YALE UNIVERSITY

MRS. HEPSA ELY SILLIMAN MEMORIAL LECTURES

LECTURES
ON
CAUCHY'S PROBLEM
IN
LINEAR PARTIAL DIFFERENTIAL EQUATIONS

SILLIMAN MEMORIAL LECTURES

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LECTURES

ON

CAUCHY'S PROBLEM

IN

LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BY

JACQUES HADAMARD, LL.D.

Member of the French Academy of Sciences
Foreign Honorary Member of the American Academy of Arts and Sciences

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THE SILLIMAN FOUNDATION

IN the year 1883 a legacy of eighty thousand dollars was left to the President and Fellows of Yale College in the city of New Haven, to be held in trust, as a gift from her children, in memory of their beloved and honored mother, Mrs. Hepsa Ely Silliman.

On this foundation Yale College was requested and directed to establish an annual course of lectures designed to illustrate the presence and providence, the wisdom and goodness of God, as manifested in the natural and moral world. These were to be designated as the Mrs. Hepsa Ely Silliman Memorial Lectures. It was the belief of the testator that any orderly presentation of the facts of nature or history contributed to the end of this foundation more effectively than any attempt to emphasize the elements of doctrine or of creed ; and he therefore provided that lectures on dogmatic or polemical theology should be excluded from the scope of this foundation, and that the subjects should be selected rather from the domains of natural science and history, giving special prominence to astronomy, chemistry, geology and anatomy.

It was further directed that each annual course should be made the basis of a volume to form part of a series constituting a memorial to Mrs. Silliman. The memorial fund came into the possession of the Corporation of Yale University in the year 1901 ; and the present volume constitutes the fifteenth of the series of memorial lectures.

PREFACE

THE present volume is a résumé of my research work on the hyperbolic case in linear partial differential equations. I have had the happiness of speaking of some parts of it to an American audience at Columbia University (1911) and also had the honour of treating some points at the Universities of Rome (1916) and Zurich (1917)*. I am much indebted to Yale for having given me the opportunity to develop the whole of it, with the recent improvements which I have been able to make.

The origin of the following investigations is to be found in Riemann, Kirchhoff and still more Volterra's fundamental Memoirs on spherical and cylindrical waves. My endeavour has been to pursue the work of the Italian geometer, and so to improve and extend it that it may become applicable to all (normal) hyperbolic equations, instead of only to one of them. On the other hand, the present work may be considered as a continuation of my *Leçons sur la Propagation des Etudes et les Equations de l'Hydrodynamique*, and, even, as replacing several pages of the last chapter. The latter, indeed, was a first attempt, in which I only succeeded in showing the difficulties of the problem the solution of which I am now able to present.

Further extensions could also be given to such researches, including equations of higher orders, systems of equations, and even some applications to non-linear equations (the study of which has been undertaken in recent times, thanks to the theory of integral equations): which subjects, however, I have deliberately left aside, as the primary one constitutes a whole by itself. I shall be happy if some geometers succeed in extending the following methods to these new cases.

After Volterra's fundamental Memoirs of the *Acta Mathematica*, vol. XVIII, and his further contributions, we should have to mention, as developing and completing Volterra's point of view, the works of Tedone, Coulon and d'Adhemar†. The latter's volume *Les equations aux dérivées partielles à caractéristiques réelles* (Scientia Collection,

* I also mention a brief note read at the International Congress of Mathematicians at Strasbourg (September 1920).

† Picard's researches—which we shall quote in their place—are also essential in several parts of the present work. Such is also the case for Le Roux.

Paris, Gauthier-Villars) includes a careful bibliographical review, and another one has been given by Volterra himself in his Lectures delivered at Stockholm (published at Hermann's, Paris). We did not think it necessary to give a third one, even to add the mention of later works, and content ourselves with eventual quotations in footnotes, apologising in advance to the authors whom we may have forgotten*.

