) and (9.4) this can be written:

$$p_2 - p_1 = \varrho\,|\bar{\gamma}|\cdot|v_m|\sin\theta \tag{9.6}$$

which gives the force in terms of $\bar{\gamma}$ and of the mean velocity.

Now the force f acting *from the airfoil element upon the fluid* has the same magnitude, but the opposite direction, *i. e.* from the upper side of the element toward the lower side. Hence we arrive at the result that the force f acting on the fluid per unit area is given by the density of the fluid multiplied by the cross product[1] of the strength of the vortex sheet (considered as a vector) into the mean velocity:

$$f = \varrho\, \bar{\gamma} \times v_m \tag{9.7}$$

This result is a special instance of a more general theorem to be developed in the course of Chapter III, which asserts that whenever "bound vortices" must be kept at a fixed position in a moving fluid, exterior forces must be applied, the intensity per unit volume of which is given by ϱ times the cross product of the vorticity into the velocity.

In the case usually considered in airfoil theory, the axes of the "bound vortices" are perpendicular to the direction of the mean velocity, so that $\sin\,\theta$ is equal to unity, while instead of the cross product the ordinary product of the absolute values appears:

$$f = \varrho\, |\gamma| \cdot |v_m| \tag{9.8}$$

The theorem shows simultaneously that free vortices, which are not acted upon by exterior forces, must either coincide with the stream-lines, in which case the cross product is zero, or otherwise they will be carried along by the motion of the fluid. In the latter case the flow can be steady only when the vortices are distributed continuously over a sheet according to a definite law, so that they succeed each other without interruption, and in such manner that the vorticity at a given point of space is independent of the time.

10. Connection between Equation (9.8) and the Kutta-Joukowski Theorem. Equation (9.8) can be considered as a special form of the Kutta-Joukowski theorem, applying to a single element of the airfoil, instead of to the whole. It will be shown later (II **3** and III **30**) how, in the case of two-dimensional motion, it is possible to deduce from (9.8) the "integral" Kutta-Joukowski theorem which was expressed by (6.1). In this deduction there is one point of special interest, which may be mentioned here. The mean velocity v_m at the element is evidently not equal to the velocity V of the air at great distances from the vortex system, but must be obtained by superposing upon it the field of motion corresponding to the entire system of vortices (bound vortices + trailing vortices) connected with the airfoil.

Now in the case of plane motion, all bound vortices have their axes exactly parallel to the span of the airfoil; moreover, their strength is constant over the span, and there are no trailing vortices. It can be demonstrated in this case that in making up the sum of the forces acting on all the elements of the airfoil, the terms due to the additional

[1] See Division C I **1**.

velocities cancel each other. In this way the result is obtained that in the final expression for the resulting force per unit span, only the velocity V occurs; the resulting force is perpendicular to V and is given by ϱ times the product of V and the total strength of all the vortices which are distributed over a section of the airfoil. The total strength of all the vortices together, however, is equal to the circulation around the whole system, that is, to the circulation around the section of the airfoil. In this way we come to the Kutta-Joukowski theorem.

In the three-dimensional case, taking an airfoil of finite span, a number of differences must be noted. In most cases all bound vortices will still be nearly parallel to the span; and the resultant force will be in the median plane of the airfoil. When the aspect ratio of the airfoil is sufficiently high, it can be shown that the part of the additional velocities due to the bound vortices does not differ greatly from the values of the additional velocities obtained in the case of plane motion (infinite span). Hence in making up the sum over all elements of the airfoil, these parts of the additional velocities again cancel (at least with a fair degree of approximation) and we come back to the Kutta-Joukowski equation for the force normal to V, that is for the lift per unit span.

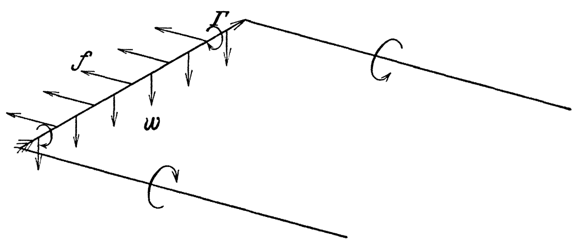

Fig. 15.

There is, however, a part of the additional velocities due to the presence of the system of trailing vortices. The terms relating to this part do *not* disappear from the final result. At the elements of the airfoil the part of the additional velocity connected with the trailing vortices is directed mainly downward. Hence there will be a component of the force acting from the airfoil upon the fluid directed against the general motion (see the diagram of directions in Fig. 15). This force evidently must be the reaction of a certain resistance experienced by the airfoil.

Hence we come to the result, that in the case of three-dimensional motion around an airfoil of finite span, there appears a certain resistance, connected with the flow due to the trailing vortices, and which is not present in the case of plane motion. As the downward velocity due to the trailing vortices can usually be considered as nearly constant over a section of the airfoil, this additional resistance per unit span is given by the product:

$$\varrho \Gamma w \tag{10.1}$$

where w represents the downward velocity.

This resistance is called the *induced resistance*, and it will be demonstrated in Chapter III that the work necessary to overcome this resistance when an airfoil is moved through air originally at rest, is equal to the energy expended in creating the flow system connected with the trailing vortices.

11. General Expression for the Induced Resistance. The notion that a certain resistance must be overcome in producing the flow system in the wake of the airfoil is, of course, self evident; it will now be shown, following a reasoning given by Prandtl, that independently of considerations regarding vortices, this idea can be used to obtain a general expression for the induced resistance.

Instead of considering the actual distribution of the velocities in the wake of the airfoil, we assume that the action of the airfoil on the air is wholly confined to that part of the air which passes through a certain area Σ perpendicular to the direction of the motion, depending in its dimensions upon those of the airfoil and connected with it. In passing through this area Σ the air experiences the reaction of the lift L; in consequence of this a downward momentum is communicated to it, and it is evident that the amount of momentum so communicated in unit of time must be equal to L.

Now the mass of air which passes through Σ in unit of time is equal to $\varrho V \Sigma$; hence calling w_0 the downward velocity communicated to it, the momentum imparted becomes $\varrho \Sigma V w_0$, and thus we obtain the equation:

$$L = \varrho \Sigma V w_0 \tag{11.1}$$

from which:

$$w_0 = \frac{L}{\varrho \Sigma V} \tag{11.2}$$

The downward motion implies kinetic energy; the amount of energy communicated to the air in unit of time has the magnitude:

$$E = \frac{1}{2} \varrho V \Sigma w_0^2 \tag{11.3}$$

A certain quantity of work must be expended in generating this energy. Writing the amount expended in unit of time $D_i V$, and calling D_i the induced resistance, we arrive at the relation:

$$D_i = \frac{E}{V} = \frac{1}{2} \varrho \Sigma w_0^2 = \frac{L^2}{2 \varrho V^2 \Sigma} \tag{11.4}$$

This is the equation to be deduced; it shows the important result that the induced resistance increases proportionally to the *square* of the lift. This could have been deduced also from the considerations in **10**: with increasing lift, the strength of the vortex system increases, and thus in the product (10.1) both Γ and w increase.

Equation (11.4) does not give a definite value for D_i unless the area Σ be known. This area cannot be obtained by elementary deductions. It will be shown later, however, that it is possible to obtain such a distribution of the load over the various elements of the airfoil, that the induced resistance assumes a *minimum* value compatible with a given lift L. In this case the area Σ will have a maximum value and it is found that this maximum value is equal to the area of the circle described with the span as diameter, that is[1]:

$$\Sigma_{max} = \pi b^2 \tag{11.5}$$

It is found, moreover, that this particular case affords a good approximation for most cases occurring in practice. Hence inserting the value (11.5) into (11.4), we obtain:

$$D_i = \frac{L^2}{2 \varrho V^2 \pi b^2} \tag{11.6}$$

This case of minimum induced resistance is characterized by the circumstance that at a sufficient distance downstream from the airfoil the downward velocity at the points of the vortex sheet has a constant value, being equal to that given in (11.2) with the value (11.5) for Σ:

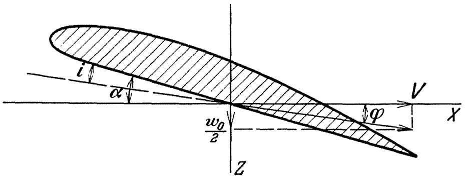

Fig. 16.

$$w_0 = \frac{L}{\varrho V \pi b^2} \tag{11.7}$$

It is evident that this downward motion cannot be generated at once, and, as will be demonstrated more exactly afterwards, it must be assumed that half of it is imparted to the air before it reaches the airfoil, the other half being generated after the air has passed the airfoil. Hence at the airfoil itself there must exist a downward velocity of the amount:

$$w = \frac{w_0}{2} = \frac{L}{2 \varrho V \pi b^2} \tag{11.8}$$

When this latter velocity is combined with the original velocity V of the air (see Fig. 16), we obtain a resultant velocity $\sqrt{V^2 + (w_0/2)^2}$ sloping downward under an angle φ, determined by

$$\tan \varphi = \frac{w_0}{2 V}$$

or, approximately:

$$\varphi = \frac{w_0}{2 V} \tag{11.9}$$

As from (11.6) and (11.7) we deduce the relation:

$$D_i = \frac{w_0}{2 V} L \tag{11.10}$$

it will be seen that the resultant of lift and induced resistance makes

[1] See IV 2.

the same angle φ with the vertical, and is thus directed perpendicular to the resultant of V and the downward velocity $w_0/2$ [1].

12. Reduction Formulae. When the induced resistance is subtracted from the total drag, we obtain the *profile drag*, that is the part of the drag which is directly due to the action of frictional forces:

$$D_0 = D - D_i$$

or introducing coefficients C_{Di}, C_{D0}, corresponding respectively to induced drag and to profile drag:

$$C_{D0} = C_D - C_{Di} \tag{12.1}$$

When in (11.6) both the lift L and the induced resistance D_i are expressed by means of the corresponding coefficients, the important equation is obtained:

$$C_{Di} = \frac{C_L^2}{\pi} \cdot \frac{S}{(2b)^2} \tag{12.2}$$

Here appears the ratio $(2b)^2/S$, already mentioned in 2 and called the aspect ratio. Denoting this by λ, (12.2) can be written:

$$C_{Di} = \frac{C_L^2}{\pi\lambda} \tag{12.3}$$

At the same time the expression for the angle φ assumes the form:

$$\varphi = \frac{D_i}{L} = \frac{C_{Di}}{C_L} = \frac{C_L}{\pi\lambda} \tag{12.4}$$

The angle φ measures the downward inclination of the resultant velocity, obtained by combining the original velocity V with the downward motion deduced from the system of trailing vortices. Hence we can do away with the trailing vortices, provided we apply this inclination to the original motion of the air. This inclination, however, has the effect of diminishing the angle of attack of the air upon the airfoil. Calling α as before the geometrical angle of incidence, we see that the *effective angle of incidence* i is given by:

$$i = \alpha - \varphi = \alpha - \frac{C_L}{\pi\lambda} \tag{12.5}$$

When now airfoils having the same profile but presenting various aspect ratios are compared, it must be expected that their lift coefficients C_L can have the same value only when the *effective angle of incidence is the same* for all these airfoils.

In this way the result is reached that when it is known that a certain airfoil has the lift coefficient C_L for the geometrical angle of incidence α,

[1] The fact that the production of lift is necessarily accompanied by the induced resistance was pointed out for the first time by F. W. LANCHESTER. Independent of LANCHESTER the problem was treated by PRANDTL, who was the first to develop a rigorous system of mathematical equations, which form the basis for all modern work upon this subject.

then another airfoil of the same profile but with a different value λ' of the aspect ratio, will give this same value of C_L at the geometrical angle of incidence α' determined by the equation:

$$\alpha' = \alpha + \frac{C_L}{\pi}\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) \tag{12.6}$$

The coefficients of induced resistance for the two airfoils are given respectively by:

$$\frac{C_L^2}{\pi\lambda}, \quad \frac{C_L^2}{\pi\lambda'}$$

Hence, as we may assume that the coefficients of profile drag will be the same (they depend on the form of the profile and on the effective angle of incidence, but not on λ), it appears that the coefficient of total drag for the second airfoil can be calculated from the corresponding coefficient for the first airfoil, referring to the same value of C_L, by means of the equation:

$$C_D' = C_D + \frac{C_L^2}{\pi}\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) \tag{12.7}$$

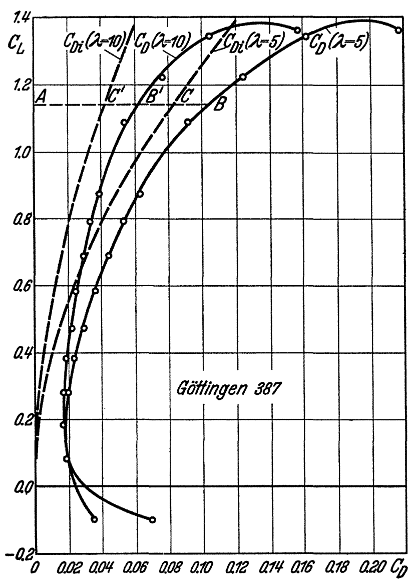

Fig. 17.

Equations (12.6) and (12.7) are called *reduction formulae*, which enable us to pass from a given airfoil to another, differing from the first only by its aspect ratio. They are used throughout aerodynamic practice in presenting experimental data obtained on model airfoils, and in making calculations relating to airfoils. They have been verified to a large extent by direct measurements; for a discussion of these measurements the reader is referred to Division J[1].

Equation (12.3) can be represented very simply in the polar diagram, (see 3) by means of a parabola, the *parabola of induced resistance*. When this parabola is drawn, together with the polar curve for the airfoil (see Fig. 17, curves labelled $\lambda = 5$), then for any particular value of C_L, for instance that given by the point A, the segment AB gives C_D, AC gives C_{Di}, and thus the segment CB represents the profile drag C_{D0}.

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[1] It must be kept in mind that these relations apply exactly only to the case of minimum induced resistance. In all other cases certain corrections must be applied, which are considered in the first part of Chapter IV. These corrections, however, are of minor importance compared to the principal terms given above.

In order to obtain the polar curve for any other aspect ratio, a new parabola must be drawn (see curves labelled $\lambda = 10$), and the point B of the original polar curve must be shifted to B', where:

$$B\,B' = C\,C' = \frac{C_L^2}{\pi}\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right)$$

Apart from this, the value of α must be changed in accordance with (12.6).

13. Concluding Remarks. Program for the Following Chapters. In the foregoing sections a general picture of the flow of the air around an airfoil has been given. In this description the airfoil was essentially replaced by a single vortex filament. No attempt was made to connect the special features of the airfoil, such as the shape of the profile, taper, etc., with the flow and with the forces acting on the airfoil. However, it is evident that wing theory for practical use must take into account the geometrical properties of the wing. The fundamental problem of wing theory is, therefore, the determination of the flow around a geometrically determined wing or wing system. In other words we have to calculate the distribution of velocities and pressures, assuming that a body having the shape of a wing is moving uniformly and with constant velocity through the fluid. Even if we restrict the investigation to ideal fluids, as is done throughout this volume, this fundamental problem can be resolved only for a body of cylindrical shape, *i. e.* for a wing with infinite span and with invariable wing section. The wing with infinite span represents an idealized case, but it has great importance because in this way we are able to obtain useful information regarding the influence of the profile form of the wing section, and we are able to use this information with some approximation also in the real case of wings with finite span. Chapter II is therefore exclusively devoted to the problem of the wing with infinite span (two-dimensional wing theory).

Chapters III and IV deal with the so-called three-dimensional wing theory. Since the solution of the fundamental problem, the uniform motion of a wing with given shape through an ideal fluid, meets with difficulties which, in the present state of our mathematical knowledge are unsurmountable, the three-dimensional wing theory uses methods which are somewhat different from those used in the theory of the wing with infinite span. In attacking the three-dimensional problem, we shall start from the idea that the influence exerted by a body immersed in a fluid upon the motion of the fluid, can be expressed as the result of the forces which the surface of the body exerts upon the fluid. Hence, supposing for a moment that these forces are known, it must be possible to obtain the flow pattern without further reference to the presence of the body.

In order to put these considerations on a more formal basis we again assume the body to be removed from the field of motion and replaced by fluid at rest. At the same time we introduce a system of forces acting at the points formerly occupied by the surface of the body, and chosen in such manner that they exactly represent the influence which the body exerted upon the fluid. We can then imagine these forces to be acting in a field of unlimited extent, which greatly simplifies the calculation of the motion produced by them.

In applying these ideas we shall investigate in Chapter III the flow of a fluid under the influence of an arbitrary system of external forces. Hence this chapter can be considered as the mathematical foundation of the general wing theory and a mathematical justification of the simplified description given in the foregoing sections.

In Chapter IV the results reached by the general mathematical theory are applied to the case of actual airfoils. Part A deals with the simple wing (monoplane), Part B with systems of wings (multiplane), and Part C with the influence of wind tunnel boundaries and the like on the airfoil. In these cases the airfoil is again represented by a single *loaded line* (vortex filament), but the results of the two-dimensional theory are used in order to take account of certain properties of the wing profiles. A still closer approximation may be realized by considering a distribution of forces over an area instead of along a line, and a limited treatment by this mode of representation will be found in IV **13**—**15**. No attempt is made to proceed with the next step through a consideration of the wing as a three-dimensional body.

Chapter V deals with the theory of airfoils in non-uniform (accelerated) and curvilinear motion. These investigations are important for certain problems of the dynamics of airplanes and for problems concerning flapping wings.

Finally in Chapters VI and VII will be found some discussion of the development of the vortex system downstream of an airfoil, or wing, and of the theory of drag and wake in ideal fluids.

CHAPTER II

THEORY OF AIRPLANE WINGS OF INFINITE SPAN

1. Introduction. We use in this chapter the idealized case of an airplane wing or a system of wings with infinite span and uniform cross section moving in a non-viscous fluid. As already pointed out in the last section, this is the only case in which the fundamental problem of wing theory, *i. e.* the calculation of velocity and pressure distribution around the moving airfoil, can be attacked in a direct way. The reason is that in the two-dimensional case, the motion around the airfoil can be assumed to be a vortexless motion, no trailing vortices leaving the airfoil. Thus, the methods of potential theory, and especially those of conformal transformation, enable us to describe fully the motion produced by a wing of certain shape. However, the direct solution in this way is rather tiresome and hence inverse methods are generally employed. We start from certain simple transformations and try to obtain families of wing profiles with a possible large variation of shape. These inverse methods facilitate a rather elegant presentation of the subject, which explains their popularity, especially with mathematicians.

In the general wing theory, three states of approximation can be established. In the first approximation we replace the airfoil by a loaded line, in the second approximation by a loaded area, and in the third stage we consider the wing as a three-dimensional body. In the three-dimensional theory, only the two first stages of approximation are feasible. Although in the two-dimensional theory an exact solution can be obtained, the two first approximations are also of great use because of their extreme simplicity. For this reason we shall start with the first approximation and in **2—4** investigate the forces acting on a system of isolated vortex filaments. The second approximation is used in **5—12**. In these investigations the airfoil is represented by a vortex sheet and we shall try to determine the distribution of vorticity along the sheet in such a way that the flow pattern is similar to that produced by an airfoil of certain profile. This method can be used in the case of thin airfoils with small inclination to the direction of flight. Sections **13—20** deal with the flow around airfoils with certain shapes. Section **21** deals with the determination of the aerodynamic characteristics of given airfoils and sections **22, 23** with problems of the biplane and the multiplane (lattices). In these sections, the case of indefinitely thin plane airfoils only is treated in detail. Some allusions are made in regard to the case of curved airfoils. Section **24** gives some examples illustrative of the preceding sections.

A. Vortex Systems and their Application in the Theory of Thin Airfoils.

2. Forces Acting on a Fluid in Two-Dimensional Motion. As in Chapter I, it is convenient to consider the wing system at rest and the fluid moving. We assume two-dimensional motion extended over the whole x, y plane; and further, that the velocity of the fluid at infinity has a constant value V, corresponding to the flying speed of the wing system. Instead of the wing system we introduce the following arrangement: Let $G_1, G_2, \ldots G_n$ denote n closed curves; inside of these curves, external forces $f_1, f_2, \ldots f_n$ act on the fluid; outside, the external forces vanish. The results of I **9**, in particular the relation expressed by I (9.8), make us expect that "bound vortices" will be present in the interior of the domains $G_1 \ldots G_n$. It may be recalled that in opposition to ordinary or "free vortices", which obey the Helmholtz law and are characterized by the fact that they are carried along by the motion of the fluid and that no resultant force acts between them and the latter, the "bound vortices" are supposed to be at rest (or, in other cases, to move in some prescribed way); the fact that they do not follow Helmholtz' law is a consequence of the circumstance that they are introduced artificially in order to represent the circulatory motion around bodies which serve as intermediary in transmitting certain forces acting upon the fluid.

Assuming irrotational motion outside of the domains $G_1 \dots G_n$ we note that the circulation vanishes if taken around any curve which does not include at least one of the sections G; on the contrary around every one of these regions there will be in general a non-vanishing circulation, the magnitude of which is connected with the magnitude of the force acting inside of the region. Reducing now the size of the G's indefinitely we obtain a system of vortex filaments. It will be the subject of the following sections to show that we can start from this system of bound vortex filaments, assigning to each of them a definite circulation; then we can determine both the flow over the whole plane and the magnitude of the forces. As the velocity distribution corresponding to a given vortex filament is given by the Biot-Savart law, we can build up the field of flow in the region left open between the domains G by superposing the flow "induced" by the individual bound vortex filaments upon the parallel flow of velocity V. The motion may also be described by introducing the complex potential function. Both methods will be applied in presenting the formulae for the forces corresponding to the system of vortices.

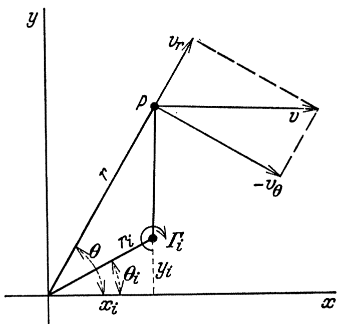

Fig. 18.

3. Forces on a System of Vortex Filaments. We assume a number of bound vortex filaments, the amount of circulation of the i-th vortex may be denoted by Γ_i, the point of intersection of the filament with the x, y plane may have the coordinates x_i, y_i. Using polar coordinates r and θ we easily express the velocity components at an arbitrary point P, see Fig. 18, taken in the radial and in the circumferential direction.

We have first the expressions:

$$\left.\begin{aligned} v_r &= \sum \frac{\Gamma_i}{2\pi} \frac{r_i \sin(\theta - \theta_i)}{r^2 + r_i^2 - 2 r r_i \cos(\theta - \theta_i)} \\ v_\theta &= -\sum \frac{\Gamma_i}{2\pi} \frac{r - r_i \cos(\theta - \theta_i)}{r^2 + r_i^2 - 2 r r_i \cos(\theta - \theta_i)} \end{aligned}\right\} \tag{3.1}$$

In these expressions it is assumed that the positive sign of the circulation corresponds to a clockwise rotation of the fluid[1]; r_i and θ_i

[1] Throughout the present Chapter, the positive circulation is taken in the *clockwise* direction, as this is the normal direction of the circulation in the case of an airfoil profile having the position indicated for instance in Fig. 6 of Chapter I or Fig. 30 of the present Chapter. It must be remarked, however, that several

are the polar coordinates corresponding to x_i, y_i. We then develop these in a series containing decreasing powers of r. The series obtained in this way are convergent, so long as r is greater than any of the values $r_1, r_2, \ldots r_n$. We thus obtain:

$$\left.\begin{aligned} v_r &= \frac{1}{r^2}\Sigma\frac{\Gamma_i}{2\pi}r_i \sin(\theta-\theta_i) + \text{higher terms} \\ v_\theta &= -\frac{1}{r}\Sigma\frac{\Gamma_i}{2\pi} - \frac{1}{r^2}\Sigma\frac{\Gamma_i}{2\pi}r_i \cos(\theta-\theta_i) + \text{higher terms} \end{aligned}\right\} \tag{3.2}$$

We now assume a uniform and parallel fluid motion with the velocity components V_x and V_y superposed on the flow given by these formulae and apply the theorem of momentum and that of the moment of momentum to the fluid enclosed within a circle C with radius $r = K$. Obviously the difference between the momentum leaving the circular boundary and that entering it is equal to the resultant of the forces acting on the fluid; and the difference between the moment of momentum leaving and entering gives the moment of the same forces.

Let us consider the forces. They are:

(*a*) the pressure p distributed along the circle C,

(*b*) the forces acting between the bound vortices and the fluid. We calculate the forces acting on the wing system for unit span. Consequently the momentum and the moment of momentum also refer to a portion of the fluid enclosed between two parallel planes at unit distance. Remembering that we define the forces as those acting from the fluid on the system of vortices, while the change of momentum and the resultant pressure over the boundary of the circle of radius K relate to the fluid as distinguished from the vortex system, we may state the momentum theorem in the form:

Change of momentum (+ leaving boundary) = — Resultant of forces on vortex system + Resultant pressure over boundary, or:

Resultant of forces = Resultant of pressures — Change of momentum,

whence:

$$X = -\oint p\cos\theta\, ds - \varrho\oint W_n W_x\, ds$$
$$Y = -\oint p\sin\theta\, ds - \varrho\oint W_n W_y\, ds\,,$$

where X and Y are the resultant forces in the x and y direction, θ locates the element ds, W_x, W_y are resultant velocity components in the x and y directions and W_n is the velocity component perpendicular to the

authors draw the airfoil profile in the opposite way, *i. e.* with the leading edge pointing to the right; in that case the velocity V comes from the right hand side. This implies a number of differences with respect to signs, etc., care of which must be taken if formulae should be compared.

We remind that in the present text, in accordance with usual mathematical practice, the polar angle θ and the component v_θ are reckoned positive in the *anti-clockwise* direction; thus for instance in Fig. 18 the component v_θ has a negative value.

element ds of the boundary. Applying these equations to the present case and remembering that:

$$W_x = V_x + v_r \cos\theta - v_\theta \sin\theta$$
$$W_y = V_y + v_r \sin\theta + v_\theta \cos\theta$$
$$W_n = V_x \cos\theta + V_y \sin\theta + v_r,$$

we obtain:

$$\left.\begin{aligned} X &= -K\int_0^{2\pi} p\cos\theta\,d\theta - \varrho K\int_0^{2\pi}(V_x\cos\theta + V_y\sin\theta + v_r)\cdot \\ &\qquad\cdot(V_x + v_r\cos\theta - v_\theta\sin\theta)\,d\theta \\ Y &= -K\int_0^{2\pi} p\sin\theta\,d\theta - \varrho K\int_0^{2\pi}(V_x\cos\theta + V_y\sin\theta + v_r)\cdot \\ &\qquad\cdot(V_y + v_r\sin\theta + v_\theta\cos\theta)\,d\theta \end{aligned}\right\} \tag{3.3}$$

The pressure p, according to Bernoulli's law for an incompressible ideal fluid, is given by:

$$\left.\begin{aligned} p = p_0 + \frac{1}{2}\varrho(V_x^2 + V_y^2) - \frac{1}{2}\varrho(V_x + v_r\cos\theta - v_\theta\sin\theta)^2 - \\ -\frac{1}{2}\varrho(V_y + v_r\sin\theta + v_\theta\cos\theta)^2 \end{aligned}\right\} \tag{3.4}$$

where p_0 is the value of the pressure at infinity. The evaluation of the integrals can be restricted to the first two terms of the order 1 and $1/K$; the contributions of the higher terms obviously vanish if we go to the limit $K = \infty$. We obtain in this way, using the expressions (3.2) for v_r and v_θ:

$$\left.\begin{aligned} X &= -\varrho V_y \Sigma\Gamma_i \\ Y &= \varrho V_x \Sigma\Gamma_i \end{aligned}\right\} \tag{3.5}$$

Thus, in accordance with the theorem of Kutta-Joukowski for the relation between circulation and lift (see I 6) the resultant force R is perpendicular to the velocity V, and has the amount:

$$R = \varrho V \Sigma \Gamma_i \tag{3.6}$$

We use now the theorem of the moment of momentum to find the moment of the forces about the origin $x = 0$, $y = 0$. We thus obtain[1]:

$$\left.\begin{aligned} M = -\varrho K^2\int_0^{2\pi}(V_x\cos\theta + V_y\sin\theta + v_r)\cdot \\ \cdot(-V_x\sin\theta + V_y\cos\theta + v_\theta)\,d\theta \end{aligned}\right\} \tag{3.7}$$

We have now to retain terms to the order $1/K^2$ and integrating along the circle we obtain the expression:

$$M = \varrho\Sigma(\Gamma_i V_x r_i \cos\theta_i + \Gamma_i V_y r_i \sin\theta_i) \tag{3.8}$$

This equation states, in effect, that the moment of the forces acting on a system of bound vortex filaments is equal to the total moment

[1] The positive direction for the moment is the anti-clockwise direction, the same as in Fig. 3 of Chapter I.

of the forces obtained by considering each one of the vortices independently in the parallel flow and neglecting their interaction. The additional forces on any one vortex due to the presence of the other vortices behave like internal forces. They obey the law of action and reaction.

This result can be proved by direct computation of the forces acting on an individual vortex filament Γ_i. We separate, in the expressions for the velocity components, contributions of the vortices $\Gamma_1, \Gamma_2, \ldots \Gamma_{i-1}, \Gamma_{i+1}, \ldots \Gamma_n$ and the contribution of the vortex filament Γ_i. Considering the components in the x and y direction, we write:

$$\left.\begin{aligned} v_x &= v_x' + v_x'' \\ v_y &= v_y' + v_y'' \end{aligned}\right\} \tag{3.9}$$

where:

$$\left.\begin{aligned} v_x' &= \Sigma' \frac{y - y_k}{(x - x_k)^2 + (y - y_k)^2} \frac{\Gamma_k}{2\pi} \\ v_y' &= \Sigma' \frac{x_k - x}{(x - x_k)^2 + (y - y_k)^2} \frac{\Gamma_k}{2\pi} \end{aligned}\right\} \tag{3.10}$$

The symbol Σ' is used in the sense that the summation is extended over all indices excepting the index i. The velocity components v_x'', v_y'' are then given by the expressions:

$$\left.\begin{aligned} v_x'' &= \frac{\Gamma_i (y - y_i)}{2\pi [(x - x_i)^2 + (y - y_i)^2]} \\ v_y'' &= \frac{\Gamma_i (x_i - x)}{2\pi [(x - x_i)^2 + (y - y_i)^2]} \end{aligned}\right\} \tag{3.11}$$

We now apply the law of momentum to the fluid enclosed in a small circle of the radius r around the center $x = x_i$, $y = y_i$ and put $x = x_i + r \cos\theta$, $y = y_i + r \sin\theta$. We thus obtain the following expressions for the components of the force acting on the vortex filament Γ_i:

$$\left.\begin{aligned} X_i = &-r \int_0^{2\pi} p \cos\theta \, d\theta - \varrho r \int_0^{2\pi} (V_x + v_x' + v_x'') \cdot \\ &\cdot [(V_x + v_x' + v_x'') \cos\theta + (V_y + v_y' + v_y'') \sin\theta] \, d\theta \\ Y_i = &-r \int_0^{2\pi} p \sin\theta \, d\theta - \varrho r \int_0^{2\pi} (V_y + v_y' + v_y'') \cdot \\ &\cdot [(V_x + v_x' + v_x'') \cos\theta + (V_y + v_y' + v_y'') \sin\theta] \, d\theta \end{aligned}\right\} \tag{3.12}$$

The value of p is given by an expression similar to (3.4). Passing to the limit $r \to 0$, we remember that v_x'', v_y'' will be infinite of the order $1/r$, all other quantities being small in comparison with them. A somewhat lengthy calculation shows that at the limit we have:

$$\left.\begin{aligned} X_i &= -\varrho \Gamma_i (V_y + v_{yi}) \\ Y_i &= \varrho \Gamma_i (V_x + v_{xi}) \end{aligned}\right\} \tag{3.13}$$

where v_{xi} and v_{yi} are the components of the velocity "induced" at the location of the vortex Γ_i by all vortices except Γ_i. Now $v_{xi} = \Sigma' v_{xik}$, $v_{yi} = \Sigma' v_{yik}$, so that

$$\left.\begin{aligned} X_i &= -\varrho\,\Gamma_i\,V_y - \Sigma'\,\varrho\,\Gamma_i\,v_{yik} \\ Y_i &= +\varrho\,\Gamma_i\,V_x + \Sigma'\,\varrho\,\Gamma_i\,v_{xik} \end{aligned}\right\} \qquad (3.14)$$

v_{xik}, v_{yik} being the velocity components "induced" at the location of the vortex Γ_i by any vortex Γ_k. Fig. 19 represents two vortices Γ_i and Γ_k, v_{ik} being the velocity induced at the location of Γ_i by Γ_k and v_{ki} the velocity induced at the location of Γ_k by Γ_i. Obviously the components of v_{ik} are:

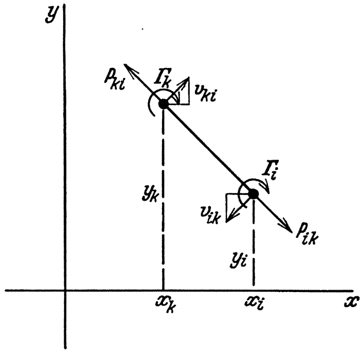

Fig. 19.

$$\left.\begin{aligned} v_{xik} &= \frac{\Gamma_k}{2\pi}\,\frac{y_i - y_k}{r_{ik}^2} \\ v_{yik} &= \frac{\Gamma_k}{2\pi}\,\frac{x_k - x_i}{r_{ik}^2} \end{aligned}\right\} \qquad (3.15)$$

so that we obtain for the interaction of the vortex Γ_k on the vortex filament Γ_i:

$$\left.\begin{aligned} X_{ik} &= \varrho\,\frac{\Gamma_i\,\Gamma_k}{2\pi}\,\frac{x_i - x_k}{r_{ik}^2} \\ Y_{ik} &= \varrho\,\frac{\Gamma_i\,\Gamma_k}{2\pi}\,\frac{y_i - y_k}{r_{ik}^2} \end{aligned}\right\} \qquad (3.16)$$

These equations show that:

$$\left.\begin{aligned} X_{ik} &= -X_{ki} \\ Y_{ik} &= -Y_{ki} \end{aligned}\right\} \qquad (3.17)$$

Hence the interaction is composed of two equal and opposite forces P_{ik} and P_{ki} and represents an attraction between vortices of opposite sense and a repulsion between vortices with circulation in the same direction. Representing wings by single bound vortex filaments, the formulae (3.16) give a first approximation for the interference between planes, for the influence of fixed walls on the lift forces, etc.

4. Calculation of the Forces Acting on a Vortex System by the Method of Complex Variables. We obtain the results of the preceding section much more easily by the method of complex variables. Every two-dimensional irrotational motion is represented by a complex potential function $F = \varphi + i\,\psi$ of a complex variable $z = x + i\,y$, x, y being the coordinates in the plane of the flow. We obtain the velocity components v_x, v_y by differentiation, introducing the complex velocity function:

$$w = v_x - i\,v_y$$

and putting:

$$w = \frac{dF}{dz} \qquad (4.1)$$

The potential function representing the vortex filament Γ_i located at the point $z_i = x_i + i\, y_i$ is given by the expression[1]:

$$F = \frac{i\,\Gamma_i}{2\,\pi}\, log\,(z - z_i) \tag{4.2}$$

The "complex velocity" at infinity in this case is obviously equal to the complex velocity corresponding to the parallel flow, *i. e.* to $W = V_x - i\, V_y$. Hence the expression for the potential function representing the parallel flow combined with the flow induced by the n "bound vortex filaments" takes the form:

$$F = (V_x - i\,V_y)\, z + \Sigma \frac{i\,\Gamma_i}{2\,\pi}\, log\,(z - z_i) \tag{4.3}$$

We apply now the law of momentum to the fluid mass enclosed by an arbitrary closed curve C. We first write the equations for the components of the resultant force in the form[2]:

$$\left.\begin{aligned} X &= -\oint_c p\,d\,y - \varrho \oint_c (v_x^2\,d\,y - v_x\,v_y\,d\,x) \\ Y &= \oint_c p\,d\,x + \varrho \oint_c (v_y^2\,d\,x - v_x\,v_y\,d\,y) \end{aligned}\right\} \tag{4.4}$$

(here v_x, v_y denote the components of the total velocity). According to Bernoulli's law:

$$p = q_0 - \frac{1}{2}\,\varrho\,(v_x^2 + v_y^2) \tag{4.5}$$

where q_0 is the "total head":

$$q_0 = p_0 + \frac{1}{2}\,\varrho\,(V_x^2 + V_y^2) \tag{4.6}$$

We obtain in this way:

$$\left.\begin{aligned} X &= -\oint_c q_0\,d\,y - \frac{\varrho}{2}\oint_c [(v_x^2 - v_y^2)\,d\,y - 2\,v_x\,v_y\,d\,x] \\ Y &= \oint_c q_0\,d\,x - \frac{\varrho}{2}\oint_c [(v_x^2 - v_y^2)\,d\,x + 2\,v_x\,v_y\,d\,y] \end{aligned}\right\} \tag{4.7}$$

We introduce now the "complex force":

$$R = X - i\,Y \tag{4.8}$$

combining the two equations and remembering that $d\,x + i\,d\,y = dz$, we obtain:

$$R = -q_0 \oint_c (d\,y + i\,d\,x) + \frac{\varrho\,i}{2}\oint_c w^2\,d\,z \tag{4.9}$$

The first integral vanishes when taken around a closed line, so that we have finally:

$$R = \frac{\varrho\,i}{2}\oint_c w^2\,d\,z \tag{4.10}$$

---

[1] Care must be taken to distinguish between the use of $i = \sqrt{-1}$ as a factor and i as a subscript implying order.

[2] The sign $\oint_c$ indicates an integration along the curve c in the anti-clockwise direction.

We may now make two applications of this simple equation: (*a*) First, we assume that the curve C encloses all vortex filaments. In this case the potential function F can be developed in a series:

$$F = Wz + A \log z + \frac{B}{z} + \frac{C}{z^2} + \dots \tag{4.11}$$

If we have no sources or sinks at infinity, A is a pure imaginary number; we put therefore: $A = \frac{i\Gamma}{2\pi}$

The coefficients B, C may be real or complex. Differentiating (4.11) we obtain:

$$w = W + \frac{i\Gamma}{2\pi z} - \frac{B}{z^2} - \frac{2C}{z^3} - \dots \tag{4.12}$$

and obviously[1]:

$$\oint_c w^2 dz = \oint_c \left(W + \frac{i\Gamma}{2\pi z} - \frac{B}{z^2} - \dots\right)^2 dz = \oint_c \frac{W i \Gamma}{\pi} \frac{dz}{z} = -2 W \Gamma \tag{4.13}$$

This gives the following value for the complex force:

$$R = X - i Y = -\varrho W i \Gamma \tag{4.14}$$

or:

$$\left.\begin{aligned} X &= -\varrho \Gamma V_y \\ Y &= \varrho \Gamma V_x \end{aligned}\right\} \tag{4.15}$$

(*b*) We let the curve C enclose the vortex filament Γ_i so that all other vortex filaments are outside of it. We write:

$$F(z) = W z + F_1(z) + \frac{i\Gamma_i}{2\pi} \log(z - z_i) \tag{4.16}$$

and

$$w = W + w_1 + \frac{i\Gamma_i}{2\pi} \frac{1}{(z - z_i)} \tag{4.17}$$

The function w_1 being regular over the whole curve C and inside of it, we obtain:

$$\oint_c w^2 dz = \frac{i\Gamma_i}{\pi} \oint_c (W + w_1) \frac{1}{(z - z_i)} dz = -2\Gamma_i (W + w_1) \tag{4.18}$$

and

$$R_i = -\varrho i \Gamma_i (W + w_1) \tag{4.19}$$

which is equivalent to (3.13) obtained in the preceding section.

To find the moment M, with reference to the origin $x = y = 0$, we apply the law of the moment of momentum to the fluid enclosed by the curve C and obtain:

$$\left.\begin{aligned} M = \oint_c p (x\,dx + y\,dy) - \varrho \oint_c (&- v_y^2\, x\,dx - v_x^2\, y\,dy + \\ &+ v_x v_y\, y\,dx + v_x v_y\, x\,dy) \end{aligned}\right\} \tag{4.20}$$

We then substitute the value of the pressure (4.5) and obtain[2]:

$$M = -Re.\left[\oint_c \left(\varrho z \frac{w^2}{2}\right) dz\right] \tag{4.21}$$

---

[1] We have $\oint_c \frac{dz}{z} = 2\pi i$; see Division A I (7.2).

[2] *Re.* [ ] = the real part of the complex expression within the bracket.

We now extend the integral (4.21) over a curve including all vortex filaments and use the development for w given in (4.12). We thus obtain:

$$\left.\begin{aligned} M &= -Re.\left[\frac{\varrho}{2}\oint_c z\,dz\left(W + \frac{i\Gamma}{2\pi z} - \frac{B}{z^2} - \ldots\right)^2\right] = \\ &= -Re.\left[\frac{\varrho}{2}\oint_c\left(-\frac{\Gamma^2}{4\pi^2} - 2\,B\,W\right)\frac{dz}{z}\right] = -2\,\pi\,\varrho\,Im.\,[B\,W] \end{aligned}\right\}\quad (4.22)^1$$

In order to find the value of B, we write:

$$w = W + \Sigma\frac{i\,\Gamma_i}{2\pi(z - z_i)}$$

and use the development:

$$\frac{1}{z - z_i} = \frac{1}{z} + \frac{z_i}{z^2} + \ldots$$

This gives for B the expression:

$$B = -\Sigma\frac{\Gamma_i z_i}{2\pi}\,i \tag{4.23}$$

and finally:

$$M = \Sigma\,\varrho\,\Gamma_i\,(V_x\,x_i + V_y\,y_i) \tag{4.24}$$

in accordance with (3.8).

5. Vortex Sheets. Let us assume that all vortex filaments are located on the straight line $y = 0$ between the points $-c/2 < x < c/2$. This is equivalent to a system of forces distributed along the line $y = 0$ between $x = -c/2$ and $x = c/2$. If we assume a continuous distribution of forces acting on the fluid, we must assume also a continuous distribution of vortex filaments instead of isolated vortices. This leads to the conception of a vortex sheet (see Fig. 20). The strength of the vortex sheet may be denoted by $\bar{\gamma}\,(x)$ and it is obvious by the definition of the circulation that (ε being a small quantity)[2]:

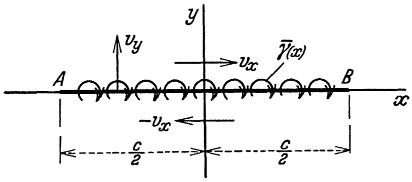

Fig. 20.

$$\bar{\gamma}\,(x) = v_x\,(x, \varepsilon) - v_x\,(x, -\varepsilon) \tag{5.1}$$

i. e. the circulation (vorticity) per unit length is equal to the difference between the velocity components taken on the two sides of the sheet

[1] *Im.* $[B\,W]$ = imaginary part of the expression $B\,W$.

[2] Compare I 9, having regard to the circumstance that in the present Chapter the positive direction for $\bar{\gamma}$ is the clockwise direction.—The forms $\bar{\gamma}\,(x)$ and $v_x\,(x, \varepsilon)$ are used to imply the vorticity at a point determined by the abscissa x, or a velocity along x at a point determined by the coordinates x, ε.

in the direction of the vortex sheet. We easily express the velocity field of such a vortex sheet by the formulae:

$$\left.\begin{aligned} v_x &= \frac{1}{2\pi}\int_{-c/2}^{c/2} \bar{\gamma}(x')\,\frac{y}{(x-x')^2+y^2}\,dx' \\ v_y &= \frac{1}{2\pi}\int_{-c/2}^{c/2} \bar{\gamma}(x')\,\frac{x'-x}{(x-x')^2+y^2}\,dx' \end{aligned}\right\} \tag{5.2}$$

The expressions for the components of the forces acting on the fluid and for their moment (with reference to the middle of the vortex sheet) are given by:

$$\left.\begin{aligned} X &= -\varrho V_y \int_{-c/2}^{c/2} \bar{\gamma}(x')\,dx' \\ Y &= \varrho V_x \int_{-c/2}^{c/2} \bar{\gamma}(x')\,dx' \\ M &= \varrho V_x \int_{-c/2}^{c/2} \bar{\gamma}(x')\,x'\,dx' \end{aligned}\right\} \tag{5.3}$$

We are especially interested in the values of the induced velocity near the line of vortices. First it is easy to show that v_x takes the value:

$$v_x = \pm\frac{\bar{\gamma}(x)}{2}\ (\text{for } y = \pm\varepsilon) \tag{5.4}$$

In fact we write:

$$v_x = \frac{1}{2\pi}\int_{-c/2}^{c/2} \bar{\gamma}(x')\,\frac{dx'/y}{1+\left(\frac{x-x'}{y}\right)^2} \tag{5.5}$$

and introducing $(x - x')/y = t$ as a new variable we obtain:

$$v_x = -\frac{1}{2\pi}\int_{\frac{x+c/2}{y}}^{\frac{x-c/2}{y}} \bar{\gamma}(x-yt)\,\frac{dt}{1+t^2} \tag{5.6}$$

If we put $y = +\varepsilon$ and go to the limit $\varepsilon \to 0$, we obviously obtain:

$$v_x = -\frac{1}{2\pi}\,\bar{\gamma}(x)\int_{\infty}^{-\infty}\frac{dt}{1+t^2} = \frac{1}{2}\,\bar{\gamma}(x) \tag{5.7}$$

while coming from $-\varepsilon$ to 0:

$$v_x = -\frac{1}{2\pi}\,\bar{\gamma}(x)\int_{-\infty}^{\infty}\frac{dt}{1+t^2} = -\frac{1}{2}\,\bar{\gamma}(x) \tag{5.8}$$

We thus obtain the result that the horizontal components of the induced velocity have equal and opposite values on the upper and lower surface of the vortex sheet.

For the following applications it is important to establish the relation between the vortex-distribution $\bar{\gamma}(x)$ and the vertical component of the induced velocity v_y. The easiest way is to use complex variables and to take advantage of the method of conformal transformation.

6. The Velocity Field of the Vortex Sheet in the Complex Form. We write the complex potential function of the vortex system formed by a continuous distribution of vortices along the line $y = 0$ between $-c/2 < x < c/2$ in the form:

$$F = \frac{i}{2\pi} \int_{-c/2}^{c/2} \bar{\gamma}(z') \log(z - z')\, dz' \quad (6.1)$$

the integral to be taken along the real axis. We transform the z plane into a ζ plane in such a way that the part of the real axis between $x = -c/2$ and $c/2$ is transformed into a circle, and the whole z plane into the domain outside of this circle. In Fig. 21 the two planes z and ζ are represented with reference to the same origin and direction of the axes. This method of representation will be used in many subsequent problems. We obtain this by using the transformation[1]:

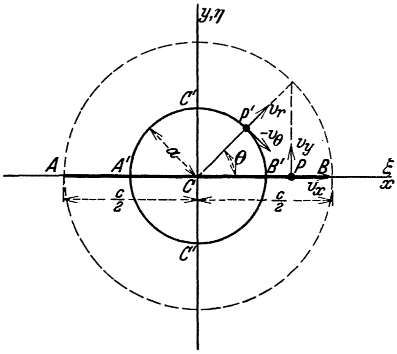

Fig. 21.

$$z = \zeta + \frac{a^2}{\zeta} \quad (6.2)$$

Introducing polar coordinates in the ζ plane:

$$\zeta = r e^{i\theta} \quad (6.3)$$

we find:

$$x + i y = r e^{i\theta} + \frac{a^2}{r} e^{-i\theta}$$

and for points $y = 0$, $-c/2 < x < c/2$:

$$\left.\begin{aligned} \theta &= \cos^{-1} \frac{2x}{c} \\ r &= \frac{c}{4} = a \end{aligned}\right\} \quad (6.4)$$

The vortex sheet is then represented in the ζ plane by singularities distributed on the circumference of radius $a = c/4$; the upper side of the vortex sheet covers the upper half circle, the lower side the lower half circle.

Let us now consider the velocity components. We call $v_\zeta = dF/d\zeta$ the "complex velocity function in the ζ plane"; v_ζ is connected with v_z, *i. e.* "the complex velocity function in the z plane", by the relation:

$$v_\zeta = v_z \frac{dz}{d\zeta} \quad (6.5)$$

[1] See Division B VI 3.

Let us assume that a line element ds in the z plane is represented by the line element $d\sigma$ in the ζ plane. Applying the principles of conformal transformation, we find the following relation between the velocity components v_s and v_σ, *i. e.* between the velocity components respectively in the s and in the σ direction,

$$v_\sigma = v_s \left| \frac{dz}{d\zeta} \right| \tag{6.6}$$

The expression $|dz/d\zeta|$ is a pure number which we may call the "measure of the conformal transformation".

Considering the velocity components on the upper and lower sides of the vortex sheet, the corresponding components will be respectively v_x and $-v_\theta$, v_y and v_r for the upper surface and upper half circle, and again v_x and v_θ, v_y and $-v_r$ for the lower surface and lower half circle. Since $v_x(x, \varepsilon) = -v_x(x, -\varepsilon)$ but $v_y(x, \varepsilon) = v_y(x, -\varepsilon)$, v_θ will be an even and v_r an odd function of θ.

From (6.2) we have: $\dfrac{dz}{d\zeta} = 1 - \dfrac{a^2}{\zeta^2}$, or, for $\zeta = ae^{i\theta}$:

$$\frac{dz}{d\zeta} = 1 - e^{-2i\theta} = 2\,i\,e^{-i\theta} \sin\theta \tag{6.7}$$

This gives $|dz/d\zeta| = |2 \sin\theta|$ and:

$$\left.\begin{aligned} v_r &= +v_y\,|2\sin\theta| \\ v_\theta &= -v_x\,|2\sin\theta| \end{aligned}\right\} \text{ for the upper half circle}$$

$$\left.\begin{aligned} v_r &= -v_y\,|2\sin\theta| \\ v_\theta &= +v_x\,|2\sin\theta| \end{aligned}\right\} \text{ for the lower half circle}$$

or for the whole circle:

$$\left.\begin{aligned} v_r &= 2\,v_y \sin\theta \\ v_\theta &= -2\,v_x \sin\theta \end{aligned}\right\} \tag{6.8}$$

We now write the potential function (6.1) in the form of a series with decreasing powers of z:

$$F = \frac{i\Gamma}{2\pi} \log z - \sum_1^\infty \frac{i A_k}{z^k} \tag{6.9}$$

We then develop $\log(z - z')$ in (6.1) and find by comparison with (6.9):

$$\Gamma = \int_{-c/2}^{c/2} \bar{\gamma}(z')\,dz' \tag{6.10}$$

and

$$A_k = \frac{1}{2\pi} \int_{-c/2}^{c/2} \frac{z'^k}{k} \bar{\gamma}(z')\,dz' \tag{6.11}$$

Introducing ζ into (6.9) by the transformation (6.2) we obtain a similar series, which we write in the following form:

$$F = \frac{i\Gamma}{2\pi} \log\zeta - \sum_1^\infty \frac{i B_k}{\zeta^k} \tag{6.12}$$

where the B_k like the A_k are real coefficients. The real part of F is the real potential:
$$\varphi = -\frac{\Gamma}{2\pi}\theta - \sum_1^\infty \frac{B_k}{r^k} \sin k\theta \tag{6.13}$$
and we obtain by differentiation the velocity components v_r and v_θ:
$$\left.\begin{aligned} v_r &= \frac{\partial \varphi}{\partial r} = \sum_1^\infty \frac{k B_k}{r^{(k+1)}} \sin k\theta \\ v_\theta &= \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{\Gamma}{2\pi r} - \sum_1^\infty \frac{k B_k}{r^{(k+1)}} \cos k\theta \end{aligned}\right\} \tag{6.14}$$
Using the relations (6.8) we obtain for $r = a$:
$$\left.\begin{aligned} v_y &= \sum_1^\infty \frac{k B_k}{a^{(k+1)}} \frac{\sin k\theta}{2 \sin\theta} \\ v_x &= \frac{\Gamma}{4\pi a \sin\theta} + \sum_1^\infty \frac{k B_k}{a^{(k+1)}} \frac{\cos k\theta}{2\sin\theta} \end{aligned}\right\} \tag{6.15}$$
Equations (6.15) represent a relation between the strength of the vortex sheet $\bar{\gamma}(x) = v_x(x, \varepsilon) - v_x(x, -\varepsilon)$ and the induced velocity v_y.

Obviously
$$\bar{\gamma} = \frac{\Gamma}{2\pi a \sin\theta} + \sum_1^\infty \frac{k B_k}{a^{(k+1)}} \frac{\cos k\theta}{\sin\theta} \tag{6.16}$$
(where $0 < \theta < \pi$) together with $x = 2a \cos\theta$ determine the vorticity distribution $\bar{\gamma}(x)$, if the coefficients B_k in the development for v_y are given and *vice versa*. However, it may be noted that in the determination of $\bar{\gamma}$ an additional term containing an arbitrary value of the circulation Γ remains undetermined.

7. The Plane Airfoil. The flow around a plane airfoil of infinite span (the basis of which is found in Division B V and VI) represents a simple example of a "bound vortex sheet". Obviously we can build up the flow corresponding to the angle of attack α by the superposition of a parallel flow in the direction of the chord with the velocity $V \cos\alpha$ at infinity and a flow perpendicular to the chord with the velocity $V \sin\alpha$ at infinity. The latter alone gives infinite velocity at the trailing edge and contributes to the ultimate circulation.

We first calculate the vorticity distribution which, combined with the rectilinear flow $V \sin\alpha$, produces the potential function corresponding to the infinite flow perpendicular to the airfoil with zero circulation. Denoting the latter by $F(z)$ and the potential corresponding to the vortex sheet by $F_1(z)$, we have:
$$F(z) = -i\,Vz \sin\alpha + F_1(z),$$
or:
$$F_1(z) = F(z) + i\,Vz\sin\alpha. \tag{7.1}$$

For this calculation it is more convenient to transform to the ζ plane, in which the airfoil appears as a circle. In this plane the expression for the potential corresponding to the flow around the circle is:

$$F(\zeta) = -i\,V\left(\zeta - \frac{a^2}{\zeta}\right) \sin\alpha \tag{7.2}$$

so that the potential function due to the vortex system is given by:

$$F_1 = F + i\,V\,z \sin\alpha = i\,V\,(z - \zeta)\sin\alpha + i\,V\frac{a^2}{\zeta}\sin\alpha \tag{7.3}$$

or with $z - \zeta = \frac{a^2}{\zeta}$:

$$F_1 = 2\,i\,V\frac{a^2}{\zeta}\sin\alpha \tag{7.4}$$

The corresponding real potential is given by:

$$\varphi = 2\,V\frac{a^2}{r}\sin\alpha\sin\theta \tag{7.5}$$

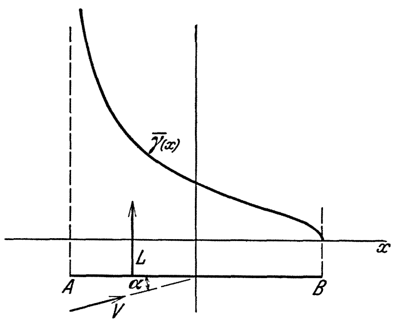

Fig. 22.

and the velocity components in the z plane, using (6.13) and (6.15) with $B_1 = -2\,V a^2 \sin\alpha$, $B_2 = B_3 = \ldots = 0$, take the form:

$$\left.\begin{aligned} v_x &= \frac{\Gamma}{4\,\pi\,a\sin\theta} - V\sin\alpha\cot\theta \\ v_y &= \qquad\qquad - V\sin\alpha \end{aligned}\right\} \tag{7.6}$$

The corresponding vorticity distribution is therefore:

$$\bar{\gamma} = \frac{\Gamma}{2\,\pi\,a\sin\theta} - 2\,V\sin\alpha\cot\theta \tag{7.7}$$

The velocity v_x has in general an infinite value for $\theta = 0$ and $\theta = \pi$, *i. e.* at the leading and the trailing edge. We assume now that the additional circulatory motion with the circulation Γ is determined by the condition of finite velocity at the trailing edge[1]. The value of Γ which compensates the infinite velocity at the trailing edge is obviously:

$$\Gamma = 4\,\pi\,V a\sin\alpha = \pi\,V c\sin\alpha \tag{7.8}$$

The resultant vorticity distribution is therefore:

$$\bar{\gamma} = 2\,V\sin\alpha\frac{1-\cos\theta}{\sin\theta} \tag{7.9}$$

or:

$$\bar{\gamma} = 2\,V\sin\alpha\sqrt{\frac{c/2 - x}{c/2 + x}} \tag{7.10}$$

The vorticity distribution according to (7. 10) is shown in Fig. 22.

The lift and the moment with respect to $x = 0$ are given by[2]:

[1] See I 8.

[2] The values for lift and moment given here (and throughout the present Chapter) refer to unit span. In the theory of the airfoil of finite span the letter L is reserved for the total lift, the lift per unit span being denoted by l (see for instance I 6).

$$\left.\begin{aligned} L &= \varrho \Gamma V = \pi \varrho c V^2 \sin \alpha \\ M &= \varrho V \int_{-c/2}^{c/2} \bar{\gamma}(x)\, x\, d x = -\frac{\pi \varrho c^2}{4} V^2 \sin \alpha \end{aligned}\right\} \qquad (7.11)$$

The center of pressure is therefore at the distance $c/4$ from the leading edge, as indicated in Fig. 22.

We introduce now the coefficients for lift and moment:

$$\left.\begin{aligned} C_L &= \frac{L}{\frac{\varrho V^2}{2} c} \\ C_M &= \frac{M}{\frac{\varrho V^2}{2} c^2} \end{aligned}\right\} \qquad (7.12)$$

and likewise the coefficient C_m for the moment with reference to the leading edge:

$$C_m = \frac{M + L \frac{c}{2}}{\frac{\varrho V^2}{2} c^2} = C_M + \frac{1}{2} C_L \qquad (7.13)$$

We thus obtain:

$$\left.\begin{aligned} C_L &= 2\pi \sin \alpha \\ C_M &= -\frac{\pi}{2} \sin \alpha \\ C_m &= \frac{\pi}{2} \sin \alpha \end{aligned}\right\} \qquad (7.14)$$

8. Theory of Thin Wing Sections (Thin Airfoils). An approximate theory of wing sections can be developed by replacing the airfoil by a curved line, approximately the mean of the upper and lower surfaces, and regarding this curved line as having small deviation from a straight line. An indefinitely thin airfoil is obviously equivalent to a sheet of vortices distributed along the line representing the section of the airfoil. If the curvature of the mean line is small, we may even assume that the flow is not very different from that due to a system of vortex filaments continuously distributed along the straight line which represents the chord of the airfoil. Of course, this conception has only practical value if we are able to find an approximate relation between the shape of the mean line of the airfoil and the vorticity distribution (or, strength of the vortex sheet) $\bar{\gamma}(x)$.

For this purpose we try to express the condition that the flow resulting from the combination of the parallel flow and the flow induced by the vortex sheet shall follow the slightly curved mean line of the airfoil.

The shape of the curved mean line may be given by the function $y = f(x)$ where for $x = \pm c/2$, $y = 0$. If we apply the conformal transformation

$$z = \zeta + \frac{a^2}{\zeta} \qquad (8.1)$$

with $a = c/4$, we obtain on the ζ plane as the representation of the slightly curved line $A\,B$ in the z plane, a curve which is slightly different from a circle with the radius a. (In Fig. 23 the curve resulting from the transformation of the arc $A\,B$ is, for convenience, represented as a circle itself. In general the transformed curve will have a slightly different shape.) It is easy to show that if y is small compared with a, the equation of the corresponding curve in the ζ plane will be:

$$r = a + \frac{y}{2\,sin\;\theta} \tag{8.2}$$

The resulting flow in the z plane is built up from the parallel flow and from the velocity field of the vortex sheet. Let V be the velocity of the parallel flow and, to take into account the angle of attack, let α be the inclination of V with respect to the real axis. We assume that α is a small angle. Considering now the flow in the ζ plane, we have the following contributions: (a) the flow around the circle, with the velocity components $\bar{v}_r$, $\bar{v}_\theta$ ($\bar{v}_r$ being zero on the circle itself); (b) an additional flow with the velocity components v'_r, v'_θ. If the curve $r = g\;(\theta)$ is a stream-line, the condition of continuity requires that

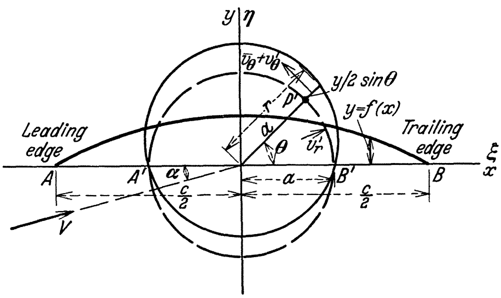

Fig. 23.

$$\int_a^r (\bar{v}_\theta + v'_\theta)\,dr = a\int_0^\theta v'_r\,d\,\theta \tag{8.3}$$

We take now for all velocity components the values corresponding to the circumference of the circle $r = a$. With this approximation, we obtain:

$$(\bar{v}_\theta + v'_\theta)\;(r - a) = a\int_0^\theta v'_r\,d\,\theta \tag{8.4}$$

or:

$$v'_r = \frac{d}{d\,\theta}\left[(\bar{v}_\theta + v'_\theta)\,\frac{r - a}{a}\right]\,. \tag{8.5}$$

Now the velocity due to the flow around the circle is given by:

$$\bar{v}_\theta = -\,2\;V\,sin\;(\theta - \alpha) \tag{8.6}$$

so that:

$$\bar{v}_\theta + v'_\theta = -\,2\;V\,sin\;\theta\;cos\;\alpha + 2\;V\,cos\;\theta\;sin\;\alpha + v'_\theta \tag{8.7}$$

Assuming $sin\;\alpha << 1$ and $v'_\theta << V$, and keeping in mind that in (8.5) $r - a << a$, we neglect the latter terms of (8.7) in comparison with the first term. This is justified at all points excepting $\theta = 0$ and $\theta = \pi$, *i. e.* at the edges. We thus obtain:

$$v'_r = -\,2\;V\,\frac{d}{d\,\theta}\left[\frac{r - a}{a}\;sin\;\theta\right] \tag{8.8}$$

Now according to (8.2), $(r - a) \sin\theta = y/2$ and we have

$$v'_r = -\frac{V}{a}\frac{dy}{d\theta} \tag{8.9}$$

We conclude therefore that to a first approximation, the additional velocities which produce the distortion of the mean airfoil line from a straight line into the actual shape are independent of the angle of attack. This result simplifies the problem to a great extent. It means that in this first approximation the influence of the angle of attack is independent of the shape of the airfoil. It is therefore allowable to superpose the flow around a curved airfoil corresponding to zero angle of attack upon the flow around a plane airfoil corresponding to the given angle of attack, and the resultant flow will represent approximately the flow around the curved shape with the same angle of attack. Hence it is sufficient to consider the case $\alpha = 0$ and to calculate the characteristics of the airfoils for this particular case. The characteristics for an arbitrary angle of attack are then obtained by adding the terms deduced in 7 for the plane airfoil.

Let us assume that the shape of the airfoil is given by the series:

$$y = f(x) = \frac{c}{2}\Sigma\beta_k \cos(k\theta) \tag{8.10}$$

The β_k are restricted by the condition that $y = 0$ for $\theta = 0$ and $\theta = \pi$.

Then (8.9) gives us the velocity distribution v'_r in terms of the β coefficients,

$$v'_r = 2V\Sigma k\beta_k \sin(k\theta) \tag{8.11}$$

and for the coefficients B_k in the expressions (6.14):

$$B_k = 2a^{(k+1)}\beta_k V \tag{8.12}$$

The corresponding vorticity distribution will be, according to (6.16):

$$\bar{\gamma}(x) = \frac{\Gamma}{2\pi a \sin\theta} + 2V\sum_1^\infty k\beta_k \frac{\cos k\theta}{\sin\theta} \tag{8.13}$$

The condition of smooth flow at the trailing edge requires the determination of the circulation in such manner that the vorticity vanishes for $\theta = 0$. We fulfill this condition by putting:

$$\Gamma = -4\pi a V\sum_1^\infty k\beta_k \tag{8.14}$$

and obtain the resulting vorticity distribution:

$$\bar{\gamma}(x) = -2V\sum_1^\infty k\beta_k \frac{1-\cos k\theta}{\sin\theta} \tag{8.15}$$

Equations (8.11) and (8.15) represent the main results of the theory of thin airfoils.

We introduce (8.14) and (8.15) in the formulae for lift and moment and, remembering that $x = 2a\cos\theta$, we obtain easily:

$$L = \varrho \Gamma V = -4\pi a \varrho V^2 \Sigma k \beta_k \qquad (8.16)$$

$$M = \varrho V \int_{-c/2}^{c/2} \bar{\gamma}(x)\, x\, dx = -8\varrho V^2 a^2 \int_0^{\pi} \Sigma k \beta_k (1 - \cos k\theta) \cos\theta\, d\theta \qquad (8.17)$$

All terms under the integral sign vanish except the term with $k = 1$ and we obtain finally: $M = 4\pi \varrho V^2 a^2 \beta_1$ (8.18)

The values of the coefficients, which we denote by $C_{L\,0}$, $C_{M\,0}$ and $C_{m\,0}$ are given by:

$$\left.\begin{aligned} C_{L\,0} &= -2\pi \sum_1^{\infty} k\beta_k \\ C_{M\,0} &= \frac{\pi}{2}\beta_1 \\ C_{m\,0} &= \frac{\pi}{2}\beta_1 - \pi \sum_1^{\infty} k\beta_k \end{aligned}\right\} \qquad (8.19)$$

For the general case of an angle of attack α, we have to add the terms obtained for a plane airfoil to the expressions (8.19). Hence the final formulae will be ($\sin\alpha \cong \alpha$):

$$\left.\begin{aligned} C_L &= 2\pi\left(\alpha - \sum_1^{\infty} k\beta_k\right) \\ C_M &= -\frac{\pi}{2}(\alpha - \beta_1) \\ C_m &= \frac{\pi}{2}(\alpha + \beta_1) - \pi \sum_1^{\infty} k\beta_k \end{aligned}\right\} \qquad (8.20)$$

We transform the last formula introducing C_L instead of α and obtain:

$$C_m = -\frac{\pi}{2}\left(\sum_1^{\infty} k\beta_k - \beta_1\right) + \frac{C_L}{4} = C_{m\,0} - \frac{C_{L0}}{4} + \frac{C_L}{4} \qquad (8.21)$$

We have thus the following results:

(*a*) The lift coefficient C_L of a thin airfoil for small angle of attack appears as the sum of a constant value $C_{L\,0}$, which is independent of the angle of attack and is determined only by the shape of the airfoil, and of the value $2\pi\alpha$, which corresponds to the lift coefficient of a plane surface.

(*b*) The coefficient of the moment C_m is composed of the constant value $C_\mu = C_{m\,0} - C_{L\,0}/4$ and the value $C_L/4$, thus:

$$C_m = C_\mu + C_L/4 \qquad (8.22)$$

Except for the case $C_L = 0$, the lift and moment can be represented by a single force acting at the distance e from the leading edge, at the so-called "center of pressure". Obviously:

$$e = c\,\frac{C_m}{C_L} \qquad (8.23)$$

If C_μ vanishes, the center of pressure is independent of the angle of attack and lies at the distance $c/4$ from the leading edge. This corresponds to a neutral equilibrium of the airfoil. If $C_\mu > 0$, the center of pressure travels toward the leading edge with increasing angle of attack, which means instability. The condition for stability is therefore, for an airfoil without tail surface:

$$C_\mu < 0, \text{ or } \left(\sum_1^\infty k\,\beta_k - \beta_1\right) > 0 \tag{8.24}$$

9. Munk's Integral Formulae for the Lift and Moment of a Thin Airfoil. It was shown in the last section, that the relation between the shape of the slightly curved thin airfoil and the velocity field of the corresponding vortex sheet produced by a parallel flow with zero angle of attack can be expressed by the equation:

$$V \frac{d\,y}{d\,\theta} = -v_r'\,a \tag{9.1}$$

Introducing again the velocity components in the z plane, we obtain:

$$V \frac{d\,y}{d\,\theta} = -2\,V \frac{d\,y}{d\,x}\,a \sin\theta = -2\,v_y'\,a \sin\theta \tag{9.2}$$

or:

$$\frac{d\,y}{d\,x} = \frac{v_y'}{V} \tag{9.3}$$

This equation can be interpreted as showing that the mean inclination of the flow on the upper and lower surface of the airfoil is measured by v_y'/V. In fact the inclination of the stream-line at the upper surface is:

$$\tan\beta_1 = \frac{v_y'}{V + v_x'},$$

and at the lower surface:

$$\tan\beta_2 = \frac{v_y'}{V - v_x'},$$

and so long as v_x' is small compared with V, the value v_y'/V may be taken as the mean value. Then (9.3) states that the mean inclination of the flow in the neighborhood of a thin airfoil is given by the inclination of the airfoil's center line.

Considering the case of a small angle of attack α, the induced velocity v_y is composed of the amount $v_y' = V\,(d\,y/d\,x)$, corresponding to the curved shape of the airfoil and of the amount $-V\alpha$, corresponding to the plane airfoil [see (7.6), second line]. Consequently we replace (9.3) by:

$$v_y = V\left(\frac{d\,y}{d\,x} - \alpha\right) \tag{9.4}$$

We can give a slightly different interpretation to this equation, expressing the condition that the velocity component in the direction normal to the curved center line of the airfoil shall vanish. Using the letter v_n for the velocity component normal to the line $y = f(x)$, the values become (with the approximation corresponding to small curvatures and small angles of attack)

$$v_{n1} = -(V + v_x)\frac{dy}{dx} + v_y + V\alpha$$

$$v_{n2} = -(V - v_x)\frac{dy}{dx} + v_y + V\alpha$$

at the upper and lower surfaces respectively. The condition that the mean value shall vanish gives, in accordance with (9.4):

$$-V\frac{dy}{dx} + v_y + V\alpha = 0 \qquad (9.5)$$

It is understood that we use the assumption that the calculation of the velocities can be carried out at the straight line, instead of at the curved mean line itself.

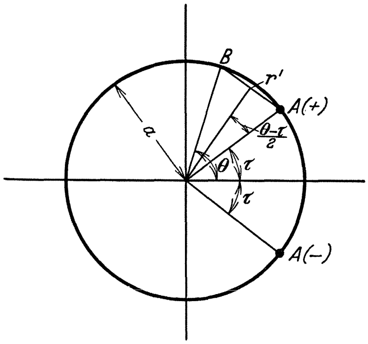

Fig. 24.

Hence the problem of finding the vorticity distribution corresponding to a given shape of airfoil is equivalent to the problem of finding a potential function $\varphi(x, y)$ extended over the whole plane x, y, and regular over the whole plane except only the part of the x axis between $-c/2 < x < +c/2$. The component $v_x = \partial\varphi/\partial x$ is discontinuous when we pass from the upper side to the lower side of this part of the x axis and the values of v_y are given by the boundary condition

$$v_y = \frac{\partial\varphi}{\partial y} = V\left(\frac{dy}{dx} - \alpha\right) \qquad (9.6)$$

This problem is an example of the so-called second type of boundary problems in potential theory. To find the solution without elaborate calculations, we refer to the ζ plane introduced in the last section and put the problem in the following way: We have to find a potential function $\varphi(r, \theta)$ which is regular outside of the circle $r = a$, the differential quotient $\partial\varphi/\partial r$ at the circumference of this circle assuming given values $v_r(\theta)$.

We easily find the solution of the problem in the following way (see Fig. 24). Let us assume a source A (+) at the point $\theta = \tau$ of the circle, and a sink of the same strength A (—) at the point $\theta = -\tau$. It is known that the circle will be one of the stream-lines of the corresponding motion, so that if we assume a continuous distribution of sources and sinks over the whole circle, with the strength per unit length equal to $2v_r$, we shall obtain a potential motion which fulfills the above mentioned conditions.

We remember that the potential corresponding to a source with the strength Q can be written in the form:

$$\varphi = \frac{Q}{2\pi} \log r' \tag{9.7}$$

where r' denotes the distance of an arbitrary point B in the plane from the source. Hence the potential for a distribution of sources with the intensity $2\, v_r$ per unit length along the circumference of the circle $r = a$ is given by:

$$\varphi = \frac{a}{\pi} \int_0^{2\pi} v_r \log r' \, d\tau \tag{9.8}$$

We notice that v_r is an odd function of τ. For points B on the circle with coordinates a, θ the distance r' is $2\, a \sin\, (\theta - \tau)/2$; and as the term $\log 2a \int_0^{2\pi} v_r\, d\tau$ vanishes, we obtain for φ:

$$\varphi = \frac{a}{\pi} \int_0^{2\pi} v_r \log \sin \frac{\theta - \tau}{2} \, d\tau \tag{9.9}$$

Hence the tangential component of the velocity v is given by:

$$v_\theta = \frac{1}{a} \frac{\partial \varphi}{\partial \theta} = \frac{1}{2\pi} \int_0^{2\pi} v_r \cot \frac{\theta - \tau}{2} \, d\tau \tag{9.10}$$

According to the assumption of smooth flow at the trailing edge, v_θ for $\theta = 0$ must be equal and opposite to the velocity induced by the circulation Γ. Hence we must have:

$$-\frac{1}{2\pi} \int_0^{2\pi} v_r \cot \frac{\tau}{2} \, d\tau = \frac{\Gamma}{2\pi a} \tag{9.11}$$

and the circulation Γ is given by:

$$\Gamma = -a \int_0^{2\pi} v_r \cot \frac{\tau}{2} \, d\tau \tag{9.12}$$

Now according to (6.8) and (9.6):

$$v_r = 2\, v_y \sin \tau = 2\, V \left(\frac{d\,y}{d\,x} - \alpha \right) \sin \tau \tag{9.13}$$

Obviously, it is sufficient to work out the case $\alpha = 0$. The solution for any given angle of attack can then be completed by adding the terms obtained for a plane airfoil. We thus put

$$v_r = 2\, V \frac{d\,y}{d\,x} \sin \tau$$

and obtain:

$$\Gamma = -2\, a\, V \int_0^{2\pi} \frac{d\,y}{d\,x} \cot \frac{\tau}{2} \sin \tau \, d\tau = -2\, a\, V \int_0^{2\pi} \frac{d\,y}{d\,x} (1 + \cos \tau)\, d\tau \tag{9.14}$$

We transform (9.14) through integration by parts. We first write:

$$\Gamma = -4aV \int_0^\pi \frac{dy}{dx}(1 + \cos\tau)\, d\tau \tag{9.15}$$

Then introducing $\frac{dy}{dx} = \frac{dy}{d\tau}\frac{d\tau}{dx} = -\frac{dy}{d\tau}\frac{1}{2a \sin\tau}$, we obtain:

$$\Gamma = 2V \int_0^\pi \frac{dy}{d\tau}\frac{1 + \cos\tau}{\sin\tau}\, d\tau \tag{9.16}$$

and then integrating by parts:

$$\Gamma = 2V \left[y \frac{1 + \cos\tau}{\sin\tau} \right]_0^\pi + 2V \int_0^\pi y \frac{d\tau}{1 - \cos\tau}$$

The first term vanishes[1] and we obtain finally:

$$\Gamma = 2V \int_0^\pi y \frac{d\tau}{1 - \cos\tau} \tag{9.17}$$

or with $t = \cos\tau = 2\, x/c$:

$$\Gamma = 2V \int_{-1}^{1} y \frac{dt}{(1 - t)(1 - t^2)^{1/2}} \tag{9.18}$$

We have now to calculate the vorticity distribution and the moment. From (5.1) and (6.8) we have as a general expression for $\bar{\gamma}$:

$$\bar{\gamma} = -\frac{v_\theta}{\sin\theta} \tag{9.19}$$

as, however, in the expression (9.10) for v_θ the amount due to the circulation, *i. e.* $-\Gamma/2\pi a$, is not included, we must add a term $\Gamma/2\pi a \sin\theta$ to the right hand side of (9.19). At the same time putting $v_r = 2\, V\, (dy/dx) \sin\tau$, we obtain:

$$\bar{\gamma} = -\frac{V}{\pi \sin\theta} \int_0^{2\pi} \frac{dy}{dx} \cot\frac{\theta - \tau}{2} \sin\tau\, d\tau + \frac{\Gamma}{2\pi a \sin\theta} \tag{9.20}$$

The moment M may be calculated by integration of the expression $M = \varrho\, V \int_{-c/2}^{c/2} \bar{\gamma}(x)\, x\, dx$, with (9.20) under the integral sign. However, it is easier to calculate the moment using the general formula (4.22).

[1] This is certainly true for $\tau = \pi$ (leading edge), as here both $y = 0$ and $(1 + \cos\tau)/\sin\tau = 0$. At the trailing edge, where $\tau = 0$, we have $1 + \cos\tau = 2$; hence it is necessary that here y vanish more rapidly than $\sin\tau = \sqrt{1 - 4\, x^2/c^2}$, or, what comes to the same, that y vanish more rapidly than $\sqrt{c - 2\, x}$. This condition is always fulfilled if, at the trailing edge, the tangent of the angle between the airfoil and the chord is not infinite.

The complex potential function corresponding to the real potential (9.8) is given by:

$$F = \frac{a}{\pi} \int_0^{2\pi} v_r(\tau) \log (\zeta - a e^{i\tau}) d\tau \tag{9.21}$$

and the velocity function is:

$$\frac{dF}{dz} = \frac{dF}{d\zeta}\frac{d\zeta}{dz} = \frac{a}{\pi}\frac{d\zeta}{dz} \int_0^{2\pi} \frac{v_r(\tau)}{\zeta - a e^{i\tau}} d\tau \tag{9.22}$$

The moment is $M = -2\pi\varrho V Im. [B]$, where $-B$ denotes the factor of $1/z^2$ in the development of dF/dz in a power series in powers of $1/z$.

We obtain easily: $B = -\frac{a^2}{\pi}\int_0^{2\pi} e^{i\tau} v_r(\tau)\, d\tau$. Hence:

$$M = 2\varrho a^2 V \int_0^{2\pi} v_r(\tau) \sin\tau\, d\tau \tag{9.23}$$

Substituting $v_r = 2V\frac{dy}{dx}\sin\tau = -\frac{V}{a}\frac{dy}{d\tau}$, we obtain:

$$\left.\begin{aligned} M &= -2\varrho a V^2 \int_0^{2\pi} \frac{dy}{d\tau} \sin\tau\, d\tau = 2\varrho a V^2 \int_0^{2\pi} y \cos\tau\, d\tau = \\ &= 4\varrho a V^2 \int_0^{\pi} y\cos\tau\, d\tau \end{aligned}\right\} \tag{9.24}$$

or introducing again $t = \cos\tau$:

$$M = 4\varrho a V^2 \int_{-1}^{1} \frac{y\,t\,dt}{(1-t^2)^{1/2}} \tag{9.25}$$

In the case of the plane airfoil it was found that the lift and moment are given by:

$$L = 4\pi\varrho a V^2 \alpha,$$
$$M = -4\pi\varrho a^2 V^2 \alpha$$

[see (7.11) and put $c = 4a$]; hence, by addition:

$$\left.\begin{aligned} L &= 2\varrho a V^2 \left[2\pi\alpha + \int_{-1}^{1} \frac{y}{a} \frac{dt}{(1-t)(1-t^2)^{1/2}}\right] \\ M &= 4\varrho a^2 V^2 \left[-\pi\alpha + \int_{-1}^{1} \frac{y}{a} \frac{t\,dt}{(1-t^2)^{1/2}}\right] \end{aligned}\right\} \tag{9.26}$$

The coefficients C_L, C_M and C_m thus will be:

$$\left.\begin{aligned} C_L &= 2\pi\alpha + 4\int_{-1}^{1} \frac{y}{c} \frac{dt}{(1-t)(1-t^2)^{1/2}} \\ C_M &= -\frac{\pi\alpha}{2} + 2\int_{-1}^{1} \frac{y}{c} \frac{t\,dt}{(1-t^2)^{1/2}} \\ C_m &= \frac{\pi\alpha}{2} + \int_{-1}^{1} \frac{1+t-t^2}{(1-t)(1-t^2)^{1/2}} \frac{2y}{c}\,dt \end{aligned}\right\} \tag{9.27}$$

Equations (9.26) are equivalent to (8.16) and (8.18), and Eqs. (9.27) are equivalent to (8.20). The methods used in 8 and **9** represent two different procedures for solving the boundary problem for the circle; in 8, Fourier-series; in **9**, sources and sinks. Of course, the results of **9** can be deduced immediately from those in 8. However, since the deduction of the first equation in (9.26) from (8.16) requires a somewhat lengthy calculation of an improper integral, preference is given in **9** to the "source and sink" method.

10. Simple Types of Thin Airfoils. General Discussion. Taking the few first terms in the development of the function which represents approximately the shape of the center line of the airfoil, we obtain certain typical cases of airfoils of simple form and we shall now discuss the resulting formulae for lift, moment and vorticity distribution. The vorticity distribution $\bar{\gamma}(x)$ determines approximately the lift distribution, *i. e.* the force Y acting on unit length of the vortex sheet, which represents the airfoil. Applying (3.13) to the case of the vortex sheet and taking into account (5.7) and (5.8) we obtain[1]:

$$Y = \varrho V \bar{\gamma} \tag{10.1}$$

Due to the approximate character of the theory great accuracy for the lift distribution cannot be claimed; however, the expressions for the resultant lift and moment are in fairly good accordance with the results of exact theory.

We use the formulae[2]:

$y = f(x) = \frac{c}{2}\Sigma \beta_k \cos(k\theta)$	for the mean line	(10.2)
$\bar{\gamma} = -2V\Sigma k\beta_k \frac{1-\cos k\theta}{\sin\theta}$	for the vorticity distribution corresponding to $\alpha = 0$	(10.3)
$C_L = 2\pi(-\Sigma k\beta_k + \alpha)$	for the lift coefficient	(10.4)
$C_m = -\pi\Sigma k\beta_k + \frac{\pi}{2}(\alpha + \beta_1)$	for the coefficient of the moment with reference to the leading edge	(10.5)

[1] See also I (9.8). To the degree of approximation as used in the present discussion the mean velocity v_m appearing in that equation is equal to V.

[2] See, respectively: (8.10), (8.15), (8.20).

Besides the plain airfoil, which has been treated in **7**, two typical simple cases are of outstanding interest.

(*a*) *The Airfoil with Parabolic Shape* (see Fig. 25). We put $\beta_0 = f/c$, $\beta_2 = -f/c$, $\beta_1 = \beta_3 = \beta_4 = \ldots = 0$. The corresponding expression $y = (f/2)(1 - \cos 2\theta)$ represents an airfoil with parabolic shape with the height f. In the following calculations we write $c/2 = 2a = b$ and, with $(1 - \cos 2\theta)/2 = 1 - x^2/b^2$,

$$y = f\left(1 - \frac{x^2}{b^2}\right) \tag{10.6}$$

The vorticity distribution corresponding to $\alpha = 0$ is given by:

$$\bar{\gamma} = 2\left(\frac{Vf}{b}\right)\frac{1 - \cos 2\theta}{\sin\theta} = 4\left(\frac{Vf}{b}\right)\sin\theta \tag{10.7}$$

and the coefficients C_L and C_m by:

$$\left.\begin{aligned} C_L &= 2\pi\left(\frac{f}{b} + \alpha\right) \\ C_m &= \pi\frac{f}{b} + \frac{\pi}{2}\alpha = \frac{\pi}{2}\frac{f}{b} + \frac{C_L}{4} \end{aligned}\right\} \tag{10.8}$$

The formulae for C_L and C_m are identical with those obtained by the exact theory for the circular airfoil, considering the curvature small (see **16**, **17**). The effective angle of attack is the angle of attack increased by f/b (b is equal to the half of the chord). The axis corresponding to the angle of attack for zero lift passes through the peak of the airfoil and the trailing edge.

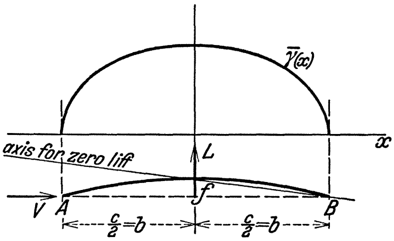

Fig. 25.

(*b*) *The Airfoil with S-shape* (see Fig. 26). We write:

$$y = -\frac{h}{2}(\cos\theta - \cos 3\theta) \tag{10.9}$$

or:

$$y = -2h\left(1 - \frac{x^2}{b^2}\right)\frac{x}{b} \tag{10.10}$$

which represents an airfoil with anti-symmetrical shape, like an S. In this case we have

$$\beta_1 = -\frac{h}{2b}, \qquad \beta_2 = 0, \qquad \beta_3 = +\frac{h}{2b}$$

The vorticity distribution corresponding to $\alpha = 0$ is given by:

$$\bar{\gamma} = \frac{Vh}{b}\left[\frac{1 - \cos\theta}{\sin\theta} - 3\,\frac{1 - \cos 3\theta}{\sin\theta}\right] \tag{10.11}$$

and the coefficients of lift and moment by

$$\left.\begin{aligned} C_L &= 2\pi\left(-\frac{h}{b} + \alpha\right) \\ C_m &= \pi\left(-\frac{5h}{4b} + \frac{\alpha}{2}\right) = -\frac{3\pi h}{4b} + \frac{C_L}{4} \end{aligned}\right\} \tag{10.12}$$

We notice that $C_\mu < 0$, so that an anti-symmetrical shape is favorable for the stability of the airfoil.

We may obtain a general view of the properties of thin airfoils, composing a series of airfoil shapes, by superposition of the parabolic arc and the S-curve of the third order.

We write: $$y = b\,(\beta_0 + \beta_1 \cos\theta - \beta_0 \cos 2\theta - \beta_1 \cos 3\theta) \qquad (10.13)$$

or: $$y = 2\,b\left(1 - \frac{x^2}{b^2}\right)\left(\beta_0 + 2\beta_1 \frac{x}{b}\right) \qquad (10.14)$$

It should be noted that while (10.13) gives the combination of a parabola and an S-curve, it is not in the form of (10.2) with regard to the sequence of the k's. Hence in applying (10.4) and (10.5), it must be remembered that the k for β_0 in the first term is zero while for the third term the k is 2 and similarly the k for β_1 in the second term is 1 and for the fourth term, 3. We may then express the constants f and h by the angles γ_1 and γ_2 between the mean line and the chord at the leading and at the trailing edge. We put $\gamma_1 = \left(\frac{dy}{dx}\right)_{\theta=\pi}$ and $\gamma_2 = -\left(\frac{dy}{dx}\right)_{\theta=0}$, both angles having positive values for an airfoil with convex center line.

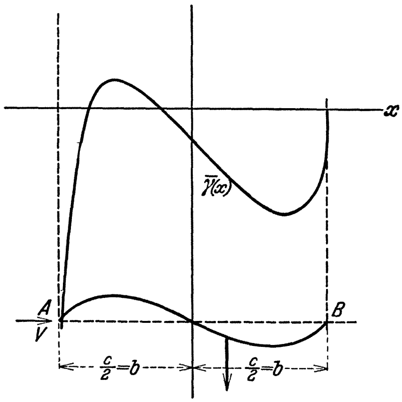

Fig. 26.

The expression $(\gamma_1 + \gamma_2)/2\,b = \varkappa$ may be considered as the mean curvature of the airfoil while the difference $\gamma_1 - \gamma_2$ divided by $(\gamma_1 + \gamma_2)$ is a measure of the asymmetrical shape. We obtain by differentiation:

$$\left.\begin{aligned}\gamma_1 &= 4\,(\beta_0 - 2\beta_1)\\ \gamma_2 &= 4\,(\beta_0 + 2\beta_1)\end{aligned}\right\} \qquad (10.15)$$

and: $$\left.\begin{aligned}\beta_0 &= \frac{\gamma_1+\gamma_2}{8} = \frac{b\,\varkappa}{4}\\ \beta_1 &= \frac{\gamma_2-\gamma_1}{16}\end{aligned}\right\} \qquad (10.16)$$

Putting the values of (10.16) in (10.4) and (10.5) and remembering the caution above with regard to the k's, we obtain the following expressions for the coefficients of lift and moment, which may be considered as approximately true for airfoils in general as far as they are not very different in shape from the "family" obtained by a development proceeding to the 3rd degree of x/a:

$$C_L = 2\pi\left(\alpha + \frac{\gamma_1+\gamma_2}{4} + \frac{\gamma_2-\gamma_1}{8}\right) = 2\pi\left(\alpha + \frac{\gamma_1}{8} + \frac{3}{8}\gamma_2\right) \qquad (10.17)$$

$$C_m = \pi\left(\frac{\alpha}{2} + \frac{\gamma_1+\gamma_2}{4} + \frac{5\,(\gamma_2-\gamma_1)}{32}\right) = \frac{C_L}{4} + \frac{3\pi}{32}(\gamma_2-\gamma_1) + \frac{\pi}{8}(\gamma_1+\gamma_2) \qquad (10.18)$$

Especially we obtain the following condition for the "intrinsic stability" of the airfoil: $\frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \geq \frac{4}{3}$ or $-\gamma_2 \geq \frac{1}{7}\gamma_1$ (10.19)

It may be noted that the results of the "thin airfoil theory" presented in these sections give valuable information in very simple form. The weak point of the theory is the rough degree of approximation in dealing with the flow around the leading edge. Even in the case of an infinitely thin airfoil, we obtain a stagnation point at some distance from the leading edge. In the approximate theory, using simultaneously the assumptions that the airfoil is indefinitely thin and that both the curvature and the angle of attack are indefinitely small, we bring the stagnation point to the leading edge; furthermore we neglect the induced velocities v_x against V even in the neighborhood of the leading edge in spite of the fact that v_x later turns out to have infinite values at the leading edge. The flow around the leading edge is represented in general by a singular point, at which $v_x = \pm \infty$, and v_y remains finite. This kind of approximation leads to the curious fact that starting from a given vorticity distribution $\bar{\gamma}(x)$, we obtain finite values for v_y at the point $x = -b$ corresponding to the leading edge, either if the function $\bar{\gamma}(x)$ vanishes at this point or if it becomes infinite of the form $1/\sqrt{x+b}$. Finite values of $\bar{\gamma}(-b)$ on the contrary lead to an infinite v_y and the corresponding airfoil center line has a vertical tangent. It seems that the approximation used in the theory of thin airfoils is really well founded in the case that $\bar{\gamma}(-b) = 0$, *i. e.* if the flow is smooth not only at the trailing, but also at the leading edge. In the case that $\bar{\gamma}(x)$ behaves like $1/\sqrt{x+b}$ near to $x = -b$, the approximation is wrong at the leading edge, but if we indefinitely diminish simultaneously thickness, curvature and angle of attack, the domain in which the error is considerable shrinks to the immediate neighborhood of the point $x = -b$.

A curious result of the presence of this singular point makes itself felt as soon as we try to calculate the resistance experienced by the airfoil. According to the general equations developed in **3**—**5**, the resultant force must be normal to the direction of the velocity at infinity, indicating that there is no resistance. But take the case of the infinitely thin plane airfoil at an angle of incidence α to the direction of the velocity at infinity. As the forces that come into play can only be pressures acting on the two sides of the infinitely thin flat plate (frictional forces are discarded throughout the theory developed in the present Chapter), we surely must expect the resultant force to be normal to the plate. If this should be the case the force would make an angle $(\pi/2) - \alpha$ with the velocity at infinity; consequently besides a lift L there would appear a resistance of the amount $L \tan \alpha$.

This resistance is of the second order with respect to α, and so it might be discarded on the ground that the theory of thin airfoils is based wholly upon first order approximations. The paradox, however, can be solved in a more satisfactory way if we assume that the infinitely thin leading edge may be considered as the limit of a rounded edge. So long as the radius of curvature at the edge is finite, pressures can act on the edge, and can give a force component more or less in the direction of the plane of the airfoil. Actually this component will be of the nature of a suction, acting in the forward direction, as for small radius of curvature the velocity will be high and the pressure very low. Now it can be shown that the limiting value of this suction, if the radius of curvature becomes infinitely small, has a finite value, which is determined wholly by the vorticity distribution in the immediate neighborhood of the leading edge. If the strength of the vortex sheet is of the form: $\bar{\gamma} = 2C/\sqrt{x+b}$, the tangential velocities on the two sides being respectively $v_x = + C/\sqrt{x+b}$ and $v_x = - C/\sqrt{x+b}$, the suctional force at the edge has the value:

$$s = \pi \varrho C^2 \tag{10.20}$$

If this suctional force is combined with the resultant of the normal pressures acting on both sides of the plane airfoil, it is found that the resistance disappears, while the resultant force in the direction perpendicular to the velocity at infinity has the value prescribed by the Kutta-Joukowski formula.

The point may be considered also in the light of the results obtained in **3** for the force acting on the separate elements of the vortex system. Equations (3.13) for the case of a vortex sheet assume the form:

$$\left.\begin{aligned} X\,dx &= -\varrho\,\bar{\gamma}\,(V_y + v_y)\,dx \\ Y\,dx &= \quad \varrho\,\bar{\gamma}\,V_x\,dx \end{aligned}\right\} \tag{10.21}$$

Now in the case of a plane airfoil lying along the x axis, $V_y = V \sin\alpha$, while according to (7.6) : $v_y = - V \sin\alpha$. Thus we should have: $V_y + v_y = 0$, $X = 0$, indicating again that the force is normal to the plane of the airfoil. The relation $v_y = - V\sin\alpha$, however, applies only to the points of the airfoil itself, that is, to the points, situated to the right of the leading edge; immediately to the left of the leading edge we have, on the contrary, a positive infinite value of v_y. As $\bar{\gamma}$ also becomes infinite at the leading edge, it is not possible to give the value of the horizontal force acting at the foremost element of the airfoil, though it may be supposed that this force will have the negative sign. In order to obtain the actual value of this force a limiting process should be used; the result must coincide with the one already given in (10.20).

In the case of an airfoil of curved shape similar considerations apply, though the precise relations are somewhat more complicated [1].

[1] The reader is referred to: R. GRAMMEL, Die hydrodynamischen Grundlagen des Fluges (Braunschweig 1917), p. 21, for the calculation of the magnitude of the suction. The paradox has been elucidated from another point of view by H. VILLAT [see: Mécanique des fluides (Paris 1930), p. 89—92], who points out that if proper

Though the point is of relatively small importance in the deductions of the present Chapter, where the Kutta-Joukowski theorem affords a sufficient basis for the calculation of the resultant force, the matter becomes different as soon as we leave the case of stationary motion, and pass over for instance to the oscillating airfoil. In that case the Kutta-Joukowski theorem is no longer valid, and the calculation of the lift and of the resistance follows more complicated lines (with non-stationary motion the resistance is not zero in general, as work must be expended to produce the changes in the flow pattern). The theory of thin airfoils still forms a very convenient approximation in this case, and then, in calculating the resistance, the suctional force must be taken into account. Some problems of non-stationary motion will be treated in Chapter V, and the calculation of the resistance is discussed shortly in V 9.

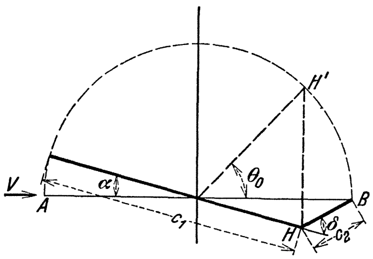

Fig. 27.

To come back to the present subject, Jeffries[1] has given a further approximation for finite thickness, with the result that the next terms in the various equations are found to be of the order $(d/c)^2$ ($d =$ thickness of the airfoil). This gives a further confirmation of the merit of the approximate theory.

11. Airfoil with Flap. We assume a broken straight line as the mean line of the airfoil (see Fig. 27). This represents the idealized case of an airfoil with a flap like an aileron, or the case of a tail surface with rudder. The chord of the mean wing may have the length c_1, the chord of the flap the length c_2. The value of θ corresponding to the intersection between the two straight portions of the airfoil, at the "hinge", may be denoted by θ_0. The angle of attack of the main wing may be equal to α, the angle of attack of the flap $\alpha - \delta$; hence δ is the relative angle between flap and main wing.

We use Munk's formula in order to find the total lift acting on the system. The circulation Γ is, according to (9.14):

$$\Gamma = -2 a V \int_0^{2\pi} \frac{d y}{d x} (1 + \cos\tau)\, d\tau \tag{11.1}$$

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notice is taken of the singularity at the leading edge, an expression for the resultant force can be obtained which differs from the Kutta-Joukowski formula by having additional terms; this gives at once the resultant of the pressures acting on both sides of the infinitely thin airfoil. VILLAT also gives references to investigations by others concerning this point.

[1] JEFFRIES, H., On Airfoils of Small Thickness, Proc. Roy. Society (London) A, **121**, p. 22, 1928.

Obviously in this case Γ will be equal to:

$$\Gamma = 2\,V a\,\alpha \int_{\theta_0}^{2\pi-\theta_0} (1 + \cos\theta)\, d\theta + 2\,V a\,(\alpha - \delta) \int_{-\theta_0}^{\theta_0} (1 + \cos\theta)\, d\theta \qquad (11.2)$$

or
$$\Gamma = 2\,V a\,\alpha \int_{0}^{2\pi} (1 + \cos\theta)\, d\theta - 2\,V a\,\delta \int_{-\theta_0}^{\theta_0} (1 + \cos\theta)\, d\theta \qquad (11.3)$$

Evaluating the integrals we obtain:

$$\Gamma = 4\,\pi\,V a \left[\alpha - \delta\left(\frac{\theta_0}{\pi} + \frac{\sin\theta_0}{\pi}\right)\right] \qquad (11.4)$$

and for the lift:

$$L = 4\,\pi\,\varrho\,a\,V^2 \left[\alpha - \delta\left(\frac{\theta_0}{\pi} + \frac{\sin\theta_0}{\pi}\right)\right] \qquad (11.5)$$

This equation shows that the inclination δ of the flap is equivalent to a decrease of the angle of attack by $\triangle\alpha = \delta\,(\theta_0 + \sin\,\theta_0)/\pi$. We remember that θ_0 is given by $\cos\,\theta_0 \cong 1 - 2\,c_2/c$. For small values of c_2/c we may write $\theta_0 \cong \sin\,\theta_0 \cong 2\,\sqrt{c_2/c}$, so that the change of the angle of attack amounts to $\triangle\alpha = 4\,\delta/\pi \cdot \sqrt{c_2/c}$. This is a simple rule for estimating the effect of the ailerons on the total lift of a wing section.

The second point of interest is the "hinge moment", *i. e.* the moment of the forces acting on the flap relative to the hinge. It is, obviously, (x_0 being the abscissa of the hinge):

$$M_h = \varrho\,V \int_{x_0}^{c/2} \bar{\gamma}(x)\,(x - x_0)\,dx \qquad (11.6)$$

or:
$$M_h = \varrho\,V \frac{c^2}{4} \int_{0}^{\theta_0} \bar{\gamma}(\theta)\,(\cos\theta - \cos\theta_0)\,\sin\theta\,d\theta \qquad (11.7)$$

Now $\bar{\gamma}(\theta)$ is given by (9.20):

$$\bar{\gamma} = -\frac{V}{\pi\sin\theta} \int_{0}^{2\pi} \frac{dy}{dx} \cot\frac{\theta - \tau}{2}\,\sin\tau\,d\tau + \frac{\Gamma}{2\,\pi\,a\sin\theta} \qquad (11.8)$$

Again putting $\frac{dy}{dx} = -\alpha$ for $\theta_0 < \theta < (2\,\pi - \theta_0)$ and $\frac{dy}{dx} = -(\alpha - \delta)$ for $-\theta_0 < \theta < \theta_0$, we have

$$\left.\begin{aligned} \bar{\gamma} = \frac{V\alpha}{\pi\sin\theta} \int_{0}^{2\pi} \cot\frac{\theta - \tau}{2}\,\sin\tau\,d\tau\, - \\ - \frac{V\delta}{\pi\sin\theta} \int_{-\theta_0}^{\theta_0} \cot\frac{\theta - \tau}{2}\,\sin\tau\,d\tau + \frac{\Gamma}{2\,\pi\,a\sin\theta} \end{aligned}\right\} \qquad (11.9)$$

Hence the hinge moment will be:

$$M_h = 4\,\varrho\,V^2 a^2\,(\eta_1\,\alpha + \eta_2\,\delta) \qquad (11.10)$$

where:

$$\left.\begin{aligned}\eta_1 = \frac{1}{\pi}\int_0^{\theta_0}(\cos\theta - \cos\theta_0)\,d\theta\int_0^{2\pi}\cot\frac{\theta-\tau}{2}\sin\tau\,d\tau\, + \\ + 2\int_0^{\theta_0}(\cos\theta - \cos\theta_0)\,d\theta\end{aligned}\right\} \quad (11.11)$$

$$\left.\begin{aligned}\eta_2 = -\frac{1}{\pi}\int_0^{\theta_0}(\cos\theta - \cos\theta_0)\,d\theta\int_{-\theta_0}^{\theta_0}\cot\frac{\theta-\tau}{2}\sin\tau\,d\tau\, - \\ -\frac{2}{\pi}(\theta_0 + \sin\theta_0)\int_0^{\theta_0}(\cos\theta - \cos\theta_0)\,d\theta\end{aligned}\right\} \quad (11.12)$$

Now $\int_0^{2\pi}\cot\frac{\theta-\tau}{2}\sin\tau\,d\tau = -2\pi\cos\theta$ and we obtain:

$$\eta_1 = -\theta_0 + \sin\theta_0\cos\theta_0 + 2\sin\theta_0 - 2\theta_0\cos\theta_0 \quad (11.13)$$

In order to calculate η_2 we have to evaluate the integrals:

$$T_1 = \int_0^{\theta_0}\int_0^{\theta_0}\cos\theta\sin\tau\cot\frac{\theta-\tau}{2}\,d\theta\,d\tau$$

$$T_2 = \int_0^{\theta_0}\int_{-\theta_0}^{0}\cos\theta\sin\tau\cot\frac{\theta-\tau}{2}\,d\theta\,d\tau$$

and:

$$T_3 = \int_0^{\theta_0}\int_0^{\theta_0}\sin\tau\cot\frac{\theta-\tau}{2}\,d\theta\,d\tau$$

We notice that:

$$T_1 = \frac{1}{2}\int_0^{\theta_0}\int_0^{\theta_0}(\cos\theta\sin\tau - \cos\tau\sin\theta)\cot\frac{\theta-\tau}{2}\,d\theta\,d\tau =$$

$$= \frac{1}{2}\int_0^{\theta_0}\int_0^{\theta_0}\sin(\tau-\theta)\cot\frac{\theta-\tau}{2}\,d\theta\,d\tau$$

or:

$$T_1 = -\frac{1}{2}\int_0^{\theta_0}\int_0^{\theta_0}[1+\cos(\theta-\tau)]\,d\theta\,d\tau = -\frac{\theta_0^2}{2} - \frac{\sin^2\theta_0}{2} - \frac{(1-\cos\theta_0)^2}{2}$$

In the same way we obtain:

$$T_2 = -\frac{\theta_0^2}{2} - \frac{\sin^2\theta_0}{2} + \frac{(1-\cos\theta_0)^2}{2}$$

The third integral can be written in the symmetrical form:

$$T_3 = \frac{1}{2} \int_0^{\theta_0} \int_0^{\theta_0} (\sin\tau - \sin\theta) \cot\frac{\theta-\tau}{2}\, d\tau\, d\theta,$$

because obviously

$$\int_0^{\theta_0} \int_0^{\theta_0} \sin\tau \cot\frac{\theta-\tau}{2}\, d\tau\, d\theta = \int_0^{\theta_0} \int_0^{\theta_0} \sin\theta \cot\frac{\tau-\theta}{2}\, d\theta\, d\tau$$

Furthermore: $\sin\tau - \sin\theta = 2\cos\frac{\tau+\theta}{2} \sin\frac{\tau-\theta}{2}$

Then: $T_3 = \frac{1}{2} \int_0^{\theta_0} \int_0^{\theta_0} (\sin\tau - \sin\theta) \cot\frac{\theta-\tau}{2}\, d\tau\, d\theta =$

$$= -\int_0^{\theta_0} \int_0^{\theta_0} \cos\frac{\theta-\tau}{2} \cos\frac{\theta+\tau}{2}\, d\tau\, d\theta = -\frac{1}{2} \int_0^{\theta_0} \int_0^{\theta_0} (\cos\tau + \cos\theta)\, d\tau\, d\theta$$

In the last integral the integration can be carried out and, noting the symmetry of the last form in τ and θ, we obtain,

$$T_3 = -\int_0^{\theta_0} \int_0^{\theta_0} \cos\tau\, d\tau\, d\theta = -\theta_0 \int_0^{\theta_0} \cos\tau\, d\tau = -\theta_0 \sin\theta_0$$

We thus obtain by introducing these expressions in (11.12):

$$\eta_2 = \frac{1}{\pi} [\theta_0^2 + 2\,\theta_0^2 \cos\theta_0 - 2\,\theta_0 \sin\theta_0 - \sin^2\theta_0] \tag{11.14}$$

For small values of θ_0, *i. e.* of c_1/c, we find:

$$\eta_1\,\alpha + \eta_2\,\delta = \frac{\theta_0^5}{15}\alpha - \frac{\theta_0^4}{3}\frac{\delta}{\pi} \tag{11.15}$$

If we write

$$M_h = C_h\, c_1^2 \frac{\varrho V^2}{2} \tag{11.16}$$

the coefficient C_h is equal to:

$$C_h \cong \frac{4}{3} \sqrt{\frac{c}{c_1}} \cdot \alpha - \frac{8}{3\pi}\delta \tag{11.17}$$

12. Two-Dimensional Approximate Biplane Theory. The approximate method used in the last sections enables us to find the mutual influence of two or more airfoils in a relatively simple way. We consider an airfoil with small curvature, as in 8, but generalize the problem in that we assume an additional velocity field v_x'', v_y'' superposed on the parallel motion with the velocity V. Using the conformal transformation

$$\zeta = z + \frac{a^2}{\zeta}$$

we obtain in the ζ plane certain velocities v_r'' and v_θ'' corresponding to v_x'' and v_y''. With the same approximation as used in 8, *i. e.* assuming v_x'' and v_y'' small compared with V, we obtain the following simple and general results: A variable velocity field $V + v_x''$, v_y'' produces the same forces on an airfoil of given shape as the parallel flow V on an airfoil with slightly distorted shape. The corresponding distortion is given by decreasing the inclination dy/dx of the airfoil mean line by the amount v_y''/V.

For instance, a uniform additional vertical velocity v_y'' is equivalent to the turning of the airfoil, *i. e.* to a change in the angle of attack, while the linear distribution along the chord $v_y'' =$ const. x/a, is equivalent to an increase or decrease of the curvature of the airfoil.

To find the mutual influence between two wings of a biplane, we have to calculate the velocities at one of the wings, due to the presence of the other. According to the approximate character of the calculations we use the first two terms of the development for the potential function (6.9):

$$F = \frac{i\,\Gamma}{2\,\pi} \log z - \frac{i\,A_1}{z}$$

We note that the coefficient A_1 is real and that it is equal to the coefficient B_1 in (6.12), as will be seen by substituting (6.2) into (6.9). Now Γ can be expressed by means of the lift L and thus also by means of the coefficient C_L. On the other hand A_1 can be expressed by means of the coefficient C_M; this can be done in the most direct way by reverting to the formulae of 4, noting that in (4.11) B is written for $-i\,A_1$, while W in the present case has the real value V. Then we have from (4.22):

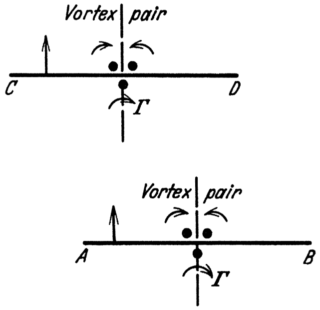

Fig. 28.

$$M = -2\,\pi\,\varrho\,Im.\,[B\,W] = 2\,\pi\,\varrho\,V A_1$$

As the coefficients C_L, C_M are given by (7.12), we obtain

$$C_L = \frac{2\,\Gamma}{V\,c}, \qquad C_M = \frac{4\,\pi\,A_1}{V\,c^2} \tag{12.1}$$

Hence we write:

$$F = \frac{i\,V\,c}{4\,\pi} \cdot C_L \log z - \frac{i\,V\,c^2}{4\,\pi} \cdot \frac{C_M}{z} \tag{12.2}$$

The potential function (12.2) represents a vortex at the center of the airfoil whose strength is $V c\, C_L/2$ and a vortex pair at the same point whose axis is parallel to the chord of the airfoil and whose strength is equal to $V c^2\, C_M/2$. The vortex and vortex pair which serve to replace the airfoil are indicated in Fig. 28.

Using the values resulting from this approximate calculation, there is no difficulty in finding v_y'' and the corrections due to the additional velocities, if size, shape, stagger, decalage and distance of the wings are given.

We do not carry out these calculations because in actual cases the additional velocities due to the trailing vortices have the same order of magnitude as those considered in this section. A treatment taking into account both effects is given in Chapter IV, Part B.

In some cases it will be found that the velocity component v''_x is relatively large compared with v''_y. The velocity v''_x did not appear in the previous calculations because the approximation was restricted to first order terms. If v''_x is relatively large, it may be necessary to take account of terms of the next order, at least of those containing the mean value of v''_x.

The exact solution of the two-dimensional biplane problem for plane airfoils is given in 22.

B. Application of the Theory of Conformal Transformation to the Investigation of the Flow around Airfoil Profiles.

13. Conformal Transformation. The exact theory of the two-dimensional flow of an incompressible ideal fluid around wing sections is based on the following principles.

We start from the flow of the fluid around a circular cylinder. The section of this circular cylinder by the ζ plane may be called the generating circle G; we denote the vector of the center by ζ_0, the radius by a, the velocity of the fluid at infinity by V, the angle between V and the ξ axis by α. We assume that the flow is a potential motion with the circulation Γ around the cylinder. We easily find the expression for the complex potential function[1]:

$$F(\zeta) = V e^{-i\alpha}(\zeta - \zeta_0) + \frac{V a^2 e^{i\alpha}}{\zeta - \zeta_0} + \frac{i\Gamma}{2\pi} \log(\zeta - \zeta_0) \tag{13.1}$$

The corresponding velocity function $dF/d\zeta = w_\zeta = v_\xi - i v_\eta$ is given by:

$$\frac{dF}{d\zeta} = V e^{-i\alpha} - \frac{V a^2 e^{i\alpha}}{(\zeta - \zeta_0)^2} + \frac{i\Gamma}{2\pi}\frac{1}{\zeta - \zeta_0} \tag{13.2}$$

The transformation function or, "mapping function", $z = f(\zeta)$ is to be an analytical function of ζ; the only restrictions introduced are:
(*a*) $dz/d\zeta$ is to be different from zero over the whole of the ζ plane outside of the circle G;
(*b*) $dz/d\zeta = 1$ for $\zeta = \infty$.

If these conditions are satisfied, the complex function $F[\zeta(z)]$ gives us the flow of an ideal fluid in the $z = x + iy$ plane; the complex velocity function will then be

$$w_z = \frac{dF}{d\zeta}\frac{d\zeta}{dz} \tag{13.3}$$

so that according to the condition (*b*) the velocity in the z plane for $z = \infty$ has the same value V as in the ζ plane for $\zeta = \infty$. Also the circulation remains unaltered. We then obtain the section corresponding

[1] See Division B III (3.5); V (2.24).

to the circle G as a stream-line and call it the profile P. According to condition (a) there are no singular points (vortices or sources) in the z plane outside of P.

We obtain in this way the flow of an ideal fluid with given circulation around all profile curves which develop from a circle by conformal transformation. The question arises, how to find the value of the circulation and the corresponding amount of the lift.

With the present state of our knowledge, we have no methods for the general solution of this problem. The flow around a cylindrical body is in general largely influenced by the effect of viscosity and even in case the flow at some distance from the body over a great range of

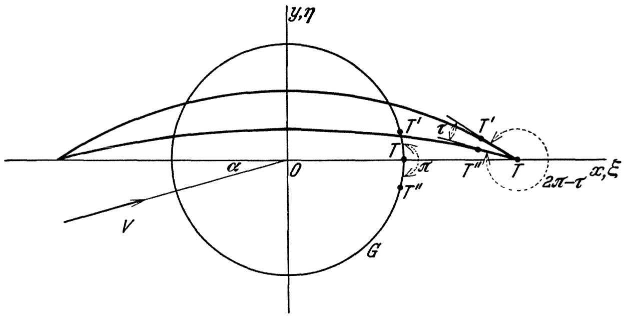

Fig. 29.

the plane is apparently similar to a potential motion of a non-viscous fluid, the circulation is probably connected with the position of the "burbling point" and with the formation of vortices at the surface of the body. The only case in which we are able to calculate the amount of circulation in fair accord with experience, is that of a profile with a "sharp trailing edge", with moderate thickness, slight curvature and small angle of attack. In this case we obtain a definite value of the circulation—independent of the viscous phenomena in the boundary layer—by the assumption of Joukowski, *viz.* the assumption of "smooth flow" at the trailing edge (see I 8). According to this assumption, the circulation has in every case just the amount necessary to secure a finite velocity of the fluid at the trailing edge.

This restriction to profiles with a sharp trailing edge has the consequence that the representation $z = f(\zeta)$ is not conformal at the point of the circle G which corresponds to the trailing edge. The point on the circumference of G representing the trailing edge may be denoted by T. In order to investigate what occurs in the vicinity of T, consider two points T' and T'' of the circle G on either side of T (see Fig. 29). The angle between TT' and TT'' in the ζ plane is equal to π. If the angle of the dihedral edge of the profile curve is denoted by τ, the corresponding angle in the z plane will be $2\pi - \tau$, so that the function $z = f(\zeta)$ at the point T has the development:

$$z - z_T = (\zeta - \zeta_T)^{\frac{2\pi - \tau}{\pi}} + \text{regular terms} \tag{13.4}$$

We notice that $\tau < \pi$ and consequently

$$\left(\frac{dz}{d\zeta}\right)_T = 0 \quad \text{or} \quad \left(\frac{d\zeta}{dz}\right)_T = \infty \tag{13.5}$$

Now the complex velocity function representing the flow in the z plane was given by (13.3). Consequently if w_z is to be finite at the trailing edge T, $(dF/d\zeta)_T$ must vanish.

The condition $(dF/d\zeta)_T = 0$ is called the "Joukowski condition" for the circulation. We determine the point T, corresponding to the trailing edge in the ζ plane, by the angle β between the ξ axis and the radius vector OT connecting the center of the generating circle with the point representing the trailing edge. We write $\zeta_T = \zeta_0 + ae^{-i\beta}$; β is supposed to be positive if T lies on the lower half of the generating circle[1]. Using (13.2) for the velocity function $dF/d\zeta$, we obtain:

$$\left(\frac{dF}{d\zeta}\right)_T = Ve^{-i\alpha} - Ve^{i(\alpha + 2\beta)} + \frac{i\Gamma}{2\pi a} e^{i\beta} = 0 \tag{13.6}$$

and consequently:

$$\Gamma = \frac{2\pi V a}{i} [e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}] = 4\pi V a \sin(\alpha + \beta) \tag{13.7}$$

14. General Expressions for Lift and Moment. Using this value for the circulation we obtain the potential function for the flow in the ζ plane:

$$F(\zeta) = Ve^{-i\alpha}(\zeta - \zeta_0) + \frac{Va^2 e^{i\alpha}}{\zeta - \zeta_0} + 2Vai \sin(\alpha + \beta) \log(\zeta - \zeta_0) \tag{14.1}$$

and introducing the mapping function $z = f(\zeta)$ in this equation, we are able to calculate the flow in the physical plane or z plane around the airfoil and also the force and moment acting on the airfoil. Assume that for large values of ζ the mapping function $z = f(\zeta)$ can be developed in a series:

$$z = \zeta + \frac{k_1}{\zeta} + \frac{k_2}{\zeta^2} + \frac{k_3}{\zeta^3} + \cdots \tag{14.2}$$

This corresponds to the assumptions made in the last paragraph, that z is without singularities at infinity and that $dz/d\zeta = 1$ for $\zeta = \infty$. We notice that there is no constant term at the right hand side of the equation. However, this does not mean any essential restriction, because the location of the generating circle is arbitrary and we can choose the coordinates ξ_0, η_0 of its center in such a way that the constant term in the development (14.2) of the mapping function vanishes.