Reasons must also be given for the change of two terms which had been previously introduced and adopted in Science. One is "fundamental solution" replaced by "elementary solution"; the other consists in replacing the word "conormal," created by the finder (d'Adhemar) himself, by "transversal." The first has been done in order to avoid confusion with the "fundamental solutions" introduced by Poincare and his successors (as solutions of homogeneous integral equations); the second for reasons of "economy of thought," as the notion in question already occurs in the Calculus of Variations, where it is denoted by the word "transversal."

I wish to express my heartiest thanks to two young American geometers, Mr Walsh and Mr Murray, whom I have been so pleased to see at Paris during the Academic year 1920—1921. They very kindly undertook to revise the English of the greater part of my manuscript. I fear many faults of language may have escaped detection, but that such errors are not more numerous is due to their useful and friendly help.

* Our own Memoirs on the subject have been inserted in the *Annales Scient. Ec. Norm. Sup.* (1904—1905) and the *Acta Mathematica* (vol. XXXI, 1908). We want to point out that the latter contains several errors in numerical coefficients, viz. in formula (30′), p. 349, where a denominator 2 must be cancelled (a factor 2 having similarly to be added in the preceding line), and in all formulæ relating to m even (corresponding to our Book IV), which must be corrected as in the present volume.

J. H.

July 1921.

I am also greatly indebted to Prof. A. L. Underhill, of Minnesota, for his kind advices in correcting faults of language during the revision of proofs, and express to him my best thanks.

May 1923.

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Several formulæ, being of general and constant use, have been denoted by special symbols, viz.:

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BOOK I

GENERAL PROPERTIES OF CAUCHY'S PROBLEM

CHAPTER I

CAUCHY'S FUNDAMENTAL THEOREM. CHARACTERISTICS

WE shall have to deal with linear partial differential equations of the hyperbolic type, and especially with Cauchy's problem concerning them.

What a linear partial differential equation is, is well known. What the hyperbolic type is, will be explained further on. Let us recall what Cauchy's problem is.

1. Boundary problems in general. A differential equation—whether ordinary or partial—admits of an infinite number of solutions. The older and classic point of view, concerning its integration, consisted in finding the so-called "general integral," i.e. a solution of the equation containing as many arbitrary elements (arbitrary parameters or arbitrary functions) as are necessary to represent any solution, save some exceptional ones.

But, in more recent research, especially as concerns partial differential equations, this point of view had to be given up, not only because of the difficulty or impossibility of obtaining this "general integral," but, above all, because the question does not by any means consist merely in its determination. The question, as set by most applications, does not consist in finding *any* solution u of the differential equation, but in choosing, amongst all those possible solutions, a particular one defined by properly given accessory conditions*. The partial differential equation ("indefinite equation" of some authors) has to be satisfied throughout the m-dimensional domain R (if we denote by m the number of independent variables) in which u shall exist; in other words, to be an identity, inasmuch as u is defined, and simultaneously the accessory conditions ("definite equations") have to be satisfied in points of the boundary of R. Examples of this will occur throughout these lectures.

If we have the general integral, there remains the question of

* This even gives, as we conceive nowadays, the true manner of obtaining the general integral, as, by varying the accessory data in every possible way, we can, as a rule, get to any solution of our equation.

choosing the arbitrary elements in its expression so as to satisfy accessory conditions. In the case of ordinary differential equations, the arbitrary elements being numerical parameters, we have to determine them by an equal number of numerical equations, so that, at least theoretically, the question may be considered as solved, being reduced to ordinary algebra; but for partial differential equations, the arbitrary elements consist of functions, and the problem of their determination may be the chief difficulty in the question. For instance, we know the general integral of Laplace's equation $\nabla^2 u = 0$; but, nevertheless, this does not enable us to solve, without further and rather complicated calculations, the main problems depending on that equation, such as that of electric distribution.