[1] In Fig. 29 the angle β happens to be zero; an instance of a positive value of β is given in Fig. 30.

We first calculate the force acting on the airfoil. According to (4.10) the resultant force $R = X - iY$ is equal to[1]:

$$R = \frac{\varrho i}{2} \oint_c w^2 \, dz \tag{14.3}$$

and we easily obtain the result, that the force is equal to $\varrho \Gamma V$ and is perpendicular to the velocity vector V. We call the resultant force the lift L and using the value of Γ given by (13.7) we obtain

$$L = 4 \pi \varrho V^2 a \sin (\alpha + \beta) \tag{14.4}$$

We represent as before the ζ plane and the z plane in the same diagram, the ζ plane containing the generating circle and the z plane the actual airfoil (see Fig. 30). In this diagram O is the origin, C the

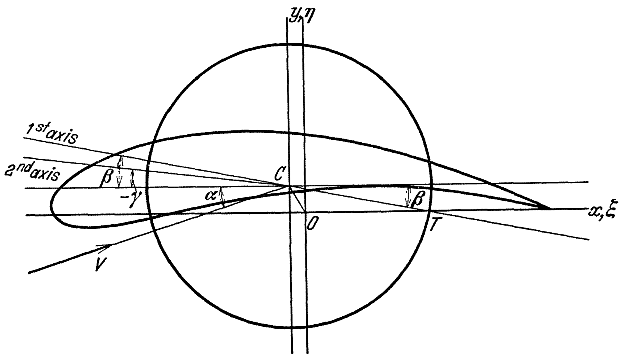

Fig. 30.

center of the generating circle and $C\,T$ its radius. It will be noted that for convenience α is counted positive when drawn from C to the left below the ξ axis and β positive when drawn from C to the right below the ξ axis. Then (14.4) shows that the lift vanishes if the direction of the wind is parallel to the line $C\,T$ connecting the center of the generating circle with the point T representing the trailing edge. This line is called "the first axis of the profile", and its direction is called the "direction of no lift". The angle $(\alpha + \beta)$ between the wind direction and the first axis is called the "effective angle of attack". The lift is proportional to the sine of the effective angle of attack or, for small angles, to the angle itself[2].

[1] The w occurring here is the w_z of (13.3).

[2] The definition of "effective angle of attack", based here upon the introduction of the "first axis" or "direction of no lift" instead of the chord of the profile as reference line, must not be confused with the definition of the effective angle of incidence in the theory of the airfoil of finite span, as indicated by I (12.5).

Both definitions may be adopted at the same time: in I (12.5) we may reckon α from the direction of no lift, then also the effective angle of incidence i is measured from the direction of no lift. As will be seen in Chapter IV this leads to more convenient formulae than are obtained by keeping to the chord as reference line.

If we introduce the chord c of the airfoil profile, and write:

$$L = C_L \frac{\varrho V^2}{2} c,$$

we obtain from (14.4): $$C_L = \frac{8 \pi a}{c} \sin(\alpha + \beta) \qquad (14.5)$$

or, for small values of α and β, approximately:

$$C_L = \frac{8 \pi a}{c} (\alpha + \beta)$$

As will be seen from the investigation of special cases in the subsequent sections, the ratio c/a in general does not differ much from 4 [compare, for instance, (17.7), (17.16), (19.15), and the discussion given in 21]. Thus we find that the slope of the curve for C_L as a function of α usually does not differ very much from the value 2π, which was obtained in the theory of thin wing sections [see (9.27)].

For the moment with reference to the point $z = 0$ we obtained in (4.21):

$$M = - Re. \left[\frac{\varrho}{2} \oint_c z w^2 dz \right]$$

We have now to calculate the moment M_C with respect to the point $z = \zeta_0$, *i. e.* the center C of the generating circle. We have then to use the formula:

$$M_C = - Re. \left[\frac{\varrho}{2} \oint_c (z - \zeta_0) w^2 dz \right] \qquad (14.6)$$

We deduce from (14.1):

$$w = \frac{dF}{dz} = \left[V e^{-i\alpha} - \frac{V a^2 e^{i\alpha}}{(\zeta - \zeta_0)^2} + \frac{2 V a i \sin(\alpha + \beta)}{(\zeta - \zeta_0)} \right] \frac{d\zeta}{dz} \qquad (14.7)$$

and we obtain with (14.2):

$$\left. \begin{aligned} M_C = - Re. \Bigg[& \frac{\varrho}{2} \oint_c \left(\zeta - \zeta_0 + \frac{k_1}{\zeta} + \ldots \right) \cdot \\ & \cdot \left[V e^{-i\alpha} - \frac{V a^2 e^{i\alpha}}{(\zeta - \zeta_0)^2} + \frac{2 V a i \sin(\alpha + \beta)}{(\zeta - \zeta_0)} \right]^2 \frac{d\zeta}{\left(1 - \frac{k_1}{\zeta^2} - \ldots \right)} \Bigg] \end{aligned} \right\} \qquad (14.8)$$

In order to evaluate the circular integral occurring in (14.8) we have to retain the terms containing $1/\zeta$.

First, we develop the fraction $\dfrac{1}{1 - \frac{k_1}{\zeta^2} - \ldots}$ in a series with ascending powers of $1/\zeta$ and write

$$\frac{1}{1 - \frac{k_1}{\zeta^2} - \ldots} = 1 + \frac{k_1}{\zeta^2} + \ldots$$

We then notice that the following products give us terms of the order $1/\zeta$:

a) The product of k_1/ζ in the first bracket with the term $V^2 e^{-2i\alpha}$ in the second and the term 1 in the third bracket.

b) The product of ζ in the first bracket with the term $V^2 e^{-2i\alpha}$ in the second bracket and k_1/ζ^2 in the third.

c) The product of ζ in the first bracket with the double product of the first two terms in the second and 1 in the third bracket.

d) The product of ζ with the square of the last term in the second and the term 1 in the third bracket.

In this way we obtain:

$$M_C = -Re.\left[\frac{\varrho}{2}\oint_c \{2V^2 k_1 e^{-2i\alpha} - 2V^2 a^2 - 4V^2 a^2 \sin^2(\alpha+\beta)\}\frac{d\zeta}{\zeta}\right]$$

We see easily that only the first term contributes to the real part of the integral and we obtain:

$$M_C = -2\pi\varrho V^2 Re.\,[i k_1 e^{-2i\alpha}] \tag{14.9}$$

As shown by (14.2), k_1 has the dimension of the square of a length. We put:

$$k_1 = k^2 e^{2i\gamma} \tag{14.10}$$

and obtain:

$$M_C = 2\pi\varrho V^2 k^2 \sin 2(\gamma-\alpha) \tag{14.11}$$

This analytical combination of γ with α requires us to count γ in the same direction as α, *i. e.* positive to the left and below the axis of ξ [1]. We remember that M_C represents the moment with respect to the center C of the generating circle. For $\alpha = \gamma$, M_C vanishes. We draw a line through the center C with the inclination γ to the ξ axis and call this line the "second axis of the profile". Obviously, if the direction of the velocity V coincides with the second axis, M_C is equal to zero, *i. e.* the line of action of the resultant force passes through the center of the generating circle.

We now introduce in (14.11) the effective angle of attack and separate the expression on the right hand side into two parts—a first part independent of the angle of attack and therefore of the lift, and a second part proportional to the lift. We make use of the identity:

$$\left.\begin{aligned}\sin 2(\gamma-\alpha) &= \sin 2[(\beta+\gamma)-(\alpha+\beta)] = \\ &= \sin 2(\beta+\gamma) - 2\sin(\alpha+\beta)\cos(2\gamma+\beta-\alpha)\end{aligned}\right\} \tag{14.12}$$

and obtain:

$$\left.\begin{aligned}M_C &= 2\pi\varrho V^2 k^2 \sin 2(\beta+\gamma) - \\ &\quad - 4\pi\varrho V^2 k^2 \sin(\alpha+\beta)\cos(2\gamma+\beta-\alpha)\end{aligned}\right\} \tag{14.13}$$

or using (14.4) for the lift:

$$M_C = 2\pi\varrho V^2 k^2 \sin 2(\beta+\gamma) - L\frac{k^2}{a}\cos(2\gamma+\beta-\alpha) \tag{14.14}$$

We may easily give a physical interpretation to this expression. Obviously the first term, to be denoted by

$$M_F = 2\pi\varrho V^2 k^2 \sin 2(\beta+\gamma) \tag{14.15}$$

[1] Thus in Fig. 30 γ has a negative value.

represents the part of the moment M_C which is independent of the angle of attack, or the moment corresponding to the angle of attack for "no lift"[1]. Now it has been shown in 8, that the stability of the airfoil depends on the sign of this quantity. If the part of the moment which is independent of the angle of attack represents a diving moment (positive sign), the airfoil is unstable; if it results in a stalling moment (negative sign), the airfoil is stable.

A wing of a given section is therefore stable if the expression (14.15) is negative. We see now that this depends entirely on the sign of the angle $(\beta + \gamma)$. If γ is negative and of such a magnitude that we have to turn the first axis clockwise in order to have coincidence with the second axis, the section is stable; in the opposite case (as in Figs. 30, 31) it is unstable. The condition of neutral equilibrium requires that lift and moment with reference to the center of the generating circle shall vanish for the same value of the angle of attack.

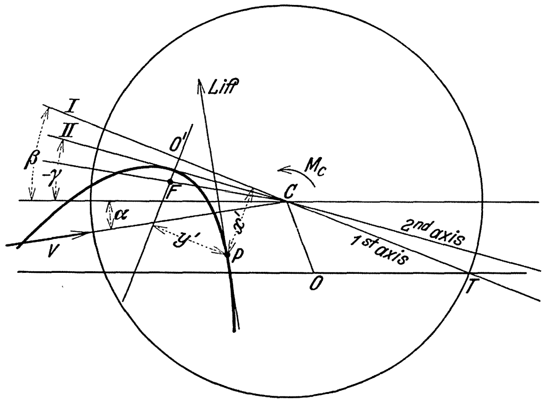

Fig. 31.

The second term permits of the following interpretation. We draw the line CF (Fig. 31) making the angle $2(\beta + \gamma)$ with the first axis and $\beta + \gamma$ with the second axis, so that the second axis is the bisectrix between the first axis and the line CF. The point F at the distance k^2/a from the center C of the generating circle may be called the "focus" of the section. Let us assume the lift L acting at this point, and note that the angle between the direction of attack and the line CF is equal to $-(2\gamma + \beta - \alpha)$. Hence the component of L normal to CF is equal to $L \cos(2\gamma + \beta - \alpha)$ and the moment with reference to C is equal to $-L \cos(2\gamma + \beta - \alpha)\, k^2/a$.

We conclude that the total system of forces is equivalent to an invariable moment of the amount $M_F = 2\pi\varrho V^2 k^2 \sin 2(\beta + \gamma)$ and to the force L, acting at the focus of the section.

15. Metacentric Parabola. Now a single force and a moment can be replaced by a single force located at a particular line of action. It is easy to show that the system of straight lines representing the lines

[1] As the moment in the case of no lift is independent of the point of reference it follows that $M_F = \frac{1}{2}\varrho V^2 c^2 C_\mu$, where C_μ is the coefficient introduced in (8.22).

of action corresponding to different angles of attack touch a parabola. The axis of this parabola is perpendicular to the first axis of the section, the focus F of the section is the focus and the first axis of the section is the directrix of the parabola.

It is convenient to introduce a coordinate system x', y', where the x' axis is perpendicular to the first axis of the section passing through the focus while the y' axis may coincide with the first axis (see Fig. 31). The distance $O'F$ of the focus F from the first axis may be denoted by p. The equation of the parabola with $O'C$ as directrix and F as focus is given by:

$$y' = \sqrt{p\,(2\,x' - p)} \tag{15.1}$$

We draw a tangent touching the parabola at an arbitrary point x', y' and denote the angle between this tangent and the x' axis by δ. It is obvious that

$$\tan\delta = \frac{d\,y'}{d\,x'} = \sqrt{\frac{p}{2\,x' - p}} \tag{15.2}$$

whence:

$$x' = \frac{p}{2}\,\frac{1}{\sin^2\delta}, \qquad y' = p\cot\delta \tag{15.3}$$

Let us now assume that the line of action of the lift L is the tangent to the parabola at the point x', y'. The moment of L with reference to the focus F will be:

$$M_F = L\,[y'\cos\delta + (p - x')\sin\delta] \tag{15.4}$$

Introducing the values of x' and y' from (15.3) we obtain easily:

$$M_F = L\,\frac{p}{2\sin\delta} \tag{15.5}$$

Now the angle δ is equal to the effective angle of attack because the lift is perpendicular to the direction of attack and the x' axis is perpendicular to the first axis; hence $L/\sin\delta = 4\,\pi\,\varrho\,V^2\,a$, and as on the other hand $p = (k^2/a)\sin 2\,(\beta + \gamma)$, the moment M_F becomes:

$$M_F = 2\,\pi\,\varrho\,V^2\,k^2\sin 2\,(\beta + \gamma) \tag{15.6}$$

i. e. equal to the invariable part of the total moment.

Hence the parabola P, characterized by the first axis of the section as directrix and by the focus F as focus, represents the envelope of the lines of action of the lift for different angles of attack. The parabola P is called the "metacentric parabola" of the airfoil.

16. The Joukowski Transformation. Classification of Airfoil Families. The method of the following sections is essentially an inverse one. We start from the flow around the circle represented in the $\zeta = \xi + i\eta$ plane and try to find relatively simple transformations between this plane and the $z = x + iy$ plane, which result in the distortion of the circle into certain shapes similar to the airfoils used in aeronautic practice. If the transformation is regular at infinity and if there are no singular points outside of the circle, the domain of the ζ plane outside of the circle will be represented by the domain of the z plane outside of the airfoil. As explained in **13**, we assume that the airfoil has a sharp trailing edge, so that the transformation is not conformal at the point representing the trailing edge.

Let us denote the point T (see Fig. 32) corresponding to the trailing edge in the "ideal plane" by ζ_T and the corresponding point T' in the "physical z plane" by z_T and the tail angle of the airfoil at the trailing edge by τ. The development of the mapping function $z = f(\zeta)$ connecting the two planes will be, at the point T [see (13.4)]:

$$z - z_T = (\zeta - \zeta_T)^{(2 - \tau/\pi)} \tag{16.1}$$

We consider first the case $\tau = 0$, *i. e.* the airfoil with vanishing tail angle and choose the transformation:

$$\frac{z - z_T}{z + z_T} = \left(\frac{\zeta - \zeta_T}{\zeta + \zeta_T}\right)^2 \tag{16.2}$$

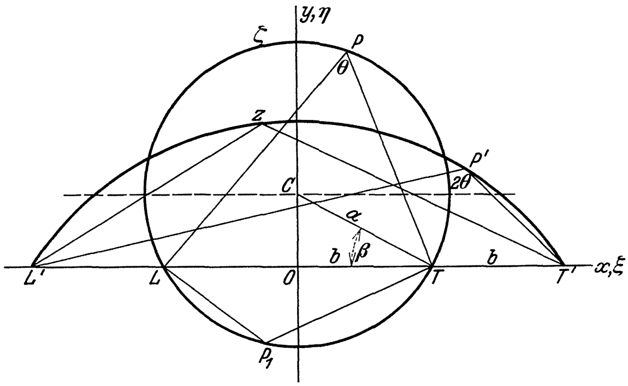

Fig. 32.

We see at once that by putting $\zeta_T = b$, where b is a real quantity, we do not restrict the generality of the transformation except that we fix the position of the airfoil in the plane.

We remember that we want to satisfy the relation $dz/d\zeta = 1$ for $\zeta = z = \infty$. This involves $z_T = 2\zeta_T$, so that we must write

$$\frac{z - 2b}{z + 2b} = \left(\frac{\zeta - b}{\zeta + b}\right)^2 \tag{16.3}$$

or:

$$\log \frac{z - 2b}{z + 2b} = 2 \log \frac{\zeta - b}{\zeta + b} \tag{16.4}$$

Let us denote by L and L' the points $\zeta = -b$ and $z = -2b$; furthermore by P and P' two corresponding points in the ζ and z planes. We introduce the angle $\theta = \angle TPL$ by the following definition: We turn the vector TP in the clockwise direction, until it will be parallel to the vector LP. Using the same definition for the angle $\theta' = \angle T'P'L'$ in the z plane, we see at once that

$$\theta' = 2\theta \tag{16.5}$$

From this equation it follows that

(*a*) Every circular arc in the ζ plane which connects the points T and L is represented in the z plane by a circular arc through T' and L'. They correspond to constant values of θ' and θ.

(*b*) Two circular arcs, which belong to the same circle in the ζ plane are represented by two coincident circular arcs in the z plane. We see this at once, if we remember that θ and $\theta + \pi$ are the values of $\angle TPL$ at the upper and lower arcs of the same circle. In the z plane we obtain 2θ and $2\theta + 2\pi$ as corresponding values for $\angle T' P' L'$, so that the two arcs coincide.

Instead of drawing the corresponding curves in two separate planes ζ and z, we prefer again to use one plane for their representation, as shown in Fig. 32. The chord LT appears in the "physical plane" enlarged in the scale 2:1. The distance between the center C of the circle and the chord LT is obviously one half of the height of the circular arc $L' T'$.

Using different circles passing through L and T in the ζ plane as "generating circles", we easily obtain all information for indefinitely thin airfoils of circular shape with different curvature. This computation is more general than the approximate theory given in **10** because we are no longer restricted to small curvatures and small angles of attack. However, the whole circulation theory breaks down for large angles and large curvatures, so that this improvement in the theory does not represent an important progress in practice as compared with previous calculations. On the other hand we can proceed in a simple way to the investigation of airfoils with finite thickness by using as generating circle any of the circles passing through T and having L as an interior point. As the points T and L are the only singular points of the transformation, there will be no singular points outside of the circle and we obtain a flow in the z plane around a curve of certain shape, which has a sharp point with a tail angle $\tau = 0$; the point L' corresponding to L lies inside of the profile obtained in this way.

The airfoils deduced by this procedure are called the "airfoils of the Joukowski family" (see Figs. 1 and 2, Plate II).

There are two essential restrictions in this process of constructing airfoil profiles. First, they all have zero tail angle at the trailing edge; second, if their thickness is relatively small, the "mean camber line" will be nearly a circular arc, so that we do not have the desirable freedom in shaping this line according to the information given by the "thin wing theory" concerning stability and the displacement of the center of pressure. We have seen in **10** that for reducing the displacement of the center of pressure we must use airfoils of a more or less definite S shape.

We can remove both restrictions by generalizing the transformation (16.3).

The simplest way to obtain a given tail angle τ is to use the transformation:

$$\frac{z - z_T}{z + z_T} = \left(\frac{\zeta - \zeta_T}{\zeta + \zeta_T}\right)^{2 - \tau/\pi}$$

We may show easily that circles through the points ζ_T and $-\zeta_T$ are transformed into an airfoil enclosed by two circular arcs with the tail angle τ; using circles passing through $\zeta = \zeta_T$ and having $\zeta = -\zeta_T$ as interior point—in the same way as explained above in the case of the transformation (16.3)—we obtain profiles with a round nose and sharp trailing edge with the desired tail angle.

The airfoils derived in this way are known as the "Kármán-Trefftz family" (see Fig. 3, Plate II).

In order to remove the second restriction we give a somewhat different form to (16.3). Solving the equation for z we obtain:

$$z = \zeta + \frac{b^2}{\zeta} \tag{16.6}$$

We see that the transformation (16.6) is identical with that used in the previous sections for the transformation of a circle into a straight line. The straight airfoil is of course a special case of the circular airfoil. From (16.6) follows

$$\frac{dz}{d\zeta} = 1 - \frac{b^2}{\zeta^2} = \left(1 - \frac{b}{\zeta}\right)\left(1 + \frac{b}{\zeta}\right) \tag{16.7}$$

The natural way to obtain a family of airfoils with zero tail angle at the point T—but with a greater variety in the shape of the center line—is to put:

$$\frac{dz}{d\zeta} = \left(1 - \frac{b}{\zeta}\right)\left(1 - \frac{\lambda_1}{\zeta}\right)\left(1 - \frac{\lambda_2}{\zeta}\right)\cdots\left(1 - \frac{\lambda_n}{\zeta}\right) \tag{16.8}$$

where λ_1, λ_2, ... are complex values satisfying the condition that the points represented by them lie inside of the generating circle.

The airfoils derived in this way are sometimes known as the "Mises family" (see Plate III).

17. The Joukowski Family of Airfoils. In order to use the transformation:

$$z = \zeta + \frac{b^2}{\zeta} \tag{17.1}$$

for the derivation of airfoils of the Joukowski type, we consider first the case where the generating circle goes through both singular points L and T. In this case—as mentioned in the last section—the shape of the airfoil is a circular arc and its chord c will be $c = 4b$ (see Fig. 32)[1]. The line TC represents the first axis of the profile, so that the angle between the first axis (direction of vanishing lift) and the chord is given by:

$$\beta = \cos^{-1}\frac{b}{a} \tag{17.2}$$

[1] The reader will notice that in consequence of the way in which the constant b was introduced in (16.3), etc., this relation differs from that occurring in **10**, etc., where we had $c = 2b$.

We remember that the angle between the ξ axis and the second axis of the airfoil is given by [see (14.10)]:

$$\gamma = \frac{1}{2} \, arg \, (k_1) \tag{17.3}$$

so that in this case γ vanishes and the second axis is parallel to the chord. Therefore the "diving moment" or the part of the moment which is independent of the angle of attack amounts to:

$$M_F = 2\,\pi\,\varrho\,V^2\,b^2 \sin 2\,\beta \tag{17.4}$$

while the lift has the value:

$$L = 4\,\pi\,\varrho\,V^2\,a \sin(\alpha + \beta) = 4\,\pi\,\varrho\,V^2\,b\,\frac{\sin(\alpha+\beta)}{\cos\beta} \tag{17.5}$$

where α is the angle of attack measured from the direction of the chord. Introducing the actual chord of the airfoil as $c = 4b$, we write:

$$\left.\begin{aligned} L &= C_L \frac{\varrho V^2}{2} c \\ M_F &= C_{MF} \frac{\varrho V^2}{2} c^2 \end{aligned}\right\} \tag{17.6}$$

and obtain easily by comparison with (17.4) and (17.5):

$$\left.\begin{aligned} C_L &= 2\,\pi \frac{\sin(\alpha+\beta)}{\cos\beta} \\ C_{MF} &= \frac{\pi}{4} \sin 2\,\beta \end{aligned}\right\} \tag{17.7}$$

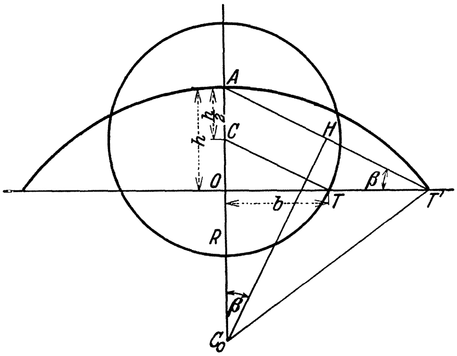

Fig. 33.

For small values of α and β we obtain $C_L = 2\,\pi\,(\alpha + \beta)$ and $C_{MF} = (\pi\,\beta)/2$ and it is easy to show that this result is identical with the results obtained in **10** for an airfoil with parabolic shape or with a circular shape of small curvature (see 10.8)[1].

It is sometimes convenient to use the maximum height h or the curvature $1/R$ of the camber line instead of the angle β. Obviously $\tan\,\beta = h/2\,b$, and $R = AH/\sin\,\beta$ (Fig. 33) or $R = b/\sin\,\beta \cos\beta$. A simple calculation then gives:

$$\left.\begin{aligned} C_L &= 2\pi\left(\sin\alpha + \frac{2\,h}{c}\cos\alpha\right) = 2\,\pi\left(\sin\alpha + \frac{c}{4R - 2h}\cos\alpha\right) \\ C_{MF} &= \pi\,\frac{h\,c}{c^2 + 4\,h^2} = \frac{\pi\,c}{8\,R} \end{aligned}\right\} \tag{17.8}$$

By way of approximation the formula for C_L may also be written:

$$C_L = 2\,\pi\left(\alpha + \frac{c}{4\,R}\right) \tag{17.9}$$

In order to produce new airfoil profiles with a rounded nose, we now remove the center of the generating circle from the point $\xi_0 = 0$,

[1] The coefficient C_{MF} is identical with the coefficient C_μ introduced in (8.22).

$\eta_0 = h/2$, keeping constant the direction of the first axis and write (see Fig. 34):

$$\xi_0 = -a\,\varepsilon \cos\beta\,, \qquad \eta_0 = \frac{h}{2} + a\,\varepsilon \sin\beta \tag{17.10}$$

The radius of the generating circle obviously will be $a\,(1 + \varepsilon)$ and we obtain for the lift acting on the modified airfoil, the expression:

$$L_1 = 4\,\pi\,\varrho\,V^2 a\,(1+\varepsilon)\sin(\alpha+\beta) \tag{17.11}$$

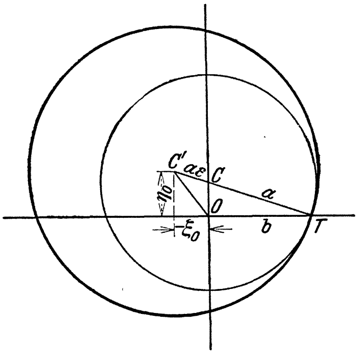

Fig. 34.

The expression for the moment remains unaltered:

$$M_{F1} = M_F = 2\,\pi\,\varrho\,V^2 b^2 \sin 2\beta \tag{17.12}$$

In order to calculate C_L we have to find the relation between the actual geometrical chord c and the chord of the circular arc which serves as the "skeleton" of the airfoil. The ratio $c/4b$ will be a function of the parameter ε, which is obviously connected with the "thickness-ratio" d/c of the airfoil, where d denotes the maximum thickness.

The calculation for the general case is somewhat complicated, and graphical methods—to be given in the following section—are generally preferred. We carry out, however, the calculation for the special case of a symmetrical airfoil[1] (see Fig. 35). In this case $\beta = 0$, $a = b$ and we write for the points of the generating circle:

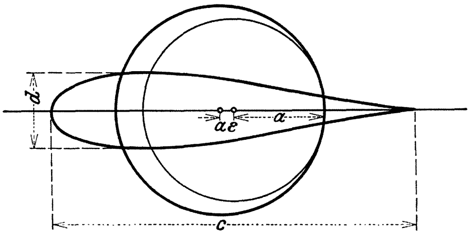

Fig. 35.

$$\zeta = -a\,\varepsilon + (1+\varepsilon)\,a\,e^{i\theta} \tag{17.13}$$

Introducing this expression in (17.1) we obtain:

$$z = -a\,\varepsilon + a\,(1+\varepsilon)\,e^{i\theta} + \frac{a}{-\varepsilon + (1+\varepsilon)\,e^{i\theta}} \tag{17.14}$$

We develop the right hand side in a series with increasing powers of ε:

$$z = 2a\cos\theta + a\,\varepsilon\,(\cos 2\theta - 1) + a\,\varepsilon\,i\,(2\sin\theta - \sin 2\theta) + \dots \tag{17.15}$$

with restriction to the terms with the first power of ε. With this approximation the actual chord c will be:

$$c = z\,(0) - z\,(\pi) = 4a \tag{17.16}$$

and the maximum value of the half thickness:

$$\frac{d}{2} = y_{max} = a\,\varepsilon\,(2\sin\theta - \sin 2\theta)_{max} \tag{17.17}$$

[1] See also Division B VI 3.

The value of θ which corresponds to the maximum of $d/2$ is given by $d/d\theta\,(2 \sin \theta - \sin 2\theta) = 0$, *i. e.* $\cos \theta = \cos 2\theta$, $\theta = 2\pi/3$. With this value we have: $y_{max} = 3/2\, a\, \varepsilon \sqrt{3}$

Hence the thickness-chord ratio will be

$$\frac{d}{c} = \frac{3\sqrt{3}}{4}\varepsilon = 1.299\,\varepsilon \qquad (17.18)$$

If we introduce d/c instead of ε in (17.11) and put $\beta = 0$ we obtain the following expressions for the lift coefficient:

$$C_{L1} = 2\pi \sin\alpha \left(1 + 0.77\frac{d}{c}\right) = C_L\left(1 + 0.77\frac{d}{c}\right) \qquad (17.19)$$

If the curvature of the skeleton (*i. e.* the curvature of the main camber line) is small, the same correction factor can be used also for the non-symmetrical Joukowski profiles. According to this calculation the slope of the lift curve, plotted as a function of the angle of attack, should increase with increasing thickness-chord ratio. This is not, however, in accordance with experimental evidence; the main influence of thickness is due to effects which are neglected in the present theory, *e. g.* to the influence of viscosity on the pressure distribution over the upper surface of the airfoil.

In order to compute the part of the moment which is variable with the lift, we remember the general formula (14.11) for the moment with respect to the point C (center of the generating circle):

$$M_C = 2\pi\varrho V^2 k^2 \sin 2(\gamma - \alpha) \qquad (17.20)$$

Putting $\gamma = 0$ and $k = b$ we have:

$$M_C = -2\pi\varrho V^2 b^2 \sin 2\alpha$$

Subtracting M_F, *i. e.* the part which is independent of the lift, we obtain:

$$M_C - M_F = -2\pi\varrho V^2 b^2 (\sin 2\alpha + \sin 2\beta)$$

and using (17.11) for the lift L (omitting the subscript 1):

$$M_C - M_F = -L\frac{b^2}{a}\frac{\cos(\alpha - \beta)}{1 + \varepsilon}$$

Introducing $c = 4b$, $b = a\cos\beta$, we have:

$$M_C - M_F = -L\frac{c}{4}\frac{\cos\beta\cos(\alpha-\beta)}{1+\varepsilon} = -L\frac{c}{4}\frac{\cos\beta\cos(\alpha-\beta)}{\left(1 + 0.77\frac{d}{c}\right)} \qquad (17.21)$$

This equation shows, that to a first approximation the system of air forces can be considered as composed of the invariable moment M_F and of the lift L located at the distance $c/4$ from the center of the profile, or approximately at the middle point between the leading edge and the center of the chord. This result is in accordance with the conclusions drawn before from the elementary theory of thin airfoils.

18. Graphical Method for Plotting Joukowski Airfoils and Computing Velocity Distribution. A simple graphical method for plotting Joukowski airfoils has been given by E. Trefftz (see Fig. 36). We employ the transformation:

$$z = \frac{1}{2}\left(\zeta + \frac{b^2}{\zeta}\right) \tag{18.1}$$

instead of (17.1), *i. e.* we draw the profile reduced in the scale 1:2.

If we let ζ take values corresponding to the points on the orbit of the generating circle, the points corresponding to b^2/ζ describe another circle. To find the center C'' and the radius of this second circle, we first note that the point $\zeta = b$ belongs to both circles. Taking into consideration also that the straight line $L'T$ goes over into itself by the transformation, we conclude from the theorem of the conservation of angles in the conformal transformation, that the two circles are tangent to each other at the point T, and hence that the center of the second circle lies on the straight line CT. We also find easily the location of the center C''. Let us denote the intersection of the generating circle with the x axis by L', the intersection of the circle representing b^2/ζ by L''. Obviously $L'O \cdot L''O = b^2$. On the other hand $L'O = L'T - OT = 2a(1+\varepsilon)\cos\beta - a\cos\beta$. Denoting the distance of the center C'' from C by $a\eta$, we obtain $L''O = 2a(1-\eta)\cos\beta - a\cos\beta$. Hence $L'O \cdot L''O = b^2(1+2\varepsilon)(1-2\eta)$ or $(1+2\varepsilon)(1-2\eta) = 1$ and finally $1/\eta - 1/\varepsilon = 2$.

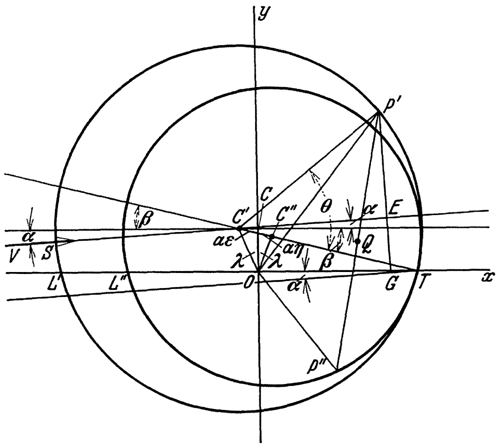

Fig. 36.

From this equation it follows that OC divides the angle $C'OC''$ in two equal parts. In fact assuming the two angles $\angle C'OC$ and $\angle COC''$ equal to λ, we obtain:

$$\left.\begin{aligned} \frac{CO}{CC'} &= \frac{\sin(90^0 - \lambda - \beta)}{\sin\lambda} \\ \frac{CO}{CC''} &= \frac{\sin(90^0 - \lambda + \beta)}{\sin\lambda} \end{aligned}\right\} \tag{18.2}$$

and subtracting the two equations:

$$CO\left[\frac{1}{CC''} - \frac{1}{CC'}\right] = \frac{\sin(90^0 - \lambda + \beta) - \sin(90^0 - \lambda - \beta)}{\sin\lambda} = 2\sin\beta \tag{18.3}$$

or with $CO = a\sin\beta$, $CC' = a\varepsilon$, $CC'' = a\eta$:

$$\frac{1}{\eta} - \frac{1}{\varepsilon} = 2, \quad q.e.d. \tag{18.4}$$

Hence, the following construction can be used for plotting points of the Joukowski profile.

We first construct the generating circle with the center C' through the trailing edge T and then draw the line OC'' symmetrical to OC' with reference to the y axis and construct a second circle through T with C'' as center. Drawing any two straight lines with equal and opposite inclination to the x axis, the points of intersection with the two circles represent points corresponding to ζ and b^2/ζ. Connecting two such points P' and P'', the point at half the distance will correspond to $1/2\ (\zeta + b^2/\zeta)$ and give a point Q on the profile curve.

It may be remarked that linkages can be constructed which enable the designer to draw Joukowski profiles without any geometrical construction[1].

In order to compute the velocity distribution we first calculate the velocity in the ideal ζ plane. Denoting by V the absolute value of the velocity at infinity, by α the angle of attack with respect to the x axis and by θ the central angle corresponding to the arc TP', the component of the velocity at any such point P' due to the velocity V will be $2\ V \sin\ P'\ C'\ S = 2\ V \sin P'\ C'\ E$, while the component due to the circulation will be $\Gamma/[2\,\pi\,a\,(1+\varepsilon)]$. But from (13.7) we shall have here $\Gamma = 4\,\pi\,V\,a\ (1+\varepsilon)\sin(\alpha+\beta)$ and the component due to the circulation will be $2\ V \sin(\alpha+\beta)$. Hence:

$$\frac{v_\zeta}{V} = 2\,[\sin\,[\theta - (\alpha+\beta)] + \sin\,(\alpha+\beta)] \tag{18.5}$$

We draw a line parallel to the velocity vector V through the trailing edge and let fall the normal from the point P' on this line. From the diagram the length of this normal is $n = P'\,E + E\,G$, or:

$$n = a\,(1+\varepsilon)\,[\sin\,[\theta - (\alpha+\beta)] + \sin\,(\alpha+\beta)] \tag{18.6}$$

Whence:
$$\frac{v_\zeta}{V} = \frac{2\,n}{a\,(1+\varepsilon)} \tag{18.7}$$

In order to obtain the velocity in the z plane, we have to multiply v_ζ by $|d\zeta/dz|$ or by $\dfrac{1}{|dz/d\zeta|} = \dfrac{1}{|1-b^2/\zeta^2|} = \left|\dfrac{\zeta}{\zeta - b^2/\zeta}\right|$

The latter quantity is represented by $OP'/P'P''$, so that we finally obtain the following expression for the velocity at the corresponding point Q of the profile:

$$v_z = V\,\frac{2\,n}{a\,(1+\varepsilon)}\,\frac{OP'}{P'P''} \tag{18.8}$$

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[1] See for instance B. N. Jurieff and N. P. Lessnikowa, Aerodynamical Investigations, Trans. Centr. Aero-Hydrodyn. Institute, Moscow, No. 33, p. 105/6, 1928 (p. 411 of english abstract) for an apparatus designed by B. V. Korostelew and consisting of a six-linked chain, having nine degrees of freedom. A mechanism designed by P. Léglise has been described in L'Aéronautique, **10**, p. 50, 1928; see also: N. Joukowski, Aérodynamique, with notes by W. Margoulis (Paris 1931), p. 192.

In the Russian literature on the subject the Joukowski profiles often are described as "sections obtained by the inversion of a parabola".

The pressure acting on the airfoil at the point Q is given by Bernoulli's relation:

$$p - p_0 = \frac{\varrho}{2}(V^2 - v_z^2) \tag{18.9}$$

19. The Kármán-Trefftz Family of Airfoils. The "skeleton" of these airfoils is composed of two circular arcs with the tail angle τ. We obtain the skeleton using the transformation:

$$\frac{z - z_T}{z + z_T} = \left(\frac{\zeta - \zeta_T}{\zeta + \zeta_T}\right)^{\varkappa} \tag{19.1}$$

where $\varkappa = 2 - \tau/\pi$. We have to satisfy the condition $dz/d\zeta = 1$ at infinity and for this purpose it is convenient to solve (19.1) for z, giving,

$$z = z_T \frac{(\zeta + \zeta_T)^{\varkappa} + (\zeta - \zeta_T)^{\varkappa}}{(\zeta + \zeta_T)^{\varkappa} - (\zeta - \zeta_T)^{\varkappa}} \tag{19.2}$$

and developing in powers of $1/\zeta$:

$$z = z_T \frac{1 + \frac{\varkappa(\varkappa - 1)}{1.2}\left(\frac{\zeta_T}{\zeta}\right)^2 + \ldots}{\varkappa\frac{\zeta_T}{\zeta} + \frac{\varkappa(\varkappa-1)(\varkappa-2)}{1.2.3}\left(\frac{\zeta_T}{\zeta}\right)^3 + \ldots} \tag{19.3}$$

or:

$$z = \frac{z_T}{\varkappa\zeta_T}\left(\zeta + \frac{\varkappa^2 - 1}{3}\frac{\zeta_T^2}{\zeta} + \ldots\right) \tag{19.4}$$

From the condition $dz/d\zeta = 1$ for $\zeta = \infty$, follows:

$$z_T = \varkappa\zeta_T \tag{19.5}$$

and thus (19.4) can be written:

$$z = \zeta + \frac{\varkappa^2 - 1}{3}\frac{\zeta_T^2}{\zeta} + \ldots \tag{19.6}$$

With (19.5) the equation of the transformation (19.1) takes the form:

$$\frac{z - \varkappa\zeta_T}{z + \varkappa\zeta_T} = \left(\frac{\zeta - \zeta_T}{\zeta + \zeta_T}\right)^{\varkappa} \tag{19.7}$$

or:

$$\log\left(\frac{z - \varkappa\zeta_T}{z + \varkappa\zeta_T}\right) = \varkappa \log\left(\frac{\zeta - \zeta_T}{\zeta + \zeta_T}\right) \tag{19.8}$$

Here ζ_T will be taken equal to b, as before.

We first consider the aerodynamic characteristics of the "skeleton" itself used as an airfoil.

We introduce (see Fig. 37) the angle β connecting the point T with the center C of the generating circle. Also, in this case the line TC indicates the direction of the first axis. We denote the second intersection between the generating circle and the chord, *i. e.* the point $\zeta = -\zeta_T$, by L, and an arbitrary point on the generating circle by P_1. The argument of $\log\ (\zeta - \zeta_T)/(\zeta + \zeta_T)$ is equal to the angle between the vectors TP_1 and LP_1, *i. e.* to $\pi/2 - \beta$, if P lies on the upper arc of the circle, and to $-\pi/2 - \beta$, if P lies on the lower arc. The angles are measured in the same way as explained in **16.** Now using the transformation (19.8) these angles will be multiplied by $\varkappa$, so that if the point P

goes over into P', T and L into T' and L', the angle $T'P'L'$ will be equal to $\varkappa(\pi/2-\beta)$ for the part of the profile corresponding to the upper arc and to $\varkappa(-\pi/2-\beta)$ for the part corresponding to the lower arc. Introducing $\varkappa = 2-\tau/\pi$, we obtain for the two angles:

$$T'P'L' = \left\{\begin{array}{c} \pi - 2\beta - \frac{\tau}{2} + \frac{\beta\tau}{\pi} \\ -\pi - 2\beta + \frac{\tau}{2} + \frac{\beta\tau}{\pi} \end{array}\right\} \tag{19.9}$$

This equation determines two circular arcs; the angles between the tangents to these two circular arcs at the trailing edge and the chord are:

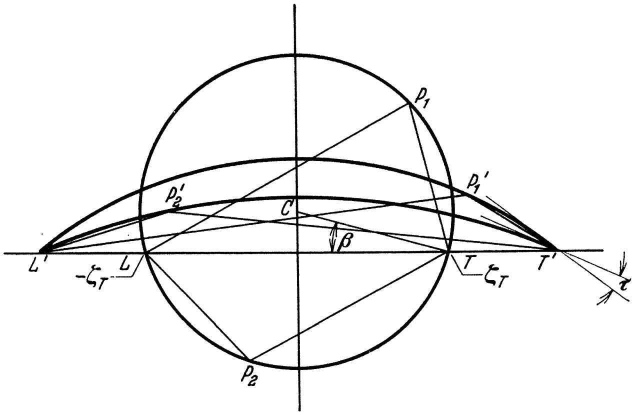

Fig. 37.

$$\left.\begin{array}{l} \theta_1 = 2\beta\left(1-\frac{\tau}{2\pi}\right)+\frac{\tau}{2} \\ \theta_2 = 2\beta\left(1-\frac{\tau}{2\pi}\right)-\frac{\tau}{2} = \theta_1 - \tau \end{array}\right\} \tag{19.10}$$

If we express β (*i. e.* the angle between the first axis and the chord) in terms of θ_1 and θ_2, we obtain:

$$\beta = \frac{\theta_1+\theta_2}{4\left(1-\frac{\tau}{2\pi}\right)} \tag{19.11}$$

The inclination of the bisector of the tail angle at the trailing edge is $(\theta_1+\theta_2)/2$; hence, the first axis divides the angle between this bisector and the chord in two nearly equal parts. The exact ratio of the two parts is $(1-\tau/\pi):1$; the angle between the first axis and the chord being slightly larger than the angle between the first axis and the bisector.

In the special case of an airfoil composed of a circular arc as the upper contour and a straight line as the lower, we must put $\theta_2 = 0$, $\theta_1 = \tau$ and (19.11) becomes:

$$\beta = \frac{\tau}{4\left(1-\frac{\tau}{2\pi}\right)} \tag{19.12}$$

i. e. the angle of attack of no lift is approximately equal to one quarter of the tail angle at the trailing edge. This rule gives a fair approximation for the zero lift direction of thick profiles as used in the design of airscrews.

The lift is given by: $L = 4\pi \varrho V^2 a \sin(\alpha + \beta)$ (19.13)

In order to obtain the slope of the C_L curve, we have to express the ratio c/a. Obviously: $c = T'L' = 2\varkappa b = 2\varkappa a \cos\beta$ (19.14)

Whence for small values of β: $\frac{c}{a} = 2\varkappa$ (19.15)

and the lift: $L = \frac{2\pi\varrho V^2}{\varkappa} c(\alpha + \beta)$ (19.16)

whence: $C_L = 4\frac{\pi}{\varkappa}(\alpha + \beta)$ (19.17)

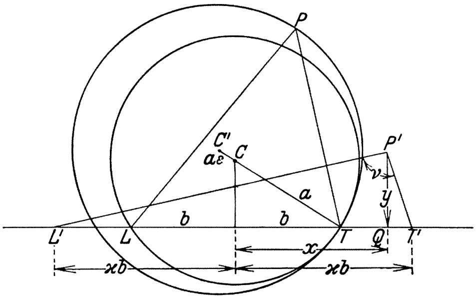

Fig. 38.

The slope of the C_L curve is slightly larger than for a single circular arc; the ratio is equal to $2/\varkappa = 2/(2 - \tau/\pi) = 1/\left(1 - \frac{\tau}{2\pi}\right)$.

The moment with reference to the focus, *i. e.* the part of the moment which is independent of the lift, is essentially given by the second coefficient in the development represented by (19.6).

As the coefficient is real, the inclination γ of the second axis to the ξ axis is equal to zero and the moment with reference to the focus is:

$$M_F = 2\pi\varrho V^2 \left(\frac{\varkappa^2 - 1}{3}\right) b^2 \sin 2\beta \qquad (19.18)$$

The coefficient C_{MF} will be:

$$C_{MF} = 4\pi \frac{\varkappa^2 - 1}{3} \frac{b^2}{c^2} \sin 2\beta = \pi \frac{\varkappa^2 - 1}{3} \frac{1}{\varkappa^2} \sin 2\beta \qquad (19.19)$$

Comparing this result with the corresponding value obtained for a Joukowski profile with the same β (*i. e.* approximately with the same mean camber line) we find:

$$\frac{C_{MF}\,(\text{Kármán-Trefftz})}{C_{MF}\,(\text{Joukowski})} = \frac{\varkappa^2 - 1}{3} \frac{4}{\varkappa^2} \cong 1 - \frac{1}{3}\frac{\tau}{\pi} \qquad (19.20)$$

The influence of the tail angle on the stability characteristics of the profile is very small. As has been pointed out, we cannot expect a radical change in the stability conditions without changing the practically constant curvature of the main camber line.

Proceeding to the generalized case (see Fig. 38), we remove the center of the generating circle from C and, keeping constant the direction of the first axis, we choose as center the point C', at the distance $(1 + \varepsilon)\,a$

from the trailing edge. The radius of the generating circle is obviously equal to $(1+\varepsilon)\,a$. Let us consider an arbitrary point P on the circle. Using the transformation (19.8) the angle TPL goes over into:

$$\angle T'P'L' = \varkappa(\angle TPL) \tag{19.21}$$

and the ratio between the distances $T'P'$ and $L'P'$ will be:

$$\frac{T'P'}{L'P'} = \left(\frac{TP}{LP}\right)^{\varkappa} \tag{19.22}$$

We introduce for the angle $T'P'L'$ the notation ν and for the ratio $T'P'/L'P'$ the notation s. Then remembering that:

$$\overline{L'P'}^2 - \overline{P'T'}^2 = \overline{L'Q}^2 - \overline{QT'}^2 = (L'Q - QT')\cdot L'T' = 2\,x\cdot L'T'$$

and also:

Area $L'P'T' = 1/2\, L'P'\cdot P'T' \sin\nu = 1/2\, L'T'\cdot y$, we readily express the Cartesian coordinates $OQ = x$ and $P'Q = y$ of the point P' by the following equations:

$$\frac{x}{\varkappa b} = \frac{1-s^2}{1+s^2-2\,s\cos\nu}$$

$$\frac{y}{\varkappa b} = \frac{2\,s\sin\nu}{1+s^2-2\,s\cos\nu}$$

These equations together with (19.21) and (19.22) enable us to plot the actual shape of the airfoil. Compared with Joukowski profiles the aerodynamic characteristics of the airfoil are only slightly changed by the finite tail angle. However, we obtain better information concerning the velocity and pressure distribution along the airfoil. We shall see later (21) that the generalized transformation (19.1) is especially useful for the computation of the aerodynamic characteristics of given airfoils.

20. The Mises Family of Airfoils. We remember that for all profiles considered in the last sections the second axis coincides with the ξ axis connecting the trailing and leading edge of the skeleton, so that the "diving moment" M_F cannot be negative. Consequently all these airfoils have an inherent instability. In the limiting case of the symmetrical skeleton, the airfoil is at the limit of instability, but unfortunately symmetrical airfoils have rather poor qualities as far as stalling angle is concerned, and their use is practically restricted to cases in which equal lift characteristics in both directions are desired. Thus they are used as fins or rudders, or as airfoils for airplanes with special features for stunt flight. If high stalling angle and inherent stability of the airfoil are both required, we must change the shape of the camber line. The development of such airfoil profiles cannot be done without experimental test, because at the present time we are not able to predict stalling characteristics by purely theoretical calculations. However, theoretically developed stable airfoil profiles can serve as starting points for experimental tests.

Let us consider the general transformation:

$$z = \zeta + \frac{k_1}{\zeta} + \frac{k_2}{\zeta^2} + \frac{k_3}{\zeta^3} + \dots \tag{20.1}$$

and assume that we cut the rest of the series after the term k_n/ζ^n. Differentiating both sides of (20.1) we obtain:

$$\frac{dz}{d\zeta} = 1 - \frac{k_1}{\zeta^2} - \frac{2\,k_2}{\zeta^3} - \dots - \frac{n\,k_n}{\zeta^{(n+1)}} \tag{20.2}$$

and we can write obviously:

$$\frac{dz}{d\zeta} = \left(1 - \frac{\lambda}{\zeta}\right)\left(1 - \frac{\lambda_1}{\zeta}\right)\left(1 - \frac{\lambda_2}{\zeta}\right)\dots\left(1 - \frac{\lambda_n}{\zeta}\right) \tag{20.3}$$

We wish a sharp point at the trailing edge, so that we must put one of the coefficients λ equal to ζ_T, where ζ_T represents the trailing edge in the ζ plane. Without restricting the generality of our investigation, we can put $\zeta_T = b$ where b is a real quantity. Now the expressions (20.2) and (20.3) are equal, and we conclude that carrying out the multiplications indicated in (20.3) the coefficient of $1/\zeta$ vanishes. Hence:

$$\lambda + \sum_1^n \lambda_i = 0 \tag{20.4}$$

or:

$$\sum_1^n \lambda_i = -b \tag{20.5}$$

a real quantity. The complex quantities $\lambda_1, \lambda_2, \dots \lambda_n$ define n points in the ζ plane. According to (20.5) the center of gravity of these points is on the real axis at the distance $-b/n$ from the origin. The Joukowski transformation is generalized in this way by using n singular points instead of the one used as "the leading edge of the skeleton". We replace now the leading edge of the skeleton by n points, so that the center of gravity of the whole system of points, including T, is invariably at the origin O.

The second term of the development (20.2) can also be easily calculated.

We obtain:

$$-k_1 = \lambda \sum_1^n \lambda_i + \frac{1}{2} \sum_1^n \sum_1^n \lambda_i \lambda_j \,(i \neq j) \tag{20.6}$$

or:

$$-k_1 = -b^2 + \frac{1}{2} \sum_1^n \sum_1^n \lambda_i \lambda_j \,(i \neq j) \tag{20.7}$$

We remember that the angle γ between the chord and second axis is equal to the half of the argument of k_1 [see (17.3)]. Denoting $1/2\, \Sigma\Sigma \lambda_i \lambda_j$ by Π, we write

$$2\,\gamma = arg.\,(b^2 - \Pi) \tag{20.8}$$

In order to obtain stable airfoils $(\gamma + \beta)$ must be negative [see (14.15)], where β denotes the angle between the first axis and the real axis. This requires γ to be negative, while its absolute magnitude must satisfy the condition:

$$|\gamma| = \left|\frac{1}{2} arg.(b^2 - \Pi)\right| > \beta \tag{20.9}$$

We consider the cases $n = 2$ and $n = 3$ (transformations with 3 or 4 terms).

a) In the case of $n = 2$ it follows from (20.5) that $(\lambda_1 + \lambda_2)$ must be real and equal to $-b$. We therefore write:

$$\left.\begin{aligned} \lambda_1 &= -\frac{b}{2} + \mu b e^{-i\theta} \\ \lambda_2 &= -\frac{b}{2} - \mu b e^{-i\theta} \end{aligned}\right\} \tag{20.10}$$

where, for convenience, the negative sign in $e^{-i\theta}$ is used.

The coefficient k_1 will be given by:

$$k_1 = b^2 - \lambda_1 \lambda_2 = b^2 \left(\frac{3}{4} + \mu^2 e^{-2i\theta}\right) \tag{20.11}$$

or:

$$k_1 = b^2 \left(\frac{3}{4} + \mu^2 \cos 2\theta - i \mu^2 \sin 2\theta\right) \tag{20.12}$$

With $\theta < \frac{\pi}{2}$ the argument 2γ of this expression will be negative. It is easily seen that:

$$|\gamma| = \frac{1}{2} \tan^{-1} \frac{\mu^2 \sin 2\theta}{3/4 + \mu^2 \cos 2\theta} \tag{20.13}$$

We draw a straight line with the inclination γ through the point T, *i. e.* through the trailing edge in the ζ plane. For a negative γ this line slopes upward to the left, and in order to obtain stability, the center of the generating circle must be located between the chord and this line, as can be inferred from Fig. 30. This condition is expressed by the equation:

$$\frac{\mu^2 \sin 2\theta}{3/4 + \mu^2 \cos 2\theta} > \tan 2\beta \tag{20.14}$$

b) In the case of $n = 3$, it is convenient to put one of the singular points at $\zeta = -b$. In this special case the two other singular points must be arranged symmetrically to the origin O. We write therefore:

$$\lambda_2 = \mu b e^{-i\theta}, \qquad \lambda_3 = -\mu b e^{-i\theta} \tag{20.15}$$

$$\frac{dz}{d\zeta} = \left(1 - \frac{b^2}{\zeta^2}\right)\left(1 - \mu^2 b^2 \frac{e^{-2i\theta}}{\zeta^2}\right) \tag{20.16}$$

and we obtain for the coefficients k_1, k_2, k_3 the following values:

$$k_1 = b^2 (1 + \mu^2 e^{-2i\theta}), \qquad k_2 = 0, \qquad k_3 = -\frac{1}{3} \mu^2 b^4 e^{-2i\theta} \tag{20.17}$$

The absolute value of the inclination of the second axis is given by:

$$|\gamma| = \frac{1}{2} \tan^{-1} \frac{\mu^2 \sin 2\theta}{1 + \mu^2 \cos 2\theta} \tag{20.18}$$

Hence the condition of stability for this family of airfoils is:

$$\frac{\mu^2 \sin 2\theta}{1 + \mu^2 \cos 2\theta} > \tan 2\beta \tag{20.19}$$

We choose $2\theta = \pi/2$. The stability condition (20.19) will be in this case $\mu^2 > \tan 2\beta$. Putting, for instance, $\tan\beta = 0.1$—corresponding to an angle of attack of about -6^0 for zero lift—we obtain $\mu^2 > 0.2$. In order to obtain intrinsic stability for the airfoil we must therefore assume two singular points at a distance equal to 0.44 b or more, in addition to the two singular points used in the Joukowski transformation. The correction terms which modify the shape of the profile are: $z_1 = \frac{b^2\mu^2 e^{-2i\theta}}{\zeta}$ and $z_3 = -\frac{1}{3}\frac{b^4\mu^2 e^{-2i\theta}}{\zeta^3}$. These corrections can be computed analytically or graphically. The reader may be referred for details to the original papers of v. Mises[1] and W. Müller[2], or to the treatise of A. Toussaint and E. Carafoli[3], which gives an excellent review of the different methods for plotting airfoils and velocity distributions. Toussaint and Carafoli have also developed further extensions of the airfoil theory, obtaining profiles of very general character.

21. Aerodynamic Characteristics of Given Airfoils (see Plate IV). Direct methods for finding the aerodynamic characteristics of airfoils with a given shape have been developed by v. Kármán and Trefftz, and later by W. Müller, by Höhndorf and by Theodorsen. The problem is identical with that of finding the conformal transformation between the given airfoil and a circle. This can be accomplished in two steps: the first step is to transform the given airfoil profile into a curve of nearly circular shape by one of the simple transformations used for the inverse method in the previous sections. The second step is to distort the curve of nearly circular shape into a circle. In the general case this second step cannot be carried out exactly; however, we are able to find successive approximations with fairly good convergence.

We start from the "physical" z plane and use the transformation of **19**:

$$\left(\frac{z - \varkappa b}{z + \varkappa b}\right)^{1/\varkappa} = \frac{z_1 - b}{z_1 + b} \tag{21.1}$$

The value of the exponent is connected with the tail angle of the profile at the trailing edge by the relation:

$$\varkappa = 2 - \frac{\tau}{\pi} \tag{21.2}$$

We put the trailing edge at the point $z = \varkappa b$ and must then choose b in such a way that the point $-\varkappa b$ lies inside of the airfoil profile. We know by the approximate computations given in **17** and **19** that $2\varkappa b$ is slightly smaller than the chord of the airfoil, hence we put the point $-\varkappa b$ near to the boundary of the profile, in the neighborhood of the point of greatest curvature.

[1] v. Mises, R., Zeitschr. f. Flugtechnik u. Motorluftschiffahrt, **11**, p. 68, 1920.

[2] Müller, W., Zeitschr. f. angew. Math. u. Mech., **3**, p. 117, 1923; **4**, pp. 213, 389, 1924.

[3] Toussaint, A. et Carafoli, E., Théorie et tracés des profils d'ailes sustentrices (Paris 1928).

Plotting values of z_1 corresponding to the points z of the given airfoil profile, we obtain a curve which may be called a "nearly circular boundary". Due to the transformation (21.1) the sharp angle at the trailing edge has disappeared and we obtain a boundary with continuous slope at every point.

We have next to find the transformation between the z_1 plane and a new ζ plane in which the curve corresponding to the nearly circular boundary appears as an exact circle. We can use successive approximations or reduce the problem to the resolution of a system of an infinite number of linear equations.

The method of successive approximations, used by Trefftz, v. Kármán and W. Müller, is based on the idea of the harmonic analysis. We draw the circle which has the same tangent at the trailing edge (corresponding to the point T) as the nearly circular boundary, and the same area as the region enclosed by the latter curve. We denote the center of this circle by C, its radius by R (see Fig. 39), and the coordinates of the nearly circular curve, measured from the center C, by x_2, y_2. The corresponding complex quantity may be denoted by $z_2 = x_2 + i\,y_2$, or using polar coordinates, $z_2 = r e^{i\theta}$. Obviously we can write:

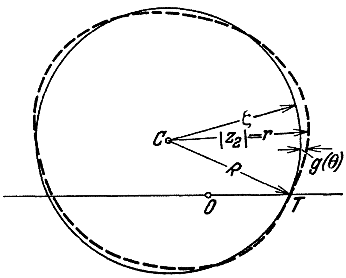

Fig. 39.

$$r = R\,[1 + g\,(\theta)] \tag{21.3}$$

as the equation of the nearly circular boundary, and $g\,(\theta)$ will be small compared with 1. The angle θ is measured from the line $C\,T$, so that the trailing edge corresponds to $z_2 = R$; hence $g\,(0) = 0$.

The function which establishes the conformal transformation of the outside of this boundary into the outside of the circle C may be denoted by:

$$\zeta = z_2\,[1 + f\,(z_2)] \tag{21.4}$$

where ζ and z_2 have their origin at the center C and the absolute value of $f\,(z_2)$ is again small compared with 1. We write:

$$\log \zeta = \log z_2 + \log\,[1 + f\,(z_2)] \tag{21.5}$$

Obviously for points on the circle the real part of $\log \zeta$ must be equal to $\log R$ and developing $\log\,[1 + f\,(z_2)]$ we write as first approximation:

$$\log R = Re.\,[\log z_2] + Re.\,[f\,(z_2)] \tag{21.6}$$

According to (21.3) the real part of $\log z_2$ is equal to $\log R\,[1 + g\,(\theta)]$ or approximately to $\log R + g\,(\theta)$ and we obtain:

$$g\,(\theta) + Re.\,[f\,(z_2)] = 0 \tag{21.7}$$

We now give the problem the following formulation: we have to determine a complex function $f(z_2)$ whose real part takes given values $-g\,(\theta)$

on the orbit of the circle C. In order to calculate $f(z_2)$ we develop the function $g(\theta)$ in a Fourier series:

$$g(\theta) = \sum_{1}^{\infty} a_n \cos n\theta + \sum_{1}^{\infty} b_n \sin n\theta \tag{21.8}$$

The constant term of the development is equal to zero, because the area of our region is equal to the area of the circle with the radius R, and hence the mean value of $g(\theta)$ vanishes.

The coefficients a_n, b_n can be determined by graphical integration, according to the well known formulae of the Fourier analysis:

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta\, d\theta \end{aligned}\right\} \tag{21.9}$$

On the other hand, the complex function $f(z_2)$ has the form, for $|z_2| > R$:

$$f(z_2) = C_0 + \sum_{1}^{\infty} \frac{C_n}{z_2^n} \tag{21.10}$$

because its derivative vanishes at infinity and has no singularities outside of the circle with the radius R. We put $z_2 = Re^{i\theta}$ and determine the coefficients C_0, $C_1 \ldots$ in such a way that the real part of $-f(Re^{i\theta})$ is equal to the right hand side of (21.8). We satisfy this condition by putting $-C_n = R^n(a_n + ib_n)$ and C_0 a pure imaginary quantity which is for the first hand arbitrary. We can choose C_0 in such a way that the point corresponding to the trailing edge shall not be changed by the transformation. We remember that $\theta = 0$ corresponds to this invariable point and obtain $C_0 - i\sum_{1}^{\infty} b_n = 0$, $\sum_{1}^{\infty} a_n$ being zero in consequence of the condition $g(0) = 0$ which is satisfied by g. Consequently the mapping function between the ζ and z_2 planes is given by:

$$\log\zeta = \log z_2 + i\sum_{1}^{\infty} b_n - \sum_{1}^{\infty} \frac{a_n + ib_n}{z_2^n} R^n \tag{21.11}$$

The sum of the coefficients $b_1 \ldots b_n$ can be expressed by the integral:

$$\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\cot\frac{\theta}{2}\, d\theta$$

We notice that the method used in this section is very similar to that used in the approximate theory of thin wings. In both cases we have to find an approximation for conformal transformation between a region outside of a nearly circular boundary and a region outside of a circle.

The formula (21.11) enables us to calculate the aerodynamic characteristics of the given airfoil. We start again from the flow around the circle in the ζ plane. Denoting the velocity at an arbitrary point of the circle by v_ζ, the velocity at the corresponding point of the airfoil in the physical z plane will be:

$$v_z = v_\zeta \left| \frac{d\zeta}{dz_2} \right| \cdot \left| \frac{dz_2}{dz_1} \right| \cdot \left| \frac{dz_1}{dz} \right| \tag{21.12}$$

Now $|dz_2/dz_1| = 1$, because the z_1 and z_2 planes differ only by a displacement of the origin, while $|d\zeta/dz_2|$ and $|dz_1/dz|$ can be calculated from (21.11) and (21.1).

Höhndorf's method is based on Bieberbach's theorems for conformal transformation of regions with nearly circular boundary into a circle[1]. In the procedure explained above, terms with higher powers of $f(z_2)$ were neglected, while Höhndorf retains the higher terms and obtains an infinite system of linear equations. Instead of the Fourier analysis this system of equations must be solved in order to compute the coefficients of the transformation. The coefficients in the system of linear equations are integrals similar to the Fourier coefficients. Höhndorf actually keeps terms with $[f(\theta)]^2$ and neglects higher powers. It is presumably easier to carry out two steps of successive approximations, each one requiring a simple Fourier analysis, than to solve a system of linear equations. Practically, the first approximation given by (21.11) will be sufficiently exact.

22. The Theory of Biplanes. In **12** some rules were given for the computation of the mutual interference between two airfoils of infinite span. The exact theory of the flow around two airfoils of given shape is of course essentially more complicated than the theory of the monoplane wing. It has been worked out for straight and circular airfoils. We restrict ourselves here to the case of two straight lines.

In the theory of biplanes we have the first example of a flow in a region which is more than doubly connected. In the case of the monoplane the circulation (the line integral of the velocity) vanishes if the path of integration is reducible while it has a given value—in general different from zero—for all paths including the airfoil. Considering the biplane we have to distinguish between the following possibilities:

(*a*) the path is reducible,

(*b*) the path includes one of the airfoils,

(*c*) the path includes both airfoils.

In case (*a*) the circulation is zero; in case (*b*) the circulation is equal to Γ_1 or to Γ_2, corresponding to the airfoil included; in case (*c*) the circulation is $\Gamma_1 + \Gamma_2$.

[1] Höhndorf, F., Verfahren zur Berechnung des Auftriebes gegebener Tragflächen-Profile, Zeitschr. f. angew. Math. u. Mech. **6**, p. 265, 1926.

Consequently in order to construct the flow around the biplane corresponding to a given angle of attack, we have to calculate and to superpose the following elementary flow pictures:

(*a*) a flow without any circulation, with given velocity at infinity;

(*b*) a flow with the circulation Γ_1 around the first airfoil but without circulation around the second airfoil, with vanishing velocity at infinity;

(*c*) a flow with the circulation Γ_2 around the second airfoil but without circulation around the first airfoil, with vanishing velocity at infinity.

Instead of (*b*) and (*c*) it is often more convenient to use the following combination which is of course equivalent:

(*b'*) a flow with circulation of equal magnitude and the same direction around each airfoil, which we shall call the "circulation flow";

(*c'*) a flow with circulation of equal magnitude and opposite direction around each airfoil, which—following the terminology of M. Munk—we call the "counter-circulation flow".

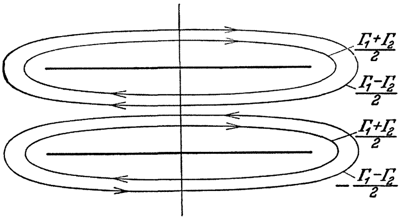

Fig. 40.

Fig. 40 shows in a schematic way the circulation and counter-circulation flow for an unstaggered biplane with wings of equal chords. The circulation is equal to $(\Gamma_1 + \Gamma_2)/2$ for each airfoil in the first case and to $\pm (\Gamma_1 - \Gamma_2)/2$ in the second case.

We now generalize the "condition of smooth flow" at the trailing edge by assuming finite velocity at the trailing edges of *both* airfoils. This assumption gives us two equations which determine the two values Γ_1 and Γ_2, and thence the circulation of the "circulation flow" and that of the "counter-circulation".

Considering the case of two parallel straight airfoils located in the x, y plane with arbitrary stagger and chord-ratio, we first show that this general case can be reduced—by conformal transformation—to the case of the flow around two straight lines located on the real axis of the plane, *i. e.* to the case of a kind of "tandem" biplane. However, while each of the wings of the "real" biplane corresponds to one of the wings of the "ideal tandem biplane", the edges of the real biplane do not go over into the edges of the tandem biplane. As in the case of the transformation used in the monoplane theory, $d\zeta/dz$ is infinite at these points, so that finite velocity at the trailing edges in the z plane means zero velocity at the corresponding points in the ζ plane. These points appear as stagnation points for the flow in the ζ plane.

The conformal transformation requires the use of elliptic functions. The easiest way to find the transformation between the physical z plane and the ideal ζ plane takes advantage of the "hydrodynamic analogy". We use the same analogy in the following section for the investigation of the flow through a "lattice". We have, therefore, to explain the idea at sufficient length.

Let us consider a potential flow in the ζ plane and a corresponding complex potential function $F(\zeta)$ with the following characteristics:

(*a*) the velocity function $w = dF/d\zeta$ has at infinity the constant value $w = e^{i\beta}$;

(*b*) two portions of the real axis $k_1 < \xi < k_2$ and $k_3 < \xi < k_4$ are portions of stream-lines. We suppose that $k_1 < k_2 < k_3 < k_4$.

We cannot find immediately the complex potential function $F(\zeta)$ but we can write directly the expression for the velocity function, putting:

$$w(\zeta) = i \sin\beta \frac{(\zeta - \lambda_1)(\zeta - \lambda_2)}{\sqrt{(\zeta - k_1)(\zeta - k_2)(\zeta - k_3)(\zeta - k_4)}} + \cos\beta \qquad (22.1)$$

where λ_1 and λ_2 denote two additional real constants. Especially we assume $k_1 < \lambda_1 < k_2$, $k_3 < \lambda_2 < k_4$ (see Fig. 41). Obviously $w = e^{i\beta}$ for $\zeta = \infty$. On the other hand, for real values of ζ the numerator of the first term on the right hand side of the equation is real, and the denominator will be real at all points except where $k_1 < \xi < k_2$ or $k_3 < \xi < k_4$, *i. e.* over the two portions mentioned above under (*b*). For points located on one of these two portions of the ξ axis, the denominator is a purely imaginary quantity, so that the velocity function w is real. Consequently, the two portions $k_1 \ldots k_2$ and $k_3 \ldots k_4$ appear as portions of stream-lines.

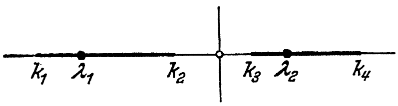

Fig. 41.

The flow picture established by (22.1) gives us immediately the conformal transformation between the ζ and the z plane. If we put $z = F(\zeta)$ as the equation of transformation and define $F(\zeta)$ as the complex potential function corresponding to w, then we may write:

$$z = x + iy = F(\zeta) = \int_0^\zeta w(\zeta)\, d\zeta = \varphi_\zeta + i\psi_\zeta \qquad (22.2)$$

Whence:

$$x = \varphi_\zeta, \qquad y = \psi_\zeta.$$

It results that the network of equipotential and stream-lines in the ζ plane ($\varphi_\zeta =$ const. and $\psi_\zeta =$ const.) will correspond to a system of vertical and horizontal lines ($x =$ const., $y =$ const.) in the z plane. The portions $k_1 \ldots k_2$ and $k_3 \ldots k_4$ of the ξ axis being portions of stream-lines, the "wings of our ideal tandem biplane" appear as horizontal lines in the z plane, and we consider them as the wings of an actual biplane in the physical z plane. The flow represented by (22.1) appears in the z plane as the flow parallel to the wings of the actual biplane corresponding to the case of an angle of attack zero. Every other flow picture in the ζ plane around the wings of the ideal tandem biplane gives us—if transferred upon the physical z plane—a flow around the actual biplane.

However, in order to apply this method to actual cases we have to compute the relations between the geometrical characteristics of the actual biplane and the constants of the transformation (22.1) or (22.2). This represents the principal mathematical work to be done, and requires the discussion of the elliptic integral

$$\int (\zeta - \lambda_1)(\zeta - \lambda_2)\, d\zeta \Big/ \sqrt{(\zeta - k_1)(\zeta - k_2)(\zeta - k_3)(\zeta - k_4)}$$

We consider in detail only the case $\beta = \pi/2$ and assume $k_1 = -1$, $k_2 = -k$, $k_3 = k$, $k_4 = 1$ and furthermore $\lambda_1 = -\lambda$, $\lambda_2 = \lambda$. In this case the two wings of the "ideal tandem" are equal and symmetrically arranged with respect to the origin $\zeta = 0$ (see Fig. 42). The velocity at infinity is equal to unity, and the direction of the flow at infinity is normal to the ξ axis or to the wings of the ideal tandem. The conformal transformation is given by:

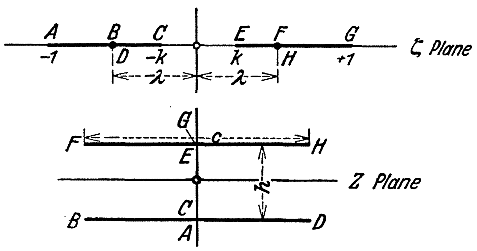

Fig. 42.

$$z = i \int_0^{\zeta} \frac{\zeta^2 - \lambda^2}{\sqrt{(\zeta^2 - 1)(\zeta^2 - k^2)}}\, d\zeta \tag{22.3}$$

For $\zeta = \infty$ we have $dz/d\zeta = i$. The origin $\zeta = 0$ corresponds to the origin $z = 0$. For $\zeta = \pm\lambda$, $dz/d\zeta = 0$, so that these points correspond to the edges of the actual biplane. The coordinates of the edges in the z plane are:

$$z_e = \pm i \int_0^k \frac{\xi^2 - \lambda^2}{\sqrt{(1 - \xi^2)(k^2 - \xi^2)}}\, d\xi \pm \int_k^{\lambda} \frac{\xi^2 - \lambda^2}{\sqrt{(1 - \xi^2)(\xi^2 - k^2)}}\, d\xi \tag{22.4}$$

We obtain four values for z_e, namely $\pm c/2 \pm ih/2$, with the following expressions for c and h:

$$c = 2 \int_k^{\lambda} \frac{\lambda^2 - \xi^2}{\sqrt{(1 - \xi^2)(\xi^2 - k^2)}}\, d\xi \tag{22.5}$$

$$h = 2 \int_0^k \frac{\lambda^2 - \xi^2}{\sqrt{(1 - \xi^2)(k^2 - \xi^2)}}\, d\xi \tag{22.6}$$

Corresponding points are indicated respectively by identical letters in Fig. 42[1].

Hence in this special case the two wings have equal chords and the biplane is unstaggered. The chord is equal to c and the distance between the wings is h. We have yet to determine the constant λ introduced in the transformation without specific conditions. However, in order to use the integral in (22.2) as a mapping function we must be sure that the integral is independent of the path. This will be the case if the integral $\int w\, d\zeta$ extended over any closed path vanishes. This obviously happens for all reducible paths, but the same must hold also for integrals including each of the wings of the "ideal tandem". In the general case this gives two conditions determining λ_1 and λ_2. In the special case under consideration we have to satisfy only one condition, namely:

$$I = \oint_c \frac{\xi^2 - \lambda^2}{\sqrt{(1-\xi^2)(\xi^2-k^2)}}\, d\xi = 0 \qquad (22.7)$$

The integral is extended from k to 1 along the lower side of the segment EG and from 1 to k along the upper side. The contributions from both portions are equal, hence:

$$I = 2\int_k^1 \frac{\xi^2 - \lambda^2}{\sqrt{(1-\xi^2)(\xi^2-k^2)}}\, d\xi = 0 \qquad (22.8)$$

The condition that this integral vanishes can be interpreted from the point of view of the hydrodynamic analogy in the sense that the flow used in order to establish the relation between the ζ and z planes has no circulation around the wings of the "ideal tandem".

The condition (22.8) can be written:

$$\int_k^1 \frac{\xi^2\, d\xi}{\sqrt{(1-\xi^2)(\xi^2-k^2)}} = \lambda^2 \int_k^1 \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-k^2)}} \qquad (22.9)$$

or, using the substitution, u being an auxiliary variable:

$$\sqrt{1-\xi^2} = u\sqrt{1-k^2}; \quad -\frac{\xi\, d\xi}{\sqrt{1-\xi^2}} = \sqrt{1-k^2}\, du; \quad \xi = \sqrt{1-(1-k^2)\,u^2},$$

[1] In order to determine the position of corresponding points in both planes it is necessary to have some data concerning the argument of the complex quantity $dz/d\zeta$. Want of space forbids us to treat this matter here in detail (the reader who is interested in such questions may be referred to the appendix to IV **17**, where a related problem is discussed); for the sake of information we give the following statements relative to the values of $dz/d\zeta$ at points of the ξ axis:

to the right of G	:	$dz/d\zeta$ is positive imaginary
segment GF, upper side	:	positive real
segment FE, upper side	:	negative real
segment EC	:	positive imaginary
segment CB, upper side	:	positive real
segment BA, upper side	:	negative real
to the left of A	:	positive imaginary

we obtain:

$$\int_0^1 \sqrt{\frac{1-(1-k^2)\,u^2}{1-u^2}}\,d\,u = \lambda^2 \int_0^1 \frac{d\,u}{\sqrt{(1-u^2)\,[1-(1-k^2)\,u^2]}} \qquad (22.10)$$

If we take k as the "modulus" of a system of elliptic functions, and introduce the "complementary modulus" $k' = \sqrt{1-k^2}$, then the definite

integrals: $$\int_0^1 \frac{\sqrt{1-k'^2\,u^2}}{\sqrt{1-u^2}}\,d\,u\,, \qquad \int_0^1 \frac{d\,u}{\sqrt{1-u^2}\,\sqrt{1-k'^2\,u^2}}$$

are called the "complete elliptic integrals" corresponding to k'. We shall use for them the notation[1]: $E\,(k')$, $F\,(k')$, or also: $E'\,(k)$, $F'\,(k)$. Hence we obtain for λ^2 the value:

$$\lambda^2 = \frac{E\,(k')}{F\,(k')} = \frac{E'\,(k)}{F'\,(k)}$$

Using this value of λ^2, the expressions for the chord c and the gap h of the biplane will be:

$$c = \frac{2\,E'}{F'} \int_k^\lambda \frac{d\,\xi}{\sqrt{(1-\xi^2)\,(\xi^2-k^2)}} - 2 \int_k^\lambda \frac{\xi^2\,d\,\xi}{\sqrt{(1-\xi^2)\,(\xi^2-k^2)}}$$

and: $$h = \frac{2\,E'}{F'} \int_0^k \frac{d\,\xi}{\sqrt{(1-\xi^2)\,(k^2-\xi^2)}} - 2 \int_0^k \frac{\xi^2\,d\,\xi}{\sqrt{(1-\xi^2)\,(k^2-\xi^2)}} \qquad (22.11)$$

Using the same transformation as for the complete integrals, we obtain:

$$\left.\begin{aligned} c &= \frac{2}{F'}\left[-E'\cdot F'\left(\sqrt{\frac{1-\lambda^2}{1-k^2}}\right) + F'\cdot E'\left(\sqrt{\frac{1-\lambda^2}{1-k^2}}\right)\right] \\ h &= \frac{2}{F'}\left[E'\cdot F'\left(\frac{1}{\sqrt{1-k^2}}\right) - F'\cdot E'\left(\frac{1}{\sqrt{1-k^2}}\right)\right] \end{aligned}\right\} \qquad (22.12)$$

where the notations $F'\,(p)$ and $E'\,(p)$ stand for the incomplete elliptical integrals F and E with the limits 0 and p, the modulus in all cases being k'.

The transformation being established, the aerodynamic characteristics of the biplane can be computed in a comparatively easy way. Instead of the complex potential function, we give the expressions for the velocity function.

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[1] For an account of the theory of elliptic functions the reader can be referred to various textbooks, *e. g.* to E. T. WHITTAKER and G. N. WATSON, Modern Analysis (Cambridge University Press).

The integral which has been denoted by $F\,(k')$ resp. $F'\,(k)$ in the text, often is also denoted by $K\,(k')$ or by $K'\,(k)$. — Tables of the values of E and F are mostly given as functions of the "modular angle" α, which is defined by: $k = \sin\alpha$.

Obviously we have: $E'\,(k) = E\left(\frac{\pi}{2} - \alpha\right)$, $\quad F'\,(k) = F\left(\frac{\pi}{2} - \alpha\right)$.

This function, for the case of the flow without circulation with the angle of attack α, can be obtained by combination of a flow with the velocity $V\cos\alpha$ at infinity in the x direction with a flow perpendicular to the wings with the velocity $V\sin\alpha$ at infinity (see Fig. 43). The first type of flow is obviously undisturbed in the z plane by the presence of the parallel wings, while the second type of flow appears in the ζ plane as an undisturbed flow parallel to the tandem wings, because, due to $(dz/d\zeta)_{\zeta=\infty} = i$, the velocities at infinity are perpendicular to each other in the two planes for the same type of flow. Hence, for the combined flow we write:

$$w_1(z) = V\cos\alpha + V\sin\alpha \frac{d\zeta}{dz} \qquad (22.13)$$

where:

$$\frac{d\zeta}{dz} = -i\frac{\sqrt{(\zeta^2-1)(\zeta^2-k^2)}}{\zeta^2-\lambda^2} \qquad (22.14)$$

Due to the relation $d\zeta/dz = -i$ for $\zeta = \infty$, we have:

$$w_1(\infty) = V\cos\alpha - i\,V\sin\alpha,$$

which corresponds to a flow at infinity with the angle of attack α.

We now proceed to find the velocity function for the circulation flow; it is easier to establish the function for the complex velocity in the ζ plane than in the z plane. This function $w_2(\zeta)$ is characterized by the following conditions:

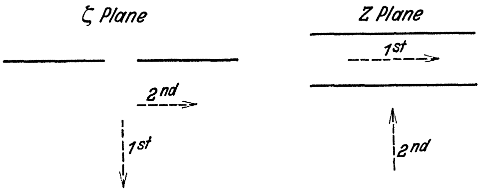

Fig. 43.

a) $w_2(\zeta)$ behaves at infinity as i/ζ;

b) $w_2(\zeta)$ is real at points on the real axis between -1 and $-k$ and between k and 1;

c) there are no singular points except at $\zeta = \pm k$ and $\zeta = \pm 1$.

In order to satisfy these conditions, we write, A being a real constant:

$$w_2(\zeta) = \frac{iA\zeta}{\sqrt{(\zeta^2-k^2)(\zeta^2-1)}} \qquad (22.15)$$

It is obvious that the flow given by (22. 15) contains the segments of the ξ axis between -1 and $-k$, resp. $+k$ and $+1$, as portions of stream-lines. For infinitely great values of ζ (22.15) approaches to:

$$w_2(\zeta) = \frac{iA}{\zeta}$$

The corresponding part of the velocity in the infinite part of the z plane is, to the same order of approximation:

$$w_2(z) = w_2(\zeta)\cdot\frac{d\zeta}{dz} = \frac{iA}{z}$$

Hence it follows that for a path in the z, or physical plane, including both wings, the circulation has the value:

$$2\pi A \qquad (22.16)$$

As the circulation for both wings is the same (as will be seen from the symmetry of the arrangement), $w_2(z)$ represents the "circulation flow" according to the definition given at the beginning of the section.

Finally:
$$w_3(\zeta) = \frac{i B}{\sqrt{(\zeta^2 - k^2)(\zeta^2 - 1)}} \tag{22.17}$$

represents the counter circulation flow. We have in this case the conditions (*b*) and (*c*) as before. At infinity the velocity function behaves as i/ζ^2 so that the total circulation vanishes. The absolute value of the circulation around each of the wings is equal to:

$$2|B| \int_k^1 \frac{d\xi}{\sqrt{(\xi^2 - k^2)(1 - \xi^2)}} = 2|B|F' \tag{22.18}$$

The coefficients A and B are determined by the condition of smooth flow at the trailing edges. We remember that at these points $d\zeta/dz = \infty$ so that the velocity in the ζ plane must vanish. The edges correspond to the points $\zeta = \pm\lambda$ on the lower side of the tandem wings, so that the conditions for smooth flow are satisfied if $w(\zeta)$ vanishes at these points. We write $w(\zeta) = w_1(z)\frac{dz}{d\zeta} + w_2(\zeta) + w_3(\zeta)$

and take in account, that $\frac{dz}{d\zeta} = 0$ at the points considered, furthermore, that (for positive A) $w_2(\zeta)$ has the same negative real value both at H and D, while $w_3(\zeta)$ has opposite signs at these two points. Thus we find:

$$\left.\begin{aligned} V \sin\alpha - \frac{A\lambda + B}{\sqrt{(\lambda^2 - k^2)(1 - \lambda^2)}} &= 0 \\ \text{and} \qquad V \sin\alpha - \frac{A\lambda - B}{\sqrt{(\lambda^2 - k^2)(1 - \lambda^2)}} &= 0 \end{aligned}\right\} \tag{22.19}$$

Hence:
$$\left.\begin{aligned} A &= \frac{V \sin\alpha \sqrt{(\lambda^2 - k^2)(1 - \lambda^2)}}{\lambda} \\ B &= 0 \end{aligned}\right\} \tag{22.20}$$

Finally the circulation around each wing has the value:

$$\Gamma_1 = \Gamma_2 = \frac{\pi V \sin\alpha \sqrt{(\lambda^2 - k^2)(1 - \lambda^2)}}{\lambda} \tag{22.21}$$

Equation (22.21) establishes the relation between the circulation and the parameters λ and k. On the other hand, (22.12) represents the relations between the same parameters and the geometrical quantities h and c.

The total lift acting on the wing system—equal to $2\varrho\Gamma V$—is always inferior to the lift which would result for the two wings, considered as monoplanes with the same chord and angle of attack. The distribution of the forces on the two wings can be found by calculating the circular integral (4.10) along paths including one of the wings. We find that the lift for the upper wing is slightly higher than that for the lower wing. Furthermore, a small drag acts on the upper wing, the lower wing receiving a small propulsive force of the same amount. The difference between the lift at the upper and the lower wing is of the order α^2, the drag and propulsion of the order α^3, α denoting the angle of attack.

The calculation is somewhat more elaborate in the case of different chords and stagger, but can be carried out without difficulty by a complete discussion of the integral in (22.2). In the case of "decalage" and circular airfoils, the hydrodynamic analogy does not lead to a simple solution of the problem, and we have to recur to the conformal transformation of the triply connected region including the two airfoils in a region of annular shape having concentric circles as boundaries. This transformation has been used in the original papers of Kutta[1] and is discussed in detail by Villat[2]. The two straight or circular airfoils go over into the circular boundaries, and the infinite point of the physical plane is mapped by an interior point in the ideal plane. We have further to assume a "doublet" (source-sink) at this point in order to establish the flow in the annular region, corresponding to the flow around the two airfoils.

Munk has given approximate solutions for the case of slightly curved airfoils and of airfoils with small decalage, using the theory of thin airfoils[3]. He first calculates the interference between the two wings of the biplane, replacing the actual airfoils by parallel straight lines as in the present section. He then calculates the distortion of the velocity field in the neighborhood of each wing according to the approximate theory explained in **12** and obtains corrections for decalage and curvature of the airfoils.

It may be noted, that while the two-dimensional theory of biplanes has surely an instructive value, its importance for practice is restricted, because the interference due to the trailing vortices is of the same order as the interaction considered in the two-dimensional theory[4].

23. Flow through a Lattice Composed of Airfoils. The flow through a system of an infinite number of identical airfoils has important applications, especially to the theory of airscrews and turbines. In the case of a "lattice" composed of straight airfoils an exact solution can be found by a generalization of the Joukowski transformation, using the hydrodynamic analogy explained in the foregoing section.

Let us consider the Joukowski transformation

$$z = f(\zeta) = \zeta + \frac{a^2}{\zeta} \tag{23.1}$$

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[1] Kutta, W. M., Sitzungsberichte bayr. Akademie, p. 65, 1911.

[2] Villat, H., Leçons sur l'hydrodynamique, Chapter II (Paris 1929).

[3] Munk, M., Nat. Adv. Comm. Aeronautics (Washington) Rep. No. 151, 1922.

[4] A theory which takes into account the influence of the trailing vortices is considered in IV **25—28**.

from the point of view of the hydrodynamic analogy. Considering $z = f(\zeta)$ as a complex potential function, (23.1) represents the flow around a circle $|\zeta| = a$, composed of the parallel flow $z_1 = \zeta$ and a flow produced by a doublet $z_2 = a^2/\zeta$ located at the center of the circle. The parallel flow can be considered as produced by a doublet at infinity. Let us now remove the source and the sink composing the doublet at infinity to the points $\zeta = -a/\varkappa$ and $\zeta = a/\varkappa$, $\varkappa$ being a positive real number, smaller than unity. In order to keep the circle as a stream-line, we must then also separate the source and the sink which compose the doublet in the center, and remove them to the points $-\varkappa a$ and $\varkappa a$ (see Fig. 44). Consequently we write:

$$z = f(\zeta) = \frac{g}{2\pi}\left[\log\frac{\zeta + a/\varkappa}{\zeta - a/\varkappa} + \log\frac{\zeta + a\varkappa}{\zeta - a\varkappa}\right] \tag{23.2}$$

where g is supposed to be a real quantity.

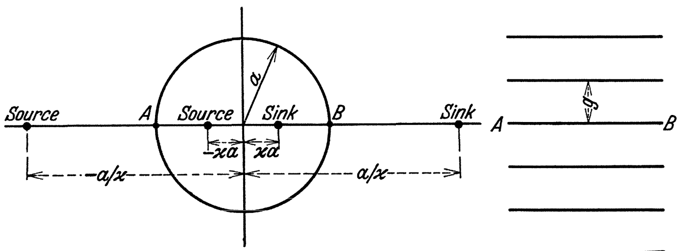


Fig. 44.

Interpreting z as a mapping function, it is obvious that the circle will be represented in the z plane by an infinite number of parallel straight lines; the distance between two consecutive lines will be g, corresponding to the difference $2\pi i$ between consecutive values of the logarithmic function. We obtain in this way the mapping function between the circle in the ζ plane and a system of straight airfoils in the z plane. This lattice is without stagger because g is supposed to be a real quantity.

In order to obtain the mapping function for the general case of a staggered lattice, we assume instead of the sources and sinks, combinations of sources and sinks with vortices (see Fig. 45). It is understood furthermore that the circle remains a stream-line if we assume at the points $a\varkappa$ and $a/\varkappa$ and also at the points $-a\varkappa$ and $-a/\varkappa$, vortices of opposite sense. We write in completion of (23.2):

$$z = \frac{1}{2\pi}\left[(g - ih)\log\frac{\zeta + a/\varkappa}{\zeta - a/\varkappa} + (g + ih)\log\frac{\zeta + a\varkappa}{\zeta - a\varkappa}\right] \tag{23.3}$$

Equation (23.3) gives us the mapping function between a circle with the radius a and a staggered lattice composed of straight airfoils. It will be seen that z is multivalued; the values differ by

$(1/2\,\pi)\;(g - ih)2\pi i n = (h + i\,g)\,n$, where n is an integer, because the leaves of the ζ plane are connected at the points $\pm\,a/\varkappa$ and so the value of $log\;(\zeta + a/\varkappa)$ or $log\;(\zeta - a/\varkappa)$ changes by $2\,\pi\,i$ or $-2\,\pi\,i$ when we proceed from one leaf to the next one. The normal distance between the airfoils is obviously g, the relative displacement of the airfoils in the x direction h. Introducing the distance between the leading edges d and the stagger ratio $\sigma = tan\,\beta$, we obtain $h = d\,sin\,\beta$, $g = d\,cos\,\beta$ and (23.3) becomes, with the introduction of an unimportant additive constant:

$$z = \frac{d}{2\,\pi}\left[e^{-i\beta}\,log\,\frac{a + \varkappa\,\zeta}{a - \varkappa\,\zeta} + e^{+i\beta}\,log\,\frac{\zeta + a\,\varkappa}{\zeta - a\,\varkappa}\right] \tag{23.4}$$

The parameter $\varkappa$ is connected with the chord-gap ratio. In order to establish the relation between them, we have to calculate the chord, *i. e.* the distance between leading and trailing edge of one of the airfoils.

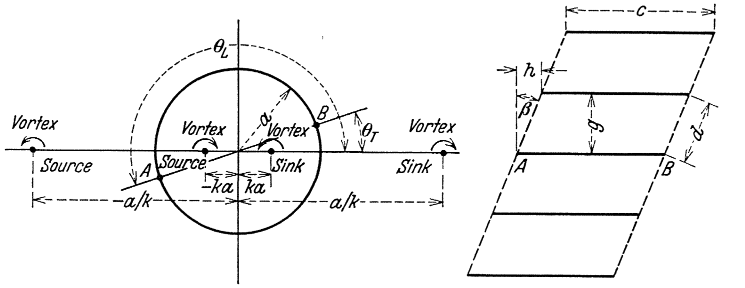


Fig. 45.

At the edges we have $dz/d\zeta = 0$; hence for $\zeta = \zeta_L$ (A, Fig. 45) and $\zeta = \zeta_T$ (B, Fig. 45):

$$e^{-i\beta}\left[\frac{\varkappa}{a + \varkappa\,\zeta} + \frac{\varkappa}{a - \varkappa\,\zeta}\right] + e^{+i\beta}\left[\frac{1}{\zeta + a\,\varkappa} - \frac{1}{\zeta - a\,\varkappa}\right] = 0 \tag{23.5}$$

Introducing $\zeta = a e^{i\theta}$, we obtain after a simple but somewhat lengthy calculation,

$$tan\,\theta = \frac{1 - \varkappa^2}{1 + \varkappa^2}\,tan\,\beta \tag{23.6}$$

Denoting the two roots of this equation by θ_L and $\theta_T = \theta_L - \pi$ we write:

$$tan\,\theta_T = tan\,\theta_L = \frac{1 - \varkappa^2}{1 + \varkappa^2}\,tan\,\beta \tag{23.7}$$

We have now to introduce $\zeta_L = a e^{i\theta_L}$ and $\zeta_T = -a e^{i\theta_L}$ in (23.4) and calculate the difference between the corresponding values x_L and x_T. We obtain in this way the expression for the chord:

$$\left.\begin{aligned} c = \frac{2d}{\pi}\Bigg[&cos\,\beta\,log\,\frac{\sqrt{\varkappa^4 + 2\,\varkappa^2\,cos\,2\,\beta + 1} + 2\,\varkappa\,cos\,\beta}{1 - \varkappa^2} + \\ &+ sin\,\beta\,tan^{-1}\,\frac{2\,\varkappa\,sin\,\beta}{\sqrt{\varkappa^4 + 2\,\varkappa\,cos\,2\,\beta + 1}}\Bigg]\end{aligned}\right\} \tag{23.8}$$

Equation (23.8) establishes the relation between the chord-gap ratio and the parameter $\varkappa$ of the transformation (23.4).

We now proceed to the calculation of the flow, first in the ideal ζ plane. It must be a potential flow with the circle as stream-line and having singularities outside of the circle only at the points $\zeta = -a/\varkappa$ and $\zeta = a/\varkappa$. The potential function

$$F = Vz = \frac{Vd}{2\pi}\left[e^{-i\beta} \log \frac{a+\varkappa\zeta}{a-\varkappa\zeta} + e^{+i\beta} \log \frac{\zeta+a\varkappa}{\zeta-a\varkappa}\right] \tag{23.9}$$

satisfies these conditions, but in the z plane it obviously represents the simple case of uniform flow with velocity V and zero angle of attack. We therefore generalize (23.9), putting:

$$F = \frac{Vd}{2\pi}\left[e^{-i(\beta+\alpha)} \log \frac{a+\varkappa\zeta}{a-\varkappa\zeta} + e^{+i(\beta+\alpha)} \log \frac{\zeta+a\varkappa}{\zeta-a\varkappa}\right] \tag{23.10}$$

In order to obtain the physical interpretation of this equation, we calculate the velocity at infinity. We must distinguish between the two sides of the lattice. The point $\zeta = -a/\varkappa$ corresponds to infinity in front of the lattice (*i. e.* to the left of it), the point $\zeta = +a/\varkappa$ to infinity in rear (*i. e.* to the right) of the lattice. We use the notations $z = -\infty$ and $z = +\infty$ for these two points. Then:

$$\left[\frac{dF}{dz}\right]_{-\infty} = \left[\frac{dF}{d\zeta}\right]_{\zeta=-a/\varkappa} \cdot \left[\frac{d\zeta}{dz}\right]_{\zeta=-a/\varkappa} = V e^{-i\alpha} \tag{23.11}$$

We thus obtain easily:

$$\left[\frac{dF}{dz}\right]_{\infty} = \left[\frac{dF}{dz}\right]_{-\infty} = V e^{-i\alpha} \tag{23.12}$$

Therefore, the potential function represented by (23.10) corresponds to a flow with equal velocities at infinity in front and at the rear of the lattice. The lattice does not produce any deflection in the flow in this case and no force acts on the airfoils. In order to investigate the general case, a "circulation flow" is to be added. We write

$$F_1 = \frac{iK}{2\pi} \log \frac{\zeta^2 - a^2\varkappa^2}{\zeta^2 - a^2/\varkappa^2} \tag{23.13}$$

The circle $|\zeta| = a$ appears also in the flow represented by (23.13) as composed of stream-lines. The flow is produced by two vortices of the same sense located at $\zeta = a/\varkappa$ and $\zeta = -a/\varkappa$ and by two opposite vortices located inside of the circle. The circulation around the circle is therefore $2K$.

Putting $F + F_1$ for the resultant flow, we calculate the resultant velocity at the trailing edge. The condition that $(dF/d\zeta)_T$ vanishes determines the amount of the circulation. We obtain by a little calculation:

$$K = \frac{2\varkappa \sin\alpha}{\sqrt{\varkappa^4 + 2\varkappa^2 \cos 2\beta + 1}} Vd \tag{23.14}$$

Considering now the velocity field in the z plane, it is obvious that the velocity will be different at infinity on the two sides of the lattice. We extend the circulation integral around one of the airfoils, taking as path two portions of straight lines parallel to the front line of the lattice and two portions composed of two straight lines parallel to the airfoils, at a distance equal to the spacing of the lattice. Obviously, the integral along the last mentioned portions of the closed path vanishes, the velocities being identical. Therefore, the circulation Γ will be equal to:

$$\Gamma = [V_t(-\infty) - V_t(\infty)] \frac{g}{\cos\beta} \tag{23.15}$$

V_t denoting the component of the velocity in the direction of the "front line" of the lattice, *i. e.* along a direction inclined by the angle β to the y axis, taken positive if directed upward. The length $d = g/\cos\beta$ is the spacing of the lattice measured along the same front line.

Denoting the velocity component normal to the front line by V_n, we obtain easily:

$$V_n(-\infty) = V_n(\infty) = V\cos(\alpha+\beta) \tag{23.16}$$

$$\left.\begin{aligned} V_t(-\infty) &= V\sin(\alpha+\beta) + \frac{2\varkappa V}{\sqrt{\varkappa^4 + 2\varkappa^2\cos 2\beta + 1}}\sin\alpha \\ V_t(+\infty) &= V\sin(\alpha+\beta) - \frac{2\varkappa V}{\sqrt{\varkappa^4 + 2\varkappa^2\cos 2\beta + 1}}\sin\alpha \end{aligned}\right\} \tag{23.17}$$

In accordance with the condition of continuity, V is equal on both sides. The component V_t changes. It may be noted that the "angle of attack" is also changed by the circulation flow and has the value:

$$\tan^{-1}\frac{\sin\alpha + \frac{2\varkappa}{R}\cos\beta\sin\alpha}{\cos\alpha + \frac{2\varkappa}{R}\sin\beta\sin\alpha}, \text{ instead of } \alpha,$$

where

$$R = \sqrt{\varkappa^4 + 2\varkappa^2\cos 2\beta + 1}$$

The forces on the lattice can be calculated directly by application of the equations for the momentum, or by application of (4.10), extending the integral along a path including one airfoil. We obtain the following values for the components of the force on a single airfoil:

$$\left.\begin{aligned} X &= -4\varrho V^2 d\frac{\varkappa}{R}\sin^2\alpha \\ Y &= 4\varrho V^2 d\frac{\varkappa}{R}\sin\alpha\cos\alpha \end{aligned}\right\} \tag{23.18}$$

Equations (23.8), (23.16), (23.17) and (23.18) thus give the complete solution of the problem.

In the case of a lattice without stagger ($\beta = 0$), (23.8) can be solved for $\varkappa$ and we obtain $\varkappa = \tanh\left(\frac{c}{d}\frac{\pi}{4}\right)$ and $2\varkappa/R = \tanh\left(\frac{c}{d}\frac{\pi}{2}\right)$. Thus the expressions for the components of the resulting force will be:

$$\left.\begin{aligned} X &= -2\varrho V^2 d\tanh\left(\frac{c}{d}\frac{\pi}{2}\right)\sin^2\alpha \\ Y &= 2\varrho V^2 d\tanh\left(\frac{c}{d}\frac{\pi}{2}\right)\sin\alpha\cos\alpha \end{aligned}\right\} \tag{23.19}$$

Obviously for small values of c/d we obtain:

$$\left.\begin{aligned} X &\cong -2\pi \frac{\varrho V^2}{2}\, c\, sin^2\alpha \\ Y &\cong 2\pi \frac{\varrho V^2}{2}\, c\, sin\,\alpha\, cos\,\alpha \end{aligned}\right\} \qquad (23.20)$$

i. e. the well-known expressions for the components of the force acting on a plane airfoil in an infinitely extended airflow. For large values of c/d, $tanh\left(\frac{c}{d}\frac{\pi}{2}\right)$ approaches unity and we obtain the following expressions for the components of the forces acting per unit length of the lattice:

$$\left.\begin{aligned} \frac{X}{d} &= -4\frac{\varrho V^2}{2} sin^2\alpha \\ \frac{Y}{d} &= 4\frac{\varrho V^2}{2} sin\,\alpha\, cos\,\alpha \end{aligned}\right\} \qquad (23.21)$$

The "angle of attack" α', *i. e.* the angle between direction of the velocity at infinity in front of the lattice and the direction of the vanes is given by: $tan\,\alpha' = \left(1 + \frac{2\varkappa}{R}\right) tan\,\alpha = \left(1 + tanh\frac{c}{d}\frac{\pi}{2}\right) tan\,\alpha$ (23.22)

The inclination α'' of the velocity vector at the rear of the lattice is given by: $tan\,\alpha'' = \left(1 - \frac{2\varkappa}{R}\right) tan\,\alpha = \left(1 - tanh\frac{c}{d}\frac{\pi}{2}\right) tan\,\alpha$ (23.23)

Hence:

$$tan\,\alpha'' = e^{-\frac{c\pi}{d}}\, tan\,\alpha' \qquad (23.24)$$

or

$$tan\,\alpha' - tan\,\alpha'' = \left(1 - e^{-\frac{c\pi}{d}}\right) tan\,\alpha'$$

If $c/d \gg 1$, $tan\,\alpha'' \to 0$, *i. e.* the velocity behind the lattice is parallel to the vanes and the stream is deflected by the angle α'. Thus we can consider the factor $\left(1 - e^{-\frac{c\pi}{d}}\right)$ as the efficiency factor of the lattice as far as the deflection of the stream is concerned.

If the ratio c/d is small, the flow around any of the airfoils can be determined approximately by replacing the rest of the airfoils by vortex filaments. This method has the advantage that it can be applied to the case of a lattice composed of slightly curved airfoils as shown by Betz[1].

24. Some Examples of the Application of Conformal Transformation to Problems Connected with Airfoils. Plates II—IV show the results of certain examples carried through numerically and by actual geometrical construction, as illustrations of the deductions given in sections **16—21**. Short explanations of these diagrams will be of use to the reader.

Plate II, Figs. 1 and 2, represent two Joukowski sections, see **17—18**. Fig. 1 gives an example of a rather thick section, Fig. 2 shows a thin section. The skeleton is in both cases a circular arc $T'\,L'$, the curvature of the skeleton is determined by the angle β, which gives the inclination between the tangent to the circular arc $T'\,L'$ and the chord $T'\,L'$. The

[1] Betz, A., Ingenieur-Archiv **3**, p. 359, 1931.

Plates II–IV
(Division E)

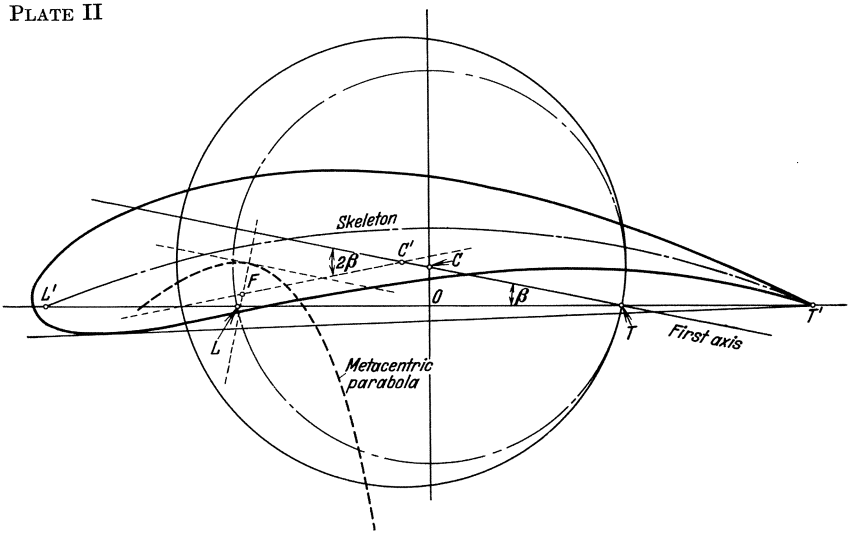


Fig. 1. Joukowski airfoil.

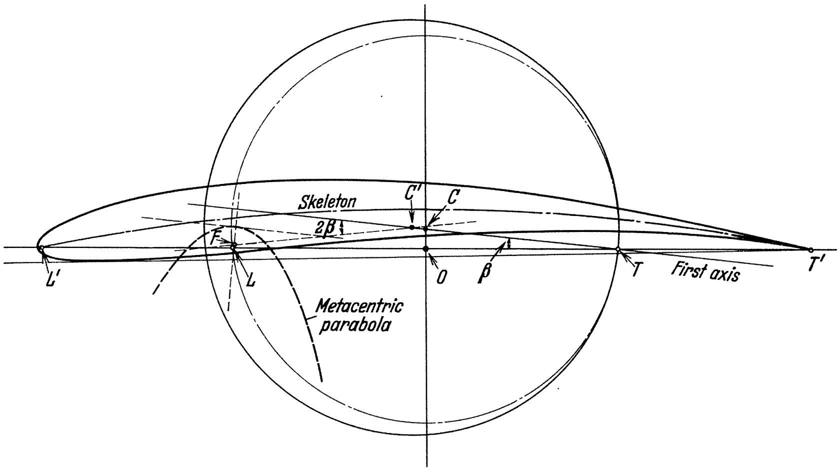


Fig. 2. Joukowski airfoil.

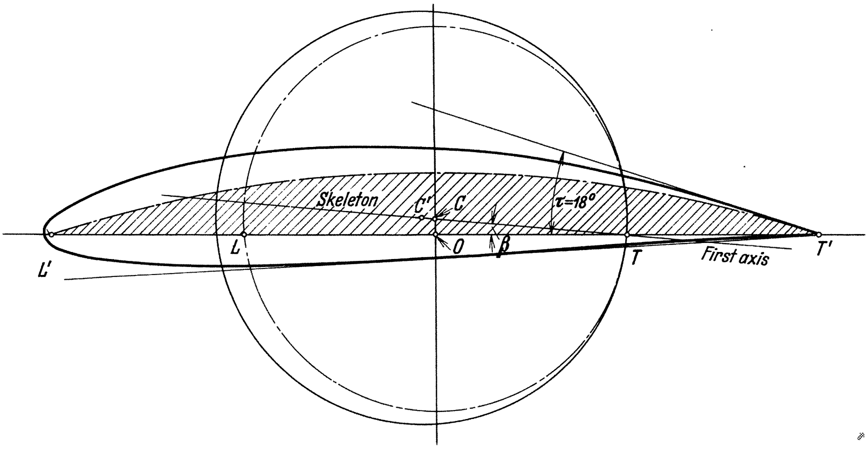


Fig. 3. Kármán-Trefftz airfoil.

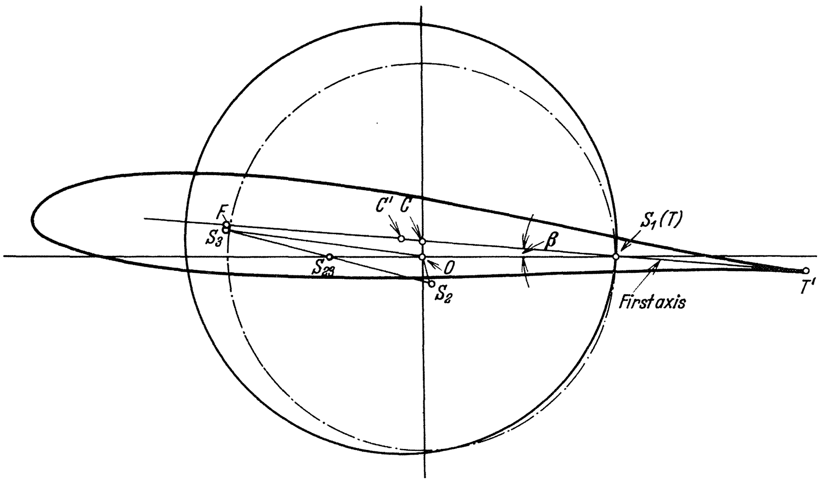


Fig. 1. Mises airfoil with 3 singular points.

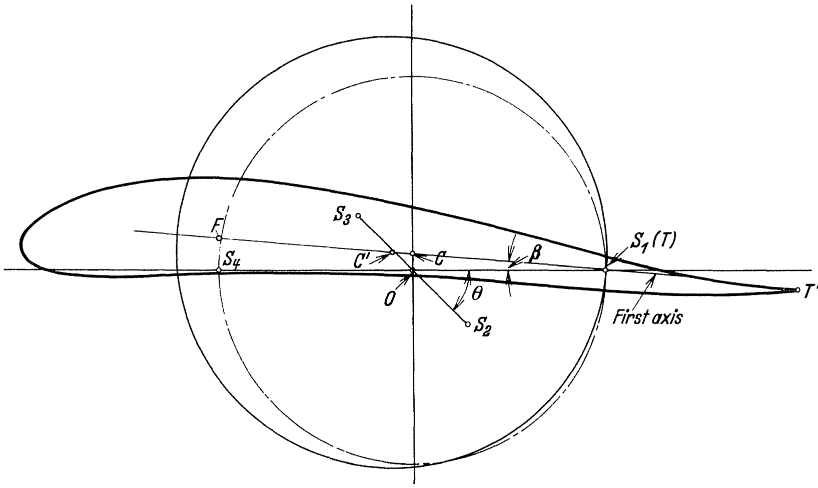


Fig. 2. Mises airfoil with 4 singular points.

PLATE IV

Fig. a

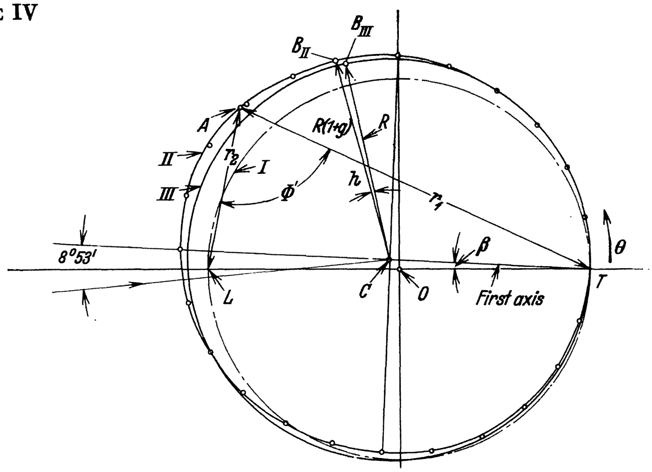


Fig. b

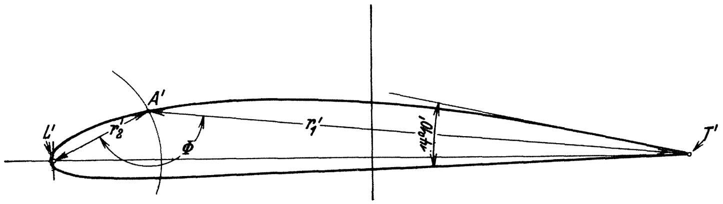


Fig. c

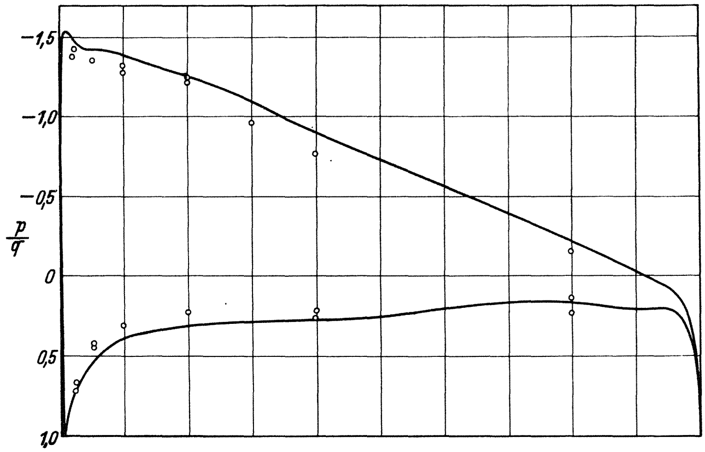
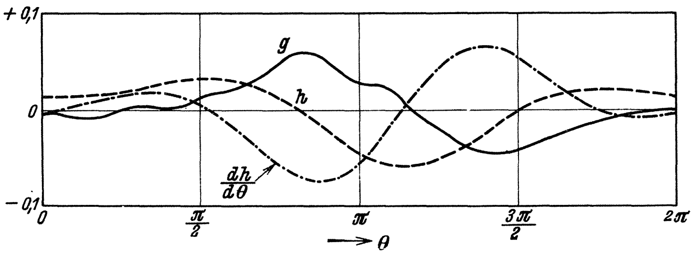


Fig. d



Figs. a—d. Computation of the pressure distribution for a Clark Y airfoil by the Kármán-Trefftz method.

angle β represents also the inclination between the first axis and the chord. For the thick section, $tan\,\beta = 0.2$; for the thin section, $tan\,\beta = 0.1$. The thickness of the airfoils is determined essentially by the eccentricity ratio $\varepsilon = CC'/CT$; this is chosen equal to 0.15 for the thick and to 0.075 for the thin section. The metacentric parabola is shown in both cases.

Plate II, Fig. 3, represents a so-called Kármán-Trefftz airfoil section (see **19**). The skeleton is a "crescent", bounded in this case by a circular arc and a straight line. The characteristics of this section are: the angle τ at the trailing edge $= 18^0$, the angle β, the inclination between the first axis and the chord $T'L'$ of the skeleton, $= tan^{-1}\,0.0825$, the eccentricity ratio $\varepsilon = CC'/CT = 0.07$.

Plate III, Figs. 1 and 2, represent two airfoils generated by the use of the generalized Joukowski transformation as suggested by v. Mises (see **20**). They are calculated in such a way that both have neutral stability, *i. e.* the metacentric parabola degenerates into the focus. The first and the second axes coincide in this case, and the focus is located on their common trace. The line of attack of the lift passes through the focus for all angles of attack, and the moment for the angle of attack corresponding to zero lift vanishes. Fig. 1 represents an airfoil generated by a transformation built up by three singular points, S_1 (coincident with the trailing edge T), S_2 and S_3. The location of the two latter singular points is given by $\zeta = b\,(-1/2 \pm \mu e^{-i\theta})$ where $b = OT$, $\mu = 0.545$, $\theta = \pi/12 = 15^0$. The eccentricity ratio $\varepsilon = CC'/CT = 0.11$ and the inclination of the first axis, which coincides in this case with the second axis, is $\beta = -\gamma = (1/2)\,tan^{-1}\,0.147$. In the second example, (Fig. 2) four singular points S_1, S_2, S_3, S_4 are used. S_1 and S_4 have the same locations as T and L in the case of the Joukowski transformation proper. S_2 and S_3 are given by $\zeta = \pm\,\mu b e^{-i\theta}$ with $b = OT$, $\mu = 0.4$, $\theta = \pi/4$. The eccentricity ratio is $\varepsilon = 0.11$ as in the first example. The inclination between $S_1 S_4$ and the first axis, which coincides also in this case with the second axis, $\beta = -\gamma = (1/2)\,tan^{-1}\,0.16$.

Plate IV shows an example of the computation of the pressure distribution for a given airfoil. The profile chosen for this purpose is the so-called Clark Y, best known in the United States and shown at b) on Plate IV. The angle enclosed between the two tangents at the trailing edge is, $\tau = 14^0\,10'$.

The method used is a slight modification of that given in **21**. The present method was found more suitable for numerical calculation.

We choose the plane represented in Fig. b as the physical z plane and transform the boundary of the airfoil into a nearly circular boundary (II) in a z_1 plane (Fig. a). For this purpose we use the transformation (21.1)

$$\left(\frac{z - \varkappa b}{z + \varkappa b}\right)^{\frac{1}{\varkappa}} = \frac{z_1 - b}{z_1 + b} \tag{24.1}$$

The value of $\varkappa$ corresponding to $\tau = 14^0\, 10'$ is $\varkappa = 2 - \tau/\pi = 1.921$. The two singular points in the z plane are T' and L'. The point T' is given by the trailing edge, the point L' is chosen inside of the boundary near the point of greatest curvature at the leading edge. By the transformation (24.1) the chord $L'T' = 2\varkappa b$ of the skeleton is transformed into LT in a). The length of LT is equal to $2b$. The circle I is the so-called basic circle of the transformation. An arbitrary point A' of the airfoil boundary is transformed into A. The two tangents at the trailing edge T' are transformed into one tangent at the nearly circular boundary II, and the first axis is perpendicular to this tangent. We then determine the position of the center C in such a way that the circle III with C as center and $CT = R$ as radius shall have the same area as the nearly circular boundary II. The polar coordinates of an arbitrary point B of II with C as origin are r and θ_2 and the corresponding complex quantity, $z_2 = r e^{i\theta_2}$. We have now to find a transformation between z_2 and $\zeta = R e^{i\theta}$ in such a way that the region outside of II in the z_2 plane is mapped on the region outside of the circle III in a ζ plane. The origin of both z_2 and ζ is C. For the points at the boundaries II and III respectively, we put

$$\left.\begin{aligned} r &= R\,[1 + g(\theta_2)] \\ \theta_2 &= \theta + h(\theta) \end{aligned}\right\} \tag{24.2}$$

where g and h are considered as small quantities. Geometrically, $(1+g)\,R$ and h are shown on Plate IV, a. As a slight deviation from the procedure sketched in 21, where the point T was kept invariable, it is more convenient for our present purpose to leave the infinity unchanged by the transformation. This transformation can be written

$$\log z_2 = \log \zeta + \sum_1^\infty \frac{a_n + i\,b_n}{\zeta^n} R^n \tag{24.3}$$

Introducing (24.2) into (24.3), we obtain, taking into account that g is small in comparison with R

$$\log R + g(\theta_2) + i\,[\theta + h(\theta)] = \log R + i\,\theta + \sum_1^\infty (a_n + i\,b_n)\, e^{-i n\theta}$$

$$\left.\begin{aligned} g(\theta) &= \sum_1^\infty (a_n \cos n\,\theta + b_n \sin n\,\theta) \\ h(\theta) &= \sum_1^\infty (b_n \cos n\,\theta - a_n \sin n\,\theta) \end{aligned}\right\} \tag{24.4}$$

The function $g(\theta)$ is determined by the shape of II (Fig. a). Exactly, $g R = r - R$ is given as a function of θ_2; however, g and h being small, we replace θ_2 by θ in the argument. After evaluating the coefficients

a_n, b_n by harmonic analysis, the functions $h(\theta)$ and $dh/d\theta$ can be plotted as in Fig. d. In the numerical example six terms are used for $g(\theta)$ and $h(\theta)$. The circles on III, Fig. a, indicate the corresponding approximate values of $g(\theta)$; it is seen that the approximation to the curve II is very close.

In order to find the pressure distribution, the velocity at an arbitrary point of the airfoil boundary must be computed. We start from the flow in the ζ plane, *i. e.* around circle III. Assuming a certain value of the angle of attack ($8^0\,53'$ in the numerical example), we can easily calculate the velocity v_ζ at an arbitrary point of the circle. Then the velocity v_z at the corresponding point of the airfoil is equal to

$$v_z = v_\zeta \left|\frac{d\zeta}{dz_2}\right| \cdot \left|\frac{dz_2}{dz_1}\right| \cdot \left|\frac{dz_1}{dz}\right|$$

The value of $\left|\frac{dz_2}{dz_1}\right|$ is 1, because z_2 and z_1 differ only by having different origins. The value of $\left|\frac{dz_1}{dz}\right|$ is to be calculated from (24.1). Obviously,

$$\log\frac{z-\varkappa b}{z+\varkappa b} = \varkappa \log\frac{z_1-b}{z_1+b}$$

and by differentiation,

$$\left(\frac{1}{z-\varkappa b} - \frac{1}{z+\varkappa b}\right) dz = \varkappa\left(\frac{1}{z_1-b} - \frac{1}{z_1+b}\right) dz_1$$

Hence
$$\frac{dz_1}{dz} = \frac{(z_1-b)(z_1+b)}{(z-\varkappa b)(z+\varkappa b)}$$

Now $|z_1-b| = r_1$, $|z_1+b| = r_2$ (Fig. a) and $|z-\varkappa b| = r_1'$, $|z+\varkappa b| = r_2'$ (Fig. b), thus

$$\left|\frac{dz_1}{dz}\right| = \frac{r_1 r_2}{r_1' r_2'}$$

Finally
$$\left|\frac{d\zeta}{dz_2}\right| = \left|\frac{d\,[R\,e^{i\theta}]}{d\,[R(1+g)\,e^{i(\theta+h)}]}\right| \cong 1 - g - \frac{dh}{d\theta}$$

Hence
$$v_z = v_\zeta\left(1 - g - \frac{dh}{d\theta}\right)\frac{r_1 r_2}{r_1' r_2'}$$

Denoting the pressure at an arbitrary point by p and assuming the values $p = 0$, $v_z = v_\infty$ at infinity, the pressure distribution will be given by $p = (\varrho/2)\,(v_\infty^2 - v_z^2)$. In Fig. c the ordinates represent

$$\frac{p}{q} = \frac{p}{(\varrho/2)\,v_\infty^2} = 1 - \frac{1}{2}\left(\frac{v_z}{v_\infty}\right)^2$$

The circles are taken from measurements (corrected for infinite aspect ratio); they seem to be in very good agreement with the computed values. To be sure, the pressure rise at the trailing edge cannot be found experimentally because, even for small angles of attack, the flow separates before reaching the edge.

CHAPTER III

MATHEMATICAL FOUNDATION OF THE THEORY OF WINGS WITH FINITE SPAN

1. Equations of Motion of the Fluid. According to the program indicated at the end of Chapter I, the theory of the flow around wing systems of finite span will be developed by taking as a starting point the investigation of the influence of external forces upon the motion of a fluid. The basis for our deductions consequently will be formed by the equations of hydrodynamics completed with terms representing these external forces (see Division B I). The influence of viscosity will be neglected. It is true, as noted in I 8, that the presence of viscosity is of prime importance in the production of the flow system that accounts for the lift; likewise the influence of viscosity is responsible for the appearance of the profile drag, for the fact that the circulation remains below the value to be deduced from the hypothesis of tangential flow at the trailing edge of the airfoil profile, and for the existence of a maximum angle of attack beyond which the normal flow system cannot exist. However, viscosity is of only minor importance in the general formulae to be developed and its inclusion would make them too cumbersome for ready application. A proper treatment moreover is impossible as yet, in consequence of the instability of laminar flow for high velocities (high Reynolds' numbers more properly) and the appearance of turbulent motion. Reference to the influence of viscosity will only be made as need may require. Furthermore, the profile drag will not be included in the equations for the motion of the fluid, as it does not materially affect the flow pattern.

Regarding the notation to be employed, we shall denote the coordinates by x, y, z, the components of the velocity by v_x, v_y, v_z and the components of the vorticity (rotation or "curl" of the velocity) by γ_x, γ_y, γ_z. The coordinate axes will form a right hand system. For dealing with problems relating to airfoils it is convenient to take the x axis in the direction of the airflow relative to the airfoil, and the z axis in the downward direction. The y axis then is in the direction of the span, and points to the left for an observer looking against the direction of the airflow.

The coordinate system thus differs from that used in Chapter II. In order to obtain the connection between the two systems it must be noted that the x, y plane of the two-dimensional theory corresponds to the plane $O\,x\,z$ of the three-dimensional case. Moreover, as the reader will have observed, in the two-dimensional theory the x axis is not always parallel to the direction of the general motion of the fluid; often it is taken along the chord of the airfoil section. The y axis of the two-dimensional theory is obtained from the x axis by a counterclockwise rotation through 90°; thus if the x axis is parallel to the direction of the general motion, the y axis points vertically upward and takes the part of the negative z axis in the three-dimensional theory.

As before, the density of the fluid will be denoted by ϱ, while the pressure, or more properly the difference between the actual pressure and its hydrostatic part, is indicated by p. In this way gravity becomes eliminated from the equations.

The components of the external forces acting immediately on the fluid will be denoted by f_x, f_y, f_z; they refer to unit volume[1].

Other notations will be explained whenever they occur for the first time.

Expressed in this notation the equations of hydrodynamics take the form:

$$\left.\begin{aligned}\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{f_x}{\varrho}\\ \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \frac{f_y}{\varrho}\\ \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{f_z}{\varrho}\end{aligned}\right\} \tag{1.1}$$

to which must be added the equation of continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{1.2}$$

The first point to be demonstrated in connection with these equations is that the action of external forces upon a fluid always produces *vortex motion*. This can be deduced most simply by considering the case of forces of very short duration (impulsive forces). In this manner we obtain a convenient approach to the more general investigation. This forms the subject of **2—5**.

In **6—15** consideration is given to the case of steady motion under the action of forces independent of the time. This case is considered more rigorously than that of impulsive forces and leads to the equations upon which all further deductions are based.

The remainder of the chapter is devoted to various developments of the results thus obtained. In **16—22** are deduced the basic theorems concerning induced resistance, while in **23—29** are developed equations for the calculation of the components of the induced velocity in various special cases. The Kutta-Joukowski theorem is considered in **30**, **31**, and finally in **32** some concluding remarks are given in connection with the problem of finding a system of forces reproducing the action of a given airfoil upon the fluid.

The problem of the motion under forces which depend on time will be deferred to Chapter V.

[1] The dimensions of the quantities f_x etc. are $(M \div L^2\, T^2)$.

A. Motion of a Perfect Fluid Produced by External Forces.

2. Motion Produced by Impulsive Forces. Reference to the action of impulsive forces has already been made in Division B I 8. When forces act during a very short time only, but with a very great intensity, the local acceleration, expressed by the terms $\partial v_x/\partial t$, $\partial v_y/\partial t$, $\partial v_z/\partial t$ in (1.1), and likewise the pressure gradients $\partial p/\partial x$, $\partial p/\partial y$, $\partial p/\partial z$ produced during that time, will be very great compared to the other terms of the equations. On account of this circumstance the equations may be simplified as follows:

$$\left.\begin{aligned} \frac{\partial v_x}{\partial t} &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{f_x}{\varrho} \\ \frac{\partial v_y}{\partial t} &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \frac{f_y}{\varrho} \\ \frac{\partial v_z}{\partial t} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{f_z}{\varrho} \end{aligned}\right\} \tag{2.1}$$

Together with the equation of continuity we then have four *linear* equations for the determination of the three components v_x, v_y, v_z and the pressure p. They may be attacked in two ways. In order to determine the pressure, we differentiate the first equation of system (2.1) with respect to x, the second with respect to y, the third with respect to z. Adding the results we obtain on the left hand side:

$$\frac{\partial^2 v_x}{\partial x\,\partial t} + \frac{\partial^2 v_y}{\partial y\,\partial t} + \frac{\partial^2 v_z}{\partial z\,\partial t}, \quad \text{or:} \quad \frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

which is zero in consequence of the equation of continuity (1.2). Hence the velocity is eliminated and, after multiplication by ϱ, we have:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{2.2}$$

from which equation the distribution of the pressure can be deduced. The right hand side of this equation represents the divergence of the vector f. Putting the result obtained into (2.1) we obtain three separate equations for v_x, v_y, v_z. This is the method to be followed in most cases.

In the present case, however, the alternate method is preferred, and we begin by eliminating the pressure by means of cross differentiation. Thus we differentiate the third equation of the system (2.1) with respect to y, the second with respect to z, and then take the difference. This gives:

$$\frac{\partial \gamma_x}{\partial t} = \frac{1}{\varrho}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)$$

and analogously:

$$\left.\begin{aligned} \frac{\partial \gamma_y}{\partial t} &= \frac{1}{\varrho}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) \\ \frac{\partial \gamma_z}{\partial t} &= \frac{1}{\varrho}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \end{aligned}\right\} \tag{2.3}$$

It will be seen that the expressions occurring on the right hand side of these equations represent the components of the rotation or "curl" (see Division A VI 9) of the vector f, divided by ϱ.

3. Generation of a Vortex Ring by an Impulsive Pressure Acting over a Circular Area. In order to obtain an interpretation of these equations we investigate a special case. Thus consider the space lying between the planes $z = 0$ and $z = h$, and bounded further by a cylindrical surface of radius a, having its axis parallel to the axis of z (see Fig. 46). Outside of this space the external forces are to be zero; within they are directed parallel to the axis of z (downward). Their intensity is to be constant within this space, with the exception of the region near the cylindrical surface, where it is to fall off rapidly to zero. The system is symmetrical with respect to the axis of z, and between the planes $z = 0$ and $z = h$ is independent of z.

It follows directly that in this case the components of *rot f*, occurring on the right hand side of (2.3), are zero everywhere, with the exception of the region in the neighborhood of the cylindrical surface. As f_z is the only force component different from zero, (2.3) reduces to:

$$\left.\begin{aligned}\frac{\partial \gamma_x}{\partial t} &= \frac{1}{\varrho}\frac{\partial f_z}{\partial y}\\ \frac{\partial \gamma_y}{\partial t} &= -\frac{1}{\varrho}\frac{\partial f_z}{\partial x}\\ \frac{\partial \gamma_z}{\partial t} &= 0\end{aligned}\right\} \qquad (3.1)$$

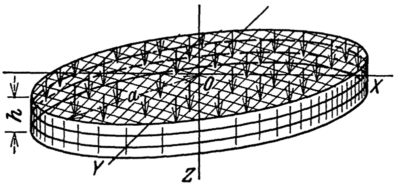


Fig. 46.

Hence we conclude that this system of forces generates vorticity only in the neighborhood of the cylindrical surface, where the forces fall off to zero. When we integrate (3.1) with respect to the time, supposing that the motion started from rest, we find:

$$\left.\begin{aligned}\gamma_x &= \frac{1}{\varrho}\frac{\partial i_z}{\partial y}\\ \gamma_y &= -\frac{1}{\varrho}\frac{\partial i_z}{\partial x}\\ \gamma_z &= 0\end{aligned}\right\} \qquad (3.2)$$

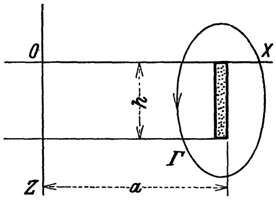


Fig. 47.

where i_z denotes the impulse or time integral (again referred to unit volume) of the forces f_z. It is readily seen that the system of vortex lines obtained in this way consists of circles, having their planes parallel to the x, y plane, and concentric with the axis of z.

If the height h of the space is supposed to be small compared with the radius a, then for many purposes it is sufficient to combine these vortex lines into one single circular vortex, lying practically in the x, y plane, and having the radius a. The total strength Γ of this vortex[1] may be found by considering its section with the x, z plane (Fig. 47,

[1] It may be remembered that the strength of a vortex is measured by the circulation or line integral of the velocity along a closed line encircling the vortex.

where the dotted area indicates the region where f_z falls off to zero) and integrating γ_y over this section. In this way we obtain:

$$\Gamma = \int\int d\,x\,d\,z\,\gamma_y = -\frac{1}{\varrho}\int d\,z\int d\,x\,\frac{\partial\,i_z}{\partial\,x} = \frac{1}{\varrho}\int d\,z\;i_z$$

and thus, as i_z is independent of z between the planes $z = 0$ and $z = h$:

$$\Gamma = \frac{1}{\varrho}\,i_z\,h \tag{3.3}$$

When h is very small, we may consider $i_z h$ as the intensity per unit area of the impulse of the external forces. Denoting this quantity by ϖ, usually called an impulsive pressure[1], we obtain:

$$\Gamma = \frac{\varpi}{\varrho} \tag{3.4}$$

In this way we come to the result that an impulsive pressure applied over a certain circular area will produce a vortex ring along the circumference of the area, the strength of which is equal to the impulsive pressure divided by the density of the fluid.

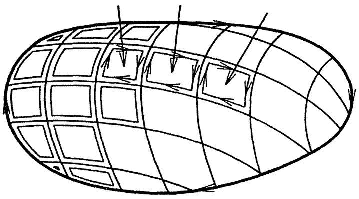


Fig. 48.

It is obvious that this result may be generalized in a great number of ways. In the first place it does not depend on the position of the circular area in space; in the second place it is true for a plane of arbitrary form. It can be generalized equally for an arbitrary curved surface. Indeed such a surface may be considered as the limit of a system of plane facets. Applying to each of these facets the same impulsive pressure (which for every particular facet must act normally to it) we obtain a vortex ring along the circumference of each facet (see Fig. 48). As the strength of all these rings has the same value, it is easily seen that they cancel each other everywhere except for those parts which lie on the outer boundary line of the surface.

4. Action of Continuous Forces. The result obtained can be considered as the key to the understanding of a large number of problems in hydrodynamics. In order to make proper use of it, still further generalizations must be made, and to this end we must revert to the investigation of the action of continuous forces.

It is easily seen that we may consider the action of a continuous force as the limit of a series of small impulses, following so to speak, immediately one after the other. An impulse has been defined as the time-integral of a force acting continuously during a short period of time. Taking the other point of view, we consider a force as the time rate of a series of impulses communicated to the fluid. Applying this

[1] The dimensions of ϖ are: $(M \div L\,T)$.

reasoning to the case before us, it might appear at first sight that when a certain pressure P acts continuously over a given area, a vortex should be generated, lying along the boundary of this area, and having a continuously increasing strength determined by the equation:

$$\frac{d\Gamma}{dt} = \frac{P}{\varrho} \tag{4.1}$$

This reasoning, however, would be incorrect, because no notice has been taken of the fact that a vortex ring, once generated, does not remain at the same place, but possesses a proper motion, and so moves away from the spot where it was formed. It is true that vortices are generated continuously at a rate determined by an equation somewhat resembling the one just mentioned, but all these vortices move away once they are formed, and it must be kept in mind that the motion of a vortex depends both on its own configuration and on the form, position and strength of *all other vortices* present in the field.

Hence the exact determination of the system of vortices generated by continuously acting forces is a matter of great difficulty. This difficulty, indeed, is precisely the one which pervades the whole science of fluid mechanics and which finds its usual mathematical formulation in the statement that the basic equations are not linear but contain terms of the second degree. Leaving aside the theory of irrotational motion, this circumstance considerably hampers a full development of the subject, and with a few exceptions progress has been made for the most part in cases where the motion of the vortices could be neglected, or could be described with sufficient approximation in a very simple way.

In the following discussions restrictions of this character will be necessary and it will be readily seen that the proper choice of the approximation in relation to the problems to be treated is a matter of great importance.

Returning to the problem of a pressure acting continuously over a given area, it has been noted that the motion of the vortices generated, especially under their mutual influence, makes the problem too complicated for an exhaustive treatment. Let us consider, however, the case where, before any external forces were applied, the fluid is assumed to possess a motion with a constant velocity, say V, in the direction of the axis of z downward. This general motion of the fluid will carry the vortices with it, and the case may now arise where this "impressed" motion is much stronger than the proper motion of the vortices themselves, a condition which will result when the applied pressure P is sufficiently small.

In this case the vortices generated by the pressure will arrange themselves on a cylindrical sheet, extending from the boundary of the area over which the pressure acts, downward in the direction of the axis of z (see Fig. 49). This cylinder will thus become what is called a *vortex sheet.* The strength $\bar{\gamma}$ of a vortex sheet is determined by the difference of the velocities on its two sides and is equal to the circulation around a strip of the sheet having unit length in the direction of the axis of the cylinder (see I 9). As the vortices move away with the velocity V, the circulation generated in unit time is given by $V\bar{\gamma}$, and instead of (4.1) we obtain:

$$V\bar{\gamma} = \frac{P}{\varrho} \tag{4.2}$$

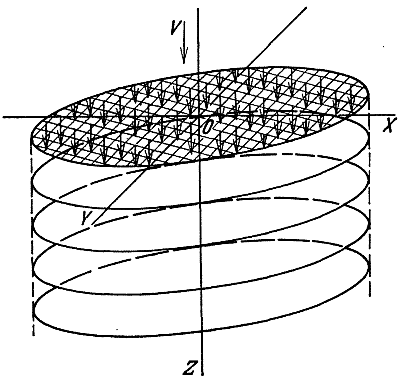


Fig. 49.

The product $V\bar{\gamma}$ at the left hand side, which measures the transfer of vorticity along the sheet in the downward direction, has now taken the place of the quantity $d\Gamma/dt$ in (4.1).

The problem here considered is related to that of the so-called "actuating disc" introduced by Rankine and R. E. Froude in their elementary theory of the propeller. Its solution can be used as a starting point for the theory of the airscrew. The condition mentioned above that the pressure P must be small, when expressed more exactly, takes the form that the ration of P to ϱV^2 must be small compared to unity. When the general motion V is absent, we have the more difficult problem of the airscrew, operating at a fixed point as a fan. If the direction of the general motion is opposite to that of the forces acting on the circle in the x, y plane, we have the case of the windmill.

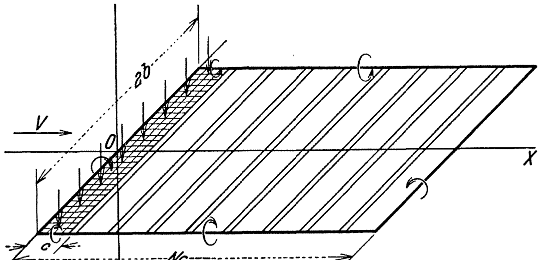


Fig. 50.

For the treatment of these problems the reader is referred to Division L.

The latter case can also be used as a starting point for a theory of the resistance (both frictional and pressure) experienced by a body, held fixed in a moving fluid.

5. Forces Directed Perpendicular to the Original Motion of the Fluid. We shall now consider a case more directly related to the problem of the airplane wing.

In the x, y plane assume a rectangle of dimension c in the direction Ox and of breadth $2b$ in the direction Oy (see Fig. 50). We shall suppose that the fluid has a general motion with the velocity V now in the direction of the axis Ox. An impulsive pressure ϖ in the direction of z is then applied periodically to the rectangle with a period equal to c/V.

At each application of this pressure a rectangular vortex is generated of strength $\Gamma = \varpi/\varrho$. If again we make the assumption that the proper motion of the vortices (both that due to themselves and that due to their mutual influence) may be neglected in comparison with the general velocity V, we obtain the result that in the period c/V these rectangular vortices are displaced over the distance c. Hence they arrange themselves one after another as indicated in the diagram, in such a manner that the downstream side of each vortex coincides with the upstream side of its neighbor. The coinciding vortex lines, being of equal but opposite strengths, cancel each other, so that the whole system is equivalent to a single rectangular vortex (indicated by the thicker lines), of which one side lies along the axis Oy, the opposite side lies at the distance Nc downstream (where N is the number of periods during which the pressure has been applied) while the remaining sides, of lengths Nc, are parallel to the axis of x.

It is not difficult to pass to the limiting case in which c is taken infinitely small. Instead of the impulsive pressure ϖ applied at intervals of time c/V over an area of width c, we arrive at a continuous force acting at the points of a segment of the axis Oy. The intensity per unit of length of this force will be denoted by Λ, which quantity evidently is determined by the relation that its time integral over the period c/V must be equal to the resultant of the impulsive pressure over an area of dimension unity in the direction of the span and c in the direction of x. Hence:

$$\Lambda \frac{c}{V} = \varpi c$$

from which we deduce:

$$\Lambda = \varpi V \tag{5.1}$$

Writing t for Nc/V, that is for the total duration of the time during which the force Λ has been applied, we obtain a rectangular vortex of breadth $2b$ and of length Vt in the direction downstream, having the intensity:

$$\Gamma = \frac{\Lambda}{\varrho V} \tag{5.2}$$

This case may be considered as the most simple representation of the flow system existing in the neighborhood of an airfoil in its first stages, as described in I 8. It will be seen that (5.2) has the same form as I (6.1) which expresses the Kutta-Joukowski theorem, provided l is read for Λ.

It may be noted that it is not necessary to take the period of the action of the impulsive pressure equal to c/V. This was done merely for convenience; the reasoning, however, can be extended to the case of any period smaller than c/V, which has some advantage in passing to the limiting case of continuous action of the pressure. However, we shall not consider this point in detail, as the case of steady motion under a continuous force will be treated on a more rigorous basis in **12.**

Returning to the problem considered, if we give to the whole system a velocity V in the direction of the negative axis of x, we have the case of an airplane wing of very small chord in a fluid originally without motion, starting at a certain instant in the direction of $-x$ from its position of rest on a segment of the axis of y. At the time t the wing lies along the line $x = -Vt$; from its ends extend downstream vortices of strength $\Lambda/\varrho V$, and besides, when t is not too great, along the segment of the axis Oy originally occupied by the wing is to be found the "starting" vortex with a circulation equal and opposite to that around the wing.

6. Steady Motion under the Action of Forces Independent of the Time. Transformation of the Hydrodynamic Equations. It will now be our purpose to develop in more detail the results thus far obtained and to consider especially the steady motion of a fluid under the action of forces independent of the time. We hold to the case where a general motion with a constant velocity V parallel to the axis of x is already present and suppose again that the disturbances caused by the action of the forces are relatively small.

The method adopted in attacking this problem starts from a solution of the hydrodynamic equations analogous to that given by Oseen, but simplified in so far as the viscosity is neglected. In developing the equations we begin by considering the case of a system of forces distributed continuously over a certain space; from this general case we may pass to cases of forces distributed over a certain surface or over a line by means of the reasoning of common use in mathematical physics.

It is convenient to change the notation by writing the expressions:

$$V + w_x,\ w_y,\ w_z$$

for the components of the velocity of the fluid, instead of v_x, v_y, v_z. Then w_x, w_y, w_z denote the components of the *additional velocity*, or as may be said, the components of the disturbance of the original motion, that is, the motion with the constant velocity V parallel to the axis Ox. With further restriction to the case of steady motion, the velocity at any given point in the field will not depend upon the time.

The equations of hydrodynamics now take the form:

$$\left.\begin{aligned}
V\frac{\partial w_x}{\partial x} + w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y} + w_z\frac{\partial w_x}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{f_x}{\varrho}\\
V\frac{\partial w_y}{\partial x} + w_x\frac{\partial w_y}{\partial x} + w_y\frac{\partial w_y}{\partial y} + w_z\frac{\partial w_y}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \frac{f_y}{\varrho}\\
V\frac{\partial w_z}{\partial x} + w_x\frac{\partial w_z}{\partial x} + w_y\frac{\partial w_z}{\partial y} + w_z\frac{\partial w_z}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{f_z}{\varrho}
\end{aligned}\right\} \quad (6.1)$$

As V is a constant, the components w_x, w_y, w_z themselves must satisfy the equation of continuity:

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} = 0 \tag{6.2}$$

We transfer to the right hand side of (6.1) the groups of terms which are of the second degree in the quantities w_x, w_y, w_z. Moreover we arrange these terms in a certain special way, indicated for the first equation by the following formula:

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} + w_z \frac{\partial w_x}{\partial z} = \frac{\partial}{\partial x}\left(\frac{w_x^2 + w_y^2 + w_z^2}{2}\right) - w_y \gamma_z + w_z \gamma_y$$

Here γ_x, γ_y, γ_z denote the components of the vector of the vorticity, as has been mentioned before. Applying the same type of transformation to the second and third equations, we obtain:

$$\left.\begin{aligned} V\frac{\partial w_x}{\partial x} &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\frac{w^2}{2} + \frac{f_x}{\varrho} + w_y \gamma_z - w_z \gamma_y \\ V\frac{\partial w_y}{\partial x} &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} - \frac{\partial}{\partial y}\frac{w^2}{2} + \frac{f_y}{\varrho} + w_z \gamma_x - w_x \gamma_z \\ V\frac{\partial w_z}{\partial x} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} - \frac{\partial}{\partial z}\frac{w^2}{2} + \frac{f_z}{\varrho} + w_x \gamma_y - w_y \gamma_x \end{aligned}\right\} \tag{6.3}$$

where $w^2 = w_x^2 + w_y^2 + w_z^2$.

We now introduce a quantity q defined by:

$$q = p + \frac{1}{2}\varrho w^2 - p_0 \tag{6.4}$$

where p_0 is the constant pressure at infinity. We further write:

$$\left.\begin{aligned} \varrho\,(w_y \gamma_z - w_z \gamma_y) &= g_x \\ \varrho\,(w_z \gamma_x - w_x \gamma_z) &= g_y \\ \varrho\,(w_x \gamma_y - w_y \gamma_x) &= g_z \end{aligned}\right\} \tag{6.5}$$

As is well known, the quantities between ( ) represent the components of the vectorial or cross product[1] of the additional velocity w into the vorticity γ. We then put:

$$\left.\begin{aligned} k_x &= f_x + g_x \\ k_y &= f_y + g_y \\ k_z &= f_z + g_z \end{aligned}\right\} \tag{6.6}$$

Our equations now take the form,

$$\left.\begin{aligned} V\frac{\partial w_x}{\partial x} &= -\frac{1}{\varrho}\frac{\partial q}{\partial x} + \frac{k_x}{\varrho} \\ V\frac{\partial w_y}{\partial x} &= -\frac{1}{\varrho}\frac{\partial q}{\partial y} + \frac{k_y}{\varrho} \\ V\frac{\partial w_z}{\partial x} &= -\frac{1}{\varrho}\frac{\partial q}{\partial z} + \frac{k_z}{\varrho} \end{aligned}\right\} \tag{6.7}$$

Equations (6.7) are the exact equations of fluid motion if we put in the values of the k's given by (6.6). On the other hand, if we equalize the k's with the f's, neglecting the g's, a first approximation may be

[1] See Division C I 1.

obtained. Thus, we may call the quantities k_x, k_y, k_z "generalized forces" or forces of the first order, and the quantities g_x, g_y, g_z, forces of the second order. Also the expression "induced forces" may be used for the latter quantities.

7. Solution of the Equations by Successive Approximations. The equations thus obtained have the outward appearance of linear equations. They would be linear if the "generalized forces" k_x etc. were known quantities; in such case we should have in (6.7), together with the equation of continuity (6.2), a system of four linear equations to determine the unknowns w_x, w_y, w_z, q. With these terms known, (6.4) would then give the actual pressure p.

In reality the terms k contain the unknowns themselves through the intermediary of the quantities g, and the problem thus becomes less simple. However, as the terms g_x, g_y, g_z are of the second degree in the unknowns, and moreover are zero in every part of the field where the vorticity is zero, we may, following the procedure indicated by Oseen, frame a method for the solution of these equations along the following lines. We begin by neglecting g_x, g_y, g_z and for k_x put f_x, etc. By solving the equations we then obtain a first approximation to the real motion. This first solution enables us to deduce approximate values for the terms neglected. By substituting these values as a new system of k's into (6.7) we can then compute a correction to the first solution. From this corrected solution we obtain more exact values for the g's; and the process can be repeated to the required degree of precision.

A rigorous treatment of the hydrodynamic equations by this method would require an investigation regarding the convergence of these successive approximations. We shall not here consider this matter, which thus far has not been made the subject of an exhaustive treatment, and shall be content with the assumption that no difficulties will arise when the actual external forces f_x, f_y, f_z are sufficiently small.

The first approximation corresponds to that considered in **4** and **5**. It is equivalent in a certain way to the assumption that the proper motion of the vortices under their mutual influence can be neglected in comparison with the general velocity V. In the following developments we shall be mainly concerned with this first approximation. At a certain point, however, it will become necessary to take into consideration the forces of the second order g_x, g_y, g_z, and the manner in which this will be done may be indicated as follows:

Due to the physical nature of the problems to be considered in the present chapter the forces f_x, f_y, f_z act only within a region of limited extent, which for convenience will usually be denoted by the letter G. Eventually this region may consist of various parts separated from each other (as in the case of multiplanes); also it may be multiply connected. The field of motion itself extends unlimited in every direction and has no internal surface boundaries.

We replace the real problem by a slightly modified one: we assume, that instead of the f's the quantities k_x, k_y, k_z are restricted to the region G, while outside of G the k's are assumed to be zero. This investigation forms the subject of **8** to **13**. The motion calculated in this way might be the motion of an actual fluid, satisfying the ordinary hydrodynamic equations, assuming the fluid acted upon by a system of external forces f_x, f_y, f_z as determined from the equations:

$$f_x = k_x - g_x,\ f_y = k_y - g_y,\ f_z = k_z - g_z \quad \text{in the interior of the region } G \tag{7.1}$$

and from the equations:

$$f_x = -g_x,\ f_y = -g_y,\ f_z = -g_z \quad \text{outside the region } G \tag{7.2}$$

Here g_x, g_y, g_z are to be calculated by means of (6.5) from the field of motion obtained.

It is evident that the result arrived at in this way does not represent exactly the solution of the original problem. It can, however, be adapted to it, when the system of k-forces in the interior of G is chosen in such a way, that the values of f_x, f_y, f_z calculated from (7.1), are equal to the value of the f's assumed in this original problem. In the region G the hydrodynamic equations are then satisfied, but in the region outside of G additional external forces are acting, which is not consistent with the original problem. In order to eliminate them we have to introduce the quantities g_x, g_y, g_z obtained outside G as a new additional system of k's in (6.7) from which a correction to the first solution must be calculated in the manner as indicated above.

In order to avoid the difficulties which are encountered when we actually try to determine a system of k-forces corresponding to a given system of f-forces, we shall accept the opposite point of view in the following sections and start from a given system of "generalized forces" k_x, k_y, k_z, from which afterwards the values of the corresponding external forces f are to be calculated from (7.1) (see **14**). This leads to a much simpler treatment, and still is sufficiently general for obtaining a picture of the various relations involved in the theory of lifting systems.

Besides, in **15**, reasons will be given for considering the influence of the corrections due to the g's present outside the region G as negligible in most problems relating to airfoil theory. This gives a further simplification of the treatment; the degree of approximation thus obtained corresponds to the so-called first order theory given by Prandtl, which forms the basis for all modern researches in this domain.

Summarizing it is thus seen that the first problem to be considered is to obtain the solution of the systems (6.2), (6.7) for the case where the k's are regarded as known quantities, differing from zero in a region of limited extent only, while the field of motion is infinite. The second step is then to calculate from (7.1) the corresponding f's in the interior of this region.

For convenience, in those cases where no ambiguity is to be feared, we shall sometimes drop the adjective "generalized", and simply speak of the forces k_x, k_y, k_z.

It may be remarked that the dimensions of the quantities k_x, k_y, k_z, like those of g_x, g_y, g_z, are the same as those of f_x, f_y, f_z, that is of forces per unit volume ($M \div L^2 T^2$).

8. Solution of the System of Equations (6.2), (6.7).—Determination of q. Following the procedure already indicated (see 2) we take the divergence of the system (6.7) and apply (6.2); that is, we differentiate the first equation of the system (6.7) with respect to x, the second with respect to y, the third with respect to z, and add the results. The velocity disappears from the sum, and after multiplication by ϱ we have:

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} = \frac{\partial k_x}{\partial x} + \frac{\partial k_y}{\partial y} + \frac{\partial k_z}{\partial z}$$

or as it may be written: $$\nabla^2 q = div\, k \tag{8.1}$$

The solution of this equation, which is of the type of Poisson's equation[1], is well known; as q is zero at infinity it has the form:

$$q = -\iiint d\xi\, d\eta\, d\zeta \frac{\dfrac{\partial k_x}{\partial \xi} + \dfrac{\partial k_y}{\partial \eta} + \dfrac{\partial k_z}{\partial \zeta}}{4 \pi r} \tag{8.2}$$

Here r is equal to $\sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$; x, y, z denote the coordinates of the point where the value of q is to be determined; ξ, η, ζ denote the coordinates of any point of space where forces k_x, k_y, k_z are applied. The integration must be extended throughout the space G where these forces are acting.

Integrating (8.2) by parts, we obtain:

$$q = \iiint d\xi\, d\eta\, d\zeta \frac{k_x (x - \xi) + k_y (y - \eta) + k_z (z - \zeta)}{4 \pi r^3} \tag{8.3}$$

In this equation the integration again extends throughout the region G where the forces k are acting.

As an example we take the case of forces acting only in a region of limited extent around the origin of the system of coordinates, and directed along $O z$. When we consider points x, y, z lying at a sufficient distance from the origin, we may simplify formula (8.3) by substituting

[1] Poisson's equation has the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = m$$

and expresses, for a vector having the potential φ, the net flux of the vector through an element of volume $dx\, dy\, dz$, as measured by $m\, dx\, dy\, dz$.

zero for ξ, η, ζ, thus making r equal to $\sqrt{x^2 + y^2 + z^2}$. In this way we have:

$$q = \frac{z}{4\pi r^3} \int\int\int d\xi\, d\eta\, d\zeta\, k_z = \frac{z}{4\pi r^3} K_z \tag{8.4}$$

where K_z has been written for the resultant of the forces k_z.

If we neglect the term $(1/2)\, \varrho\, w^2$ in (6.4), then q may be taken for the excess of the pressure over its value at infinity, and it will be seen that the distribution of the pressure excess given by (8.4) corresponds to the field of a doublet lying along the z axis (see Division B IV **10**).

9. Determination of the Components of the Velocity. We have now to put the expression obtained for q into (6.7). Before actually doing this we may note that the form of these equations suggests for the components w_x, w_y, w_z expressions in the form:

$$\left.\begin{aligned} w_x &= w'_x + \frac{\partial \varphi}{\partial x} \\ w_y &= w'_y + \frac{\partial \varphi}{\partial y} \\ w_z &= w'_z + \frac{\partial \varphi}{\partial z} \end{aligned}\right\} \tag{9.1}$$

where the quantities w'_x, w'_y, w'_z, φ are to be determined from the equations:

$$\left.\begin{aligned} V \frac{\partial w'_x}{\partial x} &= \frac{k_x}{\varrho} \\ V \frac{\partial w'_y}{\partial x} &= \frac{k_y}{\varrho} \\ V \frac{\partial w'_z}{\partial x} &= \frac{k_z}{\varrho} \end{aligned}\right\} \tag{9.2}$$

and:

$$V \frac{\partial \varphi}{\partial x} = -\frac{q}{\varrho} \tag{9.3}$$

These equations can be solved by an integration with respect to x. As regards limiting conditions we assume for $x = -\infty$, that is for infinite distances upstream from the region where the forces are acting, that the components w_x, w_y, w_z of the additional velocity, vanish. It is convenient to apply this condition separately to the quantities w'_x, w'_y, w'_z as well as to the function φ. In this way we have from (9.2):

$$\left.\begin{aligned} w_x' &= \frac{1}{\varrho V} \int_{-\infty}^{x} d\xi\; k_x \\ w_y' &= \frac{1}{\varrho V} \int_{-\infty}^{x} d\xi\; k_y \\ w_z' &= \frac{1}{\varrho V} \int_{-\infty}^{x} d\xi\; k_z \end{aligned}\right\} \tag{9.4}$$

In these expressions the integration is performed along a line parallel to the axis of x, extending from $-\infty$ to the point x, y, z where the quantities w_x' etc. are to be determined. The letter ξ is used to denote the integration variable, which runs from the value $-\infty$ to the value x. Under the sign of integration, k_x etc. must therefore be considered as functions of ξ, y and z.

For simplicity we shall denote the integrals occurring in (9.4) by Q_x, Q_y, Q_z respectively. Then

$$k_x = \frac{\partial Q_x}{\partial x}, \text{ etc.} \tag{9.5}$$

and furthermore

$$w_x' = \frac{Q_x}{\varrho V}, \text{ etc.} \tag{9.6}$$

When the point x, y, z lies beyond the region G where the forces k_x, k_y, k_z are acting, the functions Q_x, Q_y, Q_z take values independent of the coordinate x. These values will be denoted respectively by $\overline{Q}_x, \overline{Q}_y, \overline{Q}_z$. They are functions of y and z only.

In (9.3) we have still to substitute (8.3) for q, after which it must be integrated with respect to x. To prevent confusion we write ξ' for x in (8.3) and take ξ' as the integration variable, which runs from $-\infty$ to the value x, corresponding to the point where φ is to be determined. In this way we find:

$$\varphi = -\frac{1}{\varrho V}\int_{-\infty}^{x} d\xi' \int\int\int d\xi\, d\eta\, d\zeta\, \frac{k_x(\xi'-\xi) + k_y(y-\eta) + k_z(z-\zeta)}{4\pi\,[(\xi'-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} \tag{9.7}$$

The integration with respect to ξ' can be executed before the other ones and with the aid of some elementary formulae we thus obtain:

$$\varphi = \frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta \left[\frac{k_x}{4\pi r} - \frac{k_y(y-\eta) + k_z(z-\zeta)}{4\pi r\,(r - x + \xi)}\right] \tag{9.8}$$

where r as before stands for $\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$. The integration with respect to ξ, η, ζ must always be extended over the whole region G.

Appendix to Section 9.—Remark in Connection with Bernoulli's Theorem. One of the most important relations valid for the steady motion of a non-viscous fluid, is Bernoulli's theorem which connects the pressure p at any point of the field with the resultant velocity v at the same point, according to the equation:

$$p + \frac{1}{2}\varrho v^2 = \text{const.} \tag{1}$$

The equation is valid along a stream-line, the constant in general being different for the various stream-lines. When the motion is irrotational in a part of the field, then an extended form of the theorem holds, as in that case the constant has the same value for all stream-lines in this domain.

The theorem has been obtained from the general hydrodynamic equations in Division B I 6. As the deductions of the present Chapter, however, are based upon a rather special system of equations, in which account is taken of the influence of external forces, it seems desirable to investigate whether the theorem still holds under the assumptions now introduced.

By integrating with respect to x the first equation of the system (6.7) we obtain:

$$\varrho V w_x = -q + \int_{-\infty}^{x} d\xi\, k_x \tag{2}$$

The constant of integration can be omitted, as both w_x and q vanish for large negative values of x. Solving this equation for q, and introducing the result into (6.4), we have:

$$p = p_0 - \frac{\varrho}{2}(2 V w_x + w_x^2 + w_y^2 + w_z^2) + \int_{-\infty}^{x} d\xi\, k_x \tag{3}$$

Returning to the original notation of the velocity components, $v_x = V + w_x$, $v_y = w_y$, $v_z = w_z$, and making use at the same time of (9.5), this equation can be brought into the form:

$$p + \frac{\varrho}{2}(v_x^2 + v_y^2 + v_z^2) = p_0 + \frac{\varrho}{2} V^2 + Q_x \tag{4}$$

Now the value of Q_x can differ from zero only within the region G and in the wake behind it. Outside of both of these regions we thus have:

$$p + \frac{\varrho}{2} v^2 = p_0 + \frac{\varrho}{2} V^2 \tag{5}$$

which is, in fact, Bernoulli's equation again. When k_x is zero everywhere, which is the most important case in the theory of lifting systems, Q_x likewise is zero; in that case Bernoulli's theorem holds throughout the whole field.

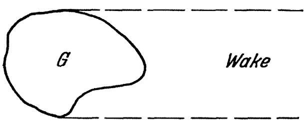


Fig. 51.

10. Discussion of the Result Obtained—Vorticity. The separation of the velocity components w_x, w_y, w_z into the parts w'_x, w'_y, w'_z on the one side, and terms depending on the function φ on the other side, which is expressed by (9.1), has an important bearing on the results. The function φ can be considered as a kind of potential, though it is not the ordinary potential usually occurring in hydrodynamics, as only a part of the velocity is dependent upon it. This part, however, must necessarily represent an irrotational motion. On the other hand the part expressed by w'_x, w'_y, w'_z in general will represent a motion with vorticity.

An interesting conclusion can be derived immediately from (9.4). These equations show that the quantities w'_x, w'_y, w'_z differ from zero only within the region G and in a cylindrical "*wake*" stretching downstream from G (see Fig. 51). Outside of this "wake" the motion is wholly determined by the function φ. Consequently outside of the "wake" the motion is irrotational, and vorticity will be present only within G and within the "wake".

The result has a certain resemblance to the well known experimental fact, that vorticity is to be found principally on the downstream side of any body which disturbs the motion of the fluid. In actual cases the wake is not bounded by a cylindrical surface; that this happens to be the case in the present result is a consequence of the circumstance that we have not taken account of the corrective terms (induced or second order forces) g_x, g_y, g_z and the solution thus obtained is only a first approximation to the solution of the actual hydrodynamic equations. In actual cases furthermore a certain turbulence sets in which has a marked influence upon the distribution of the vorticity over the field; but as yet this cannot well be introduced into the mathematical theory.

The components of the vorticity are given by the formulae:

$$\left.\begin{aligned} \gamma_x &= \frac{\partial w'_z}{\partial y} - \frac{\partial w'_y}{\partial z} \\ \gamma_y &= \frac{\partial w'_x}{\partial z} - \frac{\partial w'_z}{\partial x} \\ \gamma_z &= \frac{\partial w'_y}{\partial x} - \frac{\partial w'_x}{\partial y} \end{aligned}\right\} \tag{10.1}$$

Substituting the expressions (9.6) we obtain the formulae for the γ's in terms of the forces:

$$\left.\begin{aligned} \gamma_x &= \frac{1}{\varrho V}\left(\frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z}\right) \\ \gamma_y &= \frac{1}{\varrho V}\left(\frac{\partial Q_x}{\partial z} - k_z\right) \\ \gamma_z &= \frac{1}{\varrho V}\left(-\frac{\partial Q_x}{\partial y} + k_y\right) \end{aligned}\right\} \tag{10.2}$$

For points lying in the wake of the region G these formulae must be replaced by:

$$\left.\begin{aligned} \gamma_x &= \frac{1}{\varrho V}\left(\frac{\partial \bar{Q}_z}{\partial y} - \frac{\partial \bar{Q}_y}{\partial z}\right) \\ \gamma_y &= \frac{1}{\varrho V}\frac{\partial \bar{Q}_x}{\partial z} \\ \gamma_z &= -\frac{1}{\varrho V}\frac{\partial \bar{Q}_x}{\partial y} \end{aligned}\right\} \tag{10.3}$$

The separation of the velocity components into the parts w' and the parts depending on φ can also be viewed in another way: The quantities w'_x, w'_y, w'_z in a certain way represent the velocity components set up by the immediate action of the forces k_x, k_y, k_z. These components in general do not satisfy the equation of continuity. Hence a correction of the field is necessary, which is given by the terms depending on φ. These latter terms actually come in through the action of the pressure, and consequently extend throughout the whole field.

We shall now investigate some special examples of forces k_x, k_y, k_z and though our main problem is to be the study of the motion under forces directed normal to the general velocity V, it will be useful to take first the case of forces parallel to V.

11. Forces Parallel to the Direction of the Original Motion. Let us consider only the terms depending on k_x in (9.4) and (9.8). We then have:

$$\varphi = \frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta\, \frac{k_x}{4\pi r} \tag{11.1}$$

$$\left.\begin{aligned} w'_x &= \frac{1}{\varrho V}\int_{-\infty}^{x} d\xi\, k_x = \frac{Q_x}{\varrho V} \\ w'_y &= 0 \\ w'_z &= 0 \end{aligned}\right\} \tag{11.2}$$

Equation (11.1) can be interpreted by saying that every force directed parallel to the original motion produces the field of a (negative) *source* with strength $-k_x/\varrho V$ [1]. Hence at a great distance from the region where these forces are applied the additional motion is directed radially inward, with the velocity:

$$w_{\text{rad.}} = \frac{\partial \varphi}{\partial r} = -\frac{K_x}{4\pi \varrho V r^2} \tag{11.3}$$

where K_x denotes the resultant of the system of forces k_x. There is thus a total flow inward of the amount $K_x/\varrho V$.

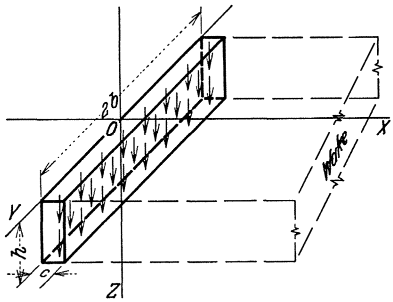


Fig. 52.

This fluid of course cannot be absorbed in the region where the forces are acting, and has to flow off again to infinity. This takes place in the wake, where, besides the motion depending on φ, we have the velocity w'_x given by (11.2). The integral of w'_x over a perpendicular section of the wake lying beyond the region G amounts to:

$$\iint dy\, dz\, w'_x = \frac{1}{\varrho V}\iint dy\, dz\, \overline{Q}_x = \frac{K_x}{\varrho V} \tag{11.4}$$

and is thus equal to the volume noted above.

When the resultant force K_x is negative, the radial flow is directed outward, and the flow in the wake is inward.

As has been indicated already in **4**, these results can be used as a starting point for the theory of the propeller and the windmill, and for the theory of hydrodynamic resistance. These problems, however, fall outside the scope of the present Division of this work.

12. Forces Directed Normal to the Original Motion—Loaded Line with Uniform Lift Distribution. This case will be considered at greater length than the preceding. We begin with the investigation of the following example: Within a region having the form of a flat rectangular box, as indicated in Fig. 52, forces act in the direction of the axis Oz, with the constant intensity k_z per unit volume. The dimensions c and h of this box are both small compared with the breadth (or span) $2b$. The

[1] See Division B IX (3.1).

wake has then the form of a rectangular parallelopiped, extending in the direction of the axis $O\,x$.

Application of (9.4) gives zero for w'_x and w'_y, while w'_z has the value:

$$w'_z = \frac{k_z c}{\varrho V} \tag{12.1}$$

Formula (9.8) gives for φ:

$$\varphi = -\frac{k_z}{\varrho V} \int_0^c d\xi \int_{-b}^{+b} d\eta \int_0^h d\zeta \frac{z-\zeta}{4\pi r (r - x + \xi)} \tag{12.2}$$

We pass to the limiting case where c becomes infinitely small. Writing $\overline{Q}_z$ for $lim\ k_z\, c$, we have:

$$w'_z = \frac{\overline{Q}_z}{\varrho V} \tag{12.3}$$

and

$$\varphi = -\frac{\overline{Q}_z}{\varrho V} \int_{-b}^{+b} d\eta \int_0^h d\zeta \frac{z-\zeta}{4\pi r (r - x)} \tag{12.4}$$

where now:

$$r = \sqrt{x^2 + (y-\eta)^2 + (z-\zeta)^2}$$

This expression for φ may be transformed in the following manner. On account of the relation:

$$\frac{1}{r(r-x)} = \int_0^\infty d\xi \frac{1}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}}$$

which can be demonstrated in an elementary way, we may write:

$$\varphi = -\frac{\overline{Q}_z}{\varrho V} \int_0^\infty d\xi \int_{-b}^{+b} d\eta \int_0^h d\zeta \frac{z-\zeta}{4\pi [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}}$$

Now the integration with respect to ζ can be worked out and gives:

$$\left.\begin{aligned} \varphi = \frac{\overline{Q}_z}{\varrho V} \int_0^\infty d\xi \int_{-b}^{+b} d\eta \Bigg[& \frac{1}{4\pi \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}} - \\ & - \frac{1}{4\pi \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-h)^2}} \Bigg] \end{aligned}\right\} \tag{12.5}$$

The latter expression can be interpreted as representing the potential of a system of sources, distributed partly over the upper surface of the wake with the intensity $-\,\overline{Q}_z/\varrho V$ (negative) and partly over the lower surface with the intensity $+\,\overline{Q}_z/\varrho V$ (positive). Sketches of the field of motion determined by this potential are shown in Fig. 53.

It will be seen that both along the upper and lower surfaces of the wake, the normal component of the velocity (the component $\partial\varphi/\partial z$) is discontinuous. The amount of the discontinuity can be found without difficulty. Using an obvious notation we have at the upper surface:

$$\left(\frac{\partial \varphi}{\partial z}\right)_{z=+0} - \left(\frac{\partial \varphi}{\partial z}\right)_{z=-0} = -\frac{\overline{Q}_z}{\varrho V} \tag{12.6}$$

and at the lower surface:

$$\left(\frac{\partial \varphi}{\partial z}\right)_{z-h=+0} - \left(\frac{\partial \varphi}{\partial z}\right)_{z-h=-0} = +\frac{\overline{Q}_z}{\varrho V} \tag{12.7}$$

Within the wake we have to superpose the velocity w_z' upon the motion determined by φ. When this superposition has been performed,

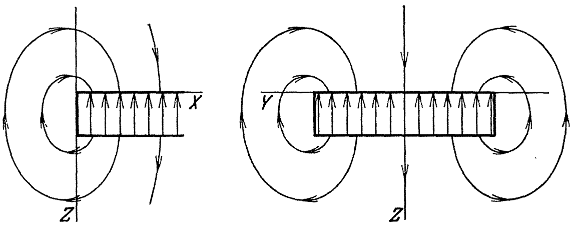


Fig. 53.

the discontinuities along the upper and lower surfaces of the wake disappear. Hence the resulting field is continuous and the general form of the stream-lines is indicated in Fig. 54.

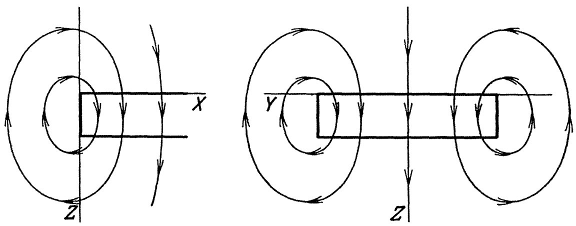


Fig. 54.

It is evident that this motion presents a *circulation* along every line that encircles one of the vertical surfaces bounding the wake. The amount of this circulation depends solely upon the components w_z', as the motion derived from φ is necessarily devoid of circulation. As the distance from the upper to the lower surface of the wake is equal to h, we find for the magnitude of the circulation:

$$\Gamma = w_z' h = \frac{\overline{Q}_z h}{\varrho V} \tag{12.8}$$

The vertical surfaces which bound the wake are to be considered as *vortex sheets,* as the tangential component w_z of the total field presents a discontinuity along these surfaces of the amount:

$$w_z' = \frac{\overline{Q}_z}{\varrho V} \tag{12.9}$$

The vortex lines are parallel to the axis of y in the front surface (that is in the region where the forces are acting) and parallel to the axis of x in the other surfaces; see Fig. 55.

It is natural next to introduce a further limiting condition, and to take the height h equal to zero, which leads to the case of the "loaded line". Writing A for $lim\, \overline{Q}_z h$ (that is for the load per unit of length[1] of the segment from $y = -b$ to $y = +b$), we obtain a vortex line of horseshoe form, consisting of the loaded line itself and of two straight lines extending downstream from its ends to infinity, the strength of the vortex being:

$$\Gamma = \frac{A}{\varrho V} \tag{12.10}$$

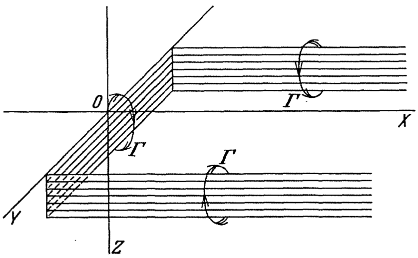


Fig. 55.

This result agrees with that obtained before in **5**, provided an infinite value is given to the time t.

When the "span" $2b$ of the supporting line is taken infinite, a single rectilinear vortex is obtained in this case, as the two trailing vortices disappear into infinity. This is of importance for the case of the two-dimensional field.

13. Loaded Line in Arbitrary Position and with Variable Lift Distribution. The result of the last section may be generalized first to the case of a loaded line of arbitrary position in space, supposing (as we always shall) that the generalized forces are perpendicular both to the direction of the original velocity V and to the direction of the line itself. In order to keep (12.10) unchanged, it is necessary in this case to define A as the load *per unit of length of the projection* of the line upon the y, z plane. We may also consider the case of a curved line, which can be obtained as the limit of a system of rectilinear segments; in that case it is necessary that at every point the force (always meaning the generalized force) is directed normal both to V and to the tangent to the curve.

When the load per unit of projected length is not constant along the loaded line, trailing vortices are formed and pass downstream from every point where the load changes. The strength of these vortices is determined by the change in load per unit of projected length, divided by ϱV.

---

[1] It must be kept in mind that with "forces" we still mean the "generalized forces", according to the definition given in **6**. Hence properly speaking A must be called the "generalized load" per unit of length (unit span). The real load or the lift per unit span will be denoted by l; in **30** and **31** we shall investigate under what conditions it is possible to put $l = A$.

The dimensions of A, like those of l, are the dimensions of a force per unit length: $(M \div T^2)$.

It is of importance to obtain this result directly from the first equation of (10.3). Assume in Fig. 56 that the curve s represents the projection of the loaded line upon the y, z plane. In the neighborhood of the element $d\,s$ take a system of coordinates s, n directed parallel respectively to the tangent and to the normal at the element considered, and let $\bar{Q}_s$, $\bar{Q}_n$ denote the components of $\bar{Q}$ along these directions. Then the equation mentioned can be replaced by:

$$\gamma_x = \frac{1}{\varrho V}\left(\frac{\partial \bar{Q}_n}{\partial s} - \frac{\partial \bar{Q}_s}{\partial n}\right) \tag{13.1}$$

Now in the case considered the tangential component $\bar{Q}_s$ is assumed to be zero, while the normal component $\bar{Q}_n$ (which in the general formula refers to an element of the y, z plane) is replaced by the load Λ acting on the element ds per unit length. Instead of γ_x we must then take the strength of the vortex sheet extending downward from ds, and denoted by $\bar{\gamma}_x$. This gives the equation

$$\bar{\gamma}_x = \frac{1}{\varrho V}\frac{d\Lambda}{d\,s} \tag{13.2}$$

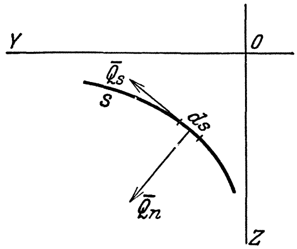


Fig. 56.

As an illustration consider a case which is of great importance in the theory of the airplane wing. It can usually be accepted that the load per unit length in the direction of the span falls off as we go from the plane of symmetry of the wing, and becomes zero at the wingtips. Assuming the span parallel to the y axis, it follows from (13.2) that behind the wing a sheet of trailing vortices must extend, the intensity of which is given by:

$$\bar{\gamma}_x = \frac{1}{\varrho V}\frac{d\Lambda}{d\,y} \tag{13.3}$$

It will be seen that this result is in conformity with the description of the vortex system behind an airfoil given in I 7. According to (12.10) the circulation Γ is equal to $\Lambda/\varrho V$, and thus we may write:

$$\bar{\gamma}_x = \frac{d\,\Gamma}{d\,y} \tag{13.4}$$

which corresponds to the statement made in the last paragraph of I 7.

As a special example take the case of an elliptic distribution of the load, which is of great importance in theoretical investigations. In that case Λ is given by:

$$\Lambda = \Lambda_0\sqrt{1 - y^2/b^2} \tag{13.5}$$

which equation represents a semi-ellipse over the span, as indicated in Fig. 57. From every element dy of the loaded line there will then issue a band of trailing vortices, having the strength:

$$\bar{\gamma}_x = \frac{1}{\varrho V}\frac{d\Lambda}{d\,y} = -\frac{y\,\Lambda_0}{\varrho\, V\, b\sqrt{b^2 - y^2}} \tag{13.6}$$

This case will receive further notice in 22.

The examples of force systems acting at the points of a line or of a system of lines mentioned in the present section form the basis for the investigation of the flow pattern around airfoil systems. As explained in I **13**, a further approximation can be obtained by distributing the forces over a certain area instead of along a loaded line. This case will be considered in **28**. However, for the following investigations we are using the first approximation represented by the loaded line. Our first aim is to consider the significance of the second order terms, g_x, g_y, g_z, representing the "induced forces". The most important results given by this investigation will be an expression for the induced resistance experienced by every load system, showing the fact that even in the case of an ideal fluid no lift can be produced by a wing system of finite span without expenditure of energy.

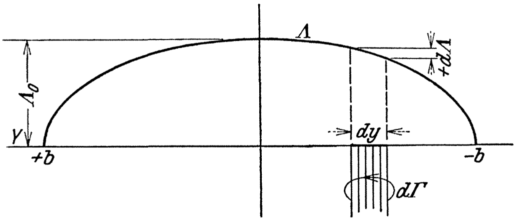


Fig. 57.

14. Introduction of the "Induced Forces" (Second Order Forces) g_x, g_y, g_z. We return to the statement made in **7** regarding the relation between the generalized forces k_x, k_y, k_z and the real forces f_x, f_y, f_z. In that section the point of view was taken that in the interior of the region G the actual forces (to be applied by external means) should be calculated from the values assumed for the k's by subtracting the g's from them. In the wake behind G the g's should be considered as a new system of generalized forces. Outside of both the region G and the wake, the vorticity is zero and hence the g's vanish; here the force field disappears entirely.

If for convenience we provisionally hold to the case where the generalized or k forces are all parallel to the z axis, we obtain from (10.2) the following values for the components of the vorticity in G:

$$\left.\begin{aligned} \gamma_x &= \frac{1}{\varrho V}\frac{\partial Q_z}{\partial y} \\ \gamma_y &= -\frac{k_z}{\varrho V} \\ \gamma_z &= 0 \end{aligned}\right\} \qquad (14.1)$$

The values of the same components in the wake are, according to (10.3):

$$\left.\begin{aligned} \gamma_x &= \frac{1}{\varrho V}\frac{\partial \bar{Q}_z}{\partial y} \\ \gamma_y &= 0 \\ \gamma_z &= 0 \end{aligned}\right\} \qquad (14.2)$$

In Chapter I the term "bound vortex" was used for the system of vortices which could be imagined as a substitute for the airfoil. The adjective "bound" indicates that these vortices are kept at a fixed position in space by the action of external forces [compare the remark made in connection with I (9.7)]. "Free" vortices on the contrary are carried along with the motion of the fluid, in accordance with Helmholtz' law.

In order to make this statement precise we now adopt as a definition that all vortex elements lying within the region G will be called "*bound vortices*", while the vortices in the wake behind G, where no external forces are acting, are free vortices. As the region G actually stands for the region occupied by the airfoil, or for the aggregate of regions occupied by the various parts of an airfoil system, the appropriateness of the definition will be evident.

The components of the external forces which must be introduced at the points of the region G in order to produce the assumed values of k_z, and thus to keep unchanged the intensity of the "bound vortices" in this region, are found to be [according to (7.1), (6.5), (14.1)]:

$$\left.\begin{aligned} f_x &= \quad\;\; -g_x = +\varrho\, w_z \gamma_y \\ f_y &= \quad\;\; -g_y = -\varrho\, w_z \gamma_x \\ f_z &= k_z - g_z = -\varrho\,(V + w_x)\,\gamma_y + \varrho\, w_y \gamma_x \end{aligned}\right\} \tag{14.3}$$

From these equations we obtain the relation:

$$(V + w_x)\, f_x + w_y\, f_y + w_z\, f_z = 0 \tag{14.4}$$

which shows that the external force (real force) is perpendicular to the resultant velocity of the fluid[1] (formerly denoted by v). It is also perpendicular to the resultant vorticity. Equations (14.3) are in fact equivalent to the statement that *the external force per unit volume f is equal to the vectorial (or cross) product of the bound vorticity into the resultant velocity multiplied by the density of the fluid.* This is the theorem which was mentioned in connection with I (9.7)[2].

We deduce from the first equation of (14.3) that in consequence of the presence of the component w_z the external forces have a *component in the direction of the axis Ox*, a fact, which has already been stated in Chapter I [see I (10.1)]. As there noted this component must be applied in order to overcome a certain resistance experienced by the object or system that transmits the forces to the fluid; this is called the *induced resistance.*

When the value $-k_z/\varrho V$ is substituted for γ_y, the following expression for f_x is obtained:

$$f_x = -\frac{w_z}{V}\, k_z \tag{14.5}$$

[1] See Division B V 7.

[2] A form of this theorem applicable to isolated vortices was given by II (3.13) for the case of two-dimensional motion.

Hence by integrating over the whole region G, the equation for the induced resistance, which has the opposite sign, assumes the form:

$$D_i = \frac{1}{V}\int\int\int dx\,dy\,dz\,k_z w_z \tag{14.6}$$

It is necessary to give also the equations for the more general case where the "generalized forces" have components k_y as well as k_z. In that case (14.1) must be replaced by:

$$\left.\begin{aligned} \gamma_x &= \frac{1}{\varrho V}\left(\frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z}\right) \\ \gamma_y &= -\frac{k_z}{\varrho V} \\ \gamma_z &= +\frac{k_y}{\varrho V} \end{aligned}\right\} \tag{14.7}$$

while instead of (14.3) we have:

$$\left.\begin{aligned} f_x &= \varrho w_z \gamma_y - \varrho w_y \gamma_z \\ f_y &= \varrho (V + w_x)\gamma_z - \varrho w_z \gamma_x \\ f_z &= \varrho w_y \gamma_x - \varrho (V + w_x)\gamma_y \end{aligned}\right\} \tag{14.8}$$

which can be put into the form:

$$\left.\begin{aligned} f_x &= -\frac{1}{V}(k_y w_y + k_z w_z) \\ f_y &= \frac{V + w_x}{V} k_y - \varrho w_z \gamma_x \\ f_z &= \frac{V + w_x}{V} k_z + \varrho w_y \gamma_x \end{aligned}\right\} \tag{14.9}$$

It remains true that the external force per unit volume is given by the cross product of the vorticity into the resultant velocity, multiplied by the density, and again the force is perpendicular both to the resultant velocity and to the vector of the vorticity.

The induced resistance now becomes:

$$D_i = \frac{1}{V}\int\int\int dx\,dy\,dz\,(k_y w_y + k_z w_z) \tag{14.10}$$

15. Continuation. Influence of the "Second Order Forces" g in the Wake. We must now consider the action of the "second order forces" g_x, g_y, g_z which according to 7 should be introduced in the wake behind the region G. As was mentioned in that Section, these forces must be considered as a new system k_x, k_y, k_z; then by the aid of (6.7) etc., we have to calculate the motion produced by them.

These forces, the same as those which were used within the region G in order to arrive at the f's, are of the second order compared with either the k's or the f's. Notwithstanding their relative smallness, they will have a certain influence on all quantities considered: pressure, potential φ, and vortex motion. As a force, however, can never produce vortex motion at a point situated upstream from it, the distribution of the vorticity within the region G (that is, of the "bound" vortices) is not affected. Hence it will be seen that the influence of the corrections to be deduced from the g's on the products occurring in (14.3) or (14.8) is only of the *third* order of magnitude.

It is customary, following the theory given by Prandtl, in calculating the f's, to neglect the correction to be deduced from the g's in the wake. The velocities w_x, w_y, w_z are then correct only to the *first* order of magnitude, while the values of the external forces f_x, f_y, f_z are obtained correctly to the *second* order of magnitude.

To the present time few attempts have been made to carry further the degree of approximation; it may even be that the entire system of successive approximations cannot be developed without going back to the equations for viscous fluids. A few questions concerning the action of the "g forces" in the wake are treated in Chapter VI. One result may be inferred from general principles: in the final motion all free vortices must satisfy Helmholtz' laws, and thus, as we are considering steady motion, they must coincide with stream-lines. In the wake the resultant motion, in the case now considered, differs from the original rectilinear motion with constant velocity, principally on account of the downward velocity w_z. In consequence the effect to be expected is that the free vortices will show a downward slope, instead of remaining parallel to the x axis. In addition, at some distance down the wake the vortex sheet will roll up at both sides into two separate vortices[1].

B. Wake Energy and Induced Drag.

16. Energy Expended in Producing the Flow Pattern. We have seen that when a system of generalized forces acts on the fluid in a certain region which has been denoted by G, vortices appear in a cylindrical wake extending downstream from this region. According to (10.1) the components of the vorticity depend on the quantities w'_x, w'_y, w'_z. On the other hand from (9.6) we see that, once we are downstream from this region, the latter quantities are independent of the coordinate x. The vorticity components, of course, show the same character, as moreover can be seen from (10.3). It is then evident that the part of the motion derived from the potential φ must likewise approach a certain limiting type, independent of x, and extending to infinity.

This limiting flow, both in the interior of the wake and in the outer space adjacent to it, possesses a certain amount of energy. As the fluid moreover moves with the general velocity V in the positive direction

[1] Compare the remarks made in Chapter I, at the ends of 7 and 9. — The fact that the free vortices must coincide with the stream-lines was pointed out by PRANDTL in his "Tragflügeltheorie" I (Nachr. Ges. Wissenschaften, Göttingen, 1918, republished in: Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen, 1927).

of the axis $O x$, we see that there is a continuous flow of energy downstream, the amount of which will be given by the formula

$$E = \frac{1}{2} \varrho V \int\int d y \, d z \, (w_{x\infty}^2 + w_{y\infty}^2 + w_{z\infty}^2) \qquad (16.1)$$

The integral is to be extended over a plane perpendicular to the axis $O x$, lying sufficiently far downstream in order that w_x, w_y, w_z may have reached their limiting values, as indicated by the suffix ∞ attached to them in (16.1). It will be seen that the integral represents the transfer across this plane per unit of time of the energy due to the additional velocities only. The expression might have been obtained also by starting from the energy due to the total motion, having the components $V + w_x$, w_y, w_z, and subtracting from it the energy already present in the motion upstream of the region G. It can be shown that the difference of these amounts is again given by (16.1).

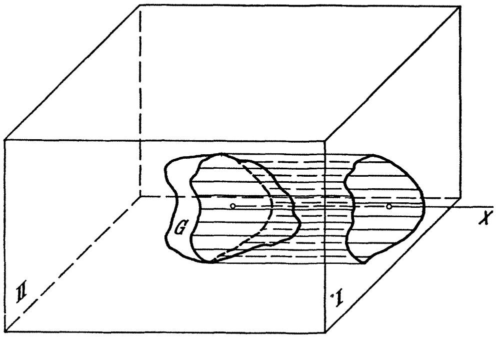


Fig. 58.

Work must be done in order to produce this energy. We may try to frame an expression for this work by starting from the scalar product[1] of the vector of the generalized force k into the vector of the additional velocity w:

$$k \cdot w = k_x w_x + k_y w_y + k_z w_z \,,$$

and writing down the integral:

$$A = \int\int\int d x \, d y \, d z \, (k_x w_x + k_y w_y + k_z w_z) \qquad (16.2)$$

taken over the region G. This integral might be considered in a certain way as the work done by the forces k on account of the additional velocity w. Though such an interpretation cannot be upheld rigorously, as the generalized forces contain terms depending on the q's, which are not real forces, still we shall find that there exists the relation:

$$E = A \qquad (16.3)$$

This relation can be obtained as follows: We multiply the first equation of (6.7) by w_x, the second by w_y, the third by w_z and add the results. We then have (after multiplication by ϱ):

$$\left.\begin{aligned} \frac{1}{2} \varrho V \frac{\partial}{\partial x} (w_x^2 + w_y^2 + w_z^2) = (k_x w_x + k_y w_y + k_z w_z) - \\ - \left[\frac{\partial (q w_x)}{\partial x} + \frac{\partial (q w_y)}{\partial y} + \frac{\partial (q w_z)}{\partial z} \right] \end{aligned}\right\} \qquad (16.4)$$

[1] See Division C I 1.

The second group of terms on the right hand side has been transformed slightly, making use of (6.2).

We now consider a volume as in Fig. 58 of very large extent, completely surrounding the region G in such a way that its sides are everywhere at a great distance from G. We then proceed to integrate (16.4) throughout this volume. The expression on the left hand side of (16.4) gives:

$$\frac{\varrho V}{2}\iint\limits_{(I)} dy\,dz\,(w_x^2 + w_y^2 + w_z^2) - \frac{\varrho V}{2}\iint\limits_{(II)} dy\,dz\,(w_x^2 + w_y^2 + w_z^2)$$

where I, II respectively denote the sides of the rectangular space marked thus in Fig. 58. As it can be shown from (9.8) that w_x, w_y, w_z decrease at least proportional to r^{-2} in all directions making a finite angle with the positive part of the axis $O\,x$, only the first integral need be taken into account, and it is immediately seen that this integral gives us the amount E.

The first term at the right hand side of (16.4) gives us the integral formerly called A.

Applying Gauss' theorem, the second term becomes:

$$-\iint dS\,q\,w_n,$$

extended over all sides of the volume, where $d\,S$ denotes an element of one of these sides and w_n the component of the additional velocity perpendicular to it. Now from (8.3) it may be seen that for large distances q decreases at least as r^{-2}, since we have assumed that there are no k forces outside the finite region G. Hence it results that by taking the dimensions of the rectangular space sufficiently large, the surface integral vanishes (even for that part of the side I where w remains finite).

In this way, from (16.4) we obtain the relation: $E = A$, in conformity with (16.3).

The theorem has been developed here for the general case where the generalized forces k have components in all directions. In the applications to airfoil theory we restrict ourselves, as noted before, to the case of generalized forces perpendicular to the direction of the original motion (to the x axis). In that case we deduce from the equations developed in **9** that w'_x is zero, while the derivative $\partial\varphi/\partial x$ assumes the value:

$$\frac{\partial\varphi}{\partial x} = \frac{-1}{\varrho V}\iiint d\xi\,d\eta\,d\zeta\,\frac{k_y\,(y-\eta) + k_z\,(z-\zeta)}{4\pi r^3} \qquad (16.5)$$

Hence at large distances from the region G the component w_x can be neglected, as it decreases as r^{-2}. The expression E in this case becomes:

$$E = \frac{1}{2}\varrho V\iint dy\,dz\,(w_{y\,\infty}^2 + w_{z\,\infty}^2) \qquad (16.6)$$

while A is reduced to:

$$A = \iiint dx\,dy\,dz\,(k_y\,w_y + k_z\,w_z) \qquad (16.7)$$

Comparing now with (14.10) we see that the quantities E and A in this case are related to the induced resistance by the equation:

$$E = A = D_i\, V \tag{16.8}$$

This result is of great importance, and makes it clear that notwithstanding the rather formal way is which it was introduced, the quantity A can actually be regarded as an amount of work to be expended in unit of time in order to produce the flow pattern behind a lifting system. To see this we consider the case where the region G with its system of forces moves through space with the velocity V in the direction of the negative axis $O\,x$, in a fluid originally at rest. The work to be expended per unit time in overcoming the induced resistance in such case is given by the product $D_i\, V$, and is therefore equal to A.

In the following sections we proceed to the deduction of a number of theorems concerning the induced resistance, which are obtained by making use of certain properties of the formulae expressing w_y and w_z in terms of the load system k_y, k_z. It is convenient to begin with the special case where the generalized forces are parallel to the axis $O\,z$, so that only k_z need be taken into account. This moreover is the most important case in view of practical applications. Return to the more general problem will be made in **19**.

17. Case of Generalized Forces all Parallel to $O\,z$. In this case the expression for the induced resistance assumes the form (14.6). The value of w_z is obtained from the equations given in **9**. According to (9.1) we have:

$$w_z = w_z' + \frac{\partial \varphi}{\partial z},$$

where from (9.4):

$$w_z' = \frac{1}{\varrho V} \int_{-\infty}^{x} d\xi\, k_z,$$

while the potential is given by [see (9.8)]:

$$\varphi = -\frac{1}{\varrho V} \int\!\!\int\!\!\int d\xi\, d\eta\, d\zeta\, \frac{k_z\,(z-\zeta)}{4\pi r\,(r - x + \xi)}$$

Hence performing the differentiation, the expression for w_z becomes:

$$\left.\begin{aligned} w_z = \frac{1}{\varrho V} \Bigg[\int_{-\infty}^{x} d\xi\, k_z & - \\ - \int\!\!\int\!\!\int d\xi\, d\eta\, d\zeta\, k_z\, & \frac{[(x-\xi)^2 + (y-\eta)^2]\,(r - x + \xi) - (z-\zeta)^2\, r}{4\pi r^3\,(r - x + \xi)^2} \Bigg] \end{aligned}\right\} \tag{17.1}$$

The fraction occurring under the triple integral sign presents a singularity for those points ξ, η, ζ which simultaneously satisfy the relations $\eta = y$, $\zeta = z$, $\xi \leqq x$, that is for the points situated on a straight segment extending from the point x, y, z in the direction of $-x$. For all such points the denominator of the fraction becomes zero.

In order to avoid this singularity we shall exclude from the domain of integration a flat prismatic region, bounded by two pairs of planes $\eta = y \pm \delta$, $\zeta = z \pm \varepsilon$, where δ and ε are both very small, while the ratio δ/ε will be taken great compared with unity. Within this slit k_z may be considered as being independent of η and ζ, and the same property, of course, applies to its integral Q_z with respect to x. In order to obtain an estimate of the inaccuracy introduced by the omission of the forces acting in the interior of the slit, we apply (10.2) combined with the results of **12**. The vortex system produced by these forces consists of filaments parallel to the y axis, having the intensity: $\gamma_y = -k_z/\varrho V$, and two vortex bands along the vertical sides of the slit, each having the strength [see (12.8)]: $\Gamma = Q_z \varepsilon/\varrho V$. The contribution of the bands to the vertical velocity in the axis of the slit is of the order ε/δ; the contribution of the vortex filaments parallel to the axis Oy is of the order $\delta \varepsilon$. Hence both amounts can be neglected when δ and ε fulfill the conditions stated.

In the case of forces distributed over a surface more or less parallel to the x, y plane, this reasoning does not apply, and another method of overcoming the difficulty is to be preferred. This will be considered in **23—28**. Provisionally we may imagine the surface distribution to be replaced by a distribution over a space of finite, but small thickness; this is sufficient for the demonstration of the main theorems expounded in the following sections[1].

When applying this procedure to the calculation of w_z, we must not only omit the domain indicated from the triple integral, but also the first term of (17.1), giving the value of w'_z. Indeed, in calculating the integrals (9.4) the line along which the integration must be performed now passes through a slit in the region G, so that no forces act at the points of this line and consequently $w'_z = 0$.

We thus obtain the following expression for w_z:

$$w_z = \frac{1}{\varrho V} \int\int\int d\xi\, d\eta\, d\zeta\, k_z \frac{-[(x-\xi)^2 + (y-\eta)^2](r-x+\xi) + (z-\zeta)^2 r}{4\pi r^3 (r-x+\xi)^2} \qquad (17.2)$$

it being understood that in working out this expression (and similar ones, occurring later), the prismatic slit has been omitted from the domain of integration[2].

---

[1] It may be noted that this singularity is not present in the original formula for the potential φ, and only arises from the circumstance that we have differentiated (9.8) under the sign of integration. No difficulty would be encountered if we should first work out the integral, and then differentiate. This, however, would require k_z to be given explicitly.

[2] It is possible to obtain other expressions in which the section of the domain to be left out has another form. For the case of a slit with square section for instance, one half of the value of w'_z, that is $Q_z/2\varrho V$, must be added to the integral.

We now consider points lying at a great distance downstream from the region G. In that case the value of $x - \xi$ (which is positive) becomes very great in comparison with all values of $y - \eta$, $z - \zeta$, which need be taken into account for points x, y, z lying either within the wake or at not too great a distance outside it. Hence we may make use of the approximations:

$$r \cong x - \xi, \qquad r - x + \xi \cong \frac{(y-\eta)^2 + (z-\zeta)^2}{2(x-\xi)}$$

Substituting these values in the fraction occurring in (17.2) we have the following expression, to be denoted by $\triangle$:

$$\triangle = \frac{-(y-\eta)^2 + (z-\zeta)^2}{2\pi[(y-\eta)^2 + (z-\zeta)^2]^2} \tag{17.3}$$

This quantity is independent of $x - \xi$. When it is substituted in (17.2),

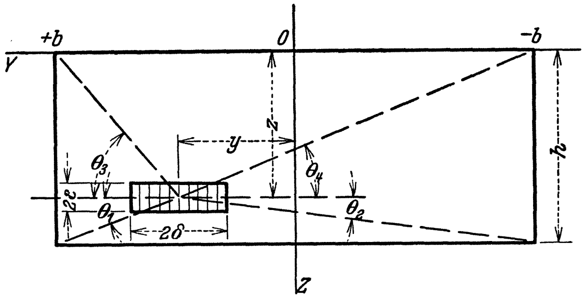


Fig. 59.

the integration with respect to ξ can be performed. In this manner we obtain for the limiting value $w_{z\infty}$ of the component w_z:

$$w_{z\infty} = \frac{1}{\varrho V} \int\int d\eta\, d\zeta\, \bar{Q}_z \triangle \tag{17.4}$$

In accordance with what has been said the quantity $\triangle$ must be replaced by zero in this integral, wherever we simultaneously have:

$$|\eta - y| < \delta, \qquad |\zeta - z| < \varepsilon \tag{17.5}$$

By way of an example we apply (17.4) to the case considered in **12.** $\bar{Q}_z$ has then to be taken constant over a rectangle in the plane Oyz, having the "span" $2b$ and the height h (see Fig. 59). Beginning with the integration with respect to the coordinate ζ, we obtain the result:

$$\int d\zeta \triangle = \frac{-1}{2\pi}\left[\frac{h-z}{(\eta-y)^2 + (h-z)^2} + \frac{z}{(\eta-y)^2 + z^2}\right]$$

for all values of η satisfying the inequality $|\eta - y| > \delta$; and:

$$\int d\zeta \triangle = \frac{-1}{2\pi}\left[\frac{h-z}{(\eta-y)^2 + (h-z)^2} + \frac{z}{(\eta-y)^2 + z^2}\right] + \frac{1}{\pi}\,\frac{\varepsilon}{(\eta-y)^2 + \varepsilon^2}$$

for values of η satisfying the relation $|\eta - y| < \delta$.

We have now to perform the integration with respect to η. The expression to be calculated may be written:

$$-\int_{-b}^{+b} d\eta \frac{1}{2\pi}\left[\frac{h-z}{(\eta-y)^2+(h-z)^2}+\frac{z}{(\eta-y)^2+z^2}\right]+\int_{y-\delta}^{y+\delta} d\eta \frac{1}{\pi}\frac{\varepsilon}{(\eta-y)^2+\varepsilon^2}$$

This gives the result:

$$-\frac{1}{2\pi}\left[tan^{-1}\frac{b-y}{h-z}+tan^{-1}\frac{b+y}{h-z}+tan^{-1}\frac{b-y}{z}+ + tan^{-1}\frac{b+y}{z}\right]+\frac{2}{\pi}tan^{-1}\frac{\delta}{\varepsilon}$$

On account of the condition imposed on the ratio δ/ε we are entitled to write:

$$tan^{-1}\frac{\delta}{\varepsilon}=\frac{\pi}{2}$$

Hence introducing the angles θ_1, θ_2, θ_3, θ_4 as indicated in Fig. 59, we have:

$$\int\int d\eta\, d\zeta\, \triangle=\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2\pi}$$

Substituting this into (17.4), the value of $w_{z\infty}$ becomes:

$$w_{z\infty}=\frac{\overline{Q}_z}{\varrho V}\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2\pi} \tag{17.6}$$

The same result is obtained by starting with the vortex system as in Fig. 55 and applying Biot and Savart's formula.

18. Reduction of the Integral for the Induced Resistance.—Munk's Theorems. As has been mentioned, the induced resistance in the case considered is given by the expression:

$$D_i=\frac{1}{V}\int\int\int d\,x\,d\,y\,d\,z\,k_z w_z \tag{18.1}$$

We substitute for w_z its value (17.2), and obtain:

$$D_i=\frac{1}{\varrho V^2}\int\int\int\int\int\int d\,x\,d\,y\,dz\,d\,\xi\,d\,\eta\,d\,\zeta\cdot \cdot k_z k_z' \frac{-[(x-\xi)^2+(y-\eta)^2](r-x+\xi)+(z-\zeta)^2 r}{4\pi r^3(r-x+\xi)^2} \tag{18.2}$$

where the notation k_z, k_z' is used to distinguish between the values of k_z at the points x, y, z and ξ, η, ζ respectively.

The order of integration in the sextuple integral is of no importance, provided we always keep in mind that the system of points satisfying the inequalities (17.5) must be excluded from the now six-dimensional domain of integration.

We interchange the set of variables x, y, z with the set ξ, η, ζ. The new integral must necessarily have the same value as the original one, and in consequence of the unimportance of the order of integration we may write it:

$$\left.\begin{aligned} D_i = \frac{1}{\varrho V^2} \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int d x\, d y\, d z\, d \xi\, d \eta\, d \zeta \cdot \\ \cdot k_z k_z' \frac{-[(\xi - x)^2 + (\eta - y)^2](r - \xi + x) + (\zeta - z)^2 r}{4 \pi r^3 (r - \xi + x)^2} \end{aligned}\right\} \quad (18.3)$$

We can now deduce a new expression for D_i by adding together (18.2) and (18.3), and taking one half of their sum; this gives:

$$D_i = \frac{1}{\varrho V^2} \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int d x\, d y\, d z\, d \xi\, d \eta\, d \zeta\, k_z k_z' \frac{-(y-\eta)^2 + (z-\zeta)^2}{4\pi\,[(y-\eta)^2 + (z-\zeta)^2]^2} \quad (18.4)$$

The fraction which occurs here is just one half of that formerly called $\triangle$. As it is independent both of x and of ξ the integrations with respect to these variables can be effected; we thus obtain:

$$D_i = \frac{1}{2 \varrho V^2} \int\!\!\int\!\!\int\!\!\int d y\, d z\, d\eta\, d\zeta\, \overline{Q}_z \overline{Q}_z' \triangle \quad (18.5)$$

where $\overline{Q}_z$ refers to the variables y, z, while $\overline{Q}_z'$ refers to η, ζ.

Comparing with (17.4) and performing the integration with respect to η, ζ, we see that this integral may be written:

$$D_i = \frac{1}{2 V} \int\!\!\int d y\, d z\, \overline{Q}_z\, w_{z\,\infty} \quad (18.6)$$

where now $\overline{Q}_z$ and $w_{z\,\infty}$ both relate to the point y, z.

It is of interest to note that the integral in (18.6) is extended *over a section of the wake* only, in contrast to the integral E, which extends over the surface I in Fig. 58 as far as there are appreciable values of $w_{y\,\infty}$, $w_{z\,\infty}$.

The relations here deduced lead to a series of theorems, known as Munk's theorems[1]. They may be stated as follows:

I. The induced resistance experienced by any lifting system does not change when the points of application of the forces k_z are shifted parallel to the direction of the original velocity. In particular the whole system of forces may be concentrated in the plane $O y z$ (so-called *systems without stagger*).

II. When by shifting the forces parallel to the direction of the axis $O x$ the entire system of forces has been concentrated into the plane $O y z$, the part of the induced resistance experienced by the "concentrated" force $\overline{Q}_z$, acting at an element $d y\, d z$, on account of the flow system produced by the "concentrated" force $\overline{Q}_z'$, acting at another element $d\eta\, d\zeta$, is equal to the part of the induced resistance experienced by $\overline{Q}_z'$ on account of the flow system produced by $\overline{Q}_z$. This is a consequence of the symmetry of the function $\triangle$ with respect to the systems of variables y, z and η, ζ.

[1] See MUNK, M., Isoperimetrische Aufgaben aus der Theorie des Fluges (Inaug.-Dissertation, Göttingen 1919).

III. In calculating the total induced resistance, once all the forces have been concentrated into the plane $O y z$, we may, instead of using the actual values of the velocity w_z in the original points of application of the forces, avail ourselves of one half the limiting value $w_{z\infty}$ for the corresponding values of the coordinates y, z.

The demonstration of these theorems can also be given in a slightly different way, without introducing the multiple integrals.

In order to obtain the first theorem we start from the equation $E = A$ deduced in **16**. It follows from (17.4) that at large distances downstream the velocity $w_{z\infty}$ is dependent on the quantity $\overline{Q}_z$ only, and not on the distribution of the forces k_z over lines parallel to the x axis. This can be seen also by remembering that at large distances downstream the velocity components $w_{y\infty}$, $w_{z\infty}$ are determined exclusively by the system of trailing vortices, while at the same time from (10.3) we know that γ_x is a function of $\overline{Q}_y$, $\overline{Q}_z$. Hence, as $w_{z\infty}$ does not change by a shift of the forces k_z parallel to Ox, the same property must apply to E, and thus likewise to A and to the induced resistance.

When by means of such shifts the system has been reduced to a system of zero stagger lying in the plane $O y z$, (17.2) gives the following expression for the value of w_z in points of this plane:

$$(w_z)_{x=0} = \frac{1}{\varrho V} \int\int\int d\xi\, d\eta\, d\zeta\, k_z \frac{-(y-\eta)^2 + (z-\zeta)^2}{4\pi\,[(y-\eta)^2 + (z-\zeta)^2]^2} \tag{18.7}$$

As the fraction occurring under the integral sign is equal to $\triangle/2$ and is independent of ξ, the relation is obtained:

$$(w_z)_{x=0} = \frac{1}{2\varrho V} \int\int d\eta\, d\zeta\, \overline{Q}_z \triangle = \frac{1}{2} w_{z\infty} \tag{18.8}$$

Hence the expression (18.1) reduces to:

$$D_i = \frac{1}{2V} \int\int\int dx\, dy\, dz\, k_z w_{z\infty}$$

which by performing the integration with respect to x becomes identical with (18.6), and thus leads immediately to the second and third theorems.

Equation (18.8), which expresses that *in the case of a system without stagger the velocity w_z in the points of the system itself is equal to one half of the limiting velocity $w_{z\infty}$ for the corresponding values of the coordinates* y, z, is a special case of a very general relation of great value, not only in the theory of lifting systems, but likewise in that of propellers, etc. It can be deduced without considering in detail the expression for w_z, by making use of certain symmetrical properties of the flow system. This becomes of importance especially in the case of loaded systems placed in a windchannel, in which case the full expression for w_z cannot be obtained without great difficulty. We return to this point in IV **38**.

19. General Case of Forces Perpendicular to the Axis Ox. When the forces k have components parallel to both the axes Oz and Oy the expression for w_z must be completed with a term depending on k_y, to be obtained by differentiating with respect to z the corresponding part of φ, that is $-\frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta \frac{k_y(y-\eta)}{4\pi r(r-x+\xi)}$

Working out the differentiation and adding the results to (17.2), we have:

$$\left.\begin{aligned} w_z = \frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta\, k_z \frac{-[(x-\xi)^2+(y-\eta)^2](r-x+\xi)+(z-\zeta)^2 r}{4\pi r^3(r-x+\xi)^2} + \\ + \frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta\, k_y \frac{(y-\eta)(z-\zeta)(2r-x+\xi)}{4\pi r^3(r-x+\xi)^2} \end{aligned}\right\} \quad (19.1)$$

Though the denominator of the fraction occurring in the second integral again becomes zero when simultaneously $\eta = y$, $\zeta = z$, $\xi \leqq x$, the integral itself remains regular, and hence it is not necessary to exclude any region from the domain of the integration[1].

The analogous expression for w_y becomes:

$$\left.\begin{aligned} w_y = \frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta\, k_y \frac{-[(x-\xi)^2+(z-\zeta)^2](r-x+\xi)+(y-\eta)^2 r}{4\pi r^3(r-x+\xi)^2} + \\ + \frac{1}{\varrho V}\int\int\int d\xi\, d\eta\, d\zeta\, k_z \frac{(z-\zeta)(y-\eta)(2r-x+\xi)}{4\pi r^3(r-x+\xi)^2} \end{aligned}\right\} \quad (19.2)$$

Here in the calculation of the first integral we have to leave out from the domain of integration the prismatic region bounded by the two pairs of planes: $\eta = y \pm \varepsilon$, $\zeta = z \pm \delta$, where δ is large compared with ε.

When we pass to large values of x, we may make use of the same approximations as those introduced in the reduction of (17.2). The limiting value of the fraction occurring in the second integral both of (19.1) and of (19.2) takes the form:

$$\triangle_1 = \frac{(y-\eta)(z-\zeta)}{\pi[(y-\eta)^2+(z-\zeta)^2]^2} \quad (19.3)$$

while the fraction in the first integral of (19.2) becomes $-\triangle$. Hence we may write:

$$w_{z\infty} = \frac{1}{\varrho V}\int\int d\eta\, d\zeta\, (\overline{Q}_z\triangle + \overline{Q}_y\triangle_1) \quad (19.4)$$

$$w_{y\infty} = \frac{1}{\varrho V}\int\int d\eta\, d\zeta\, (-\overline{Q}_y\triangle + \overline{Q}_z\triangle_1) \quad (19.5)$$

The expression for D_i becomes in this case:

$$D_i = \frac{1}{V}\int\int\int dx\, dy\, dz\, (k_z w_z + k_y w_y) \quad (19.6)$$

---

[1] Writing η for $\eta - y$, ζ for $\zeta - z$, and supposing that both η and ζ are small compared with $x-\xi$, the fraction occurring in the second integral becomes approximately: $\frac{\eta\zeta}{\pi(\eta^2+\zeta^2)^2}$. The integral of this expression over a rectangle defined by $|\eta| < \delta$, $|\zeta| < \varepsilon$ is zero independent of the order of integration.

After substitution of (19.1) for w_z and (19.2) for w_y we apply to this expression the same procedure as has been used in **18**, that is, we interchange the set of variables x, y, z with the set ξ, η, ζ, and take the half sum of the two integrals. This gives:

$$\left.\begin{aligned} D_i = \frac{1}{2\varrho V^2}\int\int\int\int d\,y\,d\,z\,d\,\eta\,d\,\zeta\,\big[\big(\overline{Q}_z\overline{Q}_z' - \overline{Q}_y\overline{Q}_y'\big)\,\triangle + \\ + \big(\overline{Q}_z\overline{Q}_y' + \overline{Q}_y\overline{Q}_z'\big)\,\triangle_1\big] \end{aligned}\right\} \qquad (19.7)$$

where, as in (19.5), $\overline{Q}_y$, $\overline{Q}_z$ refer to the variables y, z, while $\overline{Q}_y'$, $\overline{Q}_z'$ refer to the variables η, ζ. Making use of (19.4) and (19.5) we may transform this equation as follows, analogous with (18.6):

$$D_i = \frac{1}{2V}\int\int d\,y\,d\,z\big(\overline{Q}_z\,w_{z\,\infty} + \overline{Q}_y\,w_{y\,\infty}\big) \qquad (19.8)$$

This result shows that the theorems of **18** apply likewise to the more general case of forces in two directions. The demonstration can also be developed according to the method explained at the end of **18**.

20. Problems of Minimum Induced Resistance. In the theory of lifting systems the problem of obtaining the distribution of the forces which will give a minimum value of the induced resistance for a given value of the total lift, plays an important part. Owing to the circumstance that all our formulae are built upon the assumption of a given system of "generalized forces" as starting point, we shall assume that the resultant generalized load, to be denoted by K, has been given, and that under this condition it is required to make the value of D_i as small as possible. In the general form, when no other condition is implied, the solution of this problem, however, is indeterminate, as it can be demonstrated that the induced resistance of a lifting system diminishes indefinitely, when its dimensions parallel to the plane $O\,y\,z$ are continually increased.

In order to prove this we start from (19.7) and assume that all dimensions of the system parallel to the plane $O\,y\,z$ are multiplied by the same factor n. Then the differential products $d\,y\,dz$ and $d\,\eta\,d\,\zeta$ both take the factor n^2; the quantities $\triangle$ and $\triangle_1$ are multiplied by n^{-2}, while on account of the condition that the total load must remain unchanged, the quantities $\overline{Q}_z$ and $\overline{Q}_y$ determining the "concentrated" force per unit of area of the y, z plane, all take the factor n^{-2}. Hence the integral for D_i is multiplied by n^{-2}, and thus diminishes whenever n is increased.

The problem of minimum induced resistance takes a definite form only when the dimensions of the system parallel to the y, z plane are limited[1]. In such case we may take the domain of integration as being

[1] Extension in the direction of the axis $O\,x$ is of no importance on account of the first theorem of **18**.

fixed, and we then have the problem of making the integral (19.7) a minimum, taking account of the conditions:

$$\iint dy\, dz\, \overline{Q}_z = K \tag{20.1}$$

$$\iint dy\, dz\, \overline{Q}_y = 0 \tag{20.2}$$

(we suppose for simplicity that the resultant load is directed parallel to the z axis, so that there is no component K_y).

The variational expressions for the three integrals take the form:

$$\left.\begin{aligned}\frac{1}{\varrho V^2}\iiiint dy\, dz\, d\eta\, d\zeta\, \big[\big(\overline{Q}_z' \triangle + \overline{Q}_y' \triangle_1\big)\, \delta \overline{Q}_z + \\ + \big(-\overline{Q}_y' \triangle + \overline{Q}_z' \triangle_1\big)\, \delta \overline{Q}_y\big] = 0\end{aligned}\right\} \tag{20.3}$$

$$\iint dy\, dz\, \delta \overline{Q}_z = 0 \tag{20.4}$$

$$\iint dy\, dz\, \delta \overline{Q}_y = 0 \tag{20.5}$$

where again the letters $\overline{Q}_y$, $\overline{Q}_z$ refer to the variables y, z, while $\overline{Q}_y'$, $\overline{Q}_z'$ refer to the variables η, ζ. Hence the solution of the problem can be expressed by the equations:

$$\frac{1}{\varrho V^2}\iint d\eta\, d\zeta\, \big(\overline{Q}_z' \triangle + \overline{Q}_y' \triangle_1\big) = \text{const.} \tag{20.6}$$

$$\frac{1}{\varrho V^2}\iint d\eta\, d\zeta\, \big(-\overline{Q}_y' \triangle + \overline{Q}_z' \triangle_1\big) = \text{const.} \tag{20.7}$$

It is readily seen that when these equations are fulfilled, every possible distribution of variations δQ_z, δQ_y that satisfies both (20.4) and (20.5), also automatically satisfies (20.3). On the other hand any distribution of variations δQ_z, δQ_y which would not satisfy (20.3), in this case must be in contradiction either with (20.4) or with (20.5) or with both of them, and thus would not be a variation of load distribution, for which (20.1) and (20.2) remain valid.

Comparing with (19.4) and (19.5) we obtain the remarkable result that both $w_{z\,\infty}$ and $w_{y\,\infty}$ must assume *constant values throughout the domain of integration,* or what comes to the same thing, *throughout a perpendicular section of the wake.* This result was obtained by Munk[1].

In the special case of a system which is symmetrical with respect to the plane $O\,x\,z$ it is evident that $w_{y\,\infty}$ must be zero.

Writing w_0 for the value of $w_{z\,\infty}$ we then have the very simple expression for the induced resistance of a system fulfilling the minimum conditions:

$$D_i = \frac{w_0}{2\,V}\iint dy\, dz\, \overline{Q}_z = \frac{w_0}{2\,V}\, K \tag{20.8}$$

[1] Compare Munk, M., Isoperimetrische Aufgaben aus der Theorie des Fluges (Inaug.-Dissertation, Göttingen 1919), p. 17.—More or less elementary demonstrations of the theorem have been given in many textbooks. For an elaborate proof for the case of a lift distribution along a single line the reader may be referred to W. F. Durand, Nat. Adv. Comm. Aeronautics (Washington) Rep. No. 349, 1930.

21. Distribution of Generalized Forces Giving a Constant Value of $w_{z\infty}$, $w_{y\infty}$ over a Perpendicular Section of the Wake. The problem of the distribution of the forces $\overline{Q}_z$, $\overline{Q}_y$ which will ensure the constancy of $w_{z\infty}$ and $w_{y\infty}$ over a section of the wake, admits of a very elegant solution given by Munk[1].

From the constancy of $w_{z\infty}$ and $w_{y\infty}$ in the wake it follows that in the interior of the wake, vortices cannot exist. Hence vortices can be present only on the cylindrical surface which bounds the wake. Outside this boundary the motion is irrotational. Considering now a perpendicular section of the wake, sufficiently far downstream to permit neglect of the component w_x, so that the motion is two-dimensional and wholly parallel to this section, we have the problem of finding a system of rectilinear vortices, situated on a given cylindrical boundary, which shall ensure a constant velocity within this boundary. Superpose on the entire system a rectilinear motion which just annihilates the constant velocity in the interior of the cylinder. In the usual case, where $w_{y\infty}$ is zero, while $w_{z\infty}$ is directed downward and has the positive value w_0, this superposed motion has the velocity $-w_0$, directed vertically upward.

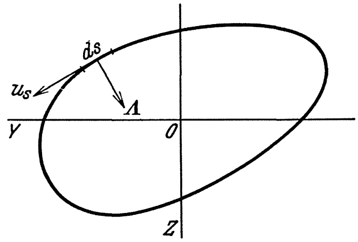


Fig. 60.

The problem then takes the form of finding an irrotational two-dimensional motion which has a given velocity $-w_0$ at infinity, and zero velocity within the interior of a given cylindrical space. It is evident from the condition of continuity, that in this case the normal component of the velocity must vanish at the surface of the cylinder. Hence we fall back on the ordinary problem of the determination of a Dirichlet-motion[2], and many solutions can be obtained from a variety of results as developed in textbooks.

When the motion outside of the cylindrical wake has been determined, the distribution of the vorticity along its circumference is to be found from the discontinuity in the tangential component of the velocity (which component, evidently, is zero in the interior of the wake, but will have a finite value outside it). The components of this motion eventually will be denoted by u_y, u_z etc. Taking the element ds of the circumference in the counterclockwise direction when looking along the direction of $-x$ (see Fig. 60) and indicating the strength of the vortex sheet by $\bar{\gamma}$, we have:

$$\bar{\gamma} = u_s \tag{21.1}$$

where u_s denotes the tangential component of the Dirichlet-motion outside of the wake.

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[1] M. Munk, l. c. p. 22.

[2] See I 8.

In order to determine the magnitude of the generalized forces, we make use of the results obtained in **13** and assume a distribution of forces directed normally and inward along the circumference of the intersection of the wake with the y, z plane. The relation between the intensity Λ of the normal force (referred to unit length) and the strength $\bar{\gamma}$ of the vortex sheet is determined by (13.2). Hence:

$$\frac{1}{\varrho V}\frac{d\Lambda}{ds} = u_s \tag{21.2}$$

Solving this equation for Λ we obtain a value which remains undetermined to the amount of an arbitrary constant, which, however, has no influence, either on the total resultant force, or on the induced resistance.

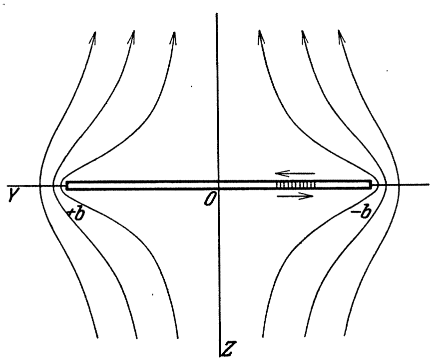

Fig. 61.

In this solution the forces will be wholly distributed over the outer circumference of the region G[1].

22. Example. Case of the Single Wing. When the load Λ is distributed over a segment of the y axis, extending from $y = -b$ to $y = +b$, the wake takes the form of a flat plate of breadth $2b$. We suppose that the forces are directed downward, as is the usual case. Then in the interior of the wake, $w_{z\infty}$ is also in the downward direction; the constant value which it must there take will again be denoted by w_0.

In order to find the distribution of the load that will ensure minimum induced resistance, we must investigate the two-dimensional irrotational motion of a fluid in the y, z plane, having at infinity the velocity $-w_0$ in the direction of the z axis, and which is obstructed by a line extending along the y axis from $y = -b$ to $y = +b$ (see Fig. 61).

As shown in Division B VI (5.9) the complex potential of this motion is given by an expression of the form:

$$w_0(\Phi + i\Psi) = i w_0 \sqrt{(y+iz)^2 - b^2} \tag{22.1}$$

where Φ itself (and likewise the stream-function Ψ) refer to unit velocity at infinity. The sign of the radical must be chosen in such a way that for large values of $y + iz$ it approximates to $+(y + iz)$; then here $w_0 \Phi \cong -w_0 z$, corresponding to the velocity $-w_0$ parallel to Oz.

The components of the velocity are then to be deduced from the equation:

$$u_y - i u_z = w_0 \frac{d(\Phi + i\Psi)}{d(y+iz)} = \frac{i w_0 (y+iz)}{\sqrt{(y+iz)^2 - b^2}} \tag{22.2}$$

[1] Certain other solutions may be deduced from this one, as has been indicated by PRANDTL (see Tragflügeltheorie II, republished in: Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen 1927).

This gives along the upper side of the segment:

$$u_y = -\frac{w_0\, y}{\sqrt{b^2 - y^2}} \tag{22.3}$$

and along the lower side:

$$u_y = +\frac{w_0\, y}{\sqrt{b^2 - y^2}} \tag{22.4}$$

The segment must be considered as a double line, as it represents the limit of a certain region in the y, z plane. It is convenient, however, to take the vortex layers on the upper and lower sides together. The intensity of the resulting vortex layer is then given by the expression:

$$\bar{\gamma}_x = -\frac{2\, w_0\, y}{\sqrt{b^2 - y^2}} \tag{22.5}$$

Inserting this into (13.2) we have:

$$\frac{d\,A}{d\,y} = -\frac{2\,\varrho\, V\, w_0\, y}{\sqrt{b^2 - y^2}} \tag{22.6}$$

from which, by integration: $A = 2\,\varrho\, V\, w_0 \sqrt{b^2 - y^2}$ (22.7)

This formula gives the elliptic distribution of the load already mentioned in **13**[1].

Other cases, relating to more complicated wing systems, will be given in IV **16—20**.

C. The Field of Induced Velocities.

23. Expressions for the Calculation of the Velocity Components when the "Generalized Forces" are given. We return to the problem of deducing expressions for the velocity components which are applicable in various cases. The general equations have already occurred at various points in the course of the deductions concerning the theorems on induced resistance. Thus the value of w_x has been given in (16.5), and that of w_z and w_y in (19.1) and (19.2). For convenience they are repeated here for the case $k_y = 0$. The results derived for this case can be generalized without difficulty if k_y should be different from zero.

Thus in this case we have:

$$w_x = \frac{1}{\varrho\, V} \int\!\!\int\!\!\int d\,\xi\, d\,\eta\, d\,\zeta\, k_z \frac{-(z-\zeta)}{4\,\pi\, r^3} \tag{23.1}$$

$$w_y = \frac{1}{\varrho\, V} \int\!\!\int\!\!\int d\,\xi\, d\,\eta\, d\,\zeta\, k_z \frac{(y-\eta)\,(z-\zeta)\,(2\,r - x + \xi)}{4\,\pi\, r^3\,(r - x + \xi)^2} \tag{23.2}$$

$$w_z = \frac{1}{\varrho\, V} \int\!\!\int\!\!\int d\,\xi\, d\,\eta\, d\,\zeta\, k_z \frac{-[(x-\xi)^2 + (y-\eta)^2]\,(r - x + \xi) + (z-\xi)^2 r}{4\,\pi\, r^3\,(r - x + \xi)^2} \tag{23.3}$$

[1] Another demonstration of this result will be obtained in IV 4.

For great distances downstream from the region G the component w_x vanishes, while w_y and w_z are given respectively by the following equations:

$$w_{y\,\infty} = \frac{1}{\varrho V} \int\int d\eta\, d\zeta\, \overline{Q}_z \frac{(y-\eta)(z-\zeta)}{\pi[y-\eta)^2+(z-\zeta)^2]^2} \tag{23.4}$$

$$w_{z\,\infty} = \frac{1}{\varrho V} \int\int d\eta\, d\zeta\, \overline{Q}_z \frac{-(y-\eta)^2+(z-\zeta)^2}{2\pi[(y-\eta)^2+(z-\zeta)^2]^2} \tag{23.5}$$

The fractions occurring under the integral sign in these expressions formerly have been denoted respectively by the letters $\triangle_1$, $\triangle$ (see **17**, **19**).

It must be remembered that in calculating the value of the integrals (23.3) and (23.5) we must exclude from the domain of integration the part cut out by the rectangular prism defined by the set of planes:

$$\eta = y \pm \delta, \qquad \zeta = z \pm \varepsilon \tag{23.6}$$

where δ and ε are both very small, while the ratio of δ to ε is great compared with unity.

In the case of an unstaggered system, lying wholly in the plane Oyz, we must put $\xi = 0$ in (23.1)—(23.3); the integration with respect to ξ can then be performed, so that under the integral sign now occurs $\overline{Q}_z$ instead of k_z. It is of interest to note the values which the various components in that case assume at points in the plane Oyz; these values are respectively:

$$w_x = \frac{1}{\varrho V} \int\int d\eta\, d\zeta\, \overline{Q}_z \frac{-(z-\zeta)}{4\pi r^3} \tag{23.7}$$

$$w_y = \frac{1}{\varrho V} \int\int d\eta\, d\zeta\, \overline{Q}_z \frac{(y-\eta)(z-\zeta)}{2\pi r^4} \tag{23.8}$$

$$w_z = \frac{1}{\varrho V} \int\int d\eta\, d\zeta\, \overline{Q}_z \frac{-(y-\eta)^2+(z-\zeta)^2}{4\pi r^4} \tag{23.9}$$

where now

$$r = \sqrt{(y-\eta)^2+(z-\zeta)^2} \tag{23.10}$$

It is seen that the values of w_y, w_z obtained in this case are exactly one half of $w_{y\,\infty}$, $w_{z\,\infty}$, a fact which was already noted at the end of **18**.

For practical calculations the values of the velocity components in most cases are obtained through the intermediary of the vortices. As the relation between the vortices and the velocity components is given by the ordinary equations, it is possible to obtain the velocity from the vortex system by applying Biot and Savart's formula[1]. In those cases

[1] The reader must not be disturbed by the form of equations (10.1) which express the vorticity by means of the components w'_x, w'_y, w'_z only. The full equations for the γ's are:

$$\gamma_x = \frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}, \text{ etc.} \tag{I}$$

Equations (10.1) have been obtained from them by making use of the circumstance that part of the components w_x, etc. depends on a potential. When the equations (I) are considered as a system of equations for the unknowns w_x, w_y, w_z, which moreover must satisfy the equation of continuity, and the condition of vanishing at infinite distances from the vortex system, their solution is unique. It is this solution which is given by Biot and Savart's formula.

in which the vortex system has a rather simple form and can be obtained easily from the distribution of the generalized load k_z, this method has many advantages, and avoids furthermore the difficulty of removing the singularity from the integrals (23.3), (23.5) and (23.9).

In the calculation of $w_{y\,\infty}$ and $w_{z\,\infty}$ according to this method we have to take account of the vortices parallel to the axis Ox only, and these now may be considered as extending indefinitely in both directions. We thus obtain the problem of a two-dimensional motion relating to a perpendicular section of the system of trailing vortices by a plane parallel to Oyz.

24. Expressions for w_x, w_y, w_z in the Case of Uniform Loading. The most simple example for the application of the method explained at the end of the foregoing section is that of an airfoil, lying along the y axis, and loaded uniformly over the whole span. When the aspect ratio

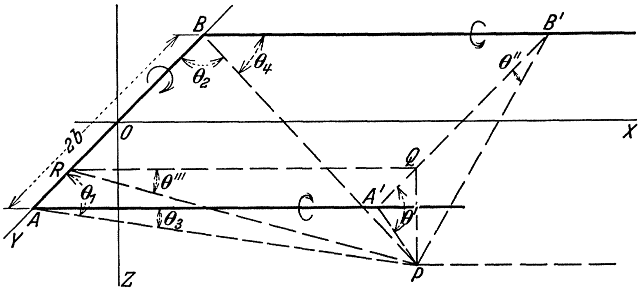

Fig. 62.

of the airfoil is not too small, then for many purposes it is sufficient to replace it by a simple loaded line, extending from $y = -b$ to $y = +b$. When the total load is K (always in the sense of resultant of the "generalized forces"), then according to the results of **12** there are only two trailing vortices, one extending from each end of the line, having the strength (irrespective of sign):

$$\Gamma = \frac{K}{2\varrho V b} \tag{24.1}$$

The method of calculating the velocity components due to the horseshoe vortex obtained in this way, can be explained with the aid of Fig. 62. Suppose it is required to determine the velocities at the point P, having the coordinates x $(= RQ)$, y $(= OR)$, z $(= PQ)$. Introduce the angles θ', θ'', θ''', θ_1, θ_2, θ_3, θ_4, which are defined by the equations:

$$\left.\begin{aligned}
&\tan\theta' = \frac{z}{b-y}, \qquad \tan\theta'' = \frac{z}{b+y}, \qquad \tan\theta''' = \frac{z}{x}\\
&\tan\theta_1 = \frac{\sqrt{x^2+z^2}}{b-y}, \qquad \tan\theta_2 = \frac{\sqrt{x^2+z^2}}{b+y}\\
&\tan\theta_3 = \frac{\sqrt{(b-y)^2+z^2}}{x}, \qquad \tan\theta_4 = \frac{\sqrt{(b+y)^2+z^2}}{x}
\end{aligned}\right\} \tag{24.2}$$

Then according to Biot and Savart's formula, the part of the velocity due to the transverse vortex extending from A to B is given by:

$$w_I = \frac{\Gamma}{4\pi\sqrt{x^2+z^2}}(\cos\theta_1 + \cos\theta_2) \tag{24.3}$$

It is directed normally to the plane passing through the points A, B and P; hence it can be decomposed into a component w_{Ix} and a component w_{Iz}, determined respectively by the equations:

$$w_{Ix} = -w_I \sin\theta''', \qquad w_{Iz} = w_I \cos\theta''' \tag{24.4}$$

The part of the velocity due to the trailing vortex extending from A to infinity parallel to the x axis is given by:

$$w_{II} = \frac{\Gamma}{4\pi\sqrt{(b-y)^2+z^2}}(1+\cos\theta_3) \tag{24.5}$$

while the part due to the trailing vortex extending from B to infinity has the magnitude:

$$w_{III} = \frac{\Gamma}{4\pi\sqrt{(b+y)^2+z^2}}(1+\cos\theta_4) \tag{24.6}$$

Both these parts lie in the plane $A'\,B'\,P$; w_{II} is perpendicular to $A'\,P$, w_{III} to $B'\,P$. They can be resolved into components $w_{II\,y}$, $w_{II\,z}$, $w_{III\,y}$, $w_{III\,z}$, determined respectively by the equations:

$$\left.\begin{aligned} w_{IIy} &= +w_{II}\sin\theta', & w_{IIz} &= w_{II}\cos\theta' \\ w_{IIIy} &= -w_{III}\sin\theta'', & w_{IIIz} &= w_{III}\cos\theta'' \end{aligned}\right\} \tag{24.7}$$

It is possible to eliminate all angles from these expressions; this can be left to the reader, as it is often a matter of mere convenience which expression is to be preferred in various cases. Likewise it seems superfluous to consider the changes brought about in the expressions when the point Q (the projection of P upon the plane $O\,x\,y$) falls outside of the region included by the horseshoe vortex.

A few special results may be mentioned, however.

The value of the component w_x (which is wholly determined by w_{Ix}) is given by the expression:

$$w_x = \frac{-\Gamma}{4\pi}\,\frac{z}{x^2+z^2}\left[\frac{b-y}{\sqrt{x^2+(b-y)^2+z^2}} + \frac{b+y}{\sqrt{x^2+(b+y)^2+z^2}}\right] \tag{24.8}$$

It is not difficult to see that this result could have been obtained equally well from (23.1), by writing it in the form:

$$w_x = \frac{-\Gamma}{4\pi}\int_{-b}^{+b} d\eta\,\frac{z}{r^3}$$

after which the integration can be performed by elementary means.

When the point P lies in the plane $O\,x\,y$, the components w_x, w_y vanish and the value of w_z is given by:

$$w_z = \frac{\Gamma}{4\pi}\left[\frac{1\pm\sqrt{1+(b-y)^2/x^2}}{b-y} + \frac{1\pm\sqrt{1+(b+y)^2/x^2}}{b+y}\right] \tag{24.9}$$

where the $+$ sign is valid for positive values of x, while the $-$ sign is to be used for x negative. For points on the axis Ox the expression becomes still more simple; it takes the form:

$$w_z = \frac{\Gamma}{2\pi b}\left(1 \pm \sqrt{1 + b^2/x^2}\right) \tag{24.10}$$

The velocity components w_y, w_z in any plane at right angles to the x axis lying sufficiently far downstream in order that $\cos\theta_3$ and $\cos\theta_4$ may be put equal to unity, while the part of the velocity due to the vortex AB can be neglected, are given by the equations:

$$\left.\begin{aligned} w_{y\infty} &= \frac{\Gamma}{2\pi}\left[\frac{z}{(b-y)^2+z^2} - \frac{z}{(b+y)^2+z^2}\right] \\ w_{z\infty} &= \frac{\Gamma}{2\pi}\left[\frac{b-y}{(b-y)^2+z^2} + \frac{b+y}{(b+y)^2+z^2}\right] \end{aligned}\right\} \tag{24.11}$$

This result evidently could have been deduced at once from the equations for plane motion in a field containing two vortices (see Fig. 63).