The true questions which actually lie before us are, therefore, the "boundary problems," each of which consists in determining an unknown function u so as to satisfy:

(1) an "indefinite" partial differential equation;

(2) some "definite" boundary conditions.

Such a problem will be "correctly set" if those accessory conditions are such as to determine one and only one solution of the indefinite equation.

The simplest of boundary problems is Cauchy's problem.

2. Statement of Cauchy's problem. It represents, for partial differential equations, the exact analogue of the well-known fundamental problem in ordinary differential equations.

The theory of the latter was founded by Cauchy on the following theorem: Given an ordinary differential equation, say of the second order,

$$\phi\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}\right) = 0 \tag{1}$$

or, solving with respect to $\frac{d^2y}{dx^2}$,

$$\frac{d^2y}{dx^2} = f\left(x,\ y,\ \frac{dy}{dx}\right) = f(x,\ y,\ y'), \tag{1'}$$

a solution of this equation is (under proper hypotheses) determined if, for $x = 0$, we know the numerical values y_0, y_0' of y and $\frac{dy}{dx}$ (or, if the equation were of order k, the numerical values of y, $\frac{dy}{dx}$, ..., $\left.\frac{d^{k-1}y}{dx^{k-1}}\right)$.

Now let us start from a partial differential equation of the second order, such as (for two independent variables)

$$(2) \qquad \phi(u, x, y, p, q, r, s, t) = 0$$

or, if the number of independent variables is m,

$$(\text{II}) \qquad \phi(u, x_i, p_i, r_i, s_{ik}) = 0,$$

where u is the unknown function; $x_1, x_2, \ldots, x_m$ the independent variables and p_i $(i = 1, 2, \ldots, m)$ stands for the first derivative $\frac{\partial u}{\partial x_i}$, r_i for the second derivative $\frac{\partial^2 u}{\partial x_i^2}$, s_{ik} for the second derivative $\frac{\partial^2 u}{\partial x_i \partial x_k}$. We especially deal with the linear case: that is, the left-hand side is linear with respect to u, p_i, r_i, s_{ik}, the coefficients being any given functions of $x_1, x_2, \ldots, x_m$. Now if we are asked to find a solution of that equation such that, for $x_m = 0$, u and the first derivative $\frac{\partial u}{\partial x_m}$ be given functions of $x_1, x_2, \ldots, x_m$, viz.

$$u(x_1, x_2, \ldots, x_{m-1}, 0) = u_0(x_1, x_2, \ldots, x_{m-1}),$$

$$\frac{\partial u}{\partial x_m}(x_1, x_2, \ldots, x_{m-1}, 0) = u_1(x_1, x_2, \ldots, x_{m-1}),$$

this will be called *Cauchy's problem* with respect to $x_m = 0$; u_0 and u_1 will be *Cauchy's data* and $x_m = 0$ the hypersurface*—here a hyperplane—which "bears" the data.

3. Of course, there is no reason to consider exclusively plane hypersurfaces. Let us imagine that the m-dimensional space be sub-

* In the m-dimensional space $(x_1, x_2, \ldots, x_m)$, we shall, for brevity's sake, call a *hypersurface* (or even a *surface*) the $(m-1)$-fold variety defined by one equation between the x's; we call an *edge* the $(m-2)$-fold variety defined by two equations. A *line* will, as usual, mean the locus of a point depending on one parameter; it will be a *straight* line if the x's are linear functions of the parameter.

mitted to a point transformation

$$\text{(T)} \qquad \begin{cases} x_1 = G_1(X_1, \ldots, X_m), \\ x_2 = G_2(X_1, \ldots, X_m), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m = G_m(X_1, \ldots, X_m) \end{cases}$$

(u not being altered by the transformation). The hyperplane $x_m = 0$ will become, in that new X-space, a certain arbitrary surface S

$$\text{(S)} \qquad G_m(X_1, \ldots, X_m) = 0.$$

Our differential equation being replaced by an analogous one

$$\text{(II}a\text{)} \qquad \Phi(u, X_1, X_2, \ldots, X_m, P_i, R_i, S_{ik}) = 0,$$