The distribution of the component $w_{z\infty}$ over the line $z = 0$ in such a plane is given by the equation:

$$w_{z\infty} = \frac{\Gamma b}{\pi(b^2 - y^2)} \tag{24.12}$$

It will be seen that this expression becomes infinite at the points $y = \pm b$.

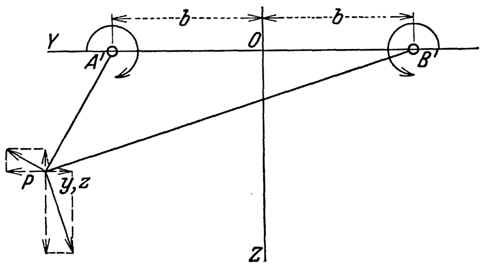

Fig. 63.

25. Approximate Calculation of Induced Velocities (Reduced Span). In many cases it is possible to obtain a good approximation to the actual distribution of the velocity in the neighborhood of any symmetrically loaded line by substituting for it a line with uniform loading. Though this substitution cannot be applied to obtain the velocities at the points of the line itself, it is of much use in problems relating to biplane systems and the like, where it is required to estimate the influence of one airfoil upon the other airfoils. In order to obtain the best approximation in such case, it is necessary to assume the trailing vortices separated by a distance not equal to the actual span $2b$, but by a somewhat reduced distance to be denoted by 2β.

We shall calculate this distance by considering especially the values of $w_{y\infty}$ and $w_{z\infty}$.

Suppose first that we have two vortices, one of strength Γ_1 at the point $y = -\eta$, $z = 0$, the other of strength $-\Gamma_1$ at the point $y = +\eta$, $z = 0$. Then the velocity components are given by equations analogous to (24.11). The expressions are simplified when $w_{y\infty}$ and $w_{z\infty}$ are combined into the complex quantity $w_{y\infty} - i\,w_{z\infty}$, which for the time being, will be denoted by w. We then obtain the equation:

$$w = \frac{i\Gamma_1}{2\pi}\left[\frac{1}{(y-\eta)+iz} - \frac{1}{(y+\eta)+iz}\right] \tag{25.1}$$

which has a form well known from the theory of plane motion.

When $\sqrt{y^2+z^2}$ is great compared with η, the expression can be developed according to inverse powers of $y+iz$:

$$w = \frac{i\Gamma_1}{\pi}\left[\frac{\eta}{(y+iz)^2} + \frac{\eta^3}{(y+iz)^4} + \dots\right] \tag{25.2}$$

In the case of an airfoil with arbitrary, but symmetrical load distribution, extending from $\eta = -b$ to $\eta = +b$, we must replace $-\Gamma_1$ at the points of the segment $0 < \eta < b$ by $\bar{\gamma}\, d\eta$, which according to (13.3) is equal to: $(1/\varrho V)\cdot dA/d\eta$, A being the load per unit span considered as a function of η. Then we must integrate with respect to η from $\eta = 0$ to $\eta = b$. If we limit the development to the first two terms, as indicated in (25.2), we obtain the following result:

$$w = \frac{-i}{\pi\varrho V}\left[\frac{1}{(y+iz)^2}\int_0^b d\eta\,\eta\frac{dA}{d\eta} + \frac{1}{(y+iz)^4}\int_0^b d\eta\,\eta^3\frac{dA}{d\eta} + \dots\right] \tag{25.3}$$

The integrals occurring in this equation can be transformed by integrating by parts. Taking into account the circumstance that the load per unit span A falls off to zero at the endpoint $\eta = b$, the following expressions are obtained:

$$\int_0^b d\eta\,\eta\frac{dA}{d\eta} = -\int_0^b d\eta\,A = -\frac{1}{2}K$$

$$\int_0^b d\eta\,\eta^3\frac{dA}{d\eta} = -3\int_0^b d\eta\,\eta^2 A$$

and thus the equation for w becomes:

$$w = \frac{i}{2\pi\varrho V}\left[\frac{K}{(y+iz)^2} + \frac{6}{(y+iz)^4}\int_0^b d\eta\,\eta^2 A + \dots\right] \tag{25.4}$$

Now instead of the actual system, again take two trailing vortices at a distance 2β from each other. As the total load must be the same as before, the load per unit span becomes $A = K/2\beta$ and the strength of these vortices will be $\Gamma = K/2\varrho V\beta$. As can be seen from (25.2) by substituting $K/2\varrho V\beta$ for Γ_1 and β for η the expression for w then assumes the following form:

$$w = \frac{iK}{2\pi\varrho V}\left[\frac{1}{(y+iz)^2} + \frac{\beta^2}{(y+iz)^4} + \dots\right] \tag{25.5}$$

Comparing with (25.4) it is found that the first terms of both expressions are identical. The second terms will become equal, when β satisfies the condition:

$$\beta^2 = \frac{6}{K}\int_0^b d\eta\,\eta^2 A \tag{25.6}$$

By means of this equation the value of β corresponding to any given load distribution can be calculated.

Taking for instance the case of elliptic load distribution we may write [see (22.7)]: $\Lambda = 2\varrho V w_0 \sqrt{b^2 - \eta^2}$

This gives: $K = \pi \varrho V w_0 b^2$,

and further: $\int_0^b d\eta\, \eta^2 \Lambda = \frac{\pi}{8} \varrho V w_0 b^4$

Whence we obtain: $\beta = b\sqrt{\frac{3}{4}} = 0.866\, b$ (25.7)

It can be shown that this same value of β can be used in obtaining approximate values for all the velocity components. In fact (23.1)—(23.3), when applied to the single loaded line, may be written:

$$\left.\begin{aligned} w_x &= \frac{1}{4\pi\varrho V}\int_{-b}^{+b} d\eta\, \Lambda \frac{-z}{r^3} \\ w_y &= \frac{1}{4\pi\varrho V}\int_{-b}^{+b} d\eta\, \Lambda \frac{(y-\eta)\, z\, (2r - x)}{r^3 (r-x)^2} \\ w_z &= \frac{1}{4\pi\varrho V}\int_{-b}^{+b} d\eta\, \Lambda \frac{-[x^2 + (y-\eta)^2](r-x) + z^2 r}{r^3 (r-x)^2} \end{aligned}\right\} \tag{25.8}$$

In any of these equations suppose the factor of Λ under the integral sign to be developed into a series according to powers of η. This is always possible when we are sufficiently far from the loaded line, and will give a development in the form:

$$A + B\eta + C\eta^2 + \ldots \tag{25.9}$$

where A, B, C, . . . are used to denote certain functions of x, y, z different for the three components.

When such a development is inserted, then on account of the symmetry of Λ, the integral takes the form:

$$\frac{1}{4\pi\varrho V}\left[A\,K + C\int_{-b}^{+b} d\eta\, \eta^2 \Lambda + \ldots\right] \tag{25.10}$$

With constant load $K/2\beta$ over a reduced span 2β we should have obtained:

$$\frac{1}{4\pi\varrho V}\left[A\,K + \frac{1}{3}\beta^2 C\,K + \ldots\right] \tag{25.11}$$

Hence in order to make both expressions equal up to terms of the order β^2, we must take: $\frac{1}{3}\beta^2 K = \int_{-b}^{+b} d\eta\, \eta^2 \Lambda$,

which condition is identical with (25.6).

This method of obtaining approximate expressions will be used frequently in Chapter IV.

It is evident that for calculations in which a first approximation only is necessary, the first term may be sufficient. This term depends upon the total load only and represents the velocity due to an airfoil of infinitesimal span, but still carrying the load K. It may be convenient for some purposes to give the values of w_x, w_y, w_z obtained in this way:

$$\left.\begin{aligned} w_x &= \frac{K}{4\pi\varrho V}\frac{-z}{r^3} \\ w_y &= \frac{K}{4\pi\varrho V}\frac{y\,z\,(2\,r-x)}{r^3\,(r-x)^2} \\ w_z &= \frac{K}{4\pi\varrho V}\frac{-(x^2+y^2)\,(r-x)+z^2\,r}{r^3\,(r-x)^2} \end{aligned}\right\} \tag{25.12}$$

At great distances behind the loaded system we have:

$$\left.\begin{aligned} w_{x\,\infty} &= 0 \\ w_{y\,\infty} &= \frac{K}{\pi\varrho V}\frac{y\,z}{(y^2+z^2)^2} \\ w_{z\,\infty} &= \frac{K}{2\pi\varrho V}\frac{-y^2+z^2}{(y^2+z^2)^2} \end{aligned}\right\} \tag{25.13}$$

26. Full Expression for the Downwash at Infinity in the Case of Elliptic Loading. The approximate expressions obtained in the foregoing section are not always satisfactory, and in some cases an exact expression is wanted. In the case of elliptic load distribution it is possible to express the exact value of $w_{z\,\infty}$ in finite terms.

Taking the formula (22.7) for $\varLambda$, inserting it in (23.9) in the place of Q_z, taking $\zeta = 0$ and multiplying by 2, the following expression is obtained:

$$w_{z\,\infty} = 2\,w_0\int\limits_{-b}^{+b} d\eta\,\sqrt{b^2-\eta^2}\,\frac{-(y-\eta)^2+z^2}{2\,\pi\,[(y-\eta)^2+z^2]^2} \tag{26.1}$$

Integration by parts leads to:

$$w_{z\,\infty} = 2\,w_0\int\limits_{-b}^{+b} d\eta\,\frac{\eta}{\sqrt{b^2-\eta^2}}\,\frac{\eta-y}{2\,\pi\,[(\eta-y)^2+z^2]} \tag{26.2}$$

This result, of course, could have been obtained just as well by starting from the system of trailing vortices, the intensity of which is given by (22.5).

The integral in (26.2) can be worked out by applying the ordinary method of substituting a new variable for η so as to make the integrand rational. As the expressions become rather complicated the following method may be of advantage.

Consider η as a variable which can assume complex values, and in the "complex η plane" take the integral:

$$I = \int d\eta\,\frac{\eta}{\sqrt{\eta^2-b^2}}\,\frac{\eta-y}{2\,\pi\,[(\eta-y)^2+z^2]} \tag{26.3}$$

performed in counterclockwise direction about a circle around the origin, of very great radius R. The singular points of the integrand are:

$$\eta = y + iz, \quad y - iz, \quad + b, - b.$$

At all of these points the integrand becomes infinite and furthermore the points $+ b$, $- b$ are branch points where the two-valued expression $\sqrt{\eta^2 - b^2}$ can change sign. No ambiguity in sign, however, will arise if it is assumed that in no case shall the path of the variable η ever cross the straight segment from $- b$ to $+ b$. To be precise, we assume that the sign of $\sqrt{\eta^2 - b^2}$ is to be determined in such manner that for η moving radially outward this expression approximates to η.

Under these conditions the limiting value of the integral along the circle mentioned for indefinitely increasing values of R, is easily obtained; it is equal to:

$$I = \int d\eta \frac{1}{2\pi\eta} = i \qquad (26.4)$$

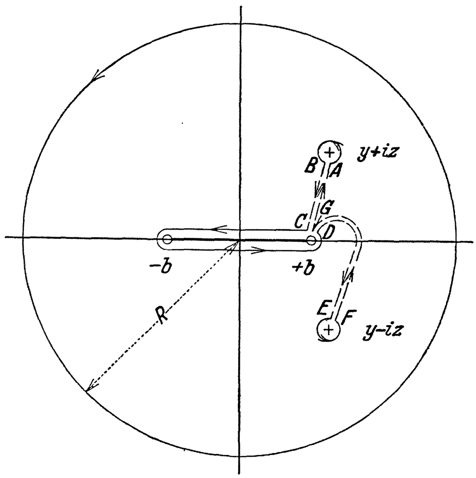

Fig. 64.

Now according to Cauchy's theorem on the integrals of analytical functions of a complex variable[1], the value of the integral I remains unchanged when the path of integration is transformed in an arbitrary manner, provided all transformations are made in such a way that the path of integration is never moved over any of the singular points, nor anywhere crosses the segment from $- b$ to $+ b$. We may take for example the contour indicated in Fig. 64, which, starting from A, consists first of a small circle around the pole $y + i z$, then goes from B to C, after which it consists of an elongated contour encircling both branch points, next goes from D to E, encircles the pole $y - i z$, and finally goes back to A along the path FGA. When the curves BC, AG are sufficiently near to each other, then the parts of the integrals relating to them cancel, since the integrand is univalued on account of the condition assumed before, while both paths are taken in opposite senses. The same reasoning can be applied to the curves DE, GF. Hence the integral relating to the original contour can be transformed into the sum of three integrals, relating respectively to two contours of small extent, each encircling one of the poles $y + i z$, $y - i z$ and a third

[1] See Division A I 8, 9.

contour which encircles both branch points. All contours again must be taken in the counterclockwise direction.

The integrals along the contour encircling the poles simply give the "residues" at those poles, which respectively amount to:

$$\left.\begin{aligned} Res.\,(y+i\,z) &= \frac{i}{2}\frac{y+i\,z}{\sqrt{(y+i\,z)^2-b^2}} \\ Res.\,(y-i\,z) &= \frac{i}{2}\frac{y-i\,z}{\sqrt{(y-i\,z)^2-b^2}} \end{aligned}\right\} \tag{26.5}$$

Finally it is easily seen that the integral along the contour encircling both branch points is equal to twice the value of the integral:

$$\int_{-b}^{+b} d\eta \frac{\eta}{\sqrt{\eta^2-b^2}}\,\frac{\eta-y}{2\,\pi\,[(\eta-y)^2+z^2]} \tag{26.6}$$

when that sign is given to $\sqrt{\eta^2-b^2}$, which is obtained on approaching the segment from $-b$ to $+b$ from *below*. Under this condition:

$$\sqrt{\eta^2-b^2} = -\,i\,\sqrt{b^2-\eta^2},$$

and with reference to (26.2), we see that the integral (26.6) is equal to:

$$i\int_{-b}^{+b} d\eta \frac{\eta}{\sqrt{b^2-\eta^2}}\,\frac{\eta-y}{2\,\pi\,[(\eta-y)^2+z^2]} = i\,\frac{w_{z\,\infty}}{2\,w_0} \tag{26.7}$$

Hence comparing with (26.4) we obtain the relation:

$$w_{z\,\infty} = w_0\left\{1-\frac{1}{2}\left[\frac{y+i\,z}{\sqrt{(y+i\,z)^2-b^2}}+\frac{y-i\,z}{\sqrt{(y-i\,z)^2-b^2}}\right]\right\} \tag{26.8}$$

This result could have been obtained also in another way. As the elliptic distribution of Λ along the span corresponds to the minimum of induced drag, it is connected with the two-dimensional Dirichlet-motion, mentioned in **21** and **22**. As the potential of this latter motion is denoted by $w_0\,\Phi$, we have the relations:

$$\left.\begin{aligned} w_{y\,\infty} &= w_0\frac{\partial\,\Phi}{\partial\,y} \\ w_{z\,\infty} &= w_0\frac{\partial\,\Phi}{\partial\,z}+w_0 \end{aligned}\right\} \tag{26.9}$$

Introducing once more the complex quantity $w = w_{y\,\infty} - i\,w_{z\,\infty}$ and the complex potential $\Phi + i\,\Psi$, the expressions can be combined into:

$$w = w_0\left[\frac{d\,(\Phi+i\,\Psi)}{d\,(y+i\,z)} - i\right] \tag{26.10}$$

As $\Phi + i\,\Psi$ is given by (22.1), we obtain:

$$w = i\,w_0\left[\frac{y+i\,z}{\sqrt{(y+i\,z)^2-b^2}} - 1\right] \tag{26.11}$$

which again leads to the expression (26.8) for $w_{z\,\infty}$.

The expression can be freed from imaginaries by introducing two new variables λ, μ, defined by the equation:

Then:
$$y + i\,z = b\,\cosh\,(\mu + i\,\lambda) \tag{26.12}$$

$$\left.\begin{aligned} y &= b\,\cosh\mu\,\cos\lambda \\ z &= b\,\sinh\mu\,\sin\lambda \end{aligned}\right\} \tag{26.13}$$

These variables λ, μ are called elliptical coordinates. It can be shown that the lines $\mu = const.$ are ellipses, while the lines $\lambda = const.$ are hyperbolae, all having their foci in the points $y = \pm\, b$, $z = 0$ [1]. After some reduction the following formula is obtained:

$$w_{z\,\infty} = w_0 \left[1 - \frac{\sinh\mu\,\cosh\mu}{\cosh^2\mu - \cos^2\lambda}\right] \tag{26.14}$$

Numerical values for $w_{z\,\infty}/w_0$ have been calculated for various points; the results are given in the accompanying Table and likewise are shown in Fig. 65. As special cases of (26.13) it may be noted that $\mu = 0$ gives: $y = b\,\cos\lambda$, $z = 0$; in this case $-\,b \leqq y \leqq +\,b$, while from (26.14):

$$w_{z\,\infty} = w_0$$

as necessarily must be. On the other hand $\lambda = 0$ gives: $y = b\,\cosh\mu$, $z = 0$; hence $|y| \geqq b$. Equation (26.14) now leads to:

$$w_{z\,\infty} = w_0 \left[1 - \frac{y}{\sqrt{y^2 - b^2}}\right]$$

The latter expression gives $w_{z\,\infty} = 0$ for $y = \infty$.

27. Calculation of the Downwash at the Points of the Load System—Wing Replaced by Loaded Line. In calculating the components of the induced velocity at the points of the load system itself it is often possible to make use of certain simplifying assumptions. In particular this is the case when the extent of the system in the direction of the x axis is very small, so that the system approaches to one having zero stagger. A further simplification is possible when the extent in the vertical direction likewise is very small, so that the principal dimension of the system is parallel to the y axis. It can be assumed that the load systems, corresponding to airfoils and airfoil combinations of the usual type, approach fairly well to this case.

It is convenient to start again from the vortex system. With a sufficient degree of approximation it can be assumed that the trailing vortices (the vortices parallel to the x axis), start from points in the y, z plane. In calculating the part of the induced velocity due to these vortices, it is sufficient to consider points of this plane only. The values obtained in this way for w_y, w_z will differ only slightly from the exact values of these components, as the procedure indicated amounts to neglecting the terms $\cos\theta_3$, $\cos\theta_4$ in expressions of the type (24.5), (24.6),

[1] See Division B VIII 2.

Tabular Values for Fig. 65 (Values of $w_{z\infty}/w_0$).

y/b \ z/b	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	1.0000	0.9005	0.8039	0.7126	0.6286	0.5528	0.4855	0.4265	0.3753	0.3310	0.2929
0.05	1.0000	0.9001	0.8032	0.7117	0.6276	0.5517	0.4845	0.4256	0.3744	0.3303	0.2922
0.20	1.0000	0.8943	0.7925	0.6976	0.6116	0.5352	0.4686	0.4109	0.3613	0.3188	0.2823
0.40	1.0000	0.8716	0.7514	0.6449	0.5541	0.4782	0.4152	0.3629	0.3192	0.2825	0.2513
0.60	1.0000	0.8102	0.6486	0.5251	0.4351	0.3692	0.3197	0.2811	0.2500	0.2243	0.2025
0.70	1.0000	0.7398	0.5446	0.4198	0.3421	0.2915	0.2557	0.2287	0.2070	0.1888	0.1731
0.80	1.0000	0.5904	0.3663	0.2686	0.2245	0.1994	0.1835	0.1712	0.1606	0.1510	0.1419
0.90	1.0000	0.1984	0.0701	0.0719	0.0874	0.1001	0.1081	0.1122	0.1134	0.1126	0.1103
0.95	1.0000	−0.2092	−0.1158	−0.0328	0.0191	0.0514	0.0714	0.0835	0.0904	0.0938	0.0947
1.00		−0.7022	−0.2922	−0.1296	−0.0444	0.0057	0.0366	0.0561	0.0683	0.0756	0.0796
1.10	−1.4004	−0.9191	−0.4856	−0.2658	−0.1442	−0.0708	−0.0239	0.0071	0.0273	0.0418	0.0511
1.20	−0.8091	−0.6909	−0.4821	−0.3148	−0.2000	−0.1224	−0.0692	−0.0321	−0.0059	0.0127	0.0259
1.30	−0.5650	−0.5193	−0.4171	−0.3092	−0.2175	−0.1505	−0.0990	−0.0607	−0.0323	−0.0112	0.0045
1.40	−0.4289	−0.4068	−0.3509	−0.2826	−0.2171	−0.1610	−0.1156	−0.0796	−0.0516	−0.0298	−0.0129
1.50	−0.3416	−0.3289	−0.2958	−0.2514	−0.2044	−0.1607	−0.1226	−0.0907	−0.0646	−0.0434	−0.0265

which is apparently allowed, as these angles are all nearly 90^0. The values of w_y, w_z then become just one half of the values of $w_{y\infty}$, $w_{z\infty}$ which can be found—as indicated at the end of **23**—by intersecting the system of trailing vortices by a plane parallel to Oyz, and calculating the velocity with the aid of the formulae valid for plane motion.

To the values obtained in this way must be added the part of the induced velocities due to the vortices which are parallel to the plane Oyz, that is due to the transverse vortices. When again we take the case $k_y = 0$, these vortices will be parallel to the axis Oy. Now in calculating the velocity components due to the transverse vortices at the points of any plane $y = const.$, intersecting them, it may be assumed that they can be replaced by rectilinear vortices of infinite extent and constant intensity. The error introduced in this way will again be small, as long as the points considered are sufficiently near to the vortices, which is implied in the condition of a very small extent of the system in both of the directions Ox, Oz. [The error amounts to substituting unity both for $\cos\theta_1$ and $\cos\theta_2$ in expressions of the type (24.3)]. In performing the calculations the vortices must be taken with the intensity they have at the points of intersection with the plane under consideration.

As an example take the case of a system of forces distributed over a plane rectangular area, forming part of the x, y plane, and satisfying the condition that its dimension c in the x direction is small compared to the span $2b$. For convenience the symbol k_z^* will be used to denote the force per unit of area. All vortices lie in the plane Oxy, and the general character of the system is indicated in Fig. 66. The connection between the transverse vortices and the trailing vortices has been pictured in an arbitrary manner. The actual distribution of the vortex elements must be calculated from equations of the type (14.1). Thus the strength of a band of transverse vortices (directed parallel to the y axis) having the breadth $d\xi$, at the distance y from the x, z plane is given by:

$$\bar{\gamma}_y \, d\xi = -\frac{k_z^*}{\varrho V} d\xi \qquad (27.1)$$

The intensity of a band of trailing vortices having the breadth dy, once it has left the loaded area, is, according to (13.3):

$$\bar{\gamma}_x \, d\eta = \frac{1}{\varrho V} \frac{dA}{d\eta} d\eta \qquad (27.2)$$

where A, the load per unit span, is given by the integral:

$$A = \int_0^c d\xi \, k_z^* \qquad (27.3)$$

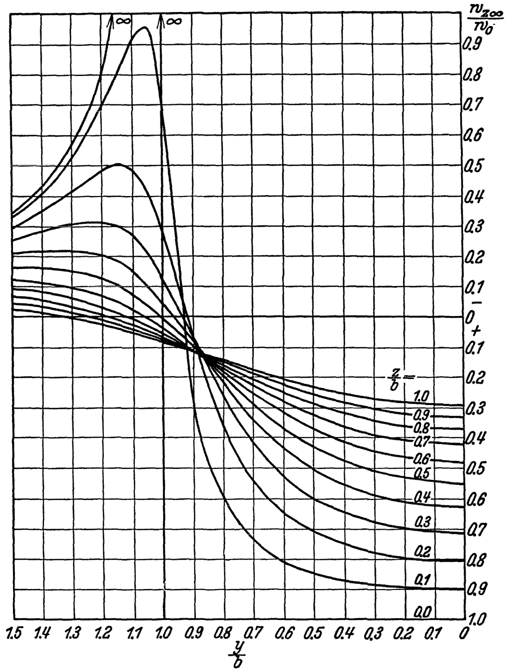

Fig. 65.

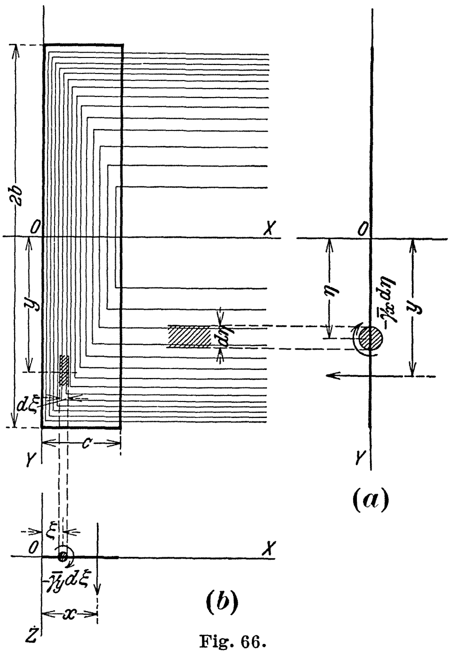

Fig. 66.

and is considered as a function of η. Then the part of the component w_z at a point of the loaded area due to the trailing vortices becomes (see Fig. 66a, where the case of a negative value of $\bar{\gamma}_x$ has been taken):

$$\frac{1}{4\pi}\int_{-b}^{+b} d\eta \, \frac{\bar{\gamma}_x}{y-\eta} \tag{27.4}$$

and the part due to the transverse vortices (see Fig. 66b, where also $\bar{\gamma}_y$ must be considered as negative):

$$-\frac{1}{2\pi}\int_{0}^{c} d\xi \, \frac{\bar{\gamma}_y}{x-\xi} \tag{27.5}$$

The difference between the numerical factors before the integrals, $1/4\pi$ in (27.4) and $1/2\pi$ in (27.5), is due to the circumstance that in the first case we are dealing with vortices extending from the plane Oyz in the direction of the positive x axis only, whereas in the other case we consider the vortices as extending indefinitely in both directions. The denominator of the fraction in (27.4) will become zero for $\eta = y$; that of the fraction occurring in (27.5) becomes zero for $\xi = x$. No difficulty, however, arises from these singularities when it is understood that in both cases the so-called principal value of the integral must be taken[1].

Inserting the values of $\bar{\gamma}_x$, $\bar{\gamma}_y$ given respectively by (27.2) and (27.1) we obtain the following expression for w_z:

$$w_z = \frac{1}{\varrho V}\left[\int_{-b}^{+b} d\eta \, \frac{d\Lambda/d\eta}{4\pi(y-\eta)} + \int_{0}^{c} d\xi \, \frac{k_z^*}{2\pi(x-\xi)}\right] \tag{27.6}$$

In this way that part of the field due to the vortices parallel to the y axis and that due to the trailing vortices (parallel to the x axis) are completely separated from each other.

Formula (27.6) can be extended without difficulty to the case of a non-rectangular area; this can be done for instance by retaining a rectangle completely surrounding the given area as the domain of integration, and taking k_z^* equal to zero for those parts of it which lie outside of the loaded area.

Though the formula derived strictly applies only under the limitation of a very great span, compared to the other dimensions of the loaded

---

[1] The principal value of the integral $\int_{-b}^{+b} d\eta \, \frac{\bar{\gamma}_x}{y-\eta}$ is defined as the limit of the sum of the integrals: $\int_{-b}^{y-\varepsilon} d\eta \, \frac{\bar{\gamma}_x}{y-\eta} + \int_{y+\varepsilon}^{+b} d\eta \, \frac{\bar{\gamma}_x}{y-\eta}$, when ε goes to zero. This limit has a finite value. An analogous remark applies to the other integral.

system, it is customary to use it as a basis for the calculation of the field of motion in the neighborhood of ordinary airfoils with aspect ratio in the usual range (from *ca.* 4 upward), assuming that the span of the airfoil is parallel to the y axis. When the span of the airfoil makes an angle with the y axis, (27.6) cannot be used. This case requires a special investigation and will not be considered here[1].

Henceforth we introduce the notation:

$$w = \frac{1}{\varrho V} \int_{-b}^{+b} d\eta \frac{dA/d\eta}{4\pi(y-\eta)} \tag{27.7}$$

then (27.6) takes the form:

$$w_z = w + \frac{1}{\varrho V} \int_0^c d\xi \frac{k_z^*}{2\pi(x-\xi)} \tag{27.8}$$

The point of importance in this equation is that the first term on the right hand side, w (*i. e.* the vertical velocity produced by the system of trailing vortices) is independent of both x and z, while the second term, representing the part due to the transverse vortices, has the same form as it would have in the case of plane motion, all transverse vortices extending indefinitely in the direction of the y axis.

28. Case of a Loaded Surface of Arbitrary Form. The simplifying assumptions introduced in the foregoing section become less satisfactory when the dimension of the loaded region in the direction of the x axis is no longer small compared to the dimension in the y direction. It may be useful in such case to have a formula for w_z derived without reference to those assumptions.

We consider the case of a system of forces distributed over a plane surface of given extent, forming part of the x, y plane. The intensity per unit area is again denoted by k_z^*. The value of w_z may be derived from the expression (17.1). We again consider points lying in the loaded region itself. In order to avoid the difficulties arising from the circumstance that in this case the term $r - x + \xi$ in the denominator may become zero for certain points ξ, η, we provisionally take points x, y, z which lie at a certain small distance z below this plane and afterward go back to the limit $z = 0$[2]. In writing down the expression for the case $z \gtrless 0$ we must omit the first term from (17.1) and thus obtain:

$$w_z = \frac{1}{\varrho V} \int\int d\xi\, d\eta\, k_z^* \frac{-[(x-\xi)^2 + (y-\eta)^2](r - x + \xi) + z^2 r}{4\pi r^3 (r - x + \xi)^2} \tag{28.1}$$

In order to deduce a more convenient formula, we divide the region of the x, y plane, where the forces k_z^* are acting, into three parts as

[1] See IV **14**, **15**.

[2] It can be demonstrated that the velocity component w_z remains continuous when z passes through zero.

indicated in Fig. 67. The quantities δ, δ' must both be very small; moreover we assume the ratio δ'/δ large compared to unity.

When the point ξ, η lies within the region I the term $r - x + \xi$ cannot become zero when z is taken equal to zero. Hence in the part of the integral (28.1) that relates to the region I, z may be put equal to zero at once; in this way the first part of w_z takes the value:

$$(w_z)_I = \frac{1}{4\pi\varrho V} \iint\limits_{(I)} d\xi\, d\eta \frac{-k_z^*}{r(r - x + \xi)} \tag{28.2}$$

The suffix (I) under the integral sign indicates that the integration is to be performed over this region only.

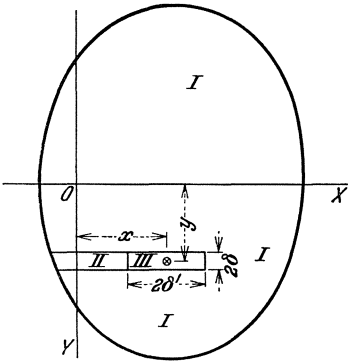

Fig. 67.

Coming to the region II we consider k_z^* as independent of η within this region, on account of its small breadth 2δ. Besides, as for all points within this region the ratio of $y - \eta$ to $x - \xi$ is small compared with unity, we may then introduce the approximations:

$$r \cong x - \xi, \qquad r - x + \xi \cong \frac{(y-\eta)^2 + z^2}{2(x-\xi)}$$

When these expressions are put into the integral (28.1), the following formula for the second part of w_z is obtained:

$$(w_z)_{II} = \frac{1}{4\pi\varrho V} \int^{x-\delta'} d\xi\, k_z^* \int_{y-\delta}^{y+\delta} d\eta \frac{2[-(y-\eta)^2 + z^2]}{[(y-\eta)^2 + z^2]^2} \tag{28.3}$$

The lower limit with respect to ξ in this integral and in the following one is determined by the extent of the loaded surface in the direction of $-x$; the upper limit, as indicated, is $x - \delta'$. The integration with respect to η can be effected without difficulty; it gives the result $4\delta/(\delta^2 + z^2)$. When now we take z equal to zero, the value of $(w_z)_{II}$ becomes:

$$(w_z)_{II} = \frac{1}{\pi\varrho V\delta} \int^{x-\delta'} d\xi\, k_z^* \tag{28.4}$$

In the interior of the region III, k_z^* will be considered as being independent both of η and ξ, so that it can be removed before the integral signs. Now if we consider the value of the fraction occurring in (28.1) for a point ξ, η and add to it the value of this fraction for the point ξ', η where $\xi' = 2x - \xi$ (so that the second point is symmetrical to the first with respect to a line drawn through the point x, y parallel to the y axis), it appears that the sum is independent of both x and ξ. If this sum is substituted in the integral, then the integration with respect to ξ must be effected between the limits $x - \delta'$ and x only; hence the third part of w_z assumes the form:

$$(w_z)_{III} = \frac{k_z^*}{4\pi\varrho V}\int\limits_{x-\delta'}^{x} d\xi \int\limits_{y-\delta}^{y+\delta} d\eta \frac{2[-(y-\eta)^2 + z^2]}{[(y-\eta)^2 + z^2]^2} \tag{28.5}$$

The integration with respect to η again gives: $4\,\delta/(\delta^2 + z^2)$; taking z equal to zero and observing that the integration with respect to ξ simply gives the factor δ', the expression for $(w_z)_{III}$ becomes:

$$(w_z)_{III} = \frac{1}{\pi\varrho V\delta}\,\delta' k_z^* \tag{28.6}$$

The expressions (28.4) and (28.6) can be combined as follows:

$$(w_z)_{II} + (w_z)_{III} = \frac{1}{\pi\varrho V\delta}\int^{x} d\xi\, k_z^* \tag{28.7}$$

Taking all parts together, we finally obtain:

$$w_z = \frac{1}{\varrho V}\left[\iint\limits_{(I)} d\xi\, d\eta \frac{-k_z^*}{4\pi r(r - x + \xi)} + \frac{1}{\pi\delta}\int^{x} d\xi\, k_z^*\right] \tag{28.8}$$

where in the first integral k_z^* is to be considered as a function of ξ, η, while in the second integral η in k_z^* is replaced by y [1].

Though the form of this expression may appear somewhat unusual, it is not difficult to show that in simple cases, as for instance that of a uniformly loaded line, it leads to the same results as are obtained by other methods. It is also possible to obtain (27.6) from (28.8), by introducing the assumption made in the preceding section, *viz.* that the span of the loaded area is much greater than its dimensions in the direction of Ox [2].

29. Remark in Connection with Equations (28.8) and (27.6). In framing the method of solving the hydrodynamic equations (see **4**, **7**), the assumption was made that the additional velocities everywhere should be small compared to the original velocity V. This is necessary

[1] If in the diagram Fig. 67 the lines $\eta = y \pm \delta$ are prolonged in the positive direction of the x axis, so that a narrow strip is formed which stretches out from the boundary of the region denoted by *III*, down to the trailing edge of the domain of integration, then it can be shown that the contribution of this strip to the first integral of (28.8) is of the order δ/δ'. Hence it represents a quantity which can be neglected if the ratio δ'/δ is chosen sufficiently large. Consequently this strip may be omitted from the region *I* without having any appreciable influence upon the value of the integral. The region *I* is then formed by cutting out from the original domain of integration a strip of width $2\,\delta$ extending *from the leading edge to the trailing edge*. This may be convenient in certain reductions of the integral.

[2] See IV **15**.—The problem of the calculation of w_z in the case of airfoils of small aspect ratio will be considered again in IV **13**—**15** principally in connection with work by Blenk.

in order to be able to apply the method of successive approximations. The question arises whether the various formulae developed for the calculation of the induced velocities actually give values which are sufficiently small.

A thorough investigation of this point is a rather complicated matter, and in many cases would require lengthy discussions of numerical values. It is moreover a matter which has received but slight attention.

Only one point will be mentioned here: As long as the integrals give finite values for the various components, it is always possible to arrive at cases where these components assume arbitrarily small values, simply by supposing the loads to be diminished proportionally in a given ratio. Though the question remains as to what magnitudes the additional velocities take in actual cases, it is seen that cases can be constructed, in which the original assumptions are valid.

An exception, however, appears, when at certain points one or more of the components of the induced velocity become infinite. Then a decrease by diminishing the loads proportionally is impossible. Thus the most important question appears to be whether all integrals have finite values throughout the whole of space? Difficulties in this respect especially may be encountered at the boundaries of the loaded area.

It seems fairly certain that a necessary condition for obtaining finite values of the induced velocity at the boundary of a loaded area is, that the load per unit area must *vanish* at the boundary. If this were otherwise, the normal component of the velocity would usually become infinite at the boundary.

The question thus arises whether the force systems which must be introduced in order to represent the action of airfoils always satisfy this condition? The answer to this question is negative. In the case of an airfoil, though the force per unit area vanishes at the trailing edge and at the tip edges, it usually does *not* vanish at the leading edge.

This circumstance is connected with the fact that in most cases the front stagnation point of the airflow does not fall on the leading edge, but on the under side of the airfoil. That, notwithstanding this fact, the flow is possible without leading to infinite velocities, is a consequence of the rounding off of the leading edge, which thus cannot be assimilated to a body of zero thickness.

This point has received attention already in Chapter II, as it occurs primarily in relation to the theory of airfoil sections (see **10**), and it appears from that Chapter that as far as we are concerned with the part of the induced velocities which is connected with the transverse vortices, the difficulties can be escaped by making use of the theory of conformal transformations, which leads at once to the field of flow around an airfoil profile of arbitrary shape.

The pecularities, however, which may arise in this respect when we pass to the general three-dimensional problem, thus far have not received an adequate treatment.

D. The Kutta-Joukowski Theorem.

30. The Kutta-Joukowski Theorem for Wings of Infinite Span. According to the program formulated at the end of **7** and to the present point in our deductions, the starting point has been the assumption of some prescribed distribution of "generalized forces" (k forces). Formulae for the relations between the "generalized forces" and the real forces (the f's) have been developed in **14**, but these formulae have as yet been applied only to calculate the induced resistance, the theory of which was given in Part B (**16—22**). In all deductions relating to the magnitude of the induced drag, the same as in the deductions of Part C (**23—29**) concerning the field of induced velocities, the term "load" was used as an abbreviation for "generalized load", *i. e.* for the resultant of the k forces. Having regard to the connection between the k forces and the "bound vortices", we might as well have started, in many cases, from a prescribed distribution of bound vortices—a procedure which indeed is followed in a number of papers and textbooks.

It now remains to consider the relation between the "generalized forces" and the actual resultant load, which is experienced as the *lift* by the airfoil or airfoil system. It will be found that in most cases of importance the resultant lift with sufficient approximation can be put equal to the resultant "generalized load", assuming that both are taken over the whole system. This result gives at once a more practical significance to the theory of induced drag, as we now see that the theorems obtained in **20—22** are no longer restricted to the case of a prescribed resultant "generalized load", but can be applied to find the minimum induced drag compatible with a prescribed total lift (combined of course, with certain geometrical conditions, as for instance a given value of the span).

Viewed from another side, we can express the result mentioned also in the form of a relation between the circulation around the system of bound vortices and the magnitude of the actual lift, and it will appear that this brings us back to the Kutta-Joukowski theorem, already considered in I 8, and in various parts of Chapter II.

It will be useful first to recapitulate the formulae for the connection between the "generalized forces" and the bound vortices. This connection is obtained from the equations deduced in **10** and **12**; for our present purpose we can take as the basic equation (12.10), which applies to the case of a single loaded line: $\Gamma = \Lambda/\varrho V$, or, as it may be written:

$$\Lambda = \varrho V \Gamma \tag{30.1}$$

From the generalization given in **13** it is evident that this relation is valid for the three-dimensional case with non uniform loading equally well as for the case of two-dimensional motion.

The theorem equally applies to the case of a *group of parallel loaded lines,* or to that of a loaded surface (or, as the case may be, of a cylindrical body), made up from such lines. In these cases the resultant generalized load Λ taken over a section of the system by a plane parallel to the x, z plane, will always be equal to the circulation around that section, multiplied by $\varrho\, V$.

Coming now to the real forces (f forces), (14.3), or more generally (14.8), give the basic relations to be considered.

By way of introduction let us first take the case of *two-dimensional motion.* This brings us back to the systems considered in Chapter II; it will be seen, moreover, that the following deductions are related to those of II **3**, but they are repeated here in order to obtain the connection with our present point of view and with the three-dimensional case, to be considered in the next section.

The two-dimensional case is obtained from our equations by assuming the component k_z to be independent of the coordinate y, while k_y is zero. Consequently there are no trailing vortices extending downstream. This does away at the same time with the second order forces (g forces) in the wake behind the region G. Hence the problem of successive approximations plays no part in the following equations which thus become exact.

From the first and third equation of (14.3) we have now the following expressions for the components of the external forces:

$$f_x = \varrho\, w_z\, \gamma_y\,, \quad f_z = -\varrho\,(V + w_x)\,\gamma_y \tag{30.2}$$

It is convenient to use temporarily the coordinates x, y in describing the two-dimensional motion. This can be done by replacing z by $-y$, y by $+z$, γ_y by γ_z, for which we shall write, however, $-\gamma$, in order to submit to the convention concerning the sign of the circulation, adopted in II **3**; x is left unchanged. The new y axis thus points upwards. Equations (30.2) thus become:

$$f_x = +\varrho\, w_y\, \gamma, \qquad f_y = -\varrho\,(V + w_x)\,\gamma \tag{30.3}$$

Now the quantities w_x, w_y can be calculated from the distribution of the vorticity by means of the well known equations[1]:

$$\left.\begin{aligned} w_x &= \frac{1}{2\pi}\iint d\xi\, d\eta\, \frac{(y-\eta)\,\gamma'}{(x-\xi)^2+(y-\eta)^2} \\ w_y &= \frac{1}{2\pi}\iint d\xi\, d\eta\, \frac{-(x-\xi)\,\gamma'}{(x-\xi)^2+(y-\eta)^2} \end{aligned}\right\} \tag{30.4}$$

[1] See Division B III (2.3), having regard to the convention concerning the sign of γ.

where γ' denotes the vorticity at the element $d\xi\, d\eta$. Hence for the components of the resultant of the forces f we have the expressions:

$$F_x = \frac{-\varrho}{2\pi} \int\int\int\int dx\, dy\, d\xi\, d\eta \frac{(x-\xi)\gamma\gamma'}{(x-\xi)^2+(y-\eta)^2}$$

$$F_y = \frac{-\varrho}{2\pi} \int\int\int\int dx\, dy\, d\xi\, d\eta \frac{(y-\eta)\gamma\gamma'}{(x-\xi)^2+(y-\eta)^2} - \varrho V \int\int dx\, dy\, \gamma$$

In these integrals the letter γ denotes the vorticity at the point x, y, while as in (30.4) γ' denotes the vorticity at the point ξ, η. The value of the quadruple integrals must remain unchanged, when the set of variables ξ, η is interchanged with the set x, y. As the factors $(x-\xi)$ and $(y-\eta)$ change sign in this case, we must conclude that both quadruple integrals vanish. Hence the first expression is zero, while the second one reduces to:

$$F_y = -\varrho V \int\int dx\, dy\, \gamma \tag{30.5}$$

The integral $\int\int dx\, dy\, \gamma$, however, is equal to the circulation Γ around the system; hence (30.5) is identical with

$$F_y = -\varrho V \Gamma \tag{30.6}$$

As F_y is the reaction of the lift l experienced by the airfoil per unit span, we may write (30.6) also in the form:

$$l = \varrho V \Gamma = \Lambda \tag{30.7}$$

This is the equation to be deduced. It gives us the Kutta-Joukowski theorem, demonstrated now for the case of a continuously distributed bound vorticity (the demonstration of II **3** assumed a system of isolated vortices), and shows that in the case of plane motion there is exact equality between the lift and the "generalized load" taken over a section of the system—all terms which are due to the additional (or induced) velocities w_x, w_y dropping out of the calculation.

It must be noted that (30.6) applies to a section of the *whole* system. In the case of a biplane or multiplane system of infinite span, the equation cannot be used to obtain the lifts of the individual airfoils separately.

31. The Application of the Kutta-Joukowski Theorem to the Three-Dimensional Case. We now pass over to the case of an airfoil of finite span, where the field of motion is essentially three-dimensional. In this case complications arise which require special consideration and it will be found that the theorem is not valid generally, but remains approximately true, when certain conditions are satisfied.

In order to investigate this subject we take first the case that everywhere $k_y = 0$, and start from the last equation of the system (14.3). Moreover we provisionally neglect the last term on the right hand side, and thus obtain[1]:

$$f_z = k_z - \varrho\, w_x \gamma_y \tag{31.1}$$

[1] We return to the coordinate system generally used throughout the present Chapter.

Substitution of the value of γ_y, given in (14.1), leads to:

$$f_z = k_z + \frac{w_x k_z}{V} \tag{31.2}$$

The value of w_x can be taken from (23.1); inserting this value into (31.2) and integrating with respect to x, y, z over the whole system, we have the following expression for the resultant of the forces f_z:

$$F_z = K - \frac{1}{\varrho V^2} \int\int\int\int\int\int dx\, dy\, dz\, d\xi\, d\eta\, d\zeta \frac{(z-\zeta)\, k_z k_z'}{4\pi r^3} \tag{31.3}$$

where—as was done formerly—the notation k_z, k_z' is used to distinguish between the values of k_z respectively at the points x, y, z and ξ, η, ζ, while K is the resultant of the "generalized forces" k_z.

The sextuple integral occurring in (31.3) has the value zero, as can be seen by interchanging the set of variables ξ, η, ζ with the set x, y, z. In such case the value of the integral must remain unchanged. However, the factor $z - \zeta$ changes sign; hence the integral must vanish. In this way we obtain:

$$-\int\int\int dx\, dy\, dz\, \varrho\, w_x \gamma_y = \int\int\int dx\, dy\, dz \frac{w_x k_z}{V} = 0 \tag{31.4}$$

and:

$$F_z = K \tag{31.5}$$

Equation (31.4) is a special case of a theorem given by Munk, which can be stated in the following form: When all bound vortices are parallel to one and the same direction in space, the part of the resultant external forces due to the mutual interaction of the bound vortices is zero.

Equation (31.5) expresses the fact that the resultant K of the generalized forces taken over the whole system is equal to the resultant F_z of the actual forces taken likewise over the whole system.

Now for k_z again substitute $-\varrho V \gamma_y$; then (31.5) leads to:

$$F_z = -\varrho V \int\int\int dx\, dy\, dz\, \gamma_y \tag{31.6}$$

The integral $-\int\int dx\, dz\, \gamma_y$ taken over any particular section of the system is equal to the circulation Γ around that section[1]. On the other hand F_z (positive if downward) is equal to the total lift L experienced by the system. Hence from (31.5) we obtain:

$$L = K \tag{31.7}$$

and from (31.6):

$$L = \varrho V \int dy\, \Gamma \tag{31.8}$$

If now we should write, as in (30.7), $l = \varrho V \Gamma$, then (31.8) gives us: $L = \int dy\, l$. Equation (31.8) thus expresses the fact that when the Kutta-Joukowski theorem is used to determine the lift per unit span in the general three-dimensional case, the total lift integrated over the

[1] According to the convention about signs ($\gamma_y = -\gamma$) adopted in **30.**—In the case of multiplane systems it must be kept in mind that a section of the system in general will comprise sections of the various airfoils together.

whole system obtains its correct value, provided the assumptions made are taken for granted.

In order to illustrate the meaning of this result, consider the case of two loaded segments, both in the y, z plane and situated as shown in Fig. 68, their lengths being very small compared with their distance. If K is the "generalized load" upon each of them, then the velocity w_x induced by B at the place of A will be, according to (23.1) or (25.12), with sufficient approximation:

$$w_x = -\frac{K\,h}{16\,\pi\,\varrho\,V\,(b^2+h^2)^{3/2}},$$

while that induced by A at the place of B has just the opposite value. In this case it is easily seen that the formula $l = \varrho\,V\Gamma$ is *not* true for a single section of the system by a plane perpendicular to the y axis; we have, instead, $l = \varrho\,(V + w_x)\,\Gamma$. Still (31.8) is valid for the *whole* system, consisting of both segments.

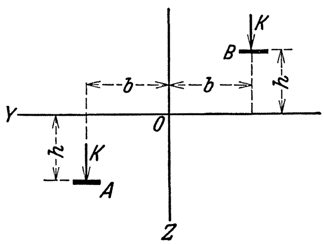

Fig. 68.

It is only *when the extension of the system in the direction of the z axis, i. e.* in the example of Fig. 68 the vertical distance h, *should be very small,* that we may neglect the value of w_x, and may put $l = \varrho\,V\Gamma$ for every section without much error.

Let us now consider the simplifying assumptions which had been introduced in order to arrive at (31.1). In the first place there is the neglected term $\varrho\,w_y\,\gamma_x$ in the full expression for f_z as given by the third equation of (14.3). This term originates from the fact that the bound vortices will not always be exactly parallel to the y axis (they may be curved perhaps, so as to have their endpoints upon the trailing edge of the airfoil). According to the definition given in **14** we may also express this by saying that certain parts of the trailing vortices will lie within the region G and thus belong to the system of bound vortices (compare the schematical case pictured in Fig. 66). These parts may become of importance in the case of a system of rather great extent in the direction of the x axis. In the ordinary case of a system of sufficiently great aspect ratio, however, we may expect that the "density" of the transverse vortices will be much greater than the "density" of the trailing vortices, which are distributed over the whole span of the system, and in consequence γ_x generally will have a much smaller value than γ_y. When besides, the extent of the system in the vertical direction is small, we may further assume that in the neighborhood of the system the velocity component w_y is much smaller than the components w_x or w_z. Hence we arrive at the conclusion that usually $w_y\gamma_x$ will be of far less significance than $w_x\gamma_y$.

It may be remarked moreover that in the somewhat artificial case of a system of forces k_z, symmetrically distributed with respect to the plane Oxy in such a way that to every force k_z acting at the point ξ, η, ζ there is present an equal force acting at ξ, η, $-\zeta$, the resultant of the terms $\varrho\, w_y\, \gamma_x$, taken over the whole system, vanishes. Hence it may be expected that in ordinary cases the neglect of this term in the expression for f_z does not lead to any serious error, though cases may occur where a more accurate calculation seems to be necessary.

Another departure from the assumption involved in Munk's theorem quoted above, *viz.* that all bound vortices should be parallel to one and the same direction, occurs when generalized force components k_y must be taken into account. Though in the greater part of the foregoing deductions we accepted the point of view that we could start from a given system of generalized forces, in which case it might seem allowable to restrict ourselves exclusively to the assumption of components k_z, it will be shown in the next section that as soon as we attempt to determine a system of generalized forces, reproducing the action of a given airfoil, such a restriction in general appears to be impossible. The components k_y will then have a certain influence upon the expression for w_x, and thus will affect the validity of our demonstration. Still in ordinary cases the values of k_y remain very small, and the correction which must be applied to the value given by (23.1) appears to be unimportant.

We will not penetrate further into these problems, nor investigate in how far various corrections may cancel each other, as in any given case definite results can always be obtained by starting from (14.3) combined with the full expressions for the induced velocities.

The conclusion, however, which we may draw from our discussion is that for an airfoil of ordinary type and not too small aspect ratio, that is for wings of the type as considered in **27**, we may with sufficient accuracy assume the equality of l and Λ for every section. This conclusion can also be deduced immediately from the results obtained in **27**, as it was shown there that for wings of this type the field of motion at a particular section can be calculated the same as with a problem in two-dimensional motion, provided a vertical velocity w is added, as given by (27.7), which is independent of both x and z.

The airfoil section is then placed in a field of motion with the velocity $\sqrt{V^2 + w^2}$, inclined at the angle φ, given by:

$$tan\ \varphi = w/V \tag{31.9}$$

When this change is taken into account, then, as was mentioned, the problem can be considered as a case of plane motion, and we can apply at once the result obtained in **30** concerning the relation between the resultant force experienced per unit span and the circulation.

We thus find that the actual force experienced per unit span has the magnitude: $$\varrho\, \Gamma \sqrt{V^2 + w^2} \tag{31.10}$$
It is perpendicular to the resultant velocity $\sqrt{V^2 + w^2}$ and is thus inclined to the vertical at the angle φ. Its component perpendicular to the original velocity V has the magnitude: $\varrho\, \Gamma V$. As this component is the reaction of the lift l experienced by the system per unit span, we have: $$l = \varrho\, \Gamma V = \Lambda \tag{31.11}$$
Hence under the assumptions made in **27** we come back to the same relation as in the two-dimensional case. It must be noted again that in the case of multiplane systems the relation does not hold for the separate airfoils.

The component of the external force parallel to V, that is, the induced drag per unit span, experienced by the section, is given by:

$$d_i = \varrho\, \Gamma w = l \frac{w}{V} = l \tan \varphi \tag{31.12}$$

These relations are of use in making the connection between the three-dimensional theory of the airfoil of finite span, and the theory of the two-dimensional flow around airfoil sections, and will form the foundation for an important part of Chapter IV (especially for Part **A**).

Remark. When the case of a viscous fluid is regarded, an interesting form of the Kutta-Joukowski theorem can be deduced, which is applicable to systems of arbitrary type. It is expressed by the equation:

$$L = \varrho\, V \int dy\, \Gamma',$$

where now Γ' is the circulation around a section of a cylinder of very large radius, having its axis along Oy, and completely surrounding the system. This theorem has been demonstrated by S. Goldstein (see Proc. Roy. Soc. London A **123**, p. 216, 1929 and A **131**, p. 198, 1931).

32. Concluding Remarks.—Inverse Problem. Having given so much attention to the properties of the field of flow produced by given forces, it remains to devote a few words to the problem of finding a system of generalized forces which will reproduce the influence of a given airfoil or of an airfoil system upon the fluid. It is obvious that in order to solve this problem we must try to determine the system of forces such that the lines of flow of the motion produced by them follow the surface of the prescribed airfoil (or airfoil system). To this condition evidently must be added the other one that the bound vortices must all fall within the space occupied by the airfoil or by the various airfoils. Taking for simplicity the case of a single airfoil, sufficiently thin in order that it may be considered as a single surface as indicated in I **9**, the conditions are that the resultant motion must be tangential to this surface, while the bound vortices must coincide with it. According to the results of **14** these conditions are equivalent to the statement that the actual external forces must have their points of application in this surface, and must everywhere be perpendicular to it.

If Z denotes the distance of a point of the surface below the x, y plane, these conditions can be expressed either by the equations:

$$(V + w_x)\frac{\partial Z}{\partial x} + w_y \frac{\partial Z}{\partial y} - w_z = 0 \tag{32.1}$$

$$\gamma_x \frac{\partial Z}{\partial x} + \gamma_y \frac{\partial Z}{\partial y} - \gamma_z = 0 \tag{32.2}$$

or by:

$$\frac{\partial Z/\partial x}{f_x} = \frac{\partial Z/\partial y}{f_y} = \frac{-1}{f_z} \tag{32.3}$$

Even without having regard to these equations it is to be seen that the simple picture of a vortex system, consisting only of transverse vortices parallel to the y axis and of trailing vortices parallel to the x axis, is not sufficient to represent the general case. Normally the boundaries of the sections of an airfoil by planes parallel to $O\,x\,z$ will present curvatures and will not be everywhere parallel to the x axis; and thus the parts of the trailing vortices which lie *in* the airfoil will be inclined to the x axis. This implies that there is a component of vorticity parallel to the z axis.

Now according to (14.7) a component γ_z can only appear when the "generalized forces" have a component k_y. This proves the remark made in the foregoing section that in all cases of force systems reproducing the action of ordinary airfoils, we must introduce components k_y. At the same time, however, we see that the influence of these components cannot be great: in fact, it must be equivalent to the effect of the slope of the trailing vortices upon the distribution of the additional velocity. The latter effect will be of the same order as that which is due to the slope of the trailing vortices behind the airfoil, which has been done away with in **15**.

It is only in the case of airfoils at exceptionally high angles of incidence or of exceptional curvature, and in the case of airfoils of very small aspect ratio (*e. g.* with an extension in the x direction greater than that in the y direction), that the influence of these points need be taken into account. As such types of airfoil differ in marked degree from those in common use, no further reference will be made to them.

Returning thus to the more ordinary type of airfoils, we can obtain an important simplification of our problem if in (32.1) we neglect the "second order terms" $w_x \partial Z/\partial x$ and $w_y \partial Z/\partial y$, so that it takes the form:

$$V\frac{\partial Z}{\partial x} - w_z = 0 \tag{32.4}$$

It thus remains necessary only to fulfil a condition concerning the component w_z. Moreover we assume that this component with sufficient accuracy is given by (27.6) or (27.8). Then as the principal unknown we take the "generalized load" per unit span Λ (as we have seen, in the case of a single wing Λ is equal to the lift l per unit span; in the case of a multiplane system this equality does not hold); Λ is to be considered as a function of the coordinate y. The distribution of the trailing vortices can then be expressed by means of Λ, and also the downward velocity w at the various sections of the airfoil. In the case of the single wing, w is given by (27.7). From w we calculate the angle φ of the downward slope of the resultant velocity according to (31.9); this angle in general will be a function of y. We then consider every section as representing the profile of an airfoil of infinite span, placed in a field with the resultant velocity $\sqrt{V^2 + w^2}$, sloping down at the angle φ. The problem of finding the distribution of the values of k_z^* over such a profile, or, what comes to the same, of finding the distribution of the system of bound vortices, can be treated by the methods of Chapter II: we may either apply the method of vortex sheets as used in the theory of thin airfoils, or the method of conformal transformation. From k_z^* we obtain the value of Λ by integrating with respect to ξ. If we do not wish to go into details we may also use a relation of the type $\Lambda = (1/2)\, \varrho\, (V^2 + w^2)\, c\, (m\, i + m')$, or with sufficient approximation for all purposes:

$$\Lambda = \frac{1}{2}\, \varrho\, V^2\, c\, (m\, i + m') \tag{32.5}$$

where i is the effective angle of incidence, measured with respect to the direction of the effective velocity, while m and m' are coefficients depending upon the shape of the profile.

In this way we first have a relation expressing w by means of Λ, and then a relation giving Λ as a function of w, through the intermediary of φ and of certain geometrical parameters. Thus a complete system of equations is obtained, from which Λ can be solved.

The treatment of this problem will form the subject of the next Chapter.

CHAPTER IV

AIRFOILS AND AIRFOIL SYSTEMS OF FINITE SPAN

1. Introduction. The subject of this Chapter is the investigation of the forces experienced by an airfoil of finite span and by airfoil systems. The basic idea for the following deductions has been indicated at the end of III **32**. The only point we have to add is that in the case of a single wing of ordinary type we may substitute the lift per unit span l for Λ (see III **31**), and consequently take l as the principal unknown quantity. Then III (27.7) becomes:

$$w = \frac{1}{\varrho\, V} \int_{-b}^{+b} d\eta\, \frac{dl/d\eta}{4\pi\, (y - \eta)} \tag{1.1}$$

and III (31.9), neglecting the difference between φ and *tan* φ:

$$\varphi = \frac{w}{V} = \frac{1}{\varrho V^2} \int_{-b}^{+b} d\eta \frac{dl/d\eta}{4\pi (y - \eta)} \tag{1.2}$$

Now suppose that the geometrical angle of incidence of an airfoil section. defined with reference to the x axis, or what comes to the same,. with reference to the horizontal velocity V, has the value α; then the *effective angle of incidence* i, measured from the downward sloping effective velocity, is given by[1]:

$$i = \alpha - \varphi \tag{1.3}$$

On the other hand from the results of Chapter II it is known that for every profile there exists a connection between the lift per unit span and the effective angle of incidence, which for all practical purposes can be written in the form of a linear function:

$$l = \frac{1}{2} \varrho V^2 c (m i + m') \tag{1.4}$$

Here c is the chord of the profile, while m and m' are two coefficients, which can be calculated for the particular profile.

In order to simplify the notation we shall assume throughout this Chapter that for each section the angles α and i are measured not from the chord, but *from the direction of zero lift*, a direction which, just as the chord, has a definite position for every airfoil profile, and which, from the aerodynamic point of view, is more important than the rather arbitrarily defined chord (see II **14**). Then m' is zero, and (1.4) becomes:

$$l = \frac{1}{2} \varrho V^2 c m i \tag{1.5}$$

According to what has been remarked in connection with II (14.5) the value of the coefficient m should in most cases be nearly 2π. Actually it generally appears to remain somewhat below this value (see Division J); as an approximation we may take it equal to 5.5. If exact values are required, they can be deduced from experimental results obtained with model airfoils according to the methods indicated in **2** and **6** below.

Substituting the value of i in (1.5) we obtain:

$$l = \frac{1}{2} \varrho V^2 c m (\alpha - \varphi) \tag{1.6}$$

Equations (1.2) and (1.6) taken together now form a system of two linear equations in the unknowns l and φ. When they have been solved, the distribution of the lift over the whole span of the airfoil will be known.

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[1] The angles α, φ, i are supposed all to be expressed in circular measure.

Equations (1.2) and (1.6) can be combined into the following integral equation for l, which is obtained by eliminating φ[1]:

$$\frac{1}{4\pi\varrho V^2}\int_{-b}^{+b} d\eta \frac{dl/d\eta}{y-\eta} + \frac{l}{\frac{1}{2}\varrho V^2 c m} = \alpha \qquad (1.7)$$

The determination of the distribution of the lift over the span will form the main subject to be considered in the first part (**A**) of this Chapter. When it has been solved, we can calculate the induced resistance experienced by the airfoil, as well as various other quantities.

It is clear that (1.7) can also be used to solve the problem: what airfoil must be taken in order to obtain a prescribed distribution of the lift? Then l is known from the beginning; hence with the aid of (1.7) and when suitable values are assumed for the coefficient m and for the angle α, the chord c necessary to produce the prescribed l can be determined; or otherwise, the angle α can be calculated for the various sections, supposing c and m to have been given.

Before entering upon the investigation of (1.2) and (1.6), or the equivalent (1.7), it is convenient to consider the more simple case of the airfoil with elliptic loading, in which case the angle φ is constant over the whole span, while at the same time the induced resistance assumes the smallest possible value for a given lift and span (see **2**).

In Part **B** of this Chapter multiplane systems will be considered. Here the problem of minimum induced resistance will first be treated. The other problem, the determination of the distribution of the lift over the various airfoils of any given system, will not be solved in an exhaustive form, as the complications involved are rather great and we must restrict ourselves rather to an indication of the methods which can be applied to its investigation.

Part **C** finally deals with the influence of boundaries which may be present in the field of motion, upon the forces experienced by airfoil systems. It is restricted to the consideration of boundaries extending indefinitely in the direction of the x axis, and consisting of generating lines parallel to Ox.

A. Single Wing.

2. Case of Elliptic Loading. It was shown in III 22 that in the case of a single airfoil, minimum induced resistance is obtained when the distribution of the lift along the span can be represented by a semi-ellipse. In that case the magnitude of the vertical velocity $w_{z\infty}$ in the

[1] In several treatises, instead of l, the circulation Γ, connected with l by the relation $l = \varrho V \Gamma$, is taken as the unknown; this, however, makes no difference for the solution.

"wake" behind the airfoil is constant[1]. For convenience we denote its value by w_0. As the downward velocities due to the trailing vortices at the points of the airfoil itself are equal to $(1/2)\, w_{z\infty}$, they are also constant, having the value $w = w_0/2$.

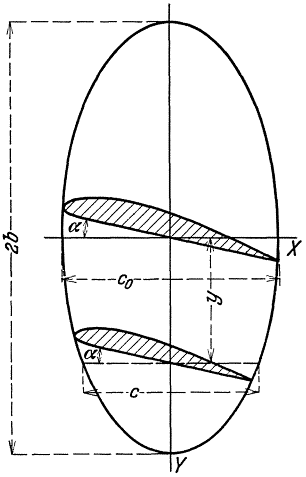

Fig. 69.

Hence in the case of elliptic distribution of lift we obtain the result that all wing sections are placed in a field of motion, which presents a general downward inclination of the amount:

$$\varphi = w_0/2\,V \tag{2.1}$$

Thus, when the geometrical angle of incidence of a wing section taken with reference to the direction of the x, y plane is denoted by α, then in order to be able to apply the results of the theory of wing sections in two-dimensional motion, we must take as the effective angle of incidence of the section the value:

$$i = \alpha - \varphi = \alpha - w_0/2\,V \tag{2.2}$$

The elliptic distribution of the lift can be obtained in the simplest way by taking a wing of elliptic plan form, having geometrically similar sections with parallel chords over the whole span (see Fig. 69). Calling the maximum chord c_0, the chord of a section lying at the distance y from the plane of symmetry becomes[2]:

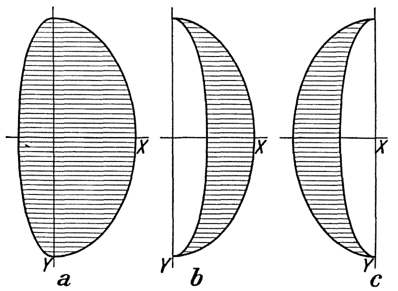

Fig. 70.

$$c = c_0 \sqrt{1 - y^2/b^2} \tag{2.3}$$

According to (1.5) the lift per unit of span is expressed by the formula:

$$l = \frac{1}{2}\,\varrho\, V^2\, c\, m\, i \tag{2.4}$$

As α is constant for all sections (on account of the assumption of parallel chords), the same applies to i in the present case, and as m is also a constant for all sections, (2.4) gives a value of l proportional to

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[1] It must be remembered that according to the definition of Chapter III by the "wake" of any region or body is meant the space lying downstream from it, bounded by a cylindrical surface touching the outer circumference of the particular region or body. In the case of an airfoil which usually has a negligible thickness in the direction of the z axis, the wake takes the form of a flat band.

[2] The contour of the airfoil may consist of two half ellipses, joined along the y axis and having different dimensions along the x axis (see Fig. 70 a, b, c). In all these cases (2.3) holds.

$\sqrt{1 - y^2/b^2}$, thus fulfilling the assumption which formed our starting point. It is of importance to remark that in the case of a wing of elliptic plan form, having everywhere similar sections with parallel chords, this result is independent of the particular value of α.

By integrating (2.4) we obtain the total lift experienced by the airfoil:

$$L = \int_{-b}^{+b} l\, d\, y = \frac{1}{2} \varrho V^2 \frac{\pi b c_0}{2} m i \tag{2.5}$$

As the area of the wing has the value:

$$S = (\pi/2) b c_0 \tag{2.6}$$

the value of the lift coefficient of the airfoil at the geometrical angle of incidence α becomes: $\quad C_L = m i \quad$ (2.7)

Now according to III (22.7) the lift per unit of span can be expressed by: $\quad l = A = 2 \varrho V w_0 \sqrt{b^2 - y^2} \quad$ (2.8)

Hence comparing with (2.4) we find:

$$w_0 = V \frac{c_0}{4 b} m i \tag{2.9}$$

Elimination of the product $c_0 m i$ by means of (2.5) gives:

$$w_0 = V \frac{L}{\varrho V^2 \pi b^2} \tag{2.10}$$

When in (2.10) L is expressed by means of the lift coefficient C_L, it can be transformed into: $\quad w_0 = V \frac{2 C_L}{\pi \lambda} \quad$ (2.11)

where λ is the aspect ratio of the airfoil, which according to I (2.1) is defined by: $\quad \lambda = \frac{(2b)^2}{S} = \frac{8b}{\pi c_0} \quad$ (2.12)

Substituting this result in (2.1) we obtain for the downward inclination: $\quad \varphi = \frac{C_L}{\pi \lambda} \quad$ (2.13)

By combining this equation with (2.7) and (1.3) the following interesting relations are found: $\quad i = \frac{\alpha}{1 + m/\pi\lambda} \quad$ (2.14)

and: $\quad C_L = \frac{m \alpha}{1 + m/\pi\lambda} \quad$ (2.15)

According to III (20.8) the airfoil experiences an induced resistance of the amount (remembering that for the single airfoil $K = L$):

$$D_i = \frac{w_0}{2V} L \tag{2.16}$$

Substituting (2.10) for w_0, we arrive at the important relation:

$$D_i = \frac{L^2}{2 \varrho V^2 \pi b^2} \tag{2.17}$$

which shows that for an airfoil of given span the induced resistance increases proportional to the square of the total lift, while for given total lift it is inversely proportional both to the square of the span and to the square of the velocity V.

Introducing the *coefficient of induced resistance* (see I **12**), we obtain the equation:

$$C_{Di} = \frac{C_L^2}{\pi \lambda} \tag{2.18}$$

Most of these results have already been considered in I **11** and **12**. Equation (2.10) above furnishes the information necessary to obtain the magnitude of the maximum value of the area Σ to be inserted into I (11.2), which area could not be obtained by elementary deductions[1].

The equations for φ and C_{Di} further lead to the so called reduction formulae, developed in I **12**, which make it possible to establish a connection between the polar curves for airfoils of the same profile, but having different aspect ratios.

It is of importance to note especially that these formulae can be used to deduce results relative to airfoil profiles in two-dimensional motion from experiments on airfoil models of finite span. Such deductions are necessary for a great number of calculations, not only in connection with airfoil problems, but also in connection with airscrews, windmills and the like. Though the equations to be employed are immediately apparent from what has been said, it may be of use to state them again in the following form:

Suppose an airfoil of elliptic plan form and aspect ratio λ mounted in a horizontal airstream of unlimited extent at the geometrical angle of incidence α, a lift coefficient C_L being observed. Then an airfoil having the same profile and infinite span would have the same lift coefficient, provided the angle of incidence had the value:

$$i = \alpha - C_L/\pi \lambda \tag{2.19}$$

If it was found moreover, that the actual airfoil for this value of C_L had the drag coefficient C_D, then the coefficient of *profile* drag, that is the drag coefficient of the airfoil of infinite span, would have the value:

$$C_{D0} = C_D - C_L^2/\pi \lambda \tag{2.20}$$

Comparison with the magnitude of the frictional resistance experienced by flat plates and by symmetrical bodies, where lift is absent, has demonstrated the correctness of this reduction (see Division J).

It must be kept in mind, that when the airfoil is not of elliptic plan form, corrections must be applied to these formulae. These corrections

[1] It may be mentioned that this maximum value of Σ, that is the value for systems having minimum induced drag, is related to the potential of the two-dimensional Dirichlet-motion considered in III **21**. We return to this question below in **16**.

can be deduced from the general theory of airfoils of arbitrary form, which will be discussed in the following sections.

3. General Problem of the Single Wing. We now return to the investigation of the system of equations (1.2) and (1.6), or what comes to the same, of the integral equation (1.7).

It is assumed that the airfoil is defined by giving the form and position of every section made by planes parallel to the median plane; thus the chord c, the angle α, the coefficient m are known functions of the variable y. If the sections are all geometrically similar, m will be independent of y; if besides, all directions of zero lift are parallel, α will likewise be independent of y. In the case of an airfoil with rectangular plan form finally, c also will be a constant.

Various methods have been developed to solve the integral equation[1]. In the method originally used by Prandtl and by Betz, a development for the unknown quantity l was assumed of the type:

$$l = \sqrt{b^2 - y^2}\,(l_0 + l_2\, y^2 + l_4\, y^4 + \ldots) \tag{3.1}$$

(taking the case of a symmetrical distribution of the lift). The factor $\sqrt{b^2 - y^2}$ brings into evidence the peculiar way in which the lift must decrease at the tips of the airfoil in order to get the proper behaviour of the angle φ in this region. If the coefficients $l_2, l_4, \ldots$ were zero, (3.1) would give elliptical distribution of the lift. In the general case, the expression between ( ) can be regarded as a measure of the deviation between the actual distribution and the elliptical.

The manipulation of expressions of the type (3.1) in solving the integral equation, however, is very cumbersome. An approximate treatment of the problem, based on the same development for l, was afterwards given by Fuchs.

Quite a different method was proposed by Trefftz, who, instead of starting from the integral equation, considers the two-dimensional field of motion which is to be found in a section of the wake at a great distance behind the airfoil. This field is generated by the system of trailing vortices, which form a band along the y axis, stretching from $y = -b$ to $y = +b$. The general conditions from which the integral equation was obtained in **1**, can be expressed in the form of a boundary condition for the potential of this two-dimensional field. With the aid of conformal transformation the field is brought into relation with the field outside of a circle of unit radius; then the potential is approximated by a trigonometric expression, and an approximate fulfillment of the boundary condition is sought.

[1] The reader is referred to a paper by IRMGARD LOTZ, Zeitschrift f. Flugtechnik u. Motorluftschiffahrt **22**, p. 189, 1931, for a review of the various methods applied to the investigation of the problem. See also **7**, **11**, **12** below.

The introduction of trigonometric series and the procedure for constructing an approximate solution given by Trefftz have been the starting point for much of the further work on this subject. In the following paragraphs we shall hold generally to the treatment given by Glauert[1], and return to (1.7).

A new variable ψ is substituted for y, defined by the relation

$$y = -b \cos \psi \tag{3.2}$$

The connection between ψ and y is such that to $y = -b$ corresponds the value $\psi = 0$; for $y = 0$, $\psi = \pi/2$ and for $y = +b$, $\psi = \pi$.

It is then assumed that l can be developed into a series proceeding according to sines of multiples of the angle ψ, of the form:

$$l = 4\,\varrho\, V^2\, b\, \Sigma A_n \sin n\psi \tag{3.3}$$

where the A_n represent a set, finite or infinite, of numerical coefficients. This series belongs to the type of Fourier series, and it is known that a great number of functions can be approximated in a satisfactory way by taking only a limited number of terms. The series written in (3.3) satisfies the condition that l becomes zero at the wing tips in a way mathematically equivalent to that assumed in (3.1).

Now substitute (3.3) in (1.2), writing $-b\cos\psi'$ for η, so that:

$$d\eta = b \sin\psi'\, d\psi', \qquad \frac{dl}{d\eta} = 4\,\varrho\, V^2 \frac{\Sigma n A_n \cos n\psi'}{\sin\psi'};$$

then (1.2) transforms into:

$$\varphi = \int_0^\pi d\psi' \frac{\Sigma n A_n \cos n\psi'}{\pi(\cos\psi' - \cos\psi)}$$

Making use of the equation:

$$\int_0^\pi d\psi' \frac{\cos n\psi'}{\cos\psi' - \cos\psi} = \pi \frac{\sin n\psi}{\sin\psi},$$

which is demonstrated in the Appendix to this Section, the expression for φ takes the form:

$$\varphi = \Sigma n A_n \frac{\sin n\psi}{\sin\psi} \tag{3.4}$$

The expressions (3.3) and (3.4) will now be inserted into (1.6), thus leading to the equation:

$$\Sigma A_n \sin n\psi = \frac{mc}{8b}\left(\alpha - \Sigma n A_n \frac{\sin n\psi}{\sin\psi}\right) \tag{3.5}$$

As noted, c, m, α are known functions of y, and thus also of the auxiliary variable ψ. Writing:

$$\mu = \frac{mc}{8b} \tag{3.6}$$

and further taking together the terms containing the A's, and multiplying by $\sin\psi$, we can transform (3.5) into:

$$\Sigma A_n \sin n\psi\,(n\mu + \sin\psi) = \mu\,\alpha \sin\psi \tag{3.7}$$

[1] Glauert, H., The Elements of Aerofoil and Airscrew Theory, p. 138, Cambridge, 1930.

This equation is the basic relation considered in the theory of the single airfoil.

The exact solution of the problem of the lift distribution would require that (3.7) should be satisfied for all values of ψ from 0 to π inclusive. This, however, in general can be obtained only by taking an infinite number of terms in the series (3.3). The method commonly used in searching for an approximate solution is to require that (3.7) shall be satisfied at a *limited* number of points only. Then the number of coefficients can be limited to the same value as the number of points. Applying (3.7) to each of these points, a sufficient set of equations is obtained, from which the values of the coefficients A_n can be found without great difficulty. The degree of approximation obtained in this way naturally becomes greater with increase in the number of points.

When the determination of the A's has been effected, the distribution of l and of the angle φ can be calculated.

Before entering upon examples, some general relations concerning the total lift, the moment about the x axis, the induced resistance and its moment about the z axis will be developed.

Appendix to Section 3.—Evaluation of the Integral:

$$I_n = \int_0^\pi d\psi' \frac{\cos n\,\psi'}{\cos\psi' - \cos\psi} \tag{1}$$

According to the statement made in III 27 (foot note), in all integrals of this type, where the denominator becomes zero at a point of the domain of integration (in the case at hand, when ψ' passes through the point ψ), the principal value must be taken. Now consider first the integral:

$$I_0 = \int_0^\pi d\psi' \frac{1}{\cos\psi' - \cos\psi} \tag{2}$$

This integral can be reduced by means of the substitution: $\tan(\psi'/2) = z$. The indefinite integral then becomes:

$$\int \frac{dz}{\sin^2\frac{\psi}{2} - z^2\cos^2\frac{\psi}{2}} = \frac{1}{\sin\psi}\log\frac{z + \tan\frac{\psi}{2}}{z - \tan\frac{\psi}{2}} \tag{3}$$

Taking account of the limits (z first increases from 0 to a quantity $\tan(\psi/2) - \delta$, then from $\tan(\psi/2) + \delta$ to ∞), it is found:

$$I_0 = 0 \tag{4}$$

Next consider:

$$I_1 = \int_0^\pi d\psi' \frac{\cos\psi'}{\cos\psi' - \cos\psi} \tag{5}$$

As this may be written:

$$I_1 = \int_0^\pi d\psi' + I_0 \cos\psi\,,$$

we get the result:

$$I_1 = \pi \tag{6}$$

As to the further integrals, on account of the well known equation:

$$\cos (n+1)\, \psi' - 2 \cos \psi' \cos n\, \psi' + \cos (n-1)\, \psi' = 0 \tag{7}$$

we can write:

$$I_{n+1} + I_{n-1} = 2 \int_0^\pi d\, \psi' \cos n\, \psi' \left(1 + \frac{\cos \psi}{\cos \psi' - \cos \psi}\right) \tag{8}$$

which leads to the following relation between three consecutive integrals (supposing that n is at least equal to 1):

$$I_{n+1} - 2\, I_n \cos \psi + I_{n-1} = 0 \tag{9}$$

As I_0 and I_1 are known already, this relation can be used first to calculate I_2, then I_3, and so on. It will be seen that the value mentioned before, *viz.*:

$$I_n = \pi \frac{\sin n\, \psi}{\sin \psi} \tag{10}$$

satisfies the recurrency equation (9) and as it gives the proper values for I_0 and I_1, it actually must represent the solution of the problem.

4. General Relations Expressed with the Aid of the Fourier Coefficients A_n. From the series (3.3) the total lift results:

$$L = \int_{-b}^{+b} d\, y\, l = \int_0^\pi d\, \psi\, l\, b \sin \psi = 2\, \pi\, \varrho\, V^2\, b^2\, A_1 \tag{4.1}$$

The total lift thus depends only on the first coefficient A_1. The lift coefficient for the airfoil takes the value[1]:

$$C_L = \frac{4\, \pi\, b^2\, A_1}{S} = \pi\, \lambda\, A_1 \tag{4.2}$$

where λ is the aspect ratio.

The moment of the lift about the x axis is given by the integral:

$$M_x = - \int_{-b}^{+b} d\, y\, y\, l \tag{4.3}$$

where the sign has been chosen in such a way that the moment experienced by the airfoil is reckoned positive, when it tends to push down the left hand part of the airfoil. Inserting (3.2) and (3.3), the following result is obtained:

$$M_x = 4\, \varrho\, V^2\, b^3 \int_0^\pi d\, \psi \sin \psi \cos \psi\, (\Sigma\, A_n \sin n\, \psi) = \pi\, \varrho\, V^2\, b^3\, A_2 \tag{4.4}$$

Hence this moment depends on the coefficient A_2 only. It vanishes for a symmetrical lift distribution. If a moment coefficient is introduced by means of the equation: $M_x = (1/2)\, \varrho\, V^2\, S\, c'\, C_{M\,x}$, where c' is the mean chord of the airfoil, the relation between $C_{M\,x}$ and A_2 is expressed by:

$$C_{M\,x} = \frac{\pi}{4} \lambda^2\, A_2 \tag{4.5}$$

---

[1] The reader must be warned not to deduce from this equation that C_L is proportional to λ. Nor is $C_{M\,x}$ as given by (4.5) or $C_{M\,z}$ as given by (4.15) proportional to λ^2. As will be seen from (3.7), the coefficients A_n are complicated functions of the parameter μ, which contains both m and λ.

The magnitude of the induced resistance can be calculated from III (31.12) by integrating with respect to y:

$$D_i = \int_{-b}^{+b} d\,y\, d_i = \int_{-b}^{+b} d\,y\, l\, \varphi \tag{4.6}$$

($tan\,\varphi$ has been replaced by φ, as before). Now for l, take the series (3.3) and for φ, take (3.4); then we obtain:

$$\left.\begin{aligned} D_i &= 4\,\varrho\, V^2\, b^2 \int_0^{\pi} d\,\psi\, (\Sigma\, A_n\, sin\, n\,\psi)\, (\Sigma\, n\, A_n\, sin\, n\,\psi) = \\ &= 2\,\pi\,\varrho\, V^2\, b^2\, \Sigma\, n\, A_n^2 \end{aligned}\right\} \tag{4.7}$$

The coefficient of induced resistance $C_{D\,i}$ thus becomes:

$$C_{D\,i} = \pi\,\lambda\,\Sigma\, n\, A_n^2 \tag{4.8}$$

or, making use of (4.2): $C_{D\,i} = \dfrac{C_L^2}{\pi\,\lambda} \sum \dfrac{n\,A_n^2}{A_1^2}$ (4.9)

Now from (4.1) we have seen that the lift depends upon the value of the first coefficient, A_1, only. Hence it appears that the smallest value of the induced resistance compatible with a given lift and a prescribed value of the span, is obtained if all coefficients other than A_1 vanish. In this case the distribution of the lift along the span is determined by: $l = 4\,\varrho\, V^2\, b\, A_1\, sin\,\psi = 4\,\varrho\, V^2\, A_1 \sqrt{b^2 - y^2}$ (4.10)

This is the elliptical lift distribution and we thus have obtained a new demonstration of the theorem that elliptic loading ensures minimum induced drag.—We return to this case at the end of this section.

In order to bring into evidence the difference between the value of $C_{D\,i}$ in the general case and that given by (2.18) for the case of elliptic loading, it is convenient to write $1 + \delta$ for the summation, where:

$$\delta = \Sigma\, \frac{n\,A_n^2}{A_1^2} \quad (n > 1) \tag{4.11}$$

Then:

$$C_{D\,i} = \frac{C_L^2}{\pi\,\lambda}\,(1 + \delta) \tag{4.12}$$

The induced resistance can have a moment about the z axis, which is given by the integral: $M_z = -\int_{-b}^{+b} d\,y\, y\, l\, \varphi$ (4.13)

The sign has been chosen in such a way, that the moment experienced by the airfoil is reckoned positive, if it tends to drive backward the right hand part of the airfoil. Inserting again equations (3.2), (3.3) and (3.4), the following result is obtained:

$$\left.\begin{aligned} M_z &= 4\,\varrho\, V^2\, b^3 \int_0^{\pi} d\,\psi\, cos\,\psi\, (\Sigma\, A_n\, sin\, n\,\psi)\, (\Sigma\, n\, A_n\, sin\, n\,\psi) = \\ &= \pi\,\varrho\, V^2\, b^3\, (3\,A_1 A_2 + 5\,A_2 A_3 + 7\,A_3 A_4 + \ldots) \end{aligned}\right\} \tag{4.14}$$

Introducing a moment coefficient by means of the equation: $M_z = (1/2)\,\varrho\, V^2 S\, c'\, C_{M\,z}$, the following formula is obtained:

$$C_{M\,z} = \frac{\pi}{4}\,\lambda^2\,(3\,A_1 A_2 + 5\,A_2 A_3 + 7\,A_3 A_4 + \ldots) \tag{4.15}$$

If the coefficients A_3, A_5 . . . are sufficiently small in order that we may write approximately:

$$C_{M\,z} \cong \frac{3\,\pi}{4}\,\lambda^2 A_1 A_2 \tag{4.16}$$

then the following relation, which was given by Munk, holds between M_x and M_z:

$$\frac{M_z}{M_x} \cong 3\,A_1 \cong \frac{C_L}{\lambda}$$

Here (4.2) is used to express A_1 with the aid of C_L; the error introduced by taking π equal to 3 in the case of the rectangular wing appears to be compensated partly by the error involved in (4.16)[1].

A few remarks may be added concerning the moment about an axis perpendicular to the x, z plane. In II (8.22) the following expression for the moment coefficient of a single section was obtained, taken with respect to the leading edge (or, more properly, with respect to the point of intersection of the reference lines assumed for that section, as indicated in I 3):

$$c_m = c_\mu + n\,c_l \tag{4.17}$$

Here small letters have been used to distinguish coefficients referring to a single section from those valid for the whole airfoil; c_μ is the moment coefficient for zero lift; $c_l = m\,(\alpha - \varphi)$ is the lift coefficient of the section for the value $i = \alpha - \varphi$ of the effective angle of incidence, while n is a constant, which can be taken equal to 0.25. It is evident that when the distribution of l and φ over the span is known, the moment experienced by the whole airfoil can be calculated from the reactions on the separate sections, provided due regard is had for the circumstance, whether all reference points are located on a single straight line parallel to the y axis, or not. In the case of the rectangular airfoil having everywhere the same profile, the matter becomes most simple as all reference points are on one line. Then the moment coefficient for the whole airfoil satisfies the relation:

$$C_m = C_\mu + n\,C_L \tag{4.18}$$

which has the same form as (4.17).

An example relating to a more complicated case is considered in **10**.

If the airfoil is symmetric with respect to the median plane, the distribution of the lift likewise will be symmetric. In that case the

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[1] See Lotz, I., l. c. and Munk, M., Nat. Adv. Comm. Aeronautics (Washington) Rep. No. 197, 1924.—In regard to the practical applicability of the relation it must be kept in mind that the asymmetrical lift distribution, as is obtained for example with an airfoil the ailerons of which are moved out of the neutral position, usually will be accompanied by an asymmetrical distribution of the profile drag, which likewise will contribute to M_z.

terms relating to *even* multiples of the angle ψ in the development (3.3) must be omitted, so that the series now takes the form:

$$l = 4\,\varrho\, V^2\, b\, [A_1 \sin\psi + A_3 \sin 3\,\psi + A_5 \sin 5\,\psi \ldots] \qquad (4.19)$$

In solving (3.7) it is then sufficient to consider one half of the airfoil only, as it will be automatically satisfied for the other half. The moment M_x and the moment M_z in this case both vanish. If the airfoil is asymmetric, as is the case for example when the ailerons are moved out of the neutral position, the even terms must be retained. We return to this point in **11**.

A few words may be devoted to the case of the *elliptical* lift distribution, which is obtained when all coefficients A_n vanish with the exception of A_1. Equation (3.4) then gives:

$$\varphi = A_1 \qquad (4.20)$$

while (3.7) reduces to:

$$A_1 \sin\psi\,(\mu + \sin\psi) = \mu\,\alpha \sin\psi,$$

which may be put into the form:

$$\frac{\mu\,\alpha}{\mu + \sin\psi} = A_1 \qquad (4.21)$$

It is evident that this relation can be true only when either μ or α, or both of them, are subject to special conditions. In the case of an airfoil with elliptic plan form having everywhere parallel chords, we have:

$$c = c_0 \sin\psi,\ \text{giving:} \quad \mu = \frac{m\,c_0}{8\,b} \sin\psi = \mu_0 \sin\psi \qquad (4.22)$$

where μ_0 is a constant, as is α in this case. Inserting in (4.21) we have:

$$A_1 = \frac{\mu_0\,\alpha}{1 + \mu_0} \qquad (4.23)$$

Now according to (4.2) $C_L = \pi\,\lambda\,A_1$; on the other hand it is easily found that the constant μ_0 can be expressed by means of the aspect ratio in the form: $\mu_0 = m/\pi\,\lambda$. Hence (4.23) brings us back to (2.15).

Another possibility for satisfying (4.21) is to take the chord constant (wing of rectangular plan form), so that μ becomes constant, and to make α vary proportionally to $\mu + \sin\psi$. This would demand a certain twist of the airfoil, the angle of incidence being a maximum in the plane of symmetry and decreasing towards the ends. It is evident that an airfoil constructed in this way would fulfill the required condition for one special angle of setting only, as the proportionality between α and $\mu + \sin\psi$ is destroyed when all angles are increased by the same amount.

5. Rectangular Wing of Constant Profile and Constant Angle of Incidence. To obtain this case, which is of importance in connection with the greater part of the experimental work performed on model airfoils, assume μ and α in (3.7) constant along the span.

As has been noted, only the uneven terms need be taken in the series on account of the symmetry of the distribution. The solution of the problem for the case of 4 coefficients A_1, A_3, A_5, A_7 has been given by Glauert under the conditions that (3.7) should be satisfied for the following values of ψ:

$$\frac{\pi}{8}\,(22.5^0)\,, \quad \frac{\pi}{4}\,(45^0)\,, \quad \frac{3\,\pi}{8}\,(67.5^0)\,, \quad \frac{\pi}{2}\,(90^0)\,.$$

The equation will then be satisfied simultaneously at the points:

$$\frac{5\,\pi}{8}\,(112.5^0)\,, \quad \frac{3\,\pi}{4}\,(135^0)\,, \quad \frac{7\,\pi}{8}\,(157.5^0)\,,$$

and furthermore it is always satisfied at the points 0 and π[1].

From the form of (3.7) it is evident that the solution will be a function of the parameter μ. For a given value of μ the A's will be proportional to α and may be expressed most conveniently in proportion to $\mu\,\alpha$.

The parameter μ is related to the aspect ratio λ (which for the rectangle has the value $2b/c$), by the equation:

$$\mu = \frac{m}{4\,\lambda} \tag{5.1}$$

For a series of values of $1/\mu$ the following values of the coefficients $A_1, \ldots A_7$ have been obtained[2] (Table 1).

TABLE 1.

$\frac{1}{\mu}$	$\frac{\lambda}{m}$	$\frac{A_1}{\mu\,\alpha}$	$\frac{A_3}{\mu\,\alpha}$	$\frac{A_5}{\mu\,\alpha}$	$\frac{A_7}{\mu\,\alpha}$	$\frac{C_L}{m\,\alpha}$	$\frac{4\,\lambda\,\alpha}{C_L}$
2	0.50	0.748	0.060	0.009	0.0014	0.587	3.40
3	0.75	0.859	0.090	0.016	0.0027	0.675	4.45
4	1.00	0.928	0.115	0.023	0.0041	0.729	5.49
5	1.25	0.976	0.136	0.030	0.0055	0.767	6.52
6	1.50	1.011	0.154	0.036	0.0070	0.794	7.56
7	1.75	1.038	0.169	0.042	0.0084	0.815	8.58

In considering these results it is of importance to remember that according to (4.2) the lift coefficient of the airfoil is given by $C_L = \pi\,\lambda\,A_1$. The lift coefficient for a single section in a two-dimensional field of motion (*i. e.* for an airfoil of infinite span) for an effective angle of incidence equal to the geometrical angle of incidence α in the case of finite span, has the value $m\,\alpha$. The ratio of C_L to $m\,\alpha$ is given by the equation:

$$\frac{C_L}{m\,\alpha} = \frac{\pi\,\lambda\,A_1}{m\,\alpha} = \frac{\pi}{4}\,\frac{A_1}{\mu\,\alpha} \tag{5.2}$$

This ratio has been tabulated in column 7 of Table 1, which clearly shows how the lift coefficient for the whole airfoil falls off as the aspect ratio is decreased, the geometrical angle of incidence being kept constant.

The corresponding load grading curves have been given in Fig. 71. For all cases the span $2b$ has been taken the same; the quantity represented is $l/4\,\varrho\,V^2\,b\,\mu\,\alpha$, which also can be written $l/(1/2)\,\varrho\,V^2\,c\,m\,\alpha$.

[1] See, however, **11.**

[2] See GLAUERT, H., The Elements of Aerofoil and Airscrew Theory, p. 147, Cambridge, 1930. GLAUERT in the first column of his Table 11 gives a quantity which is equal to $1/2\,\mu$; likewise the quantity in his seventh column is half of that tabulated here as $4\,\lambda\,\alpha/C_L$.

For a comparison with actually observed load grading curves the reader is referred to Division J.

6. Effective Angle of Incidence. Induced Resistance. As most of the experimental work done on airfoils has been based upon rectangular models of constant section and constant angle of attack, it is of importance to indicate how values relating to a single section in two-dimensional motion may be deduced from the results obtained with such models.

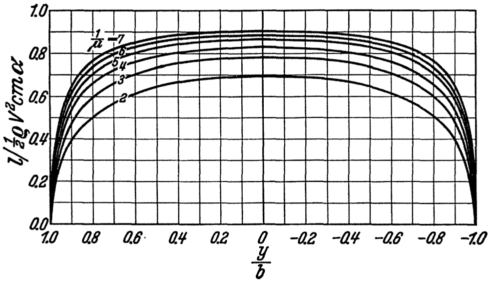

Fig. 71.

In performing the experiments, the values of α and of λ are known[1]; the measurements then furnish the values of C_L and C_D. The first problem now becomes to find the value of the parameter μ for the model airfoil. This can be done most conveniently by considering the quantity $4\,\lambda\,\alpha/C_L$, which on account of (4.2) can be expressed in the form:

$$\frac{4\,\lambda\,\alpha}{C_L} = \frac{4}{\pi\,\mu}\,\frac{\mu\,\alpha}{A_1} \tag{6.1}$$

By means of the data given in Table 1 this can be calculated as a function of $1/\mu$; it has been tabulated in column 8 of that Table and is represented graphically in Fig. 72. Glauert has remarked that the following approximate relation holds:

$$\frac{4\,\lambda\,\alpha}{C_L} \cong 1.32 + 1.04\,\frac{1}{\mu} \tag{6.2}$$

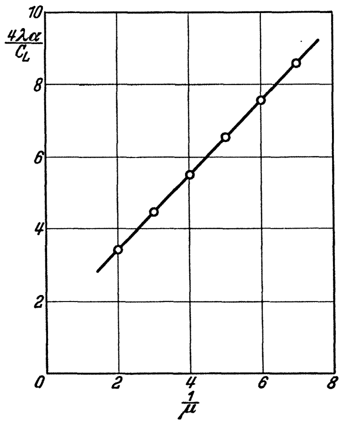

Fig. 72.

As on the other hand the quantity $4\,\lambda\,\alpha/C_L$ can be calculated from the experimentally determined value of C_L, it is possible to find the value of $1/\mu$, and if necessary that of m, and thus the solution of (3.7) appropriate to the given case can be obtained.

The *effective angle of incidence* $i = \alpha - \varphi$ evidently differs for the various sections. We define a mean value according to Glauert by taking the angle $\bar{i}$ for which a single section in two-dimensional motion would give a lift coefficient equal to the observed value of C_L. Then:

[1] As the airfoil has constant profile and parallel chords everywhere, the position of zero lift is the same both for the single section in two-dimensional motion and for the airfoil of finite aspect ratio and thus can be found experimentally.

$$\bar{i} = \frac{C_L}{m} = \frac{C_L}{4\lambda\mu} \tag{6.3}$$

On the other hand we may write:

$$\alpha = \frac{A_1}{\mu}\frac{\mu\alpha}{A_1} = \frac{C_L}{\pi\lambda\mu}\frac{\mu\alpha}{A_1} \tag{6.4}$$

and thus:

$$\bar{\varphi} = \alpha - \bar{i} = \frac{C_L}{\pi\lambda}\frac{1}{\mu}\left(\frac{\mu\alpha}{A_1} - \frac{\pi}{4}\right) \tag{6.5}$$

Writing:

$$\frac{1}{\mu}\left(\frac{\mu\alpha}{A_1} - \frac{\pi}{4}\right) = 1 + \tau \tag{6.6}$$

the quantity τ is a measure of the relative difference between the angle of deflection in the case of elliptic loading [see (2.13)] and the mean value of the angle in the case of the rectangular airfoil.

The equation for $\bar{i}$ may now be written:

$$\bar{i} = \alpha - \bar{\varphi} = \alpha - \frac{C_L}{\pi\lambda}(1 + \tau) \tag{6.7}$$

Formula (6.6) can also be obtained by observing that the decrease of the angle of incidence of any particular section by the angle φ causes a decrease of lift coefficient amounting to $m\,\varphi$. The lift per unit span at that section is then diminished by: $(1/2)\,\varrho\,V^2 c\,m\,\varphi$, and the decrease of the total lift amounts to:

$$\triangle L = \frac{1}{2}\varrho V^2 m \int_{-b}^{+b} c\,\varphi\,dy \tag{6.8}$$

Now the definition of $\bar{\varphi}$ adopted by Glauert is equivalent to the statement that a part of a wing of the same profile and of infinite span, having the area S, experiences the same decrease $\triangle L$ in lift, when the angle of incidence is diminished by $\bar{\varphi}$. Hence: $\triangle L = (1/2)\,\varrho\,V^2 S\,m\,\bar{\varphi}$,

from which we obtain:

$$\bar{\varphi} = \frac{1}{S}\int_{-b}^{+b} c\,\varphi\,dy \tag{6.9}$$

In order to show that this equation for $\bar{\varphi}$ again leads to (6.5), we write (1.6) in the form: $c\,\varphi = c\,\alpha - \frac{l}{(1/2)\,\varrho\,V^2 m}$. Multiplying by dy and integrating over the span we find after division by S:

$$\bar{\varphi} = \alpha - \frac{L}{(1/2)\,\varrho\,V^2 S\,m} = \alpha - \frac{C_L}{m} = \alpha - \bar{i}$$

The *coefficient of the induced resistance* is given by:

$$C_{D\,i} = \frac{C_L^2}{\pi\lambda}(1 + \delta) \tag{6.10}$$

where now δ is obtained from:

$$\delta = \frac{3A_3^2}{A_1^2} + \frac{5A_5^2}{A_1^2} + \frac{7A_7^2}{A_1^2} \tag{6.11}$$

which again can be calculated from the data collected in Table 1.

The values of τ and δ for the cases considered are given in Table 2 and in Fig. 73.

It will be seen that the increase in induced resistance due to the deviation from elliptic loading is relatively small. The correction to be applied to the formula for $\overline{\varphi}$ is more important. Both τ and δ increase with increasing aspect ratio; they can be considered as being nearly, though not quite, proportional to λ. Hence with the equations for $\overline{\varphi}$ and C_{Di} in the form:

$$\overline{\varphi} = \frac{C_L}{\pi\lambda} + \frac{C_L}{\pi}\frac{\tau}{\lambda}, \qquad C_{Di} = \frac{C_L^2}{\pi\lambda} + \frac{C_L^2}{\pi}\frac{\delta}{\lambda},$$

the second terms on the right hand side change only slightly with aspect ratio.

Table 2.

$\frac{1}{\mu} = \frac{4\lambda}{m}$	τ	δ
2	0.10	0.019
3	0.14	0.034
4	0.17	0.049
5	0.20	0.063
6	0.22	0.076
7	0.24	0.088

The reduction formulae I (12.6) and I (12.7) now take the form:

$$\alpha' = \alpha + \frac{C_L}{\pi}\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) + \frac{C_L}{\pi}\left(\frac{\tau'}{\lambda'} - \frac{\tau}{\lambda}\right) \tag{6.12}$$

$$C_D' = C_D + \frac{C_L^2}{\pi}\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) + \frac{C_L^2}{\pi}\left(\frac{\delta'}{\lambda'} - \frac{\delta}{\lambda}\right) \tag{6.13}$$

On account of the small values of the last term in both equations it is seen that no great error is made when the original formulae, which are strictly true for elliptic loading only, are also used for rectangular airfoils. It seems fairly safe to assume that the formulae I (12.6) and I (12.7) can be used without further correction in all cases where the general character of the load distribution does not change. However, as soon as we pass from one type of distribution to another, as for instance from the rectangular wing to the wing with elliptic loading, the correction indicated by the letters τ and δ must not be neglected[1].

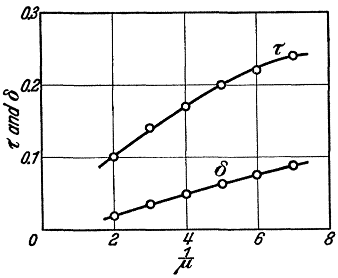

Fig. 73.

Likewise the quantities τ and δ must be observed, when it is required to deduce data for a single section in two-dimensional motion from the results obtained with rectangular model airfoils. The equations appropriate to this case are given in (6.7) and (6.10).

It should be noted that with the definition of the angle $\overline{\varphi}$ assumed by Glauert, the relation $D_i = L\,\overline{\varphi}$ no longer holds. In order that this relation might be true, it would be necessary to adopt another definition expressed by the equation: $\overline{\varphi} = \frac{1}{L}\int_{-b}^{+b} l\,\varphi\,dy$.

[1] See Reid, E. G., Applied Wing Theory, p. 103, New York, 1932.

Then, however, the relation $C_L = m(\alpha - \overline{\varphi})$ would no longer be valid.

7. Comparison with Other Calculations. In view of the importance of the case of the rectangular wing, it is of interest to compare some of the results given by Glauert with those of Betz, Fuchs and Trefftz.

From the data given in Betz' paper we deduce the following Table, expressed in the notation introduced in the foregoing sections[1]:

$1/\mu$	$C_L/m\,\alpha$	$1+\delta$
1	0.427	1.007
2	0.588	1.020
4	0.728	1.051
9	0.847	1.124

It will be seen that the values given by Betz for $1/\mu = 2$ and $1/\mu = 4$, which can be compared with those given by Glauert, are in close agreement with the latter.

From Fuchs' paper it may be noted that in the case $\lambda = 1/6$ the induced drag is about 5% higher than for elliptic lift distribution; while in the case $\lambda = 1/10$ it is about 10% higher[2]. As Fuchs takes the constant m equal to 2π, these values of λ correspond resp. to $1/\mu = 3.8$ and 6.4; hence they seem to be somewhat in excess of the values given by Glauert.

Trefftz' calculations, when viewed apart from the formulation as a potential problem, are essentially the same as those of Glauert[3]. The following values of λ are considered: 2, 4, 10, 20; m is taken equal to 2π.

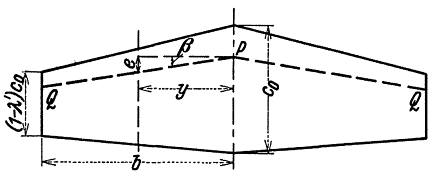

Fig. 74.

Trefftz has also investigated the case of a semi-infinite wing, which can be considered as an approximation to the tip of a wing of very great aspect ratio.

8. Tapered Airfoils. Examples relating to tapered and to twisted airfoils have been worked out by Glauert.

In order to obtain the first case in (3.5) assume for the chord c a formula of the type (see Fig. 74):

$$c = c_0\left(1 \pm \lambda' \frac{y}{b}\right) = c_0\left(1 \mp \lambda' \cos\psi\right) \tag{8.1}$$

where the upper sign is to be taken for $y < 0$, $\psi < \pi/2$ and the lower sign for $y > 0$, $\psi > \pi/2$. The quantity μ defined in (3.6) now becomes a function of ψ, which can be written:

$$\mu = \mu_0 (1 \mp \lambda' \cos\psi) \tag{8.2}$$

where:

$$\mu_0 = \frac{m\,c_0}{8\,b} \tag{8.3}$$

[1] Betz, A., Inaugural-Dissertation Göttingen 1919, Tables 4, 6.

[2] Fuchs, R., Zeitschr. f. angew. Math. u. Mechanik **1**, p. 106, 1921.

[3] Trefftz, E., Zeitschr. f. angew. Math. u. Mechanik **1**, p. 206, 1921.

In order to express this latter quantity as a function of the aspect ratio we observe that the area of the airfoil has the magnitude: $S = (2 - \lambda')\, b c_0$;

hence: $$\lambda = \frac{4b}{(2-\lambda')\, c_0} \tag{8.4}$$

and thus: $$\mu_0 = \frac{m}{(4 - 2\lambda')\,\lambda} \tag{8.5}$$

This equation takes the place of (5.1) for the rectangular airfoil.

Equation (3.7) now takes the form (for $\psi < \pi/2$):

$$\Sigma A_n \sin n\psi\, [n \mu_0 (1 - \lambda' \cos\psi) + \sin\psi] = \mu_0\, \alpha\, (1 - \lambda' \cos\psi) \sin\psi \tag{8.6}$$

There are thus two parameters in the equation, μ_0 and λ'. When their value has been given, the coefficients A_n again will be proportional to α.

Glauert[1] has given results for the cases: $\lambda = 0.5\, m$; $\lambda = m$; $\lambda = 1.5\, m$, with the following values of λ': 0, 0.50, 1.00; 0, 0.25, 0.50, 0.75, 1.00; 0, 0.50, 1,00. The cases with $\lambda' = 0$ correspond to those considered in 5 and 6. As an example the results in Table 3 are given for the case $\lambda = m$.

TABLE 3.

λ'	$\frac{A_1}{\alpha}$	$\frac{A_3}{\alpha}$	$\frac{A_5}{\alpha}$	$\frac{A_7}{\alpha}$	$\frac{C_L}{m\alpha}$	τ	δ
0.00	0.232	0.029	0.006	0.001	0.729	0.17	0.049
0.25	0.236	0.020	0.008	0.000	0.742	0.10	0.026
0.50	0.240	0.007	0.010	—0.001	0.754	0.03	0.011
0.75	0.241	—0.012	0.010	—0.002	0.757	0.01	0.016
1.00	0.232	—0.050	0.002	—0.004	0.729	0.17	0.141

The ratio of the lift coefficient C_L for the whole airfoil to $m\alpha$, the lift coefficient for a single section in two-dimensional motion, is now to be obtained from the equation:

$$\frac{C_L}{m\alpha} = \frac{\pi \lambda A_1}{m\alpha} = \pi \frac{A_1}{\alpha} \quad \text{(for the case } \lambda = m) \tag{8.7}$$

The mean angle of deviation, according to the definition adopted by Glauert, becomes, for the same case,

$$\bar{\varphi} = \alpha - \bar{i} = A_1 \frac{\alpha}{A_1} - \frac{C_L}{m} = \frac{C_L}{\pi\lambda}\left(\frac{\alpha}{A_1} - \pi\right);$$

hence we have: $$1 + \tau = \frac{\alpha}{A_1} - \pi \tag{8.8}$$

The parameter δ can be calculated with the aid of (4.11), which as in the foregoing case reduces to the form (6.11). The values of $C_L/m\alpha$, τ and δ have been given in columns 6—8 of Table 3. It will be seen that values of λ' from 0.50 to 0.75 (tip chord equal to 1/2 — 1/4 of c_0) give the smallest amounts both for τ and for δ.

For comparison a few load grading curves have been collected in Fig. 75. The span $2b$ has been taken the same in all cases; also for all

[1] See GLAUERT, H., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1226, 1928, and: The Elements of Aerofoil and Airscrew Theory, p. 151, Cambridge, 1930. Table 3 of the text is taken from Table 14 of the latter book; λ' corresponds to λ as used by GLAUERT, who takes the symbol A for the aspect ratio.

curves the aspect ratio λ is equal to m. The quantity represented is $l/\varrho\, V^2\, b\, \alpha = l/(1/2)\, \varrho\, V^2\, c'\, m\, \alpha$, where c' is the mean chord ($c' = 2b/\lambda$).

9. Twisted Airfoils. In (3.7) take α variable, *e.g.* as defined by the equation:

$$\alpha = \alpha_0 \mp \varepsilon \cos \psi \tag{9.1}$$

(the upper sign again is to be taken for $y < 0, \psi < \pi/2$ and the lower sign for $y > 0, \psi > \pi/2$); then, for $\psi < \pi/2$:

$$\Sigma A_n \sin n \psi\, (n \mu + \sin \psi) = \mu\, \alpha_0 \sin \psi - \mu\, \varepsilon \sin \psi \cos \psi \tag{9.2}$$

On solving this equation in the same way as before, the coefficients A_n become expressed as the sum of two parts, one proportional to α_0, the other proportional to ε. This remains true both for constant μ and for variable μ. When the chord of the airfoil is constant, the first part of the solution is the same as that obtained in 5; in the case of a tapered airfoil it can be taken from the calculations mentioned in 8. It thus remains to obtain the second part of the solution. Values have been given by Glauert for the same set of cases as mentioned in 8[1].

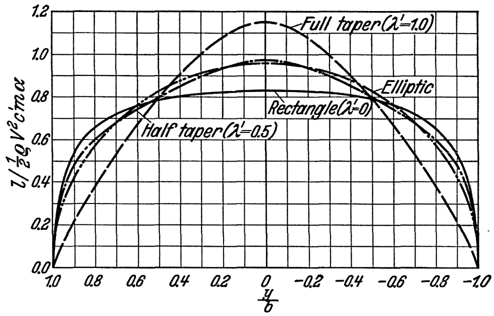

Fig. 75.

We give here the results for the case $\lambda' = 0$, $1/\mu = 4$:

$$A_1 = 0.232\, \alpha_0 - 0.102\, \varepsilon$$
$$A_3 = 0.029\, \alpha_0 - 0.060\, \varepsilon$$
$$A_5 = 0.006\, \alpha_0 + 0.003\, \varepsilon$$
$$A_7 = 0.001\, \alpha_0 - 0.006\, \varepsilon$$

The lift coefficient for the whole airfoil in this case is determined by the equation: $C_L = \pi \lambda A_1 = m\, (0.729\, \alpha_0 - 0.320\, \varepsilon)$. It is no longer proportional to α_0, and becomes zero for $\alpha_0 = 0.44\, \varepsilon$.

In this case therefore the angle of zero lift for the airfoil differs from the angle of zero lift for the central section taken alone in a field of two-dimensional motion; this, of course, was to be expected in advance.

The coefficient of induced resistance C_{Di} again can be calculated from (4.9); C_{Di} now depends on the ratio of α_0 to ε. The following values of the quantity δ are mentioned:

$\alpha_0/\varepsilon = 1.0$	$\delta = 0.205$
$= 1.5$	$= 0.027$
$= 2.0$	$= 0.009$
$= 2.5$	$= 0.003$

[1] GLAUERT, H., The Elements of Aerofoil and Airscrew Theory, p. 152, Cambridge, 1930.

For comparison it may be noted that in the case: $\varepsilon = 0$ $(\alpha_0/\varepsilon = \infty)$, with $\lambda = m$ as before, it was found: $\delta = 0.049$.

10. Influence of Sweep-Back on Pitching Moment. Glauert[1] has considered also the value of the pitching moment of a tapered and twisted airfoil, when regard is taken of the *sweep-back*. It is accepted that for angles of sweep-back less than 30^0 the assumptions of III 27, which form the basis for the deduction of (3.7), are still valid. The sweep-back then has no influence upon the distribution of the lift, but comes in when the moment of the forces about an axis parallel to $O\,y$ is calculated.

The moment of the forces acting on a single section can be expressed by means of the moment coefficient c_m which is given by: $c_m = c_\mu + n\, c_l$, where $n = 0.25$ [see (4.17)]. The moment is then taken with respect to the leading edge of the profile. When it is taken with respect to a point, lying at 1/4 chord from the leading edge, the term $n\, c_l$ disappears, and the coefficient is given simply by c_μ, which is independent of the angle of incidence.

Now define the angle of sweep-back β as the angle which the line $P\,Q$ of the points at 1/4 chord for one half of the airfoil makes with the y axis (see Fig. 74). The backward displacement of the 1/4 chord point of an arbitrary section as compared with that of the median section is then given by:

$$e = y \tan\beta \quad (\text{for } y > 0) \tag{10.1}$$

The moment experienced by a single section (referred to unit span) with respect to its own 1/4 chord point is: $(1/2)\, \varrho V^2 c^2 c_\mu$; with respect to the 1/4 chord point of the median section it becomes:

$$\frac{1}{2}\, \varrho\, V^2\, c^2\, c_\mu + e\, l \tag{10.2}$$

Hence, when c_μ is assumed constant (which requires geometrically similar sections over the whole span), the moment experienced by the whole airfoil has the value:

$$M = \frac{1}{2}\, \varrho\, V^2\, c_\mu \int_{-b}^{+b} c^2\, d\,y + \int_{-b}^{+b} e\, l\, d\, y \tag{10.3}$$

After working out the integrals, the following equation is obtained for the moment coefficient C_M of the whole airfoil, taken about an axis through the 1/4 chord point of the median section, and with the mean chord as standard length:

$$C_M = c_\mu \frac{1 - \lambda' + \lambda'^2/3}{1 - \lambda' + \lambda'^2/4} + 2\,\lambda^2 \tan\beta \left(\frac{A_1}{3} + \frac{A_3}{5} - \frac{A_5}{21} + \frac{A_7}{45}\right) \tag{10.4}$$

It is possible to express this in the form:

$$C_M = C_\mu + N\, C_L \tag{10.5}$$

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[1] See reference in 8.

where N is a new constant. Glauert then proceeds to investigate under what conditions C_μ vanishes, in which case a constant position of the center of pressure is obtained.

11. Airfoil with Ailerons Moved out of Neutral Position. Discontinuous Change of Angle of Incidence at Certain Points of the Span. When the ailerons are moved out of their neutral position the characteristic properties of the airfoil profile show an abrupt change at particular values of y. Moreover, in the ordinary case, when the ailerons are moved in opposite directions, the lift distribution ceases to be symmetrical.

The problem of determining the distribution of the lift in such cases has been attacked from various sides. The method of solution treated in the foregoing sections, in which only four terms are used, becomes less satisfactory in this case, and it is necessary to take a greater number. Wieselsberger[1] has remarked that as now the influence of the wing tips becomes rather important (*e. g.* in the calculation of the moments M_x and M_z), care must be taken that (3.5) is satisfied at the points $\psi = 0$ and $\psi = \pi$. This equation is fully equivalent to (3.7) at all points where $\sin\psi$ is different from zero. However, at the points $\psi = 0$ and $\psi = \pi$, (3.7) is satisfied automatically, while (3.5) still imposes a condition on the A's. Taking the first point, $\psi = 0$, it is seen that this condition assumes the form:

$$\Sigma n^2 A_n = \alpha \tag{11.1}$$

At the point $\psi = \pi$ a similar equation is obtained where the terms of even index, however, have the minus sign.

In treating the aileron problem for the rectangular wing it is assumed that both c and m remain constant along the span, while the angle of incidence has the following values:

$$\left.\begin{array}{lll} -b < y < -a & (0 < \psi < \psi_a) : & \alpha + \varepsilon \\ -a < y < +a & (\psi_a < \psi < \pi - \psi_a) : & \alpha \\ +a < y < +b & (\pi - \psi_a < \psi < \pi) : & \alpha - \varepsilon \end{array}\right\} \tag{11.2}$$

Here $b - a$ denotes the length of the ailerons, while the angle ψ_a is defined by:

$$a = b \cos\psi_a \tag{11.3}$$

As (3.7) is linear both in the A's and in the angle of incidence, it is possible to build up the solution as the sum of two special solutions, one relating to a constant angle of incidence of the magnitude α, the other to an angle of incidence having the values:

$$\left.\begin{array}{ll} 0 < \psi < \psi_a : & +\varepsilon \\ \psi_a < \psi < \pi - \psi_a : & 0 \\ \pi - \psi_a < \psi < \pi : & -\varepsilon \end{array}\right\} \tag{11.4}$$

The first solution is the symmetrical solution for the rectangular wing with constant angle of incidence, which has been considered in

[1] WIESELSBERGER, C., Rep. Aeron. Res. Instit. Tokyo, No. 30, Vol. II, p. 421, 1927.

5—7. Hence it remains to consider the other solution, which has an antisymmetrical lift distribution to be represented by a series containing only the sines of *even* multiples of the angle ψ. Wieselsberger takes 8 terms in this series, with coefficients A_2, A_4, .. A_{16}, determined by means of the conditions (*a*) that (11.1) must be satisfied (relating to the point $\psi = 0^0$) and (*b*) that (3.7) must be satisfied at the points $\psi =$ resp. 20^0, 35^0, 45^0, 55^0, 65^0, 75^0, 85^0. It is then automatically satisfied also for the other half of the airfoil.

Four cases are taken, in which the angle ψ_a respectively has the values 40^0, 60^0, 70^0, 90^0. As these points lie between the points where (3.7) must be satisfied, the problem treated does not actually represent the case of an abrupt change of the angle of incidence, but corresponds to a certain gradual change.

From Wieselsberger's results, note may be made of the values obtained for the moments M_x and M_z and for the increase in induced drag; they are expressed in the following form[1]:

$$M_x = \frac{1}{2} \varrho V^2 (2b)^3 \zeta \varepsilon$$

$$M_z = \frac{1}{2} \varrho V^2 (2b)^3 \xi \alpha \varepsilon$$

$$\delta D_i = \frac{1}{2} \varrho V^2 (2b)^2 \eta \varepsilon^2$$

For the case $m = 2\pi$, $\lambda = 2\pi$, making $1/\mu = 4$, the coefficients ζ, ξ, η are given in the accompanying Table[2].

Table 4.

ψ_a	$\frac{b-a}{b}$	ζ	ξ	η
40^0	0.234	0.047	0.0492	0.0462
60^0	0.500	0.100	0.0895	0.1114
70^0	0.658	0.114	0.0965	0.144
90^0	1.000	0.135	0.1089	0.198

Wieselsberger further considers the case where both ailerons are moved in the *same* direction. In that case the second solution has a symmetrical lift distribution, just as the original one, and must be represented by a series containing the sines of *uneven* multiples of ψ. No moments are obtained in this case; there is, however, an increase both of lift and of induced drag, which are expressed by equations of the form:

$$\delta L = \frac{1}{2} \varrho V^2 (2b)^2 \varkappa_L \varepsilon$$

$$\delta D_i = \frac{1}{2} \varrho V^2 (2b)^2 \varkappa_D \varepsilon^2$$

The coefficients $\varkappa_L$, $\varkappa_D$, for the same case as mentioned above, are given in the following Table 5.

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[1] The notation ζ, ξ, η has been taken over from Wieselsberger. For $\varkappa_L$, $\varkappa_D$ occurring below, Wieselsberger uses resp. λ, $\varkappa$.

[2] $(b-a)/b$ is the ratio of the length of an aileron to half span of the airfoil.

In a subsequent paper Wieselsberger and Asano[1] have given an approximate method for obtaining the values of the coefficients ζ, ξ, η, $\varkappa_L$, $\varkappa_D$ corresponding to other values of the parameter $1/\mu$ (4.5, 5.0, 5.5, 6.0).

Another set of calculations has been developed by Gates[2]. It differs from the method described in **3** by the way in which the coefficients of the Fourier series are determined. Instead of applying the condition that (3.7) must be satisfied at a finite number of points, equal to the number of terms retained in the series, it is required that the square of the error remaining in the fulfillment of this equation, integrated over the whole span, shall be a minimum. This condition is expressed by the set of equations:

TABLE 5.

ψ_a	$\frac{b-a}{b}$	$\varkappa_L$	$\varkappa_D$
40^0	0.234	0.130	0.056
60^0	0.500	0.326	0.116
70^0	0.658	0.442	0.137
80^0	0.826	0.587	0.163
90^0	1.000	0.729	0.178

$$\frac{\partial}{\partial A_n} \int_0^\pi d\psi \, [\Sigma A_n \sin n\psi \, (n\mu + \sin\psi) - \mu \alpha \sin\psi]^2 = 0 \qquad (11.5)$$

It is readily seen that the number of equations obtained in this way is equal to the number of coefficients to be calculated, and that again all these equations are linear in the unknowns. It seems probable, judging from the results obtained in other domains of mechanics, that this procedure with the same number of terms affords a better approximation than the original method of satisfying (3.7) at a set of isolated points.

Finally mention should be made of some work by Betz and Petersohn[3] relating to the same problem. These authors have investigated the distribution of the lift along an airfoil of infinite span, which at a certain point presents an abrupt change in the angle of incidence, *e. g.* from the value α_1 to the value α_2. For this case they deduce a mathematical expression describing the gradual change of l in the neighborhood of this point, from the value $(1/2)\,\varrho\, V^2 c\, m\, \alpha_1$ to $(1/2)\,\varrho\, V^2 c\, m\, \alpha_2$, and develop a method for the adaptation of this case to an approximate treatment of the aileron problem.

12. Iteration Method Proposed by Irmgard Lotz. The method of Trefftz and Glauert, and also Gates' method, all have the serious inconvenience that when it is desired to obtain a higher degree of accuracy by taking more terms in the series for l, it becomes necessary to repeat the whole series of calculations. Besides, when the number of terms increases the process of solving the equations becomes very laborious.

[1] WIESELSBERGER, C. and ASANO, T., Zeitschr. f. Flugtechnik und Motorluftschiffahrt **19**, p. 289, 1928.

[2] GATES, S. B., Techn. Rep. Aeron. Research Committee (Teddington), R. & M. No. 1175, 1928.

[3] BETZ, A. and PETERSOHN, E., Zeitschr. f. angew. Math. u. Mechanik 8, p. 253, 1928; E. PETERSOHN, Luftfahrtforschung **2**, p. 40, 1928.

A great advantage would be gained, if the calculations could be made in the form of a series of successive approximations, where each new step is based upon the results of the preceding steps, so that the degree of approximation can be steadily pushed farther along.

A method satisfying this requirement has been given by Irmgard Lotz[1]. The starting point is again the Fourier series for l, which is written in a form slightly different from (3.3):

$$l = \frac{1}{2}\varrho V^2 c_0 m \Sigma a_n \sin n\psi \qquad (12.1)$$

where c_0 is the chord of the median section[2]. The equation for φ then becomes, instead of (3.4):

$$\varphi = \frac{m c_0}{8b} \Sigma n a_n \frac{\sin n\psi}{\sin\psi} \qquad (12.2)$$

The new features of the process now are that Fourier series are introduced for the quantities $\alpha \sin\psi$ and $(c_0/c)\sin\psi$ (α, as before, being the geometrical angle of incidence; c the chord of a particular section), as follows:

$$\alpha \sin\psi = \Sigma \alpha_n \sin n\psi \qquad (12.3)$$

$$\frac{c_0}{c}\sin\psi = \Sigma \beta_n \cos n\psi \qquad (12.4)$$

By means of the ordinary method of Fourier analysis the coefficients[3] α_n, β_n of these series can be calculated as soon as both α and c are known as functions of ψ; moreover each coefficient can be calculated separately, so that it is possible to increase the accuracy by taking more terms in the development without changing the values of the coefficients already obtained. When the plan form of the airfoil is symmetric, the series (12.4) contains cosines of even multiples of ψ only. Similar simplifications may occur in the series for $\alpha \sin\psi$.

In the case of the rectangular wing the coefficients β_n have the following values:

$$\beta_0 = \frac{2}{\pi}, \quad \beta_{2n} = -\frac{4}{\pi(4n^2-1)}, \quad \beta_{2n+1} = 0 \qquad (12.5)$$

Equation (3.5), or (3.7), is now replaced by:

$$(\Sigma a_n \sin n\psi)(\Sigma \beta_n \cos n\psi) + \mu_0 \Sigma n a_n \sin n\psi = \Sigma \alpha_n \sin n\psi \qquad (12.6)$$

where:

$$\mu_0 = \frac{m c_0}{8b} \qquad (12.7)$$

In the product of two series, occurring in the left hand member of (12.6), we write provisionally as indices, k and l instead of n; the product can then be brought into the form of a double series:

$$\sum_k \sum_l a_k \beta_l \sin k\psi \cos l\psi,$$

[1] See footnote to 3.

[2] The coefficient m is regarded as a constant. If this should not be the case, the method may be changed by taking for m in (12.1) the value m_0 relating to the median section, while in (12.4) c_0/c is replaced by $m_0 c_0/mc$.

[3] The reader should not confuse the coefficients α_n with the letter α (without subscript), used for the angle of incidence.

which transforms into:

$$\frac{1}{2}\sum_{k}\sum_{l} a_k \beta_l \left[\sin(k+l)\psi + \sin(k-l)\psi\right].$$

In this double series we pick out the coefficient of $\sin n\psi$, which is found to be:

$$a_n \beta_0 + \frac{1}{2} a_{n-1}\beta_1 + \frac{1}{2} a_{n-2}\beta_2 + \dots + \frac{1}{2} a_1 \beta_{n-1} +$$
$$+ \frac{1}{2} a_{n+1}\beta_1 + \frac{1}{2} a_{n+2}\beta_2 + \dots \text{ ad inf.}$$
$$- \frac{1}{2} a_1 \beta_{n+1} - \frac{1}{2} a_2 \beta_{n+2} - \dots \text{ ad inf.},$$

or after scme rearrangement:

$$\frac{1}{2} a_1 (\beta_{n-1} - \beta_{n+1}) + \frac{1}{2} a_2 (\beta_{n-2} - \beta_{n+2}) + \dots +$$
$$+ \frac{1}{2} a_{n-1} (\beta_1 - \beta_{2n-1}) + a_n \left(\beta_0 - \frac{1}{2}\beta_{2n}\right) +$$
$$+ \frac{1}{2} a_{n+1} (\beta_1 - \beta_{2n+1}) + \frac{1}{2} a_{n+2} (\beta_2 - \beta_{2n+2}) + \dots$$

Now the coefficients of $\sin n\psi$ on both sides of (12.6) must be the same; hence we obtain the system of equations:

$$\left.\begin{aligned} &\frac{1}{2} a_1 (\beta_{n-1} - \beta_{n+1}) + \dots + \frac{1}{2} a_{n-1} (\beta_1 - \beta_{2n-1}) + \\ &+ a_n \left(\beta_0 - \frac{1}{2}\beta_{2n} + n\mu_0\right) + \frac{1}{2} a_{n+1} (\beta_1 - \beta_{2n+1}) + \\ &+ \frac{1}{2} a_{n+2} (\beta_2 - \beta_{2n+2}) + \dots \text{ ad inf.} = \alpha_n \end{aligned}\right\} \quad (12.8)$$

In its exact form the system contains an infinite number of equations with an infinite number of unknowns a_1, a_2, ... It is seen, however, that in the n'th equation the unknown a_n has the greatest coefficient, while the coefficients of the other terms usually decrease rather rapidly [see the values of the β's given in (12.5) for the rectangular wing]. On account of this circumstance the solution of the system can be approximated by successive steps as follows: First consider the system of equations:

$$\left.\begin{aligned} &\frac{1}{2} a_1 (\beta_{n-1} - \beta_{n+1}) + \dots + \frac{1}{2} a_{n-1} (\beta_1 - \beta_{2n-1}) + \\ &+ a_n \left(\beta_0 - \frac{1}{2}\beta_{2n} + n\,\mu_0\right) = \alpha_n \end{aligned}\right\} \quad (12.9)$$

which are obtained from (12.8) by rejecting the terms with a_{n+1}, a_{n+2}, ... The first equation of the system (12.9) appears to be:

$$a_1\left(\beta_0 - \frac{1}{2}\beta_2 + \mu_0\right) = \alpha_1$$

and thus gives a value for a_1; then from the second one, which contains only a_1 and a_2, a value of a_2 is obtained; a_3 is obtained from the third, and so on. In this way a first approximation is calculated.

Now write the original system (12.8) in the form:

$$\left.\begin{aligned}\frac{1}{2} a_1(\beta_{n-1} - \beta_{n+1}) + \ldots + \frac{1}{2} a_{n-1}(\beta_1 - \beta_{2n-1}) + \\ + a_n\left(\beta_0 - \frac{1}{2}\beta_{2n} + n\mu_0\right) = \\ = \alpha_n - \frac{1}{2} a_{n+1}(\beta_1 - \beta_{2n+1}) - \ldots \text{ ad. inf.}\end{aligned}\right\} \qquad (12.10)$$

and insert on the right hand side the values of the a's obtained from the first approximation. The system obtained in this way, as it were with "corrected" right hand members, can be solved in the same way as the system (12.9), and so a second approximation is obtained. Then this second approximation can be introduced on the right hand side of the system (12.10), and a third approximation can be found.

In this way the process can be carried on until the desired degree of accuracy is obtained. In starting the work as many equations are taken as may appear to give values of a_n of sufficient importance in the right hand member of (12.10); if required, it is always possible to increase the number of equations afterward, so as to obtain further coefficients, using the values already calculated as a provisional approximation, and correcting them according to the procedure indicated.

If the plan form of the airfoil is symmetric, so that the β's of uneven order vanish, the system of (12.8), as well as the systems (12.9), (12.10), can be separated into one set of equations for the a's of uneven order, and another set for the a's of even order.

The method has been applied by Irmgard Lotz to the case of the rectangular airfoil with ailerons. The angle of incidence changes along the span in the manner indicated in (11.2), and as was noted in **11**, the solution can be built up from two separate parts, one relating to the ordinary rectangular wing with constant angle of incidence, the other relating to the antisymmetrical distribution of the angle of incidence, given in (11.4). The coefficients of α_n, β_n of the series (12.3), (12.4) corresponding to the various cases must be prepared in advance, and then the system (12.8) can be attacked.

The results are given in the form of diagrams for the coefficients ζ, ξ, η, $\varkappa_L$, $\varkappa_D$ mentioned in **11** (though defined somewhat otherwise) as functions of the ratio $(b - a)/2b$ (aileron length divided by total span). The coefficient m is taken equal to 5.05; for λ the values 5 and 8 are chosen, so that the parameter $1/\mu$ has the values 3.96 and 6.34 resp. The results are compared with those obtained by Wieselsberger for the case $1/\mu = 4$, and though differences are apparent, the general agreement of the results is not bad.

For further particulars the reader is referred to the original papers and to Division J where comparison with experimental values is treated.

13. Airfoils of Moderate or Small Aspect Ratio.—Summary of Blenk's Theory for the Rectangular Airfoil. The calculations of the foregoing sections have been based upon the expression III (27.6) for the downward vertical velocity w_z at the points of the airfoil. As has been mentioned, this expression was obtained by a reasoning which can be justified only by assuming a large value for the ratio of span to chord, *i. e.* for the aspect ratio λ. However, cases may occur in practice where the actual value of λ is such that doubt may arise as to the validity of the simplifying assumptions of III 27. Hence it is of importance to search for expressions for w_z which may be valid also in the case of small λ's, or at least make possible an estimate of the errors introduced by the application of III (27.6).

A system of equations for this purpose has been developed by Blenk, and some indications concerning his method may be given at this point[1].

Taking first the case of the rectangular wing of span $2b$ and chord c, the direction of the motion lying in the median plane, it is assumed that the distribution of the "generalized load" per unit area k_z^* over the plane of the wing can be represented by a formula of the type:

$$k_z^* = \varkappa \sqrt{b^2 - \eta^2} \tag{13.1}$$

where: $$\varkappa = \alpha \sqrt{\frac{c-\xi}{\xi}} + \beta \sqrt{\xi(c-\xi)} + \gamma\left(\xi - \frac{c}{2}\right)\sqrt{\xi(c-\xi)} \tag{13.2}$$

α, β, γ being provisionally arbitrary coefficients. It must be remarked that the notation of the equations has been adapted to the one used in the present Division of the work, and deviates from that applied in Blenk's paper; especially it is to be noted that in this paper the coordinate x and a non-dimensional variable ξ are measured in the direction of the span, while the coordinate y and a non-dimensional variable η are measured in the direction of the chord. In (13.2) the coordinate ξ is measured along the chord from the leading edge; if we put: $\xi = (c/2)(1 + \cos\theta)$, then (13.2) may be transformed into:

$$\varkappa = \alpha \frac{1-\cos\theta}{\sin\theta} + \frac{\beta c}{4}\,\frac{1-\cos 2\theta}{\sin\theta} + \frac{\gamma c^2}{16}\left[\frac{1-\cos\theta}{\sin\theta} - \frac{1-\cos 3\theta}{\sin\theta}\right]$$

Now if we remember that according to III (27.1) the "bound vorticity" $\bar{\gamma}_y$ is connected with k_z^* by the relation: $\bar{\gamma}_y = -k_z^*/\varrho V$, then it will be seen that this expression yields a vorticity distribution which is of the type indicated by II (10.3)[2]. The factor $\sqrt{b^2 - \eta^2}$ by which $\varkappa$ is multiplied in (13.1) above, corresponds to an elliptical distribution of the lift over the span; hence (13.1) can be considered as an immediate

[1] Blenk, H., Zeitschr. f. angew. Math. u. Mechanik, **5**, p. 36, 1925.

[2] Expressions of the type (13.2) for the distribution of the vorticity had been introduced into the theory of thin airfoils of infinite span by W. Birnbaum, Zeitschr. f. angew. Math. u. Mechanik, **3**, p. 290, 1923. The trigonometric series given in II **8** and **10**, however, are more convenient for the calculation.

extension of the formulae for the strength of the vortex sheet as used in the theory of thin airfoils, to the case of airfoils of finite span.

Blenk himself does not introduce the load k_z^*, but the "bound vorticity" $\bar{\gamma}_y$ and the system of free vortices connected with the "bound vortices". Then applying Biot and Savart's formula it is possible to express the value of w_z in the form of a double integral extended over the area of the airfoil. The integration with respect to η is evaluated in an approximate way and yields an expression of the form:

$$\varkappa\left[A + \frac{B}{\xi - x} + C\,(\xi - x) + D\,log\,|\xi - x| + E\,(\xi - x)\,log\,|\xi - x|\right] \qquad (13.3)$$

where A, B, ... E indicate certain functions of y. Substituting for $\varkappa$ in succession each of the three terms occurring in (13.2), and integrating with respect to ξ, the expression for w_z assumes one of the three forms:

$$\left.\begin{aligned} w_z &= \alpha\,[A_\alpha + B_\alpha\,x + C_\alpha\,x^2] \\ w_z &= \beta\,[A_\beta + B_\beta\,x + C_\beta\,x^2 + D_\beta\,x^3] \\ w_z &= \gamma\,[A_\gamma + B_\gamma\,x + C_\gamma\,x^2 + D_\gamma\,x^3 + E_\gamma\,x^4] \end{aligned}\right\} \qquad (13.4)$$

where again the capital letters indicate various functions of y, for which the reader must be referred to the original paper, taking care of the change in notation mentioned before.

From the value of w_z the wing profile is calculated by means of the equation:

$$Z = \frac{1}{V}\int_0^x w_z\,d\,x \qquad (13.5)$$

which is obtained from III (32.3) by integration with respect to x. The results of this calculation have been compared with those which are obtained if III (27.6) is applied. It appears that the deviations between the two sets of results increase as a section of the airfoil nearer to the tips is considered. Blenk, however, states that his equations, so far as they have been developed, do not permit a decision as to whether this phenomenon is real, or whether it is due to a lack of convergence of his procedure, appearing in the neighborhood of the wingtips.

From the practical point of view, the most interesting conclusions are that Blenks' results lead both to a greater angle of incidence and to a greater curvature of the wing profile than are obtained from the theory based upon III (27.6). The mean increase of the angle of incidence is given by:

$$\triangle\,\alpha = 0.059\,\frac{c}{2\,b}\,C_L\,,$$

the mean increase of curvature by:

$$\triangle\frac{1}{R} = 0.056\,\frac{C_L}{2\,b}\,,$$

C_L being the lift coefficient.

The value of the induced drag is not influenced, a result which could be expected in consequence of the first of Munk's theorems (see III 18).

14. Application to the Inverse Problem. Calculation of the Distribution of the Lift for a Given Airfoil. The distribution of lift represented by (13.1) and (13.2) is not sufficiently general for the treatment of the inverse problem; it has been generalized by writing:

$$k_z = \Sigma\, \varkappa_n \eta^n \sqrt{b^2 - \eta^2} \tag{14.1}$$

where now:

$$\varkappa_n = \alpha_n \sqrt{\frac{c-\xi}{\xi}} + \beta_n \sqrt{\xi\,(c-\xi)} + \gamma_n \left(\xi - \frac{c}{2}\right) \sqrt{\xi\,(c-\xi)} \tag{14.2}$$

so that for every value of n a particular set of coefficients α_n, β_n, γ_n is introduced. In the case of the rectangular wing with the direction of motion lying in the median plane, Blenk reduces the series (14.1) to the terms $n = 0$ and $n = 2$. There are thus six arbitrary coefficients α_0, β_0, γ_0, α_2, β_2, γ_2 in the equations, and it is possible to assign such values to them, that the direction of the airflow, as determined by the equation:

$$\varphi^* = \frac{w_z}{V} \tag{14.3}$$

assumes a prescribed value at six given points.

This procedure has been applied by Blenk in order to investigate the distribution of the lift over a given airfoil. The airfoil chosen had the profile Göttingen No. 389, with constant chord and constant angle of incidence over the whole span. As this profile has a finite thickness, the first point to be considered was the determination of the profile of an infinitely thin airfoil, by which the given one could be replaced. This problem was solved by comparing it to a series of crescent shaped profiles of the Kármán-Trefftz family (see II **19**), having equal chord but various thicknesses, all, however, giving the same lift. Having obtained the appropriate infinitely thin profile, its inclination $\partial Z/\partial x$ to the chord was determined at the points:

$$x = (c/2)\,(1 - \sqrt{3/4}), \qquad x = c/2, \qquad x = (c/2)\,(1 + \sqrt{3/4}).$$

Then values of α_0, β_0, γ_0, α_2, β_2, γ_2 were calculated satisfying the equation:

$$\varphi^* = \alpha + \frac{\partial Z}{\partial x} \tag{14.4}$$

(α being the angle of incidence measured from the chord) at the six points:

$$y = +\, b/4, \qquad y = +\, 3b/4; \qquad x = c/2, \qquad x = (c/2)\,(1 \pm \sqrt{3/4})$$

On account of the symmetry of the airfoil, (14.4) is also satisfied at the points:

$$y = -\, b/4, \qquad y = -\, 3b/4; \qquad x = c/2, \qquad x = (c/2)\,(1 \pm \sqrt{3/4}).$$

From the coefficients $\alpha_0, \ldots \gamma_2$ the values of C_L and C_D were obtained as functions of the angle α.

The calculations have been performed for four values of the aspect ratio λ: 6, 4, 2, 1, and the results were compared with the reduction formulae I (12.6) and I (12.7), or IV (2.19) and IV (2.20). It was found that no deviations appeared in the case of the formula for the coefficient of induced drag. As to the formula for the effective angle of incidence, it was known that I (12.6) is not in conformity with the experimental results in the cases $\lambda = 2$ or $\lambda = 1$. The deviations, however, appeared to be cleared away for the greater part by the application of the new calculations.

There remains a certain disturbing effect, causing a curvature of the line giving C_L as a function of α, which for its explanation probably will require an extension of the ordinary Prandtl theory to terms of higher order.

Other cases investigated by Blenk. Besides the symmetrical case of the rectangular wing, Blenk has also given formulae for a wing in the form of a parallelogram, with the short sides parallel to the direction of motion; a wing with sweep-back; and a rectangular wing, the median plane of which makes an angle with the direction of motion (yawed wing). The various cases have been sketched in Fig. 76. Special attention has been given to the latter case, and the calculations have been effected for a lift distribution of the type given by (14.1) with the terms $n = 0, 1, 2, 3$ in the series. The inverse problem has been solved for a plane rectangular wing of aspect ratio 6, the tangent of the angle β between the velocity V and the median plane (the angle of yaw) being resp. 0, 0.1, 0.3, 0.5. As there are 12 coefficients in the series in this case, the condition for φ^* could be satisfied at the 12 points:

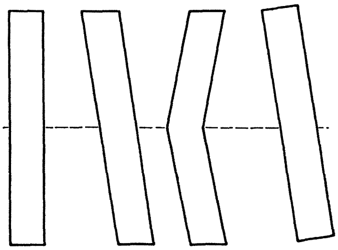

Fig. 76.

$$y = \pm\, b/4, \quad y = \pm\, 3b/4; \quad x = c/2, \quad x = (c/2)\,(1 \pm \sqrt{3/4}).$$

The distribution of the lift over the span becomes asymmetric if $\beta \neq 0$, causing a moment about the x axis. However, as the formulae, as Blenk states, are not quite rigorous, and especially do not apply to the wing tips (which may have a considerable influence upon the magnitude of this moment), it was not possible to obtain a reliable value[1].

15. Application of Equation III (28.8) to the Calculation of w_z.—Formulae for Yawed Rectangular Airfoil. A general expression for the calculation of w_z in the case of a loaded surface of arbitrary plan form has been deduced in III 28. The equation obtained does not require the introduction of the vortex system, and gives the value of w_z immediately

[1] The reader is referred to an interesting experimental investigation by D. H. Williams and A. S. Batson, Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1203, Oct. 1928.

in the form of a double integral extended over the loaded area. Though the denominator occurring in the integral has a relatively simple form, a direct evaluation in finite terms in general will be not easy. The integral, however, can be subjected to a similar method of approximation as was used by Blenk.

The general idea of this method is as follows: Assuming that the value of w_z is required at a point P with coordinates x, y, a first division of the domain of the integration is introduced by means of the lines $\eta = y \pm a$. This division precedes that considered in III 28, which will be introduced at a later stage of the process. The quantity a must be greater than the greatest value of $x - \xi$ occurring in the integral. If this condition is satisfied, then for all points ξ, η of the region outside of the lines $\eta = y \pm a$, it is possible to develop the fraction: $1/[r\,(r - x + \xi)]$ into a series proceeding according to ascending powers of $(\xi - x)$. In the region between the lines on the other hand, it is assumed that the load per unit area k_z^* (to be considered as a function of ξ and η) can be developed into a series proceeding according to ascending powers of $(\eta - y)$, with coefficients depending both on y and on ξ. It is evident that such a procedure will be practical only if both developments can be restricted to a small number of terms. Strictly this requires that a should be large compared with the dimension of the loaded region in the direction of the x axis (that is, with the chord in the ordinary case of an airfoil with zero yaw), while at the same time a should be small compared with the span of the airfoil. Such a condition could be satisfied only in the case of an airfoil of large aspect ratio. It will be evident thus (as is stated already by Blenk), that the expressions obtained have only a limited domain of validity. They are useful in so far as they give some idea of the nature of the corrections which must be applied to the approximate formula III (27.6), on which the ordinary theory of the finite airfoil is based. But they do not give reliable information in the case of a point P situated in the neighborhood of the wing tips, or in the immediate neighborhood of the median plane if a wing with sweep-back is considered, since for points P situated in these regions, the domain between the two parallel lines becomes incomplete, or assumes a more complicated form, for which the integrations have not been effected. Also the assumption that k_z^* can be developed according to powers of $(\eta - y)$ may not be applicable in such cases.

By way of example we shall give a few indications concerning the way in which the developments can be effected for the case of the yawed rectangular airfoil. In this case it is more convenient to make a change in the coordinate system, and to take x and y parallel to the median chord and to the span respectively, instead of parallel and perpendicular to the direction of the motion, as was done formerly. Denoting the angle of yaw by β, it will be seen that III (28.8) can be written in the form:

$$w_z = \frac{1}{\varrho V}\left[\iint\limits_{(I)} d\xi\, d\eta \frac{-k_z^*}{4\pi r\,[r-(x-\xi)\cos\beta+(y-\eta)\sin\beta]} - \text{correction term}\right] \tag{15.1}$$

The first division indicated above will now be made by lines $\eta = y \pm a$ parallel to the median chord. In the domain lying outside of these lines the first term of (15.1) only has to be considered. In the domain between them, on the other hand, we must introduce the further division used in III 28 and illustrated in Fig. 67, which—also in this case—is based upon lines parallel and perpendicular to the direction of V. Then also the correction term in (15.1) comes into play. A diagrammatic sketch of the yawed airfoil and of both divisions is given in Fig. 77.

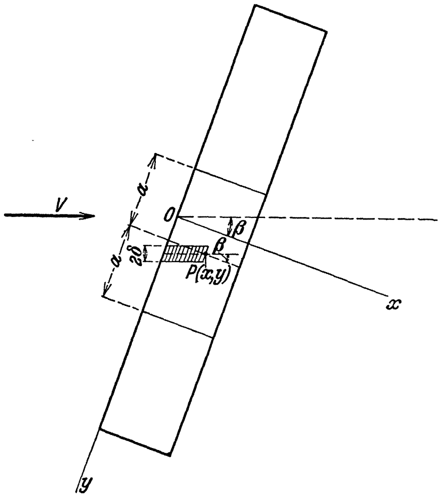

Fig. 77.

(*a*) In the regions lying outside of the lines $\eta = y \pm a$ we develop the denominator of the integral as follows:

$$\frac{1}{r\,[r-(x-\xi)\cos\beta+(y-\eta)\sin\beta]} = \frac{1}{1\mp\sin\beta}\cdot\frac{1}{(\eta-y)^2} \mp \frac{\cos\beta}{(1\mp\sin\beta)^2}\,\frac{\xi-x}{(\eta-y)^3} \pm \frac{3}{2}\,\frac{\sin\beta}{(1\mp\sin\beta)^2}\,\frac{(\xi-x)^2}{(\eta-y)^4}\cdots \tag{15.2}$$

The upper sign must be taken if $\eta > y$; the lower sign if $\eta < y$.

If this expression is substituted in the integral (15.1), the integration with respect to ξ can be effected by introducing the following notation:

$$\left.\begin{aligned}&\int_0^c d\xi\, k_z^* = A \ [\text{see III (27.3)}]\\ &\int_0^c d\xi\,\xi\, k_z^* = M\\ &\int_0^c d\xi\,\xi^2\, k_z^* = N, \text{ etc.}\end{aligned}\right\} \tag{15.3}$$

The integral then assumes the form:

$$-\frac{1}{4\pi\varrho V}\int d\eta\left[\frac{1}{1\mp\sin\beta}\,\frac{A}{(\eta-y)^2} \mp \frac{\cos\beta}{(1\mp\sin\beta)^2}\,\frac{M-xA}{(\eta-y)^3} \pm \frac{3}{2}\,\frac{\sin\beta}{(1\mp\sin\beta)^2}\,\frac{N-2Mx+Ax^2}{(\eta-y)^4}\cdots\right] \tag{15.4}$$

and it remains to effect the integration with respect to η, the limits being resp. $-b$ and $y-a$ (with the lower sign everywhere); $y+a$, $+b$ (with the upper sign everywhere).

In case the derivatives of the functions $A, M, \ldots$ are sufficiently continuous, the integrals can be transformed by partial integration. For instance taking the first term of the expression between [], and assuming that A is zero at the wingtips (that is, for $\eta = \pm b$), we obtain:

$$\left.\begin{aligned} -\frac{1}{4\pi\varrho V}\Bigg[\int_{-b}^{y-a} d\eta \frac{dA/d\eta}{(1+\sin\beta)(\eta-y)} + \int_{y+a}^{+b} d\eta \frac{dA/d\eta}{(1-\sin\beta)(\eta-y)} + \\ + \frac{2A}{a\cos^2\beta} + \frac{2A'\sin\beta}{\cos^2\beta} + \frac{aA''}{\cos^2\beta} + \ldots\Bigg] \end{aligned}\right\} \quad (15.5)$$

where $A', A'', \ldots$ resp. denote the first, second, ... derivatives of A with respect to η, taken at the point $\eta = y$.

(*b*) We next consider the region lying between the lines $\eta = y \pm a$ and develop k_z^* in the form:

$$k_z^* = k_0 + k_1(\eta - y) + \frac{1}{2}k_2(\eta - y)^2 + \ldots \quad (15.6)$$

where k_0, k_1, $k_2 \ldots$ are functions of y as well as of ξ.

It is convenient to consider separately the domain where $\xi < x$ and the domain where $\xi > x$. Beginning with the former one, we can get rid of the radical in the denominator of the integral (occurring in r), by introducing a new variable σ, by the substitution:

$$\eta - y = |\xi - x| \frac{2\sigma}{1-\sigma^2} = -(\xi - x)\frac{2\sigma}{1-\sigma^2} \quad (15.7)$$

The integral then assumes the following form:

$$\left.\begin{aligned} -\frac{1}{4\pi\varrho V}\int d\xi \int d\sigma \frac{-2}{(1+\cos\beta)(\xi-x)}\Bigg[\frac{k_0}{(\sigma-p)^2} - \\ - \frac{2k_1(\xi-x)\sigma}{(1-\sigma^2)(\sigma-p)^2} + \frac{2k_2(\xi-x)^2\sigma^2}{(1-\sigma^2)^2(\sigma-p)^2}\ldots\Bigg] \end{aligned}\right\} \quad (15.8)$$

where $p = \sin\beta/(1+\cos\beta)$. The question of the limits of the integration with respect to σ now becomes of great importance. At the boundaries defined by the lines $\eta = y \pm a$ we have: $\sigma = \pm\sigma_1$, where:

$$\sigma_1 = \sqrt{1 + \frac{(\xi-x)^2}{a^2}} + \frac{\xi - x}{a} \cong 1 + \frac{\xi-x}{a} + \frac{(\xi-x)^2}{2a^2}$$

According to the deductions of III 28 we must exclude from the domain of integration a narrow region of (infinitesimal) breadth 2δ, bounded by two lines parallel to the direction of the original velocity V, and lying at the distance δ respectively to the left and to the right of the point P. In terms of the coordinate η the limits of this strip are defined by:

$\eta = y - (\xi - x) \tan\beta \pm \delta \sec\beta$; the corresponding limits of σ become (taking account of the smallness of δ):

$$\sigma = \frac{\sin\beta}{1+\cos\beta} \mp \frac{\delta}{\xi - x}\,\frac{\cos\beta}{1+\cos\beta} = p \mp \frac{\delta}{\xi - x}\,\frac{\cos\beta}{1+\cos\beta}.$$

The integrals with respect to σ are all elementary, and the result of the integration (developed with respect to $\xi - x$ to the first positive power) becomes:

$$\left.\begin{aligned}
&-\frac{1}{4\pi\varrho V}\int d\xi\Big\{k_0\Big[\frac{4}{\delta\cos\beta} + \frac{2}{(\xi-x)\cos\beta} - \frac{2}{a\cos^2\beta} +\\
&+ \frac{(1+\sin^2\beta)}{\cos^3\beta}\,\frac{\xi - x}{a^2}\dots\Big] + k_1\Big[-\frac{4\sin\beta}{\cos^2\beta}\,\frac{\xi-x}{\delta} +\\
&+ \frac{2\sin\beta}{\cos^2\beta}\log\frac{2a}{|\xi - x|} + \frac{2}{\cos^2\beta}\left(\log\frac{1-\sin\beta}{\cos\beta} - \sin\beta\right) +\\
&+ \frac{4\sin\beta}{\cos^3\beta}\,\frac{\xi - x}{a}\dots\Big] + k_2\Big[\frac{2\sin^2\beta}{\cos^3\beta}\,\frac{(\xi-x)^2}{\delta} -\\
&- \frac{1+\sin^2\beta}{\cos^3\beta}(\xi - x)\log\frac{2a}{|\xi - x|} + \frac{a}{\cos^2\beta} +\\
&+ (\xi - x)\left(\frac{2\sin^2\beta + \cos\beta}{2\cos^3\beta} - \frac{2\sin\beta}{\cos^3\beta}\log\frac{1-\sin\beta}{\cos\beta}\right) + \dots\Big]\Big\}
\end{aligned}\right\} \quad (15.9)$$

To this expression must be added, according to III (28.8), the integral of k_z^* along a line parallel to the direction of V and extending from the leading edge of the airfoil to the point P. In terms of the coordinates used here this integral assumes the form:

$$\frac{1}{\pi\delta\varrho V}\int \frac{d\xi}{\cos\beta}k_z^* \qquad (15.10)$$

The value of k_z^* refers to the points of the line just mentioned. As along this line we have: $\eta - y = -(\xi - x)\tan\beta$, this value can be expressed by means of the series:

$$k_z^* = k_0 - (\xi - x)\,k_1 \tan\beta + \frac{1}{2}(\xi - x)^2 k_2 \tan^2\beta - \dots \qquad (15.11)$$

where the quantities k_0, k_1, $k_2\dots$ are those defined already in (15.6). Hence the integral becomes:

$$\frac{1}{4\pi\varrho V}\int d\xi\left[\frac{4k_0}{\delta\cos\beta} - \frac{4(\xi - x)\,k_1\sin\beta}{\delta\cos^2\beta} + \frac{2(\xi - x)^2 k_2 \sin^2\beta}{\delta\cos^3\beta}\dots\right] \qquad (15.12)$$

It is seen that this expression exactly cancels all those terms occurring in (15.9), which have δ in the denominator.

The treatment of the integral in the domain where $\xi > x$ is quite similar; instead of (15.7) the substitution is used:

$$\eta - y = +(\xi - x)\frac{2\sigma}{1-\sigma^2} \qquad (15.13)$$

and as it is not necessary to exclude any strip from the domain, it is superfluous to consider the internal boundaries. It is found that the result of the integration can be obtained from (15.9) by omitting the terms with δ in the denominator, and changing simultaneously the signs of both $\xi - x$ and $\cos\beta$, which, however, appears not to influence the result.

Taking together the results both for the domain $\xi < x$ and for the domain $\xi > x$, the following expression is obtained:

$$\left.\begin{aligned} -\frac{1}{4\pi\varrho V}\int d\xi\Big\{k_0\Big[\frac{2}{(\xi-x)\cos\beta}-\frac{2}{a\cos^2\beta}+ \\ +\frac{(1+\sin^2\beta)}{\cos^3\beta}\frac{\xi-x}{a^2}\dots\Big]+k_1\Big[\frac{2\sin\beta}{\cos^2\beta}\log\frac{2a}{|\xi-x|}+ \\ +\frac{2}{\cos^2\beta}\left(\log\frac{1-\sin\beta}{\cos\beta}-\sin\beta\right)+\frac{4\sin\beta}{\cos^3\beta}\frac{\xi-x}{a}\dots\Big]+ \\ +k_2\Big[-\frac{1+\sin^2\beta}{\cos^3\beta}(\xi-x)\log\frac{2a}{|\xi-x|}+\frac{a}{\cos^2\beta}+ \\ +(\xi-x)\left(\frac{2\sin^2\beta+\cos\beta}{2\cos^3\beta}-\frac{2\sin\beta}{\cos^3\beta}\log\frac{1-\sin\beta}{\cos\beta}\right)+\dots\Big]\Big\} \end{aligned}\right\} \quad (15.14)$$

The integration with respect to ξ in this expression is to be extended from $\xi = 0$ (leading edge) to $\xi = c$ (trailing edge). As to the first term of the expression, which contains $(\xi - x)$ in the denominator, the so-called principal value must be taken (see footnote to III 27); in the other terms this is not necessary.

It will be understood that the result applies to such points P only, as are lying at a distance from the wing tips greater than a (preferably much greater).

(*c*) The result (15.14) must be added to the integral (15.4) in which the third term will be omitted, as the development of (15.14) has been carried out to a degree corresponding with the second term of (15.4). The integration with respect to ξ in (15.14) for those terms which do not contain $(\xi - x)$ in the denominator or under the *log* sign, leads to the quantities Λ, M, introduced in (15.3), and their derivatives with respect to η, taken at the point $\eta = y$. As the result does not present an especially simple form, we shall not give it in full, the more so as all necessary operations are elementary and can be left to the reader. The same applies to the continuation of the developments to higher powers of $\xi - x$. It will be seen that the integrand of (15.14) is an expression of the type indicated in (13.3). Blenk's results can be obtained by assuming for k_z^* an expression of the form (13.1), (13.2); then Λ, M, k_0, k_1, k_2 .... can be calculated and the integrations in (15.4) and in (15.14) effected.

It is of importance to note that the sum of (15.4) and (15.14) is independent of the value of the quantity a, as can be checked by differentiation with respect to it.

For the general case, $\beta \neq 0$, we shall give the most important terms (zero order terms) of the final result:

$$
\left.\begin{aligned}
w_z = \frac{-1}{2\pi\varrho V\cos\beta}\Bigg[\int_0^c d\xi\,\frac{k_z^*}{\xi - x} + {}&\\
+\frac{\sin\beta}{\cos\beta}\int_0^c d\xi\,\frac{\partial k_z^*}{\partial y}\,\log\frac{b}{|\xi - x|}\Bigg] - {}&\\
-\frac{1}{4\pi\varrho V}\left[\frac{1}{1+\sin\beta}\int_{-b}^{y-a} d\eta\,\frac{dA/d\eta}{\eta - y} + \frac{1}{1-\sin\beta}\int_{y+a}^{+b} d\eta\,\frac{dA/d\eta}{\eta - y}\right] - {}&\\
-\frac{dA/dy}{2\pi\varrho V\cos^2\beta}\left[\sin\beta\log\frac{2a}{b} + \log\frac{1-\sin\beta}{\cos\beta}\right]&
\end{aligned}\right\}\quad(15.15)
$$

If β is taken zero, we fall back upon the ordinary symmetrical case of the rectangular airfoil. In that case the sum of (15.4) and (15.14) can be written as follows (up to terms of the first positive power in x):

$$
\left.\begin{aligned}
w_z = -\frac{1}{4\pi\varrho V}\Bigg[\int_0^c d\xi\,\frac{2k_z^*}{\xi - x} + \int_{-b}^{+b} d\eta\,\frac{dA/d\eta}{\eta - y} + {}&\\
+\int_0^c d\xi\,\frac{\partial^2 k_z^*}{\partial y^2}(\xi - x)\log\frac{|\xi - x|}{b} + \int_{-b}^{y-a} d\eta\,\frac{dM/d\eta - x\,dA/d\eta}{2(\eta - y)^2} - {}&\\
-\int_{y+a}^{+b} d\eta\,\frac{dM/d\eta - x\,dA/d\eta}{2(\eta - y)^2} + \left(\frac{d^2M}{dy^2} - x\frac{d^2A}{dy^2}\right)\log\frac{b}{2a}\Bigg]&
\end{aligned}\right\}\quad(15.16)
$$

In these expressions instead of k_0, we have written k_z^* again, as k_0 actually means the value of k_z^* at the points of the line $\eta = y$. As noted before, the principal value of the first integral must be taken. In the second integral of (15.16) the interior limits $\eta = y \pm a$ have been omitted; it is assumed instead that the principal value is to be taken also of this integral. This made necessary the omission of a term $2a\,d^2A/dy^2$ which otherwise would have appeared in the expression between [ ].

It will be seen that the most important terms of the expression, which are represented by the first and the second integral, together give the expression III (27.6), which is used in the ordinary form of the airfoil theory.

B. Multiplane Systems.

16. Minimum Induced Drag of Multiplane Systems. In studying the properties of multiplane systems it is convenient to start with the question whether for a given span and height of the system, it is possible to obtain a lower value of the induced drag than in the case of a single wing, the total lift being unchanged. According to the result deduced in III 21, if the circumference of the projection of the system upon the y, z plane is given, minimum induced drag can be obtained by distributing a system of generalized forces in a definite way along this circumference, the direction of the force being everywhere normal to it. It follows from this result that for a given span and height the most effective distribution of forces will be obtained with a system in the form of a *closed rectangle* having the prescribed height and span. In order to realize such a system it would be necessary to construct a biplane with two horizontal wings of equal span, with the corresponding tips of the upper and lower wing united by vertical wings and with a special distribution of angles of incidence to be deduced form detailed calculations. According to the theory, such a system for the same value of the total lift would have a smaller induced resistance than an arbitrary multiplane combination confined wholly within the boundaries of the rectangle.

In the following sections attention will be given to some of the properties of such a system, and likewise to those of various other types.

It follows from III **21** that the distribution of load which will ensure minimum induced resistance for an arbitrary given system can be calculated as soon as we have succeeded in determining the two-dimensional Dirichlet-motion along the projection of the system upon the plane Oyz, the velocity at infinity of this motion having the value w_0 directed parallel to $-Oz$ (*i.e.* upward). Suppose the potential of this motion to be denoted by $w_0 \Phi$, then from III (21.2) we have:

$$\frac{1}{\varrho V}\frac{d\Lambda}{ds} = u_s = w_0 \frac{\partial \Phi}{\partial s} \tag{16.1}$$

where Λ is the load (in the sense of resultant of the generalized forces) per unit length acting on the circumference of the projection of the system, directed normally and inward. On integrating we obtain:

$$\Lambda = \varrho\, V w_0 \Phi + \text{const.} \tag{16.2}$$

Taking the positive direction of ds counterclockwise when viewing from the positive side of the axis Ox, as was done before (see Fig. 60), and denoting by (n, z) the angle which the positive direction of the normal makes with the positive z axis (or, what comes to the same, the angle between ds and Oy), we have for the resultant parallel to Oz:

$$\int ds\, \Lambda \cos(n, z) = \int \Lambda\, dy \tag{16.3}$$

Inserting from (16.2) we obtain the following equation for the total lift L, which according to III (31.7) is equal to the total resultant K of the generalized forces:

$$L = K = \varrho\, V w_0 \int \Phi\, dy \tag{16.4}$$

In applying this equation it must be kept in mind that the integral $\int \Phi\, dy$ is taken along the circumference of the projection of the lifting system upon the y, z plane. For example in the case of a single wing the integral is taken first along the upper side (in the direction of $+y$), and then along the lower side (in the direction of $-y$); in the case of a biplane system this process must be carried out for both wings, etc.

Now according to III (20.8) the induced resistance is given by: $D_i = (w_0/2V)K = (w_0/2V)L$. Elimination of w_0 gives:

$$D_i = \frac{L^2}{2\varrho V^2 \int \Phi\, dy} \tag{16.5}$$

It is evident from this equation that the induced drag will be smaller as the integral $\int \Phi\, dy$ takes larger values. On comparing (16.5) with I (11.4) we see that this integral may be considered as representing the maximum value of the area Σ introduced in I **11**.

For convenience we shall henceforth denote this integral by the letter Σ.

The integral in some cases can be represented geometrically, as will be seen from the following example relating to the case of the single wing. Here the function Φ is to be deduced from III (22.1); hence at the upper surface we have: $\Phi = \sqrt{b^2 - y^2}$, while at the lower: $\Phi = -\sqrt{b^2 - y^2}$. In Fig. 78 these values have been laid off normally from the projection of the wing upon the y, z plane. The area obtained in this way is equal to the integral to be calculated. In the case considered this area is a circle and has the magnitude πb^2 already mentioned in **2**.

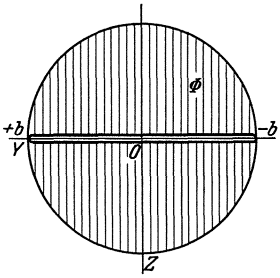

Fig. 78.

17. Closed Rectangular System. For several types of systems the function Φ can be obtained by applying the theory of complex variables. A few of the results may be noted, without going into the details of the calculations which involve elliptic functions, as in the more important cases sufficiently accurate results can be obtained by other methods to be mentioned below (see **21**, *seq.*).

We begin with the system having the smallest possible induced resistance for given span and height: the *closed rectangle* (Fig. 79a). As the system is symmetrical with respect to the x, z plane, the problem can be stated as follows: to find an irrotational two-dimensional motion, one stream-line of which follows the course A-B-C-O-G-H-A'. Assume a second plane with the potential Φ and the stream function Ψ as coordinates (Fig. 79b), in which the stream-lines become straight lines parallel to the Φ axis. The particular stream-line mentioned may then be taken as the zero stream-line ($\Psi = 0$) and the point O, lying symmetrically between C and G, can be made to correspond to the point $\Phi = 0$, $\Psi = 0$. The points B, C, G, H will then correspond to certain points of the line $\Psi = 0$, to be denoted respectively by:

$$\Phi = -\beta, \quad -\alpha, \quad +\alpha, \, +\beta.$$

Writing now: $y + iz = \zeta, \qquad \Phi + i\Psi = \chi$ (17.1)

we have to find a relation between ζ and χ which will give χ as an analytic function of ζ, or inversely, subject to the correspondence of the points mentioned and to the condition that at infinity we shall have:

$$\frac{d\chi}{d\zeta} = +i \tag{17.2}$$

The latter condition is obtained by observing that as Φ refers to unit velocity at infinity, the absolute value of the derivative must be unity there, while the argument is determined by the circumstance that to great positive real values of χ ($\Psi = 0$, $\Phi > 0$) correspond great negative imaginary values of ζ ($y = 0, z < 0$).

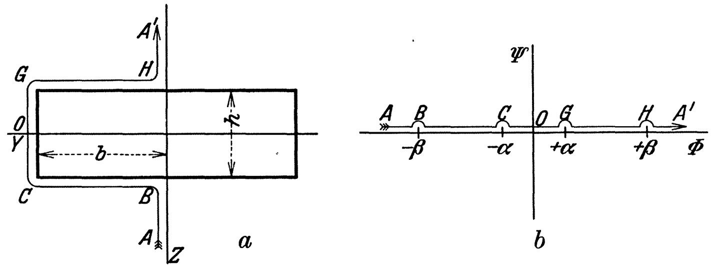

Fig. 79.

The Schwarz-Christoffel theorem[1] furnishes the desired relation by considering not particular values of the variables ζ and χ themselves, but the relations existing between the differentials $d\zeta$, $d\chi$ and especially by investigating the relations between the arguments of these differentials, which can be read off from the diagram by comparing the tract A-B-C-O-G-H-A' in the ζ plane with the corresponding tract in the χ plane. In the present case the following expression is obtained for $d\zeta/d\chi$ [adjusting the constant factor so as to satisfy (17.2)]:

$$\frac{d\zeta}{d\chi} = -i\sqrt{\frac{\chi^2 - \alpha^2}{\chi^2 - \beta^2}} \tag{17.3}$$

α and β being as defined above. It will be easily seen that the expression on the right hand side everywhere has the proper argument[2].

Equation (17.3) can be integrated with the aid of elliptic functions. We write:

$\alpha/\beta = k$ (the modulus of the elliptic functions) (17.4)

$$\chi = \alpha\, sn\, \xi \tag{17.5}$$

[1] See Appendix to this section.

[2] The corresponding expression in the case of the single wing is: $d\zeta/d\chi = -i\chi/\sqrt{\chi^2 - \beta^2}$. This can be obtained either directly by the same type of reasoning, or from (17.3) by taking α equal to zero. Integrating we have: $\zeta = -i\sqrt{\chi^2 - \beta^2}$, which leads back to III (22.1), when β is taken equal to b.

In this expression, ξ is an auxiliary variable while $sn\,\xi$ is the common notation for the function $sin\,am\,\xi$ introduced by Jacobi; then the integral takes the form:

$$\zeta = -i\beta\left[E(\xi) - \left(1 - \frac{\alpha^2}{\beta^2}\right)\xi\right] + \text{const.} \qquad (17.6)$$

where $E(\xi)$ denotes the elliptic integral of the second kind[1].

In order to determine the values of α and β and of the additive constant in (17.6) we apply this equation to the points O, G, H. Omitting the details of the calculations the following relations are obtained:

$$h/2 = \beta\,[E - (1 - k^2)\,K] \qquad (17.7)$$

$$b = \beta\,[E' - k^2\,K'] \qquad (17.8)$$

where K (not to be confused with the resultant generalized load), E are the complete elliptic integrals of the first and second kinds respectively, while K', E' are the complementary integrals. When the dimensions h and $2b$ of the rectangle have been given, these equations, taken together with (17.4), can be solved for α and β.

It is convenient to start with a few values of the modulus k, and to calculate the corresponding ratios $h/2b$ and β/b. In this way Table 6 has been constructed.

TABLE 6.

k	$h/2b$	β/b	$\Sigma/\pi b^2$
$sin\ 20^0 = 0.342$	0.113	1.21	1.295
$30^0 = 0.500$	0.302	1.49	1.66
$35^0 = 0.574$	0.460	1.70	1.94

The integral Σ which determines the induced drag can be obtained by observing that, in consequence of the symmetry of the system, Σ is equal to four times the integral $\int_H^G \Phi\,dy$ [*i. e.* the integral in (16.4) taken in the counterclockwise sense]. As along the line GH we have $\Psi = 0$, hence $\chi = \Phi$, while further $dz = 0$, so that $d\zeta = dy$, and the latter integral can be transformed as follows:

$$\int_H^G \Phi\,dy = \int_H^G \chi\,d\zeta = \int_\beta^\alpha \chi\frac{d\zeta}{d\chi}d\chi = \int_\alpha^\beta \chi\sqrt{\frac{\chi^2 - \alpha^2}{\beta^2 - \chi^2}}\,d\chi = \frac{\pi}{4}(\beta^2 - \alpha^2) \qquad (17.9)$$

Hence:

$$\Sigma = \pi\beta^2\,(1 - k^2) \qquad (17.10)$$

The ratio $\Sigma/\pi b^2$ is given in the last column of Table 6. The reciprocal values of this quantity give the ratio of the induced drag of the rectangular system to the induced drag of a single wing of the same span, assuming elliptic lift distribution and the same value of the total

[1] The functions $sn\,\xi$, $E(\xi)$, etc. are explained in every treatise on elliptic functions, and even in many handbooks on analysis. E. T. WHITTAKER and G. N. WATSON, Modern analysis (Cambridge University Press) can be mentioned. Tables of numerical values are given for instance in E. JAHNKE u. F. EMDE, Funktionentafeln, Leipzig, 1933.

lift. Hence they give an answer to the question mentioned at the beginning of **16**. These reciprocal values have been represented graphically in Fig. 80, curve A.

The detailed distribution of the generalized load over the sides of the rectangle can be obtained by calculating the value of Φ for the various points of these sides. We shall not work out this problem, nor investigate the form of the wings (both horizontal and vertical) which are necessary to realize this load distribution. A rough sketch has been given in Fig. 81; the load has been laid off in the direction in which the force is experienced by the wings, and it is seen that the generalized force on the vertical wings is directed outside for the upper parts, inside for the lower. The distribution of the load over the horizontal wings becomes more and more uniform, as the ratio $h/2b$ increases. The forces

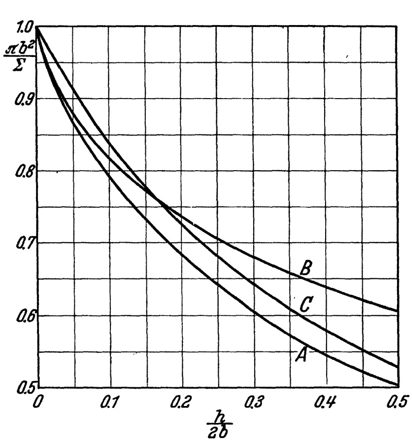

Fig. 80.

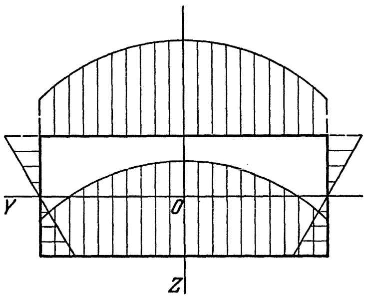

Fig. 81.

on the vertical wings are necessary to reduce the vortices which would appear at the wingtips, if they were not united.

Appendix to Section 17.—The Schwarz-Christoffel Theorem. Given two functions ζ and χ connected by the relation:

$$d\zeta/d\chi = P\,f(\chi) \tag{1}$$

or:

$$d\zeta = P\,f(\chi)\,d\chi \tag{2}$$

where P is some constant factor.

Assume further that AD, Fig. 82a, represents an axis of χ along which values of χ progress from A to D. In going along this line, then, $d\chi$ is always positive and real.

Assume again that we propose to lay off values of ζ, starting at A, Fig. 82b, and taking the initial direction vertical, that is, along $-iz$ (z positive downwards).

Now suppose that $f(\chi)$ is simply $(\chi+\beta)^n$. Then for all points to the right of B, Fig. 82a, $(\chi+\beta)$ will be a positive quantity, as between B and 0, χ, though being negative, is numerically smaller than β, while to the right of 0, χ itself is positive already. On the other hand for all points on the left of B, χ is negative and greater than β numerically, and so $(\chi+\beta)$ will be a negative quantity. In order to bring the sign into evidence, we write:

for points on the right of B: $(\chi+\beta) = +\sigma$
for points on the left: $(\chi+\beta) = -\sigma$.

The latter quantity may be also written:

$$\sigma e^{\pi i} \quad \text{or:} \quad \sigma i^2 .$$

To remove any ambiguity as to the question whether we should write $e^{+\pi i}$ or $e^{-\pi i}$, that is, whether the argument of $(\chi + \beta)$ for points on the left of B, has the value $+\pi$ or the value $-\pi$, we observe that instead of taking the path of the variable χ exactly through the point B, it is allowed to deform this path slightly, for instance as is indicated by the small semi-circle in Fig. 82a. We introduce the assumption that such deviations from the straight path may be made only towards the *upper* half of the diagram; it shall not be allowed on the contrary to make any deviations towards the lower half.

Now for any point χ on the small semi-circle, the quantity $(\chi + \beta)$ is represented by the vector from the point B to the point χ; when χ moves along the semi-circle from left to right, this vector turns in the clockwise direction, and thus its argument

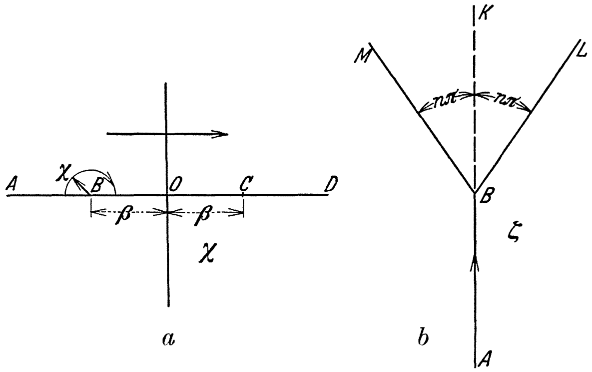

Fig. 82.

decreases by π. As it has the value zero for points to the right of B, the value for points on the left of B, from which it started, must be $+\pi$.

Returning now to (2) for points between A and B we shall have the relation:

$$d\zeta = P(-\sigma)^n \, d\chi = P\sigma^n i^{2n} \, d\chi \tag{3}$$

In order to meet the requirements in the ζ plane, $d\zeta$ between A and B, Fig. 82b, must be negative and lie along the vertical axis. That is, it must be in the form: $d\zeta = -i\,Q\,d\chi$, Q being a real, positive quantity. Hence P in (3) must contain a factor which, combined with the factor i^{2n} will give the product $-i$. Then writing P in the form $|P|\; i^{-2n-1}$, we may write (3):

$$d\zeta = |P|\, i^{-2n-1} \sigma^n i^{2n} \, d\chi \tag{4}$$

or:

$$d\zeta = -|P|\, i\, \sigma^n \, d\chi \tag{5}$$

In this form $d\zeta$ will be continuously laid off in the direction along z upward. That is, AB, Fig. 82a, transformed by (5), will give AB, Fig. 82b.

Next consider a point on the right of B, Fig. 82a; then (4) for this point becomes:

$$d\zeta = |P|\, i^{-2n-1} \sigma^n \, d\chi \tag{6}$$

Comparing (4) with (6) it is clear that to obtain (6) from (4) we must multiply (4) by the factor i^{-2n}. That is, the new direction of the line in Fig. 82b, starting from B, will be given by turning AB through an angle corresponding to i^{-2n} as a vector operator. But i as a vector operator means a turn through $+\pi/2$. Hence i^{-2n} means a turn through $-n\pi$. Hence the line in Fig. 82b will start off in some direction BL such that the angle $KBL = n\pi$.

The same reasoning will show that if $f(\chi)$ is $(\chi + \beta)^{-n}$, the result will be a turn through an angle $+ n\pi$ or along some direction BM.

If the case is restricted to turns less than π (less than 180^0), the exponent n must be smaller than 1. This at the same time is the condition which guarantees that the integral:

$$\int P(\chi + \beta)^{-n} d\chi \qquad (7)$$

shall be convergent when χ approaches to $-\beta$.

Cases with $n > 1$ do not occur in the applications of the theorem with which we shall be concerned. Cases with $n = 1$ are rather common; if the exponent is positive, no peculiarity appears in this case. With a negative exponent, however, some special considerations are necessary, but we shall leave aside this point as it does not occur in the applications to airfoil theory.

Next suppose that $f(\chi)$ consists of two factors, *e. g.* $f(\chi) = (\chi + \beta)^n (\chi - \beta)^n$. As long as we are to the left of the point C, the second factor will always be of the same sign. Hence nothing will be changed in the foregoing argument, provided the proper complex factor is given to P. However, in passing through the point C, it is the factor $(\chi - \beta)^n$ which changes, and thus implies a turn through $-n\pi$ of the vector $d\zeta$. As is easily seen, a factor $(\chi - \beta)^{-n}$ the same way would imply a turn through $+ n\pi$. Hence in this case there are two turns, which either may be in the same direction (both exponents positive or both exponents negative), or may be in different directions (one exponent positive and the other negative).

In this way it is seen that if the function $f(\chi)$ consists of a number of factors, in such a way that equation (1) assumes the form:

$$d\zeta/d\chi = P(\chi - \alpha)^{n_1} (\chi - \beta)^{n_2} (\chi - \gamma)^{n_3} \dots \qquad (8)$$

where the constants α, β, $\gamma \dots$ are all of them real, then the transformation indicated by this equation will change the real axis of the χ plane into a broken line in the ζ plane, consisting of a number of straight segments.

The result obtained allows us to derive the function $\zeta(\chi)$ or $\chi(\zeta)$ by which a given domain in the ζ plane, bounded by a closed polygonal contour, can be transformed conformally upon the upper half of the χ plane. It is then necessary to write down an expression of the type (8), with as many factors as there are angular points in the polygon, and to choose the exponents n_1, n_2, ... in accordance with the prescribed values of the angles. The real constants α, β, $\gamma \dots$ are unknown provisionally, as is also the complex constant factor P; they must be determined after carrying out the integration by taking account of the lengths of the sides of the polygon. The argument of P is obtained by considering the direction of one of the sides of the polygon; the other sides then obtain their respective directions on account of the turns determined by the exponents n_1, n_2, ... It must be mentioned further that it is always possible to give arbitrary values to three of the real constants α, β, $\gamma \dots$; it is evident moreover that in many cases relations of symmetry can help us in making a proper choice.

As to the term "closed polygon in the ζ plane" used above, it must be remembered that in the theory of complex variables "infinity" is considered as a single point, and there is no objection to the passing of one of the sides of the polygon through this point, or to this point being one of the corners of the polygon. Also one of the constants α, β, $\gamma \dots$ may be made to correspond with the point at infinity of the χ plane; in that case the corresponding factor disappears from (8).

18. Biplane System with Equal Span for Both Wings. Formulae for this case can be obtained by a similar procedure to that of **17**. The projection of the system upon the y, z plane is given in Fig. 83a; the corresponding Φ, Ψ plane is given in Fig. 83b. The stream-line which follows the course A-B-C-D-O-F-G-H-A' in the y, z plane is taken as the zero stream-line; the points B, C, D, O, F, G, H are assumed to correspond respectively to the following values of Φ:

$$\Phi = -\beta, \quad -\gamma, \quad -\alpha, \quad 0, \quad +\alpha, \quad +\gamma, \quad +\beta.$$

Having regard to the various changes of direction along this line, it is found that the equation appropriate to this case becomes[1]:

$$\frac{d\zeta}{d\chi} = -i\,\frac{\chi^2-\gamma^2}{\sqrt{(\chi^2-\alpha^2)(\chi^2-\beta^2)}} \tag{18.1}$$

We write again (ξ being a new variable):

$$\alpha/\beta = k \tag{18.2}$$

$$\chi = \alpha\, sn\, \xi \tag{18.3}$$

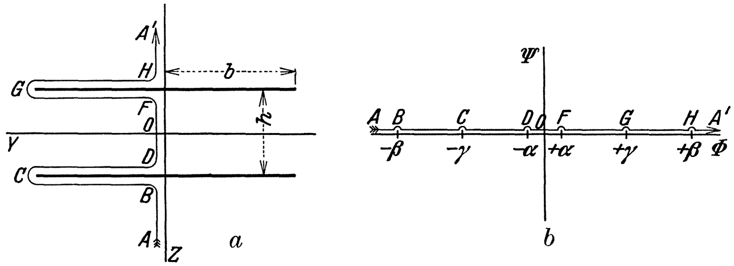

Fig. 83.

and furthermore put:

$$\gamma = \alpha\, sn\,(K + i\,\eta) = \alpha/dn'\eta \tag{18.4}$$

where η is a new constant, while $dn'\eta$ is written to denote the function[2] $\triangle\, am\, \eta$, corresponding to the conjugate modulus $k' = \sqrt{1-k^2}$. The integral of (18.1) then becomes:

$$\zeta = -i\,\beta\left[E(\xi) - \left(1-\frac{\gamma^2}{\beta^2}\right)\xi\right] + \text{const.} \tag{18.5}$$

This equation must be applied to the points O, F, G, H in order to obtain the following relations, which serve for the determination of the quantities α, β, γ:

$$\gamma^2/\beta^2 = E'/K' \tag{18.6}$$

$$h/2 = \pi\beta/2K' \tag{18.7}$$

$$b = \beta\left[\frac{\gamma^2}{\beta^2}\eta - E'(\eta) + \sqrt{\left(\frac{\gamma^2}{\beta^2}-\frac{\alpha^2}{\beta^2}\right)\left(\frac{\beta^2}{\gamma^2}-1\right)}\right] \tag{18.8}$$

In the latter equation the notation $E'(\eta)$—instead of $E(\eta, k')$—has been used for the incomplete elliptic integral of the second kind, relating

[1] The reader will observe that (18.1) shows a close analogy with II (22.3). There exists indeed a connection between the mathematical problem treated here and that considered in II **22**, though the physical problems are different in the two cases.

[2] In many treatises this function is written $dn(\eta, k')$.

to the argument η and to the complementary modulus k'. If in (18.5) we substitute for γ^2/β^2 the value given in (18.6), it assumes the form:

$$\zeta = -i\beta\left[E(\xi) - \frac{K'-E'}{K'}\xi\right] + \text{const.} \tag{18.9}$$

It appears to be sufficient to consider small values of k only, as larger values give systems with very large ratios $h/2b$. On account of this circumstance it is allowable to make use of a number of approximate formulae which are derived for this case in the theory of elliptic functions. In the first place we may write: $E' = 1$, $K' \cong \log(4/k)$, which leads to:

$$\frac{\gamma^2}{\beta^2} = \frac{1}{\log(4/k)} \tag{18.10}$$

Further instead of (18.4) we may take:

$$\gamma/\alpha = 1/dn'\,\eta \cong \cosh\eta \cong e^{\eta}/2,$$

giving:

$$\eta = \log\frac{2\gamma}{\alpha} = \log\frac{2}{k\sqrt{\log(4/k)}} \tag{18.11}$$

Finally we have:

$$E'(\eta) \cong 1, \quad \sqrt{\left(\frac{\gamma^2}{\beta^2} - \frac{\alpha^2}{\beta^2}\right)\left(\frac{\beta^2}{\gamma^2} - 1\right)} \cong 1 - \frac{\gamma^2}{2\beta^2}.$$

In this way the following expressions are obtained:

$$\frac{h}{2} = \beta\frac{\pi/2}{\log(4/k)} \tag{18.12}$$

$$b = \frac{\beta}{\log(4/k)}\left[\log\frac{2}{k\sqrt{\log(4/k)}} - \frac{1}{2}\right] \tag{18.13}$$

A few numerical examples are given in Table 7.

TABLE 7.

k	$\log(4/k)$	$h/2b$	β/b	$\Sigma/\pi b^2$
0.0001	10.60	0.191	1.29	1.35
0.001	8.29	0.260	1.37	1.43
0.005	6.68	0.346	1.47	1.51
0.01	5.98	0.402	1.53	1.57

The calculation of the integral Σ can be effected in the same way as in the case of the foregoing section, as it is equal to four times the integral $\int_H^F \Phi\,dy$. The value of the latter integral is found to be:

$$\int_H^F \Phi\,dy = \int_\beta^\alpha \chi\frac{d\zeta}{d\chi}d\chi = \int_\alpha^\beta \chi\frac{(\chi^2-\gamma^2)\,d\chi}{\sqrt{(\chi^2-\alpha^2)(\beta^2-\chi^2)}} = \frac{\pi}{4}(\beta^2+\alpha^2-2\gamma^2) \tag{18.14}$$

Keeping in mind that $\alpha^2/\beta^2 = k^2$ can be neglected, and substituting the approximate value (18.10) for γ^2/β^2, we obtain:

$$\Sigma = \pi\beta^2\left(1 - \frac{2}{\log(4/k)}\right) \tag{18.15}$$

The values of $\Sigma/\pi b^2$ are given in the last column of Table 7 and their reciprocals are represented in Fig. 80, curve B. Comparison with curve A gives an idea of the differences between the minimum induced resistance of a biplane system and that of the corresponding closed rectangular system.

Without calculations it will be evident that by taking h infinite we get two single wings, each carrying half of the load and thus having one fourth of the induced resistance of a single wing with full load. The induced resistance of the two together is then half of the latter value. Hence the limit of $\Sigma/\pi b^2$ for infinite values of h must be 2.

19. Single Wing with Shields at Ends. This case has received some attention in connection with theoretical investigations, and a number of experiments have been performed on models of this kind in wind channels. It may be expected that the flow around a wing provided with such "protecting" shields will approximate more closely to the two-dimensional flow than that around the ordinary wing.

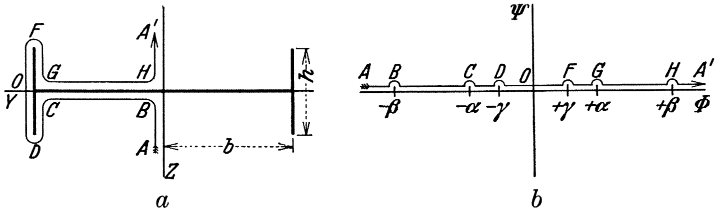

Fig. 84.

The projection of the system upon the y, z plane is shown in Fig. 84a. The treatment follows the same lines as before; the stream-line A-B-C-D-O-F-G-H-A' must correspond to the line $\Psi = 0$ of the Φ, Ψ plane (Fig. 84b). The points B, C, D, O, F, G, H are assumed to correspond respectively to the following values of Φ:

$$\Phi = -\beta, \quad -\alpha, \quad -\gamma, \quad 0, \quad +\gamma, \quad +\alpha, \quad +\beta.$$

As in this case, the same as in that of **18**, we have four turns to the left of 90^0, and two turns to the right of 180^0, the differential equation for ζ retains the form (18.1), now, however, with γ smaller than α. We put again:

$$\alpha/\beta = k \tag{19.1}$$

$$\chi = \alpha\, sn\, \xi \tag{19.2}$$

and further:

$$\gamma = \alpha\, sn\, \eta \tag{19.3}$$

η being a new constant. The integrated equation for ζ becomes as before:

$$\zeta = -i\beta\left[E(\xi) - \left(1 - \frac{\gamma^2}{\beta^2}\right)\xi\right] + \text{const.} \tag{19.4}$$

Applying this equation to the points O, F, G, H we obtain:

$$\frac{\gamma^2}{\beta^2} = \frac{K - E}{K} \tag{19.5}$$

$$\frac{h}{2} = \beta\left[E(\eta) - \left(1 - \frac{\gamma^2}{\beta^2}\right)\eta\right] \tag{19.6}$$

$$b = \beta\frac{\pi}{2K} \tag{19.7}$$

Some numerical examples are given in Table 8.

The integral Σ is given by:

$$\Sigma = 4 \int_H^G \Phi \, d y = 4 \int_\beta^\alpha \chi \frac{d \zeta}{d \chi} d \chi = \pi (\beta^2 + \alpha^2 - 2 \gamma^2) \tag{19.8}$$

which expression may be transformed into:

$$\Sigma = \pi \beta^2 \left[\frac{2 E}{K} - k'^2 \right] \tag{19.9}$$

Table 8.

k	$h/2b$	β/b	$\Sigma/\pi b^2$
$sin\ 30^0 = 0.500$	0.072	1.07	1.14
$45^0 = 0.707$	0.174	1.18	1.34
$60^0 = 0.866$	0.347	1.37	1.64
$75^0 = 0.966$	0.690	1.77	2.23

The values of $\Sigma/\pi b^2$ are given in the last column of Table 8 and their reciprocals are represented in Fig. 80, curve C.

It is of interest to consider the distribution of the load per unit span as a function of y in this case. According to (16.2) we have: $\Lambda = \varrho V w_0 \Phi + \text{const.}$ Hence taking the resultant of the forces acting along $G H$ and along $C B$, we obtain:

$$2 \varrho V w_0 \Phi \tag{19.10}$$

where Φ denotes the value along $G H$. Now at G we have: $\Phi = \alpha$, and at H: $\Phi = \beta$, and the ratio of these quantities is equal to the modulus k. Looking at the table we see that a reasonably close approximation to a constant lift per unit of length can be obtained, for instance, by taking $h/2b = 0.347$ ($k = sin\ 60^0 = 0.866$).

In Fig. 85 examples of the variation of Φ/β along the span are given for various dimensions h of the shields, beginning with $h = 0$ (simple wing).

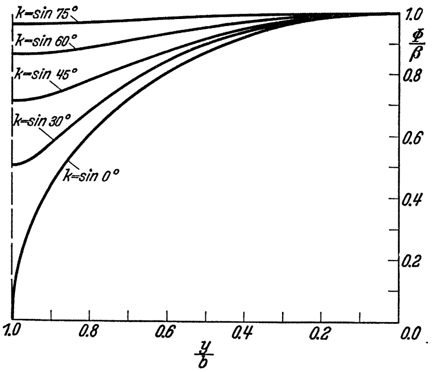

Fig. 85.

20. Airfoil with Gap. The case of a single airfoil with an interruption or gap in the middle part has been considered in some treatises. Though it is immediately evident that the presence of the gap will cause an increase of induced resistance, so that this case does not belong to the systems of minimum induced resistance, it may be best treated here, as it leads to a problem which again can be solved by the aid of elliptic functions.

The projection of the system upon the y, z plane is given in Fig. 86a, with a few stream-lines sketched in. The corresponding Φ, Ψ plane is represented in Fig. 86b. In Fig. 86a a stream-line is shown, which at the stagnation point C' on the underside of the left hand portion of the airfoil divides into two branches, one passing through B, the other through D, and uniting again at the stagnation point C on the upperside. In the Φ, Ψ diagram, this stream-line is represented as a straight line, the segment $C' C$ of which must be counted as a double line. It will be seen that the relation between these planes is opposite in nature to that for the corresponding planes in Figs. 83a and 83b.

On account of this fact the relation between the variables ζ and χ can be written down at once by starting from the equations of 18, and making the appropriate changes as follows:

$$\chi = i\, b \left[E(\xi) - \frac{K' - E'}{K'}\, \xi \right] \tag{20.1}$$

[see (18.9)]. The constants occurring in the elliptic functions again will be related to the ratios of certain lengths in Figs. 86a and 86b; as will

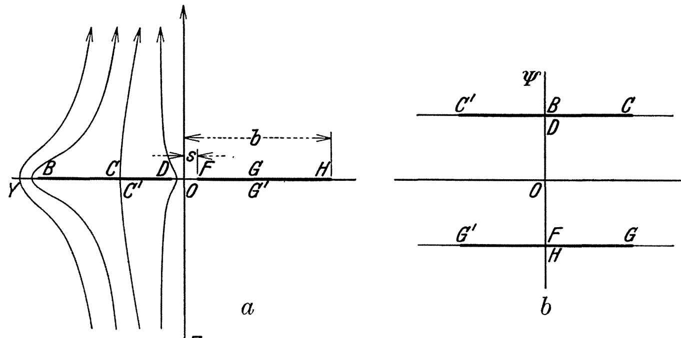

Fig. 86.

be seen subsequently, the ratio E'/K' is connected with the position of the stagnation points C', C [see (20.5)], while a comparison with Fig. 83b and (18.2), (18.3) shows that the modulus k must be given by

$$k = s/b \tag{20.2}$$

while the complex variable ξ is determined by:

$$\zeta = s\, sn\, \xi \tag{20.3}$$

The differential relation between χ and ζ, which of course could be obtained from the geometrical relations between Fig. 86a and 86b, can now be deduced from (20.1) and thus has the form:

$$\frac{d\chi}{d\zeta} = \frac{\dfrac{d\chi}{d\xi}}{\dfrac{d\zeta}{d\xi}} = i\, b \frac{\left(1 - \dfrac{\zeta^2}{b^2}\right) - \dfrac{K' - E'}{K'}}{s \sqrt{\left(1 - \dfrac{\zeta^2}{s^2}\right)\left(1 - \dfrac{\zeta^2}{b^2}\right)}} \tag{20.4}$$

Hence Φ obtains its maximum value corresponding to any real value of ζ, as for instance at the point C, when y satisfies the relation:

$$y/b = \sqrt{E'/K'} \tag{20.5}$$

which makes (20.4) equal to zero.

In order to calculate the integral Σ we observe that, by reason of symmetry:

$$\Sigma = 2 \int\limits_{DCBC'D} \Phi \, dy \tag{20.6}$$

As Φ and y are both univalent, we may transform this integral into the following:

$$\left.\begin{aligned}\Sigma = -2 \int\limits_{DCBC'D} y \, d\Phi = -2 \int\limits_{DCBC'D} \zeta \, d\chi = -4 \int\limits_{DCB} \zeta \frac{d\chi}{d\zeta} d\zeta = \\ = \pi \left[b^2 \frac{K' - 2E'}{K'} + s^2\right]\end{aligned}\right\} \tag{20.7}$$

For small values of k we again make use of the approximation mentioned in **18**, thus giving:

$$\Sigma \cong \pi b^2 \left[1 - \frac{2}{\log(4/k)} + k^2\right] \tag{20.8}$$

A few values are given in the accompanying Table.

TABLE 9.

$k = s/b$	$\Sigma/\pi b^2$	$\Sigma/\pi (b-s)^2$
0	1	1
0.001	0.759	0.761
0.01	0.666	0.679
0.1	0.461	0.568
0.5	0.127	0.507

In the last column the values of Σ are compared with those relating to a single wing of span $2(b - s)$, the wing which would be obtained by simply moving together the two parts, formerly separated by the gap $2s$.

It must be remarked that under actual circumstances in the case of very narrow gaps the flow through the gap is reduced by the action of resistances (frictional and the like) which are not accounted for in this theory[1]. In consequence of this the actual induced resistance in such cases is lower than that deduced from the value of $\Sigma/\pi b^2$.

The problem of the airfoil with an open gap or slit must be distinguished from the case where the wing is interrupted by the fuselage, as occurs in many of the present airplanes. The distribution of the lift for such a case must be determined by other methods; we return to this question in **45**, **46**.

21. Direct Method for the Calculation of Biplane Systems. The method explained in **17**—**20** can be applied likewise to the case of a biplane system having wings of unequal span, though the calculation becomes more complicated. For triplane systems, however, the method fails. In this case the stream-line analogous to the line A-B- . . . . A' of the former cases presents 6 turns of 90^0 (besides 3 of 180^0), in consequence of which there appears in the equation for $d\zeta/d\chi$ the square root of an expression of the 6^{th} degree leading to hyperelliptic integrals, for which convenient methods of computation do not exist.

[1] Compare BETZ, A., Handbuch der Physik, VII, p. 256.

It thus becomes important to consider another method of calculation, which, although it does not lead to the exact minimum values of D_i for the various cases, is of great use for the investigation of many practical problems and can be adapted to systems of largely different shapes. As the first example of the application of this method we again consider the case of the biplane. The wings will be treated as loaded lines (*i. e.* as airfoils of infinitely small chord). Provisionally it will be assumed that they are lying vertically one above the other, so that a system of zero stagger is to be considered (the more general case of arbitrary stagger will be treated in **25**, *seq.*). The spans of the wings may differ, and will be denoted respectively by $2b_1$ (upper wing) and $2b_2$ (lower wing). The vertical distance or gap between the wings is denoted by h (see Fig. 87).

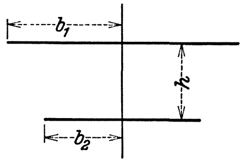

Fig. 87.

The method consists in assuming a certain distribution of the generalized load over both wings and calculating the induced resistance by the application of III (18.6). We introduce the resultant Λ of the generalized forces (resultant of the components k_z) acting on a section of the airfoil and referred to unit span; this quantity takes the place of $\overline{Q}_z$ in III (18.6)[1]. The actual lift l can be deduced from Λ afterwards (see **25**). Instead of $w_{z\,\infty}$ we introduce the vertical velocity induced by the system of trailing vortices at the points of the airfoils, to be denoted by w; from III (18.8): $w = (1/2)\, w_{z\,\infty}$. Finally the integration over the y, z plane occurring in III (18.6), etc., in the present case is to be replaced by integrations along the spans of the two wings. Denoting quantities relating to the upper wing by the suffix 1, those relating to the lower wing by the suffix 2, we obtain the following equation for the induced resistance:

$$D_i = \frac{1}{V}\left[\int d\,y_1\,\Lambda_1\,w_1 + \int d\,y_2\,\Lambda_2\,w_2\right] \tag{21.1}$$

The velocity w can be obtained from III (17.4), which in the present case gives:

$$w = \frac{1}{2}\,w_{z\,\infty} = \frac{1}{2\,\varrho\,V}\left[\int d\,\eta_1\,\Lambda_1\,\triangle + \int d\,\eta_2\,\Lambda_2\,\triangle\right] \tag{21.2}$$

where $\triangle$ is the function defined by III (17.3).

As noted in III **17** a singularity occurs in these integrals which calls for special attention. We can, however, avoid the difficulties arising from this point by writing the expression for w at the points of the upper wing as follows:

$$w_1 = w_{11} + w_{12} \tag{21.3}$$

where w_{11} denotes that part which is derived from the flow system caused by the trailing vortices of the upper wing, while w_{12} denotes

[1] In many treatises instead of the generalized load Λ the circulation Γ around the airfoil section is introduced. As $\Lambda = \varrho\,V\,\Gamma$ the necessary changes in the equations can be carried out without difficulty.

the part derived from the flow system due to the trailing vortices of the lower wing. In like manner we have for points on the lower wing:

$$w_2 = w_{21} + w_{22} \tag{21.4}$$

where now w_{21} is due to the trailing vortices of the upper wing, and w_{22} to the trailing vortices of the lower wing. Putting these expressions into (21.1) the following equation for D_i is obtained:

$$D_i = D_{11} + D_{12} + D_{21} + D_{22} \tag{21.5}$$

where

$$\left.\begin{aligned} D_{11} &= \frac{1}{V}\int d\,y_1\,A_1\,w_{11} \\ D_{12} &= \frac{1}{V}\int d\,y_1\,A_1\,w_{12} \\ D_{21} &= \frac{1}{V}\int d\,y_2\,A_2\,w_{21} \\ D_{22} &= \frac{1}{V}\int d\,y_2\,A_2\,w_{22} \end{aligned}\right\} \tag{21.6}$$

It is seen that D_{11} represents the drag induced on the upper wing by the load system of this wing itself and similarly D_{22} represents the drag due to the "self induction" of the lower wing, while the parts D_{12} and D_{21} represent the drag due to the mutual induction of the two wings. As the system considered is of zero stagger, the latter quantities must be equal, according to Munk's second theorem mentioned in III 18, and hence[1]:

$$D_{12} = D_{21} \tag{21.7}$$

Consequently we obtain the equation:

$$D_i = D_{11} + 2\,D_{12} + D_{22} \tag{21.8}$$

Now D_{11} and D_{22} can be calculated for various load distributions by applying the methods explained in Part **A** of the present chapter. In this way we overcome the difficulties arising from the singular behavior of the function $\triangle$ at certain points. It thus remains to calculate the value of D_{12}, which is given by the integral

$$D_{12} = \frac{1}{2\varrho\,V^2}\int_{-b_1}^{+b_1} d\,y \int_{-b_2}^{+b_2} d\,\eta\,A_1\,A_2\,\frac{h^2-(y-\eta)^2}{2\pi\,[h^2+(y-\eta)^2]^2} \tag{21.9}$$

It will be noticed that here in the expression for $\triangle$ as given in III (17.3) the quantity $(z-\zeta)^2$ has been replaced by h^2, where h is the vertical distance (gap) between the two airfoils. Instead of y_1 (for the points of the upper wing) and η_2 (points of the lower wing) we have written y, η; it is to be understood that A_1 refers to y and A_2 to η.

22. Elliptic Distribution of the Generalized Load for Both Wings. When elliptic distribution of load is assumed for both wings, the expressions for A_1 and A_2 become respectively:

---

[1] This can be deduced also by comparing (21.9) below with the analogous expression for D_{21}.

$$\Lambda_1 = \frac{2\,K_1}{\pi\, b_1^2}\sqrt{b_1^2 - y^2}\,, \quad \Lambda_2 = \frac{2\,K_2}{\pi\, b_2^2}\sqrt{b_2^2 - \eta^2} \tag{22.1}$$

where K_1, K_2 are the integrated values over the whole span. In that case the values of D_{11} and D_{22} are determined by (2.17):

$$\left.\begin{aligned} D_{11} &= \frac{K_1^2}{2\,\varrho\, V^2 \pi\, b_1^2} \\ D_{22} &= \frac{K_2^2}{2\,\varrho\, V^2 \pi\, b_2^2} \end{aligned}\right\} \tag{22.2}$$

In calculating the value of D_{12}, instead of starting from (21.9), we go back to (21.6); inserting the expression for Λ_1, we obtain:

$$D_{12} = \frac{2\,K_1}{V \pi b_1^2} \int_{-b_1}^{+b_1} d y \sqrt{b_1^2 - y^2}\, w_{12} \tag{22.3}$$

Now the value of w_{12} for this case is to be deduced from III (26.14), provided we divide by 2. Inserting for w_0 its value $K_2/(\varrho V \pi\, b_2^2)$ [see (2.10) with K_2 substituted for L], we have:

$$D_{12} = \frac{K_1\, K_2}{\varrho\, V^2 \pi^2\, b_1^2\, b_2^2} \int_{-b_1}^{+b_1} d\,y \sqrt{b_1^2 - y^2} \left\{1 - \frac{\sinh \mu \cosh \mu}{\cosh^2 \mu - \cos^2 \lambda}\right\} \tag{22.4}$$

Following Prandtl this expression will be written in the form:

$$D_{12} = \frac{K_1\, K_2}{2\,\varrho\, V^2 \pi\, b_1\, b_2}\,\sigma \tag{22.5}$$

where σ is a coefficient defined by:

$$\sigma = \frac{2}{\pi\, b_1\, b_2} \int_{-b_1}^{+b_1} d y \sqrt{b_1^2 - y^2} \left[1 - \frac{\sinh \mu \cosh \mu}{\cosh^2 \mu - \cos^2 \lambda}\right] \tag{22.6}$$

Its value can be obtained by applying the methods for numerical or for graphical calculation, for instance making use of the results collected in Fig. 65. The following table for σ has been published by Prandtl, while a graphical representation is given in Fig. 88a[1].

TABLE 10. Values of σ.

$s = b_2/b_1$	Values of $h/(b_1 + b_2)$										
	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
1.0	1.000	0.780	0.655	0.561	0.485	0.420	0.370	0.327	0.290	0.258	0.230
0.8	0.800	0.690	0.600	0.523	0.459	0.401	0.355	0.315	0.282	0.252	0.225
0.6	0.600	0.540	0.485	0.437	0.394	0.351	0.315	0.285	0.255	0.231	0.210

[1] PRANDTL, L., Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen II (1923), p. 11. PRANDTL writes b_1, b_2 for the spans instead of $2b_1$, $2b_2$.—The fraction $h/(b_1 + b_2)$ in Table 10 gives the ratio gap/mean span. The ratio b_2/b_1 is denoted by s. For simplicity it has been assumed in Table 10 and also in Tables 11a, 11b occurring below, that the lower wing is the smaller one, as is the usual case Should this be otherwise, then take the lower wing as No. 1 and the upper as No. 2.

It may be remarked that (22.5) can be written in the form:

$$D_{12} = \frac{K_1 (w_{12})\text{mean}}{V},$$

where the "*mean value of* w_{12}", usually written $\overline{w}_{12}$, is defined by:

$$\overline{w}_{12} = \frac{K_2}{2\varrho V \pi b_1 b_2} \sigma \qquad (22.7)$$

In the same way the mean value of w_{21} is:

$$\overline{w}_{21} = \frac{K_1}{2\varrho V \pi b_1 b_2} \sigma \qquad (22.8)$$

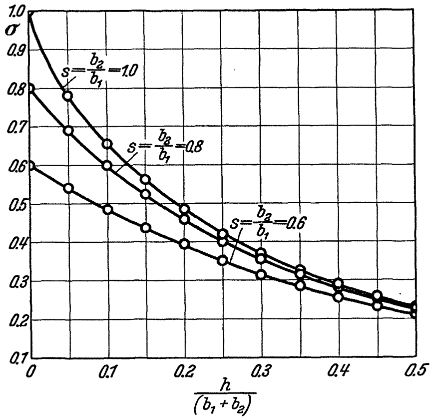

Fig. 88a.

Similar mean values will be considered at greater extent in **25**.

An approximate value for σ can be obtained by applying the method mentioned in III **25**, which consists in taking the load per unit span constant for every airfoil, reducing at the same time the span to a value 2β as given by III (25.6)[1].

On the assumption of uniform loading the value of w_{12} is to be deduced from III (24.11), provided we divide by 2 and substitute $K_2/2\varrho V\beta$ for Γ:

$$w_{12} = \frac{K_2}{8\pi\varrho V \beta_2}\left[\frac{y+\beta_2}{(y+\beta_2)^2+h^2} - \frac{y-\beta_2}{(y-\beta_2)^2+h^2}\right]$$

---

[1] It is of interest to note that the ratio β/b can be expressed by means of the coefficients of the Fourier series for the lift, introduced in **3** of the present Chapter, assuming, as will be understood, that it now refers to A instead of l. Writing as before $\eta = -b\cos\psi'$, $d\eta = b\sin\psi'\,d\psi'$ and putting:

$$A = 4\varrho V^2 b\,\Sigma A_n \sin n\psi',$$

it is readily found that:

$$\int_0^b d\eta\,\eta^2 A = \frac{\pi}{4}\varrho V^2 b^4 (A_1 + A_3).$$

As on the other hand [see (4.1)]: $K = 2\pi\varrho V^2 b^2 A_1$, we obtain:

$$\beta = b\sqrt{\frac{3(A_1+A_3)}{4A_1}}.$$

In the case of the elliptic distribution $A_3 = 0$, and we fall back on the value given before [III (25.7)]. In most cases, however, A_3 will be positive, the ratio β/b lying somewhat nearer to unity.

The quantity D_{12} is then obtained from the integral:

$$D_{12} = \frac{K_1}{2 V \beta_1} \int_{-\beta_1}^{+\beta_1} d y \, w_{12},$$

which can be evaluated at once, giving:

$$D_{12} = \frac{K_1 K_2}{8 \pi \varrho V^2 \beta_1 \beta_2} \log \frac{\sqrt{(\beta_1 + \beta_2)^2 + h^2}}{\sqrt{(\beta_1 - \beta_2)^2 + h^2}} \tag{22.9}$$

In this way the following approximate expression for σ is obtained:

$$\sigma = \frac{b_1 b_2}{4 \beta_1 \beta_2} \log \frac{\sqrt{(\beta_1 + \beta_2)^2 + h^2}}{\sqrt{(\beta_1 - \beta_2)^2 + h^2}} \tag{22.10}$$

A few results obtained from this formula for the case of elliptic distribution of load on both wings, so that $\beta_1 = b_1 \sqrt{3/4}$, $\beta_2 = b_2 \sqrt{3/4}$, are given in the accompanying table and can be compared with the exact values given in Table 10.

$h/(b_1 + b_2)$	0.2	0.3
$s = b_2/b_1 = 1.0$	$\sigma = 0.497$	0.373
0.8	0.463	0.356
0.6	0.368	0.302

For the case $b_1 = b_2$, again assuming elliptic load distribution, another approximate formula is given by Prandtl:

$$\sigma \cong \frac{1 - 0.33\, h/b}{1.055 + 1.85\, h/b} \tag{22.11}$$

23. Final Expression for the Induced Resistance. The full expression for the induced resistance of a biplane system can now be written:

$$D_i = \frac{K_1^2}{2 \varrho V^2 \pi b_1^2} + \frac{K_2^2}{2 \varrho V^2 \pi b_2^2} + \frac{2 \sigma K_1 K_2}{2 \varrho V^2 \pi b_1 b_2} \tag{23.1}$$

The first problem to be considered is to determine the distribution of load over the two airfoils giving the minimum value of D_i for a given total load $K = K_1 + K_2$ (equal to the total lift L). On working out the minimum problem, it is found that the ratio K_1/K_2 must satisfy the equation:

$$\frac{K_1}{K_2} = \frac{b_1/b_2 - \sigma}{b_2/b_1 - \sigma} \tag{23.2}$$

The value of D_i then becomes, writing s for b_2/b_1:

$$D_i = \frac{K^2}{2 \varrho V^2 \pi b_1^2} \frac{1 - \sigma^2}{1 - 2 \sigma s + s^2} \tag{23.3}$$

which may be put in the form:

$$D_i = \frac{L^2}{2 \varrho V^2 \pi b_1^2} \varkappa \tag{23.4}$$

The coefficient $\varkappa$ gives the ratio of the induced resistance of the biplane to that of a single wing having the same total load L and the span b_1. Values of this coefficient and of the ratio K_2/K, deduced from (23.2), are given in Tables 11a and b, and in Fig. 88b[1].

TABLE 11a. Values of $\varkappa$.

$s=b_2/b_1$	Values of $h/(b_1+b_2)$										
	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.6	1.000	0.990	0.974	0.954	0.932	0.911	0.892	0.875	0.861	0.848	0.839
0.7	1.000	0.982	0.956	0.926	0.897	0.871	0.849	0.830	0.812	0.797	0.783
0.8	1.000	0.974	0.932	0.892	0.855	0.825	0.800	0.778	0.758	0.740	0.728
0.9	1.000	0.950	0.893	0.847	0.807	0.773	0.744	0.719	0.699	0.683	0.671
1.0	1.000	0.890	0.827	0.779	0.742	0.710	0.684	0.662	0.645	0.629	0.615

TABLE 11b. Values of K_2/K.

$s=b_2/b_1$	Values of $h/(b_1+b_2)$										
	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.6	0	0.060	0.104	0.134	0.157	0.176	0.191	0.202	0.211	0.218	0.224
0.7	0	0.105	0.164	0.202	0.228	0.248	0.262	0.272	0.281	0.288	0.294
0.8	0	0.172	0.246	0.285	0.310	0.327	0.338	0.347	0.355	0.361	0.364
0.9	0	0.303	0.359	0.387	0.402	0.412	0.419	0.425	0.429	0.431	0.433
1.0	—	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500

It is seen that the smallest values for $\varkappa$ are obtained when $s = 1$, that is, for wings of equal span. We then have[2]: $K_1 = K_2 = L/2$, and:

$$\varkappa = (1 + \sigma)/2 \tag{23.5}$$

A comparison between the curve for $s = 1$ of Fig. 88b and curve B of Fig. 80 shows that the minimum values calculated by starting from the elliptical distribution of lift do not appreciably differ from those obtained for the exact minimum value of D_i.

Equation (23.5) at the same time shows that $\varkappa$ never can decrease below 1/2, which is in accordance with the remark made at the end of **18**.

Now let S_1, S_2 be the areas of the separate airfoils; then we may define the lift and induced drag coefficients of the biplane system by the equations:

$$L = \frac{1}{2}\varrho V^2 (S_1 + S_2)\, C_{L\,B} \tag{23.6}$$

$$D_{i\,B} = \frac{1}{2}\varrho V^2 (S_1 + S_2)\, C_{D\,i\,B} \tag{23.7}$$

When (23.6) and (23.7) are substituted in (23.4) we obtain the relation:

$$C_{D\,i\,B} = \varkappa \frac{(C_{L\,B})^2}{\pi} \frac{S_1 + S_2}{(2\,b_1)^2} \tag{23.8}$$

[1] PRANDTL, L., Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen II (1923), p. 13.

[2] It must be remembered that this equation does not imply $L_1 = L_2$.

This relation may be compared with (2.18). In order to better harmonize the outward appearance of the two formulae, we write $(2b_M)^2/S_M$ for λ in (2.18). When equal values of C_L are taken both for the biplane and for the monoplane, (2.18) can be subtracted from (23.8), giving the result:

$$C_{DiB} - C_{DiM} = \frac{C_L^2}{\pi}\left[\varkappa \frac{S_1 + S_2}{(2\,b_1)^2} - \frac{S_M}{(2\,b_M)^2}\right] \tag{23.9}$$

As we may assume that for equal values of C_L the profile drag will be the same in both cases, this equation also can be used for determining the difference between the total drag coefficients.

When the load distribution is not elliptical, it is necessary to introduce the quantity δ given by (4.11) or (6.11).

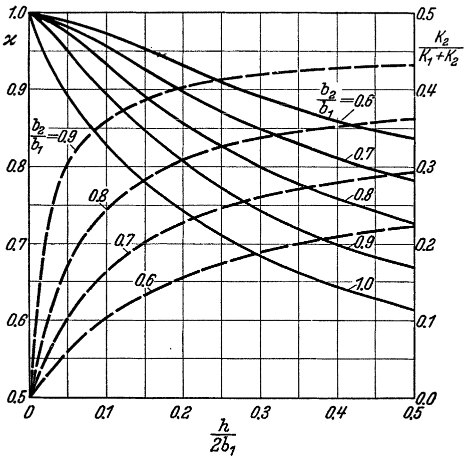

Fig. 88b.

24. Induced Resistance of Triplane Systems. The method of calculating the induced drag of biplane systems, indicated in **21—23**, can be extended without difficulty to other cases. A few equations relating to unstaggered triplane systems have been developed by Prandtl[1]. When elliptical load distribution is assumed for each separate wing, the induced velocities can be calculated by applying III (26.14) with a proper substitution of the various gaps. Introducing three numbers σ_{12}, σ_{13}, σ_{23} determined by means of (22.6) applied first to wings 1 and 2, then to 1 and 3, and finally to 2 and 3, the following expression is obtained for the total induced resistance:

$$\left.\begin{aligned} D_i = \frac{1}{2\pi\varrho V^2}\left[\frac{K_1^2}{b_1^2} + \frac{K_2^2}{b_2^2} + \frac{K_3^2}{b_3^2} + 2\,\sigma_{12}\frac{K_1 K_2}{b_1 b_2} + \right. \\ \left. + 2\,\sigma_{13}\frac{K_1 K_3}{b_1 b_3} + 2\,\sigma_{23}\frac{K_2 K_3}{b_2 b_3}\right] \end{aligned}\right\} \tag{24.1}$$

In the simpler case where the gap between the upper wing and the middle wing is equal to the gap between the middle wing and the lower wing, all wings being taken moreover of equal span, σ_{12} becomes equal to σ_{23}. Let us assume $K_1 = K_3$ and for convenience write (K being equal to $K_1 + K_2 + K_3$):

$$\left.\begin{aligned} \sigma_{12} = \sigma_{23} = \sigma_1, \qquad \sigma_{13} = \sigma_2 \\ K_2 = x\,K, \quad K_1 = K_3 = (1 - x)\,K/2 \end{aligned}\right\} \tag{24.2}$$

[1] Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen II (1923), p. 14.

Then the expression for the induced resistance can be simplified into:

$$D_i = \frac{K^2}{4\pi\varrho V^2 b^2}[(1+\sigma_2) - 2x(1+\sigma_2-2\sigma_1) + x^2(3+\sigma_2-4\sigma_1)] \quad (24.3)$$

This expression has its minimum value for:

$$x = \frac{1+\sigma_2-2\sigma_1}{3+\sigma_2-4\sigma_1} \quad (24.4)$$

The same as in the case of the biplane we put:

$$D_i = \frac{L^2}{2\varrho V^2 \pi b^2}\varkappa \quad (24.5)$$

In Table 12 (also from Prandtl) certain values of x with the corresponding values of $\varkappa$ have been given for various values of $h/2b$, h being

TABLE 12.

	Values of $h/2b$										
	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
x (for $\varkappa_{min.}$)	0	0.161	0.177	0.190	0.202	0.212	0.222	0.231	0.238	0.244	0.251
$\varkappa_{min.}$	1.000	0.885	0.819	0.767	0.724	0.687	0.656	0.630	0.607	0.585	0.565
$\varkappa$ (for $x = 1/3$)	1.000	0.889	0.824	0.774	0.732	0.695	0.663	0.637	0.612	0.591	0.571

the distance between the outer wings. Besides the values of $\varkappa$ obtained on the supposition $x = 1/3$ have been added.

In Fig. 89 a comparison has been made between the values of $\varkappa$ for the biplane of equal span (given in Table 11a), the triplane with $x = 1/3$, the triplane with least induced resistance, and the rectangular system treated in 17.

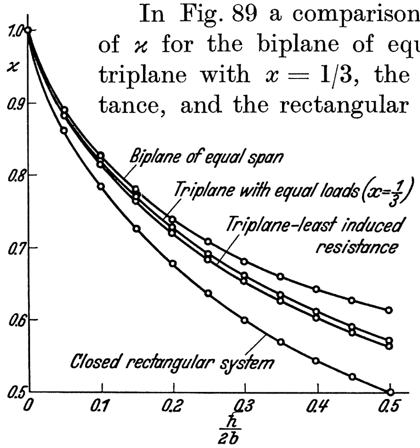

Fig. 89.

25. Detailed Investigation of the Forces Acting on the Wings of a Biplane System.—Mean Values of the Velocity Components along the Wings. We now proceed to a more detailed investigation of the forces which are experienced by the wings of a biplane system, considering especially their relation to the angles of incidence. It will be evident that again a number of simplifying assumptions must be introduced, as the complications of the problem otherwise become too great. Thus for the main part of the calculation of the interaction of the wings we shall assume that they may be reduced to airfoils of infinitely small chord, *i. e.* to loaded lines. We suppose that these lines extend along the lines of the centers of pressure of the two wings; further it is supposed that these lines are straight and parallel to the y axis; should they prove to be curved in any actual case, then a suitable mean position must be taken. Though actually the position of the center of pressure of a profile is not independent of the angle of incidence, we shall provisionally neglect this circumstance and consider the lines as fixed. In 27, 28 indications will be given concerning a more refined theory in which corrections can be introduced both for the finite dimensions of the chords (which usually are of the same order of magnitude as the gap between the wings) and for the displacement of the centers of pressure.

A section of the system by the plane of symmetry is pictured in Fig. 90. The points P_1, P_2 denote the centers of pressure of the two profiles. The distance $P_2 Q$ measured along the perpendicular from P_2 to the chord of the upper wing is called the *geometric gap* h_g, while the angle $P_1 P_2 Q$ measures the *geometric angle of stagger*. In calculations it is more convenient to introduce the *aerodynamic gap* $P_2 R = h$, and the *aerodynamic stagger*, determined by the length $P_1 R = f$. The angle $P_1 P_2 R$ is known as the *aerodynamic angle of stagger* (β). When α_{g1} denotes the geometrical angle of incidence of the upper wing measured between the chord and the original velocity V[1], the following relations exist:

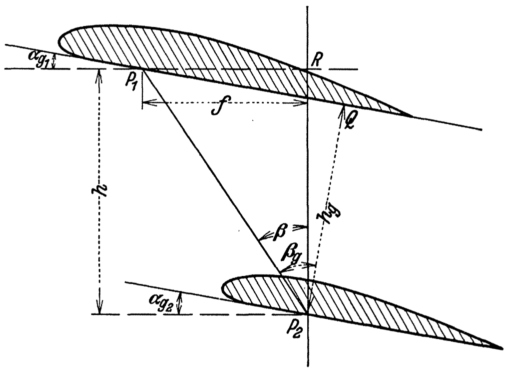

Fig. 90.

$$\sqrt{f^2 + h^2} = h_g / \cos \beta_g, \qquad \beta = \beta_g - \alpha_{g1} \tag{25.1}$$

A further simplification will be obtained in all calculations concerning interaction by replacing the actual load distribution over the wings by uniform loading over a reduced span, to be denoted by $2\beta_1$ for the upper wing and by $2\beta_2$ for the lower wing[2], which quantities can be determined in each case by means of III (25.6) (see also footnote, p. 218).

The theory of the biplane was originally developed along these lines by Betz, and has since been refined in various respects by other authors, especially by Millikan[3].

The first point to be investigated is the relation between the actual forces and the generalized forces, which thus far have been the prime

[1] This is not to be confused with the angle α_1 introduced in (26.4), which is measured from the direction of zero lift.

[2] It is assumed that no confusion will arise between the quantities β_1 and β_2, denoting the reduced semispans of the wings, and the angles β, β_g.

[3] BETZ, A., Zeitschr. f. Flugtechnik u. Motorluftschiffahrt, **5**, p. 253, 1914; MILLIKAN, C. B., Nat. Advis. Comm. Aeronautics (Washington) Rep. No. 362 (1930).

object of our considerations. On similar grounds as exposed in III **31**, we assume that the generalized forces have a component k_z only, and further that the bound vortices will be everywhere parallel to the y axis, so that there is no component γ_x to be taken into account. Consequently the relations which determine the magnitude of the components of the actual forces are to be obtained from III (14.3) with the omission of the terms containing γ_x, thus giving:

$$f_x = \varrho\, w_z\, \gamma_y = -\, w_z\, k_z/V\,,$$
$$f_y = 0\,,$$
$$f_z = -\,\varrho\, (V + w_x)\, \gamma_y = k_z + w_x\, k_z/V\,.$$

In the present case these formulae will be applied to the calculation of the mutual interaction of the wings only, as the parts depending upon the "self induction" can be obtained by the methods applied in the theory of monoplane wings. If now, in order to avoid the need of introducing a special notation, we assume that the letters w_x, w_z henceforth are used exclusively to denote the velocity components induced at the location of one wing by the action of the other, then the expressions $-\, w_z\, k_z/V$, $+\, w_x\, k_z/V$ give us the terms to be introduced respectively into the horizontal and the vertical forces in order to keep account of the interaction.

Denoting by Λ the resultant of k_z over a section of the airfoil, by l the lift per unit span, by d_i' the part of the induced resistance per unit span which is due to the self induction of the wing, by d_i the total induced resistance per unit span, it is seen that in the case of no interaction we should have: $l = \Lambda$, $d_i = d_i'$, and as soon as interaction is taken into account[1]:

$$\left.\begin{aligned} l &= \Lambda + \iint d\,x\,d\,z\,w_x\,k_z/V \\ d_i &= d_i' + \iint d\,x\,d\,z\,w_z\,k_z/V \end{aligned}\right\} \tag{25.2}$$

Here the integrals are to be extended over a section of the wing. However, in order to avoid difficulties in performing such integrations, we shall consider the quantities w_x, w_z as constant over the section of the wing over which the integration is extended. This assumption is not sufficient for an exact treatment of the problem of interaction and it will be necessary to introduce a correction afterwards (which can be done by calculating the curvature of the stream-lines in the neighborhood of a section), but for the present we will content ourselves with it. Equations (25.2) then assume the form:

$$\left.\begin{aligned} l &= \Lambda + w_x\, \Lambda/V \\ d_i &= d_i' + w_z\, \Lambda/V \end{aligned}\right\} \tag{25.3}$$

[1] In the formula for d_i the sign has been changed, as d_i represents the reaction experienced by the wing, while f_x was the force acting upon the air. In the expression for Λ no such change of sign is necessary, as Λ is reckoned positive if directed upward, while the positive direction for f_z is downward (according to the downward direction of the z axis).

The ratio w_z/V at the same time determines the slope of the streamlines in so far as they have been influenced by the presence of the other wing; this quantity thus must be taken into account in determining the effective angle of incidence.

In consequence of the circumstance that we do not restrict the treatment to the case of zero stagger, the expression for w_z will contain terms due to the field of the bound vortices along with terms due to the presence of the system of trailing vortices. We thus no longer have the relation $w_z = (1/2)\, w_{z\infty}$, but must go back to more general formulae of Chapter III.

Finally it is to be expected that the values of w_x, w_z will be different at the various points of the span, and thus will be functions of y. It is customary, however, to introduce still another simplification, and to replace them by mean values $\overline{w}_x$, $\overline{w}_z$, valid over the whole span. These mean values can be obtained as follows:

(*a*) $\overline{w}_x$. We start from III (24.8), in which b must be replaced by β_2. The mean value of this expression over a line parallel to the y axis and extending from $y = -\beta_1$ to $y = +\beta_1$, can be found by integrating and dividing the result by $2\beta_1$. The integrals are elementary, and the result of the calculation is found to be:

$$\left.\begin{aligned}\frac{\overline{w}_x}{V} = \frac{K_2}{8\pi\varrho V^2\beta_1\beta_2}\,\frac{-z}{x^2+z^2}\Big[&\sqrt{x^2+(\beta_1+\beta_2)^2+z^2}\,-\\ &-\sqrt{x^2+(\beta_1-\beta_2)^2+z^2}\Big]\end{aligned}\right\} \tag{25.4}$$

We put this expression into the form:

$$\frac{\overline{w}_x}{V} = \frac{K_2}{2\pi\varrho V^2 b_1 b_2}\,\sigma' \tag{25.5}$$

where σ' is a numerical coefficient. In order to simplify notation Millikan has introduced the following quantities:

$$r = \sqrt{1+\frac{(\beta_1+\beta_2)^2}{x^2+z^2}}\,, \qquad r' = \sqrt{1+\frac{(\beta_1-\beta_2)^2}{x^2+z^2}} \tag{25.6}$$

Considering in particular the action experienced by the upper wing from the lower one, as is the case assumed in (25.4) and (25.5), we have:

$$x = -f = -h\tan\beta\,, \qquad z = -h\,, \qquad \sqrt{x^2+z^2} = \frac{h}{\cos\beta} = \frac{h_g}{\cos\beta_g}\,.$$

Hence in this case σ' becomes:

$$\sigma' = \frac{b_1 b_2}{4\beta_1\beta_2}(r-r')\cos\beta \tag{25.7}$$

For the action experienced by the lower wing from the upper one we must replace K_2 by K_1 in (25.4) and (25.5), while besides:

$$x = +f = +h\tan\beta, \qquad z = +h,$$

and thus:

$$\sigma' = -\frac{b_1 b_2}{4\beta_1\beta_2}(r-r')\cos\beta \tag{25.8}$$

To distinguish between these two values of σ', we shall consider (25.7) as $+\sigma'$ and (25.8) as $-\sigma'$.

(*b*) $\overline{w}_z$. The full expression for w_z was not given in III 24, but can be deduced without difficulty from III (24.2)—(24.7). Then the mean value over the upper wing can be calculated in the same way as in the case of w_x, leading to an expression which will be written:

$$\frac{\overline{w}_z}{V} = \frac{K_2}{2\pi\varrho V^2 b_1 b_2}\sigma \tag{25.9}$$

Here σ is a numerical coefficient, which for the case $x = 0$ is equal to the coefficient σ given by (22.10) above, and in the more general case has the value:

$$\sigma = \frac{b_1 b_2}{4\beta_1\beta_2}\left[\frac{x}{\sqrt{x^2+z^2}}(r-r') + \log\frac{r-\dfrac{x}{\sqrt{x^2+z^2}}}{r'-\dfrac{x}{\sqrt{x^2+z^2}}}\right] \tag{25.10}$$

There are again two values, which now, however, are not opposites. They will be denoted by σ_1, σ_2 respectively. For the action experienced by the upper wing from the lower we have:

$$\sigma_1 = \frac{b_1 b_2}{4\beta_1\beta_2}\left[-(r-r')\sin\beta + \log\frac{r+\sin\beta}{r'+\sin\beta}\right] \tag{25.11}$$

and for that experienced by the lower wing from the upper:

$$\sigma_2 = \frac{b_1 b_2}{4\beta_1\beta_2}\left[(r-r')\sin\beta + \log\frac{r-\sin\beta}{r'-\sin\beta}\right] \tag{25.12}$$

It is not difficult to deduce the relation:

$$(\sigma_1+\sigma_2)/2 = \sigma_0 \tag{25.13}$$

where σ_0 is used to denote the value given by (22.10).

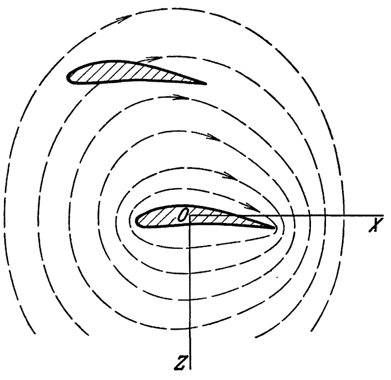

Fig. 91.

When the angle β is positive (the usual case of forward stagger of the upper wing) we have:

$$\sigma_2 > \sigma_0 > \sigma_1.$$

These relations are most easily remembered by noticing the picture of the circulatory flow around a wing (see Fig. 91); it is seen that this flow diminishes the value of w_z at all points lying in front of the wing, and increases it at points lying behind. The same picture shows that w_x is positive above the wing and negative below, which determines the sign of σ'.

To facilitate calculations two more auxiliary quantities are used:

$$\mu = \frac{\beta_1+\beta_2}{hg}\cos\beta_g \qquad \mu' = \frac{\beta_1-\beta_2}{hg}\cos\beta_g \tag{25.14}$$

Then: $r = \sqrt{1 + \mu^2}, \qquad r' = \sqrt{1 + \mu'^2}$ (25.15)

Further a function e is introduced, given by:

$$e = \frac{1}{2} [r \sin \beta - \log (r + \sin \beta)] \qquad (25.16)$$

Writing $\bar{e}$ for the same function in which β is replaced by $-\beta$; e', $\bar{e}'$ for similar functions with r' instead of r, we then have the equations:

$$\sigma_1 = \frac{b_1 b_2}{2 \beta_1 \beta_2} (e' - e), \qquad \sigma_2 = \frac{b_1 b_2}{2 \beta_1 \beta_2} (\bar{e}' - \bar{e}) \qquad (25.17)$$

A graph for the functions e and $\bar{e}$ against β with μ (or μ') as parameter has been given by Millikan, *loc. cit.* Fig. 12.

26. Continuation. Calculation of L_1 and L_2 when the Geometrical Angles of Incidence of both Wings Are Given. Integration of (25.3) with respect to y gives the following equation for either of the two wings:

$$L_i = K_i (1 + \bar{w}_x/V) \qquad (26.1)$$

Inserting (25.5) for $\bar{w}_x$ and taking notice of the remark made after (25.8) concerning the sign of σ', we have:

$$\left.\begin{aligned} &\text{for the upper wing: } L_1 = K_1 \left(1 + \frac{K_2}{2 \pi \varrho V^2 b_1 b_2} \sigma'\right) \\ &\text{for the lower wing: } L_2 = K_2 \left(1 - \frac{K_1}{2 \pi \varrho V^2 b_1 b_2} \sigma'\right) \end{aligned}\right\} \qquad (26.2)$$

These equations determine the *distribution of the actual lift* over the wings of the biplane, when the appropriate distribution K_1, K_2 has been found, for instance from (23.2). Equations (26.2) satisfy the condition:

$$L_1 + L_2 = K_1 + K_2 \qquad (26.3)$$

as is necessary in view of the results obtained in III **31**.