Cauchy's problem for that equation, with respect to the surface S, will consist in finding a solution of (IIa), satisfying, at every point of this surface, two conditions such as

$$u = u_0, \quad \frac{du}{dN} = U_1.$$

N is a direction given arbitrarily at each point of S, but not tangent to it; u_0 and U_1 (a quantity suitably deduced from u_0 and the primitive u_1) are given numerical values at each point of S, these again being called *Cauchy's data* for the present case.

4. Physical examples. We immediately remind the reader that Cauchy's problem occurs in several physical applications. For instance, let us consider a cylindrical pipe, indefinite in both senses, full of a homogeneous gas which may be subjected to small disturbances. Let us admit Bernoulli's hypothesis of parallelism of sections, so that we have to deal with the motions of a one-dimensional medium; the displacement u of any molecule being always longitudinal and a function of the initial abscissa x and the time t, u must satisfy the equation (where ω is a constant)

$$\text{(e}_1\text{)} \qquad \frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2}.$$

The motion will be determined entirely if, at the instant $t = 0$, we know the initial positions (i.e. the initial disturbances from the positions of equilibrium) and the initial velocities of all the molecules; this knowledge will be analytically expressed by the conditions

$$(C_1) \qquad u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x).$$

Similarly for the motion of electricity in a homogeneous conducting cable, indefinite in both senses, the distribution of intensities and potentials all over the cable at the initial instant being given: the only difference will be that the problem is not governed by equation (e_1) but by the so-called "telegraphist's equation."

If we now come to a three-dimensional medium, that is, to ordinary space, let us consider a homogeneous gas filling that space indefinitely in every direction, and without any gap.

Small motions of such a gas will be governed by the *equation of sound* or *of spherical waves*

$$(e_3) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

u being a properly chosen unknown function (the so-called "velocity potential") of x, y, z, t, and ω again a constant (the velocity of sound in the gas). Knowing initial disturbances and initial speeds at the instant $t = 0$ will be equivalent to knowing the conditions (Cauchy's conditions)

$$(C_3) \quad u(x, y, z, 0) = u_0(x, y, z), \quad \frac{\partial u}{\partial t}(x, y, z, 0) = u_1(x, y, z),$$

u_0 and u_1 being given functions of x, y, z.

4 *a.* We have been speaking of one-dimensional and three-dimensional mediums; of course we may also conceive two-dimensional ones. Let us, for instance, imagine that the state of an aerial mass happens at every instant to be the same all along each vertical line, so that pressures, densities, velocities (the latter being horizontal) are all independent of the vertical coordinate z. Such a motion will be governed by the *equation of cylindrical waves*

$$(e_2) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

which is deduced from (e_3) by supposing that u is independent of z.

This case being evidently a sub-case of the preceding one, we again can complete the determination of u by Cauchy's conditions

$$(C_2) \qquad u(x, y, 0) = u_0(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = u_1(x, y).$$

Of course, we can also conceive the same problem as corresponding to the preceding one for beings living in a space with only two dimensions. But it will be very important for us to remember that this two-dimensional problem may be considered as a mere special case of the three-dimensional one.

We note that, in each case, the number of independent variables is greater by one than the number of dimensions of the medium, the time t constituting a supplementary variable or, as we may say, playing the part of a new coordinate*. It is known that physicists in recent times have fully adopted this point of view, the combination of a point in space and value of t being called by them an "event" or "universe point," the ensemble of all points of space combined with all values of t, a "universe."

5. Geometric configurations. Graphically, taking again a one-dimensional medium, we shall represent the combination of a value of x and a value of t (that is, a given point of the medium considered at a given instant) by a point in an xt plane.

Similarly, we may study the motion of a two-dimensional medium by introducing coordinates x, y, and t in a space analogous to our ordinary one, the medium at the instant $t=0$ being represented by a certain plane in that space, while other instants (especially later ones) would be represented by displacing that plane normally to itself. Everything takes place as if, at the same time in which our two-dimensional motion occurs, the horizontal plane in which it takes place possessed a vertical velocity equal to 1.

6. The case of motion in ordinary space will present a little more difficulty as, adding t, we have to introduce four-dimensional space. We do it, as it seems to me, as clearly as possible by imitating exactly the method of ordinary descriptive geometry. We simultaneously draw two systems of axes x, y, z and x, y, t (fig. 1): each four-dimen-

* This conception was beautifully illustrated a good many years ago by the novelist Wells in his "Time Machine."

sional point, or "universe point," (x, y, z, t) shall be represented by two simultaneous points (x, y, z) and (x, y, t). The plane of xy shall play the part of the "ground plane," the only difference from ordinary descriptive geometry being that, for clearness' sake, this ground plane will often be drawn twice, as in fig. 1a*.

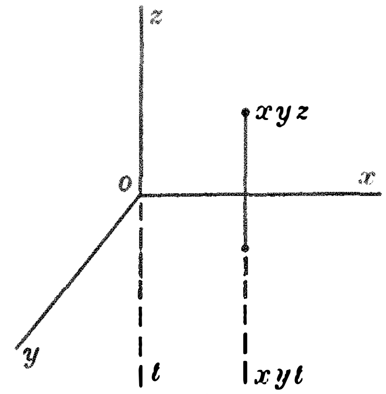

Fig. 1.

7. Cauchy-Kowalewsky's theorem. Now, concerning Cauchy's problem, the following three questions evidently arise:

1. Has Cauchy's problem a solution?

2. Has it only one solution? (in other words, is that problem correctly set?); and lastly

3. How is that solution to be calculated?

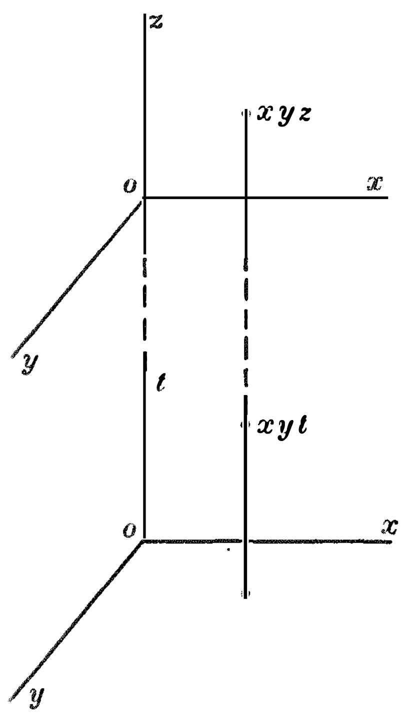

Fig. 1a.

Though the first two questions will be considered here as merely introductory†, we shall begin by seeing how we must answer them.

It is well known that Cauchy himself, then Sophie Kowalewsky and, at the same time, Darboux‡ considered the case in which (2) or (II) can be solved with respect to r (or r_m), viz.

$$(2') \qquad r = f(u, x, y, p, q, s, t)$$

$$\text{or (II')} \qquad r_m = f(u, x_1, \ldots),$$

which is the case in (2) or (II) if

$$(3) \qquad \frac{\partial \phi}{\partial r} \neq 0 \text{ or } \frac{\partial \phi}{\partial r_m} \neq 0;$$

* We shall also quite frequently limit ourselves to drawing one "projection," viz. the (x, y, t) diagram, or even more simply (whatever m may be) the section of the m-dimensional diagram by a two-dimensional space.

† For some further details, we refer to our Columbia Lectures (1911), New York, Columbia University Press (1915), Lecture I.