We must now investigate the connection between the lift and the angle of incidence. Taking the upper wing, the lift L_1 is generated by an air flow of velocity $V + \bar{w}_x$, the mean effective angle of incidence being:

$$\bar{i} = \alpha_1 - \bar{\varphi}_1 - \bar{w}_z/V \qquad (26.4)$$

Here α_1 is assumed to be measured from the direction of zero lift; $\bar{\varphi}_1$ is the mean reduction of the angle of incidence due to the vortex system of the wing itself (the self induction), while $\bar{w}_z/V$ determines the reduction due to the presence of the other wing. (It will be understood that for $\bar{w}_x$, $\bar{w}_z$ in this case are to be taken the values, at the upper wing, of the field generated by the lower wing). Hence the lift coefficient, taken with respect to the effective velocity $V + \bar{w}_x$, has the value:

$$m_1 (\alpha_1 - \bar{\varphi}_1 - \bar{w}_z/V) \qquad (26.5)$$

where m_1 is the coefficient introduced in (1.5). Then the lift L_1 becomes:

$$L_1 = \frac{1}{2} \varrho (V + \bar{w}_x)^2 S_1 m_1 (\alpha_1 - \bar{\varphi}_1 - \bar{w}_z/V) \qquad (26.6)$$

It is convenient to go back once more to the resultant K_1, given by:

$$K_1 = \frac{1}{2} \varrho V (V + \overline{w}_x) S_1 m_1 (\alpha_1 - \overline{\varphi}_1 - \overline{w}_z / V) \tag{26.7}$$

To shorten notation we introduce the abbreviations:

$$K_1/(1/2) \varrho V^2 S_1 = k_1, \qquad K_2/(1/2) \varrho V^2 S_2 = k_2$$

and further write:

$$S_1/4 \pi b_1 b_2 = s_1, \qquad S_2/4 \pi b_1 b_2 = s_2 \tag{26.8}$$

Then from (26.7) we obtain for the upper wing:

$$k_1 = m_1 (1 + k_2 s_2 \sigma') (\alpha_1 - \overline{\varphi}_1 - k_2 s_2 \sigma_1)$$

while the corresponding equation for the lower wing runs:

$$k_2 = m_2 (1 - k_1 s_1 \sigma') (\alpha_2 - \overline{\varphi}_2 - k_1 s_1 \sigma_2) \tag{26.9}$$

The angles $\overline{\varphi}_1$, $\overline{\varphi}_2$ are to be calculated from the equations:

$$\overline{\varphi}_1 = k_1 (1 + \tau_1)/\pi \lambda_1, \qquad \overline{\varphi}_2 = k_2 (1 + \tau_2)/\pi \lambda_2 \tag{26.10}$$

where the quantities τ_1, τ_2 must be obtained for each separate wing by means of the methods explained in Part **A** of the present Chapter. It is true that the distribution of the load over each wing will be influenced by the presence of the other wing, a circumstance which should be taken account of if very exact calculations are required. In view of the great complication of such calculations, together with the fact that the advantage gained would probably be small, other simplifications having already been introduced in the deductions, it is customary to neglect this influence.

In many cases the quantities τ_1, τ_2 are neglected altogether, which amounts to assuming elliptical lift distribution over each wing.

The coefficients m_1, m_2 can be deduced from measurements effected with the separate wings. If we write C_{L0} for the lift coefficient obtained with the separate wing, we have from (6.3) and (6.7):

$$m = \frac{C_{L0}}{\overline{i}} = \frac{C_{L0}}{\alpha - C_{L0}(1+\tau)/\pi \lambda} \tag{26.11}$$

It will thus be seen that if for a biplane with given spans a certain distribution of loads K_1, K_2 is assumed, corresponding for example to the case of minimum induced resistance, then as soon as suitable values have been assigned to the wing chords, it is possible to calculate the angles of incidence for both wings by means of (26.9). The quantities σ' etc. are obtained from the formulae of the preceding section; k_1, k_2, s_1, s_2 are given by (26.8), $\overline{\varphi}_1$, $\overline{\varphi}_2$ by (26.10), so that α_1 and α_2 can be found at once.

The most important quantity connected with the angles of incidence is the *decalage* $\alpha_1 - \alpha_2$. As in the ordinary type of biplane cellules the decalage must be fixed once for all, it must be given that value which corresponds to the state of loading of most importance for the special biplane considered.

A connection exists between the decalage and the stagger. This connection can be obtained by observing that in (26.9) the only quantities which to a marked degree depend on stagger are σ_1, σ_2. Hence solving for α_1, α_2 and taking the difference, we may write:

$$\alpha_1 - \alpha_2 \cong (k_2\, s_2\, \sigma_1 - k_1\, s_1\, \sigma_2) + \text{amount independent of } \beta.$$

On account of the relation (25.13), taken in connection with the circumstance that σ_0 is independent of β, this may be put in the form:

$$\alpha_1 - \alpha_2 \cong -\frac{1}{2}\,(k_1\, s_1 + k_2\, s_2)\,(\sigma_2 - \sigma_1) + \text{amount independent of } \beta \quad (26.12)$$

From this equation we easily deduce that an increase of forward stagger of the upper wing requires a diminution of decalage, or a change of the decalage to negative values, while a decrease of stagger of the upper wing requires an increase of decalage. In other words the forward wing in general must have the smaller angle of incidence.

Equations (26.9), however, can also be applied to the calculation of the loads if the angles of incidence, etc. of the wings have been given. They must, however, be solved by means of successive approximations, starting with some assumed values for k_1, k_2, and correcting them gradually. From k_1, k_2 we obtain K_1, K_2 with the aid of (26.8), and then L_1, L_2 from (26.2).

In order to be able to obtain an estimate of the effects due to the mutual interference of the wings, instead of (26.9), equations sometimes are given for the differences between the values of k_1, k_2 for the biplane and the values these quantities would have if both wings were separate, provided the geometric angles of incidence were kept unchanged. Denoting the latter values by k_{10}, k_{20}, and writing $\overline{\varphi}_{10}$, $\overline{\varphi}_{20}$ for the corresponding reductions of the angle of incidence on account of self induction, we have:

$$k_{10} = m_1\,(\alpha_1 - \overline{\varphi}_{10}), \qquad \overline{\varphi}_{10} = k_{10}\,(1 + \tau_1)/\pi\,\lambda_1 \quad (26.13)$$

with analogous relations for the other wing. To shorten notation we further write:

$$1 + m\,(1 + \tau)/\pi\,\lambda = p \quad (26.14)$$

with the subscript 1 for the upper wing and the subscript 2 for the lower wing. Then, neglecting a term $m_1\,(k_2\, s_2)^2\,\sigma'\,\sigma_1$ on the ground that it contains the product of several small quantities, the first equation of (26.9), if for $\overline{\varphi}_1$ we substitute the expression given by (26.10), can be brought into the form:

$$k_1\, p_1 = m_1\,(1 + k_2\, s_2\, \sigma')\,\alpha_1 - m_1\, k_2\, s_2\, \sigma'\, k_1\,(1 + \tau_1)/\pi\,\lambda_1 - m_1\, k_2\, s_2\, \sigma_1$$

Similarly from (26.13) we have:

$$k_{10}\, p_1 = m_1\,\alpha_1$$

Subtracting we find:

$$(k_1 - k_{10})\, p_1 = k_2\, s_2\,[m_1\,\alpha_1\,\sigma' - m_1\,\sigma'\, k_1\,(1 + \tau_1)/\pi\,\lambda_1 - m_1\,\sigma_1]$$

Thus, if we write: $\triangle k_1 = k_1 - k_{10}$, and in the second term between the [ ] neglect the difference between k_1 and k_{10}, we obtain:

$$\triangle k_1 = k_2 s_2 \frac{k_{10}\sigma' - m_1 \sigma_1}{p_1} \tag{26.15}$$

It is convenient in these calculations to make use of the relation $C_{L0} = m\alpha/p$, which is a consequence of (26.11). Differentiating with respect to α we have: $\quad d\,C_{L0}/d\,\alpha = m/p \quad$ (26.16)

The quantity $d\,C_{L0}/d\,\alpha$ can be obtained from experimental investigations performed upon a single wing. Then (26.15) can be replaced by:

$$\triangle k_1 = k_2 s_2 \left[\frac{k_{10}\sigma'}{p_1} - \sigma_1 \left(\frac{d\,C_{L0}}{d\,\alpha}\right)_1\right]$$

In a similar way: (26.17)

$$\triangle k_2 = k_1 s_1 \left[-\frac{k_{20}\sigma'}{p_2} - \sigma_2 \left(\frac{d\,C_{L0}}{d\,\alpha}\right)_2\right]$$

From the equations for $\triangle k_1$, $\triangle k_2$, equations for the changes of the lift coefficients C_{L1}, C_{L2}, which are assumed to be defined with respect to the original velocity V, can be deduced by making use of (26.2).

We have: $\quad C_{L1} = k_1(1 + k_2 s_2 \sigma'), \qquad C_{L10} = k_{10},$

and thus: $\quad \triangle C_{L1} = k_2 s_2 \left[\left(k_1 + \frac{k_{10}}{p_1}\right)\sigma' - \sigma_1 \left(\frac{dC_{L0}}{d\,\alpha}\right)_1\right]$

and similarly: $\quad \triangle C_{L2} = k_1 s_1 \left[-\left(k_2 + \frac{k_{20}}{p_2}\right)\sigma' - \sigma_2 \left(\frac{dC_{L0}}{d\,\alpha}\right)_2\right] \quad$ (26.18)

It is possible in these formulae to replace k_1, k_{10} etc. by expressions containing C_{L1}, etc., and further simplifications can be introduced by neglecting certain small amounts. But this can be left to the reader[1].

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[1] If it should be wished to make a comparison with the expressions given by MILLIKAN in the paper mentioned before, it will be necessary to give due regard to the following points: First there are some differences of notation, and especially it must be remarked that MILLIKAN uses the subscript 2 for the upper wing, and the subscript 1 for the lower wing. Then in the final results given at pp. 22, 23 of MILLIKAN's paper, the corrections indicated by $\triangle_c C_L$, $\triangle_d C_L$ give the influence of the curvature and of the double or S-shaped curvature of the stream-lines, effects which will be considered in 27 and 28 below; further the terms depending upon the moment coefficient C_M are connected with a different assumption concerning the position of the loaded lines, a point which likewise will be considered in 28. It thus remains to consider the following formulae:

$$\triangle C_L = \triangle_m C_L + \triangle_s C_L;$$

$$\triangle_m C_L = \triangle_x C_L + \triangle_y C_L, \qquad \triangle_s C_L = -\left(\frac{2\,\eta/A}{1 + 2\,\eta/A}\right)\triangle_m C_L.$$

Here the letter A is used by MILLIKAN for the aspect ratio and corresponds to our λ; the coefficient η corresponds to $m/2\pi$ in our notation. It must further be remarked that MILLIKAN does not introduce the reduced spans; consequently in making a comparison we must put $\beta_1 = b_1$, $\beta_2 = b_2$; finally the quantities τ_1, τ_2

27. Refinement of the Theory.—Correction for Curvature of Stream-Lines. The assumption that for the calculation of the mutual interaction the airfoils may be replaced by wings of infinitely small chords, by loaded lines so to speak, deviates rather far from actual circumstances, as in most biplane cellules the gap/chord ratio is of the order of unity. Hence various corrections must be applied to the foregoing calculations, of which the most important are the following[1]:

1. The field of motion induced by one wing at the place of the other wing is not the same at all points of the chord of the latter. The stream-lines of the induced field usually will be curved, and this curvature may reduce the effect of the camber of the wing placed in it, when it is in the same sense as the camber. Compared with this curvature effect the change of absolute velocity along the chord is of minor importance.

2. The assumption that the induced field might be calculated as if it were due to the action of a loaded line is not correct in itself. With the usual small gaps the induced field will be seriously affected by the circumstance that the chord of the inducing wing is not infinitely small.

3. On account of the finite thickness of the wings the velocity of the air between the wings will be increased slightly over the value obtained from the assumption of loaded lines.

In the present section we consider the first point, keeping to the treatment given by Bose and Prandtl, in which the inducing wing is still considered as a loaded line[2].

First consider the influence of a given curvature of the stream-lines upon the lift coefficient of a wing. An estimate of this influence is obtained by taking the case of a wing in the form of a circular arc, the lift

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are neglected. Taking now the upper wing, and changing the notation so as to remain in conformity with our text, we find:

$$\triangle_x C_{L1} = C_{L2} C_{L10} \left(\frac{b_2}{b_1} \frac{r' - r}{2\pi\lambda_2} \right) \cos\beta = 2\, C_{L2}\, C_{L10}\, s_2\, \sigma',$$

$$\triangle_y C_{L1} = C_{L2} \left(\frac{b_2}{b_1} \frac{m_1}{2\pi\lambda_2} \right) (e - e') = - C_{L1}\, m_1\, s_2\, \sigma_1 .$$

Hence, having regard to (26.14) and (26.16):

$$\triangle C_{L1} = \frac{\triangle_x C_{L1} + \triangle_y C_{L1}}{p_1} = C_{L2}\, s_2 \left[\frac{2\, C_{L10}\, \sigma'}{p_1} - \sigma_1 \left(\frac{d C_L}{d\alpha} \right)_1 \right]$$

This result can be compared with that given in (26.18). The differences between the two expressions are due to differences in the simplifying assumptions which have been introduced in the course of the deductions, and which in MILLIKAN's calculations have been applied to a somewhat greater extent than in the calculations given in the present text.

[1] Corresponding problems for wings of infinite span have been considered in II **12.**

[2] BOSE, N. K. and PRANDTL, L., Zeitschr. f. angew. Math. u. Mechanik **7**, p. 1, 1927.

coefficient of which for two-dimensional flow is approximately given by the equation[1]:

$$C_L = m\left(\alpha + \frac{c}{4R}\right) \tag{27.1}$$

Here c is the chord, $1/R$ is the curvature of the circular arc (inverse value of the radius), and α is the angle of incidence, measured from the chord. Hence if the stream-lines have a radius of curvature R', turned the same way as the radius of the circular arc, there will be a reduction of lift coefficient of the amount:

$$\delta C_L = -mc/4R' \tag{27.2}$$

A second question must also be considered, *viz.* which tangent to the curved stream-line must be taken as representing the mean direction of the flow? Holding to the same case of an airfoil in the shape of a circular arc, mounted in a field with uniform curvature of the stream-lines, it is seen that the mean direction of the flow is given by the tangent at the *geometric center* of the airfoil. Now the reduction of angle of incidence determined by the term $\overline{w}_z/V$ in the equations of the preceding section, was measured at the *aerodynamic center* of the wing profile. The difference in direction at the geometric center as compared with that at the aerodynamic center is given by:

$$\frac{c/2 - e}{R'} \tag{27.3}$$

$c/2$ being the distance of the geometric center from the leading edge, and e the distance of the center of pressure from this edge, so that $c/2 - e$ is the distance between the two centers[2]. In the ordinary case, in which the stream-lines are convex upwards, this quantity means a further reduction of the angle of incidence.

As the reduction of lift coefficient given by (27.2) likewise can be interpreted as a reduction of the angle of incidence by the amount $c/4R'$, the total reduction of angle of incidence becomes:

$$\delta\alpha = -\frac{3/4\,c - e}{R'} \tag{27.4}$$

In consequence of the approximate relation $e \cong c\,C_M/C_L$ [see I (4.4)], we may write:

$$\delta\alpha = -\frac{c}{R'}\left(\frac{3}{4} - \frac{C_M}{C_L}\right) \tag{27.5}$$

It now remains to determine the value of R'. The curvature of an individual stream-line is obtained from the equation:

$$\frac{1}{R_1} = \frac{1}{V}\frac{\partial w_z}{\partial x} \tag{27.6}$$

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[1] See II (17.9), the coefficient 2π being replaced by m for the reasons indicated before.

[2] The ordinary notation e for the distance of the center of pressure from the leading edge should not be confused with the quantities introduced in (25.16), (25.17).

The derivative $\partial w_z/\partial x$ has been calculated by Bose for a number of points lying in the vertical plane through the loaded line to which the induced field is due, assuming elliptic load distribution. From the results obtained in this way, mean values have been deduced relating to lines parallel to the axis Oy, applying the equation:

$$\frac{1}{R'} = \frac{\int\limits_{-b}^{+b} dy\,(\sqrt{b^2-y^2})/R_1}{\int\limits_{-b}^{+b} dy\,\sqrt{b^2-y^2}},$$

in order to give greater weight to the values at the points of greatest load. Passing over the details of the calculation we note the following approximate formula which is given for $1/R'$:

$$\frac{1}{R'} = 0.0875\,C_L\,\frac{c}{h^2} \tag{27.7}$$

where C_L and c refer to the inducing wing. In order to obtain consistency with the notation introduced in the preceding section it would be necessary to replace the coefficient C_L by the quantity k_1 when the inducing wing is the upper one and by k_2 when it is the lower one.

Hence the reduction of the angle of incidence of the upper wing due to the curvature effect becomes:

$$\delta\,\alpha_1 = -0.0875\,k_2\,\frac{c_1 c_2}{h^2}\left[\frac{3}{4} - \left(\frac{C_M}{C_L}\right)_1\right]$$

and the analogous quantity for the lower wing: (27.8)

$$\delta\,\alpha_2 = -0.0875\,k_1\,\frac{c_1 c_2}{h^2}\left[\frac{3}{4} - \left(\frac{C_M}{C_L}\right)_2\right]$$

These corrections must be introduced into (26.4)—(26.7) and (26.9); they will also have their influence upon (26.15), (26.17), (26.18).

As an application to a special case, consider a biplane with equal wings, having zero stagger and zero decalage. In this case $\lambda_1 = \lambda_2 = \lambda$; $s_1 = s_2 = 1/\pi\,\lambda$; $\sigma_1 = \sigma_2 = \sigma$. Suppose $K_1 = K_2$, hence $k_1 = k_2$, and for simplicity neglect the difference between these quantities and C_L or, in other words, neglect the correction terms depending on the quantity σ'. Then the mean effective angle of incidence takes the same value for both wings, which value is given by the equation:

$$\bar{i} = \alpha - \frac{C_L}{\pi\,\lambda}(1 + \tau + \sigma) - 0.0875\,\frac{c^2}{h^2}\left(\frac{3}{4}C_L - C_M\right) \tag{27.9}$$

As in the case considered, the aspect ratio λ can be written in the form $(S_1 + S_2)/8b_1^2$, while according to (23.5), $1 + \sigma = 2\varkappa$; this expression may then be brought into the form:

$$\bar{i} = \alpha - \frac{C_L}{\pi}\,\frac{S_1 + S_2}{(2\,b_1)^2}\left(\varkappa + \frac{\tau}{2}\right) - 0.0875\,\frac{c^2}{h^2}\left(\frac{3}{4}C_L - C_M\right) \tag{27.10}$$

If now one of the wings is taken separately, then for the same value of C_L we must have the same value of the effective angle of incidence, which is now connected with the geometric angle α_M by the equation:

$$\bar{i} = \alpha_M - \frac{C_L}{\pi} \frac{S_M}{(2\,b_M)^2}(1+\tau) \tag{27.11}$$

Subtracting (27.11) from (27.10), in which α_B will be written for α, the following formula is obtained, which expresses the increase of geometric angle of incidence to be applied in passing from the single wing to the biplane:

$$\left.\begin{aligned} \alpha_B - \alpha_M = \frac{C_L}{\pi}\left[\left(\varkappa + \frac{\tau}{2}\right)\frac{S_1 + S_2}{(2\,b_1)^2} - (1+\tau)\frac{S_M}{(2\,b_M)^2}\right] + \\ + 0.0875\,\frac{c^2}{h^2}\left(\frac{3}{4}\,C_L - C_M\right) \end{aligned}\right\} \tag{27.12}$$

If this formula is compared with (23.9), then it will be seen that even for the case of elliptical load distribution [which was assumed in (23.9) and which makes τ equal to zero], the relation

$$C_{D\,i\,B} - C_{D\,i\,M} = C_L\,(\alpha_B - \alpha_M)$$

does not hold.

28. Further Refinement of the Theory. The correction for the curvature effect, considered in the preceding section, takes account of the finite magnitude of the wing chord only in so far as the influence experienced by the wing from the surrounding field is to be determined. The effect of the finite magnitude of the chord of the *inducing* wing upon the velocities of this field are neglected. In the more elaborate theory which is given by Millikan[1] the latter point has not been neglected. Following von Kármán in his treatment for calculating the induced field, the inducing airfoil is replaced by a loaded line, combined with a *double vortex*, that is by a pair of vortices lying infinitely near to each other, and having infinite strength. This arrangement moreover has the advantage that both the loaded line itself, which of course is equivalent to a single vortex, and the vortex pair can be located at the geometric center of the wing profile, thus having a fixed position for all angles of attack of the biplane cellule.

The expressions for the strength of the single vortex and for that of the vortex pair have been given in II **12**; we thus have:

strength of the vortex: $$\Gamma = V c\, C_L/2 \tag{28.1}$$

strength of the vortex pair: $$2\pi A_1 = V c^2\, C_M/2 \tag{28.2}$$

In the latter expression, in conformity with the notation adopted in Chapter II, the moment coefficient C_M is taken with respect to the geometric center of the airfoil (center of the chord). If we return to the ordinary definition of C_M with respect to the leading edge of the

[1] See footnote to **25**.

profile (in Chapter II this quantity is denoted by C_m), we must replace (28.2) by[1]:

strength of the vortex pair: $V c^2 (C_M/2 - C_L/4)$ (28.3)

In applying these equations to the biplane problem it would be necessary to take account of the difference between actual forces and generalized forces, as it is the latter which are directly connected with the vortex system. Besides in general it would be necessary to introduce the ratio of the reduced span to the actual span, in order to take account of the influence of non uniform loading. In Millikan's paper these corrections are omitted; it is assumed that they are of small importance.

When a wing has been replaced by a vortex system of the type indicated, combined with the trailing vortices associated with the single vortex (the system of trailing vortices is not affected by the presence of the double vortex), the velocity field in its neighborhood can be calculated by the application of Biot and Savart's formula. The reader is referred to Millikan for the details of the calculations; we note here simply that expressions have been obtained for the mean values $\overline{w}_x/V$, $\overline{w}_z/V$, which were introduced in **25**.

By differentiation of $\overline{w}_x/V$ with respect to x two other quantities are obtained, *viz.* $\frac{\partial}{\partial x}\left(\frac{\overline{w}_x}{V}\right)$, $\frac{\partial^2}{\partial x^2}\left(\frac{\overline{w}_x}{V}\right)$,
the first of which determines the curvature of the stream-lines, while the second is a measure of the S-shaped or double curvature, which the stream-lines may present.

Having found these quantities the next problem becomes to determine the lift experienced by a given airfoil in a field, the velocity, direction, curvature and double curvature of which have been calculated. This is done by an application of the method indicated in II **12**. Finally the change of the value of C_L experienced in passing from a separate airfoil to a wing of a biplane system, having the same geometrical angle of incidence, is obtained as an expression of the form:

$$\triangle C_L = \frac{\triangle_x C_L + \triangle_z C_L + \triangle_c C_L + \triangle_d C_L}{p} \tag{28.4}$$

where $\triangle_x C_L$ means the increase of C_L due to the increase of velocity, $\triangle_z C_L$ that due to the change of direction, $\triangle_c C_L$ that due to the curvature, and $\triangle_d C_L$ that due to the double curvature of the stream-lines. The division by p, *i. e.* by the expression given in (26.14), is necessary to take account of the change of self induction in consequence of the change of C_L. Every term $\triangle_x C_L, \ldots \triangle_d C_L$ contains a part derived from the single vortex with the trailing vortices, and a part derived from the vortex pair, which together replace the other wing.

[1] In MILLIKAN's paper the moment is defined with respect to the center of the chord, so that (28.2) can be applied.

Simultaneously with the change of the lift coefficient the change obtained in the moment coefficient C_M has been calculated.

The results are expressed by a rather long series of equations, which again can be solved by successive approximations.

Though want of space prevents the reproduction of Millikan's theory in more detail, it will be seen that this treatment of the influence of the finite magnitude of the chords of both wings is much more satisfactory than that of former theories. In minor details it seems possible to make the treatment somewhat more exact, by taking notice of the difference between the lift coefficient and the quantities k_1, k_2 and by introducing the reduced spans. Other refinements could be added, as for instance a detailed investigation of the distribution of load along the wing span, but such corrections would require long and complicated calculations.

Finally there remains the problem mentioned under 3. in 27, *viz.* the influence of the finite thickness of the wing profiles. An estimate of this influence might be obtained if the two-dimensional theory of biplane systems, given in II 22, had been extended to the case of airfoils of arbitrary profile. Also another mode of treatment seems possible, by introducing a second vortex pair, one vortex line of which lies below the other. A system of equations would then result, of the same type as those given by Millikan, for the terms due to the moment of the load. Limitation of space, however, prevents any consideration of these questions in detail.

C. Influence of Boundaries in the Field of Motion around Airfoil Systems.

29. General Considerations Concerning the Influence of Boundaries[1]. The investigation of the motion around airfoils and airfoil systems in the presence of boundaries of the field of flow is of great importance both in theoretical and in practical aerodynamics. One of the most interesting cases relates to the experimental determination of airfoil characteristics in a windchannel, as it will be seen that the channel walls have an appreciable influence upon the results of the measurements, which must be taken into account in order to obtain values applicable to free air conditions. Another important example is that of the airplane near the ground. We can also consider the case of internal boundaries, such as are formed for example by the fuselage of an airplane, by the motor with its cowling, etc. In all such cases the motion will differ from that obtained from the equations valid for a field of unlimited extension.

[1] In connection with the problems of Part C the reader may be referred to H. GLAUERT, Wind Tunnel Interference on Wings, Bodies and Airscrews, Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1566, 1933.

The presence of the boundaries makes itself felt through the conditions which it imposes on the velocity components or on the pressure. In the case of so-called *fixed* boundaries, formed by solid walls and the like, it is clear from the conditions of continuity that the normal component of the velocity at the boundary must vanish. It might be expected that there is also a certain influence, due to friction, upon the tangential component. In most cases, however, this influence appears to be relatively unimportant, and it is customary to neglect it. Along with the case of fixed boundaries there is that of a so-called *free* surface, which occurs when the original motion of the air is present in a part of the field only, and is bounded by air at rest. Such circumstances occur for example when experiments are performed in a windchannel with an open working section. As there can be no discontinuity in the pressure along a free surface, the pressure at the side where the air is in motion must be equal to that at the other side where the air is at rest, and thus must assume a constant value. A third case, more general, may be considered, in which the current of air to be investigated is bounded by another current with a different velocity. Such cases can also occur under practical circumstances, as for instance when an airfoil, moving through the air at a given velocity, at the same time experiences the influence of the propeller slip stream, where the velocity is different. As will appear from the treatment given in **31** it is possible to start from the general case and to obtain the equations appropriate to the cases first mentioned by specialization.

It is not the object in the following sections, to treat the boundary problem in its most general aspect. The treatment will be restricted to the case of boundaries consisting of generating lines parallel to the direction of the original motion, that is parallel to the x axis. The original rectilinear motion of the air then remains undisturbed by the presence of the boundaries. The following cases will be investigated: one plane surface, two parallel plane surfaces, four plane surfaces forming a channel with rectangular cross section, and a cylindrical channel with circular cross section; it will be assumed in each case that the boundary surface extends unlimited both in the upstream and in the downstream direction. From the practical point of view it would be of interest to consider also boundary surfaces of limited extent, as this would give a better representation of various cases occurring in experimental arrangements. The case presented by a windchannel with an open jet for instance might be approximated by a boundary system consisting of fixed walls say from $x = -\infty$ up to $x = 0$, representing the mouth of the channel, passing over then into a system of free surfaces extending onward in the positive x direction, and representing the boundaries of the free jet. A still better approximation would be obtained, if these free surfaces extended up to a limited distance only, say up to $x = a$, and then were replaced again by a fixed boundary, representing the collector of the channel. However interesting such cases may appear, little attention has been given to them in consequence of the great complications their analysis presents, and we shall leave them aside[1].

A number of problems relating to the influence of boundaries may be treated with the aid of the theory of two-dimensional motion. Though such treatment is limited to the case of airfoils of infinite span and to certain problems which can be reduced to this case, it can throw light upon various details, *e. g.* upon the change in the effective angle of incidence, the curvature of the stream-lines in the x, z plane, and in general upon all questions relating to changes in the distribution of the velocity and of the pressure around the airfoil profile. But also these problems (properly speaking they would fall within the scope of Chapter II, and some analogy exists with the questions treated in II 12 and II 22, 23) will be discarded here[2], as we shall be concerned almost exclusively with cases where the airfoil is of finite span, and can be treated as a loaded line (the chord being assumed infinitely small).

From the mathematical point of view the method to be applied in order to find the influence of the boundary system upon the airfoil consists in the determination of an additional potential motion, which must be superposed upon the field determined by the equations of III 9 in order that the resulting motion shall satisfy the boundary conditions. In a number of cases when the boundary is formed by certain simple systems of plane surfaces we can, however, leave aside this general mathematical formulation and deduce a solution of the problem in a direct way by introducing a so-called *image*, or eventually a set of images, of the given system. In the case of a complicated boundary formed by a less simple system of plane surfaces, the method becomes impracticable, and in the case of a cylindrical boundary it fails, at least

[1] The reader is referred to TH. VON KÁRMÁN, Beitrag zur Theorie des Auftriebes, Vorträge aus dem Gebiete der Aerodynamik usw. (Aachen 1929), p. 98. Also for cases treated with the aid of the theory of two-dimensional motion to T. SASAKI, On the effect of the walls of a wind tunnel upon the lift coefficient of a model, Rep. Aeron. Res. Instit. Tokyo No. 77 (Vol. VI, p. 315, 1931; Japanese with English abstract).

[2] Cases of two-dimensional motion have been considered by various authors, who nearly always make use of the methods of conformal transformation. By way of example the following papers are mentioned: T. SASAKI, On the effect of the wall of a wind tunnel upon the lift coefficient of a model, Rep. Aeron. Res. Instit. Tokyo No. 46 (Vol. IV, p. 149, 1928; Japanese with English abstract) and No. 77 (see footnote above); L. ROSENHEAD, The lift on a flat plate between parallel walls, Proc. Roy. Soc. London A 132, p. 127, 1931; S. TOMOTIKA, On the moment of the force acting on a flat plate placed in a stream between two parallel walls, Rep. Aeron. Res. Instit. Tokyo No. 94 (Vol. VII, p. 357, 1933); S. TOMOTIKA, T. NAGAMIYA and Y. TAKENOUTI, The lift on a flat plate near a plane wall etc., ibidem No. 97 (Vol. VIII, p. 1, 1933; further papers are following).

when it is required to determine the channel influence at every point of the field. In order to solve certain important special questions in the case of a channel with circular cross-section, there is, however, a possibility of procedure by the use of images, as with the simpler cases.

It appears to be most convenient to start with those cases where the method of images can be applied and to delay the introduction of the additional potential field until we come to the cylindrical channel.

30. Example.—Image of a System with Respect to a Single Plane Boundary. In order to obtain a preliminary picture of the influence of a boundary, consider the case where the field is bounded by a single plane fixed boundary parallel to the axis Ox. The solution of the hydrodynamic equations valid for the infinite field must be corrected in such a way that, at the boundary, the normal component of the velocity vanishes. Now imagine the field of motion to extend beyond the boundary plane, and assume a second system of forces obtained by reflecting the given system in this plane. The new system is called the *image* of the original (actual) system with respect to the plane. Taking together the actual force system and its image, we have then a system of forces symmetrical with respect to the given plane. Hence the resulting motion (calculated by means of the ordinary equations for the unlimited field) will be symmetrical as well, and thus the normal component of the velocity will vanish at every point of the plane of symmetry, so that the prescribed condition is fullfilled.

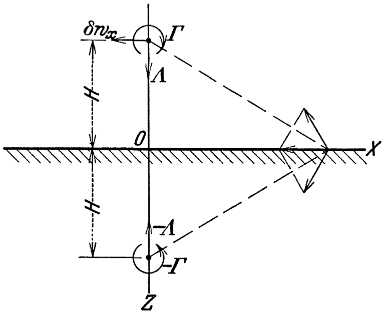


Fig. 92.

It is clear that the velocity and the pressure in the actual part of the field will now have values differing from those obtained when the boundary plane was absent, since everywhere they include elements arising from the forces of the reflected system.

As a simple case take a uniformly loaded line of infinite extent parallel to the y axis, while the field is bounded below by a plane perpendicular to Oz. A section of this system with a plane parallel to the plane xOz is given in Fig. 92. Let A denote the generalized load per unit span, directed parallel to Oz; then in the reflected system we have the same load with the opposite sign. Each load system produces a vortex of the strength $\Gamma = A/\varrho V$, with circulations in opposite directions. At the original loaded line the reflected vortex induces a horizontal velocity $\delta w_x = -\Gamma/4\pi H$, where $2H$ is the distance between the two systems, or twice the distance H of the given system from the boundary plane.

The problem can now be compared to that of the biplane and thus, applying the first equation of (25.3), we find that the actual force per unit span corresponding to the generalized load Λ has the value:

$$l = \Lambda \, (1 + \delta w_x/V) = \Lambda \left(1 - \frac{\Lambda}{4 \pi \varrho V^2 H}\right) \tag{30.1}$$

If there were no reflected system we should have $l = \Lambda$. Hence for the same value of the circulation around the line, the lift is changed in the proportion $(1 - \Lambda/4 \pi \varrho V^2 H) : 1$.

If we replace the line by an airfoil of infinite span, having a chord c sufficiently small compared with H in order that we may take the velocity δw_x as constant over the airfoil profile, then, taking account of this change of the velocity, the circulation will be given by:

$$\Gamma = \frac{1}{2} (V + \delta w_x) \, c \, m \, \alpha \tag{30.2}$$

and the lift per unit span by:

$$l = \frac{1}{2} \varrho \, (V + \delta w_x)^2 \, c \, m \, \alpha \tag{30.3}$$

Here α is the (ordinary, or geometrical) angle of incidence, measured between the direction of zero lift and the x axis, while m is the coefficient introduced in (1.5). Hence for the same value of the angle of incidence, the lift is reduced in a ratio given by the square of $(1 - \Lambda/4 \pi \varrho V^2 H)$, or approximately by $(1 - l/2 \pi \varrho V^2 H)$[1].

Instead of a fixed wall we also might have considered a *free* surface, in which case we must apply the condition that the pressure at all points of the boundary has the same value (being equal to the constant pressure of the air at rest beyond the boundary). To the degree of approximation which has been accepted throughout this theory, it is permissible to apply this condition to the quantity q defined by III (6.4), instead of the actual pressure p. Indeed q and p differ by terms of the second degree in the w's and as noted in III **15**, the velocities w are correct only to the first order.

Now according to the first of III (6.7) and taking notice of the fact that k_x is zero, we have: $V \dfrac{\partial w_x}{\partial x} = -\dfrac{1}{\varrho} \dfrac{\partial q}{\partial x}$,

and hence, as q is constant along the boundary, it appears that w_x likewise must be constant. As it is zero for $x = -\infty$ it must be zero

[1] In the case of an airfoil of finite span (see **32**) there is also a change in the angle of incidence, causing an increase of the lift. Other effects causing an increase of lift come into play as soon as H is of the order of the chord, or smaller than the chord of the airfoil [see: TOMOTIKA and others, Rep. Aeron. Res. Instit. Tokyo No. 97 (Vol. VIII, p. 1, 1933); further: G. DÄTWYLER, Untersuchungen über das Verhalten von Tragflügelprofilen sehr nahe am Boden, Mitt. Inst. f. Aerodynamik E. T. H. Zürich, 1934].

everywhere along the boundary, and thus the boundary condition assumes the form[1]:

$$w_x = 0 \tag{30.4}$$

Now it is easily seen that the introduction of a symmetrical image of the force system, as was done above, has the effect of doubling the values of the tangential velocity components at the boundary, as in consequence of the symmetry the amounts deduced from the image are exactly equal to those deduced from the actual system. When, however, after having obtained the reflection of the actual system we reverse the directions of all the forces in the image, all velocity components obtained from it change sign simultaneously, so that along the boundary plane the tangential components caused by the image will exactly annihilate the tangential components caused by the actual system and (30.4) is satisfied. It can be deduced also directly from the equations of III 8 that now q is zero along the boundary. The normal component of the velocity no longer vanishes but is doubled at the boundary plane; hence the resulting velocity is no longer exactly parallel to this plane. This implies a slight deviation of the boundary surface from its original form, which is possible as it is a free (not fixed) surface; as long as the deviations are small, they are of little consequence in comparison with the principal phenomena.

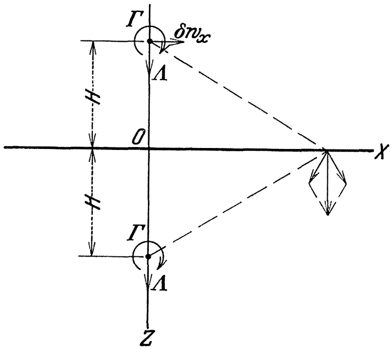


Fig. 93.

When the two cases are compared, it is seen that in the first case (simple reflection) the normal components of the forces (taken with respect to the boundary plane) are reversed, while the components parallel to this plane retain their original directions. In the second case on the contrary, where the reflection is followed by a reversion, the normal components retain their direction, while the parallel components are reversed. The same applies to the velocity components. As to the vorticity, the component normal to the boundary plane keeps the same sign in the first case, and changes sign in the second; the components parallel to the boundary plane show the opposite behavior.

In the case of the infinite loaded line parallel to Oy in the presence of a free surface boundary perpendicular to the z axis, the image is a loaded line with the force in the same direction as it has for the actual line, while also the circulation is in the same direction around both lines (see Fig. 93). The disturbing velocity δw_x at the original loaded

[1] The same result can be deduced from Bernoulli's equation if again terms of the second order in the w's are neglected. In the general treatment of the next section this method will be employed.

line now has the opposite direction to that which it had in the case of a fixed wall. Hence in the equations obtained before, (30.1)—(30.3), the sign of the correction must be reversed.

It must be noted that this rule is valid for a single reflection only; in the case of a complicated system of boundaries, when multiple reflections must be introduced (see **34**, **35**), the resulting corrections for fixed and free boundaries, though differing in sign, are no longer of the same absolute magnitude.

31. General Treatment of the Influence of a Plane Boundary. The following case will be considered: a system of generalized forces all parallel to Oz is acting in a fluid bounded by a plane perpendicular to this axis. According to a method given by von Kármán[1] it will be supposed that all the air above the plane $z = 0$ is moving with the velocity V, while the air lying below this surface is moving with another velocity V', both velocities being in the positive direction of the axis Ox. The given system of forces k_z acts in the first region (that is the part of space for which z is negative). As noted, the surface of separation between the two regions will, in general, not remain plane. We shall not enter into a discussion of the exact form of this surface, but assume that the deviations from the original position $z = 0$ remain small.

The motion in both parts of the space is assumed to be steady, as was done in all foregoing considerations; hence the form of the surface of separation will not depend on the time. For convenience, quantities relating to the lower part of the space will be distinguished by primes. The conditions to be fulfilled by the motion of the air along the surface of separation are that the normal components on both sides shall be zero, and that the pressures on both sides shall be equal. When the form of the surface is given by the equation

$$\psi(x, y, z) = 0$$

then the first condition is expressed by the pair of equations:

$$\left.\begin{aligned} (V + w_x)\frac{\partial \psi}{\partial x} + w_y \frac{\partial \psi}{\partial y} + w_z \frac{\partial \psi}{\partial z} &= 0 \\ (V' + w'_x)\frac{\partial \psi}{\partial x} + w'_y \frac{\partial \psi}{d y} + w'_z \frac{\partial \psi}{\partial z} &= 0 \end{aligned}\right\} \tag{31.1}$$

Both $\partial\psi/\partial x$ and $\partial\psi/\partial y$ will be of the same order of magnitude as the w's; hence neglecting squares and products of small quantities, these equations can be simplified into:

$$\left.\begin{aligned} V \frac{\partial \psi}{\partial x} + w_z \frac{\partial \psi}{\partial z} &= 0 \\ V' \frac{\partial \psi}{\partial x} + w'_z \frac{\partial \psi}{\partial z} &= 0 \end{aligned}\right\} \tag{31.2}$$

---

[1] von Kármán, Th., Beitrag zur Theorie des Auftriebes, Vorträge aus dem Gebiete der Aerodynamik usw., p. 95, Aachen, 1929.

Eliminating the ratio of $\partial\psi/\partial x$ to $\partial\psi/\partial z$, we have[1]:

$$w_z/V = w_z'/V' \tag{31.3}$$

Though this condition strictly applies at the points of the surface of separation, we shall assume that it holds with sufficient accuracy when for w_z and w_z' we take the values obtained by putting $z = 0$ in the expressions for these components.

Coming to the second condition, we remark that as the surface of separation is formed by stream-lines and as the motion is steady, the pressure can be calculated with the aid of Bernoulli's equation. If p_0 denotes the pressure at infinity, we have along the upper side of the surface:

$$p + \frac{\varrho}{2}\left[(V + w_x)^2 + w_y^2 + w_z^2\right] = p_0 + \frac{1}{2}\varrho V$$

and along the lower side:

$$p' + \frac{\varrho}{2}\left[(V' + w_x')^2 + w_y'^2 + w_z'^2\right] = p_0 + \frac{1}{2}\varrho V'^2 \tag{31.4}$$

In connection with the simplifications already introduced, we neglect the second powers of the w's; then the condition of equality of pressure reduces to:

$$V w_x = V' w_x' \tag{31.5}$$

We assume again that this condition can be applied to the plane $z = 0$.

In passing it may be remarked that when the points of application of the forces k_z are all at a finite distance from the surface of separation, then the motion in the neighborhood of this surface will be irrotational and will depend wholly on a potential. Writing φ for the potential in the upper part of the space, φ' for the potential in the lower part, we deduce from (31.5), on integrating with respect to x:

$$V\varphi = V'\varphi' \tag{31.6}$$

There is no constant of integration, as at very large distances upstream of the force system both φ and φ' tend to zero. This result will be of use in later deductions (see **36**, **38**, *seq.*).

It is easy now to return from the general case to that of a fixed boundary. In this case we have the condition that the normal component of the velocity must become zero. Hence, as the boundary is perpendicular to the z axis:

$$w_z = 0 \tag{31.7}$$

Now it will be immediately seen that this condition can be obtained from (31.3) by taking an infinite value for the velocity V'. In that case (31.5) leads to $w_x' = 0$, which may be interpreted as indicating that no disturbances can penetrate into the space below the plane $z = 0$.

When on the other hand V' is taken equal to zero, we come to the case of a free surface between moving air and air at rest. In that case (31.5) gives:

$$w_x = 0 \tag{31.8}$$

[1] In the paper quoted before (p. 242), VON KÁRMÁN uses the boundary condition $w_z = w_z'$; equation (31.3) represents the correct relation.

while (31.3) reduces to $w_z' = 0$, indicating again that no motion is present below the plane $z = 0$.

Having obtained in this way the general form of the boundary conditions to be applied, we proceed to investigate how it is possible to satisfy them. In the absence of any surface of separation the motion would be determined by the equations developed in Chapter III for the infinite field, the values of the components w_x, w_y, w_z being given by III (23.1)—(23.3). Applying them to the plane $z = 0$, the values of w_x, w_z, to be denoted in this case by w_{x0} and w_{z0}, would then be given by:

$$\left.\begin{aligned} w_{x0} &= \frac{1}{\varrho V}\int\!\!\int\!\!\int d\xi\, d\eta\, d\zeta\, k_z \frac{\zeta}{4\pi r^3} \\ w_{z0} &= \frac{1}{\varrho V}\int\!\!\int\!\!\int d\xi\, d\eta\, d\zeta\, k_z \frac{-[(x-\xi)^2+(y-\eta)^2](r-x+\xi)+\zeta^2 r}{4\eta r^3 (r-x+\xi)^2} \end{aligned}\right\} \quad (31.9)$$

In order to obtain the motion in the upper part of the space when the boundary is present, we assume a symmetrical image of the system of forces k_z in the lower part, the intensity of which is reduced in the proportion ν. This fictitious system will produce certain components δw_x, δw_y, δw_z in the upper space. The values of δw_x, δw_z at the points of the plane $z = 0$ are obtained from the expressions (31.9) by replacing ζ by $-\zeta$, k_z by $-k_z$ and introducing the factor ν. The values of x, y, ξ, η must not be changed; hence also r retains its value, which is evident, as the distance of a point of the image to a point of the surface $z = 0$ is equal to the distance of the corresponding point of the actual system to the same point of the boundary. The simultaneous change of sign applied to ζ and k_z leaves the value of w_x the same as before, while the sign of w_z is reversed. Hence we obtain:

$$\delta w_x = \nu w_{x0}, \qquad \delta w_z = -\nu w_{z0} \qquad (31.10)$$

Regarding the motion in the lower half of the space, we assume that it is produced by a system of forces identical in aspect and position with the original system, but having an intensity reduced in the proportion ν'. We further substitute V' for V in the denominators of the general formulae, in order to bring them into the necessary relation with the motion originally present in this part of space. In this way we obtain at the surface $z = 0$:

$$w_x' = \nu' (V/V')\, w_{x0}, \qquad w_z' = \nu' (V/V')\, w_{z0} \qquad (31.11)$$

If now we put in the upper space:

$$w_x = w_{x0} + \delta w_x, \qquad w_z = w_{z0} + \delta w_z \qquad (31.12)$$

and substitute the results in the boundary conditions (31.3), (31.5), the following equations are obtained:

$$\left.\begin{aligned} &\text{from (31.3)} \qquad (1-\nu)/V^2 = \nu'/V'^2 \\ &\text{from (31.5)} \qquad 1+\nu = \nu' \end{aligned}\right\} \qquad (31.13)$$

These equations can be solved for ν and ν'; they give:

$$\nu = \frac{V'^2 - V^2}{V'^2 + V^2}\,, \qquad \nu' = \frac{2\,V'^2}{V'^2 + V^2} \tag{31.14}$$

In the case of a fixed wall, which is obtained by taking $V' = \infty$, we have:

$$\nu = 1 \tag{31.15}$$

indicating simple reflection both of the position and of the direction of the forces, without change of intensity. In the case of a free surface, to be obtained by taking $V' = 0$, we find:

$$\nu = -1 \tag{31.16}$$

In this case the points of application of the forces are reflected into the boundary, but the direction of the forces is the same as in the actual system.

When V' is equal to V, which means absence of any surface of discontinuity, we have $\nu = 0$, $\nu' = 1$.

32. Disturbing Velocities Experienced by the Original System. We restrict ourselves to the consideration of the motion in the upper half of the space, and ask for the magnitude of the disturbing velocities which are observed at the place of the original system. For simplicity we take the case of the single wing.

As mentioned in **30** the combination of the actual wing and the image can be considered as a biplane system and the formulae developed in **21** *seq.* can be applied. The system is of zero stagger. In calculating the influence experienced by the actual wing we replace both wings by uniformly loaded lines of reduced span 2β (the same for both). Having regard to the position of the reflected system and to the reversal of k_z, and inserting the factor ν, we obtain for the mean value $\delta\,\overline{w}_x$ of the horizontal additional velocity at the place of the actual wing [see (25.5)]:

$$\delta\,\overline{w}_x = \frac{-\nu K}{2\,\pi\,\varrho\,V\,b^2}\,\sigma' \tag{32.1}$$

where according to (25.7):

$$\sigma' = \frac{b^2}{4\beta^2}\left(\sqrt{1 + \frac{\beta^2}{H^2}} - 1\right) \tag{32.2}$$

since we must put: $b_1 = b_2$, $\beta_1 = \beta_2$, $h = 2H$, $f = 0$, $\cos\beta = 1$. The mean value $\delta\,\overline{w}_z$ of the vertical velocity due to the image of the actual system is given by (25.9):

$$\delta\,\overline{w}_z = \frac{-\nu K}{2\,\pi\,\varrho\,V\,b^2}\,\sigma \tag{32.3}$$

where on account of the zero stagger σ reduces to the value given by (22.10). In the present case it takes the form:

$$\sigma = \frac{b^2}{4\beta^2}\,\log\sqrt{1 + \frac{\beta^2}{H^2}} \tag{32.4}$$

For small values of the ratio β/H (or what comes to the same, of the ratio semi-span/distance from the boundary), σ and σ' are nearly equal.

When in (32.1) and (32.3) the difference between K and the actual lift L is neglected, they can be brought into the form:

$$\frac{\delta \overline{w}_x}{V} = -\nu \sigma' \frac{C_L}{\pi \lambda} \tag{32.5}$$

$$\frac{\delta \overline{w}_z}{V} = -\nu \sigma \frac{C_L}{\pi \lambda} \tag{32.6}$$

The same as in the problem investigated in **26**, the latter quantity represents a correction to be applied in calculating the effective angle of incidence. If we take the case of a fixed wall, so that $\nu = +1$, then the effective angle of incidence is increased by the amount $\sigma C_L/\pi \lambda$; in the case of a free surface it is decreased by the same amount. Keeping to the first case the effective angle of incidence assumes the value:

$$\bar{i} = \alpha - \frac{C_L}{\pi \lambda} (1 + \tau - \sigma) \tag{32.7}$$

The change of the lift coefficient of the wing in this case can be deduced approximately from (26.18), if we replace k_{10} and k_1 by C_L, k_2 by $-C_L$, σ_1 by σ and s_2 by $1/\pi \lambda$. With $p \cong 1 + m/\pi \lambda$ we obtain:

$$\triangle C_L = \frac{C_L}{\pi \lambda} \left\{ \sigma \frac{d C_L}{d \alpha} - \frac{2 + m/\pi \lambda}{1 + m/\pi \lambda} \sigma' C_L \right\} \tag{32.8}$$

The change of the lift is found from this expression after multiplication by $(1/2)\ \varrho\ V^2 S$.

The induced resistance is calculated most conveniently by starting from the generalized forces, as in the considerations of **21** *seq.* Taking account of the quantity δ introduced in **4**, the following expression is obtained:

$$D_i = \frac{K^2}{2 \varrho V^2 \pi b^2} (1 + \delta - \sigma) \tag{32.9}$$

As the relation between K and L is determined by:

$$L = K (1 + \delta \overline{w}_x / V) = K (1 - \sigma' C_L/\pi \lambda),$$

the coefficient of induced drag assumes the value:

$$C_{Di} = \frac{C_L^2}{\pi \lambda} \left(1 + 2 \sigma' \frac{C_L}{\pi \lambda}\right) (1 + \delta - \sigma) \tag{32.10}$$

Hence the increase of the coefficient of induced drag as compared with the value in the absence of the boundary is given:
(*a*) for the same value of the lift coefficient by:

$$(\triangle C_{Di})_{C_L = \text{const.}} = -\frac{C_L^2}{\pi \lambda} \left(\sigma - 2 \sigma' \frac{C_L}{\pi \lambda}\right) \tag{32.11}$$

(*b*) for the same value of the geometrical angle of incidence by:

$$(\triangle C_{Di})_{\alpha = \text{const.}} = -\frac{C_L^2}{\pi \lambda} \left[\sigma \left(1 - \frac{2}{\pi \lambda} \frac{d C_L}{d \alpha}\right) + \frac{2 \sigma' C_L}{\pi \lambda} \frac{1}{1 + m/\pi \lambda}\right] \tag{32.12}$$

In practical calculations, to these amounts must be added the change of the coefficient of profile drag corresponding to the change of the effective angle of incidence.

The corrections depending on σ' in the expressions for $C_{D\,i}$ are of relatively small importance, as they are multiplied by $2\,C_L/\pi\,\lambda$, which usually will be of the order 1/10. Also in (32.8) the term $\sigma\, d\, C_L/d\,\alpha$ is of more importance than the second term.

It may be of interest to note that in the present case of a fixed boundary the integral of the pressure over the boundary is equal to the load supported by the system. To the degree of approximation employed in the present work, this can be readily demonstrated from III (8.3) for the value of q. It can also be obtained as a consequence of the general theorem on the conservation of momentum.

It must be remarked finally that if instead of a fixed wall *below* the airfoil a fixed wall *above* it should be taken, σ' *changes sign*. This is due to the presence of the factor $-z$ in (25.4), which is positive when the "inducing system" lies below the one where the additional velocity is experienced, but is negative in the opposite case.

33. Case of a Plane Boundary Perpendicular to the Axis $O\,y$. The calculation of the additional velocities in the case of a boundary perpendicular to the y axis, the k forces remaining as before parallel to the axis $O\,z$, proceeds in the same manner as for the preceding case.

Equation (31.3) in this case must be replaced by:

$$w_y/V = w'_y/V' \tag{33.1}$$

while (31.5) can be taken over without change:

$$V\,w_x = V'\,w'_x \tag{33.2}$$

For convenience we take the boundary in the plane $y = 0$, supposing that the whole system of forces, or the airfoil, lies at one side of this plane, say on the positive side. From III (23.1)—(23.3) we obtain the following expressions for the values of w_x and w_y at the points of the plane $y = 0$, in so far as they depend on the actual forces:

$$\left.\begin{aligned} w_{x\,0} &= \frac{1}{\varrho\,V}\int\!\!\int\!\!\int d\xi\,d\eta\,d\zeta\,k_z\,\frac{-(z-\zeta)}{4\,\pi\,r^3} \\ w_{y\,0} &= \frac{1}{\varrho\,V}\int\!\!\int\!\!\int d\xi\,d\eta\,d\zeta\,k_z\,\frac{-\eta\,(z-\zeta)\,(2\,r - x + \xi)}{4\,\pi\,r^3\,(r - x + \xi)^2} \end{aligned}\right\} \tag{33.3}$$

Now introduce the reflection of the system in the plane $y = 0$. This reflection leaves the direction of the forces k_z unchanged, while for every η is substituted $-\eta$. Hence the additional velocity components become, taking account of the factor ν:

$$\delta\,w_x = \nu\,w_{x\,0}\,, \qquad \delta\,w_y = -\,\nu\,w_{y\,0} \tag{33.4}$$

We again derive the motion on the other side of the plane $y = 0$ from a system having the position of the actual system, with intensities reduced in the proportion ν', and acting in a fluid moving with the velocity V'. Then at the plane $y = 0$ the following values of w'_x and w'_y are obtained:

$$w'_x = \nu'\,(V/V')\,w_{x\,0}\,, \qquad w'_y = \nu'\,(V/V')\,w_{y\,0} \tag{33.5}$$

As before we put:

$$w_x = w_{x\,0} + \delta\, w_x, \qquad w_y = w_{y\,0} + \delta\, w_y \tag{33.6}$$

Substituting the results in (33.1) and (33.2) it is seen that:

$$\left.\begin{aligned}(1 - \nu)/V^2 &= \nu'/V'^2\\ 1 + \nu &= \nu'\end{aligned}\right\} \tag{33.7}$$

from which: $$\nu = \frac{V'^2 - V^2}{V'^2 + V^2}, \quad \nu' = \frac{2\,V'^2}{V'^2 + V^2} \tag{33.8}$$

as before. In the case of a fixed wall we again have $\nu = +1$; and in the case of a free surface $\nu = -1$.

As an example take the case of a single loaded line (a single airfoil), with its plane of symmetry at the distance B from the plane $y = 0$ (see Fig. 94). The only additional velocity present at the points of the actual airfoil in that case is the vertical velocity $\delta\, w_z$ due to the trailing vortices from the reflected system. In order to obtain an estimate of the magnitude of this additional velocity, we assume constant load per unit length for both airfoils, reducing, as before, the span to 2β.

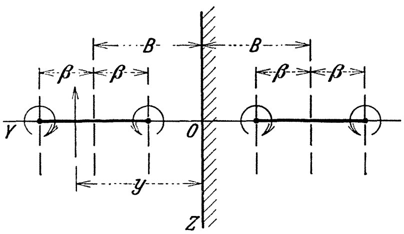


Fig. 94.

The vertical velocity at a point y of the actual airfoil due to the two trailing vortices of the image is (taking account of the factor ν):

$$\frac{-\nu K}{8\,\pi\,\beta\,\varrho\, V}\left[\frac{1}{y + B - \beta} - \frac{1}{y + B + \beta}\right],$$

where K is the total load, which is equal to the lift L. The mean value of this expression over the segment from $y = B - \beta$ to $y = B + \beta$ is:

$$\delta\, \overline{w}_z = \frac{-\nu L}{16\,\pi\,\beta^2\,\varrho\, V}\, log\, \frac{B^2}{B^2 - \beta^2} \tag{33.9}$$

Introducing a new constant σ'', defined by:

$$\sigma'' = \frac{b^2}{8\,\beta^2}\, log\, \frac{1}{1 - \beta^2/B^2} \tag{33.10}$$

we have: $$\frac{\delta\, \overline{w}_z}{V} = -\nu\, \frac{C_L}{\pi\,\lambda}\, \sigma'' \tag{33.11}$$

The effective angle of incidence thus becomes:

$$\bar{i} = \alpha - \frac{C_L}{\pi\,\lambda}\,(1 + \tau - \nu\,\sigma'') \tag{33.12}$$

while the coefficient of induced resistance takes the value:

$$C_{D\,i} = \frac{C_L^2}{\pi\,\lambda}\,(1 + \delta - \nu\,\sigma'') \tag{33.13}$$

The velocity in the direction of the x axis is not affected by the presence of the image in this case, as we have considered only a single loaded line. In the general case of a system of forces, distributed arbitrarily through space, it must be expected, however, that a component $\delta \overline{w}_x$ will arise.

Equation (33.13) shows that the induced resistance is diminished in the case of a fixed wall ($\nu = 1$), whereas it is increased in the case of a free surface.

When we compare the result given by (33.13) *e.g.* for $\nu = 1$, with that expressed by (32.10) for a horizontal boundary, taking the value of B in the one case equal to that of H in the other and neglecting the term with σ' in (32.10), it will be seen that the difference is small, as σ'' does not much differ from σ.

34. Boundaries Composed of Systems of Plane Surfaces. When the boundary of the field of motion is composed of more than one plane surface, all being parallel of course to the direction of the general motion, it is necessary to introduce repeated reflections, in the same way as if the surfaces were a system of plane mirrors. It will be seen that in general an infinite system of images will be obtained in this way. If the problem is taken in the general form considered in **31—33**, it must be noted that each reflection introduces a factor ν. Hence an image which is obtained by two reflections has its intensity multiplied by ν^2, an image which is obtained by three reflections has its intensity multiplied by ν^3, etc. For the applicability of the method it is necessary that no image falls into the actual field; but even then the determination of the resultant effect on the flow near the original system may require the calculation of the sum of a very complicated infinite series.

The cases usually considered are those of two parallel plane walls, in which case we have a linear system of images, leading to a simple infinite series, and the case of four boundaries, forming a rectangular prism, in which case the system of images extends to infinity in two directions and the corresponding series becomes a so-called doubly infinite series. The boundaries will be assumed either fixed or free, so that ν is either $+ 1$ or $- 1$, as for arbitrary values of ν difficulties arise.

In the following examples the force system will consist of a single loaded line only, parallel to the axis Oy and lying symmetrically between the walls[1].

A. Two horizontal boundaries. Assume the loaded line to be a part of the y axis, and suppose that the boundaries are given respectively by the planes $z = + H$ and $z = - H$. An infinite series of images due to repeated reflection is then obtained (see Fig. 95). These images lie in the plane Oyz at the heights:

$$z = 2\,nH \tag{34.1}$$

Uneven values of n correspond to images obtained by an uneven number of reflections and consequently having the forces in the opposite direction

[1] Compare GLAUERT, H., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 867, 1923.

if $\nu = +1$, whereas even values of n correspond to images obtained by an even number of reflections, having the forces in the original direction for $\nu = +1$. The number n can assume all positive and negative integer values with the exception of zero.

The additional horizontal velocities due to images with equal positive and negative values of n cancel at the place of the actual loaded line[1]; hence there is no resultant horizontal velocity δw_x. It follows that $L = K$. In order to calculate the additional vertical velocity, put:

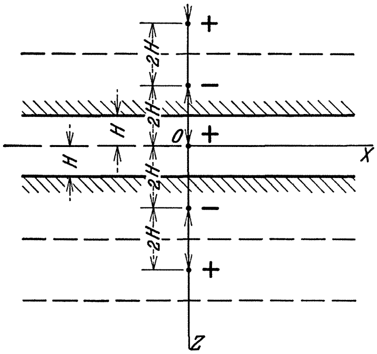


Fig. 95.

$$\sigma_n = \frac{b^2}{8\beta^2} \log \left(1 + \frac{\beta^2}{n^2 H^2}\right) \qquad (34.2)$$

[see (32.4)]. Taking account of the factors $\nu, \nu^2, \ldots$, the resultant additional vertical velocity at the place of the loaded line is then determined by the series:

$$\delta \overline{w}_z = \frac{L}{2\pi\varrho V b^2} 2 \sum_1^\infty (-\nu)^n \sigma_n \qquad (34.3)$$

as, on account of the symmetry of the whole system, the images for equal positive and negative values of n can be taken together. Instead of the exact summation of a series of this type, the following procedure may be employed. For large values of n we put approximately:

$$\sigma_n \cong \frac{b^2}{8 n^2 H^2} \qquad (34.4)$$

If these values are used instead of the exact ones, the series reduces to:

$$\sum_1^\infty (-\nu)^n \frac{b^2}{8 n^2 H^2}.$$

In mathematical treatises it is demonstrated that $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence for $\nu = -1$ (free surfaces) we have immediately:

$$\sum_1^\infty \frac{b^2}{8 n^2 H^2} = \frac{\pi^2}{48} \frac{b^2}{H^2} \qquad (34.5)$$

When $\nu = +1$, so that the terms of the series have alternating signs, we write:

[1] In the case of an airfoil of finite thickness there is, of course, an effect due to the decrease of the passage of the air above and below the airfoil. This point will not be considered; an estimate of its effect might be obtained by investigating the two-dimensional case.

$$\sum_1^\infty (-1)^n \frac{1}{n^2} = -\sum_1^\infty \frac{1}{n^2} + 2\sum_1^\infty \frac{1}{(2n)^2} = -\frac{1}{2}\sum_1^\infty \frac{1}{n^2} = -\frac{\pi^2}{12},$$

and thus:
$$\sum_1^\infty (-1)^n \frac{b^2}{8\,n^2 H^2} = -\frac{\pi^2}{96}\frac{b^2}{H^2} \tag{34.6}$$

Now the differences between the actual values of the σ's and the approximation (34.4) can be of importance for low values of n only. It is always possible to calculate these differences numerically, and their sum can be determined in a direct way with sufficient accuracy. This sum can then be added to the results given in (34.5) and (34.6).

In order to obtain an estimate of the influence of the system of images we neglect this correction, and use (34.6), or (34.5). The effective angle of incidence will then be given in the case of fixed boundaries by:

$$i = \alpha - \frac{C_L}{\pi\lambda}\left(1 + \tau - 0.206\,\frac{b^2}{H^2}\right) \tag{34.7}$$

and in the case of free boundaries by:

$$i = \alpha - \frac{C_L}{\pi\lambda}\left(1 + \tau + 0.411\,\frac{b^2}{H^2}\right) \tag{34.8}$$

A corresponding change is obtained in the coefficient of induced resistance[1].

B. Vertical boundaries. In the case of two vertical boundaries given respectively by the planes $y = +B$ and $y = -B$, we have a series of images which for positive ν are all of the same sign. In determining the additional vertical velocity, coefficients σ_n'' must be used, given by:

$$\sigma_n'' = \frac{b^2}{8\,\beta^2}\,log\,\frac{1}{1 - \beta^2/n^2 B^2} \tag{34.9}$$

---

[1] The above results are not valid if b is too great (*e. g.* greater than H), as in such cases neither formula (34.2) for σ, nor the approximation (34.4) can be used.

The limiting case of an airfoil of infinite span between two horizontal plane boundaries is of interest in connection with certain experimental arrangements. It has been investigated by Prandtl (Tragflügeltheorie, II, No. 13), who has considered also the curvature of the stream-lines in a plane parallel to Oxz, which is given by:
$$\frac{1}{R} = \frac{\partial}{\partial x}\left(\frac{w_z}{V}\right).$$

The following results are taken over from Prandtl:

fixed boundaries: $\delta w_z = 0$, $\quad \dfrac{1}{R} = -\dfrac{\pi C_L c}{96\,H^2}$;

free boundaries: $\delta w_z = \dfrac{l}{4\varrho V H}$, $\quad \dfrac{1}{R} = +\dfrac{\pi C_L c}{48\,H^2}$,

l being the lift supported per unit span. The value of δw_z in the latter case is obtained by the application of the theorem of momentum: far behind the airfoil the whole airstream will possess the downward vertical velocity $2\,\delta w_z$, so that the downward momentum generated in unit time, per unit span, amounts to $4\,\varrho\,VH\,\delta w_z$ (the distance of the boundaries being $2\,H$).

For large values of n these approximate to:

$$\sigma_n'' \cong \frac{b^2}{8\,n^2 B^2} \tag{34.10}$$

The additional velocity now has the value:

$$\delta \overline{w}_z = -\frac{L}{2\pi\varrho V b^2}\, 2 \sum_1^\infty \frac{\nu^n b^2}{8\,n^2 B^2} \tag{34.11}$$

apart from a correction due to the differences between the actual values of the quantities σ_n'' and the values (34.10). The summation of the series proceeds as before. When the correction is neglected, the effective angle of incidence becomes:

for the case of fixed boundaries:

$$i = \alpha - \frac{C_L}{\pi\lambda}\left(1 + \tau - 0.411\,\frac{b^2}{B^2}\right) \tag{34.12}$$

and for the case of free boundaries:

$$i = \alpha - \frac{C_L}{\pi\lambda}\left(1 + \tau + 0.206\,\frac{b^2}{B^2}\right) \tag{34.13}$$

35. Case of Four Boundaries Forming a Rectangular Prism. This case is of great importance in windchannel work, as in various laboratories channels are used having a rectangular section. The system of images now becomes doubly infinite; a diagrammatic sketch of such a system is given in Fig. 96, where the plus and minus signs respectively denote the images with the forces in the original direction and those with the forces in the reversed direction. If the point O at the center of the cell is considered as representing a vortex pair, then considering the reflections in the walls of the cell, the distribution of (+) and (—) units will be readily seen to develop as in the diagram. Provisionally we take $\nu = +1$.

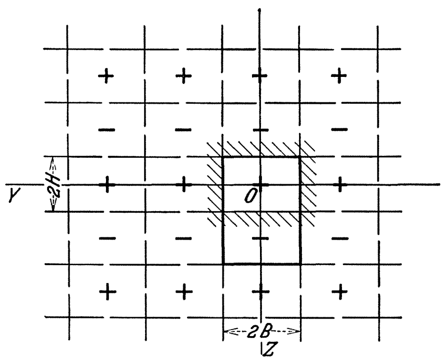


Fig. 96.

We need now an expression for the mean induced velocity at the place of the original system, set up by any one of these images. If we take the case of an airfoil with very small span in comparison with the dimensions of the channel, the velocity at the point $y = 0$, $z = 0$ induced by the image having its center at:

$$\eta = 2\,m\,B, \qquad \zeta = 2\,n\,H \tag{35.1}$$

is approximately:

$$(-1)^n \frac{L}{4\pi\varrho V}\, \frac{-\eta^2 + \zeta^2}{(\eta^2 + \zeta^2)^2} = (-1)^n \frac{L}{4\pi\varrho V}\, \frac{-m^2 B^2 + n^2 H^2}{4\,(m^2 B^2 + n^2 H^2)^2} \tag{35.2}$$

as can be deduced from III (23.9) or from III (25.13). Summing the terms due to all images, we have for the ratio of the additional velocity to V, the expression:

$$\frac{\delta w_z}{V} = \frac{-L}{2\varrho V^2 \pi B^2} \sum_{(m)} \sum_{(n)} (-1)^n \frac{m^2 - n^2 (H/B)^2}{8\,[m^2 + n^2 (H/B)^2]^2} \tag{35.3}$$

where the summation must be extended over all positive and negative values of both m and n, with the exception of the pair of values 0,0.

The sum of this doubly infinite series has been evaluated by Glauert by making use of the development for the cotangent and applying a number of transformations, leading finally to the equation:

$$\Sigma\,\Sigma\,(-1)^n \frac{m^2 - n^2 (H/B)^2}{8\,[m^2 + n^2 (H/B)^2]^2} = \frac{\pi^2}{24} + \pi^2 \sum_{p} \frac{p}{1 + e^{2\pi p H/B}} \tag{35.4}$$

The series remaining in the last expression must be extended over all positive integer values of p, but it converges so rapidly, that it is sufficient to retain its first term only, hence numerical values can be obtained without difficulty. In this way the expression for the correction to be applied in calculating the effective angle of incidence becomes:

$$\delta\varphi = \frac{\delta w_z}{V} = -C_L \frac{S}{S'} \frac{\pi H}{B} \left[\frac{1}{24} + \frac{1}{1 + e^{2\pi H/B}}\right] \tag{35.5}$$

where S is the area of the airfoil and $S' = 4BH$ is the cross-section of the channel.

Instead of reproducing the reduction given by Glauert it may be of interest to give a somewhat different one, making use of the properties of elliptic functions and leading to a result in finite terms. We first note that the field of motion taken together with its doubly infinite system of images as pictured in Fig. 96 is doubly periodic, the periods being $2B$ in the horizontal direction and $4H$ in the vertical direction. As moreover in the case considered the circulation along the circumference of every cell (and even of every half cell of height $2H$ and breadth $2B$) is zero, we must expect that the potential of the field likewise will be doubly periodic. The stream function must show the same character, as there is no outflow from any of the cells. Hence it can be concluded that the complex potential χ of the motion must be a doubly periodic function of the complex variable $y + iz$, having the real period $2B$ and the imaginary period $4iH$. Now according to the theory of elliptic functions such a function is completely determined by the position of its poles and by the values of the coefficients associated with these poles.

In the case at hand, taking the period cell consisting of the section of the actual channel and of the image of this region lying immediately below it (in Fig. 96 this cell is marked by a heavy contour line) the complex potential has a singularity at the point O (representing the original infinitesimal loaded element), and a second one at the point $+2iH$ (the image of the element in the lower boundary of the channel). In the immediate neighborhood of the first point the principal term of the vertical velocity is given by the expression [see III (25.13)]:

$$\frac{L}{4\pi\varrho V}\frac{-y^2+z^2}{(y^2+z^2)^2} \tag{35.6}$$

The potential of the field thus presents a term:

$$\frac{L}{4\pi\varrho V}\frac{-z}{y^2+z^2} \tag{35.7}$$

and the corresponding term of the complex potential χ must be

$$\frac{L}{4\pi\varrho V}\frac{-i}{y+iz} \tag{35.8}$$

The latter expression defines the nature of the singularity at the point O, and shows that it is a simple pole, the coefficient associated with it having the value $-iL/4\pi\varrho V$. At the other pole, $y+iz=2iH$, the same coefficient occurs with the opposite sign.

Now from the theory of elliptic functions it is known that, taking ξ as a complex variable, the function $sn\,\xi$ is a doubly periodic function of ξ, with the periods $4K$, $2iK'$. This function changes sign when ξ is increased by an uneven multiple of $2K$, and has simple poles at all points $\xi = iK' + 2mK + 2niK'$. Hence according to general theorems deduced in the theory of such functions it must be possible to express the complex potential χ as an sn function. On account of the periodicity relations and the position of the poles the connection between ξ and $y+iz$ then must be chosen in such a way that

$$\xi = i\frac{y+iz}{B}K' + iK' \tag{35.9}$$

provided that the modulus k of the elliptic function is adjusted to fulfill the relation

$$H/B = K/K' \tag{35.10}$$

K and K' being the complete elliptic integrals of the first kind. The expression for χ thus becomes:

$$\chi = A\,sn\left(iK' + i\frac{y+iz}{B}K'\right) \tag{35.11}$$

where A is a constant factor.

The development of the function $sn\,\xi$ in the neighborhood of its poles, for instance in the neighborhood of the point $\xi = iK'$, is given in various treatises[1]. If for convenience we put $\xi = iK' + \xi_1$, it begins with the terms: $sn(iK'+\xi_1) = \frac{1}{k\xi_1} + \frac{1+k^2}{6k}\xi_1 + \dots$

Substituting for ξ_1 the value $i(y+iz)K'/B$ and multiplying by A, this development takes the form:

$$-i\frac{AB}{(y+iz)kK'} + \frac{i(y+iz)(1+k^2)K'A}{6kB} + \dots \tag{35.12}$$

[1] See WHITTAKER, E. T. and WATSON, G. N., Modern Analysis, p. 504, Cambridge University Press.

Hence comparing with (35.8) we see that in order to obtain full equality between the function $A\, sn\, \xi$ and the complex potential it is necessary to take

$$A = \frac{L\, k\, K'}{4\,\pi\,\varrho\, V\, B} \tag{35.13}$$

Now the first term of the development (35.12) evidently represents the part of the field due to the actual system at the origin. When this term is removed from the series the remainder must represent the complex potential of the field due to the combined action of all the images. By taking the real part of this remainder we obtain the ordinary potential, from which the velocity components are derived by differentiating with respect to y and z.

As we only need the induced velocity at the point $y = 0$, $z = 0$, we need not go beyond the first term of this potential, which is given by:

$$-z\,\frac{L\,(1+k^2)\,K'^2}{24\,\pi\,\varrho\,V\,B^2} = -z\,\frac{L\,(1+k^2)\,K\,K'}{24\,\pi\,\varrho\,V\,B\,H},$$

thus leading to the vertical velocity:

$$\delta\, w_z = -\frac{L\,(1+k^2)\,K\,K'}{24\,\pi\,\varrho\,V\,B\,H} \tag{35.14}$$

Hence the correction to be applied to the angle of incidence becomes:

$$\delta\,\varphi = -\frac{L\,(1+k^2)\,K\,K'}{24\,\pi\,\varrho\,V^2\,B\,H} = -C_L\,\frac{S}{S'}\,\frac{(1+k^2)\,K\,K'}{12\,\pi} \tag{35.15}$$

In the case of free boundaries ($\nu = -1$) the calculation can be done in an analogous way. It is found that (35.14) and (35.15) change sign, while at the same time the modulus k of the elliptic function must be determined from the condition:

$$H/B = K'/K \tag{35.16}$$

which is the inverse of (35.10).

In the following table a few numerical results have been given for the factor of $C_L\, S/S'$ in (35.15).

TABLE 13.

H/B	1/4	1/2	$1/\sqrt{2}$	1	$\sqrt{2}$	2	4
Fixed boundaries	—0.262	—0.137	—0.119	—0.137	—0.183	—0.262	—0.523
Free boundaries	+0.523	+0.262	+0.183	+0.137	+0.119	+0.137	+0.262

In consequence of a theorem known as Landen's transformation formula for elliptic functions, the expression $(1 + k^2)\, K\, K'$ in the case of fixed boundaries has the same value both for the ratio $H/B = \varkappa$ and for the ratio $H/B = 1/2\,\varkappa$ ($\varkappa$ being an arbitrary number); in the case of free boundaries it has the same value both for the ratio $H/B = \varkappa$ and for the ratio $H/B = 2/\varkappa$.

The foregoing deductions apply to the case of a loaded line of infinitesimal span placed in the axis of the channel. They can be extended to a loaded element placed at an arbitrary position; then by means of an integration we might pass to the case of any loaded system. The equations, however, become rather complicated and we shall not develop this phase of the problem[1]. In the case of a channel with circular section the problem of a loaded line of finite span assumes a more simple form; it will be discussed in **40**.

It is of interest to remark that in all cases of fixed boundaries the velocity δw_z at the loaded line is negative, in consequence of which the effective angle of incidence is increased, while the induced resistance is decreased. With free boundaries the contrary takes place. The same effect is observed with a channel of circular cross-section (see **40**). A popular explanation of this effect can be given by noting that the presence of fixed walls gives a support to the air, in consequence of which it takes up a smaller amount of kinetic energy than under ordinary conditions. In the case of an air jet bounded by free surfaces, on the contrary, the support derived from the surrounding air, now at rest, is less than if moving along with the air of the jet. Hence more kinetic energy is taken up, and the induced resistance is increased.

36. General Considerations on the Influence of Cylindrical or Prismatic Boundaries. The results developed thus far relate to relatively simple systems of plane boundaries, in which the additional field could be determined in a direct way by introducing a system of images. When the number of boundary planes increases the difficulties to be surmounted augment considerably. It is seen moreover that in the case of a cylindrical boundary with a circular section the method does not apply at all. It is of importance therefore to investigate the motion of a fluid in the presence of a boundary, from a different point of view of more general applicability.

We limit consideration again to the case of a boundary formed by generating lines parallel to the x axis (the direction of the original velocity V of the fluid). The fluid is acted upon by external forces and as before we take the case where the generalized forces are perpendicular to the x axis. We then return to III (6.7) and ask for that solution of these equations which shall fulfill the boundary conditions deduced in **31** above, substituting the normal velocity component w_n for w_z and likewise w_n' for w_z' in (31.3). The boundary conditions thus become:

[1] See Terezawa, K., On the interference of wind tunnel walls of rectangular cross section on the aerodynamical characteristics of a wing, Rep. Aeron. Res. Instit. Tokyo No. 44 (Vol. IV, p. 69, 1928); Rosenhead, C., The effect of wind tunnel interference on the characteristics of an aerofoil, Proc. Roy. Soc. London **A** 129, p. 135, 1930; and Glauert, H., Interference on the characteristics of an aerofoil in a wind tunnel of rectangular section, Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1459, 1932, where also numerical results, both for uniform and for elliptic loading, have been derived.

$$w_n/V = w_n'/V' \tag{36.1}$$
$$V\,w_x = V'\,w_x' \tag{36.2}$$

Assuming that the points of application of the generalized forces are all at a finite distance from the boundary, the second condition can also be written in the form (31.6):

$$V\varphi = V'\varphi' \tag{36.3}$$

As the theory of boundary corrections has been developed principally in relation to wind channel work, it may be assumed in the following deductions that the boundary is in the form of a cylindrical or prismatic body, completely surrounding the field of motion. Most of the results, however, apply equally to the case of a cylindrical or of a prismatic boundary lying outside of the force system; the field then extends to infinity and the cylinder or the prism forms an *internal* boundary. Such a case is obtained for example when we consider an airfoil in the presence of a cylindrical fuselage, assuming that this fuselage is composed of generating lines parallel to the velocity V and extends to infinity in both directions. As a limiting case of a cylinder we might even take a flat plate of infinite length in the directions of $+x$ and $-x$, but having a finite breadth in the perpendicular direction. In such a case it is necessary to consider both surfaces of the plate as separate parts of the boundary.

In constructing a solution of III (6.7) we start again from III (9.1), where the quantities w_y', w_z' are obtained from the second and third of III (9.4). As k_x is assumed zero, w_x' vanishes. The vorticity is not changed by the presence of the boundary and behind the region G where the generalized forces are acting, we have a system of trailing vortices γ_x, determined by the first equation of III (10.3).

The potential φ, however, is no longer given by the integral III (9.8); or, as we may better say, to this integral must be added a term which accounts for the changes caused by the presence of the boundaries and which is to be determined with the aid of the boundary conditions. As noted already in 29, the determination of the potential becomes the main object of the theory.

When x increases indefinitely in the positive direction, the velocity components w_y, w_z tend to limiting values (independent of x), which will be denoted by $w_{y\infty}$, $w_{z\infty}$ as before. The component $w_{x\infty}$ must vanish: indeed it cannot assume a constant value, nor be a function of x only, as this would be inconsistent with the equation of continuity; if it should depend on y and z, there would be vorticity components γ_y or γ_z, which is impossible, as $k_x = 0$.

The limiting values $w_{y\infty}$, $w_{z\infty}$ do not depend on the x (or ξ) coordinates of the points of application of the forces k_y, k_z from which they are derived. Indeed they are linear functions of k_y, k_z and as the contribution of every separate force is independent of x, it remains unchanged when the point of application of this force is shifted parallel to the x axis; and the same must hold for arbitrary shifts of all forces. Hence $w_{y\infty}$, $w_{z\infty}$ retain the same value, when by means of such shifts all points of application are brought into the y, z plane (unstaggered system). At large distances upstream from the region G all three components w_x, w_y, w_z vanish.

As to the quantity q which is related to the pressure p, it is best calculated from the first equation of the system III (6.7), where k_x is zero. On integrating, this equation gives:

$$q = -\varrho V w_x \tag{36.4}$$

As both q and w_x must be zero at large distance upstream of the system, there is no additive constant in this equation. In consequence of the remark made in connection with w_x it follows that q also vanishes for increasing positive values of x.

Before turning to the determination of the potential φ we shall investigate a generalization of the theorems on induced resistance given in III **16** and III **18**. It will be seen that for calculating the induced resistance of any load system in the presence of a boundary, it is sufficient to know the values of $w_{y\infty}$, $w_{z\infty}$, and hence it is only necessary to determine the limiting form of the function φ for infinite positive values of x, which will be written φ_∞.

A knowledge of the full form of the function φ is required only in cases where the additional velocities at an arbitrary point of the field must be determined; this problem will be considered in **42**.

37. Extension of the Theorem of III 16. We start from III (16.4) and integrate this equation through a space bounded as before by two planes I, II perpendicular to the x axis (the first lying at a large distance downstream from the region G, the other at a large distance upstream) and further by the boundaries already present in the field which have generating lines parallel to the x axis. When the boundaries are not formed by fixed walls, but are free surfaces, then they must be taken in the slightly distorted form, so that they consist wholly of stream-lines. For convenience these boundaries will momentarily be denoted by S; $d\,S$ will be an element of them and (n, x) the angle between the outward normal to $d\,S$ and the x axis.

The integral of III (16.4) taken over the space indicated becomes:

$$\left.\begin{aligned} &\frac{\varrho V}{2}\iint\limits_{(I)} dy\,dz\,(w_x^2+w_y^2+w_z^2) - \frac{\varrho V}{2}\iint\limits_{(II)} dy\,dz\,(w_x^2+w_y^2+w_z^2) + \\ &\qquad + \frac{\varrho V}{2}\iint d\,S\,(w_x^2+w_y^2+w_z^2)\cos(n,x) = \\ &= \iiint dx\,dy\,dz\,(k_x w_x + k_y w_y + k_z w_z) - \iint\limits_{(I)} dy\,dz\,q\,w_x + \\ &\qquad + \iint\limits_{(II)} dy\,dz\,q\,w_x - \iint d\,S\,q\,w_n \end{aligned}\right\} \tag{37.1}$$

In the first integral on the left hand side w_x is zero, while for w_y and w_z we must take the limiting values $w_{y\infty}$, $w_{z\infty}$. The second integral vanishes altogether.

On the right hand side in the first integral, the term $k_x w_x$ must be omitted as k_x is zero. The second and third integrals vanish, since both q and w_x vanish at large distances from the load system. Finally the fourth integral vanishes, as w_n is everywhere zero along the boundary S.

It remains to discuss the third integral on the left hand side, which represents a quantity not present in the case of the infinite field. It is evidently exactly zero in the case of fixed walls, as then *cos* (n, x) is zero all along S. In the case of a free surface we shall neglect it on account of its smallness; the angle (n, x) can deviate from 90^0 only by quantities of the order of magnitude of the forces; *cos* (n, x) is of the same order, and the whole integral is of the third order.

Hence we obtain the result:

$$\frac{\varrho V}{2} \iint_{(I)} dy\, dz\, (w_{y\infty}^2 + w_{z\infty}^2) = \iiint dx\, dy\, dz\, (k_y w_y + k_z w_z) \qquad (37.2)$$

or as it may be written:

$$E = A \qquad (37.3)$$

As noticed at the end of III 18 this result at once leads to Munk's first theorem, the so-called stagger theorem, which asserts that, as the value of E remains unchanged, when the points of application of the forces k_y, k_z are shifted parallel to the x axis, the same property must hold for the integral A.

As according to III (16.8) the integral A is equal to the value of the induced resistance of the load system multiplied by V, it is seen that the same as in the case of the unlimited field, the induced resistance is not affected by shifts of the points of application of the forces, parallel to the x axis.

38. Equation for the Induced Resistance. The result obtained in the preceding section makes it possible to express the induced resistance of any system in the presence of walls formed by generating lines parallel to the x axis, in the form of an integral containing the velocity components $w_{y\infty}$, $w_{z\infty}$, the same as in III 18 and III 19 for the case of the unlimited field.

In order to obtain the same equation for the case at hand, it is necessary to demonstrate again, that when by means of shifts parallel to the x axis the whole load system is reduced to a system without stagger, lying in the y, z plane, the velocities w_y, w_z in this plane are just one half of the limiting values $w_{y\infty}$, $w_{z\infty}$ for points having the same coordinates y, z. It is possible to demonstrate this theorem with the aid of certain symmetrical properties of the vortex system.

When the whole load system has been concentrated in the plane Oyz, the "transverse vortices" (bound vortices) are all located in this same plane, while the trailing vortices (γ_x) extend from this plane in the downstream direction. The field of flow corresponding to the vortex system must satisfy the boundary conditions (36.1) and (36.2) or (36.3). Now superpose on this field (which for convenience will be denoted as field I) a system of rectilinear vortices, extending from $-\infty$ to $+\infty$, coinciding for $x > 0$ with the trailing vortices of field I, but having $-1/2$ their intensity. The field of flow corresponding to these rectilinear vortices, to be denoted as field II, must be adjusted in such a way that it fulfills the same boundary conditions as the original field. In this superposed field the component w_x is zero everywhere, while the components w_y, w_z are independent of x, and are equal resp. to $-1/2$ times the values of $w_{y\,\infty}$, $w_{z\,\infty}$ of the original field for the same values of y and z.

The resulting field (field III) obtained after carrying out the superposition, is characterized by the property that its vortex system is *antisymmetrical* with respect to the plane $O\,y\,z$: it is exactly reproduced when it is reflected in this plane, with the condition that the sign of the vortices is changed at the same time. Now as the velocity components are completely determined by the vortex field taken in conjunction with the boundary conditions, the distribution of the velocity components must have a similar property: it will be reproduced when it is reflected in the plane $O\,y\,z$, after which the sign everywhere must be reversed. This produces a change in the sign of the components w_y, w_z, the direction of which was unaffected by the reflection, while the sign of the component w_x is restored to its original form. Hence it follows that the resulting field (III) satisfies the conditions:

$$w_{y\,III}(-x) = -w_{y\,III}(x), \quad w_{z\,III}(-x) = -w_{z\,III}(x) \tag{38.1}$$

$$w_{x\,III}(-x) = w_{x\,III}(x) \tag{38.2}$$

it being understood that the quantities compared refer to the same values of y and z.

From (38.1) it is immediately deduced that in the plane $O\,y\,z$ the values of w_y, w_z corresponding to field III must be *zero*. This is equivalent to the statement that the components w_y, w_z of the original field (field I) in this plane are exactly compensated by the components of field II. As the latter were equal to $-1/2\,w_{y\,\infty}$, $-1/2\,w_{z\,\infty}$ respectively, we obtain the following relation for field I:

$$(w_y)_{x=0} = \frac{1}{2}\,w_{y\,\infty}, \qquad (w_z)_{x=0} = \frac{1}{2}\,w_{z\,\infty} \tag{38.3}$$

This was the result to be demonstrated. It allows us to write the equation for the induced resistance in the form:

$$D_i = \frac{1}{2\,V}\int\!\!\int dy\,dz\,(\overline{Q}_y\,w_{y\,\infty} + \overline{Q}_z\,w_{z\,\infty}) \tag{38.4}$$

The problem of determining the induced resistance of an arbitrary system in the presence of boundaries, formed by generating lines parallel to the x axis, thus again becomes reduced to a problem of two-dimensional motion, concerning a plane parallel to the y, z plane, now bounded by a certain closed curve—the intersection of this plane with the original boundary system.

The problem can now be stated as follows: it is required to determine a plane motion, corresponding to the system of trailing vortices γ_x of field I (which vortices can be considered as known) and satisfying at a certain boundary curve the conditions:

$$w_{n\,\infty}/V = w'_{n\,\infty}/V' \tag{38.5}$$

$$V\varphi_\infty = V'\varphi'_\infty \tag{38.6}$$

It will be evident that the second boundary condition must be taken in this last form, and not in the form (36.2), as the component w_x has vanished.

We apply these conditions to the investigation of the induced resistance of a system placed in a current bounded by a cylindrical surface with circular cross section. From the given distribution of the forces we calculate the vortex system by means of III (10.3) or (13.3). We then consider a section of the field by a plane perpendicular to the x axis and determine the additional field due to the presence of a circular boundary. This can be done by introducing an image of the vortex system, as will be shown in the next section[1].

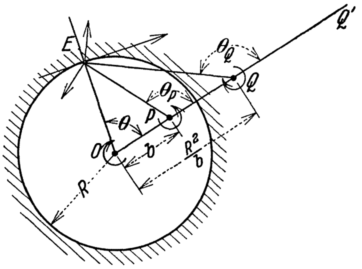


Fig. 97.

39. Image of a Vortex System with Respect to a Circular Boundary in Two-Dimensional Motion. In Fig. 97 let O be the center of the circular boundary, having the radius R; P represents the intersection of a vortex with the plane of motion. We write $OP = b$, and take the point Q on the same radius as P, so that $QO = R^2/b$. For convenience it is assumed that at P we have a vortex with strength 2π; we must then try to obtain the field within the circle by supposing that it is due to this vortex combined with another one of strength $-2\pi\nu$ at Q, and a third one

[1] Some general theorems concerning wind tunnel interference for the case of an arbitrary cross section (with special reference to elliptic sections) have been deduced by H. Glauert, Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1470, 1932. Glauert proves that the interference experienced by a very small airfoil in an open tunnel of any shape is of the same magnitude, but of opposite sign, as the interference experienced by the same airfoil, rotated through a right angle, in a closed tunnel of the same shape. This theorem, and others, are obtained by introducing the complex potential associated with the velocity field in a plane perpendicular to the x axis, and applying the methods of two-dimensional potential theory and of conformal transformation.

of the strength $+ 2\pi\nu$ at the center O. As to the field outside of the circle we shall try to obtain it by assuming a vortex of strength $2\pi\nu'$ at P.

Calculating the normal components $w_{n\infty}$, $w'_{n\infty}$ of the velocity at a point E of the circumference, and substituting them in (38.5), the following relation is obtained:

$$\frac{1}{V}\left(\frac{-\sin PEO}{PE} + \nu\frac{\sin QEO}{QE}\right) = -\frac{\nu'}{V'}\frac{\sin PEO}{PE} \tag{39.1}$$

Now on account of the similarity of the triangles OEQ and OPE: $QE/PE = R/b$, $\angle QEO = \angle OPE$; and $\sin OPE/\sin PEO = R/b$. Hence (39.1) reduces to: $(1 - \nu)/V = \nu'/V'$ (39.2)

Next consider the potentials. Apart from an additive constant the potential at the point E of the vortex at P is determined by the angle QPE, to be denoted by θ_P. In the same way the potential at E of the vortex at Q is determined by $-\nu\,\theta_Q$. Now: $\theta_Q = \angle Q'QE = \angle QEO + \angle QOE = \angle OPE + \angle QOE = \pi - \theta_P + \theta$; hence this potential becomes equal to $-\nu(\pi - \theta_P + \theta)$. The potential of the vortex at O is given by $\nu\,\theta$. Finally the potential of the exterior field due to the vortex of reduced strength assumed in P is given by $\nu'\theta_P$. Hence we obtain:

$$\left.\begin{aligned}\varphi_\infty &= \theta_P - \nu(\pi - \theta_P + \theta) + \nu\,\theta + \text{const.} = (1+\nu)\,\theta_P + \text{const.}\\ \varphi'_\infty &= \nu'\,\theta_P + \text{const.}\end{aligned}\right\} \tag{39.3}$$

Substitution in (38.6) leads to the equation:

$$V(1+\nu) = V'\nu' \tag{39.4}$$

Solving (39.2) and (39.4) for ν and ν' we obtain:

$$\nu = \frac{V'^2 - V^2}{V'^2 + V^2}, \qquad \nu' = \frac{2V'V}{V'^2 + V^2} \tag{39.5}$$

The values of ν and ν' appear to be independent of the position of the particular vortex considered. It may appear at first sight that the introduction of a vortex of strength $2\pi\nu$ at O is not allowable, as the additional field which represents the influence of the boundary must be a field of irrotational motion. However, in all cases occurring in the theory of load systems, the algebraic strength of the whole system of vortices is zero; hence when the image of the whole system is constructed, the vortices in O cancel.

The results obtained here are similar to those deduced in **31** and **33** (the difference in the factor ν' is due to the fact that in **31**, *seq.* this factor referred to the magnitude of the forces, whereas here it is used for the strength of the vortex).

In the limiting cases we obtain:

for $V' = \infty$, corresponding to the case of a *fixed wall:*

$$\nu = 1 \tag{39.6}$$

(strength of the image equal and opposite in sign to that of the original system);

for $V' = 0$, which gives the case of the *free boundary:*

$$\nu = -1 \tag{39.7}$$

(strength of the image equal and of the same sign as that of the original system).

The corrections due to the additional field in these two cases consequently are of opposite sign.

40. Application to the Case of an Airfoil with Uniform Loading. In the case of an airfoil having constant load A per unit span the trailing vortices have strength $A/\varrho V$. Hence we must introduce two vortices of absolute strength $\nu A/\varrho V$ situated at the points A', B' (see Fig. 98) at the distance R^2/b from the center O. The upward velocity due to the vortex at the left hand side at a point of the airfoil, situated at the distance y from O, is given by:

$$\frac{\nu A}{4\pi\varrho V}\frac{1}{R^2/b - y} \tag{40.1}$$

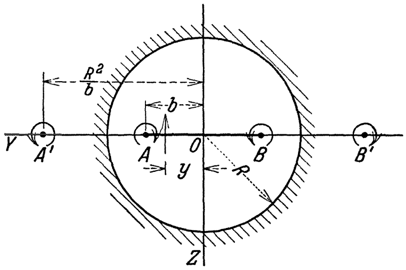


Fig. 98.

The mean value of this expression over the span amounts to:

$$\frac{\nu A}{8\pi\varrho V b}\log\frac{R^2+b^2}{R^2-b^2} \tag{40.2}$$

Hence taking together the effect due to both vortices we obtain a mean additional velocity of the magnitude:

$$\delta\overline{w}_z = -\frac{\nu A}{4\pi\varrho V b}\log\frac{R^2+b^2}{R^2-b^2} \tag{40.3}$$

When the span is not too large compared with the radius of the channel, we can use this result also for the case of an airfoil with arbitrary load distribution, remembering that we must give the equivalent airfoil of uniform loading a reduced span 2β. When K is the total generalized load, the load per unit span then becomes $A = K/2\beta$. Though we have not thus far considered the value of w_x it can be demonstrated from reasons of symmetry that in this case of a loaded line with its middle point on the axis of the channel, w_x is zero at all points of the line. Thus we may conclude that L is equal to K, and $A = L/2\beta$. Hence (40.3) may be written:

$$\delta\overline{w}_z = -\frac{\nu L}{8\pi\varrho V\beta^2}\log\frac{R^2+\beta^2}{R^2-\beta^2}$$

or, on developing the logarithm:

$$\delta\overline{w}_z = -\frac{\nu L}{4\pi\varrho V R^2}\left(1 + \frac{\beta^4}{3R^4}\cdots\right) \tag{40.4}$$

Writing $\pi R^2 = S'$ = the area of a section of the channel, we obtain for the change in the induced resistance:

$$\delta D_i = -\frac{\nu L^2}{4 \varrho V^2 S'} \left(1 + \frac{\beta^4}{3 R^4} \ldots\right) \tag{40.5}$$

while the correction to be applied to the angle of incidence has the value:

$$\delta \varphi = -\frac{\nu L}{4 \varrho V^2 S'} \left(1 + \frac{\beta^4}{3 R^4} \ldots\right) \tag{40.6}$$

Introducing the lift coefficient C_L we have:

$$\left.\begin{aligned} \delta C_{Di} &= -\nu C_L^2 \frac{S}{8 S'} \left(1 + \frac{\beta^4}{3 R^4} \ldots\right) \\ \delta \varphi &= -\nu C_L \frac{S}{8 S'} \left(1 + \frac{\beta^4}{3 R^4} \ldots\right) \end{aligned}\right\} \tag{40.7}$$

Taking especially the case of elliptic loading, for which $\beta = b\sqrt{3/4}$, we obtain the following expressions for the induced resistance coefficient and for the angle φ:

$$\left.\begin{aligned} C_{Di} &= \frac{C_L^2}{\pi \lambda} \left[1 - \nu \left(\frac{b^2}{2 R^2} + \frac{3 b^6}{32 R^6}\right)\right] \\ \varphi &= \frac{C_L}{\pi \lambda} \left[1 - \nu \left(\frac{b^2}{2 R^2} + \frac{3 b^6}{32 R^6}\right)\right] \end{aligned}\right\} \tag{40.8}$$

In the case $b = 0.5\ R$, taking $\nu = 1$, the second term amounts to 12.5% of the first term, while the third term amounts to 0.15%. When $b = 0.75\ R$, the second term amounts to 28.1 %, the third to 1.7 %. Hence the third term usually plays a rather unimportant part, and it may be safely assumed that the second term in most cases is sufficient. As this term is independent of the relation between β and b it is valid for all cases, not only for elliptic loading but also when the load distribution is arbitrary[1].

It is of interest to compare (40.7) with the result for a channel of rectangular section. Taking the case of a fixed wall ($\nu = +1$) and neglecting the term $\beta^4/3\,R^4$ in (40.7) we have for the channel with circular section: $\delta \varphi = -0.125\,(S/S')\,C_L$;

while from (35.14) and Table 13 for a channel with square section:

$$\delta \varphi = -0.137\,(S/S')\,C_L,$$

---

[1] The reader is referred to an investigation by H. GLAUERT, The Interference on the Characteristics of an Aerofoil in a Wind Tunnel of Circular Section, Techn. Rep. Aeron. Res. Committee (Teddington) R. & M. No. 1453, 1931, for a more detailed investigation. GLAUERT remarks that as soon as notice is taken of the variation of w_z along the span of the airfoil, the calculation should also include the effect of the change of lift distribution due to the tunnel interference. From a mathematical analysis it is deduced, however, that if the tunnel correction is required with an accuracy of 10% only, it is unnecessary to take account of the latter effect. See also: C. B. MILLIKAN, On the Lift Distribution for a Wing of Arbitrary Plan Form in a Circular Wind Tunnel, Trans. Amer. Soc. Mech. Engin. **54**, p. 197, 1932.

and for a rectangular channel with height to breadth in the ratio $1/\sqrt{2}$:

$$\delta\varphi = -0.119\,(S/S')\,C_L\,.$$

The differences thus appear to be rather small.

41. Symmetrical Biplane. As a second example of the calculation of wind channel corrections for circular section, the case of the symmetrical biplane may be taken. It is assumed that the center of symmetry of the biplane lies on the axis of the channel, and that the "total loads" K_1, K_2 are equal, so that the four trailing vortices all have the same strength. The theory of the images does not give any information about the additional horizontal velocity at the wings of the biplane, and thus the exact relation between K_1, K_2 and the lifts L_1, L_2 remains unknown for the present. From reasons of symmetry, however, it is apparent that the additional horizontal velocity at the upper wing must be equal and opposite to that at the lower; hence we can conclude $K_1 + K_2 = L_1 + L_2$, and thus $K_1 = K_2 = L/2$. From results obtained in other cases it may be inferred moreover that a small inequality between K_1 and K_2 will have only a slight influence upon the final result.

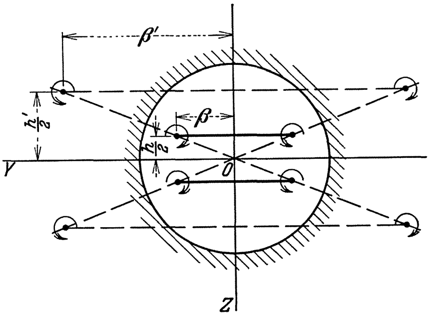


Fig. 99.

As pictured in Fig. 99 there now are four actual trailing vortices and four images. If β is the reduced semi-span and h the gap of the biplane, the coordinates of the points of intersection of the actual vortices with the plane Oyz are $\pm\beta$, $\pm h/2$. The coordinates of the points of intersection of the images will be denoted by $\pm\beta'$, $\pm h'/2$, where:

$$\beta' = \beta\,R^2/a^2, \qquad h' = h\,R^2/a^2 \tag{41.1}$$

R, as before, being the radius of the channel, while

$$a^2 = \beta^2 + (h/2)^2 \tag{41.2}$$

The mean value of the vertical additional velocity over the span of one of the wings due to the presence of the system of images can be put into the form:

$$\frac{\delta\overline{w}_z}{V} = -\frac{v\,L}{2\,\pi\,\varrho\,V^2\,b^2}\,\sigma_i \tag{41.3}$$

where the coefficient σ_i is to be obtained in an analogous way as in the former case, or from (22.10) of the biplane theory, applied first to calculate the action of the upper pair of images upon the wing considered, then that of the lower pair. It must be noticed in applying this equation that in the factor before the logarithm, $2\beta^2$ must be taken for $\beta_1\beta_2$, since all vortices are of the same strength $L/4\,\varrho\,V\,\beta$. Likewise in the expression under the logarithm, β must be taken for β_1, while β'

must be taken for β_2. Finally, h is to be replaced by $(h' - h)/2$ in one case and by $(h' + h)/2$ in the other. In this way the following result is obtained:

$$\sigma_i = \frac{b^2}{16\,\beta^2}\left[\log\frac{(\beta'+\beta)^2+\frac{1}{4}(h'-h)^2}{(\beta'-\beta)^2+\frac{1}{4}(h'-h)^2} + \log\frac{(\beta'+\beta)^2+\frac{1}{4}(h'+h)^2}{(\beta'-\beta)^2+\frac{1}{4}(h'+h)^2}\right] \tag{41.4}$$

After some reduction and after expansion of the logarithms, neglecting terms of the order β^6/R^6, as proved to be allowable in the case of the monoplane, the equation assumes the rather simple form:

$$\sigma_i = b^2/2\,R^2 \tag{41.5}$$

Whence we obtain:

$$\frac{\delta \overline{w}_z}{V} = -\frac{\nu L}{4\,\varrho\, V^2 \pi R^2} \tag{41.6}$$

which is the same as in the case of the monoplane.

42. Calculation of the Windchannel Corrections at an Arbitrary Point of the Field. Following the indication given at the end of **36**, the investigation of the influence of the channel walls has thus far been restricted to the part of the field lying so far downstream that the motion could be regarded as two-dimensional. The general problem, the determination of the motion throughout the whole field, is much more difficult. Want of space makes it impossible to consider this problem at length, and thus the treatment must be restricted to a few indications.

As noted in **36** the problem can be stated as follows: If φ_0 is the potential calculated from the force system by applying III (9.8)—for convenience this potential will be denoted as the "uncompensated" potential—it is necessary to find an "additional" potential φ_a, which satisfies the following conditions: (*a*) it must be a solution of Laplace's equation:

$$\nabla^2 \varphi_a = 0 \tag{42.1}$$

and regular throughout the field of motion; (*b*) it must vanish for $x = -\infty$; (*c*) the sum:

$$\varphi = \varphi_0 + \varphi_a \tag{42.2}$$

must satisfy the boundary conditions [see (36.1), (36.3)]:

$$\frac{1}{V}\frac{\partial \varphi}{\partial n} = \frac{1}{V'}\frac{\partial \varphi'}{\partial n} \tag{42.3}$$

$$V\varphi = V'\varphi' \tag{42.4}$$

As we regard φ_0 as a known quantity, these conditions can also be expressed in the form:

$$\frac{1}{V}\frac{\partial \varphi_a}{\partial n} - \frac{1}{V'}\frac{\partial \varphi_a'}{\partial n} = -\left(\frac{1}{V} - \frac{1}{V'}\right)\frac{\partial \varphi_0}{\partial n}$$

$$V\varphi_a - V'\varphi_a' = -(V - V')\,\varphi_0$$

In the case of a cylindrical boundary with circular cross section, a function satisfying these conditions can be constructed with the aid of the theory of Bessel functions. It is convenient to introduce cylindrical coordinates x, ω, θ, so that: $y = \omega \cos \theta$, $z = \omega \sin \theta$. Then it can be demonstrated that every expression of the type:

$$J_m(\lambda \omega) \cdot \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (m \theta) \cdot e^{\pm \lambda x} \tag{42.5}$$

where m and λ are constants (m usually being an integer), while $J_m(\lambda \omega)$ is a so-called Bessel function of the order m, satisfies Laplace's equation. Various solutions of this kind can be added together and search must be made for a combination which fulfills the required boundary conditions. For the methods to be applied in general cases the reader must be referred to treatises on potential theory or on Bessel's functions.

There are cases in which the required additional potential can be found by means of a very elegant method developed by Dougall. Want of space makes it impossible to explain this method at length. In view of the importance of the problem in connection with experimental wind channel work, however, it seems worth while to give a few examples of its application, reducing the details as far as possible[1].

Some preliminary considerations are necessary. In the first place the conditions resulting from symmetry as demonstrated in **38** must be recalled. Compare two points, lying symmetrically with respect to the plane Oyz, and having the coordinates x, y, z, and $-x, y, z$. Then the field *III*, mentioned in that section, satisfies the relations (38.1). According to the second relation we have:

$$w_{z\,III}(-x) = -w_{z\,III}(x)$$

If the velocity components of the original field in the same way are distinguished by the index *I*, we have:

$$w_{z\,III}(-x) = w_{z\,I}(-x) - \frac{1}{2} w_{z\,\infty}$$

$$w_{z\,III}(x) = w_{z\,I}(x) - \frac{1}{2} w_{z\,\infty}$$

Hence it follows that:

$$w_{z\,I}(-x) = w_{z\,\infty} - w_{z\,I}(x) \tag{42.6}$$

In the present case it is convenient to calculate the value of w_z (*i. e.* $w_{z\,I}$) for a *negative* value of x; then (42.6) makes it possible to obtain also the value for positive x, as $w_{z\,\infty}$ can be determined by the method explained in **39**.

Now consider the case of an infinitesimal loaded element, having its direction parallel to the axis Oy, and situated somewhere in the plane Oyz. Taking the total load upon it for convenience equal to 1,

[1] The reader is referred to the "Treatise on Bessel functions and their applications to physics" by A. GRAY, G. B. MATHEWS and T. M. MACROBERT (London 1922), in which an account of DOUGALL's method is to be found at pp. 101—110.

it is readily deduced from III (9.8) that the uncompensated potential φ_0 assumes the form:

$$\varphi_0 = -\frac{1}{4\pi\varrho V}\frac{z-\zeta}{r(r-x)} \tag{42.7}$$

Applying the transformation which was mentioned after III (12.4), this expression may be written:

$$\varphi_0 = -\frac{1}{4\pi\varrho V}\int_0^\infty d\xi \frac{z-\zeta}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{3/2}},$$

which finally can be transformed into:

$$\varphi_0 = -\frac{1}{4\pi\varrho V}\int_0^\infty d\xi \frac{\partial}{\partial\zeta}\left(\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\right) \tag{42.8}$$

This equation shows that the potential φ_0, corresponding to the case of an infinitesimal loaded element parallel to $O\,y$, can be expressed in the form of an integral, the integrand being the derivative with respect to ζ of the reciprocal distance from the point ξ, η, ζ. Hence if we first determine the compensated potential in the case where φ_0 is simply this reciprocal distance:

$$\varphi_0 = \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} \tag{42.8a}$$

it will not be difficult to obtain the compensated potential also for the case where φ_0 is given by (42.8).

Now it is known from the theory of Bessel functions that the reciprocal of the distance from a point ξ, η, ζ to a point x, y, z can be expressed by means of a certain definite integral, involving a special type of Bessel functions. This integral can be transformed in various ways, and it can be brought into a form very convenient in connection with the boundary conditions (42.3), (42.4). The case considered by Dougall is the one which applies to a free boundary surface, that is the case of a wind channel with open working section. As was noted before in **31**, the boundary condition (42.3) then must be omitted, and it remains necessary to take into account the condition (42.4) only, which now assumes the form:

$$\varphi = 0 \tag{42.9}$$

It appears to be relatively simple to determine the additional potential φ_a which must be added to the function φ_0 given by (42.8a) in this case; the sum $\varphi = \varphi_0 + \varphi_a$ after some transformations can be brought into the form:

$$\frac{4}{R^2}\sum_m{}' \sum_s \cos m(\theta-\theta')\, e^{-\lambda_s(\xi-x)} \frac{J_m(\lambda_s\omega)\,J_m(\lambda_s\omega')}{\lambda_s\,[J_m'(\lambda_s R)]^2} \tag{42.10}$$

Here the summation with respect to m extends over all positive integers; the prime added to the Σ sign indicates that a factor 1/2 must be inserted before the term corresponding to $m = 0$. The summation with respect to s for every m extends over all positive roots of the equation:

$$J_m(\lambda_s R) = 0 \tag{42.11}$$

$J'_m(\lambda_s R)$ is the derivative of the function $J_m(\lambda_s R)$ with respect to its argument. Finally ω' and θ' are defined by $\eta = \omega' \cos \theta'$, $\zeta = \omega' \sin \theta'$. The expression is valid whenever $\xi - x$ is positive; it is thus certainly applicable when ξ is positive and x is negative, as was the case in (42.8).

The expression (42.10) represents the compensated potential function corresponding to the uncompensated potential of (42.8a); hence in order to deduce the expression for the compensated potential corresponding to (42.8) it is necessary to obtain the derivative of (42.10) with respect to ζ, and to integrate it with respect to ξ from 0 to ∞, inserting finally the factor $-1/4\pi\varrho V$. The derivative with respect to ζ can be expressed in polar coordinates without difficulty. We pass over the details of the calculation; the result, after being specialized for the case $\zeta = 0$, so that $\theta' = 0$, while $\omega' = \eta$, takes the form[1]:

$$\varphi = -\frac{1}{\pi \varrho V R^2} \sum_m \sum_s \sin m\theta \, e^{\lambda_s x} \frac{m J_m(\lambda_s \omega) J_m(\lambda_s \eta)}{\eta \lambda_s^2 [J'_m(\lambda_s R)]^2} \tag{42.12}$$

From this expression we deduce the vertical velocity by means of the ordinary formula $w_z = \partial\varphi/\partial z$, which again can be expressed by means of cylindrical coordinates. Taking in the result $z = 0$, so that $\theta = 0$, and writing y for ω, the following equation is obtained:

$$w_z = \frac{\partial \varphi}{\partial z} = -\frac{1}{\pi \varrho V R^2} \sum_m \sum_s e^{\lambda_s x} \frac{m^2 J_m(\lambda_s y) J_m(\lambda_s \eta)}{y \eta \lambda_s^2 [J'_m(\lambda_s R)]^2} \tag{42.13}$$

It is seen that in (42.12) as well as in (42.13) the term for $m = 0$ vanishes, so that the prime over the first Σ sign can be omitted.

43. Application to a Special Case. The result obtained can be applied to the calculation of the vertical velocity at a point in the axis of the channel, caused by a loaded line or airfoil. This vertical velocity is of importance when performing measurements on the tailsetting of an airplane.

For a point on the axis, that is for $y = 0$, the expression (42.13) can be simplified by making use of the relations:

$$\lim. \frac{J_m(\lambda_s y)}{\lambda_s y} = \begin{cases} \frac{1}{2} & (m = 1) \\ 0 & (m > 1) \end{cases} \tag{43.1}$$

Thus the summation with respect to m vanishes, the terms corresponding to $m = 1$ being the only one remaining. On the other hand, in consequence of (42.11) we have the relation: $J'_1(\lambda_s R) = J_0(\lambda_s R)$; hence the denominator takes the form: $\eta \lambda_s [J_0(\lambda_s R)]^2$. The value of

[1] The case $\theta' = \pi$ is included in (42.12), by taking η negative.

the roots of the equation $J_1(\lambda_s R) = 0$ and the corresponding values of $J_0(\lambda_s R)$ are given in the tables of Bessel functions, as published in many textbooks. Now the limit of the expression $\lambda_s R [J_0(\lambda_s R)]^2$ for $s \to \infty$ is $2/\pi$, while by direct calculation from the tables the following values are obtained:

$$\frac{1}{\lambda_s R [J_0(\lambda_s R)]^2} \quad \text{for } s = 1{:}\ 1.0242\frac{\pi}{2}$$

$$2{:}\ 1.0075\frac{\pi}{2}$$

$$3{:}\ 1.0036\frac{\pi}{2} \quad \text{etc.}$$

Writing $(\pi/2)(1 + \delta_s)$ for these expressions, we finally obtain the following result, which can be evaluated without difficulty, supposing that x is not too small:

$$w_z = -\frac{1}{4\varrho V R^2}\frac{R}{\eta}\left[\sum_s (1 + \delta_s)\, e^{\lambda_s x} J_1(\lambda_s \eta)\right] \tag{43.2}$$

When η is zero, it must be remembered that:

$$lim.\ \frac{R}{\eta} J_1(\lambda_s \eta) = \frac{1}{2}\lambda_s R \tag{43.3}$$

From (43.2) values have been calculated for the cases $x = -0.25\,R$, $x = -0.50\,R$, $x = -0.75\,R$, $x = -1.00\,R$. Those corresponding to the first and the second cases are given in the following Table, together with the values for a field of unlimited extent, which are obtained from the uncompensated potential (42.7) by means of the formula:

$$w_z = \left(\frac{\partial \varphi_0}{\partial z}\right)_{y=0,\ z=0} = -\frac{1}{4\varrho V R^2}\frac{R^2}{\pi r (r - x)} \tag{43.4}$$

The constant factor $1/4\,\varrho\, V R^2$ has been omitted in the Table; the numbers given are the values of $4\,\varrho\, V\, R^2\, w_z$.

TABLE 14. Values of $4\,\varrho\, V R^2 w_z$.

$\frac{\eta}{R}$	$x = -0.25\,R$			$x = -0.50\,R$		
	from (43.2)	from (43.4)	ε	from (43.2)	from (43.4)	ε
0.0	—2.291	—2.547	+0.256	—0.438	—0.637	+0.199
0.1	—2.021	—2.277	+0.256	—0.419	—0.618	+0.199
0.2	—1.488	—1.744	+0.256	—0.371	—0.569	+0.198
0.3	—1.016	—1.272	+0.256	—0.307	—0.504	+0.197
0.4	—0.680	—0.935	+0.255	—0.240	—0.436	+0.196
0.5	—0.451	—0.704	+0.253	—0.179	—0.373	+0.194

It will be seen that the differences ε between the compensated and the uncompensated values can be regarded as practically constant in both cases over the range indicated. Hence they can be replaced by mean values; these mean values, together with those for $x = -0.75\,R$ and $x = -1.00\,R$ are as follows:

$$
\begin{aligned}
x = -0.25\,R&: \quad \varepsilon = +0.256 \\
-0.50\,R& \qquad\quad +0.197 \\
-0.75\,R& \qquad\quad +0.148 \\
-1.00\,R& \qquad\quad +0.110.
\end{aligned}
$$

It is not difficult now to pass over to the case of a loaded line, extending along the y axis from $y = -b$ to $y = +b$, the load per unit span being l. Then the correction to be applied to the vertical velocity at a point situated on the axis of the channel in front of the loaded line is given by the integral:

$$\delta w_z = \frac{1}{4\,\varrho\, V R^2} \int_{-b}^{+b} l\,\varepsilon\, d\eta \tag{43.5}$$

As ε is approximately independent of η its mean value may be brought before the integral sign, at least as long as b does not much surpass $0.5\,R$.

Hence we obtain: $$\delta w_z = \frac{L\,\varepsilon}{4\,\varrho\, V R^2} = \frac{L}{2\,\pi\,\varrho\, V R^2} \cdot \frac{\pi\,\varepsilon}{2} \tag{43.6}$$

In order to obtain the magnitude of the additional velocity at the central point of the loaded line itself, we consider the value of $w_{z\,\infty}$. The part of this quantity which is due to the channel boundary is to be found, according to **39**, by introducing two imaginary vortices of strength $L/2\,\beta$ at the points $y = \pm R^2/\beta$, the directions of rotation being the same as for the corresponding actual vortices. In the axis of the channel far behind the loaded line they give the (downward) velocity:

$$\delta w_{z\,\infty} = \frac{L}{2\,\pi\,\varrho\, V R^2} \tag{43.7}$$

This result has been obtained on the supposition of uniform loading over a span 2β; as it is independent of β, it can be expected that it is valid for an arbitrary load distribution, which in fact can be demonstrated easily.

The magnitude of the additional velocity at $x = 0$ is equal to one half of $\delta w_{z\,\infty}$, while that at points of the axis situated downstream from the loaded line can be calculated by means of (42.6). The results are given in the next section (Table 16) together with those for the case of a fixed boundary (channel with closed working section).