‡ Cauchy, *C. R. Acad. Sc.* vol. XIV, p. 1020; vol. XV, pp. 44, 85, 131 (1842);

upon that hypothesis, they proved (or are most frequently said to have proved) that *Cauchy s problem,* with respect to $x=0$ (or $x_m=0$), always admits of one and only one solution.

8. Analytic functions. The proof of this theorem has been simplified by Goursat* in such a way that we can give it in a few lines: before which, however, we have to recall what the conception of an analytic function is.

The function $f(x)$ of the (real) variable x is said to be analytic or, more exactly†, *analytic* and *regular* or also *holomorphic* in the interval (a, b) if, x_0 being any number in that interval, f can be represented, for x sufficiently near to x_0, by a Taylor series in powers of $(x-x_0)$, the convergence radius of which is therefore not zero.

If so, f can be defined, and will admit of derivatives of every order, not only for the just mentioned real values of x, but also for imaginary ones, provided their representative points are near enough to the segment (a, b) of the real axis.

But Cauchy's theory of functions shows us that this second property—viz. existence in the imaginary domain with continuity and differentiability—conversely implies Taylor's expansion, thus giving a second definition, fully equivalent to the first one, for an analytic function.

The interval of convergence of the Taylor series for f may be limited by singularities of f in (a, b); but is usually without any apparent relation to them and much smaller than would be

Sophie Kowalewsky, *Thesis*, Göttingen (1874); *Journal für Math.* t. LXXX (1875), pp. 1—32; Darboux, *C. R. Acad. Sc.* vol. LXXX (1875), pp. 101—104 and p. 317. S. Kowalewsky seems not to have known the work of Cauchy (which was also unknown to Darboux and was pointed out by Genocchi in the same vol. LXXX). She even attributes to Weierstrass, *Journal für Math.* t. LI (1856), p. 43, the first formulation of the theorem concerning ordinary differential equations, which seems to be puzzling, as she quotes Briot and Bouquet (*Journ. Ec. Polytechnique*, vol. XXI), and these begin by referring to Cauchy (though without giving a precise quotation). The theorem was again proved in other later works, such as Meray and Riquier's.

* *Bulletin de la Société Mathématique de France*, vol. XXVI (1898), p. 129; *Cours d'Analyse mathématique*, vol. II, p. 360; see Hedrick and Dunkel's translation (Ginn and Co.), vol. II, part II, pp. 53 ff.

† Analysts frequently do not cease to call a function an analytic one even if its domain of existence contains points of discontinuity (poles, essential points, etc....).

obtained by their consideration (being connected with imaginary singularities).

All this may be extended at once to the case of several variables, an analytic function of x, y, z being characterized by one of the two (equivalent) definitions:

(A) $f(x, y, z)$ is analytic in the volume $\mathcal{V}$ if, (x_0, y_0, z_0) being any point in $\mathcal{V}$, f can be represented by a convergent Taylor series in powers of $(x - x_0)$, $(y - y_0)$, $(z - z_0)$ for every position of (x, y, z) within a certain sphere with centre (x_0, y_0, z_0);

(B) $f(x, y, z)$ is analytic in the volume $\mathcal{V}$ if it can be defined, so as to be continuous and differentiable, not only for the (real) points of $\mathcal{V}$, but for any point $x = x + x''i, y = y' + y''i, z = z + z''i$ such that (x', y', z') lies in $\mathcal{V}$ and $|x''|, |y''|, |z''|$ are sufficiently small.

Analytic functions are the ones usually given by our mathematical procedure; but they are really very special ones amongst functions in general*. This is readily seen by the simple (and important) fact that *the continuation of an analytic function is determined.* If $f(x)$ is analytic in (a, b), the knowledge of its values in any—however small—sub-interval (a', b') of (a, b) enables us to calculate it all over (a, b).

For non-analytic functions, continuation has, generally, no meaning. Such a function being only given in $(0, \frac{1}{2})$, its values in $(\frac{1}{2}, 1)$ can be chosen in ∞ ways, no reason existing, as a rule, to prefer any one of these continuations to any other one.

9. Regular functions. We shall have, in the future, to deal with several kinds of functions which will not be assumed to be analytic; they will frequently be restricted by some hypothesis of regularity.

A function of one or several variables will be called *regular* if it is continuous and admits of continuous derivatives up to a certain order p. This order will vary according to the nature of the question. Strictly speaking, it should be precisely indicated in each case: I must own, however, that I shall most often omit to do this, such precision not seeming to me to be worth the somewhat tedious precautions which it would require. It will be sufficient for us to realize that such an order exists, which fact is generally obvious in each question.

* For further details, we refer to our work *La série de Taylor et son prolongement analytique*, Paris, Gauthier-Villars.

A regular function admits of Taylor's expansion, limited to terms of a certain order, and, as its derivatives also admit of corresponding expansions, all operations based on such an expansion, and in general all operations of Differential Calculus which are valid for analytic functions, hold good also for "regular" ones, provided no higher derivatives are concerned than those of order p. For instance, such a punctual transformation as (T) (§ 3) will not alter regularity if the functions G are themselves regular (with the condition, of course, that the Jacobian does not vanish).

As to calling a function "analytic and regular," this is synonymous with saying that it is holomorphic.

10. The proof of Cauchy-Kowalewsky's theorem. For the fundamental theorem concerning *ordinary* differential equations, we remind the reader that two kinds of proof have been given by Cauchy and his successors.

I. One of them is what Cauchy calls "*Calcul des Limites**," and modern writers "method of dominant functions." Taking the given differential equation in the form (1′) (§ 2), it essentially assumes that its right-hand side is *holomorphic* in x, y, y in the neighbourhood of $(x = 0,\ y = y_0,\ y = y_0')$. Using the fact that any convergent Maclaurin expansion in powers of x, y, z admits of a "dominant" expansion of any of the forms

$$\frac{K}{1 - \frac{x+y+z}{\rho}},\quad \frac{K}{\left(1-\frac{x}{\rho}\right)\left(1-\frac{y}{\rho}\right)\left(1-\frac{z}{\rho}\right)},\quad \frac{K}{\left(1-\frac{x}{\rho}\right)\left(1-\frac{y+z}{\rho_1}\right)},\ \ldots,$$

K, ρ, ρ_1 being, in each case, properly chosen positive constants, the proof establishes (upon the aforesaid hypothesis) that there exists a (unique) convergent Maclaurin expansion in powers of x satisfying the given equation and initial conditions.

II. In the second kind of methods (successive approximations), the differential equation is no longer assumed to be an analytic one. Only very simple properties (continuity and "Lipschitz's condition") are assumed concerning its right-hand side. Nevertheless, the same result—viz. existence and uniqueness of the solution—is obtained as

* See Goursat's *Cours d'Analyse*, translated by Hedrick and Dunkel, vol. II, part II, chap. II, pp. 45 ff.

in the former method, except that, of course, the solution itself is no longer analytic.

The proof of the theorem concerning partial differential equations corresponds to the *first* of the above-mentioned classes of methods. We shall present it under Goursat's form*.

Reducing the number of independent variables to two, in order to simplify the notation, we start from the equation

$$(2') \qquad r = f(u, x, y, p, q, s, t)$$

and the corresponding Cauchy problem, consisting in the determination of u by that equation and the definite conditions

$$(5) \qquad u(0, y) = u_0(y), \quad \frac{\partial u}{\partial x}(0, y) = u_1(y).$$

Let us try to satisfy all these conditions by choosing for u a power series in x

$$(4) \qquad u = u_0 + u_1 x + \ldots + \frac{u_h}{h!} x^h + \ldots.$$

Each $u_h = \left(\frac{\partial^h u}{\partial x^h}\right)_{x=0}$ will be a function of y, which we must find.

u_0 and u_1 are given. To find $u_2, u_3, \ldots$, we notice that each derivative $\frac{\partial^{h+k} u}{\partial x^h \partial y^k}$, for $x = 0$, will be a derivative of u_h, whatever k may be. Therefore, making, in (2′), $x = 0$, the right-hand side will contain, besides y itself, only the functions u_0, u_1 and their derivatives $p = u_1$, $q = u_0'$, $s = u_1'$, $t = u_0''$, so that the left-hand side $(r)_{x=0} = u_2$ can be considered as known.