44. Case of a Channel with Fixed Cylindrical Boundary (Closed Working Section). This case can be treated along the same lines as the preceding. Instead of (42.10), the following, somewhat more complicated equation is obtained for the compensated potential function corresponding to the simple reciprocal distance (valid again for $\xi - x > 0$):

$$\left.\begin{aligned}
\frac{4}{R^2} \sum_m{}' \sum_s \cos m\,(\theta - \theta')\, e^{-\lambda_s (\xi - x)} \frac{J_m(\lambda_s\,\omega)\, J_m(\lambda_s\,\omega')}{(1 - m^2/\lambda_s^2 R^2)\,\lambda_s\,[J_m(\lambda_s R)]^2} - \\
- \frac{2\,(\xi - x)}{R^2}
\end{aligned}\right\} \tag{44.1}$$

The summation with respect to s in this expression for every m extends over all positive roots of the equation:

$$J'_m(\lambda_s R) = 0 \tag{44.2}$$

The reductions can be made according to the same process as before and the final equation for the vertical velocity due to an infinitesimal loaded element becomes, instead of (43.2):

$$w_z = -\frac{1}{4\varrho V R^2}\frac{R}{\eta}\left[\sum_s (1+\delta'_s)\, e^{\lambda_s x} J_1(\lambda_s \eta)\right] \tag{44.3}$$

where now: $$1+\delta'_s = \frac{2}{\pi}\frac{1}{\lambda_s R\,(1-1/\lambda_s^2 R^2)\,[J_1(\lambda_s R)]^2} \tag{44.4}$$

For the evaluation of the series (44.3) it is necessary to know the roots of the equation $J'_1(\lambda_s R) = 0$, or of the equivalent equation: $J_1(\lambda_s R) = \lambda_s R J_0(\lambda_s R)$. They can be obtained either by interpolation from the tables, or from a general formula due to M'Mahon[1]. It is found that the expression (44.4) rapidly converges to 1.

The value of w_z has been calculated from (44.3) for the same series of values of x as before. The results for $x = -0.25\,R$ and $x = -0.50\,R$ have been given in Table 15, together again with the values for the field of unlimited extent deduced from (43.4). It will be seen that in this case also, the differences ε are nearly independent of η.

TABLE 15. Values of $4\varrho V R^2 w_z$.

η/R	$x=-0.25\,R$			$x=-0.50\,R$		
	from (44.3)	from (43.4)	ε	from (44.3)	from (43.4)	ε
0.0	—2.787	—2.547	—0.240	—0.806	—0.637	—0.169
0.1	—2.516	—2.277	—0.239	—0.787	—0.618	—0.169
0.2	—1.983	—1.744	—0.239	—0.737	—0.569	—0.168
0.3	—1.512	—1.272	—0.240	—0.671	—0.504	—0.167
0.4	—1.173	—0.935	—0.238	—0.601	—0.436	—0.165
0.5	—0.940	—0.704	—0.236	—0.536	—0.373	—0.163

The differences ε for the four values of x/R have been collected below; they are now of sign opposite to those which were obtained for the open channel:

$$\begin{aligned} x &= -0.25\,R: & \varepsilon &= -0.239\\ &\quad -0.50\,R & &\quad -0.167\\ &\quad -0.75\,R & &\quad -0.107\\ &\quad -1.00\,R & &\quad -0.064. \end{aligned}$$

The magnitude of the additional velocity δw_z in the case of a loaded line extending along the y axis from $y = -b$ to $y = +b$, is given again by (43.6). The value of $\delta w_{z\infty}$ is obtained from (43.7) taken with the

[1] See p. 260 of the Treatise mentioned in the foot note to 42.

opposite sign. In the following Table the values of the additional vertical velocity, expressed as a fraction of $|\delta w_{z\,\infty}| = \frac{L}{2\pi\varrho V R^2}$ have been given for values of x/R ranging from -1.00 to $+1.00$, both for the case of the tunnel with a fixed boundary and for a tunnel with a free surface. The values for $x > 0$ have been obtained by applying (42.6).

TABLE 16.
Values of $\delta w_z / |\delta w_{z\,\infty}|$.

x/R	closed tunnel	open tunnel
—1.00	—0.100	+0.173
—0.75	—0.168	+0.233
—0.50	—0.262	+0.309
—0.25	—0.375	+0.402
0	—0.500	+0.500
+0.25	—0.625	+0.598
+0.50	—0.738	+0.691
+0.75	—0.832	+0.767
+1.00	—0.900	+0.827
$+\infty$	—1.000	+1.000

It is of interest to compare the results for the tunnel of circular section with those given by Glauert and Hartshorn for a tunnel of rectangular section with fixed walls[1]. From equations (7) of their paper we deduce for a tunnel with square section: $\delta w_z/\delta w_{z\,\infty} = 0.5 + 0.876\, x/h$, where h is the height of the tunnel. The formula is deduced on the supposition that x and b are both small compared with the tunnel dimensions. If h is taken equal to $R\sqrt{\pi}$, thus making the sectional area of the square tunnel equal to that of the circle, then for $x = R/4 = h/4\sqrt{\pi}$ this formula gives $\delta w_z/\delta w_{z\,\infty} = 0.624$ which shows good agreement with the value for the circular tunnel. For a rectangular tunnel with height to breadth ratio 1/2, Glauert gives the formula:

$$\delta w_z/\delta w_{z\,\infty} = 0.5 + 1.067\, x/h\,.$$

The expressions deduced for the compensated potential in **42**—**44** can be used also for the calculation of other quantities, for example the axial velocity w_x. Not much work has thus far been done in this particular direction, but it may be assumed that there is still a wide field of application for this type of equations.

45. Influence of an Internal Cylindrical Boundary upon the Field of Motion around a Loaded Line. It has been mentioned in **29** and **36** that under the problems of boundary influence we may also consider the case of an *internal* cylindrical boundary, such as that formed *e. g.* by the fuselage of an airplane, supposing that it has the form of an infinite cylinder with its axis parallel to Ox. Without going into a detailed investigation it may be of interest to indicate a few characteristic points[2].

[1] GLAUERT, H. and HARTSHORN, A. S., The Interference of Wind Channel Walls on the Downwash Angle and the Tailsetting to Trim, Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 947, 1924.

[2] The reader is referred to a paper by J. LENNERTZ, Beitrag zur theoretischen Behandlung des gegenseitigen Einflusses von Tragfläche und Rumpf, Abhandl. Aerodyn. Institut Aachen 8, p. 3, 1928. See also Nation. Adv. Committee Aeronautics (Washington), Technical Memorandum No. 400.

Consider the case of a cylinder of infinite length, concentric with the x axis, and of radius R, while along the y axis, from $y = -b$ to $y = +b$ there extends a loaded line or airfoil of small chord. To be precise, it must be noted that the line supports a load from $y = -b$ to $y = -R$ and from $y = +R$ to $y = +b$ only. In general the load per unit length l may be a function of y; for simplicity we assume the load distribution to be symmetric. In consequence of the presence of the cylinder there will be an additional velocity δw_z at the points of the loaded line (there is no component δw_x as can be seen from the symmetry of the field) which will cause a change both in the effective angle of incidence and in induced resistance. An estimate of this effect can be obtained by assuming a uniform load l' over a reduced span 2β. Considering a section of the field at a great distance downstream (see Fig. 100) we must then introduce two images at distances $\pm R^2/\beta$ from the axis, corresponding to the vortices which extend from the ends of the loaded line. At a point y these images give a downward velocity:

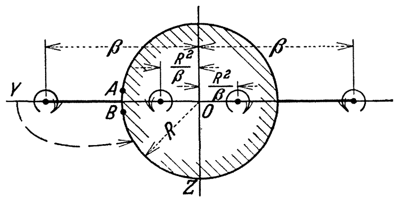


Fig. 100.

$$\delta w_{z\infty} = \frac{l'}{2\pi\varrho V}\left(\frac{1}{y - R^2/\beta} - \frac{1}{y + R^2/\beta}\right) \tag{45.1}$$

the mean value over the segment from $y = R$ to $y = \beta$ being:

$$\delta \overline{w}_{z\infty} = \frac{l'}{2\pi\varrho V(\beta - R)} \log \frac{(\beta + R)^2}{\beta^2 + R^2} \tag{45.2}$$

At the loaded line itself the velocity has half this value, hence:

$$\frac{\delta \overline{w}_z}{V} = \frac{l'}{4\pi\varrho V^2(\beta - R)} \log \frac{(\beta + R)^2}{\beta^2 + R^2} \tag{45.3}$$

Now it must be remarked that in calculating the total lift supported by the system, account must be taken of the *resultant pressure upon the cylinder*. The pressure at an arbitrary point of the cylinder is to be found either from Bernoulli's equation or from (36.4) for q, both of which lead to:

$$p = p_0 - \varrho V w_x \tag{45.4}$$

neglecting, as before, quantities of the second order in the w's. The resultant pressure is then given by the integral:

$$L_c = \iint p \cos(n, z)\, dS = -\varrho V \iint w_x \cos(n, z)\, dS \tag{45.5}$$

where dS is an element of the cylindrical surface, and (n, z) the angle between the outward normal to dS and the z axis. We write: $dS = dx\, ds$, where ds is an element of a section of the cylinder by a plane parallel to yOz. Performing the integration with respect to x, we find:

$$L_c = -\varrho V \int \varphi_\infty \cos(n, z)\, ds = \varrho V \int \varphi_\infty\, dy \tag{45.6}$$

where in the last integral the integration is to be performed in the counter-clockwise sense. It will be seen that this result resembles that given by (16.4); it can be shown also, that there is a simple connection between φ_∞ and the potential Φ considered in **16—20**.

The integral $\int \varphi_\infty \, d y$ can be calculated in the following way. The potential φ_∞ is a function of y and z only, and corresponds to a two-dimensional field of motion. The stream function ψ_∞ of this field is constant along the section of the cylinder (as this is a stream-line for the two-dimensional motion); hence writing $\varphi_\infty + i\,\psi_\infty = \chi$, we have:

$$\int \varphi_\infty d y = \int \chi \, d y$$

($\int \psi_\infty \, d y$ is zero, since the integration is performed along a closed line). Putting $y + i\,z = \zeta$, the latter equation can be written:

$$\int \varphi_\infty \, d y = Re. \int \chi \, d \zeta \tag{45.7}$$

where $Re.$ indicates that the real part of this expression is to be taken.

From the configuration of the vortex system (see Fig. 100) we deduce:

$$\chi = \frac{-l'}{2 \pi i \varrho V} \log \left(\frac{\zeta - \beta}{\zeta + \beta} \cdot \frac{\zeta + R^2/\beta}{\zeta - R^2/\beta} \right) \tag{45.8}$$

We must be careful, however, with this expression, as the logarithm is a many valued function. When we consider two points A, B lying very near to each other on the cylinder, but at opposite sides of the plane $z = 0$, then from (45.8) it might appear that φ_∞ would be approximately the same for both, while actually we have:

$$(\varphi_\infty)_A - (\varphi_\infty)_B = l'/\varrho \, V,$$

as will be seen by considering the two generating lines corresponding to these points, which pass on opposite sides of the loaded line. Hence, if the logarithm is defined by stating that it is real for ζ real, positive and greater than β, and that its value at the section of the cylinder is determined by a movement along the dotted path, it will be necessary to add the amount $l'/\varrho \, V$ for all points on the upper half of the circumference. In consequence of this, instead of (45.7) we shall write:

$$\int \varphi_\infty \, d y = Re. \int \chi \, d \zeta + 2\, l' R/\varrho \, V \tag{45.9}$$

Inserting the expression (45.8) for χ and integrating by parts we have:

$$\int \chi \, d \zeta = - \int \zeta \frac{d \chi}{d \zeta} d \zeta =$$

$$= \frac{l'}{2 \pi i \varrho V} \int \left(\frac{\zeta}{\zeta - \beta} - \frac{\zeta}{\zeta + \beta} + \frac{\zeta}{\zeta + R^2/\beta} - \frac{\zeta}{\zeta - R^2/\beta} \right) d \zeta,$$

from which by the theory of residues:

$$\int \chi \, d \zeta = - 2\, l' \, R^2/\varrho \, V \beta$$

Hence:
$$\int \varphi_\infty \, d y = (2\, l'/\varrho \, V)\, (R - R^2/\beta),$$

and thus finally:
$$L_c = 2\, l' \, (R - R^2/\beta) \tag{45.10}$$

The lift supported by the two segments of the loaded line has the magnitude:

$$L_0 = 2\,l'\,(\beta - R) \tag{45.11}$$

Consequently the total lift becomes:

$$L = 2\,l'\,(\beta - R^2/\beta) \tag{45.12}$$

This result shows that when β is much greater than R, the presence of the cylinder (the "fuselage") causes a practically negligible loss of lift, compared with the case of an uninterrupted airfoil of span 2β.

The additional vertical velocity δw_z given by (45.3) causes an increase of induced resistance of the amount:

$$2\,(R - \beta)\,l'\,\delta w_z/V = \frac{l'^2}{2\pi\varrho V^2}\,log\,\frac{(\beta + R)^2}{\beta^2 + R^2} \tag{45.13}$$

However, it is not possible to use this expression in order to obtain a comparison between the induced resistance of the combination airfoil + cylinder and that of an ordinary airfoil, having the same total span and supporting the same total lift. This is due to the circumstance that in the case of the system airfoil + cylinder, the part L_c of the lift, that is, the part supported by the cylinder has no induced resistance corresponding to it, as the cylinder cannot take up any tangential force. In making up the balance it thus would become necessary to introduce also a reduction of induced resistance, along with the increase given by (45.13). For an exact calculation, however, the picture of constant load with vortices starting from the wing tips is not sufficient and a more detailed consideration of the lift distribution would be required. It is clear also that the relation between β and the actual semi-span b cannot be found without such a detailed consideration, since III (25.6) does not apply.

We shall leave this problem, and pass over to the investigation of the minimum drag obtainable with this type of arrangement.

46. The Problem of Minimum Induced Resistance for a Loaded Line Connected with an Infinite Cylinder. If a vortex of strength Γ_1 rotating in the clockwise sense, intersects the y axis in the point η, then the value at a point y of the vertical velocity $w_{z\,\infty}$ due to the combined action of this vortex and its image with respect to the cylinder, is given by the expression:

$$w_{z\,\infty} = -\frac{\Gamma_1}{2\pi}\left(\frac{1}{y-\eta} - \frac{1}{y - R^2/\eta}\right) \tag{46.1}$$

Differentiation with respect to η gives the velocity due to a vortex pair, the vortex with the clockwise rotation lying at the positive side of the vortex with the anticlockwise rotation. Now a loaded element lying at the point η on the y axis, and supporting the load $l\,d\eta$ over the length $d\eta$, has behind itself two parallel vortices of absolute strength $l/\varrho\,V$ and of opposite sign, so arranged that the vortex with the clockwise rotation lies at the positive side of the other (it is supposed that the observer looks in the direction of $-x$). Thus it is found that the vertical velocity at the point y (z being zero) obtained at an infinite distance behind the loaded element is given by:

$$w_{z\infty} = -\frac{l\,d\eta}{2\pi\varrho V}\left[\frac{1}{(y-\eta)^2}+\frac{R^2}{(y\eta-R^2)^2}\right] \tag{46.2}$$

The expression between [ ] is the analogue for this case of the coefficient $\triangle$ introduced in III 17. As it is symmetrical in y and η, it might be expected perhaps at first sight (the same as in III 18), that the condition of minimum induced resistance for a loaded line is the constancy of $w_{z\infty}$ along this line. However, in extending the demonstration of III 18 to the present case, account must be taken of the resultant pressure upon the cylinder, which contributes toward the lift. This can be done as follows: Assuming symmetrical lift distribution, to the loaded element at η there will correspond a similar loaded element at $-\eta$. Such a system can be considered as the difference of a constant negative load $-l$ up to $\pm\eta$, and a constant positive load $+l$ up to $\pm(\eta+d\eta)$. The lift upon the cylinder corresponding to the former is given by (45.10), provided we write $-l$ for l' and η for β; the lift upon the cylinder corresponding to the latter is also given by (45.10), if we write $+l$ for l' and $\eta+d\eta$ for β. Making up the resultant we obtain:

$$\frac{d}{d\eta}\left[2l(R-R^2/\eta)\right]d\eta = 2lR^2/\eta^2\cdot d\eta$$

Adding this amount to the lift $2l\,d\eta$ supported by the two elements, we have:

$$2l(1+R^2/\eta^2)\,d\eta \tag{46.3}$$

This expression now must be integrated from $\eta = R$ to $\eta = b$, thus giving for the total lift supported by the system (if we write y instead of η):

$$L = 2\int_R^b l(1+R^2/y^2)\,dy \tag{46.4}$$

If this expression is used instead of III (20.1) in the deduction of the condition for minimum induced resistance, we have:

$$w_{z\infty} = w_0(1+R^2/y^2) \tag{46.5}$$

where w_0 is a constant. This condition takes the place of Munk's condition of a constant value of $w_{z\infty}$ along the loaded line, which was obtained in the case of the unlimited field.

As the induced resistance is given by:

$$D = 2\int_R^b l(w_{z\infty}/2V)\,dy \tag{46.6}$$

we obtain:

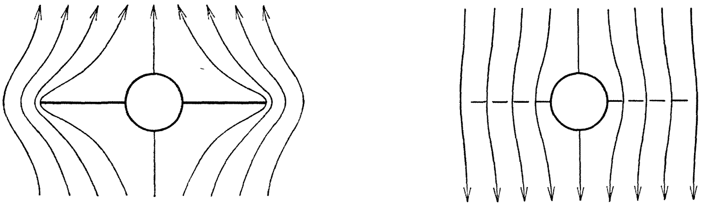
$$D_i = w_0 L/2V \tag{46.7}$$

which is the same equation as III (20.8).

Instead of the problem considered in III 21 it is now necessary to find an irrotational plane motion which along the circumference of the cylinder fulfills the condition $w_{n\infty} = 0$, while along the segments from $y = -b$ to $y = -R$ and from $y = +R$ to $y = +b$ it satisfies (46.5). This problem is solved by Lennertz by superposing the following two motions:

(*a*) an upward motion which fulfills the condition $w_{n\infty} = 0$ both at the circle and at the two segments (see Fig. 101a), having at infinity the velocity w_0;

(*b*) a downward motion which is tangential to the circle only (see Fig. 101b). If the velocity at infinity has the same value w_0, both motions annihilate each other at infinity, while it is easily seen that at the segments of the loaded line the condition (46.5) is satisfied.



a *b*

Fig. 101.

The complex potential corresponding to the first motion is:

$$w_0 \chi_I = i w_0 \sqrt{\left(\zeta + \frac{R^2}{\zeta}\right)^2 - \left(b + \frac{R^2}{b}\right)^2} \tag{46.8}$$

where $\zeta = y + iz$, while the sign of the radical is determined in such a way that for infinite values of ζ it approaches to $+\zeta$. This expression is deduced by Lennertz by means of a conformal transformation. It is not difficult to show that if we consider the stream-line $\Psi_I = 0$, so that $\chi_I = \Phi_I$, the values of ζ are of the following form:

$\Phi_I < -(b + R^2/b)$:	ζ positive imaginary; $\lvert\zeta\rvert > R$
$\Phi_I = -(b + R^2/b)$:	$\zeta = +iR$
$-(b + R^2/b) < \Phi_I < -(b - R^2/b)$:	ζ is of the form $R e^{i\theta}$, where θ moves from $\pi/2$ to 0
$\Phi_I = -(b - R^2/b)$:	$\zeta = +R$
$-(b - R^2/b) < \Phi_I < 0$:	ζ real and positive; $R < \zeta < b$
$\Phi_I = 0$:	$\zeta = +b$, etc.

The complex potential corresponding to the second motion is:

$$w_0 \chi_{II} = -i w_0 \left(\zeta - \frac{R^2}{\zeta}\right) \tag{46.9}$$

If now the value of Φ is deduced from the resulting function $\chi = \chi_I + \chi_{II}$, and further the same procedure is applied as used in III **22** and IV **16**, the following expression is obtained for l (or Λ):

$$l = 2\,\varrho\,V\,w_0 \sqrt{\left(b + \frac{R^2}{b}\right)^2 - \left(y + \frac{R^2}{y}\right)^2} \tag{46.10}$$

Inserting this into (46.4) the total lift becomes (we pass over the details of the calculation as given by Lennertz):

$$L = \varrho\ V\ w_0\,\pi\,(b - R^2/b)^2 \tag{46.11}$$

Elimination of w_0 between (46.7) and (46.11) gives:

$$D_i = \frac{L^2}{2\,\pi\,\varrho\,V^2\,(b - R^2/b)^2} \tag{46.12}$$

It is seen that the expression $\pi\,(b - R^2/b)^2$ in this case takes the part of the area $\int \Phi\,d\,y = \Sigma$ in **16**, *seq.*

The relative increase of induced resistance compared with that of an uninterrupted loaded line (with elliptic lift distribution) is thus given by:

$$\frac{\pi\,b^2}{\Sigma} = \frac{1}{(1 - R^2/b^2)^2} \tag{46.13}$$

In the following Table a few values of the expression $\Sigma/\pi\,b^2$ have been given, together with those for an airfoil with a gap, considered in **20**.

It will be seen that the differences between the two cases are very great.

TABLE 17.

Airfoil combined with infinite cylinder		Airfoil with gap	
R/b	$\Sigma/\pi\,b^2$	s/b	$\Sigma/\pi\,b^2$
0	1	0	1
0.001	1.000	0.001	0.759
0.01	1.000	0.01	0.666
0.1	0.980	0.1	0.461
0.5	0.563	0.5	0.127

The foregoing considerations may suffice to give an idea of the treatment of problems relating to the influence of internal boundaries upon the flow around a lifting system. Through want of space it is impossible to investigate other questions, as for example the effect of a vertical displacement of the loaded line with respect to the axis of the cylinder, the distribution of the lift experienced by the cylinder over its length, and the distribution of lift over the span of the airfoil.

It is evident, moreover, that there are a great number of other problems awaiting treatment; such as the influence of a cylinder of finite length, of a cylinder the axis of which is not parallel to the direction of the general motion, of bodies of arbitrary form (fuselage), etc. From the mathematical standpoint these cases appear to be very difficult, though investigations are advancing in this direction. Moreover a considerable amount of data concerning interaction between airfoils and other bodies have been obtained from experimental results.

For further particulars in this respect the reader may be referred to Divisions K—Aerodynamics of the Airplane Body (non-lifting system);

Drag and Influence on Lifting system—and P—Airplane as a Whole; General View of Mutual Interactions among Constituent Parts. Problems relating to the interaction between airfoil and airscrew, which are of a different nature, are considered in Divisions L and M.

CHAPTER V

PROBLEMS OF NON-UNIFORM AND OF CURVILINEAR MOTION

The object of the present chapter is to give a condensed review of some of the developments of the theory of airplanes relating to conditions of non-uniform and of curvilinear motion. No attempt has been made to treat these matters exhaustively, and the reader is referred to the original papers for more complete details. It will be seen, moreover, that these subjects are in the midst, or even at the beginning of their development, so that no final results are to be expected. It has, however, seemed desirable to give some attention to these subjects, since they are connected with practical problems of importance in the domain of applied aerodynamics.

A. Problems of Non-Uniform Motion.

1. Introduction.—Vortex System Associated with the Variations of the Circulation around an Airfoil. The problems of variable motion of an airfoil which have been thus far treated are not numerous. Nearly all authors have restricted themselves to the consideration of the two-dimensional case (airfoil of infinite span), a restriction which will be adopted in the following deductions[1].

It will be evident that in general every change of the state of motion of an airfoil will be accompanied by a change of the circulation Γ around it. Also from the general considerations concerning the origin of the circulation, as set forth in I 8, it appears that for every change of Γ a vortex must leave the trailing edge of the airfoil. The strength of

[1] The following references may be mentioned:

PRANDTL, L., Vorträge a. d. Gebiete der Hydro- u. Aerodynamik p. 18 (Innsbruck 1922).

BIRNBAUM, W., Zeitschr. f. angew. Math. u. Mech. 4, p. 277, 1924.

KONING, C., Proc. 1st Intern. Congress Appl. Mech. p. 414, Delft, 1924.

WAGNER, H., Zeitschr. f. angew. Math. u. Mech. 5, p. 17, 1925.

KÜSSNER, G., Luftfahrtforschung 4, p. 41, 1929.

GLAUERT, H., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. Nos. 1215, 1242, 1929; Vorträge a. d. Gebiete der Hydro- u. Aerodynamik p. 88, Aachen, 1929.

MÉTRAL, A., Proc. 3rd Intern. Congress f. appl. Mech. I, p. 403, Stockholm, 1930.

WALKER, P. B., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1402, 1931.

this vortex (*i. e.* the circulation along a closed line which encircles the vortex) is equal to the change of the circulation; the direction of rotation is opposite to that of Γ if the circulation is increasing, whereas it is the same as that of Γ if the circulation is decreasing.

In the case of a continuously changing circulation, a continuous band of vortices develops behind the airfoil.

These phenomena account for the peculiar character of the problems of unsteady motion. As the vortices detached from the trailing edge produce a vertical component of velocity at the place of the airfoil, they have a marked influence upon the flow around the airfoil and thus upon the magnitude of the circulation.

The vortices considered, being all parallel to the span of the airfoil, the most important features of their influence are not affected by the restriction to the two-dimensional case. We shall suppose further that the general motion of the airfoil, considered with respect to the air at rest, is rectilinear, apart from oscillations of small amplitude in the vertical direction, which are considered in some problems. The velocity of the airfoil with respect to the air will be in the direction of $-x$; its magnitude will be denoted by V, which quantity in general will be a function of the time. The picture of the airfoil moving with respect to the air is very useful for a number of purposes; afterwards we return to the more common picture, in which the system of coordinates is connected with the airfoil.

It is convenient to introduce the distance s travelled by the trailing edge of the airfoil in the direction of $-x$, measured from a given point of the x axis. The connection between s and the velocity V is given by the equation:

$$V = ds/dt \tag{1.1}$$

If in the interval from t to $t + dt$ the circulation around the airfoil changes by the amount $d\Gamma$, then from the trailing edge there separates a vortex of strength $-d\Gamma$. In the same interval the trailing edge of the airfoil moves from the point s to the point $s + ds$; hence this vortex must more properly be considered as a series of small vortices, distributed over the element of length ds. Introducing the strength $\bar{\gamma}$ of the vortex sheet obtained by this process, it will be seen that the strength of the newly formed element is given by[1]:

$$\bar{\gamma} = -\frac{d\Gamma}{ds} = -\frac{1}{V}\frac{d\Gamma}{dt} \tag{1.2}$$

In similar manner as in the first order theory of steady motion, developed in Chapter III, it will be assumed that the motions which the various vortices impart to each other may be neglected in comparison with the velocity V. In other words, in the picture here used of an

[1] The definition of $\bar{\gamma}$ is the circulation around a strip of the vortex sheet having unit length in the direction of the x axis (see I **9**).

airfoil moving through air at rest, it will be assumed *that the vortices remain at the places where they are formed.* In that case, if the airfoil has moved to the point determined by s_1, the strength of the vortex band at any point s, satisfying the condition $s < s_1$, is given by (1.2), where t refers to the time the trailing edge of the airfoil passed s.

Integration of (1.2) gives:

$$\int_{s_0}^{s_1} \bar{\gamma}\, d s = -(\Gamma_1 - \Gamma_0) \tag{1.3}$$

indicating that the total strength of the vortex band, integrated between two points s_0 and s_1 is equal to the total change of the circulation in the interval of time during which the trailing edge of the airfoil moved from the first point to the second.

If the airfoil started from rest, we can choose s_0 as determining the starting point, then Γ_0 will be zero. It is convenient to introduce this assumption in our further calculations; it can be used even in the case of periodic motion, provided the instant to which s_0 refers is taken sufficiently remote in the past. In order to cover all cases we write $-\infty$ instead of s_0; omitting at the same time the index 1 referring to the upper limit, (1.3) then assumes the form:

$$\int_{-\infty}^{s} \bar{\gamma}\, d s = -\Gamma \tag{1.4}$$

The problems to be investigated now are the following: In the first place we must find the influence of the vortices which constitute the band, upon the circulation around the airfoil. It has been mentioned already that these vortices produce a vertical velocity in the neighborhood of the airfoil. If this velocity is calculated by the ordinary formula, an infinity appears at the trailing edge. We cannot therefore proceed in the same way as with the vertical velocity derived from the trailing vortices in the case of the airfoil of finite span, where the influence could be expressed with sufficient accuracy by a correction to be applied to the angle of attack.

This difficulty can be overcome, according to the treatment given by Wagner, by calculating the influence of a free vortex upon the flow around an airfoil with the aid of the method of conformal transformation[1].

The basic assumption made in the calculation of the circulation is, as formerly, that its value at every moment produces *tangential flow at the trailing edge* (see I 8).

By combining (1.2) with the mathematical expression for the influence of the free vortices upon the circulation, a system of two equations is obtained, from which both Γ and $\bar{\gamma}$ can be solved.

[1] A different method is followed by BIRNBAUM and by KÜSSNER, leading, however, to rather cumbersome formulae.

Having obtained the circulation around the airfoil and the distribution of the vorticity in the wake, the next problem becomes the determination of the force and of the moment acting on the airfoil. This requires a special investigation, as the Kutta-Joukowski theorem in its usual form cannot be applied in the case of variable motion.

Before starting with the problems indicated, it is necessary first to consider the theory of variable motion from another point of view, and to establish connections with the deductions of Chapter III.

2. Equations for the Motion of a Fluid under the Influence of External Forces, if both the Latter and the General Velocity V are Functions of the Time. We return to the point of view adopted in III **6** and consider a fluid having originally a rectilinear motion with velocity V, parallel to $O\,x$, the same throughout the whole field, but possibly depending on the time; the fluid experiencing the influence of a system of external forces, which also may be functions of the time. An acceleration or a deceleration of the velocity V over the whole field can be obtained only by the action of a uniform pressure gradient in the direction of the x axis of the amount $-\varrho\, d\,V/d\,t$ throughout the field; we are not, however, concerned with this part of the problem, as in most cases the velocity V is only apparent, being introduced in consequence of the circumstance that it is convenient to work with a system of coordinates connected with the force system. Hence in the first equation of the system III (1.1) we shall diminish the left hand side by the term $d\,V/d\,t$, and simultaneously the right hand side by the corresponding pressure gradient; this having been done we shall keep the letter p for the part of the pressure which is connected only with the action of the forces to be investigated.

With the notations of III **6** the equations of motion then take the form:

$$\left.\begin{aligned}
\frac{\partial w_x}{\partial t} + V\frac{\partial w_x}{\partial x} + w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y} + w_z\frac{\partial w_x}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{f_x}{\varrho}\\
\frac{\partial w_y}{\partial t} + V\frac{\partial w_y}{\partial x} + w_x\frac{\partial w_y}{\partial x} + w_y\frac{\partial w_y}{\partial y} + w_z\frac{\partial w_y}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \frac{f_y}{\varrho}\\
\frac{\partial w_z}{\partial t} + V\frac{\partial w_z}{\partial x} + w_x\frac{\partial w_z}{\partial x} + w_y\frac{\partial w_z}{\partial y} + w_z\frac{\partial w_z}{\partial z} &= -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{f_z}{\varrho}
\end{aligned}\right\} \qquad (2.1)$$

They differ from III (6.1) by the presence of the derivatives $\partial w_x/\partial t$, $\partial w_y/\partial t$, $\partial w_z/\partial t$.

The transformations applied in III **6** can be taken over without change, as they do not involve differentiations with respect to time. Instead of the system III (6.7) we thus obtain:

$$\left.\begin{aligned}
\frac{\partial w_x}{\partial t} + V\frac{\partial w_x}{\partial x} &= -\frac{1}{\varrho}\frac{\partial q}{\partial x} + \frac{k_x}{\varrho}\\
\frac{\partial w_y}{\partial t} + V\frac{\partial w_y}{\partial x} &= -\frac{1}{\varrho}\frac{\partial q}{\partial y} + \frac{k_y}{\varrho}\\
\frac{\partial w_z}{\partial t} + V\frac{\partial w_z}{\partial x} &= -\frac{1}{\varrho}\frac{\partial q}{\partial z} + \frac{k_z}{\varrho}
\end{aligned}\right\} \qquad (2.2)$$

It will be readily seen that the equation for q and the formulae derived in III 8 remain valid in the present case. Also in solving for the velocity components, the same decomposition can be used as was introduced in III 9. The equations for the quantities w'_x, w'_y, w'_z, from which depends the vorticity, now take the form:

$$\left.\begin{aligned} \frac{\partial w'_x}{\partial t} + V\frac{\partial w'_x}{\partial x} &= \frac{k_x}{\varrho} \\ \frac{\partial w'_y}{\partial t} + V\frac{\partial w'_y}{\partial x} &= \frac{k_y}{\varrho} \\ \frac{\partial w'_z}{\partial t} + V\frac{\partial w'_z}{\partial x} &= \frac{k_z}{\varrho} \end{aligned}\right\} \tag{2.3}$$

Restricting the treatment to the case of plane motion, k_y will be taken zero, while k_x and k_z are assumed independent of y. Then:

$$w'_y = 0 \tag{2.4}$$

It remains to solve the equations for w'_x and w'_z. As both are of the same form, it is sufficient to consider one of them. For simplicity we omit the suffixes and write:

$$\frac{\partial w'}{\partial t} + V\frac{\partial w'}{\partial x} = \frac{k}{\varrho} \tag{2.5}$$

In order to overcome the complication arising from the circumstance that V may be a function of t, we divide the equation by V and introduce a new independent variable τ, connected with t by the relation:

$$d\tau = V dt \tag{2.6}$$

In the case of constant V, we can simply take $\tau = Vt$; in other cases τ is obtained by an integration. Equation (2.5) now takes the form:

$$\frac{\partial w'}{\partial \tau} + \frac{\partial w'}{\partial x} = \frac{k}{\varrho V} \tag{2.7}$$

The same as in the case of the equations of III 9, we require a solution which vanishes for infinite negative values of x. This solution is given by:

$$w' = \int_{-\infty}^{x} d\xi \left(\frac{k}{\varrho V}\right)_{\tau - x + \xi} \tag{2.8}$$

The suffix $(\tau - x + \xi)$ added to $(k/\varrho\, V)$ indicates that in order to calculate the value of w' at a point x, y, z at the time corresponding to a given value of the variable τ, we must integrate the values of $(k/\varrho\, V)$ over all points ξ, y, z satisfying $-\infty < \xi < x$, *not, however, at the time corresponding with* τ, but at some earlier time, determined by the value $(\tau - x + \xi)$ of the auxiliarly variable for every point ξ.

That the integral (2.8) actually is a solution of (2.7) can be shown by differentiation. Making use of the relation:

$$\frac{\partial}{\partial x}\left(\frac{k}{\varrho V}\right)_{\tau - x + \xi} = -\frac{\partial}{\partial \tau}\left(\frac{k}{\varrho V}\right)_{\tau - x + \xi}$$

we have:
$$\frac{\partial w'}{\partial \tau} = \int_{-\infty}^{x} d\xi \frac{\partial}{\partial \tau}\left(\frac{k}{\varrho V}\right)_{\tau - x + \xi}$$

$$\frac{\partial w'}{\partial x} = \frac{k}{\varrho V} - \int_{-\infty}^{x} d\xi \frac{\partial}{\partial \tau}\left(\frac{k}{\varrho V}\right)_{\tau - x + \xi}$$

Substituting these results in (2.7) it will be seen that the equation is satisfied.

The physical interpretation of the result expressed by (2.8) can be summed up as follows: The action of any generalized force k, having its point of application at ξ, y, z, needs a certain lapse of time in order to make felt its influence at the point x, y, z (where $x > \xi$); this time, if measured by the variable τ, is equal to the distance $x - \xi$. As it is evident from (2.6) that the auxiliary variable τ is the path described by the fluid at infinity with respect to the force system, we may also say that the action produced by any force is carried along by the general motion of the fluid with the velocity V. This is equivalent to the statement made in the preceding section, *viz.* that in the picture of an airfoil (or, of a force system) moving through air at rest, the vortices remain at the spots where they are formed.

3. Continuation. Equation for the Vorticity. Reintroducing the suffices x and z, the equations for w'_x and w'_z become:

$$\left.\begin{aligned} w'_x &= \int_{-\infty}^{x} d\xi \left(\frac{k_x}{\varrho V}\right)_{\tau - x + \xi} \\ w'_z &= \int_{-\infty}^{x} d\xi \left(\frac{k_z}{\varrho V}\right)_{\tau - x + \xi} \end{aligned}\right\} \tag{3.1}$$

The vorticity in the field has a component parallel to the y axis only; this component is given by[1]:

$$\gamma = \frac{\partial w'_x}{\partial z} - \frac{\partial w'_z}{\partial x} \tag{3.2}$$

from which:

$$\gamma = -\frac{k_z}{\varrho V} + \int_{-\infty}^{x} d\xi \left[\frac{\partial}{\partial \tau}\left(\frac{k_z}{\varrho V}\right) + \frac{\partial}{\partial z}\left(\frac{k_x}{\varrho V}\right)\right]_{\tau - x + \xi} \tag{3.3}$$

Apart from the restriction to plane motion, we have kept the force system quite general thus far. We now must consider the question, which type of system of generalized forces is suitable for the representation of the action of an airfoil upon the air. In the theory of steady

[1] We are concerned here with continuously distributed vorticity; return to the vortex sheet will be made below.

motion k_x was put equal to zero, thus considering only such generalized forces as are perpendicular to the original motion of the fluid. In the case before us the same restriction can be introduced, if it is required to represent the action of an airfoil which is kept fixed with respect to our system of coordinates. It is necessary, however, to consider also the case of an airfoil describing small oscillations, either transverse or rotational. In this case the generalized force introduced at any point must be chosen *perpendicular to the relative motion of the point of application with respect to the fluid at infinity*, which relative motion (to be denoted by V') is obtained by geometrically subtracting the general velocity V from the motion of the point taken with respect to the system of coordinates applied.

The reason for this choice becomes apparent if V is taken constant and a transverse motion with constant velocity u parallel to Oz is considered. The problem can then be reduced to the old type by a change of coordinate axes, taking the origin at the point of application of the force and the x axis parallel to the relative velocity of this point with respect to the fluid at infinity. If the force acting at the point is assumed independent of the time, we then fall back upon a case of steady motion, as discussed in Chapter III, in which case the generalized force must be perpendicular to the relative velocity.

The argument is supported by the circumstance that according to the deductions of Chapter III, the only vorticity present in this case is the bound vortex passing through the point of application of the force. Returning to the original system of coordinates and applying (3.3), it can be shown (by an analysis which will be omitted) that this requires the resultant of k_x and k_z to be perpendicular to the relative motion $V' = \sqrt{V^2 + u^2}$ of the point of application with respect to the fluid at infinity. Without going into a mathematical demonstration, this can also be seen by assuming the constant force to be replaced by a series of periodic impulses, as was done in III 5. Every impulsive application of the force produces a vortex pair, the plane of which is perpendicular to the direction of the force. These vortex pairs are carried along by the general motion with the velocity V. When now the point of application of the force is shifted in the interval between two consecutive impulses, the successive vortices will lie in one plane and cancel, as in III 5, provided the geometrical difference of the motion of the point of application and the motion of the fluid is contained in the plane of the vortex pair, and thus is perpendicular to the direction of the force. This is illustrated in Fig. 102, where 1, 2, ... 6 indicate a series of consecutive positions of the periodically applied impulse, while (1), (2), ... (6) indicate the positions of the vortex pairs, generated by the periodic impulses, at the instant of the application of the impulse 6. In the diagram at the right hand side, the vector — V indicates the horizontal component of the velocity of the force system with respect to the fluid at infinity; u is the vertical component, which is directed downward in the case considered; their resultant is the vector V', which thus describes the motion of the force system with respect to the fluid at infinity.

We shall not dwell further on this point, as in all cases to be considered subsequently the transverse motion u of the point of application (or of the airfoil) will be assumed to be small in comparison with V. As moreover the most important quantity to be investigated is the z component of the generalized force, which determines the lift, we shall usually leave the component k_x out of consideration. It must be noted, however, that this component is important where it is required to determine the induced resistance experienced by an airfoil in variable motion; in this connection we shall return to this point in **9**.

There is one more question which deserves some attention, *viz.* the definition of the "*bound vorticity*". On account of what has been explained in the foregoing lines, the most appropriate definition of "bound vortices", in the present case, appears to be the vortices whose axes pass through the points of application of the forces and which have their intensity equal to the magnitude of the generalized force divided by the product of the density into the relative velocity V'. For steady motion this definition coincides with that given in III (14.1), if it is remembered that k_y and Q_y are zero in the case of plane motion, while Q_z is independent of y.

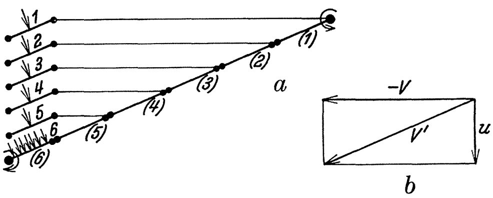


Fig. 102.

We return to the consideration of (3.3) and neglect k_x. The equation then assumes the form:

$$\gamma = -\frac{k_z}{\varrho V} + \int_{-\infty}^{x} d\xi \frac{\partial}{\partial \tau}\left(\frac{k_z}{\varrho V}\right)_{\tau - x + \xi} \tag{3.4}$$

If we neglect the small difference between V and V', the first term may be considered as representing the "bound vorticity". The second represents a vorticity which appears only if the bound vortices are changing. Following up the interpretation of the suffix $(\tau - x + \xi)$ given in the preceding section, we conclude that this part of the vorticity is carried along by the fluid with the velocity V; hence we must call this part of the vorticity the "free vorticity". It is the only part which is present downstream of the region where the generalized forces are applied; it is absent in the case of steady motion.

If the system of generalized forces is intended to represent the action of an airfoil of very thin profile, it is convenient instead of the force per unit volume, to introduce the (generalized) load per unit area (more exactly, per unit of projected area upon the x, y plane), to be denoted by k_z^* as in III 27. The vortices then will be arranged in a vortex sheet and instead of the vorticity γ we must take the strength of this sheet $\bar{\gamma}$.

We take the origin at the trailing edge of the airfoil, and assume the system of generalized forces stretching out from $x = -c$ to $x = 0$ (c being the chord of the airfoil). For nearly all purposes sufficient accuracy is obtained if it is assumed that the vortex sheet stretches out along the x axis[1]. For points with x positive, that is, downstream of the airfoil, (3.4) now becomes:

$$\bar{\gamma} = -\int_{-c}^{0} d\xi \, \frac{\partial}{\partial \tau}\left(\frac{k_z^*}{\varrho V}\right)_{\tau - x + \xi} \tag{3.5}$$

As the values of $\partial/\partial \tau \, (k_z^*/\varrho V)$ occurring under the integral sign are not changed if both τ and x are changed simultaneously by the same amount, it is seen that in this region, $\bar{\gamma}$ is a function of the single variable $\tau - x$ only. This point is of great importance. It is useful, however, first to give the equations for the vorticity in a slightly different form, by starting from (2.7) and differentiating with respect to x. As k_x is neglected, we must neglect also w_x'; hence $\gamma = -\partial w_z'/\partial x$, and thus:

$$\frac{\partial \gamma}{\partial \tau} + \frac{\partial \gamma}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{k_z}{\varrho V}\right),$$

which in the case of a vortex sheet assumes the form:

$$\frac{\partial \bar{\gamma}}{\partial \tau} + \frac{\partial \bar{\gamma}}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{k_z^*}{\varrho V}\right) \tag{3.6}$$

Downstream of the force system the right hand member is zero:

$$\frac{\partial \bar{\gamma}}{\partial \tau} + \frac{\partial \bar{\gamma}}{\partial x} = 0, \text{ for } x > 0 \tag{3.7}$$

which again expresses that for x positive, $\bar{\gamma}$ is a function of the single variable $\tau - x$ only[2].

We may assume that the distribution of k_z^* over the segment from $x = -c$ to $x = 0$ is such, that k_z^* is zero at both ends of this interval. Cases in which the force is assumed to begin or end abruptly can always be obtained by a limiting process. We may also assume that $\bar{\gamma}$ is zero at $x = -c$, while the value of $\bar{\gamma}$ at the point $x = 0$ is to be obtained from (3.5) and may be different from zero.

---

[1] An exception is made only in the consideration of the impulse of the vortex system at the end of **9**.

[2] A proof of this statement is obtained by introducing two new independent variables $\sigma' = x$, $\sigma'' = \tau - x$. Then by the ordinary rules for the change of variables we have: $\frac{\partial \bar{\gamma}}{\partial x} = \frac{\partial \bar{\gamma}}{\partial \sigma'} - \frac{\partial \bar{\gamma}}{\partial \sigma''}$; $\frac{\partial \bar{\gamma}}{\partial \tau} = \frac{\partial \bar{\gamma}}{\partial \sigma''}$. Substituting in (3.6) we find $\partial \bar{\gamma}/\partial \sigma' = 0$. Hence $\bar{\gamma}$ can depend only on the variable σ''.

The circulation around the segment from $x = -c$ to $x = 0$, that is, the circulation around the airfoil which is represented by the system of generalized forces, is given by the integral of $\bar{\gamma}$:

$$\Gamma = \int_{-c}^{0} \bar{\gamma}\, d\, x \tag{3.8}$$

The circulation thus in general will be different from the integral of the "bound vorticity" $-(k_z^*/\varrho V)$. As will be seen in 7, this difference is the cause of the failure of the ordinary form of the Kutta-Joukowski theorem to apply to the case of variable motion.

Integrating (3.6) between the limits $x = -c$ and $x = 0$ we have:

$$\frac{d\,\Gamma}{d\,\tau} + \bar{\gamma}_{(x=0)} = 0 \tag{3.9}$$

Combining this result with the statement that for x positive $\bar{\gamma}$ is a function of the variable $\tau - x$ only, we obtain:

$$\bar{\gamma}_{\tau,\,x} = \bar{\gamma}_{\tau - x,\,0} = -\left(\frac{d\,\Gamma}{d\,\tau}\right)_{\tau - x} \tag{3.10}$$

This equation is equivalent to (1.2), as the quantity $\tau - x$ can be assimilated to s in that equation.

By a second integration with respect to x, assuming that Γ was zero for $\tau = -\infty$, we find:

$$\int_{0}^{\infty} \bar{\gamma}\, d\, x = -\int_{0}^{\infty} \left(\frac{d\,\Gamma}{d\,\tau}\right)_{\tau - x} d\, x = -\int_{-\infty}^{\tau} \frac{d\,\Gamma}{d\,\tau}\, d\tau = -\Gamma \tag{3.11}$$

which is equivalent to (1.4).

4. Circulation around an Airfoil in the Presence of Free Vortices. We return to the problems mentioned in **1** and start with the investigation of the influence of free vortices upon the circulation around an airfoil.

From the deductions of II 8 it follows that in many cases the field of flow around an airfoil can be obtained with sufficient approximation by superposing the effects which are due to the influence of the angle of incidence and to the curvature, each considered separately. Extending this method to the present case, it may be assumed that to obtain an expression for the influence of the free vortices, it is sufficient to derive its value for the most simple type of airfoil; that is, for a flat plate lying in the horizontal plane. Then in order to arrive at the case of an airfoil of arbitrary profile, eventually making vertical or rotatory oscillations, we simply add this expression to the contributions due to other causes, *i. e.* due to the curvature, the angle of incidence and the oscillations.

It will be convenient to start with the expressions for the contributions last mentioned. Partly they can be deduced from the results obtained in Chapter II; in connection with this we shall change our notation, and introduce the ordinary x, y system of coordinates used commonly in the theory of two-dimensional motion. This is obtained from the system as used in **2** and **3** by changing z into $-y$, so that the positive y axis points upward. Further, in accordance with the definition adopted in the footnote to II **3**, positive circulation will be taken in the clockwise direction. The origin of the coordinate system will be taken in the trailing edge of the airfoil, as assumed in **3**.

If the airfoil is kept in a fixed position, then, as results from Chapter II, the combined influence of the shape of the airfoil profile and of the angle of incidence can be taken into account by writing the lift per unit span in the form: $l = (1/2)\, \varrho\, V^2 \cdot 2\pi \alpha c$, where α is the angle of incidence reckoned from the position of zero lift[1]. Dividing by $\varrho\, V$, we obtain the circulation:

$$\Gamma = \pi c V \alpha \tag{4.1}$$

If the airfoil is not at rest, but is making small vertical oscillations, the vertical velocity at any moment being u (reckoned positively downward), the effective angle of incidence is increased by the amount u/V (see II **12**); hence we put:

$$i = \alpha + u/V,$$

and instead of (4.1) we obtain, writing now Γ' in order to distinguish this part of the circulation from other contributions:

$$\Gamma' = \pi c V i = \pi c (V\alpha + u) \tag{4.2}$$

The contribution due to the rotatory oscillations is obtained by noting that a rotatory motion about the center of the chord with the angular velocity $d\alpha/dt$ in the clockwise direction, imparts to the elements of the airfoil a normal velocity of the amount:

$$v_y = -\left(x + \frac{c}{2}\right)\frac{d\alpha}{dt}$$

If we write $x = (c/2)(\cos\theta - 1)$ [see II (6.4), keeping in mind that we have taken the origin in the trailing edge] then:

$$v_y = -\frac{c}{2}\frac{d\alpha}{dt}\cos\theta$$

Hence, by II (6.8):

$$v_r = -c\frac{d\alpha}{dt}\sin\theta\cos\theta \tag{4.3}$$

Substituting this in II (9.12), writing τ for θ under the integral sign and replacing a by $c/4$, it is found that the contribution of the rotatory oscillations to the circulation is:

$$\Gamma'' = +\frac{c^2}{4}\frac{d\alpha}{dt}\int_0^{2\pi} \cot\frac{\tau}{2}\sin\tau\cos\tau\, d\tau = \frac{\pi c^2}{4}\frac{d\alpha}{dt} \tag{4.4}$$

---

[1] In Chapter IV **1** we introduced a coefficient m instead of 2π, in order to take into account the deviation between the experimental values and the value obtained from the theory of thin airfoils. However, as all following calculations are of an approximate character, there is no use in retaining m.

Coming to the contribution to be derived from the presence of a free vortex, we assume, as indicated above, an airfoil in the form of a horizontal flat plate of breadth c and consider an isolated vortex of strength Γ_1 at a distance ξ from the trailing edge (see Fig. 103a). By means of the well known conformal transformation (see II 6) the motion in the x, y plane can be represented upon the motion around a circle of radius $c/4$ (see Fig. 103b). After the transformation the distance of the vortex from the center of the circle has the value:

$$r' = \frac{c}{4} + \frac{\xi}{2} + \frac{1}{2}\sqrt{c\,\xi + \xi^2} \tag{4.5}$$

In order to obtain the flow along the circle produced by this vortex, it is necessary to introduce an imaginary vortex of opposite strength at the distance r'' from the center of the circle, given by:

$$r'' = \frac{c^2}{16\,r'} = \frac{c}{4} + \frac{\xi}{2} - \frac{1}{2}\sqrt{c\,\xi + \xi^2} \tag{4.6}$$

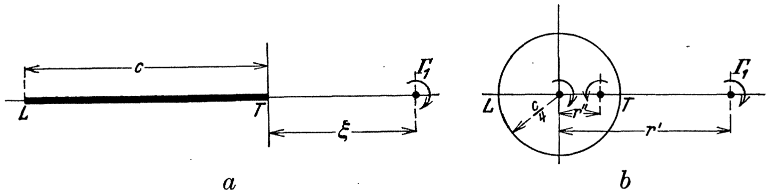


Fig. 103.

and besides, a second imaginary vortex of the same strength as the original at the center of the circle itself. A simple calculation then gives the following value for the velocity at the point T of the circle (corresponding to the trailing edge of the airfoil):

$$\frac{\Gamma_1}{2\pi}\left[\frac{-1}{c/4} + \frac{1}{r' - c/4} + \frac{1}{c/4 - r''}\right] = \frac{2\,\Gamma_1}{\pi\,c}\left[\sqrt{\frac{c+\xi}{\xi}} - 1\right] \tag{4.7}$$

The circulation around the airfoil must have such a value that the flow is tangential at the leading edge, which requires zero velocity at the point T of the circle. Thus the magnitude of the circulation around the circle (which is equal to the circulation around the airfoil) is obtained from the condition that it must annul the velocity indicated by (4.7); that is, it must be equal to the product of this velocity into the circumference of the circle. As the velocity (4.7) is reckoned upward, the circulation must be in the clockwise sense, that is positive, and its amount is:

$$\Gamma_1\left[\sqrt{\frac{c+\xi}{\xi}} - 1\right] \tag{4.8}$$

Returning to the case of a continuous band of vortices, it is necessary in (4.8) to replace the intensity Γ_1 of the vortex by $\bar{\gamma}\,d\,\xi$, and to integrate with respect to ξ from zero (trailing edge) to infinity. This gives as the circulation which is due to the presence of the vortex system:

$$\Gamma''' = \int_0^\infty \bar{\gamma} \left[\sqrt{\frac{c+\xi}{\xi}} - 1\right] d\xi \tag{4.9}$$

Adding together (4.2), (4.4) and (4.9), we obtain the following result for the total circulation around the airfoil:

$$\left.\begin{aligned} \Gamma &= \Gamma' + \Gamma'' + \Gamma''' = \\ &= \pi c (V\alpha + u) + \frac{\pi c^2}{4}\frac{d\alpha}{dt} + \int_0^\infty \bar{\gamma}\left[\sqrt{\frac{c+\xi}{\xi}} - 1\right] d\xi \end{aligned}\right\} \tag{4.10}$$

This expression is valid both for the case of constant and for that of variable V. Taken in conjunction with (3.10) (which remains valid, as both Γ and $\bar{\gamma}$ are reckoned positive in the clockwise direction), it can now be used for the determination of Γ and of $\bar{\gamma}$ in various cases. Two of these cases will be considered in the following sections.

5. Accelerated Rectilinear Motion, Starting from Rest at $t = 0$. It will be assumed that the angle of incidence is constant, and that the airfoil does not make vertical oscillations. Then (4.10) reduces to:

$$\Gamma = \pi c V \alpha + \int_0^\infty \bar{\gamma}\left[\sqrt{\frac{c+\xi}{\xi}} - 1\right] d\xi \tag{5.1}$$

On account of (3.11) this equation can be simplified into:

$$\int_0^\infty \bar{\gamma} \sqrt{\frac{c+\xi}{\xi}}\, d\xi = -\pi c V \alpha \tag{5.2}$$

Here V is to be considered as a given function of the time, which is zero for $t < 0$.

Introducing the auxiliary variable τ defined by (2.6), from which:

$$\tau = \int_0^t V\, dt \tag{5.3}$$

we have from (3.10), provided $\xi > 0$:

$$\bar{\gamma}_{\tau,\xi} = -\left(\frac{d\Gamma}{d\tau}\right)_{\tau-\xi} \tag{5.4}$$

As Γ was zero for $\tau < 0$ (which is the same as $t < 0$), we deduce that $\bar{\gamma}$ must be zero for values of $\xi > \tau$. Hence in (5.2) the integration with respect to ξ can be restricted up to the value $\xi = \tau$. We introduce a variable:

$$s = \tau - \xi \tag{5.5}$$

then $\bar{\gamma}$ becomes a function of s only, while (5.2) assumes the form[1]:

$$\int_0^\tau \bar{\gamma} \sqrt{\frac{c+\tau-s}{\tau-s}}\, ds = -\pi c V \alpha \tag{5.6}$$

[1] The lower limit of this integral, $s = 0$, corresponds to the limit $\xi = \tau$ in (5.2); the upper limit, $s = \tau$, corresponds to $\xi = 0$.

This is an integral equation, which must be solved for the unknown function $\bar{\gamma}$. At the right hand side the velocity V, which is supposed to be given as a function of the time t, must be expressed as a function of the variable τ.

Want of space prevents consideration in detail of the mathematical problem presented by this equation. Solutions have been obtained by Wagner for various cases of accelerated motion by means of special developments for the unknown function[1].

It is of importance to note that a remarkable experimental confirmation of Wagner's results has been obtained by Farren and by Walker, who have measured directly the magnitude of the circulation from photographs of the flow, taken with the aid of a special type of water tank[2]. A number of very beautiful photographs has been made and great care has been given to the investigation of all factors (as, for instance, the finite dimensions of the experimental tank) which might influence the result. These experiments have given a very valuable light upon the origin and the growth of the circulation about an airfoil set in motion.

6. Airfoil Moving with Constant Velocity Describing Harmonic Oscillations. This case has been treated by Glauert[3]. In order to make the example as simple as possible, we assume that the group of terms: $\pi c\,(V\alpha + u) + (\pi c^2/4)\, d\alpha/dt$, occurring in (4.10), by a suitable choice of the origin for t has been reduced to the single expression:

$$\beta_0 + \beta_1 \sin \nu t \tag{6.1}$$

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[1] WAGNER, H., l. c. footnote to p. 280.—The value of $\bar{\gamma}$ for the first stages of the motion can be estimated as follows: If τ is small compared with c, we may write approximately: $\int_0^\tau \bar{\gamma}\, \sqrt{c/(\tau - s)}\, ds = -\pi c V \alpha$.

This is an integral equation of known type (ABEL's integral equation). For instance taking the case of motion with constant velocity, starting impulsively from rest, in which case $V =$ constant for $\tau > 0$, the solution is:

$$\bar{\gamma} = - V \alpha \sqrt{c/s}\,,$$

showing that the initial value of γ is infinite. This solution, however, may be applied to the first moments of the motion only; as s increases, it decreases too slowly to zero and it would lead to an ever increasing value of Γ, whereas the final value of Γ must be $\pi c V \alpha$. To a certain extent this solution might be corrected by inserting the factor $(1 + s/c)$ in the denominator, thus giving:

$$\bar{\gamma} \cong - V \alpha \sqrt{c/s}\big/(1 + s/c)$$

The final value of Γ is now correct; substituting the corrected solution in (5.6) and evaluating the integral (which is elliptic), it is found, however, that for $\tau = c$ the result is 12% too low; and in connection with this, Γ reaches its final value too slowly.

[2] FARREN, W. S., Proc. 3rd Intern. Congress f. appl. Mech. I, p. 323, Stockholm, 1930; WALKER, P. B., see footnote to p. 280.

[3] GLAUERT, H., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1242, March 1929.

It is natural to suppose that the circulation Γ can be represented by an expression of the form:

$$\Gamma = \Gamma_0 + \Gamma_1 \sin \nu t + \Gamma_2 \cos \nu t \tag{6.2}$$

As the velocity V is assumed constant, we can put:

$$\tau = V t \tag{6.3}$$

Then from (3.10) we obtain, writing ξ instead of x:

$$\bar{\gamma} = -\left(\frac{d\Gamma}{d\tau}\right)_{\tau-\xi} = -\frac{1}{V}\left(\frac{d\Gamma}{dt}\right)_{t-\xi/V} \tag{6.4}$$

Substituting (6.2):

$$\left.\begin{aligned}\bar{\gamma} = -\frac{\nu}{V}\Big[&(\Gamma_1 \sin \nu t + \Gamma_2 \cos \nu t) \sin \frac{\nu \xi}{V} + \\ &+ (\Gamma_1 \cos \nu t - \Gamma_2 \sin \nu t) \cos \frac{\nu \xi}{V}\Big]\end{aligned}\right\} \tag{6.5}$$

This expression must be introduced into the integral occurring on the right hand side of (4.10); then

$$\left.\begin{aligned}&\int_0^\infty \bar{\gamma}\left[\sqrt{\frac{c+\xi}{\xi}} - 1\right] d\xi = \\ &= -\lambda\,[S\,(\Gamma_1 \sin \nu t + \Gamma_2 \cos \nu t) + C\,(\Gamma_1 \cos \nu t - \Gamma_2 \sin \nu t)]\end{aligned}\right\} \tag{6.6}$$

where:

$$\lambda = \frac{\nu c}{V} \tag{6.7}$$

while C and S resp. represent the integrals:

$$\left.\begin{aligned}C &= \int_0^\infty \cos \lambda \xi_1 \left[\sqrt{\frac{1+\xi_1}{\xi_1}} - 1\right] d\xi_1 \\ S &= \int_0^\infty \sin \lambda \xi_1 \left[\sqrt{\frac{1+\xi_1}{\xi_1}} - 1\right] d\xi_1\end{aligned}\right\} \tag{6.8}$$

These integrals are transcendental functions of the parameter λ. They have been calculated by Glauert by means of a numerical method; it is possible, however, to express them by means of certain Bessel functions[1]. In this way the following values have been obtained:

$\lambda =$	0	0.2	0.4	0.6	0.8	1.0
$C =$	∞	1.745	1.429	1.254	1.135	1.047
$S =$	0.785	0.711	0.667	0.635	0.608	0.586

[1] The necessary formulae are obtained in the following way. We put: $\lambda = 2\mu$, $\xi_1 = (\xi_2 - 1)/2$, then:

$$C + iS = \int_0^\infty e^{2i\mu\xi_1}\left[\sqrt{\frac{1+\xi_1}{\xi_1}} - 1\right] d\xi_1 = \frac{e^{-i\mu}}{2}\int_1^\infty e^{i\mu\xi_2}\left[\sqrt{\frac{\xi_2+1}{\xi_2-1}} - 1\right] d\xi_2 .$$

If λ is very small, we have approximately: $C = 1/2\,[\log 4/\lambda + 1 - \gamma] = 0.905 - 1/2 \log \lambda$ (the *log* being the natural logarithm, while γ is Euler's constant $= 0.5772$). Hence the limit of the product $C\lambda$, as λ approaches zero, is zero.

Substituting (6.2), (6.1) and (6.6) into (4.10), we obtain:

$$\left.\begin{array}{l} \Gamma_0 + \Gamma_1 \sin \nu t + \Gamma_2 \cos \nu t = \beta_0 + \beta_1 \sin \nu t - \\ \qquad - \lambda\,[S\,(\Gamma_1 \sin \nu t + \Gamma_2 \cos \nu t) + C\,(\Gamma_1 \cos \nu t - \Gamma_2 \sin \nu t)] \end{array}\right\} \quad (6.9)$$

from which, by equating respectively the coefficients of $\sin \nu t$, $\cos \nu t$ and the constant terms on both sides:

$$\begin{aligned} \Gamma_0 &= \beta_0 \\ (1 + \lambda S)\,\Gamma_1 - \lambda C\,\Gamma_2 &= \beta_1, \\ \lambda C\,\Gamma_1 + (1 + \lambda S)\,\Gamma_2 &= 0. \end{aligned} \quad (6.10)$$

The latter equations can be solved for Γ_1 and Γ_2 and the result is given by Glauert in the form:

$$\Gamma_1 = A_1\,\beta_1, \qquad \Gamma_2 = A_2\,\beta_1 \quad (6.11)$$

where: $$A_1 = \frac{1 + \lambda S}{(1 + \lambda S)^2 + \lambda^2 C^2} \qquad A_2 = \frac{-\lambda C}{(1 + \lambda S)^2 + \lambda^2 C^2} \quad (6.12)$$

Values of A_1 and A_2 are given in the following table[1]:

$\lambda =$	0	0.2	0.4	0.6	0.8	1.0
$A_1 =$	1.000	0.801	0.656	0.559	0.490	0.439
$A_2 =$	0	— 0.245	— 0.296	— 0.304	— 0.299	— 0.290

7. Expressions for the Force and the Moment Acting upon the Airfoil. The magnitude of the resulting lift and the moment experienced by

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Now from GRAY, MATHEWS and MACROBERT, A treatise on Bessel functions, p. 50, form. (29), we deduce without difficulty:

$$\int_1^\infty e^{-z\xi_2}\left[\frac{\xi_2 + 1}{\sqrt{\xi_2^2 - 1}} - 1\right] d\xi_2 = K_0(z) + K_1(z) - \frac{e^{-z}}{z}.$$

In this formula it is assumed that z is positive and real. However, both sides of the equation, *qua* functions of z, are holomorphic for $-\pi/2 \leqslant \arg z \leqslant +\pi/2$, and thus the equality holds also for pure imaginary values of z. Putting $z = -i\mu$ and making use of form. (24), p. 23 of the Treatise mentioned, we find:

$$C + i\,S = (1/2)\,e^{-i\mu}[G_0(\mu) + i\,G_1(\mu)] - i/(2\mu).$$

The functions G_0 and G_1 can be expressed by means of the more common functions J_0, Y_0, J_1, Y_1 [l. c. form. (26)]. Tables of numerical values of the latter functions are given for instance in E. JAHNKE und F. EMDE, Funktionentafeln (Leipzig u. Berlin, Teubner, 1933), and thus it is possible to obtain the values of C and S without going into the process of direct numerical computation of the integrals (6.8).

[1] These values have been obtained from the equation:

$$A_1 - i\,A_2 = \frac{e^{i\mu}}{\mu\,[G_1(\mu) - i\,G_0(\mu)]}$$

which is a consequence of the relations deduced in the preceding footnote.

the airfoil can be calculated most conveniently by considering the system of forces by which the action of the airfoil upon the air can be represented. As the theory is confined to airfoils of ordinary type with rather low camber, and to small angles of incidence, we assume, as in **3**, that the points of application of the forces lie upon the x axis, along the segment from $x = -c$ to $x = 0$. As the vortex sheet likewise lies along the x axis, the velocity induced by the vortex system at the points of the segment is everywhere vertical. There are, of course, horizontal velocity components immediately below and immediately above the vortex sheet, the magnitudes of which are given by $\pm (1/2)\,\bar{\gamma}$ respectively (see II **5**), but the mean value of these velocities is zero.

The actual, or f force, is deduced from the generalized force by the subtraction of the g term, according to III (6.6). This point, however, deserves some careful consideration. In III **14** and III **15** a distinction was made between the g forces arising from the presence of the free vorticity in the wake, and those arising from the bound vorticity present in the region where the external forces are acting. A similar distinction must be made in the present cases. The g forces arising from free vorticity will be considered as a new system of generalized forces, which have a certain influence upon the motion of the fluid, to be calculated as soon as it is required to go to higher approximations, a procedure which will not be carried out here. The g forces arising from the bound vorticity on the other hand must be subtracted from the k forces, in order to obtain the actual forces to be applied by external means.

According to the statement following (3.4) the bound vorticity has the intensity $-k_z/\varrho V$ (temporarily we return to the system of coordinates as used in treating three-dimensional motion). We have, therefore, in general:

$$g_x = w_z k_z/V, \qquad g_z = -w_x k_z/V,$$

and thus, from III (6.6):

$$f_x = k_x - w_z k_z/V, \qquad f_z = k_z + w_x k_z/V.$$

In the case of a vortex sheet of neglegible thickness we introduce the forces per unit area f_x^*, f_z^*, k_x^*, k_z^* instead of forces per unit volume; then at the same time we must replace w_x, w_z by their mean values, which for w_x is zero. In this way we obtain:

$$f_x^* = k_x^* - w_z k_z^*/V, \qquad f_z^* = k_z^* \tag{7.1}$$

The component f_x^*, which determines a resistance, will not be considered for the present (see **9**). The component f_z^* represents the vertical reaction per unit area of an element of the airfoil upon the air, and is reckoned positive if directed downward. Coming back now to the notation as used in the two-dimensional motion, we replace f_z^* by f, the lift per unit area experienced by the airfoil from the air. Then (3.6), if at the same time we change the sign of $\bar{\gamma}$ so as to reckon clockwise circulation positive, assumes the form:

$$\frac{\partial}{\partial x}\left(\frac{f}{\varrho V}\right) = \frac{\partial \bar{\gamma}}{\partial x} + \frac{\partial \bar{\gamma}}{\partial \tau} \tag{7.2}$$

We now introduce the potential φ of the motion around the airfoil. Distinguishing the lower and the upper sides of the airfoil by the indices 1 and 2 respectively, we shall have at the lower surface: $V + w_{x1} = \partial \varphi_1/\partial x$, and at the upper surface: $V + w_{x2} = \partial \varphi_2/\partial x$; hence:

$$\bar{\gamma} = w_{x2} - w_{x1} = \frac{\partial}{\partial x}(\varphi_2 - \varphi_1) \tag{7.3}$$

On account of the presence of the circulation, the potential φ is a many-valued function and in order to make the equations definite, we assume that at the leading edge of the airfoil (where $x = -c$) the values of φ_1 and φ_2 are equal. In that case integrating (7.3) we obtain:

$$\int_{-c}^{x} \bar{\gamma}\, d x = \varphi_2 - \varphi_1 \tag{7.4}$$

For $x = 0$ this expression gives the circulation Γ around the airfoil.

Now integrating (7.2) with respect to x, between the limits $-c$ and x, we have:

$$\frac{f}{\varrho V} = \bar{\gamma} + \frac{\partial}{\partial \tau}(\varphi_2 - \varphi_1) \tag{7.5}$$

or, multiplying both members by ϱV and taking notice of (2.6):

$$f = \varrho V \bar{\gamma} + \varrho \frac{\partial}{\partial t}(\varphi_2 - \varphi_1) \tag{7.6}$$

For stationary motion, where $\partial(\varphi_2 - \varphi_1)/\partial t$ is zero, this equation is the same as II (10.1)[1].

From f we obtain the lift per unit span and its moment with respect to the leading edge by the integrals:

$$l = \int_{-c}^{0} f\, d x = \varrho V \Gamma + \varrho \frac{d}{d t} \int_{-c}^{0} (\varphi_2 - \varphi_1)\, d x \tag{7.7}$$

$$m = \int_{-c}^{0} f (c + x)\, d x = \varrho V \int_{-c}^{0} \bar{\gamma}\, (x + c)\, d x + \varrho \frac{d}{dt} \int_{-c}^{0} (\varphi_2 - \varphi_1)(c + x)\, d x \tag{7.8}$$

---

[1] Equation (7.6) can be deduced directly from a form of Bernoulli's equation which is valid for unsteady irrotational motion, *viz.:*

$$p = \text{const.} - (1/2)\, \varrho\, v^2 - \varrho\, \partial \varphi/\partial t.$$

This formula gives for the difference of pressures:

$$p_1 - p_2 = \frac{1}{2} \varrho\, (v_2^2 - v_1^2) + \varrho \frac{\partial}{\partial t}(\varphi_2 - \varphi_1).$$

As $p_1 - p_2 = f$, further: $v_2 - v_1 = \bar{\gamma}$, $(v_2 + v_1)/2 = V$, we obtain:

$$f = \varrho V \bar{\gamma} + \varrho \frac{\partial}{\partial t}(\varphi_2 - \varphi_1).$$

In the expression (7.7) the first term corresponds to the result given by the Kutta-Joukowski theorem. The second term indicates the correction which must be applied on account of the variability of the flow.

Before going further it is necessary to obtain the integrals

$$I = \int_{-c}^{0} (\varphi_2 - \varphi_1)\, d x \tag{7.9}$$

$$J = \int_{-c}^{0} (\varphi_2 - \varphi_1) \left(\frac{c}{2} + x\right) d x \tag{7.10}$$

which are evaluated in the Appendix to this section. Then l and m can be put into the form:

$$l = \varrho V \Gamma + \varrho \frac{d I}{d t} \tag{7.11}$$

$$m = \varrho V (\Gamma c - I) + \varrho \left(\frac{c}{2} \frac{d I}{d t} + \frac{d J}{d t}\right) \tag{7.12}$$

Appendix to Section 7.—Calculation of the Integrals *I* and *J*. The calculation of the integrals I and J is performed most conveniently with the aid of equations derived from the theory of conformal transformation.

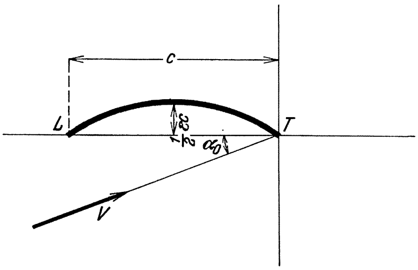

Fig. 104.

The integrals to be obtained may be written in the form:

$$I = -\oint \varphi\, d x \tag{1}$$

$$J = -\oint \left(\frac{c}{2} + x\right) \varphi\, d x \tag{2}$$

where it is assumed that the integration is performed along the whole circumference of the profile in the counterclockwise sense (that is, going to the left along the upper surface and then to the right along the lower surface).

The airfoil will be taken in the form of a circular arc, of chord c and camber $h = (1/2)\, \varepsilon c$, where ε is assumed to be small with respect to unity, so that powers higher than the first may be neglected. The angle between the chord and the direction of the general motion of the air will be denoted by α_0. The angle of incidence reckoned from the position of zero lift is then [see II (10.8) and (17.7)]:

$$\alpha = \alpha_0 + \varepsilon \tag{3}$$

This angle and the camber, however, are of importance only in so far as we are concerned with the part of the flow that depends upon the general motion of the air with the velocity V, including the part of the circulation connected with V. In all other respects (influence of transverse and rotatory oscillations, and of the vortices lying downstream) a sufficiently close approximation can be obtained by neglecting both α_0 and ε. Consequently we may conveniently picture the airfoil as lying with its chord along the x axis, while the general velocity V makes an angle α_0 with this axis (see Fig. 104). The vortices downstream are put all on the x axis as before.

The circular arc can be transformed conformally into a circle of radius $(c/4) \sqrt{1 + \varepsilon^2}$ by means of the equation (see II **17**):

$$z + c/2 = \zeta + c^2/(16\, \zeta) \tag{4}$$

where $z = x + i\,y$ is the complex variable describing the x, y plane and ζ is an auxiliary variable. On the circle we have:

$$\zeta = (c/4)\,[\sqrt{1+\varepsilon^2}\cdot e^{i\theta} + i\,\varepsilon] \tag{5}$$

Neglecting second and higher powers of ε, we obtain:

$$\zeta = (c/4)\,(e^{i\theta} + i\,\varepsilon), \qquad c^2/(16\,\zeta) = (c/4)\,(e^{-i\theta} - i\,\varepsilon\,e^{-2i\theta}) \tag{6}$$

hence:

$$x + c/2 = (c/2)\,(\cos\theta - \frac{1}{2}\,\varepsilon\,\sin 2\,\theta) \tag{7}$$

$$dx = -(c/2)\,(\sin\theta + \varepsilon\cos 2\,\theta)\,d\theta \tag{8}$$

The upperside of the airfoil is described if θ moves from $-\varepsilon$ to $\pi+\varepsilon$; the lower side if θ moves from $\pi+\varepsilon$ to $2\pi-\varepsilon$.

The various parts of the potential φ and the corresponding contributions to the integrals I and J can now be calculated as follows.

(1) The part of φ due to the general motion with velocity V making the angle α_0 with the x axis at the points of the circle can be obtained from II (13.1). On the circle we have, from (6) above, with $\zeta_0 = i\,c\,\varepsilon/4$; $\zeta - \zeta_0 = (c/4)\,e^{i\theta}$; hence, replacing α by α_0, we find:

$$F = \frac{c}{2}\,V\cos(\theta-\alpha_0) + \frac{i\,\Gamma}{2\pi}\,\log\left(\frac{c}{4}\,e^{\,i\theta}\right),$$

from which:

$$\varphi = \frac{c}{2}\,V\cos(\theta-\alpha_0) - \frac{\Gamma}{2\pi}\,\theta$$

If we first take the part depending upon V, the contribution to the integral I amounts to:

$$I_V = \frac{c^2\,V}{4}\int_{-\varepsilon}^{2\pi-\varepsilon} d\,\theta\,(\sin\theta + \varepsilon\cos 2\,\theta)\cos(\theta-\alpha_0) = \frac{\pi}{4}\,c^2\,V\,\alpha_0 = \frac{\pi}{4}\,c^2\,V\,(\alpha-\varepsilon) \tag{9}$$

while the contribution to the integral J is given by:

$$J_V = \frac{c^3\,V}{8}\int_{-\varepsilon}^{2\pi-\varepsilon} d\,\theta\,(\sin\theta + \varepsilon\cos 2\,\theta)\left(\cos\theta - \frac{\varepsilon}{2}\,\sin 2\,\theta\right)\cos(\theta-\alpha_0) = \frac{\pi}{32}\,c^3\,V\,\varepsilon \tag{10}$$

(2) From the part of φ depending upon Γ in the expression given above, we obtain:

$$I_\Gamma = \frac{1}{2}\,c\,\Gamma \tag{11}$$

$$J_\Gamma = \frac{1}{16}\,c^2\,\Gamma \tag{12}$$

In all further calculations ε will be replaced by zero; hence instead of (7) and (8) we have: $x + c/2 = (c/2)\cos\theta$, $dx = -(c/2)\sin\theta\,d\theta$.

(3) The part of φ due to the transverse motion of the airfoil with the velocity u downward, at the points of the circle assumes the value:

$$\varphi_u = \frac{c}{2}\,u\cos\left(\theta - \frac{\pi}{2}\right) = \frac{c}{2}\,u\sin\theta.$$

Contribution to the integrals:

$$I_u = \frac{\pi}{4}\,c^2\,u \tag{13}$$

$$J_u = 0 \tag{14}$$

(4) The part of φ due to the rotation of the airfoil with the angular velocity $d\alpha/dt$ about the point $x = -\frac{c}{2}$ in the clockwise direction is found as follows.

We have from (4.3) in this case: $v_r = -(c/2)\,(d\alpha/dt)\,\sin 2\theta$; hence by comparison with II (6.14), remembering that $r = a = c/4$, we obtain:

$$B_1 = B_3 = B_4 = \ldots = 0;\quad B_2 = -\left(\frac{c}{4}\right)^4 \frac{d\alpha}{dt}.$$

Then from II (6.13), with omission of the first term as the part depending upon Γ has already been considered:

$$\varphi_\Omega = \left(\frac{c}{4}\right)^2 \frac{d\alpha}{dt} \sin 2\theta$$

Contribution to the integrals:

$$I_\Omega = 0 \tag{15}$$

$$J_\Omega = \frac{\pi}{128}\, c^4\, \frac{d\alpha}{dt} \tag{16}$$

(5) The contributions due to the vortices lying downstream of the airfoil will be deduced in a slightly different way. The field around the circle, due to the presence of a single vortex, has already been considered in 4; it is obtained by introducing two imaginary vortices together with the actual one. As the flow is directed tangentially, the stream function ψ corresponding to it has a constant value at the points of the circle. Hence the integrals to be calculated can be written in the form: $\int \varphi\, d x = \int (\varphi + i\psi)\, dx = \int \chi\, dx$,

$$\int \varphi\left(x + \frac{c}{2}\right) dx = \int (\varphi + i\psi)\left(x + \frac{c}{2}\right) dx = \int \chi\left(x + \frac{c}{2}\right) dx.$$

The complex potential χ is single valued in this case (the actual vortex lying outside of the circle, while the two imaginary vortices assumed in the interior of the circle being of opposite strength, do not give any circulation around it); hence the integrals can be transformed by integration by parts, thus giving:

$$\int \chi\, dx = -\int \left(x + \frac{c}{2}\right) d\chi \tag{17}$$

$$\int \chi \left(x + \frac{c}{2}\right) dx = -\frac{1}{2}\int \left(x + \frac{c}{2}\right)^2 d\chi \tag{18}$$

As in these calculations the camber of the airfoil is neglected, we have the relations:

$$x + \frac{c}{2} = 2\, Re.\, \zeta;\qquad \frac{1}{2}\left(x + \frac{c}{2}\right)^2 = \frac{c^2}{16} + Re.\, \zeta^2,$$

where *Re.*, as formerly, indicates that the *real* part only of the expressions is to be taken. Hence our integrals can be further transformed into:

$$\int \chi\, dx = -2\, Re. \int \zeta\, d\chi \tag{19}$$

$$\int \chi\left(x + \frac{c}{2}\right) dx = -Re. \int \zeta^2\, d\chi \tag{20}$$

In this way the quantities to be calculated are made to depend on circular integrals in the complex ζ plane, which can be obtained by means of the method of residues.

The expression for the complex potential of the field in the ζ plane can be written for a single vortex of strength Γ_1:

$$\chi = \frac{-\Gamma_1}{2\pi i} \log \frac{\zeta(\zeta - r')}{\zeta - r''} \tag{21}$$

where r' and r'' are the quantities given resp. by (4.5) and (4.6). The integral (19) consequently assumes the form:

$$2\, Re.\, \frac{\Gamma_1}{2\pi i} \int \left[1 + \frac{\zeta}{\zeta - r'} - \frac{\zeta}{\zeta - r''}\right] d\zeta,$$

which by the theory of residues yields the result:

$$\int \chi \, d x = -2 \, \Gamma_1 r'' \tag{22}$$

Replacing Γ_1 by $\bar{\gamma} \, d \, \xi$ and integrating with respect to ξ, the last part of the integral I becomes:

$$I_\gamma = \int_0^\infty \bar{\gamma} \left[\frac{c}{2} + \xi - \sqrt{c \xi + \xi^2} \right] d \, \xi \tag{23}$$

The integral (20) assumes the form:

$$Re. \frac{\Gamma_1}{2 \pi i} \int \left[\zeta + \frac{\zeta^2}{\zeta - r'} - \frac{\zeta^2}{\zeta - r''} \right] d \zeta,$$

which gives:

$$\int \chi \left(x + \frac{c}{2} \right) d x = -\Gamma_1 r''^2 \tag{24}$$

Replacing Γ_1 by $\bar{\gamma} \, d \, \xi$ and integrating with respect to ξ, the last part of the integral J becomes:

$$J_\gamma = \int_0^\infty \bar{\gamma} \left[\frac{c^2}{16} + \frac{c \xi + \xi^2}{2} - \left(\frac{c}{4} + \frac{\xi}{2} \right) \sqrt{c \xi + \xi^2} \right] d \xi \tag{25}$$

Collecting the various terms the final results respectively are given by the following formulae:

$$I = \frac{\pi c^2}{4} (V \alpha - V \varepsilon + u) + \frac{c}{2} \Gamma + \int_0^\infty \bar{\gamma} \left[\frac{c}{2} + \xi - \sqrt{c \xi + \xi^2} \right] d \xi \tag{26}$$

$$J = \frac{\pi c^3}{32} V \varepsilon + \frac{c^2}{16} \Gamma + \frac{\pi c^4}{128} \frac{d \alpha}{d t} + \int_0^\infty \bar{\gamma} \left[\frac{c^2}{16} + \frac{c \xi + \xi^2}{2} - \left(\frac{c}{4} + \frac{\xi}{2} \right) \sqrt{c \xi + \xi^2} \right] d \xi \tag{27}$$

8. Calculation of the Force Experienced by the Airfoil. The expression for I must be substituted in (7.11) for the lift l.

(*a*) First take the case of the airfoil in *accelerated rectilinear motion*, u being zero and α being kept constant. The derivative of I with respect to t then becomes:

$$\frac{d I}{d t} = \frac{\pi c^2}{4} (\alpha - \varepsilon) \frac{d V}{d t} + \frac{c}{2} \frac{d \Gamma}{d t} + \int_0^\infty \frac{\partial \bar{\gamma}}{\partial t} \left[\frac{c}{2} + \xi - \sqrt{c \xi + \xi^2} \right] d \xi \tag{8.1}$$

In the last integral we replace $\partial \bar{\gamma} / \partial t$ by $- V \partial \bar{\gamma} / \partial \xi$ in consequence of (3.7); the integral can then be transformed by integrating by parts, giving:

$$\frac{c}{2} V \bar{\gamma}_{(x=0)} + V \int_0^\infty \bar{\gamma} \left[1 - \frac{\xi + c/2}{\sqrt{c \xi + \xi^2}} \right] d \xi$$

On account of (3.9) the integrated term cancels the term $(c/2) \, d\Gamma / dt$ in (8.1). Substituting the result in (7.11), replacing Γ at the same time by the value given by (5.1), we obtain, after some reduction:

$$l = \pi c \varrho V^2 \alpha + \frac{\pi c^2}{4} (\alpha - \varepsilon) \varrho \frac{d V}{d t} + \varrho V \frac{c}{2} \int_0^\infty \frac{\bar{\gamma}}{\sqrt{c \xi + \xi^2}} d \xi \tag{8.2}$$

If the motion started from rest at $\tau = 0$, the integration in the last integral must be extended not farther than up to $\xi = \tau$.

For very small values of τ we may neglect the difference between:

$$c\int_0^\tau \frac{\bar{\gamma}}{\sqrt{c\,\xi+\xi^2}}\,d\xi \quad \text{and} \quad \int_0^\tau \bar{\gamma}\sqrt{\frac{c+\xi}{\xi}}\,d\xi$$

According to (5.2), in which the upper limit likewise must be replaced by τ, the latter expression is equal to $-\pi\, c\, V\,\alpha$. Hence for τ small compared with c, (8.2) becomes, approximately:

$$l \cong \frac{\pi}{2}\,c\,\varrho\, V^2\,\alpha + \frac{\pi\,c^2}{4}\,(\alpha-\varepsilon)\,\varrho\,\frac{d\,V}{d\,t}$$

In the first instants after the beginning of the motion the lift thus consists of two parts, one proportional to the acceleration, the other proportional to V^2, and having a value equal to half the value for steady motion.

(b) *Oscillating airfoil, V being constant.* In this case:

$$\frac{d\,I}{d\,t} = \frac{\pi\,c^2}{4}\left(V\frac{d\,\alpha}{d\,t}+\frac{d\,u}{d\,t}\right)+\frac{c}{2}\,\frac{d\,\Gamma}{d\,t}+\int_0^\infty \frac{\partial\bar{\gamma}}{\partial\,t}\left[\frac{c}{2}+\xi-\sqrt{c\,\xi+\xi^2}\right]d\xi \tag{8.3}$$

The integral can be transformed the same way as in the other case; substituting the result in (7.11) and replacing Γ by the expression (4.10), we obtain:

$$l=\pi\,c\,\varrho\,(V^2\alpha+V\,u)+\frac{\pi\,c^2}{4}\,\varrho\left(2\,V\frac{d\,\alpha}{d\,t}+\frac{d\,u}{d\,t}\right)+\frac{\varrho\,V\,c}{2}\int_0^\infty \frac{\bar{\gamma}}{\sqrt{c\,\xi+\xi^2}}\,d\,\xi \tag{8.4}$$

The value of $\bar{\gamma}$ must be taken from (6.5). The evaluation of the integral can be made to depend on that of the transcendental expressions:

$$\left.\begin{aligned} P &= \int_0^\infty \cos\lambda\,\xi_1\,\frac{1}{\sqrt{\xi_1^2+\xi_1}}\,d\,\xi_1 \\ Q &= \int_0^\infty \sin\lambda\,\xi_1\,\frac{1}{\sqrt{\xi_1^2+\xi_1}}\,d\,\xi_1 \end{aligned}\right\} \tag{8.5}$$

These quantities again can be expressed by means of Bessel functions[1]; a few numerical values are given in the accompanying table:

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[1] As in footnote 1 to p. 294, we put: $\lambda=2\mu$, $\xi_1=(\xi_2-1)/2$; then:

$$P+i\,Q=\int_0^\infty e^{2\,i\,\mu\,\xi_1}\frac{1}{\sqrt{\xi_1^2+\xi_1}}\,d\,\xi_1=e^{-i\mu}\int_1^\infty e^{i\mu\,\xi_2}\frac{1}{\sqrt{\xi_2^2-1}}\,d\,\xi_2$$

Now again from GRAY, MATHEWS and MACROBERT, A treatise on BESSEL functions, p. 50, form. (29), we have for z positive and real:

$$\int_1^\infty e^{-z\,\xi_2}\frac{1}{\sqrt{\xi_2^2-1}}\,d\,\xi_2=K_0(z),$$

which formula remains valid also for pure imaginary values of z. In this way we obtain:

$$P+i\,Q=e^{-i\mu}\,G_0(\mu).$$

$\lambda =$	0	0.2	0.4	0.6	0.8	1.0
$P =$	∞	2.554	1.973	1.665	1.464	1.320
$Q =$	1.571	1.318	1.187	1.092	1.019	0.959

For very small values of $\lambda : P = 0.809 - \log \lambda$.

(*c*) *Calculation of the moment m in the case of the oscillating airfoil.* Omitting the details of the reduction, we obtain from (7.12):

$$\left.\begin{aligned} m = \frac{\pi c^2}{4} (\varrho V^2 \alpha + \varrho V^2 \varepsilon + \varrho V u) + \frac{\pi c^3}{8} \left(2 \varrho V \frac{d\alpha}{dt} + \varrho \frac{du}{dt}\right) + \\ + \frac{\pi c^4}{128} \varrho \frac{d^2 \alpha}{dt^2} + \frac{\varrho V c^2}{8} \int_0^\infty \frac{\bar{\gamma}}{\sqrt{c \xi + \xi^2}} d\xi \end{aligned}\right\} \quad (8.6)$$

After elimination of the integral with the aid of (8.4) it is possible to write:

$$m = \frac{c l}{4} + \frac{\pi c^2}{4} \varrho V^2 \varepsilon + \frac{\pi c^3}{16} \left(2 \varrho V \frac{d\alpha}{dt} + \varrho \frac{du}{dt}\right) + \frac{\pi c^4}{128} \varrho \frac{d^2 \alpha}{dt^2} \quad (8.7)$$

As remarked by Glauert, this result proves that the moment, taken with respect to the point at one quarter of the chord from the leading edge, is independent of the vorticity of the wake and thus of the past history of the motion: it depends exclusively upon the instantaneous velocity and the acceleration of the airfoil.

It must be remembered that in the calculations referring to the oscillating airfoil the rotatory oscillations were assumed to take place about the center of the chord; hence the vertical velocity u occurring in several terms of the equations is the velocity of this center. It is not difficult to give the equations for the case of an airfoil oscillating about an arbitrary point of the chord, lying at a distance $h\,c$ from the leading edge, the hinge point itself making no vertical oscillations. It is natural then at the same time to transform the expression for the moment m in such a way, that it refers to the hinge point. The moment obtained in that way may be denoted by m_h. Glauert, after having obtained the full expression for m_h, has investigated more particularly that part of it which is proportional to and in phase with the angular velocity. If this part is written in the form: $-\mu \Omega c^3 \varrho V$ (where $\Omega = - d\alpha/dt$), we can regard μ as a non-dimensional damping coefficient of the oscillatory motion.

Curves have been calculated giving μ as a function of the parameters λ and h. It appears that for a hinge position further forward than 0.25 of the chord, the damping moment changes sign at very low frequencies, and the oscillation of the airfoil will be maintained by an unstable damping moment. A diagram has also been prepared to show a comparison with an experimental result referring to an airfoil of aspect ratio 4.5 with the center of rotation at the point $h = 0.10$. As the case of an airfoil of finite span is not considered in the mathematical deductions, it was assumed that an approximate value of the damping coefficient for this case could be obtained by diminishing the part of μ which depends directly on the lift, proportionally to the reduction of the slope of the lift curve due to the finite span. The damping coefficient so calculated was in fair agreement with the observed values.

In a paper by Birnbaum the problem of the free oscillations of an airfoil hinged elastically has been attacked. In the case investigated the airfoil had two degrees of freedom, the vertical oscillation of the hinge point and the rotatory oscillation of the airfoil about this point; both oscillations are opposed by springs. Assuming simple harmonic motion it appeared possible to investigate the frequencies of the free oscillations of the system and to obtain the conditions for damped and for increasing vibrations. It must be noted that the starting point of Birnbaum's calculations is different from that used by Glauert, and that his results are given in the form of series, arranged with respect to powers of λ and of $\lambda \log \lambda$, which makes a comparison with the formulae given by Glauert rather difficult.

An important extension of Birnbaum's work has been made by Küssner. His deductions include the effect of a hinged flap or rudder attached to the airfoil, the effects due to the elastic properties of the material and to the damping caused by elastic hysteresis, and the influence of mass distribution. Küssner has directed his attention especially to the calculation of the flying velocity, at which the airfoil comes into an oscillation of constant or of increasing amplitude. Numerical calculations have been made for 52 examples. Further an approximate method has been given for the investigation of the three-dimensional problem of airfoil vibrations, which made it possible to extend the results obtained for the case of plane motion to the case of the actual airplane wing, and to compare them with experimental results. From the various results mentioned in Küssner's paper the most important one from the theoretical point of view appears to be that oscillations of increasing amplitude are possible only if the center of gravity of the airfoil lies behind the center of pressure. The motion is a combination of torsional and transverse vibrations, as the necessary energy can be taken out of the air only in case of an oscillatory motion with at least two degrees of freedom.

9. Energy Expended in Producing the Vortex System. A number of interesting problems are connected with the investigation of the energy expended in producing the vortex system behind the airfoil.

In order to have a definite case, we consider a plane airfoil and assume that the translational velocity V and the angle α are constant, while the airfoil describes vertical oscillations according to the formula:

$$u = u_1 \sin \nu t \tag{9.1}$$

In (6.1) we then have:

$$\beta_0 = \pi\, c\, V\, \alpha, \qquad \beta_1 = \pi\, c\, u_1 \tag{9.2}$$

Formula (6.2) combined with (6.10), (6.11) gives:

$$\Gamma = \pi\, c\, V\, \alpha + \pi\, c\, u_1\, (A_1 \sin \nu\, t + A_2 \cos \nu\, t) \tag{9.3}$$

while the distribution of the vorticity in the wake as determined by (6.5) becomes:

$$\left.\begin{aligned} \bar{\gamma} = -\frac{\pi\, \nu\, c\, u_1}{V}\Big[&(A_1 \sin \nu\, t + A_2 \cos \nu\, t) \sin \frac{\nu\, \xi}{V} + \\ &+ (A_1 \cos \nu\, t - A_2 \sin \nu\, t) \cos \frac{\nu\, \xi}{V}\Big] \end{aligned}\right\} \tag{9.4}$$

The lift experienced by the airfoil can be calculated from (8.4), in which the integral with respect to ξ must still be worked out. With the introduction of the functions P and Q, determined by (8.5), we obtain (with $\lambda = \nu\, c/V$):

$$\left.\begin{aligned} l = \pi\, c\, \varrho\, V^2\, \alpha + \frac{\pi\, c^2}{2}\, \varrho\, u_1\, \nu\, \Big[&\left(\frac{2}{\lambda} - A_1 Q + A_2\, P\right) \sin \nu t + \\ &+ \left(\frac{1}{2} - A_1\, P - A_2 Q\right) \cos \nu\, t\Big] \end{aligned}\right\} \tag{9.5}$$

It is seen that the expression for the lift contains three terms: one of them is a constant, connected with the angle α; the others are proportional to the amplitude of the oscillations, the first of them being in phase with u, while the second shows a difference of phase of 90^0. The circumstance that part of l is in phase with u, shows that work must be expended in order to keep up the oscillations. As l is directed upward, the force that must balance l and that does the work is directed downward, that is, in the same direction in which the velocity of the oscillation u is counted. The amount of work to be expended in unit of time can be calculated by determining the mean value of the product of l into u:

$$W = \overline{l\, u} = \frac{\pi}{4}\, \varrho\, u_1^2\, c^2\, \nu \left(\frac{2}{\lambda} - A_1 Q + A_2\, P\right) \tag{9.6}$$

We now turn to the calculation of the resistance experienced by the airfoil in its forward motion. In **3** it was pointed out that the generalized force must be normal to the relative velocity of the point of application with respect to the fluid at infinity. In the case before us this relative velocity is the resultant of the velocity $-V$ in the direction of the negative x axis and the velocity u in the downward direction. The vector of the relative velocity thus makes an angle with the x axis, the tangent of which is equal to u/V (see Fig. 102 b). Consequently in the notation used in the beginning of **7** the x component of the generalized force will be given by: $k_x = u\, k_z/V$ in the general case, or by:

$$k_x^* = u\, k_z^*/V \tag{9.7}$$

in the case of forces acting on a sheet of negligible thickness. Substituting the value of k_x^* into (7.1), the x component of the actual force becomes:

$$f_x^* = (u - w_z)\, k_z^*/V \tag{9.8}$$

Now as the airfoil is impenetrable to the air, we have the condition:

$$V\alpha - w_z = -u$$

at all points of the airfoil. Hence, as $k_z^* = f_z^* = f$:

$$f_x^* = -\alpha f \tag{9.9}$$

If now we integrate with respect to x, we obtain as a first part of the resistance experienced by the airfoil:

$$d' = -\int_{-c}^{0} f_x^* \, d\, x = \alpha\, l \tag{9.10}$$

The minus sign has been introduced, as f_x^* represents the force acting on the fluid, the force experienced by the airfoil being thus the opposite of f_x^*.

The expression (9.10), however, does not represent the whole resistance experienced by the airfoil. According to the explanation given at the end of II **10**, it is necessary to take account of the suctional force which may arise at the leading edge of the airfoil. This suctional force is present if the vorticity becomes infinite at the leading edge, the infinity being of the type $\overline{\gamma} = 2\, C/\sqrt{x + c}$ (in the present notation, the leading edge being at the point $x = -c$). The suctional force then has the magnitude [see II (10.20)]: $s = \pi\, \varrho\, C^2$.

Now, as deduced in the Appendix to this section, in the case before us the coefficient C of the infinity is given by the formula:

$$C = \left[V\alpha + \frac{1}{2} u_1 (b_1 \sin \nu t + b_2 \cos \nu t) \right] \sqrt{c} \tag{9.11}$$

where:

$$\left.\begin{aligned} b_1 &= 1 + A_1 - \lambda A_1 (Q - S) + \lambda A_2 (P - C) \\ b_2 &= A_2 - \lambda A_2 (Q - S) - \lambda A_1 (P - C) \end{aligned}\right\} \tag{9.12}$$

The total resistance experienced by the airfoil thus is found to be:

$$d = \alpha\, l - \pi\, \varrho\, C^2 \tag{9.13}$$

We consider the mean value of d with respect to the time. After some elementary calculations it is found:

$$\overline{d} = -\frac{\pi}{4} \varrho\, u_1^2\, c\, \frac{b_1^2 + b_2^2}{2} \tag{9.14}$$

The quantity $\overline{d}$ appears to be *negative*. Hence the airfoil does not experience a resistance, but a *propelling force*. The energy of the propulsion clearly is derived from the work done by the force l in producing the periodic transverse oscillations of the airfoil. The case treated thus affords the simplest example of flapping flight.

In order to make up the balance of the work expended and the work gained it is necessary also to consider the kinetic energy present in the field of flow behind the airfoil. This energy can be calculated if again we assume that the displacements which the vortices impart to each other are negligible. In order to abbreviate notation, we provisionally write $\overline{\gamma} = B \sin(\lambda\, x/c)$ for the strength of the vortex layer; then the vertical velocity w_z at a point of the layer sufficiently far from the airfoil has the value $-(1/2)\,B\cos(\lambda x/c)$, while the difference of the potential at both sides of the vortex sheet is given by $\varphi_2-\varphi_1=-(Bc/\lambda)\cos(\lambda x/c)$. The kinetic energy present in a part of the field is determined by the integral:

$$\frac{1}{2}\varrho\int\!\!\int dx\,dz\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right].$$

This integral can be transformed into another integral, to be extended along the contour of the region considered, *viz.:*

$$\frac{1}{2}\varrho\int\varphi\frac{\partial\varphi}{\partial n}\,ds\,,$$

ds being an element of the contour and n a normal directed outward from the region. If this formula is applied to a strip between two vertical lines at the distance $2\pi c/\lambda$ (the "wavelength" of the vortex system), it is found that the integral reduces to:

$$\frac{1}{2}\varrho\int w_z(\varphi_2-\varphi_1)\,dx=\frac{\pi}{4}\varrho B^2c^2/\lambda^2\,.$$

Now from (9.4) we deduce:

$$B^2=(\pi\nu c\,u_1/V)^2\,(A_1^2+A_2^2)\,.$$

Further, as the number of periods by which the vortex system increases in unit time is equal to $V\lambda/2\pi c$, the increase of kinetic energy in the field, in unit time, is found to be:

$$E=\frac{\pi}{4}\varrho u_1^2c^2\nu\cdot\frac{\pi}{2}(A_1^2+A_2^2) \tag{9.15}$$

We now must have the relation:

$$W=-\bar{d}\,V+E \tag{9.16}$$

By means of the expressions for C, S, P, Q in terms of Bessel functions, as given above, it can be verified that this relation is satisfied[1].

[1] If we insert the values of W, $\bar{d}$, E given respectively by (9.6), (9.14), (9.15), the relation to be demonstrated assumes the form:

$$\frac{2}{\lambda}-A_1Q+A_2P=\frac{b_1^2+b_2^2}{2\lambda}+\frac{\pi}{2}(A_1^2+A_2^2)\,,$$

which, in consequence of (9.12) and (6.12) can be reduced to the following equation:

$$Q+\lambda(PC+QS)-\frac{1}{2}\lambda(P^2+Q^2)=\frac{\pi}{2}\,.$$

Now we have (with $\lambda=2\mu$):

$$P+iQ=e^{-i\mu}G_0(\mu);\qquad C+iS=\frac{1}{2}e^{-i\mu}[G_0(\mu)+i\,G_1(\mu)]-i/2\mu$$

Hence, if we denote the conjugate complex of a quantity by a bar over it:

$$P^2+Q^2=G_0(\mu)\cdot\overline{G}_0(\mu)\,,$$

$$2\mu\,[PC+Q(S+1/2\mu)]=2\mu\,Re.\,[(P-iQ)(C+iS+i/2\mu)]=$$
$$=\mu\,Re.\,\{\overline{G}_0(\mu)\cdot[G_0(\mu)+i\,G_1(\mu)]\}.$$

Thus: $$Q+\lambda(PC+QS)-\frac{1}{2}(P^2+Q^2)=\mu\,Re.\,[i\,\overline{G}_0(\mu)\cdot G_1(\mu)]\,.$$

Making use of Eq. (26), p. 23 and Eq. (48), p. 25 of the treatise on Bessel functions mentioned before, we obtain:

$$\mu\,Re.\,[i\,\overline{G}_0(\mu)\cdot G_1(\mu)]=\mu\,\frac{\pi}{2}[J_1(\mu)\,Y_0(\mu)-J_0(\mu)\,Y_1(\mu)]=\frac{\pi}{2}\,,$$

and thus the equation is demonstrated.

The fact that the vertical oscillations in the case considered lead to the appearance of a propelling force can be elucidated by means of the following consideration. During the downward stroke of the airfoil the circulation is greater than during the upward stroke, at least in so far as it is determined by the term $\pi\, c\, u_1\, A_1 \sin \nu\, t$ in (9.3), which is in phase with u as given by (9.1). Hence when the airfoil is at its highest point, the circulation will be increasing, and thus a vortex will detach itself from the trailing edge with rotation opposite to the direction of the circulation, that is, counterclockwise in the ordinary form of picture. On the other hand, when the airfoil is at its lowest position, the circulation is decreasing and a vortex will detach with clockwise circulation. It is to be expected that the vortex sheet will not exactly coincide with the x, y plane, but will have the form of a series of parallel waves, in such

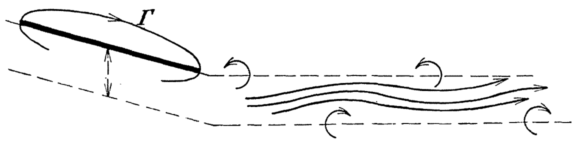

Fig. 105.

a way that at the top of the waves $\overline{\gamma}$ is negative (counterclockwise circulation), while at the bottom of the waves $\overline{\gamma}$ is positive (clockwise circulation). A picture then is obtained as sketched in Fig. 105. If now the velocity distribution due to this vortex system is calculated, it is found that there is a velocity in the direction of the positive x axis in the region bounded by two horizontal lines, one passing through the tops of the waves, the other passing through the bottoms. The velocity distribution thus is similar to the one which would be obtained if a jet of fluid was expelled from the airfoil in the direction of the positive x axis. It is apparent that in both cases the fluid gets an impulse in the direction of $+\,O\,x$; the reaction of the impulse communicated to the fluid is experienced in the form of a force in the direction of $-\,O\,x$ acting on the airfoil.

An exact deduction of the magnitude of the propelling force on the basis of this picture is rather difficult, as it would involve exact data on the form of the vortex sheet—the deviations from the x, y plane being no longer negligible in these calculations. For this reason we shall not pursue this problem further.

Still the reader will have seen, both from 8 and from 9, that the theory of variable motion leads up to a great number of problems of the highest interest. Only the most simple cases have been mentioned here, but actually the subject can be pursued much further, *e. g.* by including also rotational oscillations and by studying the various energy relations which may arise in the general case. In actual cases further complications may arise in consequence of the fact that as soon as the effective angle of incidence reaches, or surpasses, the so-called stalling angle, the flow pattern changes, and vortex motions of much greater intensity may appear. Such cases are of great importance in the problem of flutter, but as theoretical considerations do not afford much help when the stalling angle is surpassed, they are studied best with the aid of experimental methods.

Appendix to Section 9.—Calculation of the Coefficient C in the Expression $\bar{\gamma} = 2\,C/\sqrt{x+c}$ for the Vorticity in the Neighborhood of the Leading Edge. The distribution of the vorticity in the neighborhood of the leading edge can be deduced with the aid of the conformal transformation already used before (see 4 and Appendix to 7; as we are considering a flat airfoil, ε is to be replaced by zero in the formulae of that Appendix). If the velocity at the point L of the circle (see Fig. 103 b; this point corresponds to the leading edge of the airfoil) is denoted by U, then the velocities along the upper and lower sides of the airfoil are given by $\pm\, U\,|d\zeta/dz|$, and thus the vorticity becomes: $\bar{\gamma} = 2\,U\,|d\zeta/dz|$. Now: $z + c/2 = \zeta + c^2/(16\,\zeta)$, $dz/d\zeta = 1 - c^2/(16\,\zeta^2)$, and if ζ is very near to $-c/4$, it is found by some simple calculations, that $dz/d\zeta = 4\,i\,\sqrt{(x+c)/c}$. Hence $\bar{\gamma} = (1/2)\,U\,\sqrt{c/(x+c)}$, and thus: $C = (1/4)\,U\,\sqrt{c}$.

It thus remains to calculate U. By means of the ordinary methods we find that the general motion with the velocity V gives a contribution $2\,V\alpha$, while the vertical motion of the airfoil with the velocity u (positive if downward) gives the contribution $2\,u$. Further there is the amount $2\,\Gamma/(\pi\,c)$ derived from the circulation Γ around the circle. All these contributions are directed upward. In order to find the contribution due to the vortex system, we follow the same method as was used in 4, and first consider a single vortex of strength Γ_1. If in the x, y plane this vortex is situated at the distance ξ from the trailing edge, the corresponding vortex in the ζ plane will be situated at the distance r', given by (4.5), from the center of the circle. This vortex, together with its images, produces the following velocity at L:

$$\frac{\Gamma_1}{2\,\pi}\left[\frac{1}{c/4} + \frac{1}{r' + c/4} - \frac{1}{r'' + c/4}\right] = \frac{2\,\Gamma_1}{\pi\,c}\left[1 - \sqrt{\frac{\xi}{c+\xi}}\right],$$

(positive, if directed upward). If we replace Γ_1 by $\bar{\gamma}\,d\xi$ and integrate with respect to ξ, the contribution of the whole vortex sheet is found to be:

$$\frac{2}{\pi\,c}\int_0^\infty \bar{\gamma}\left[1 - \sqrt{\frac{\xi}{c+\xi}}\right] d\,\xi\,.$$

As rotatory oscillations have not been considered in the present problem, there will be no term depending upon $d\,\alpha/dt$. Thus adding together the various amounts, we obtain:

$$U = 2\,V\alpha + 2\,u + \frac{2\,\Gamma}{\pi\,c} + \frac{2}{\pi\,c}\int_0^\infty \bar{\gamma}\left[1 - \sqrt{\frac{\xi}{c+\xi}}\right] d\,\xi\,.$$

Now $\bar{\gamma}$ is given by (6.5), where A_1 and A_2 are to be determined from (6.11) and (6.12). If it is observed that:

$$1 - \sqrt{\frac{\xi}{c+\xi}} = 1 - \sqrt{\frac{c+\xi}{\xi}} + \frac{c}{\sqrt{c\,\xi + \xi^2}}\,,$$

then it will be seen that the integration with respect to ξ leads to terms which can be expressed by means of the quantities C, S, P, Q already introduced before. In this way, having regard to (9.1)—(9.3), we finally obtain:

$$U = 4\,V\,\alpha + 2\,u_1\,sin\,\nu\,t\,[1 + A_1 - \lambda\,A_1\,(Q - S) + \lambda\,A_2\,(P - C)] + \\ + 2\,u_1\,cos\,\nu\,t\,[A_2 - \lambda\,A_2\,(Q - S) - \lambda\,A_1\,(P - C)]\,.$$

As we have found $C = (1/4)\,U\,\sqrt{c}$, it is readily seen that this result leads up to (9.11).

B. Curvilinear Motion of an Airfoil.

10. General Remarks Concerning the Vortex System in the Case of Curvilinear Flight. The problem of the motion of the air in the neighborhood of an airfoil in curvilinear flight forms another departure from the theory of Chapter III. While in the case of variable motion, investigated in **1**—**9** of the present Chapter, the general velocity V of the air, considered from a coordinate system connected with the airfoil, depended exclusively upon the time, in the case now before us it is a function of the coordinates. It would be a problem in itself to investigate the changes which have to be applied to the equations of III **6** on account of this circumstance. However, in the case of an airfoil with aspect ratio not too small and moving along a path, the radius of curvature of which is large compared with the span of the airfoil, it appears possible to solve the problem of the distribution of the lift across the span without going into such details. We introduce the assumptions (1) that, the same as in the case of rectilinear motion, a system of trailing vortices is generated, the intensity of which is connected in the ordinary way with the circulation around the airfoil; and (2) that these vortices stretch out along the paths which have been described by the elements of the airfoil from which they started.

These assumptions appear to be the most natural generalizations of the mathematical results obtained in Chapter III and the latter assumption fits in with the idea, mentioned in **1** of the present Chapter, that in the first order theory the vortices or the vortical elements remain at the places where they have been formed.

On the basis of these assumptions we shall treat the problem of an airfoil, moving in a horizontal plane *along a circular path*, which has been investigated by Wieselsberger[1]. It will be assumed that the direction of the span of the airfoil likewise is horizontal, and always at right angles to the direction of the motion. Introducing a coordinate system connected rigidly with the airfoil, having its origin in the median section, the y axis along the span and the z axis downward, we suppose that the median section moves along a circle in the x, y plane, having its center at the point $-R$ on the y axis. The angular velocity of the motion will be ω

[1] WIESELSBERGER, C., Zeitschr. f. angew. Math. u. Mechanik, **2**, p. 325, 1922.

(independent of the time). Then the velocity of a section lying at the distance y from the median section is given by:

$$U = \omega\,(R + y) \tag{10.1}$$

We write $\omega R = V$ (velocity of the median section); then:

$$U = V\,(1 + y/R) \tag{10.2}$$

Around every section of the airfoil there will be a certain circulation Γ. In the case considered, the velocity at the airfoil due to the system of trailing vortices has a component parallel to the z axis only, to be denoted by w. Hence we may assume that the relation between Γ and the lift per unit span experienced by the airfoil will be given by the Kutta-Joukowski theorem:

$$l = \varrho\, U\, \Gamma \tag{10.3}$$

which in this form takes the place of III (31.11). The induced drag per unit span will be given by III (31.12)

$$d_i = \varrho\, w\, \Gamma \tag{10.4}$$

The connection between the lift per unit span and the angle of incidence of the section will be determined by the equation:

$$l = \frac{1}{2}\varrho\, U^2\, c\, m\, i \tag{10.5}$$

which takes the place of IV (1.5). Here i is the effective angle of incidence, which is given by: $\quad i = \alpha - w/U \quad$ (10.6)

Combining (10.3), (10.5) and (10.6) we have:

$$\Gamma = \frac{1}{2}\, U\, c\, m\,(\alpha - w/U) \tag{10.7}$$

Finally if the strength of a band of trailing vortices issuing from an element dy of the airfoil, is denoted by $\overline{\gamma}\, dy$, then in virtue of the equation of continuity for the vorticity we must assume:

$$\overline{\gamma} = d\Gamma/dy \tag{10.8}$$

which is the analogue of III (13.4).

From the strength of the trailing vortices we calculate the magnitude of w, having regard to the curved form of the vortices. This will be done in the next section.

11. The Downward Velocity at the Airfoil, Due to Slightly Curved Vortices. Consider a vortex line of strength Γ_1, starting from the point A and bent in the form of a circular arc with its center at M, as indicated in Fig. 106, the radius of the arc being R'. The element $R'd\,\theta$ of this arc, situated at the point Q, will produce a vertical velocity at the point P, which velocity according to Biot and Savart's formula is given by:

$$\frac{\Gamma_1}{4\pi}\,\frac{\sin\psi}{(Q\,P)^2}\,R'd\theta \tag{11.1}$$

The angle ψ occurring here is to be obtained from:

$$\psi = \angle Q'\,Q\,P - \theta,$$

where:
$$\tan Q'QP = \frac{e + R'(1 - \cos\theta)}{R' \sin\theta} \tag{11.2}$$
e being the distance PA. We write:
$$R' \cdot \theta = \xi \tag{11.3}$$
It will be assumed that the radius R' is so great, that for all elements of the vortex line which contribute appreciably to w, the value of ξ/R' is small. Then the various terms depending on θ can be developed according to inverse powers of R'. In carrying out these developments, terms of the order $1/R'$ only will be retained; it is then permissible in all these terms to replace R' (which is different for the various points A) by the value R applying to the median section of the airfoil.

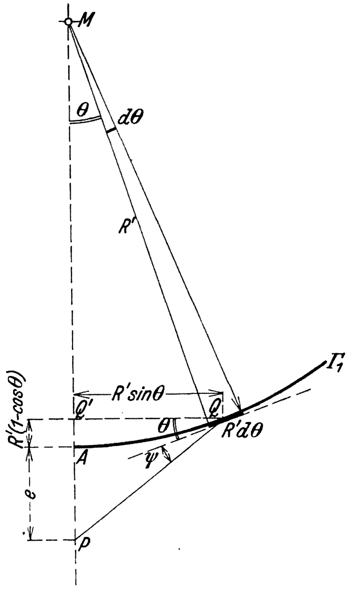

Fig. 106.

In this way the following expressions are obtained:
$$\sin\psi \cong \sin Q'QP - \theta \cos Q'QP \cong \frac{e - \xi^2/2R}{QP},$$
and:
$$\frac{\sin\psi}{(QP)^2} \cong \frac{e - \xi^2/2R}{[\xi^2 + (e + \xi^2/2R)^2]^{3/2}} \cong \frac{e}{(\xi^2 + e^2)^{3/2}} -$$
$$- \frac{\xi^2}{2R}\left[\frac{1}{(\xi^2 + e^2)^{3/2}} + \frac{3e^2}{(\xi^2 + e^2)^{5/2}}\right] \tag{11.4}$$

The latter expression must be substituted in (11.1), $R'd\theta$ being replaced by $d\xi$, and must be integrated with respect to ξ. As the integral does not converge if ξ goes to infinity, we provisionally integrate between the limits 0 and M, where M denotes a length, great compared with e. The result of the integration is:
$$\frac{\Gamma_1}{4\pi}\left\{\frac{M}{e(M^2 + e^2)^{1/2}} + \frac{1}{2R}\left[\frac{Me^2}{(M^2 + e^2)^{3/2}} - \log\frac{M + \sqrt{M^2 + e^2}}{e}\right]\right\},$$
which can be replaced by the approximation:
$$\frac{\Gamma_1}{4\pi}\left(\frac{1}{e} - \frac{1}{2R}\log\frac{2M}{e}\right).$$

This expression will be applied to the calculation of w. As w is to be obtained at the point y, while the vortex starts from η, e must be replaced by $y - \eta$. Under the *log* sign it must be replaced by the absolute value $|y - \eta|$, as the value of e here was obtained from a radical expression in which always the positive root must be taken. Replacing at the same time Γ_1 by $\overline{\gamma}\, d\eta = (d\Gamma/d\eta)\, d\eta$, and integrating with respect to η from $-b$ to $+b$, $2b$ being the span of the airfoil, we obtain:

$$w = \frac{1}{4\pi} \int_{-b}^{+b} d\eta \frac{d\Gamma}{d\eta} \left(\frac{1}{y-\eta} - \frac{1}{2R} \log \frac{2M}{|y-\eta|} \right) \tag{11.5}$$

As the integral of $d\Gamma/d\eta$ over the whole span is zero, $\log 2M$ disappears from the result, so that it can be discarded. Dividing finally by U, the expression for the downward slope of the motion of the air becomes:

$$\varphi = \frac{w}{U} = \frac{1}{4\pi U} \int_{-b}^{+b} d\eta \frac{d\Gamma}{d\eta} \left(\frac{1}{y-\eta} + \frac{\log|y-\eta|}{2R} \right) \tag{11.6}$$

This expression now takes the place of IV (1.2)[1].

12. Determination of the Distribution of the Lift over the Span. From (10.7) and (11.6) it is possible to calculate Γ and φ as functions of y or η, assuming that c and α are given, while U is known from (10.2).

Wieselsberger has considered the case of an airfoil of elliptic plan form; then:

$$c = c_0 \sqrt{1 - y^2/b^2} \tag{12.1}$$

while α is assumed constant. It is sought to find a solution for Γ of the form:

$$\Gamma = \Gamma_0 \sqrt{1 - y^2/b^2}\,(1 + \beta y/R) \tag{12.2}$$

where β is a constant provisionally unknown. With this expression (written as a function of η) substituted in (11.6) it is found, taking $U = V(1 + y/R)$ and neglecting terms of the order R^{-2}, etc.:

$$\varphi = \frac{w}{U} = \frac{\Gamma_0}{4Vb} \left[1 + \frac{y}{R} \left(2\beta - \frac{1}{2} \right) \right] \tag{12.3}$$

This result is substituted in (10.7), which must be satisfied for all values of y. This leads to the following results:

$$\Gamma_0 = \frac{m c_0 V \alpha}{2(1 + \mu_0)} \tag{12.4}$$

$$\beta = \frac{1 + \mu_0/2}{1 + 2\mu_0} \tag{12.5}$$

where μ_0 has been written for $m c_0/8b$. The first result is identical with that expressed by IV (2.15), as $\mu_0 = m/\pi\lambda$. From (12.5) it will be seen that $\beta < 1$, hence the change of Γ is smaller than the change of the velocity, which is given by $U/V = 1 + y/R$.

Wieselsberger proceeds to the calculation of the forces and moments experienced by the airfoil. It is found that the lift and the induced resistance differ from their respective values for rectilinear motion by quantities of the order b^2/R^2 only.

There is a moment of the lifting forces about the x axis of the amount:

$$M_x = \frac{L b^2 (1 + \beta)}{4R},$$

[1] A similar expression is obtained by WIESELSBERGER, who starts by calculating w for a vortex in the form of a semicircle, making use of elliptic functions, and then proceeds to the introduction of approximations of the same nature as those applied here.

and of such a sign, that it tends to lift up the wing which is at the greatest distance from the center of curvature of the path. Further, the induced drag produces a moment about the z axis of the magnitude:

$$M_{zi} = \frac{L^2 (1 + 6\beta)}{16 \pi \varrho V^2 R}.$$

There is also a moment about the same axis due to the difference of the profile drag experienced by the various sections (a consequence of the differences of the velocity U), of the magnitude:

$$M_{z0} = C_{D0} S \varrho V^2 \frac{b^2}{4R}.$$

Both these moments have such a sign that they tend to retard the part of the wing which is at the greatest distance from the center of curvature of the path.

The mathematical deductions are supplemented in Wieselsberger's paper by an experimental investigation, which notwithstanding great technical difficulties, agrees fairly well with the theoretical results.

Wieselsberger's analysis has been extended by Glauert[1] to the case of the *rectangular wing*, the calculations for which are based upon the development of the circulation in a Fourier series, of the type indicated in IV (3.3). As the distribution of the lift over the span becomes asymmetric, terms of both even and uneven orders must be included in the series. The coefficients of even order are proportional to b/R and thus are small compared with those of uneven order.

The moments M_x and $M_z = M_{zi} + M_{z0}$ are given in the form:

$$M_x = \frac{L b^2}{R} F_1, \quad M_z = \frac{L^2}{2 \pi \varrho V^2 R} F_2 + C_{D0} S \varrho V^2 \frac{b^2}{3R}.$$

The latter expression also may be written:

$$M_z = \left(D_i F_2 + \frac{2}{3} D_0\right) \frac{b^2}{R},$$

where D_i is the induced drag and D_0 the profile drag of the airfoil in rectilinear flight. The numerical factors F_1, F_2 are functions of the aspect ratio, or rather of the ratio $1/\mu = 4\lambda/m$. The following values have been calculated by Glauert.

$\frac{1}{\mu}$	F_1	F_2
2	0.435	0.681
3	0.467	0.788
4	0.489	0.864
5	0.508	0.927
7	0.532	1.020

The interest of the problem lies in the fact that the moments M_x, M_z (which in the notation used in the English R. & M. respectively are denoted by $L_r r$ and $-N_r r$; $r = V/R$ being the angular velocity of rotation about the z axis) occur in the theory of the stability of the motion of an airplane. In the elementary theory of stability the influence of the curvature of the trailing vortices is neglected,

[1] Glauert, H., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 866, 1928.

and the coefficient β in (12.2) for Γ is taken equal to unity. This gives for elliptic lift distribution: $F_1 = F_2 = 1/2$, and for the rectangular airfoil, assuming uniform load distribution: $F_1 = F_2 = 2/3$. These values must now be replaced by those given above.

The case of an airfoil describing a circular path in the x, z plane also has been investigated by Glauert[1]. It was found that the influence of the curvature of the trailing vortices upon the value of w was inappreciable. On the other hand the airflow in any section perpendicular to the span appears to be affected by the rotation of the airfoil. This effect is related to some of the problems treated in Part A of the present chapter. On the assumption of two-dimensional flow, the magnitude of the lift and the moment can be deduced from (8.4) and (8.6), by introducing the substitutions:

$$u = 0; \quad d\,u/d\,t = -\Omega\, V; \quad d\,\alpha/d\,t = \Omega; \quad \overline{\gamma} = 0.$$

It is assumed that the center of curvature of the flight path lies above the airfoil, and that the angle of incidence α is defined at the center of the chord. Hence the instantaneous value of u is zero; but as the path of the airfoil is constantly turning upward, there is an acceleration $d\,u/d\,t$ in the direction of $-z$, and of the amount $-\Omega\, V$. As the motion is steady, Γ is constant, and no vortices detach themselves from the trailing edge of the airfoil; thus $\overline{\gamma}$ is zero.

The equations mentioned now give:

$$l = \pi\, c\, \varrho\, V^2 \alpha + \frac{\pi\, c^2}{4} \varrho\, V \Omega\,,$$

$$m = \frac{\pi\, c^2}{4} \varrho\, V^2 (\alpha + \varepsilon) + \frac{\pi\, c^3}{8} \varrho\, V \Omega\,.$$

The second result can also be written in terms of the moment coefficient and the lift coefficient:

$$c_m = \frac{1}{4} c_l + c_\mu + \frac{\pi}{8} \frac{c\,\Omega}{V}\,,$$

where $c_\mu = (\pi/2)\,\varepsilon$. In this form the equation remains valid for an airfoil of finite span. For some further particulars and the expression for the rotary derivative "M_q" the reader is referred to Glauert's papers.

CHAPTER VI

THE DEVELOPMENT OF THE VORTEX SYSTEM DOWNSTREAM OF THE AIRFOIL

1. Introductory Considerations. Following the indications laid down in III 15 regarding the degree of approximation applied in the usual development of the airfoil theory, the distribution of the velocity and of the vorticity in the wake has been calculated to the first order of

[1] Glauert: H., Techn. Rep. Aeron. Res. Committee (Teddington), R. & M. No. 1216, 1928; see also R. & M. No. 1242, 1929.

small quantities only. The values of the velocity components w_x, w_y, w_z and of the components of the vorticity γ_x, γ_y, γ_z are then given respectively by III (9.1) in conjunction with III (9.4), III (9.8) and by III (10.2). The quantities k_x, k_y, k_z in these equations then represent a system of so-called "generalized forces", which are considered as known.

In III 14 it has been shown how the components f_x, f_y, f_z of the actual forces corresponding to such an assumed system of generalized forces could be deduced. It followed that a convenient course for the solution of a great number of hydrodynamic problems was, first to determine a system of generalized forces of such a type that the motion produced by them satisfies the conditions of the problem as well as possible, and then to calculate the corresponding actual forces. The latter step can always be executed, provided the velocity is known with sufficient accuracy, and hence the actual difficulty of the task is embodied in the calculation of the velocity components for a given system of k's.

By way of example, taking the case of a single loaded line (a single wing), extending along the y axis from $y = -b$ to $y = +b$, the first approximation gives a vortex sheet forming part of the x, y plane and stretching out between the lines $z = 0$, $y = -b$ and $z = 0$, $y = +b$ in the direction of $+x$. At the same time, however, in all points of this region of the x, y plane, we obtain a (downward) vertical velocity w_z, which thus is directed perpendicular to the plane of the vortex sheet. This of course is in contradiction with hydrodynamic laws, which assert that in steady motion a sheet of free vortices must be formed by a system of stream-lines, and can not be cut by them.

The question thus arises, in what way can the first order result, used thus far, be corrected, and what will be the actual form of the sheet?

Let us begin with the first question. According to the results of the first approximation, the vortex sheet consists of vortex lines parallel to Ox. From III (6.5) it will be seen that the simultaneous presence of a velocity component w_z and a vorticity component γ_x at a given point of space determines the appearance of a component g_y of the second order or "induced" forces, the magnitude of which is given by:

$$g_y = \varrho\, w_z\, \gamma_x \tag{1.1}$$

For convenience let us assume that the vortex sheet has a certain finite thickness δ and that the vorticity is distributed uniformly over this thickness. The strength of the vortex layer is then determined by:

$$\overline{\gamma} = \gamma_x\, \delta \tag{1.2}$$

Now consider a rectangular element of the layer, with sides dx, dy, respectively parallel to Ox and Oy, and with the dimension δ parallel to Oz. A section of this element by a plane parallel to Oyz is represented in Fig. 107. Following the procedure indicated in III 7, the force g_y obtained in this element must be treated as a k_y force, and the motion produced by it must be calculated.

This motion can be obtained from the example treated in III **12**, the only difference being in the direction of the force (which in III **12** was parallel to Oz), and in a few changes of notation. It will be readily seen that the force $g_y = \varrho\, w_z\, \gamma_x$, acting throughout the interior of the element, produces two vortex sheets, extending in the direction of $+\,x$ respectively from the upper and under surface of the element and having

the strength: $$\overline{\gamma}' = \frac{g_y\, d\,x}{\varrho\, V} = \frac{w_z\, \gamma_x}{V}\, d\,x \qquad (1.3)$$

[see III (12.9), where $\overline{Q}_z$ is to be replaced by $g_y\, d\,x$].

If now the thickness δ is made to decrease to zero, these two sheets taken together may be considered as constituting a *vortex double sheet.* The strength of such a double sheet is determined by the limit of the product of $\overline{\gamma}'$ into δ (the distance of the sheets by which it is formed); making use of (1.2) we obtain for the strength of the double sheet:

$$M' = \overline{\gamma}'\, \delta = \frac{w_z}{V}\, \overline{\gamma}\, d\,x \qquad (1.4)$$

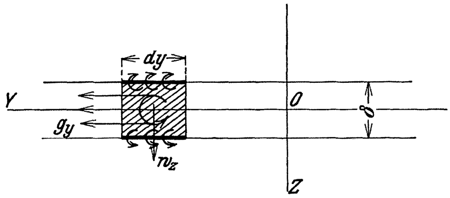

Fig. 107.

It must be noted that this quantity M' is the strength of the double sheet, produced by the g force acting in one element of the vortex layer only. The original vortex layer, however, contains an infinity of such elements, and from each of them a similar double sheet will emanate. Considering in particular a series of elements, lying one behind the other in the direction of $O\,x$, it will be seen that the resulting vortex double sheet is *continually increasing in strength*, as every element adds its contribution to it. If the strength of the vortex double sheet, produced in this way, is denoted by M, then instead of the quantity M' in (1.4) we ought to write dM, *i. e.* the increase of M over the length $d\,x$ of the element considered. Hence (1.4) can be replaced by:

$$\frac{d\,M}{d\,x} = \frac{w_z}{V}\, \overline{\gamma} \qquad (1.5)$$

As $\overline{\gamma}$ is independent of x in the first approximation, the integral of this

equation can be written: $$M = \overline{\gamma} \int_0^x d\,x\, \frac{w_z}{V} \qquad (1.6)$$

Hence we obtain the result that in the second approximation, the original (simple) vortex sheet of strength $\overline{\gamma}$ is accompanied by a vortex double sheet of strength M. In many respects such a combination of a simple sheet with a double sheet is equivalent to a *displacement of* *the simple sheet* in the direction of the axis Oz, over a distance ζ, determined by the equation:

$$\zeta = \frac{M}{\bar{\gamma}} = \int_0^x d\,x \frac{w_z}{V} \tag{1.7}$$

In order to explain what is meant by this statement, the following example relating to a case of plane motion may be noted. Consider a rectilinear vortex of strength Γ, its axis (assumed parallel to Oz) passing through the point $x = \xi$, $y = 0$. Then the x component of the velocity at a point P (coordinates x, y) is given by:

$$v_x = -\frac{\Gamma\, y}{2\,\pi\,[(x-\xi)^2 + y^2]} \tag{1.8}$$

If the distance $r = \sqrt{x^2 + y^2}$ of the point P from the origin is sufficiently great, this expression may be developed into the series:

$$v_x = -\frac{\Gamma\, y}{2\,\pi\, r^2} - \frac{\Gamma\,\xi\, x\, y}{\pi\, r^4} - \text{terms depending on } \xi^2, \xi^3, \ldots \tag{1.9}$$

The first term can be considered as representing the velocity due to a vortex of strength Γ with its axis passing through the origin; the second term can be considered as representing the velocity due to a *double vortex* of moment $\Gamma\,\xi$, coinciding with the first one. The other terms can be considered as representing the velocity due to multiple vortices of higher order. Neglecting the latter terms for the present, we obtain the result that the field due to the combined action of a single vortex and a double vortex, both passing through the origin, is approximately the same as the field due to a single vortex passing through the point ξ, 0.

This result can be easily extended to the case of vortex sheets.

It must be noted, however, that in consequence of the fact that the development (1.9) is possible only provided r is sufficiently great, the mathematical equivalence between the two expressions (1.8) and (1.9), or what comes to the same, between the two systems of vortices which were compared, is not an absolute one, but is restricted to certain conditions necessary for the convergence of (1.9).

2. Continuation. In the field of motion, as obtained in the first approximation, the differential equation for a stream-line has the form:

$$\frac{d\,x}{V + w_x} = \frac{d\,y}{w_y} = \frac{d\,z}{w_z} \tag{2.1}$$

Neglecting in the first term the small quantity w_x in comparison with V, these equations can be integrated in the form:

$$y = \int_0^x \frac{w_y}{V}\, d\,x + \text{const.}, \qquad z = \int_0^x \frac{w_z}{V}\, d\,x + \text{const.} \tag{2.2}$$

The most important for our purpose is the second one of these equations. If the constant is taken zero, it will be seen that the value of z, obtained from it, is the same as the distance ζ, determined by (1.7).

Hence the result of the preceding section may be expressed by saying, that the correction due to the introduction of the second approximation is equivalent in a certain way to a deformation of the vortex sheet so that it *fits the stream-lines passing through the loaded line.*

This result, however, must be considered only as a provisional one. Further researches concerning higher approximations (which in general will require an elaborate system of calculations) are necessary before it can be stated in its full form. Besides it must not be forgotten that, instead of an actual displacement of the vortex sheet, a double sheet, *i. e.* a singularity of higher order is introduced into the field of flow by the process indicated, so that the aspect of the solution greatly differs from what we should expect from the physical point of view. There is an indication that if we do not start from a concentrated vortex sheet, but from a more continuous distribution of vorticity through space, the latter difficulty may disappear (see Appendix to this section).

The method indicated in the preceding section for obtaining the second order corrections in the velocity components, can also be used for the investigation of other cases than the one considered here, and for correcting the pressure distribution. We shall not, however, consider these points, as their importance for the present problem is not great. The reader will already have obtained the impression that the method of constructing successive approximations will be useful only in cases where the deviations of the vortex lines or vortex sheets from the position determined by the first order equations remains small. On the other hand, for the investigation of a problem such, for instance, as the *rolling up of the vortex sheet*, it does not appear very promising. We therefore pass, in the next section, to a short review of a paper by Kaden, who has attacked the latter problem in a more direct way, and which has proved rather successful.

Appendix to Section 2.—On the Influence of Higher Approximations in the Case of a Continuous Distribution of Vorticity. Assume that the value of the vorticity component as given by the first approximation is expressed by the formula:

$$\gamma_x = A\, e^{-\beta z^2} \,(\text{for } x > 0) \tag{1}$$

and that w_z is independent of z. The value of g_y is then:

$$g_y = \varrho\, w_z\, \gamma_x = \varrho\, A\, w_z\, e^{-\beta z^2} \tag{2}$$

The vorticity γ_x' produced by g_y can be calculated from the first equation of the system III (10.2). According to III (9.5) we have:

$$Q_y = \int_0^x g_y\, d\, x = \varrho\, A\, e^{-\beta z^2} \int_0^x w_z\, d\, x \tag{3}$$

Writing for abbreviation:
$$\zeta = \int_0^x \frac{w_z}{V}\, d\,x \tag{4}$$
we may substitute for (3):
$$Q_y = \varrho\, A\, V\, \zeta\, e^{-\beta z^2} \tag{5}$$
Then from III (10.2) we obtain:
$$\gamma'_x = -A\, \zeta \frac{d}{d\,z}\, (e^{-\beta z^2}) \tag{6}$$
For simplicity we shall neglect any corrections that ought to be applied to the value of w_z in consequence of the action of the force g_y. The correction introduced into the distribution of the vorticity, if multiplied into w_z, again gives rise to a corrective term of the third order:
$$g'_y = \varrho\, w_z\, \gamma'_x = -\varrho\, A\, w_z\, \zeta \frac{d}{d\,z}\, (e^{-\beta z^2}) \tag{7}$$

The corresponding quantity Q'_y is given by the integral:
$$Q'_y = -\varrho\, A \frac{d}{d\,z}\, (e^{-\beta z^2}) \int_0^x w_z\, \zeta\, d\,x \tag{8}$$

Now:
$$\frac{1}{V} \int_0^x w_z\, \zeta\, d\,x = \int_0^x \zeta \frac{d\,\zeta}{d\,x}\, d\,x = \frac{1}{2}\, \zeta^2 \tag{9}$$
and hence:
$$Q'_y = -\frac{1}{2}\, \varrho\, A\, V\, \zeta^2 \frac{d}{d\,z}\, (e^{-\beta z^2}) \tag{10}$$
Substituting this value again in III (10.2), we obtain a second correction to the distribution of the vorticity, expressed by the equation:
$$\gamma'' = \frac{1}{2}\, A\, \zeta^2 \frac{d^2}{d\,z^2}\, (e^{-\beta z^2}) \tag{11}$$

It is not difficult to see that we can proceed indefinitely in this way. Adding up all terms, the final value of the vorticity will be given by:
$$\gamma_x + \gamma'_x + \gamma''_x + \cdots = A \sum_0^\infty \frac{(-\zeta)^n}{n!} \frac{d^n}{d\,z^n}\, (e^{-\beta z^2}) \tag{12}$$

The series appearing here represents the development of the function $e^{-\beta (z-\zeta)^2}$ into a Taylor series. Hence we obtain:
$$\gamma_x + \gamma'_x + \gamma''_x + \ldots = A\, e^{-\beta (z-\zeta)^2} \tag{13}$$
The expression arrived at in this way represents a vortex layer, distributed symmetrically on both sides of the surface:
$$z = \zeta = \int_0^x \frac{w_z}{V}\, d\,x \tag{14}$$

The singularities appearing if we start from an infinitely thin vortex sheet, have now disappeared altogether. The series (12), considered as a power series in ζ, is convergent in the whole complex ζ plane, with the exception of the point at infinity.

3. The Rolling up of the Vortex Sheet behind an Airfoil. The problem of the development of the vortex sheet behind an airfoil has a practical interest as well as a theoretical one, as the downward velocity experienced by the tail surface of an airplane is dependent on the exact form of the vortex system.

The motion of the vortex sheet has been investigated both experimentally and theoretically by Kaden[1]. In the following paragraphs a short résumé will be given of the most important points of his calculations, referring the reader to the original paper for more details.

The treatment has been simplified, first by assuming that the motion could be regarded as a two-dimensional one, wholly confined to a plane perpendicular to the x axis. In other words, consider a strip of the vortex sheet cut out by two planes perpendicular to the x axis, at a small distance apart from each other. If we consider the motion of this strip, not with respect to the airfoil, but with respect to the air at infinity, it does not possess any translation in the direction of the x axis. Initially having the form of a straight segment lying along the y axis from $y = -b$ to $y = +b$, its ends gradually roll up into two vortex cores; in this process the central part is drawn out more and more, and at last is absorbed wholly into the cores. It will be assumed that this motion is approximately the same as it would be in the case of a vortex sheet having a form independent of the coordinate x, so that all strips which could be cut out from it simultaneously would present the same stage of development.

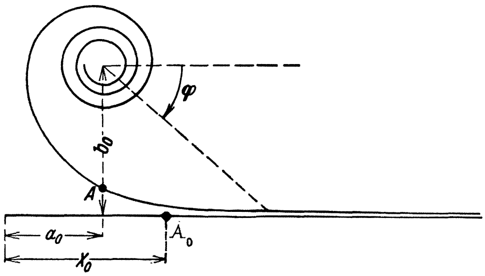

Fig. 108.

As a second simplification it is assumed that instead of treating the two ends together, it is sufficient to investigate the motion of *one* end only, assuming that toward the other side the vortex sheet extends to infinity. This will produce no great error in the results, provided the distance between the two cores is sufficiently large.

From the fact that in the latter form of the problem, relating to the semi-infinite vortex sheet, there is no characteristic length in the field, Kaden deduces that the development of the curve, in which the vortex sheet cuts the plane perpendicular to the x axis, consists in a *proportional expansion* of this curve. The curve has the form of a spiral with straight asymptote (see Fig. 108). In the initial state the dimensions of the spiral formed part are infinitely small, so that the curve can be assimilated to a semi-infinite straight line. As the motion of the sheet diminishes

[1] KADEN, H., Ingenieur-Archiv, Bd. II, p. 140, 1931.

indefinitely if we recede from the edge, the asymptote of the spiral must always coincide with the initial position of the sheet[1].

The distribution of the motion in the initial position is obtained from the equations describing the flow about the edge of a thin semi-infinite plane. From these equations it is known that the velocities at points lying on the same radius through the edge are parallel, and decrease inversely proportionally to the square root of the distance from the edge. On account of the property of the field of expanding proportionally, it must be expected generally that when the dimensions of the field have been increased n times, the velocities will have been multiplied by $n^{-1/2}$. If in the first state a motion over a certain distance δl took place in the interval of time δt, then in the second state the similar motion over the corresponding distance $\delta l' = n\,\delta l$, with velocity $\sqrt{n}$ times smaller, will take place in the interval of time $\delta t' = n^{-3/2}\,\delta t$. It follows that the time elapsed since the beginning of the process is proportional to the 3/2-power of the dimensions of the field, or in other words, that the dimensions of the field increase proportionally to the 2/3-power of the time.

Kaden then considers the interior part of the vortex core. The motion here will be approximately in circles about the center of the core; the tangential velocity u_t will be very nearly independent of the time, and consequently must be inversely proportional to the square root of the distance from the center. Denoting this distance by r, Kaden writes:

$$u_t = \frac{\varkappa}{\pi\sqrt{r}} \tag{3.1}$$

$\varkappa$ being a constant, provisionally unknown. Then the circulation along a circle of radius r has the value:

$$\Gamma = 2\pi r u_t = 2\varkappa\sqrt{r} \tag{3.2}$$

Now a polar angle φ is introduced, in such a way that $\varphi = 0$ at the point at infinity of the spiral, from which point φ increases as we go to the interior. The radius r can be considered as a function of φ, the derivative $\partial r/\partial\varphi$ being negative. The distance between two adjacent windings of the spiral is approximately:

$$\triangle r = -2\pi\frac{\partial r}{\partial\varphi} \tag{3.3}$$

and hence the quantity of fluid flowing inward between two windings in unit time becomes:

$$Q = u_t \triangle r = -\frac{2\varkappa}{\sqrt{r}}\frac{\partial r}{\partial\varphi} \tag{3.4}$$

[1] In the problem of the semi-infinite vortex sheet, the field of flow has been defined in such a way that there is no motion at infinity. If the same definition is adopted in the case of the actual sheet of finite breadth, the sheet itself has a downward motion, which is to be superposed on the motions considered here.

This quantity must be equal to the increase in area included by the spiral up to the point considered, or, what comes approximately to the same, to the increase of the area of a circle of radius r. As r is proportional to $t^{2/3}$, this increase is given by:

$$\frac{\partial}{\partial t}(\pi r^2) = 2\pi r \frac{\partial r}{\partial t} = \frac{4\pi}{3}\frac{r^2}{t} \tag{3.5}$$

Equating the expressions (3.4) and (3.5), we obtain:

$$\frac{\partial r}{\partial \varphi} = -\frac{2\pi}{3\varkappa}\frac{r^{5/2}}{t} \tag{3.6}$$

from which:

$$r = \left(\frac{\varkappa t}{\pi \varphi}\right)^{2/3} \tag{3.7}$$

4. Continuation.—Further Approximations. The result thus obtained is valid for great values of φ only. As soon as φ approaches zero, it becomes more and more inexact, as can be seen by the fact that the spiral determined by (3.7) has no asymptote. In order to arrive at a better approximation, Kaden puts:

$$r = \left(\frac{\varkappa t}{\pi}\right)^{2/3}\frac{1}{\varphi^{\nu}} \tag{4.1}$$

where the exponent ν is assumed to be a function of φ. The calculations are now repeated in a more exact way, taking account of the dependence of u_t both on r and on φ, and of the change of Γ as a function of the time. A connection between ν and φ is obtained, which for instance gives the following values:

$$\varphi = \infty: \quad \nu = 2/3$$
$$\varphi = 4\pi \quad \nu \cong 0.65$$
$$\varphi = 2\pi \quad \nu \cong 0.61.$$

For $\varphi < 2\pi$ the exponent must decrease rather rapidly, as it is shown that for $\varphi = 1$ it becomes $-\infty$.

Then the part of the curve is considered which lies in the neighborhood of the asymptote. A system of formulae is developed, giving the velocity components in the form of integrals extended over this part of the sheet. They are not evaluated, but it is deduced from them that the normal velocity at the points of the sheet which are situated at a sufficient distance d from the center of the core, must be proportional to d^{-3}. The same applies to the deviations of the actual form of the line from the straight asymptote. The factor of proportionality is deduced by means of certain theorems on the center of gravity of the vortex system. Combining the result obtained for the core with that obtained for the part of the curve situated near to the asymptote, a sufficiently exact picture of the form of the sheet could be obtained.

The following results may be mentioned in which it is assumed that in the initial state the strength of the vortex sheet at a distance d from the edge is given by:

$$\overline{\gamma} = \frac{\sigma}{\sqrt{d}} \tag{4.2}$$

The coordinates of the center of the core with respect to the original edge of the vortex sheet are (see Fig. 108):

$$a_0 = 0.57\, X_0\,, \qquad b_0 = 0.88\, X_0 \tag{4.3}$$

where:

$$X_0 = \left(\frac{9}{2\,\pi^2}\,\sigma t\right)^{2/3} \tag{4.4}$$

The point A which lies on the outer winding at the intersection with the perpendicular from the center upon the asymptote, in the initial state was situated at A_0 at the distance X_0 from the edge. In the interior part of the core the form of the spiral is determined by (3.7) and more exactly by (4.1) with the proper values of ν. The constant $\varkappa$ occurring in these equations is related to σ by the formula:

$$\varkappa = \sigma\,\sqrt{3/2} \tag{4.5}$$

5. Application to the Vortex Sheet behind an Airfoil. The equation assumed for the strength of the vortex sheet in the initial stage, (4.2), is the one which is obtained by considering the motion of a fluid about the edge of a semi-infinite plane; it follows at once from the property, mentioned in 3, that the velocity decreases inversely proportional to the distance from the edge.

It applies with sufficient accuracy to the region in the neighborhood of the edge of the vortex sheet behind an airfoil. Taking the case of elliptic distribution of lift, $\overline{\gamma}_x$ is given by III (22.5):

$$\overline{\gamma}_x = -\frac{2\,w_0\,y}{\sqrt{b^2 - y^2}} \tag{5.1}$$

Writing $y = b - d$, assuming that d is very small compared with b, we have approximately:

$$\overline{\gamma}_x = -w_0\,\sqrt{\frac{2\,b}{d}} \tag{5.2}$$

Hence in this case:

$$\sigma = w_0\,\sqrt{2\,b} \tag{5.3}$$

The final stage of the process of the rolling up of the edges consists of two vortices, of strength equal to the value of the circulation around the median section of the airfoil, that is, equal to:

$$\Gamma_0 = \frac{A_{\text{max.}}}{\varrho\,V} = 2\,w_0\,b \tag{5.4}$$

These vortices have a definite distance from each other, which can be calculated from the condition that the impulse of the vortex system is conserved in the process (see Appendix to this section). Denoting the distance between the vortices by $2\,b'$, we have:

$$2\,\varrho\,b'\,\Gamma_0 = 2\,\varrho \int_0^b d\,y\,\overline{\gamma}_x\,y = \pi\,w_0\,b^2\,,$$

from which:

$$b' = \frac{\pi}{4}\,b \tag{5.5}$$

In order to obtain an estimate of the time in which the final stage will be reached, we apply the results deduced for the semi-infinite vortex sheet, and assume that the motion described in the preceding sections goes on until the vortex cores have reached the distance $2b'$. The time T necessary for this is given by the equation:

$$b - b' = 0.57 \left(\frac{9\,\sigma}{2\,\pi^2}\,T\right)^{2/3} \tag{5.6}$$

from which[1]:

$$T = 0.18\,\frac{(2\,b)^2}{\Gamma_0} \tag{5.7}$$

If the velocity of the airfoil with respect to the air is denoted by V, the distance travelled in the time T is:

$$e = V\,T = 0.18\,V\,\frac{(2\,b)^2}{\Gamma_0} \tag{5.8}$$

This expression can be simplified by introducing the lift coefficient for the airfoil. As the mean value of the circulation over the span for elliptic loading is given by $(\pi/4)\,\Gamma_0$, we have $(\pi/2)\,b\,\varrho V\Gamma_0 = (1/2)\,\varrho V^2 S C_L$. Introducing further the aspect ratio of the airfoil by the equation $\lambda = (2\,b)^2/S$, we obtain:

$$\frac{e}{2\,b} = 0.28\,\frac{\lambda}{C_L} \tag{5.9}$$

It will be assumed that from the distance e onward, the process of rolling up can be considered as being finished, and two separate vortices are obtained instead of the original vortex sheet. It appears that e decreases with increasing values of the lift coefficient.

The separate vortices once being formed, they impart to each other a downward velocity of the amount:

$$w_z' = \frac{\Gamma_0}{4\,\pi\,b'} = \frac{2}{\pi^2}\,w_0 \tag{5.10}$$

This is much less than the downward velocity of the vortex sheet in its original state (supposed to extend to infinity in both directions), which is given by w_0.

The theory developed by Kaden does not give the value of the downward velocity of the vortex system in the intermediate stages, though it seems possible, on the basis of his results, to calculate it approximately. This would enable us at the same time to obtain a picture of the form assumed by the system in space.

Appendix to Section 5.—Impulse of a System of Vortices. The considerations of III 3 regarding the generation of vortex rings by the application of an impulsive pressure over a certain area, lead to the conception of the *impulse of a vortex ring*, by which is meant the resultant impulse of the system of forces by which the ring could be generated. In the case of a plane vortex ring the impulse is given by the expression:

$$I = \varrho\,\Gamma A \tag{1}$$

where A is the area enclosed by the ring. Its direction is normal to the plane of the ring.

In the case of a vortex ring which does not lie in one plane, the three components of the vector of the impulse are given by the formulae:

$$I_x = \varrho\,\Gamma A_x\,, \qquad I_y = \varrho\,\Gamma A_y\,, \qquad I_z = \varrho\,\Gamma A_z \tag{2}$$

[1] A numerical error in Kaden's equation (81) has been corrected.

where A_x, A_y, A_z represent the areas of the projections of the vortex ring respectively on the planes Oyz, Ozx, Oxy (taken with properly chosen signs).

Coming to the case of two-dimensional motion, it is hardly possible to speak of the impulse of one single cylindrical vortex. We can, however, introduce the notion of the impulse of a vortex pair, consisting of two opposite vortices of equal strength. It is convenient then to calculate the impulse per unit of height measured in the direction normal to the plane of the motion (*i. e.* parallel to the axis of the vortices). By an obvious generalization of the results for the three-dimensional case, we get for this quantity:

$$I = \varrho \Gamma a \tag{3}$$

where a is the distance of the vortices which constitute the pair.

The impulse of a vortex system is conserved during its motion, provided no exterior forces are acting upon the fluid. This is a consequence of the general theorem of conservation of momentum. In the case of a system of parallel rectilinear vortices, where the motion is two-dimensional and in a plane perpendicular to their axes, the theorem can also be demonstrated directly by writing down the equations for the motion of the vortex cores.

If the plane of motion is the y, z plane, and if the coordinates of the point of intersection of the n-th vortex with this plane are denoted by η_n, ζ_n, the strength of this vortex being Γ_n, we can define the *center of gravity* of the vortex system, having the coordinates:

$$\eta_0 = \frac{\Sigma \Gamma_n \eta_n}{\Sigma \Gamma_n}, \quad \zeta_0 = \frac{\Sigma \Gamma_n \zeta_n}{\Sigma \Gamma_n} \tag{4}$$

The position of the center of gravity remains unchanged during the motion of the system.

For further particulars the reader is referred to H. LAMB, Hydrodynamics (Cambridge) Chap. VII, and to a paper by A. BETZ, Verhalten von Wirbelsystemen, Zeitschr. f. angew. Math. u. Mech. **12**, p. 164, 1932.

6. On the Calculation of the Downward Velocity Experienced by a Tailplane Placed behind a Single Airfoil. Though general equations for the calculation of the velocity components in the neighborhood of an airfoil have been given already in III **23**—**26**, it may be of interest to come back to this problem and to investigate further details. The following considerations have been taken over for the greater part from a paper by Helmbold, who also gives a comparison with experimental values[1].

From the practical point of view it is required to determine the *angle of downwash*, that is the angle which the direction of the airflow makes with the x, y plane. Denoting this angle at the tailplane by φ_T, we have:

$$\varphi_T = w_z / V \tag{6.1}$$

The angle φ_T can be compared with the downward slope φ of the airflow at the points of the airfoil itself. In the case of elliptic lift distribution, this slope is connected with the velocity w_0 by means of the equation:

$$\varphi = w_0 / 2\, V \tag{6.2}$$

[see IV (2.1) or I (11.9)]. In all cases we may assume with sufficient accuracy that its mean value over the span is connected with the lift coefficient by the relation:

$$\varphi = C_L / \pi \lambda \tag{6.3}$$

[see I (12.4) and for further particulars, IV **6**].

[1] HELMBOLD, H. B., Zeitschr. f. Flugtechnik u. Motorluftschiffahrt, **16**, p. 291, 1925.

Helmbold first considers the case of an airfoil with elliptic lift distribution, neglecting any rolling up of the vortex sheet. In connection with the result obtained in the preceding section, this will apply provided the value of the lift coefficient is small, as the distance e then becomes great. This conclusion appears to be confirmed by experimental results. The value of w_z can be obtained from III (23.3), replacing k_z by the load per unit span A, given by III (22.7), and changing the triple integration into a single integration with respect to η. As the value of w_z is required in the plane of symmetry only, we can put $y = 0$. Another procedure is to consider the vortex system associated with the elliptic lift distribution, and to calculate w_z with the aid of Biot and Savart's formula. The integral is an elliptic one, even if z is taken zero. The following values for the latter case have been given by Helmbold, who also mentions an approximate formula which is useful if x/b is near unity.

$\frac{x}{b}$	$\frac{\varphi_T}{\varphi}$
0.6	2.52
0.8	2.32
1.0	2.21
1.2	2.15
1.4	2.12
∞	2.00

If instead of starting from the elliptic lift distribution, the case of a rectangular wing is taken, of very great aspect ratio, we should have two separate vortices, extending from the wingtips. The value of w_z is then given by III (24.10), provided we substitute for Γ its value $\frac{1}{2}\varrho V^2 S C_L/2\varrho b V = b V C_L/\lambda$. We thus obtain:

$$\frac{\varphi_T}{\varphi} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{b^2}{x^2}}\right) \tag{6.4}$$

The result of the preceding section shows that the picture of two separate vortices appears fairly adequate in all cases where the value of C_L is sufficiently great to make e small. It would be inaccurate, however, to take the distance of the vortices always equal to the span of the airfoil; on the contrary we must calculate this distance by applying the theorem of conservation of the impulse of the vortex system, as mentioned before. It is readily found that the impulse of the vortex system (per unit of length in the direction of the x axis) is equal to the lift experienced by the airfoil, divided by V; while the strength of the vortices is equal to the circulation around the median section of the airfoil. Applying the Fourier series of IV 3, restricted (in accordance with IV 5), to the terms with the coefficients A_1, A_3, A_5, A_7, we have:

$$L = 2\pi\varrho V^2 b^2 A_1$$

$$\Gamma_0 = 4 V b (A_1 - A_3 + A_5 - A_7),$$

and hence:

$$\frac{b'}{b} = \frac{L}{2\varrho b V \Gamma_0} = \frac{\pi}{4}\,\frac{A_1}{A_1 - A_3 + A_5 - A_7} \tag{6.5}$$

Values of this ratio can be deduced by means of the data given in IV 5 Table 1 and are collected in the following table.

$\frac{1}{\mu}$	$\frac{b'}{b}$
2	0.843
3	0.863
4	0.875
5	0.887
6	0.896
7	0.902

In the limiting case ($1/\mu = \infty$) the ratio becomes unity. If $1/\mu$ decreases to zero, we approach the case of elliptic lift distribution; in that case the ratio becomes $\pi/4 = 0.785$.

Introducing the appropriate distance of the vortices, the calculations can be made more exact. Helmbold shows that it makes a relatively small difference, whether for the bound vortex along the airfoil, elliptic load distribution is assumed, or a constant loading.

A further refinement of the theory is obtained by considering the position of the bound vortex in the airfoil. The same point occurred in the biplane theory, and in IV 28 it was noted that in order to account for the effect due to the finite magnitude of the chord of the airfoil, Millikan assumed the bound vortex passing through the geometric center of the wing profile (*i. e.* through the center of the chord) to be combined with a double vortex (vortex pair), the strength of which depended on the moment of the air forces acting upon the airfoil. A similar procedure is used by Helmbold, who takes one vortex at the center of the chord, and another midway between the center and the leading edge, *i. e.* at one quarter chord from the leading edge.

Finally it is necessary to take account of the deviation of the vortex system from the x, y plane. In the case of two separate trailing vortices, we can calculate their inclination with respect to this plane from the downward velocity they impart to each other; an integration with respect to x then gives the deviation of the vortices from this plane for various values of x. We pass over the details of these calculations, for which the reader is referred to Helmbold's paper. Some further refinements, perhaps, could be introduced by making use of Kaden's results regarding the gradual formation of these vortices from the original vortex sheet.

7. Conclusion. Other details concerning the vortex system at a large distance behind an airfoil are given by Prandtl, who *i. a.* calculates the suction in the cores of the vortices, and the diameter of the cores[1]. The latter quantity is obtained by calculating the energy present in the field of flow determined by two vortex cores of radius r' at the distance $2b'$ from each other. Per unit of length in the direction of the section, this energy is given by the expression (see Appendix below):

$$E_1 = \frac{1}{2\pi}\,\varrho\,\Gamma_0^2\left(\log\frac{2b'}{r'} + \frac{1}{4}\right) \tag{7.1}$$

This amount must be equal to the induced resistance experienced by the airfoil. Taking the case of elliptic loading we have:

$$D_i = \frac{L^2}{2\varrho V^2 \pi b^2}, \qquad \Gamma_0 = \frac{2L}{\varrho V \pi b}$$

[1] Prandtl, L., Tragflügeltheorie, II. Mitteilung (republished in Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen, 1927), Abschnitt D.

from which is obtained[1]: $r' = \frac{b'}{4.58} = 0.171\, b$ (7.2)

Other interesting problems are those relating to the development of the vortex system behind multiplane combinations. No researches thus far seem to have been published on this subject.

Appendix to Section 7.—Energy of a Vortex Pair. The expression for E_1 is obtained as follows. First consider the part of the field outside of the vortex cores. The motion in this part is irrotational; it has a stream function Ψ given by:

$$\Psi = \frac{\Gamma_0}{2\pi} \log \frac{\sqrt{(y+b)^2 + z^2}}{\sqrt{(y-b)^2 + z^2}} \tag{1}$$

from which the velocity components are obtained by the formulae:

$$w_y = \partial \Psi / \partial z, \quad w_z = -\partial \Psi / \partial y \tag{2}$$

The energy of this part of the field is given by the integral:

$$E' = \frac{\varrho}{2} \int\int d y\, d z\, (w_y^2 + w_z^2) \tag{3}$$

which may also be written:

$$E' = \frac{\varrho}{2} \int\int d y\, d z \left(w_y \frac{\partial \Psi}{\partial z} - w_z \frac{\partial \Psi}{\partial y} \right) \tag{4}$$

The latter expression can be transformed by partial integration. Remembering that $\partial w_y / \partial z = \partial w_z / \partial y$, we obtain:

$$E' = -\frac{\varrho}{2} \int w_s \Psi\, d s \tag{5}$$

where ds denotes an element of the boundary of the field and w_s the tangential velocity along this boundary. To make the formula rigorous it would be necessary to take account of the signs and of the direction in which the boundary is described. In the case before us we can omit these details, as on account of the symmetry of the field the circumference of each vortex core must contribute the same amount, while the integral itself necessarily must have a positive value. At the circumference of a vortex core, Ψ approximately assumes the constant value:

$$\Psi = \frac{\Gamma_0}{2\pi} \log \frac{2\, b'}{r'} \tag{6}$$

while the integral of w_s around the circumference is simply the circulation around the core, which is equal to Γ_0. Multiplying by 2 to take account of both vortices, we obtain:

$$E' = \frac{\varrho\, \Gamma_0^2}{2\pi} \log \frac{2\, b'}{r'} \tag{7}$$

To this amount must be added the energy of the motion in the interior of the cores. Assuming for simplicity that this is a motion with constant angular velocity $\Gamma_0 / 2\pi r'^2$, the energy contained in the two cores is found to be:

$$E'' = \varrho \int_0^{r'} 2\pi r\, d r \left(\frac{\Gamma_0\, r}{2\pi r'^2} \right)^2 = \frac{\varrho\, \Gamma_0^2}{8\pi^2} \tag{8}$$

[1] Kaden's calculations lead to a result of the same order of magnitude. From his diagram (see Fig. 108 above) the horizontal cross section of the spiral by a line through its center appears to be approximately equal to $0.78\, X_0$. Combining this with (4.4) and (5.6) it is found that this corresponds to $0.29 b$. The vertical cross section becomes approximately $0.42\, b$; mean $0.36 b \cong 2\, r'$.

Adding this amount to the value (7), we obtain:

$$E_1 = E' + E'' = \frac{\varrho \Gamma_0^2}{2\pi}\left(log\frac{2b'}{r'} + \frac{1}{4}\right) \tag{9}$$

On comparing with the equations of Chapter III, *e. g.* with III (16.6), it must be noted that the quantity E introduced there is the energy expended during unit of time. As the airfoil describes the path V in unit of time, we have the relation:

$$E = E_1 V \tag{10}$$

CHAPTER VII

THEORY OF THE WAKE

1. Introductory Remarks. The validity of the airfoil theory as presented in the foregoing chapters is limited to the range of the angle of attack in which the airfoil is "unstalled", *i. e.* no "separation" of the flow from the surface of the airfoil occurs. It is known that "stalling" or separation is, in general, connected with a certain decrease of the lift and with a very definite increase of the drag. We do not deal in this volume with the causes and the mechanism of the separation itself, because this problem belongs chiefly to the theory of viscous fluids. However, the motion in the "wake" produced behind a moving body in the case of separation can be approximately described by different methods using only the equations of ideal fluids. These methods are based on the assumption that although the viscous forces are in most cases essential for the separation, their influence on the flow in the wake itself can be neglected in a first approximation.

Three different methods have been advanced to describe the motion in the wake of a moving body as an ideal fluid flow:

a) The method of discontinuous potential motion assumes that the fluid contained in the wake follows the moving body as if it were rigidly connected with the latter. Potential flow is assumed outside of the wake. The two regions are separated by vortex sheets.

b) The theory of vortex streets assumes that the vortex sheets mentioned under a) are disintegrated into a system of individual vortices and the motion in the wake can be derived from this system of vortices following the body.

c) The theory of Oseen assumes that the wake can be considered as a continuous vortex field, while outside of the wake potential motion prevails.

These methods, based on the equations of non-viscous fluids, can describe the motion only at a comparatively small distance behind the body; in order to obtain information as to the flow in the wake at greater distances, frictional forces must be taken into account. Oseen suggested the use of the results obtained by neglecting the viscous stresses as a first approximation, in order to obtain further successive approximations which take into account the viscosity. However, in this way we are restricted to the case of laminar or molecular friction. In most cases of practical importance, the stresses due to the turbulent momentum exchange or turbulent friction, are overwhelming in magnitude as compared with those arising from the laminar friction. An attempt to calculate the velocity distribution in a "turbulent wake" at a large distance from the body was made by L. Prandtl; his method was developed by Tollmien and Schlichting; the same problem was discussed recently by G. I. Taylor and by Mattioli. Prandtl's investigations will be described in Division G.

We give in the following sections a short account of the methods mentioned under a)—c), *i. e.* those using the theory of ideal fluids. In most cases we shall assume two-dimensional motion. However, we do not restrict ourselves to the case of an airfoil, but consider rather the theory of the wake as a contribution to the general theory of the form drag of bodies moving through a fluid.

2. The Method of Discontinuous Potential Motion. Let us consider two-dimensional flow around a cylindrical body and assume that the flow separates at two points S_1 and S_2. Such separation will always occur at sharp corners, but it can also occur at other points of the surface, due to distortion of the velocity distribution in the boundary layer, as will be explained in Division G. The method of the discontinuous potential motion assumes that two lines of discontinuity start from the points S_1 and S_2, so that the velocity is different on the two sides of these lines. The fluid enclosed between the two lines of discontinuity, the fluid in the wake, moves with the same velocity as the body; if we consider the body at rest and the fluid in steady motion, the fluid in the wake will be at rest. Outside of the wake we assume steady vortexless motion. Obviously in neglecting the weight of the fluid, we must assume constant pressure in the wake and therefore along the line of discontinuity. It follows from the equilibrium condition for a volume element including a portion of the line of discontinuity, that the pressures on both sides of the latter are equal. Thus we have constant pressure, and, according to Bernoulli's theorem for ideal fluids, a velocity of constant magnitude along the stream-line following the surface of discontinuity. If the wake extends to infinity, the pressure in the wake and along the line of discontinuity will be equal to the pressure at infinity; hence the velocity of the fluid outside of the wake and along this line will be equal to the velocity at an infinite distance from the body.

Denoting the stagnation point by O, we formulate the conditions to be satisfied by the motion in the following way:

(*a*) the velocity at infinity is equal to U;

(*b*) the portions OS_1 and OS_2 of the boundary of the body (fixed boundary) are portions of stream-lines;

(c) along the continuations of these stream-lines extending from S_1 and S_2 to infinity (so-called free boundaries), the magnitude of the velocity is constant and equal to the velocity at infinity.

The solution of this problem is comparatively easy in the case of bodies having straight boundaries. Hence we consider first a special example of this type.

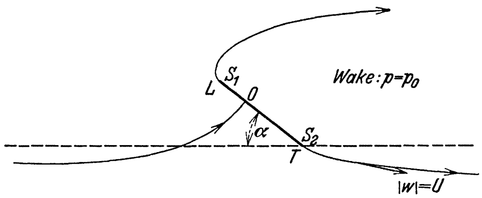

Fig. 109.

3. Discontinuous Potential Motion in the Case of a Straight Airfoil. We introduce, as in the theory of continuous potential motion, two complex functions, namely the complex potential function $f = \varphi + i\psi$, and the complex velocity function $w = u - iv$, both as functions of the complex quantity $z = x + iy$, where x and y are the coordinates in the plane in which the flow takes place. In the case of the flow without separation, the boundaries in the z plane are given, and the solution can be obtained by conformal transformation between the z and f planes. In the present case the shape of the free boundaries in

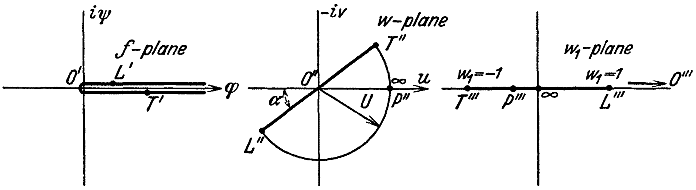

Fig. 110.

the z plane is unknown, so that the method used in the case of continuous potential motion cannot be applied. However, we notice that the free boundaries are represented by circular arcs in the w plane. On the other hand, if the fixed boundaries OS_1 and OS_2 are straight lines in the z plane, they appear as straight lines also in the w plane, so that in this case all boundaries in the w plane are given and a conformal transformation can be established between the f and w planes.

For our present example, let us consider the problem of the straight airfoil with an angle of attack α in an indefinitely extended fluid (see Fig. 109). The velocity of the fluid at infinity outside of the wake will be denoted by U. Let us assume that the stream-line starting at infinity and passing through the stagnation point is the stream-line $\psi = 0$. Then the flow outside the wake is represented in the $f = \varphi + i\psi$ plane by the whole plane. We choose the stagnation point as the origin O' of the φ, ψ coordinate system (see Fig. 110). Then for $\varphi > 0$ the stream-line $\psi = 0$ undergoes a bifurcation; the upper side of the ψ axis corresponds to the stream-line following the airfoil toward the leading edge, the lower side represents the portion of the stream-line between stagnation point and trailing edge. The points L' and T' correspond to the two edges and the portions of the φ axis from $L' \to \infty$ and $T' \to \infty$, to the free boundaries.

Mapping the f plane upon the w plane, we notice that the origin of the f plane corresponds to the origin O'' of the w plane, because at the stagnation point $u = v = 0$; that the portions $O'T'$ and $O'L'$ are represented by straight lines $O''T''$ and $O''L''$ passing through the origin, because the direction of the velocity vector is invariable along the straight airfoil. Finally, the free boundaries $L' \to \infty$ and $T' \to \infty$ are represented by a circle with the radius U, because the magnitude of the velocity $\sqrt{u^2 + v^2}$ is constant along these free boundaries. Hence the portion of the z plane outside of the wake and the whole f plane are represented in the w plane by the interior of the half circle $O''L''P''T''$. The infinite point of the f plane is represented by the point P'' of the real u axis, where $u = U$.

Thus the problem is solved if we find the function $f = F(w)$ mapping the f plane upon the interior of the half circle $O''L''P''T''$ in the w plane, so that the origin O' remains invariable; the portions $O'L'$ and $O'T'$ are transformed into the straight lines $O''L''$ and $O''T''$ and the portions $L' \to \infty$, $T' \to \infty$ into the circular arc $L''P''T''$. Let us start from the w plane. Using the conformal transformation

$$w_1 = -\frac{1}{2}\left(\frac{we^{-i\alpha}}{U} + \frac{U}{we^{-i\alpha}}\right) \tag{3.1}$$

the interior of the semi-circle $O''L''P''T''$ will be transformed into the lower half plane in the w_1 plane. In fact the points L'' and T'' are represented by $w_1 = 1$ and $w_1 = -1$, respectively; the straight line $L''O''T''$ is represented by the portions of the real axis for which $|w_1| > 1$, the circumference of the half circle by the portion between $w_1 = 1$ and $w_1 = -1$ of the latter. For $w = U$, we have $w_1 = -\cos\alpha$ (point P''' of the w_1 plane), so that the portions of the real axis $T'''P'''$ ($-1 < w_1 < -\cos\alpha$) and $P'''L'''$ ($-\cos\alpha < w_1 < 1$) represent the two boundaries of the wake. That the interior of the semi-circle corresponds to the lower half of the w_1 plane is readily seen if we remember that for an observer, travelling in the direction $L''P''T''$ the domain under consideration lies on the left hand side. This relation is conserved in the transformation. We now write:

$$f = \frac{C}{(w_1 + \cos\alpha)^2} \tag{3.2}$$

where C is a constant. By this transformation the point $w_1 = \pm\infty$ is displaced to the origin, the point $w_1 = -\cos\alpha$ moves to infinity. For real values of w_1 we obtain $\psi = 0$, $\varphi > 0$, points of the real axis of the w_1 plane between $w_1 = -\cos\alpha$ and $w_1 = \infty$ being represented by points at the upper and those between $w_1 = -\infty$ and $w_1 = -\cos\alpha$ by points at the lower side of the positive φ axis. Hence the conformal transformation

$$f = \frac{C}{\left(\cos\alpha - \frac{1}{2}\frac{we^{-i\alpha}}{U} - \frac{1}{2}\frac{U}{we^{-i\alpha}}\right)^2} \tag{3.3}$$

represents the relation between the f and the w plane, and thus solves the problem. The constant factor C is determined by the size of the plane airfoil. In order to compute the value of C, we have to express the condition that the distance between the points T and L, *i. e.* between the trailing and the leading edges in the z plane, is equal to the chord of the airfoil c.

We remember that $w = df/dz$ and write $z = \int_0^f df/w$. Then obviously $z_T - z_L = ce^{-i\alpha} = \int_{L'}^{T'} df/w$. The values of f corresponding to T' and L' are obtained from (3.3) by substituting $w = Ue^{i\alpha}$ and $w = -Ue^{i\alpha}$ respectively. Hence:

$$f_T = \varphi_T = \frac{C}{(\cos\alpha - 1)^2}, \qquad f_L = \varphi_L = \frac{C}{(\cos\alpha + 1)^2} \tag{3.4}$$

Let us introduce $t = \sqrt{f/C}$ as parameter. From (3.3) we obtain:

$$\frac{1}{t} = \cos\alpha - \frac{1}{2}\left(\frac{we^{-i\alpha}}{U} + \frac{U}{we^{-i\alpha}}\right) \tag{3.5}$$

and,

$$\left.\begin{aligned} \frac{we^{-i\alpha}}{U} &= \cos\alpha - \frac{1}{t} + \frac{1}{t}\sqrt{1 - 2t\cos\alpha - t^2\sin^2\alpha} \\ \frac{U}{we^{-i\alpha}} &= \cos\alpha - \frac{1}{t} - \frac{1}{t}\sqrt{1 - 2t\cos\alpha - t^2\sin^2\alpha} \end{aligned}\right\} \tag{3.6}$$

Denoting the values of t corresponding to φ_T and φ_L by t_T and t_L, we have:

$$t_T = \frac{1}{\cos\alpha - 1}, \qquad t_L = \frac{1}{\cos\alpha + 1}$$

and, $\quad 1 - 2t\cos\alpha - t^2\sin^2\alpha = (t_L - t)(t - t_T)\sin^2\alpha$

Hence

$$\left.\begin{aligned} w &= Ue^{i\alpha}\left[\cos\alpha - \frac{1}{t} + \frac{\sin\alpha}{t}\sqrt{(t_L - t)(t - t_T)}\right] \\ \frac{1}{w} &= \frac{e^{-i\alpha}}{U}\left[\cos\alpha - \frac{1}{t} - \frac{\sin\alpha}{t}\sqrt{(t_L - t)(t - t_T)}\right] \end{aligned}\right\} \tag{3.7}$$

Introducing this expression for $1/w$ in the integral $\int df/w$ and putting $df = 2\,Ct\,dt$, we obtain:

$$\left.\begin{aligned} z_T - z_L = \frac{Ce^{-i\alpha}}{U}\Bigg[& 2\cos\alpha\int_{t_L}^{t_T} t\,dt - 2\int_{t_L}^{t_T} dt - \\ & -2\sin\alpha\int_{t_L}^{t_T}\sqrt{(t_L - t)(t - t_T)}\,dt\Bigg] \end{aligned}\right\} \tag{3.8}$$

The evaluation of the integrals gives the following expressions:

$$\int_{t_L}^{t_T} t\,dt = \frac{1}{2}(t_T^2 - t_L^2) = \frac{2\cos\alpha}{\sin^4\alpha}, \qquad \int_{t_L}^{t_T} dt = t_T - t_L = -\frac{2}{\sin^2\alpha}$$

$$\int_{t_L}^{t_T} \sqrt{(t_L - t)(t - t_T)}\,dt = -\frac{\pi}{8}(t_L - t_T)^2 = \frac{-\pi}{2\sin^4\alpha}$$

Consequently $z_T - z_L = c e^{-i\alpha} = \dfrac{4 + \pi\sin\alpha}{\sin^4\alpha}\cdot\dfrac{Ce^{-i\alpha}}{U}$, and hence:

$$C = \frac{Uc\sin^4\alpha}{4 + \pi\sin\alpha}$$

We are interested in the dynamic action of the flow on the airfoil. The pressure at any point of the flow outside of the wake is given by Bernoulli's equation

$$p - p_0 = \varrho\frac{U^2 - |w|^2}{2} \tag{3.9}$$

where p_0 is the pressure at infinity and in the wake. The pressure on the upper surface is obviously equal to p_0, so that, denoting the length element along the airfoil by ds, the integral $\int_L^T (p - p_0)\,ds$ represents the resultant dynamic action[1] R. Along the airfoil $ds = dz\cdot e^{i\alpha}$, $|w^2| = w^2 e^{-2i\alpha}$. Hence:

$$R = \varrho\int\frac{U^2 - |w^2|}{2}ds = \frac{1}{2}\varrho U^2\int ds - e^{-i\alpha}\varrho\int_L^T\frac{w^2}{2}dz \tag{3.10}$$

Putting $\int ds = c$ and $w\,dz = df$, we obtain:

$$R = \frac{1}{2}\varrho U^2 c - \frac{1}{2}\varrho e^{-i\alpha}\int_{L'}^{T'} w\,df$$

The integral in the second term can be computed in an analogous manner as for the integral $\int df/w$, using the expression for w from the first of (3.7). We thus obtain easily:

$$\int w\,df = \frac{4 - \pi\sin\alpha}{\sin^4\alpha}\cdot C U e^{i\alpha}$$

and,

$$R = \frac{1}{2}\varrho U^2 c\,\frac{2\pi\sin\alpha}{4 + \pi\sin\alpha} \tag{3.11}$$

The coefficient of the resultant force is therefore equal to:

$$C_R = \frac{2\pi\sin\alpha}{4 + \pi\sin\alpha} \tag{3.12}$$

[1] There is no suctional force of the type considered in II (10.20), since the fluid does not turn around the leading edge with infinite velocity, but flows off tangentially.

The force R is perpendicular to the airfoil, so that the coefficients for lift and drag are given by

$$C_L = \frac{2\,\pi\, sin\,\alpha\, cos\,\alpha}{4+\pi\, sin\,\alpha}, \qquad C_D = \frac{2\,\pi\, sin^2\,\alpha}{4+\pi\, sin\,\alpha} \tag{3.13}$$

The corresponding values obtained by the circulation theory are:

$$C_L = 2\,\pi\, sin\,\alpha, \qquad C_D = 0.$$

It is obvious that for the unstalled condition the C_L values derived from the theory of wake are much too small; they are also considerably smaller than the measured values in the stalled range, but the general shape of the experimental curves is similar to that given by the theory. The reason for the discrepancy in the values for the stalled regime is chiefly due to the fact that the pressure in the wake is, in general, substantially lower than the pressure at infinity, because the wake does not extend to infinity and it is not in relative rest to the airfoil. For $\alpha = 90^0$, *i. e.* for the case of a plate exposed perpendicular to the airstream, the theory gives $C_D = 2\,\pi/\,(4+\pi) = 0.879$. The measurements for plates with large aspect ratio show that C_D has about the value 1.8. The observation of the flow shows that instead of a wake filled by fluid at rest, a system of vortex-rows is developed behind the plate. The vortex system produces a suction at the rear surface of the plate which has nearly the same magnitude as the maximum pressure at the forward surface. The high value of the drag coefficient is due to this considerable suction.

The weak point of the theory described in this section is obviously the restriction that the pressure in the wake is equal to the pressure at infinity. Riabouchinsky suggested a modification of the theory by assuming a second plate downstream and computing the shape of the wake enclosed between the two obstacles. The size and location of the second plate can be chosen in such a way that the pressure in the wake is equal to the value found empirically. Riabouchinsky assumes that if the distance of the two obstacles is large in comparison to the size of the body located upstream, the flow in the neighborhood of the body is fairly well described by his modified theory of the wake.

4. Extension of the Theory of the Discontinuous Potential Motion to Curved Boundaries. Method of Levi-Civita. Let us introduce the following new variables:

(*a*) Instead of the complex potential function f, we introduce a function ζ of f, such that the f plane is transformed into the interior of the upper half of the unit circle in the ζ plane (see Fig. 111). Starting from the ζ plane we first write:

$$\zeta_1 = -\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \tag{4.1}$$

Obviously the points $\zeta = 1$ and $\zeta = -1$ are represented by $\zeta_1 = -1$ and $\zeta_1 = 1$, respectively. The circular arc $|\zeta| = 1$ of the upper half circle is represented in the ζ_1 plane by the portion of the real axis between the points -1 and 1; the diameter of the circle is represented by the portions of the real axis outside the points -1 and 1, and the interior of the circle corresponds to the upper half of the ζ_1 plane. We want the circular arc $|\zeta| = 1$ to correspond to the fixed boundary and the diameter $\eta = 0$, $-1 \leqslant \xi \leqslant 1$ to the free boundaries; the point $\zeta = 0$ must correspond to infinity and $\zeta = e^{i\sigma_0}$ to the stagnation point. We can manage this by putting:

$$f = (a\,\zeta_1 + b)^2 \tag{4.2}$$

where a and b are real constants. For the stagnation point we want to have $f = 0$, so that $-a \cos\sigma_0 + b = 0$ and

$$f = a^2 (\zeta_1 + \cos\sigma_0)^2 \tag{4.3}$$

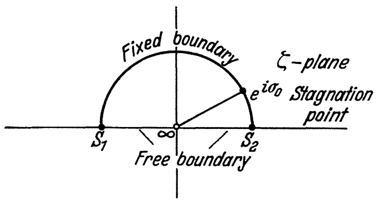

Fig. 111.

Carrying out the transformation determined by (4.1) and (4.3) we easily see that the boundary of the half circle in the ζ plane covers twice the positive portion of the real axis in the f plane. To the point $\zeta = e^{i\sigma_0}$ corresponds $f = 0$; to the point $\zeta = 1$, the point $f = a^2(-1 + \cos\sigma_0)^2$ at the lower side of the real axis and to the point $\zeta = -1$, the point $f = a^2 (1 + \cos\sigma_0)^2$ at the upper side of the real axis.

We calculate for later use the derivative of f, and obtain

$$d f = a^2 \left(\frac{1}{2}\zeta + \frac{1}{2\zeta} - \cos\sigma_0\right)\left(\zeta - \frac{1}{\zeta}\right)\frac{d\zeta}{\zeta}$$

For points on the circumference of the unit circle $\zeta = e^{i\sigma}$,

$$df = -2\,a^2 (\cos\sigma - \cos\sigma_0) \sin\sigma\, d\sigma \tag{4.4}$$

(*b*) Instead of the complex velocity function $w = u - iv$, we introduce $\omega = i \log w$. For simplicity we assume the velocity U at infinity outside of the wake to be unity. Let us denote the magnitude of the velocity by $q = e^{\tau}$, and its inclination to the x axis by θ. Then obviously $w = q e^{-i\theta}$ and $\omega = \theta + i \log q = \theta + i\tau$. Hence the real part of ω determines the inclination of the stream-lines in the physical z plane, and the imaginary part determines the magnitude of the velocity. For the free boundaries the imaginary part is zero, so that ω is real.

Let us consider again the problem of the straight airfoil with the angle of attack α, using the new variables ζ and ω. The function $\omega(\zeta)$ which solves the problem must satisfy the following conditions: (1) at the circumference of the half circle $|\zeta| = 1$, $\theta = -\alpha$ for $0 < \sigma < \sigma_0$ and $\theta = \pi - \alpha$ for $\sigma_0 < \sigma < \pi$; (2) at the real axis ω is real for $-1 < \zeta < 1$. We investigate the behavior of the function

$$\omega_0 = i \log \frac{\zeta - e^{i\sigma_0}}{\zeta - e^{-i\sigma_0}} + (\text{real constant}) \tag{4.5}$$

For $\zeta = e^{i\sigma}$ we have:

$$i\log\frac{\zeta - e^{i\sigma_0}}{\zeta - e^{-i\sigma_0}} = i\log\frac{\left(e^{i\frac{\sigma-\sigma_0}{2}} - e^{-i\frac{\sigma-\sigma_0}{2}}\right)e^{i\frac{\sigma+\sigma_0}{2}}}{\left(e^{i\frac{\sigma+\sigma_0}{2}} - e^{-i\frac{\sigma+\sigma_0}{2}}\right)e^{i\frac{\sigma-\sigma_0}{2}}} =$$

$$= i\log\frac{\sin\frac{\sigma-\sigma_0}{2}}{\sin\frac{\sigma+\sigma_0}{2}} - \sigma_0$$

The real part of this expression is equal to $-\sigma_0$ if $\sin\frac{\sigma-\sigma_0}{2}/\sin\frac{\sigma+\sigma_0}{2}$ is positive, *i.e.* $\sigma_0 < \sigma < \pi$, and is equal to $\pi - \sigma_0$ if $\sin\frac{\sigma-\sigma_0}{2}/\sin\frac{\sigma+\sigma_0}{2} < 0$, *i.e.* $0 < \sigma < \sigma_0$. Consequently ω_0 satisfies the condition (1), if we put:

$$\omega_0 = \sigma_0 - \alpha - \pi + i\log\frac{\zeta - e^{i\sigma_0}}{\zeta - e^{-i\sigma_0}}$$

or,

$$\omega_0 = -\alpha + i\log\frac{\zeta - e^{i\sigma_0}}{1 - \zeta e^{i\sigma_0}} \tag{4.6}$$

Obviously ω_0 is real for real values of ζ because for real values ξ of ζ the absolute values of the denominator and numerator are equal.

We now write $\omega = \omega_0 + \Omega$, where $\Omega(\zeta)$ is an analytical function of ζ, regular inside and at the boundary of the half circle. In order to satisfy the condition that ω is real for real values of ζ, we put $\Omega = \sum_0^\infty c_n \zeta^n$ where the c_n are real constants. We choose the c_n's so that $\sum_0^\infty c_n e^{i\sigma_0 n} = 0$. By this condition we ascertain that the stagnation point is represented as before by $\zeta = e^{i\sigma_0}$ and the tangent to the fixed boundary at the stagnation point has the inclination $-\alpha$.

The "inverse problem", *i. e.* the problem of determining the shape of the fixed boundary corresponding to a certain function $\Omega(\zeta)$, can be solved without difficulty. The easiest way is to compute the curvature $\varkappa$ of the boundary as a function of the inclination θ. The curvature $\varkappa$ is given by the derivative of θ with respect to the length of the arc s measured along the stream-line $\psi = 0$; hence[1]

$$\varkappa = \frac{d\theta}{ds} = |w|\frac{d\theta}{d\varphi} = e^{\tau}\frac{d\theta}{d\varphi} = e^{\tau}\frac{d\theta}{d\sigma}\frac{d\sigma}{d\varphi}$$

where according to (4.4)

$$\frac{d\sigma}{d\varphi} = -\frac{1}{2a^2(\cos\sigma - \cos\sigma_0)\sin\sigma}$$

Hence:

$$\varkappa = -\frac{d\theta}{d\sigma}\cdot e^{\tau}\frac{1}{2a^2(\cos\sigma - \cos\sigma_0)\sin\sigma} \tag{4.7}$$

[1] The curvature is taken to be positive if the center of curvature lies on the left side of the observer moving with the fluid along the stream-line.

Let us write $\omega_0 = \theta_0 + i\,\tau_0$ and $\Omega = \Theta + i\,T$; then $\tau = \tau_0 + T = \log\left|\frac{\sin(\sigma - \sigma_0)/2}{\sin(\sigma + \sigma_0)/2}\right| + T$ and $e^\tau = e^T\left|\frac{\sin(\sigma - \sigma_0)/2}{\sin(\sigma + \sigma_0)/2}\right|$. Furthermore $d\,\theta_0/d\sigma = 0$ and therefore $d\theta/d\sigma = d\,\Theta/d\sigma$. Hence we obtain

$$\varkappa = -\frac{d\,\Theta}{d\,\sigma}\cdot e^T \frac{|\sin(\sigma - \sigma_0)/2|}{2\,a^2\,|\sin(\sigma + \sigma_0)/2|\cdot(\cos\sigma - \cos\sigma_0)\sin\sigma}$$

With $\cos\sigma - \cos\sigma_0 = -2\sin\frac{\sigma + \sigma_0}{2}\sin\frac{\sigma - \sigma_0}{2}$, this becomes:

$$\varkappa = \pm\frac{d\,\Theta}{d\,\sigma}\cdot e^T \frac{1}{4a^2\sin^2\left(\frac{\sigma + \sigma_0}{2}\right)\cdot\sin\sigma} \tag{4.8}$$

where the positive sign applies to $\sigma < \sigma_0$ and the negative sign to $\sigma > \sigma_0$.

The last equation serves to formulate the direct problem, *i. e.* the problem of finding the flow for a given shape of the body. Let us determine the boundary curve by a kind of natural equation, giving the curvature as a function of the tangent. Then $\varkappa$ is a given function of θ or of Θ, so that we have to solve a non-linear boundary problem of the type:

$$g(\Theta)\frac{d\,\Theta}{d\,\sigma}\cdot e^T = 4a^2\cdot\sin^2\frac{\sigma + \sigma_0}{2}\cdot\sin\sigma \tag{4.9}$$

In the case of a circular boundary $\varkappa$ is constant. Denoting the radius of the circular boundary by r and taking r positive for a convex boundary, $\varkappa = 1/r$ for $\sigma < \sigma_0$ and $\varkappa = -1/r$ for $\sigma > \sigma_0$, so that

$$\frac{1}{r} = \frac{d\,\Theta}{d\,\sigma}\cdot e^T \frac{1}{4\,a^2\sin^2\left(\frac{\sigma + \sigma_0}{2}\right)\sin\sigma}$$

or

$$e^T\frac{d\,\Theta}{d\,\sigma} = \frac{4a^2}{r}\cdot\sin^2\frac{\sigma + \sigma_0}{2}\sin\sigma \tag{4.10}$$

For the case of an airfoil with small curvature we can replace e^T by unity and obtain

$$\frac{d\,\Theta}{d\,\sigma} = \frac{4a^2}{r}\cdot\sin^2\frac{\sigma + \sigma_0}{2}\cdot\sin\sigma \tag{4.11}$$

This equation can be solved without difficulty.

Several mathematicians have calculated the wake behind a circular cylinder. Assuming an arbitrary separation point, we find that the free boundary has, in general, an infinite curvature at the separation point. If the angular distance between the stagnation point and the assumed separation point is smaller than $55^0\,2'\,15''$, the curve obtained as free boundary cuts the cylindrical section, so that in the case of a full cylinder the separation cannot occur for a value of the angular distance smaller than this. In the limiting case the free boundary starts with zero curvature; if the angular distance between stagnation point and separation point becomes larger, the curvature of the free boundary at the separation point is infinite again, but in the opposite direction, so that the free boundary has a convex shape considered from the wake. In reality the separation point is determined by the behavior of the fluid in the so-called boundary layer.

5. The Instability of Vortex Sheets. The theory of discontinuous potential motion supposes the existence of a stationary surface of discontinuity with a finite difference between the velocities on the two sides. Such a surface of discontinuity is in equilibrium, if the pressure is equal on both sides; however, it can be shown that the equilibrium is unstable, in that small perturbations of the steady motion are increased spontaneously without action of external forces.

Let us consider a surface of discontinuity or, what is the same thing, a straight vortex sheet between two parallel streams; the velocity in the upper half plane ($y > 0$) is U_1, in the lower plane ($y < 0$) U_2. The corresponding velocity potentials are $\varphi_1 = U_1 x$ and $\varphi_2 = U_2 x$. The vortex sheet is situated along the x axis. We now assume that this vortex sheet is slightly perturbed, so that it has the shape $\eta = \eta(x, t)$. The corresponding motions have the potentials $\varphi_1 = U_1 x + \varphi_1'$, and $\varphi_2 = U_2 x + \varphi_2'$ where the derivatives of φ_1' and φ_2' are assumed small in comparison with U_1 and U_2. We can assume potential motion on both sides of the vortex sheet, because, as it is shown, no vorticity can be produced in the interior of an ideal fluid which is initially in vortexless motion; hence the vorticity remains concentrated in the vortex sheet.

We first express the condition that the velocity components of the fluid perpendicular to the vortex sheet are equal on both sides and equal to the movement of the vortex sheet itself in that direction.

Denoting the inclination between the curve $\eta = \eta(x, t)$ and the x axis by θ, we obtain the velocities perpendicular to the vortex sheet:

$$\left.\begin{aligned}\frac{\partial \varphi_1}{\partial n} &= \frac{\partial \varphi_1'}{\partial y} \cos\theta - \left(U_1 + \frac{\partial \varphi_1'}{\partial x}\right) \sin\theta \quad \text{for} \quad y = \eta + \varepsilon \\ \frac{\partial \varphi_2}{\partial n} &= \frac{\partial \varphi_2'}{\partial y} \cos\theta - \left(U_2 + \frac{\partial \varphi_2'}{\partial x}\right) \sin\theta \quad \text{for} \quad y = \eta - \varepsilon\end{aligned}\right\} \tag{5.1}$$

where ε is a small quantity. The derivatives $\partial\varphi_1/dn$ and $\partial\varphi_2/\partial n$ are both positive taken in the direction from the fluid 2 toward the fluid 1. Both quantities are equal to the velocity of the displacement of the vortex sheet perpendicular to its tangent $(\partial\eta/\partial t) \cos\theta$.

Hence with $\tan\theta = \partial\eta/\partial x$ and neglecting $\partial\varphi_1'/\partial x$ and $\partial\varphi_2'/\partial x$ in comparison with U_1 and U_2,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi_1'}{\partial y} - U_1 \frac{\partial \eta}{\partial x} = \frac{\partial \varphi_2'}{\partial y} - U_2 \frac{\partial \eta}{\partial x} \tag{5.2}$$

A further boundary condition for the vortex sheet is given by the fact that the pressures on both sides are equal. Assuming that the perturbation vanishes at an infinite distance from the vortex sheet, we obviously obtain the following two equations for the pressures p_1 and p_2 on the two sides of the vortex sheet (p_0 = pressure at infinity):

$$\left.\begin{aligned}\frac{p_1}{\varrho} + \frac{1}{2}\left(U_1 + \frac{\partial \varphi_1'}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi_1'}{\partial y}\right)^2 + \frac{\partial \varphi_1'}{\partial t} &= \frac{p_0}{\varrho} + \frac{U_1^2}{2} \\ \frac{p_2}{\varrho} + \frac{1}{2}\left(U_2 + \frac{\partial \varphi_2'}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi_2'}{\partial y}\right)^2 + \frac{\partial \varphi_2'}{\partial t} &= \frac{p_0}{\varrho} + \frac{U_2^2}{2}\end{aligned}\right\} \tag{5.3}$$

From $p_1 = p_2$ follows, neglecting terms of second order in the derivatives of φ_1' and φ_2':
$$U_1 \frac{\partial \varphi_1'}{\partial x} + \frac{\partial \varphi_1'}{\partial t} = U_2 \frac{\partial \varphi_2'}{\partial x} + \frac{\partial \varphi_2'}{\partial t} \tag{5.4}$$
We try to satisfy (5.2) and (5.4) by assuming:
$$\varphi_1' = Re.\,[C_1 e^{-\lambda y} e^{i\lambda x}]\,, \quad \varphi_2' = Re.\,[C_2 e^{\lambda y} e^{i\lambda x}]\,, \quad \eta = Re.\,[C_3 e^{i\lambda x}] \tag{5.5}$$
where C_1, C_2 and C_3 are functions of the time t. Obviously φ_1' and φ_2' satisfy the Laplacian equation and their derivatives vanish for $y = \infty$ and $y = -\infty$, respectively. Introducing these expressions in (5.2) and (5.4) we obtain, with y replaced by η:
$$\left.\begin{aligned} \frac{dC_3}{dt} &= -\lambda C_1 e^{-\lambda\eta} - U_1 C_3 \lambda i = \lambda C_2 e^{\lambda\eta} - U_2 C_3 \lambda i \\ i\lambda U_1 C_1 e^{-\lambda\eta} &+ \frac{dC_1}{dt} e^{-\lambda\eta} = i\lambda U_2 C_2 e^{\lambda\eta} + \frac{dC_2}{dt} e^{\lambda\eta} \end{aligned}\right\} \tag{5.6}$$
With the same approximation as before we can replace $e^{-\lambda\eta}$ and $e^{\lambda\eta}$ by unity and obtain from the first of (5.6):
$$C_1 = -\frac{1}{\lambda}\frac{dC_3}{dt} - i U_1 C_3, \qquad C_2 = \frac{1}{\lambda}\frac{dC_3}{dt} + i U_2 C_3$$
Substituting these values in the second of (5.6) we have:
$$\frac{1}{\lambda^2}\frac{d^2 C_3}{dt^2} + \frac{U_1 + U_2}{\lambda} i \frac{dC_3}{dt} - \frac{U_1^2 + U_2^2}{2} = 0 \tag{5.7}$$
The solution of this equation is of the form $C_3 = (\text{const.})\, e^{\mu t}$ and we obtain for μ/λ the equation:
$$\left(\frac{\mu}{\lambda}\right)^2 + (U_1 + U_2)\, i \frac{\mu}{\lambda} - \frac{U_1^2 + U_2^2}{2} = 0 \tag{5.8}$$
or,
$$\frac{\mu}{\lambda} = -\frac{1}{2}(U_1 + U_2)\, i \pm \frac{U_1 - U_2}{2} \tag{5.9}$$
Hence the final solution for $\eta\,(x)$ is given by:
$$\eta\,(x) = e^{i\lambda\left(x - \frac{U_1 + U_2}{2} t\right)}\left[A\, e^{\frac{U_1 - U_2}{2}\lambda t} + B\, e^{-\frac{U_1 - U_2}{2}\lambda t}\right] \tag{5.10}$$
where A and B are constants.

It is obvious that for arbitrary values of U_1 and U_2, unless $U_1 = U_2$, one of the two terms in the bracket will increase with the time. Hence the magnitude of the perturbation increases. The factor with the imaginary exponent shows that the perturbation travels along the sheet with a velocity equal to $(U_1 + U_2)/2$, *i. e.* to the mean value of the velocities of the fluid on both sides.

Recently L. Rosenhead computed the further development of the vortex sheet, taking into account the higher terms in the perturbations. He found that starting from a sinusoidal perturbation the vortex sheet obtains an asymmetric shape and gradually the vorticity becomes more and more concentrated in spiral shaped portions of the line; practically the vortex sheet is transformed into an equi-distant series of vortex nuclei. These considerations justify the method used in the next section, namely the derivation of the flow in the wake from a system of isolated vortex filaments.

6. Stability of Double Rows of Vortices. The observations of Mallock and Bénard showed that the two vortex sheets behind a body are in general replaced by a double row of vortices with opposite circulation in each row. These observations, together with his own, suggested to v. Kármán the investigation of the stability of such vortex rows in different arrangements and the expression of the magnitude of the drag by the characteristics of the vortex system.

We assume two-dimensional flow, *i. e.* systems composed of parallel vortex columns or filaments. The complex potential function for a single row of equi-distant vortices with the interval l and with the circulation Γ is given by:

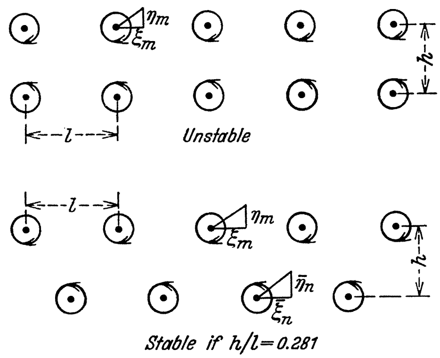

Fig. 112.

$$f(z) = \frac{i\,\Gamma}{2\,\pi} \log \sin \frac{\pi z}{l} \qquad (6.1)$$

There are two possible arrangements for a system of two rows with equal and opposite circulation, moving with constant speed so that the relative position of the vortices remains invariable, namely:

(*a*) An arrangement in which the vortices are located symmetrically with respect to the middle line of the two rows.

(*b*) An asymmetrical arrangement in which each vortex in one row is opposite the center of the interval between two consecutive vortices in the other row (see Fig. 112).

If the distance between the two rows is denoted by h, the velocity of advance of the entire system is $-U_1 = \frac{-\Gamma}{2\,l} \coth \frac{\pi h}{l}$ in the first and $-U_2 = \frac{-\Gamma}{2\,l} \tanh \frac{\pi h}{l}$ in the second case.

First we shall show that a simple row of vortices is unstable. Let us assume that the vortex whose equilibrium position is given by $x = ml$, $y = 0$ is displaced by ξ_m and η_m in the x and y directions respectively. Then the velocity components of the vortex filament with the subscript zero are:

$$\left.\begin{aligned} \frac{d\,\xi_0}{d\,t} &= \frac{\Gamma}{2\,\pi} \sum_{m=-\infty}^{\infty} \frac{\eta_0 - \eta_m}{r_{0m}^2} \\ \frac{d\,\eta_0}{d\,t} &= -\frac{\Gamma}{2\,\pi} \sum_{m=-\infty}^{\infty} \frac{\xi_0 - \xi_m - m\,l}{r_{0m}^2} \end{aligned}\right\} \qquad (6.2)$$

where $r_{0m}^2 = [(\xi_m - \xi_0 + m\,l)^2 + (\eta_m - \eta_0)^2]$. The summation includes all positive and negative integer values of m, $m = 0$ excluded. If we neglect second order terms in ξ and η we obtain:

$$\left.\begin{aligned} \frac{d\xi_0}{dt} &= \frac{\Gamma}{2\pi l^2}\sum_{m=-\infty}^{\infty}\frac{\eta_0-\eta_m}{m^2} \\ \frac{d\eta_0}{dt} &= \frac{\Gamma}{2\pi l^2}\sum_{m=-\infty}^{\infty}\frac{\xi_0-\xi_m}{m^2}\,. \end{aligned}\right\} \tag{6.3}$$

Let us assume perturbations of the type $\xi_m = \xi_0\, e^{i m \varphi}$, $\eta_m = \eta_0\, e^{i m \varphi}$ where $0 < \varphi < 2\pi$. This means that if $\varphi = 2\pi/N$, the "wave-length" of the perturbation is $N\, l$, *i. e.* each N-th vortex has the same perturbation.

We find:

$$\left.\begin{aligned} \frac{d\xi_0}{dt} &= \frac{\Gamma}{2\pi l^2}\,\eta_0\sum_{-\infty}^{\infty}\left(\frac{1-e^{i m \varphi}}{m^2}\right) \\ \frac{d\eta_0}{dt} &= \frac{\Gamma}{2\pi l^2}\,\xi_0\sum_{-\infty}^{\infty}\left(\frac{1-e^{i m \varphi}}{m^2}\right) \end{aligned}\right\} \tag{6.4}$$

and thus:

$$\frac{d^2\xi_0}{dt^2} - \lambda^2\xi_0 = 0, \qquad \frac{d^2\eta_0}{dt^2} - \lambda^2\eta_0 = 0$$

where:

$$\lambda = \frac{\Gamma}{\pi l^2}\sum_{1}^{\infty}\frac{1-\cos m\varphi}{m^2} = \frac{\Gamma}{4\pi l^2}\,\varphi\,(2\pi-\varphi)$$

It is evident that one of the particular solutions for ξ_0 and η_0 shows increase with the time t as $e^{\lambda t}$, so that the configuration is unstable.

We now investigate the case of a double row in the symmetrical arrangement. Obviously it is sufficient to write the differential equation for two vortices, namely one in each row with the subscript zero. Then we use the subscript m for the upper and the subscript n for the lower row. Let us calculate first the velocity components for the vortex in the upper row. The contributions of the vortices in the upper row to the velocity components of the vortex considered will be $\frac{\Gamma}{2\pi l^2}\sum_{m=-\infty}^{\infty}\frac{\eta_0-\eta_m}{m^2}$ and $\frac{\Gamma}{2\pi l^2}\sum_{m=-\infty}^{\infty}\frac{\xi_0-\xi_m}{m^2}$. The contribution of the lower row consists of the velocity $-U_1 = -\frac{\Gamma}{2l}\coth\frac{\pi h}{l}$ and of the contributions due to the perturbations $\bar{\xi}_n$, $\bar{\eta}_n$. The latter result in a velocity component in the x direction of the amount

$$\frac{\Gamma}{2\pi}\sum_{n=-\infty}^{\infty}\frac{n^2 l^2-h^2}{(n^2 l^2+h^2)^2}(\bar{\eta}_n-\eta_0) + \frac{\Gamma}{2\pi}\sum_{n=-\infty}^{\infty}\frac{2nhl}{(n^2 l^2+h^2)^2}(\bar{\xi}_n-\xi_0)$$

and in a velocity component in the y direction equal to

$$\frac{\Gamma}{2\pi}\sum_{n=-\infty}^{\infty}\frac{n^2 l^2-h^2}{(n^2 l^2+h^2)^2}(\bar{\xi}_n-\xi_0) - \frac{\Gamma}{2\pi}\sum_{n=-\infty}^{\infty}\frac{2nhl}{(n^2 l^2+h^2)^2}(\bar{\eta}_n-\eta_0)$$

It is to be remarked that the sums with the subscripts n include $n = 0$, while in the sums with the subscript m, $m = 0$ is to be left out.

Putting $\xi_m = \xi_0 e^{i m \varphi}$, $\eta_m = \eta_0 e^{i m \varphi}$, $\bar{\xi}_n = \bar{\xi}_0 e^{i n \varphi}$, $\bar{\eta}_n = \bar{\eta}_0 e^{i n \varphi}$, we obtain the following differential equations for ξ_0 and η_0:

$$\left.\begin{aligned} \frac{2\pi l^2}{\Gamma}\frac{d\xi_0}{dt} &= A\eta_0 + B\bar{\xi}_0 + C\bar{\eta}_0 \\ \frac{2\pi l^2}{\Gamma}\frac{d\eta_0}{dt} &= A\xi_0 + C\bar{\xi}_0 - B\bar{\eta}_0 \end{aligned}\right\} \tag{6.5}$$

where the coefficients A, B, C as functions of φ are given by the expressions:

$$A(\varphi) = \sum_{m=-\infty}^{\infty} \frac{1 - e^{i m \varphi}}{m^2} - \sum_{n=-\infty}^{\infty} \frac{n^2 - (h/l)^2}{[n^2 + (h/l)^2]^2} = \frac{1}{2}\varphi(2\pi - \varphi) + \frac{\pi^2}{\sinh^2(\pi h/l)}$$

$$B(\varphi) = \sum_{n=-\infty}^{\infty} \frac{2n(h/l)e^{i n \varphi}}{[n^2 + (h/l)^2]^2} \tag{6.6}$$

$$C(\varphi) = \sum_{n=-\infty}^{\infty} \frac{[n^2 - (h/l)^2]e^{i n \varphi}}{[n^2 + (h/l)^2]^2}$$

In order to obtain the differential equations for $\bar{\xi}_0$ and $\bar{\eta}_0$, *i. e.* for the vortices in the lower row, we have to reverse the sign of Γ and h, and furthermore change m into n, ξ into $\bar{\xi}$, η into $\bar{\eta}$ and *vice versa*. Thus we obtain:

$$\left.\begin{aligned} \frac{2\pi l^2}{\Gamma}\frac{d\bar{\xi}_0}{dt} &= -A\bar{\eta}_0 + B\xi_0 - C\eta_0 \\ \frac{2\pi l^2}{\Gamma}\frac{d\bar{\eta}_0}{dt} &= -A\bar{\xi}_0 - C\xi_0 - B\eta_0 \end{aligned}\right\} \tag{6.7}$$

We notice that the second pair of differential equations is identical with the first pair, if we put $\xi_0 = \bar{\xi}_0$ and $\eta_0 = -\bar{\eta}_0$, or $\xi_0 = -\bar{\xi}_0$ and $\eta_0 = \bar{\eta}_0$. The first assumption corresponds to perturbations symmetrical with respect to the center line, the second assumption corresponds to antisymmetrical perturbations. The general solution is composed of particular solutions and stability can only be attained if no particular solution shows increase with the time.

For $\xi_0 = \bar{\xi}_0$ and $\eta_0 = -\bar{\eta}_0$ the equations are:

$$\left.\begin{aligned} \left(\frac{2\pi l^2}{\Gamma}\frac{d}{dt} - B\right)\xi_0 &= (A - C)\eta_0 \\ \left(\frac{2\pi l^2}{\Gamma}\frac{d}{dt} - B\right)\eta_0 &= (A + C)\xi_0 \end{aligned}\right\} \tag{6.8}$$

Putting $\xi_0 \sim e^{\lambda t}$, $\eta_0 \sim e^{\lambda t}$ and $(2\pi l^2/\Gamma)\lambda = \mu$, we obtain:

$$(\mu - B)^2 = A^2 - C^2$$

For $\xi_0 = -\bar{\xi}_0$, $\eta_0 = \bar{\eta}_0$ we obtain in a similar way:

$$(\mu + B)^2 = A^2 - C^2$$

We first notice that if a value of μ satisfies the first equation, $-\mu$ satisfies the second. Hence we always obtain instability if μ has a real part. B being a purely imaginary quantity, in order to make μ purely imaginary, it is necessary that $A^2 < C^2$. However, it is easy to show that at least over a certain range of φ between 0 and 2π, $A^2 > C^2$. For instance putting $\varphi = \pi$, we obtain

$$A^2 - C^2 = \frac{\pi^4}{4}$$

Hence $A^2 - C^2$ will be positive at least in a finite range near $\varphi = \pi$. The perturbations corresponding to this range increase with the time, so that the arrangement is unstable.

Proceeding to the case of a double row with asymmetrically arranged vortices, we find that (6.5) and (6.7) hold also for this case, provided we substitute for A, B, C the following expressions:

$$\left.\begin{aligned}
A(\varphi) &= \sum_m \frac{1 - e^{m i \varphi}}{m^2} - \sum_n \frac{(n + 1/2)^2 - (h/l)^2}{[(n + 1/2)^2 + (h/l)^2]^2} = \\
&= \frac{1}{2}\varphi(2\pi - \varphi) - \frac{\pi^2}{\cosh^2(\pi h/l)} \\
B(\varphi) &= \sum_n \frac{2(n + 1/2)(h/l)\, e^{i(n + 1/2)\varphi}}{[(n + 1/2)^2 + (h/l)^2]^2} = \\
&= i\left[\frac{\pi\varphi \sinh(h/l)(\pi - \varphi)}{\cosh(\pi h/l)} + \frac{\pi^2 \sinh(\pi h/l)}{\cosh^2(\pi h/l)}\right] \\
C(\varphi) &= \sum_n \frac{[(n + 1/2)^2 - (h/l)^2]\, e^{i(n + 1/2)\varphi}}{[(n + 1/2)^2 + (h/l)^2]^2} = \\
&= \frac{\pi^2 \cosh(h\varphi/l)}{\cosh^2(\pi h/l)} - \frac{\pi\varphi \cosh(h/l)(\pi - \varphi)}{\cosh^2(\pi h/l)}
\end{aligned}\right\} \qquad (6.9)$$

The condition for stability is again $A^2 < C^2$. Now $C = 0$ for $\varphi = \pi$, so that this arrangement is also unstable except for $A(\pi) = 0$, *i. e.* $\cosh^2 \frac{\pi h}{l} = 2$, or $h/l = 0.281$. It can be shown that for this value of h/l, $A^2 < C^2$ for all values of φ between zero and 2π, so that in this case no perturbation has an increasing trend. The solutions for ξ_0, η_0, $\bar{\xi}_0$, $\bar{\eta}_0$ have the character $e^{i\nu t}$ where ν is a real frequency. For the perturbations corresponding to $\varphi = 0$ or $\varphi = 2\pi$, the frequency is zero, the period of oscillation infinite. In this case all the vortices of each row move with the same phase; except this special type, all other perturbations lead to unstable oscillations.

Hence we conclude that theoretically in a perfect fluid the double row vortex system is stable only in the case when the two rows are displaced relative to each other by one half of the interval between consecutive vortices, and the ratio of the interval between the rows to the interval between consecutive vortices is equal to $h/l = (1/\pi) \cosh^{-1}\sqrt{2}$.

7. The Expression for the Drag. We now proceed to express the drag of the body producing the vortex wake by the characteristics of the vortex system. We assume that the motion at a great distance behind the body is practically the same as in the case of the infinitely extended vortex rows. It was shown that the velocity of advance of the asymmetric vortex system in the direction of $-x$ is equal to $U_2 = \frac{\Gamma}{2l} \tanh \frac{\pi h}{l}$. Observation shows that $U_2 < U$, where U is the velocity of advance of the body (likewise in the direction of $-x$) producing the vortex system, so that new vortices are permanently created with equal and alternating circulations. Obviously a new pair of vortices is produced

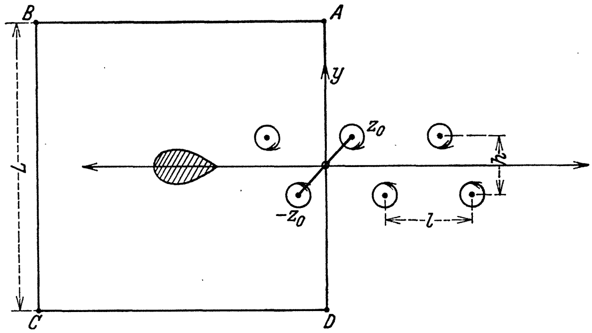

Fig. 113.

in the time interval $l/(U - U_2)$, so that the number of vortex pairs produced in unit time is $(U - U_2)/l$.

In order to obtain the expression for the drag we consider a sheet of unit thickness in the z direction. We choose the x, y coordinate system moving in the plane of flow with the vortex system and apply the momentum theorem to a fluid volume contained in the square $ABCD$, see Fig. 113, with the side length L, where L is large in comparison to the dimensions of the body and to the characteristic dimensions l and h of the vortex rows. We especially assume that the section AD is so far behind the body that the velocity components along AD can be approximated by the expressions which are exactly valid for the vortex system of infinite length. We notice that the fluid as a whole moves with the velocity U_2 in the direction of the positive x axis, while the body is assumed to move with the velocity $U - U_2$ in the direction of the negative x axis. We shall denote by u and v the velocity components additive to the uniform velocity U_2 and we note that according to the above mentioned assumption, the values of u and v along AD can be identified with the velocity components derived from the indefinitely extended vortex system. Let us choose AD in such a way that it cuts the center line in its point of intersection with a straight line connecting two neighboring vortices of the upper and of the lower row. Choosing this point as origin of a coordinate system x, y, the complex potential function of the vortex system is given by

$$f(x+iy) = f(z) = \frac{\Gamma}{2\pi} i \log \frac{\sin(z-z_0)(\pi/l)}{\sin(z+z_0)(\pi/l)} \tag{7.1}$$

where $z = x + iy$ and $z_0 = l/4 + ih/2$. The velocity components are obtained by the formula $w = u - iv = df/dz$.

Applying now the theorem of momentum to the fluid contained in $ABCD$, we first note that the flow is not a steady one, because the body is moving with the velocity $U - U_2$ in the negative x direction and due to the newly produced vortex pairs the momentum of the fluid contained in $ABCD$ is permanently increasing. The process being a periodical one, the average increase of momentum in unit time is equal to $\frac{\varrho(U-U_2)}{l} \Gamma h$. We remember that $\varrho \Gamma h$ is the horizontal component of the momentum corresponding to one vortex pair. In order to obtain the drag we have to add to this quantity the momentum leaving the square $ABCD$ in unit time through its boundaries and the resultant of the pressures acting along $ABCD$ in the direction opposite to the movement of the body. Applying the method described in II 4, we easily show that the sum of the two latter items is equal to the imaginary part of the circular integral $(1/2)\varrho \oint w^2 dz$ taken along $ABCD$. The terms containing U_2 drop out, because of the continuity condition. Now the values of u and v at the boundaries of the square $ABCD$ are of the order $1/L$ or of higher order, except at the side AD, so that the integral is reduced to $\frac{1}{2}\varrho \int_{-iL/2}^{iL/2} w^2 dz$, or at the limit, to $\frac{1}{2} i\varrho \int_{-\infty}^{\infty} w^2 dy$.

From (7.1) we deduce by differentiation,

$$w = \frac{\Gamma}{2l} i \left[\cot(z-z_0)\frac{\pi}{l} - \cot(z+z_0)\frac{\pi}{l}\right] \tag{7.2}$$

and,

$$w^2 = -\left[\frac{\Gamma}{2l}\right]^2 \left\{\frac{1}{\sin^2(z-z_0)\pi/l} + \frac{1}{\sin^2(z+z_0)\pi/l} - 2 - 2\cot\left[(z-z_0)\frac{\pi}{l}\right]\cot\left[(z+z_0)\frac{\pi}{l}\right]\right\} \tag{7.3}$$

Hence:

$$\frac{1}{2}\int_{-i\infty}^{i\infty} w^2 dz = \frac{\Gamma^2}{8\pi l}\left[\cot(z-z_0)\frac{\pi}{l} + \cot(z+z_0)\frac{\pi}{l} - 2\cot\frac{2z_0\pi}{l}\log\frac{\sin(z-z_0)\pi/l}{\sin(z+z_0)\pi/l}\right]_{-i\infty}^{i\infty} \tag{7.4}$$

By a simple calculation we finally obtain:

$$\frac{1}{2}\int_{-i\infty}^{i\infty} w^2 dz = \frac{\Gamma^2 i}{2\pi l}\left[1 - \frac{2z_0\pi}{l}\cot\frac{2z_0\pi}{l}\right] = \frac{\Gamma^2}{2\pi l}\left[i - \left(\frac{\pi}{2} + \frac{i\pi h}{l}\right)\tanh\frac{\pi h}{l}\right]$$

since

$$\cot\frac{2z_0\pi}{l} = \cot\left(\frac{\pi}{2} + i\frac{\pi h}{l}\right) = -i\tanh\frac{\pi h}{l}.$$

Introducing again $U_2 = \frac{\Gamma}{2l} \tanh \frac{\pi h}{l}$, we write:

$$Im. \left[\frac{1}{2} \oint w^2 \, dz\right] = \frac{\Gamma^2}{2\pi l} - \Gamma U_2 \frac{h}{l}$$

Hence the drag amounts to:

$$D = \varrho \Gamma (U - U_2) \frac{h}{l} + \varrho \frac{\Gamma^2}{2\pi l} - \varrho \Gamma U_2 \frac{h}{l}$$

or,

$$D = \varrho \Gamma (U - 2 U_2) \frac{h}{l} + \varrho \frac{\Gamma^2}{2\pi l} \tag{7.5}$$

With $\cosh \frac{\pi h}{l} = \sqrt{2}$, $\tanh \frac{\pi h}{l} = \frac{1}{\sqrt{2}}$, $U_2 = \frac{\Gamma}{2l\sqrt{2}}$, we find:

$$D = \varrho l U^2 \left[0.794 \frac{U_2}{U} - 0.314 \left(\frac{U_2}{U}\right)^2\right] \tag{7.6}$$

for the drag, and for the drag coefficient ($d =$ the frontal projection of the body),

$$C_D = \frac{l}{d} \left[1.588 \frac{U_2}{U} - 0.628 \left(\frac{U_2}{U}\right)^2\right] \tag{7.7}$$

Concerning the experimental evidence, the instability of the symmetric arrangements is easily observed; also the production of the vortices with alternate circulation occurs in a regular period, as required by the theory. The ratio h/l is of the order predicted by the stability criterion; however, an exact accordance can hardly be expected, because the stability criterion is only valid for a vortex row of infinite length, and near to the body the condition might be somewhat different. At great distances the influence of viscosity modifies and finally destroys the vortices. However, the formula for the drag checks fairly well with measurements.

The drag formula contains two unknown ratios: l/d and U_2/U. Instead of the latter we can consider $\Gamma/U l$ as unknown. Unfortunately, the theory has not, as yet, permitted further extension in such manner as to predict these quantities for a body of given shape. Also the extension for the three-dimensional problem has not as yet been achieved. It has been shown, however, that in the case of two-dimensional flow, the vortex columns are stable also in the case of small three-dimensional perturbations. Furthermore it is known that a series of circular vortex rings, which corresponds to some extent to the symmetrical case in the two-dimensional problem, is unstable. Very little experimental evidence and no theoretical information is available regarding the stable arrangement of individual vortex lines in the wake of three-dimensional non-cylindrical bodies such as plates, spheres or ellipsoids.

Many interesting special investigations have been carried out concerning the two-dimensional case dealt with in the present section. The stability of the vortex rows between fixed walls has been investigated; the similarity of the vortex arrangement and the change of spacing and frequency of the vortices with Reynolds' number has been thoroughly studied.

It appears that in the case of bodies with sharp corners, the vortex system is geometrically similar over the whole range of Reynolds' numbers which has been investigated, except for very small values of the latter quantity. In the case of blunt and stream-lined bodies without sharp corners, where the wake is produced by the separation of the boundary layer, it is found that the regular vortex rows disappear at about the critical value of Reynolds' number, *i. e.* when the boundary layer becomes turbulent. In the range of higher Reynolds' numbers, the vortex system is replaced by a turbulent wake with continuous vorticity distribution.

8. Oseen's Theory of the Wake. Oseen has developed general methods for the integration of the differential equations of viscous fluids, which can be applied to the problem of drag, especially in two limiting cases: in the range of small Reynolds' numbers where the Stokes' theory of pure frictional flow has been essentially improved by Oseen's method, and in the case of large Reynolds' numbers where the frictional forces are small in comparison with the inertia forces. Oseen especially investigated the transition $\mu \to 0$ (μ = coefficient of viscosity). We give in this section a short account of his results.

Oseen's approximate computations of the wake behind a body moving with constant velocity can easily be understood in connection with the investigations of Chapter III concerning the effect of forces on an ideal fluid. It is there shown that if the external forces are restricted to a region G, the flow is vortexless except in the region G and in its wake, *i. e.* a cylindrical space behind the body, the section of the cylinder being given by the "master section" of the region G perpendicular to the direction of the flow. Denoting the undisturbed velocity at infinity by U, the components of the additional velocities by u, v, w, the components of the vorticity by γ_x, γ_y, γ_z, the pressure by p and the density of the fluid by ϱ, the quantity $p + (\varrho/2)(u^2 + v^2 + w^2)$ by $\bar{p}$, the hydrodynamic equations for steady motion can be written in the following form (see III 6)[1].

$$\left.\begin{aligned} U\frac{\partial u}{\partial x} &= -\frac{1}{\varrho}\frac{\partial \bar{p}}{\partial x} + (v\gamma_z - w\gamma_y) \\ U\frac{\partial v}{\partial x} &= -\frac{1}{\varrho}\frac{\partial \bar{p}}{\partial y} + (w\gamma_x - u\gamma_z) \\ U\frac{\partial w}{\partial x} &= -\frac{1}{\varrho}\frac{\partial \bar{p}}{\partial z} + (u\gamma_y - v\gamma_x) \end{aligned}\right\} \qquad (8.1)$$

For the first approximation Oseen neglects the terms $(v\gamma_z - w\gamma_y)$ etc., *i. e.* those representing the "induced forces" as they were called in Chapter III. The physical significance of this approximation can be visualized by eliminating the pressure. Neglecting these terms we obtain the following system of equations for the vorticity components:

[1] In the present case the f_x etc. of the reference are zero, as the part of the field outside G is alone considered. The reader will note slight differences in notation. These are due to special adaptations to the subject matter involved and should cause no difficulty in the reading.

$$\left.\begin{aligned} U\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) &= U\frac{\partial \gamma_x}{\partial x}=0 \\ U\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) &= U\frac{\partial \gamma_y}{\partial x}=0 \\ U\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) &= U\frac{\partial \gamma_z}{\partial x}=0 \end{aligned}\right\} \tag{8.2}$$

According to this equation, the vorticity components have constant values along lines $y = const.$, $z = const.$, *i. e.* along lines parallel to the x axis. In an ideal incompressible fluid the vorticity is transported invariably with the matter, so that the equations do not satisfy exactly the so-called Helmholtz laws of vorticity; the terms corresponding to the vortex transport by the additional velocities u, v, w are neglected.

So far we have considered the wake behind a region inside of which forces are acting on the fluid. Oseen replaces this region by the actual body and deduces boundary conditions from the equations for a viscous fluid, discussing especially the transition $\mu \to 0$. He finds that at the frontal surface an arbitrary value of the tangential velocity component can occur; the velocity perpendicular to the surface is, of course, zero if the body is at rest and equal to the normal velocity of the surface element if the body is in motion. As for the rear surface, *i. e.* at the portion of the boundary which is in contact with the wake, he finds that the velocity of the fluid relative to the body is zero.

The computation of the velocity field—vortexless outside the wake and having a vortex distribution $\bar{\gamma}(y, z)$ in the wake—is rather complicated in the general three-dimensional case. We restrict ourselves to the two-dimensional problem in the x, y plane. In this case $\gamma_x = \gamma_y = 0$ and γ_z is a function of y only. We satisfy the condition that γ_z is zero outside and is independent of x inside the wake, by the assumption of a velocity field built up by superposition of the following constituent parts:

a) The parallel flow with constant velocity U;

b) A potential field $\varphi(x, y)$ extended over the whole plane outside of the body;

c) A field of parallel velocities $u^+(y)$ over the wake.

The velocity components resulting from the superposition can be written $U + \partial\varphi/\partial x$, $\partial\varphi/\partial y$ outside of the wake and $U + u^+ + \partial\varphi/\partial x$, $\partial\varphi/\partial y$ within the wake.

In order to satisfy the boundary conditions we have to determine φ in such a way that $\partial\varphi/\partial x$, $\partial\varphi/\partial y$ vanish at infinity, $\partial\varphi/\partial n + U\cos\theta = 0$ at the front of the body (θ = angle between the x axis and the normal from the surface), and $\partial\varphi/\partial y = 0$ at the rear surface. In order to satisfy the condition $U + u = 0$ at the rear surface, we choose u^+ such that $u^+ = -U - (\partial\varphi/\partial x)_r$, where $(\partial\varphi/\partial x)_r$ is the value of the derivative at the surface.

Thus the problem is reduced to a special kind of boundary problem for a solution of the potential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

The special feature in the problem is that along one portion of the boundary the magnitude of the normal derivative, *i. e.* of the normal velocity is given, while over the other portion of the boundary $\partial \varphi / \partial y$ vanishes, *i. e.* the direction of the velocity is given.

Oseen has given the general solution of this boundary problem; Zeilon computed and discussed the solutions for different cases, *e. g.* for the case of a circular cylinder and of a plane airfoil. Burgers suggested a very simple method using polar coordinates r, α for the circular cylinder, putting $\varphi = A_0 \log r + \sum_{n=1}^{\infty} A_n \cos n \alpha$ and trying to determine such values for the constants A that the boundary conditions are approximately satisfied. The method can be extended to boundaries of other shape because the problem can be reduced to the case of the circle by conformal transformation of the given boundaries into a circle.

Oseen has also established general expressions for the drag of the body producing the wake. He calculates the pressure at an arbitrary point, using the equations (8.1) and neglecting the quadratic terms containing the products of the vorticity and the velocity components. In this way we obtain:

$$\bar{p} = p + \frac{1}{\varrho}(u^2 + v^2) = -\varrho U \frac{\partial \varphi}{\partial x} \tag{8.3}$$

and

$$p = -\varrho U \frac{\partial \varphi}{\partial x} - \frac{1}{2} \varrho \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] \tag{8.4}$$

outside the wake, and

$$p = -\varrho U \frac{\partial \varphi}{\partial x} - \frac{1}{2} \varrho \left[\left(\frac{\partial \varphi}{\partial x} + u^+ \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] \tag{8.5}$$

inside the wake.

It may be noted that p is discontinuous at the transition surface from the outside region into the wake; furthermore p is variable over the cross-section of the wake even at an infinite distance behind the body in spite of the fact that the stream-lines in the wake at great distances from the body are parallel straight lines. Hence the calculation of the pressure appears somewhat dubious.

We now express the drag D as the sum of the x components of the pressures. Obviously, $D = -\oint p \, dy$ where the integral is to be extended counter-clockwise around the boundary of the body. Using the values for the pressure given by (8.5),

$$\left.\begin{aligned} D = \varrho \oint \left[U \frac{\partial \varphi}{\partial x} + \frac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] d y \, + \\ + \varrho \int_R \left(\frac{\partial \varphi}{d x} u^+ + \frac{1}{2} u^{+2} \right) d y \end{aligned}\right\} \tag{8.6}$$

The latter integral is only to extend over the rear portion of the boundary which is in contact with the wake. Let us consider the expression

$$\oint \left[\left\{ \left(\frac{\partial \varphi}{\partial x} \right)^2 - \left(\frac{\partial \varphi}{\partial y} \right)^2 \right\} d\,y - 2 \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} d\,x \right] \tag{8.7}$$

Obviously this integral is the imaginary part of the complex circular integral $\oint w^2\,dz$, where $w = \partial\varphi/\partial x - i\,\partial\varphi/\partial y$. The integral $\oint w^2\,dz$ is equal to zero if the function w is regular at infinity and of the order $1/z$ or higher. In the present case w satisfies this condition and the integral (8.7) is therefore equal to zero. We thus obtain:

$$\left.\begin{aligned} D = \varrho \oint \left[\left(U + \frac{\partial \varphi}{\partial x} \right) d\,y - \frac{\partial \varphi}{\partial y} d\,x \right] \frac{\partial \varphi}{\partial x} + \varrho \int_R \frac{\partial \varphi}{\partial x} u^+ \,d\,y \,+ \\ + \frac{\varrho}{2} \int_R u^{+2}\,d\,y \end{aligned}\right\} \tag{8.8}$$

The integrand of the first integral is zero at the frontal portion of the boundary, because the resultant velocity composed of the components $U + \partial\varphi/\partial x$ and $\partial\varphi/\partial y$ is perpendicular to the line element ds. At the rear side $\partial\varphi/\partial y = 0$, so that only the following terms remain:

$$D = \varrho \int_R \left(U + \frac{\partial \varphi}{\partial x} \right) \frac{\partial \varphi}{\partial x} d\,y + \varrho \int_R \frac{\partial \varphi}{\partial x} u^+ \,d\,y + \frac{\varrho}{2} \int_R u^{+2}\,d\,y \tag{8.9}$$

Now $U + \partial\varphi/\partial x + u^+ = 0$ at the rear surface so that

$$D = \frac{\varrho}{2} \int_R u^{+2}\,d\,y \tag{8.10}$$

If we consider the fluid at rest at infinity in front of the body, u^+ is the velocity at infinity in the wake, *i. e.* the velocity with which the fluid follows the body. Hence the drag is equal to the kinetic energy transported through the cross section of the wake in unit time.

In the case of the circular cylinder the drag formula (8.10) gives reasonable values. In the case of the plane airfoil the values are too high; the experimental values obtained for the stalled airfoil are between those corresponding to the theory of discontinuous potential motion and those given by Oseen's theory.

Zeilon suggested further developments of Oseen's theory. He corrected the vorticity distribution assuming that the vorticity is constant along stream-lines of a certain primary motion instead of being constant along parallel straight lines. Furthermore, he tried to eliminate the discontinuity in the pressure at the transition between the frontal portion of the body and the wake, by assuming gliding at a certain portion of the rear surface. The pressure distributions calculated for a circular cylinder in this way under reasonable assumptions are in good agreement with the measurements.

BIBLIOGRAPHY

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In addition to the titles listed below, the following more extended bibliographies may be mentioned.

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The following abbreviations are used in the bibliographical notes:

Z.A.M.M.	Zeitschrift für angewandte Mathematik und Mechanik, V.d.I.-Verlag, Berlin.
Z.F.M.	Zeitschrift für Flugtechnik und Motorluftschiffahrt, Oldenbourg, München.
Ing.-Arch.	Ingenieur-Archiv, Julius Springer, Berlin.
U.S. N.A.C.A.	National Advisory Committee for Aeronautics Reports, U.S. Gov. Printers Office, Washington, D.C.
A.R.C. R. and M.	Aeronautical Research Committee Reports and Memoranda, His Maj. Stat. Office, London.
Proc. Roy. Soc.	Proceedings of the Royal Society of London.
Phil. Trans.	Philosophical Transactions of the Royal Society of London.
Phil. Mag.	Philosophical Magazine, London.

The rest of the abbreviations are self explanatory.

Apart from a few items, the bibliography does not comprise a great number of papers written by Russian authors on theoretical hydrodynamics (JOUKOWSKI, RIABOUCHINSKY, TCHAPLIGUINE and their collaborators and successors). At present papers are appearing in the Transactions of the Central Aero-Hydrodynamical Institute, Moscow; the Journal of the Air Fleet, Moscow; Applied Mathematics and Mechanics, Leningrad, and other scientific publications in the U.S.S.R. Also reprints of papers by N. E. JOUKOWSKI are forthcoming. A paper by S. A. TCHAPLIGUINE

on the General Theory of a Monoplane Wing and a Theory of Slotted Aeroplane Wing is published in English at Paris (Gauthier-Villars, 1929). The "Bulletin de l'Institut Aérodynamique de Koutchino", which appeared until 1920 (in French; fasc. VI published at Paris) contains mainly papers on experimental subjects, though a few theoretical investigations have been inserted as well. A number of the later papers by D. RIABOUCHINSKY have been listed below under Chapter VII.

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Printed by Universitätsdruckerei H. Stürtz A.G., Würzburg.

JULIUS SPRINGER • BERLIN

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Die Neubearbeitung wird drei Bände umfassen. Der jetzt vorliegende erste Band befaßt sich mit den aerodynamischen Forschungsergebnissen und ihrer Verwendung beim Flugzeugentwurf. Er gibt einesteils möglichst umfassend die Zahlenwerte der aerodynamischen Größen, die bei der Berechnung von Flugleistungen und bei der Dimensionierung der einzelnen Flugzeugteile wesentlich sind, andererseits enthält er die Gedankengänge über Stabilität und Steuerung, die für die Entwicklung der Flugtechnik immer stärker bestimmend werden. Er wird innerhalb des Gesamtwerkes „Aerodynamik“ ergänzt durch den von Professor Dr. Fuchs bearbeiteten zweiten Band über die „Theorie der Luftkräfte“ (erscheint im Frühjahr 1935) und den von Dr. Seewald bearbeiteten dritten Band über „Luftschrauben“ (erscheint im Herbst 1935). Ein zusammenfassendes, dabei aber knapp gehaltenes Lehrbuch über diese praktischen aerodynamischen Grundlagen der Flugtechnik existiert sonst zur Zeit nicht. Das Werk ist hervorgegangen aus dem früher viel benutzten Lehrbuch von Fuchs und Hopf, ist aber vollkommen neu geschrieben und auf die Höhe der heutigen Erkenntnisse gebracht.