Furthermore, differentiating (2′) once with respect to x and then making $x = 0$, we obtain $\left(\frac{\partial r}{\partial x}\right)_{x=0} = u_3$ in terms of u_0, u_1, u_2 and their derivatives; and, in the same way, successive differentiations with respect to x will give us the values of $u_4, u_5, \ldots$, each u_h being a polynomial in $u_0, u_1, u_2, \ldots, u_{h-1}$ and their derivatives, and also in f and its derivatives.

We can also consider each u_h as expanded in powers of $(y - y_0)$ (where y_0 is some fixed value of y) so as to replace (4) by

$$(4a) \qquad u = S = \Sigma\Sigma \frac{u_{hk}}{h!\,k!} x^h (y - y_0)^k:$$

* See Goursat-Hedrick, *loc. cit.* pp. 61 ff.

then each numerical coefficient u_{hk} will, on account of the preceding operations, be expressed in terms of the preceding ones (that is, u_{hk} with smaller h and not greater k) and of the coefficients in the Taylor expansion* of f, by a polynomial P.

We see that conditions (2′) and (5) determine every coefficient of (4) or (4 a). Therefore, we can already assert that *our Cauchy problem cannot admit of more than one solution represented by a convergent series; that is, of one solution holomorphic in x.*

We have now to show that a solution actually exists. Assuming f to be holomorphic in the variables which it contains and making the same hypothesis for the functions u_0 and u_1 in the neighbourhood of some fixed value $y = y_0$, we shall show that *the series* (4) *is convergent* for $|x|$ sufficiently small†, and *that such is the case even for the double series* (4 a), provided $|x|$ and $|y - y_0|$ lie below properly chosen positive limits.

The first step will consist, as for ordinary differential equations, in noting that each successive operation for the determination of our u_h only implies differentiations, multiplications and additions (without any use of the sign $-$): in other words, that the polynomial denoted above by P has only positive terms. Therefore, we shall have a dominant of the series (4 a) if we replace each of the expansions of f, u_0, u_1 by a dominant one. The whole question is reduced to finding such dominant expansions that the corresponding problem is certain to have a solution.

For that purpose, we may at first assume that the given functions u_0, u_1 are zero, and even that zero is also the value of u_2 deduced from the equation; for, in the general case (u_0, u_1, $u_2 \neq 0$), we could, instead of u, introduce a new unknown u' by the transformation

$$u' = u - u_0 - u_1 x - u_2 x^2,$$

the new problem in u' satisfying the above requirement. Under such

* We mean an expansion in powers of x, $(y - y_0)$, $[u - u_0(y_0)]$, $[p - u_1(y_0)]$, $[q - u_0'(y_0)]$, $[s - u_1'(y_0)]$, $[t - u_0''(y_0)]$.

† That the series (4 a), when convergent, certainly defines a solution of the problem, can be shown as in the case of ordinary differential equations (and as we shall show in Book II for similar purposes).

conditions (and y_0 being taken $= 0$) a dominant of f will be

$$\frac{K}{\left(1-\frac{x+y+u+p+q}{\rho}\right)\left(1-\frac{s+t}{\rho_1}\right)}-K$$

(as the initial values of x, y, u, p, q, s, t are all zero, and the corresponding value of f is also zero), and we can replace u_0, u_1 by *any* Maclaurin expansions with positive coefficients, as any such expansions are obviously dominant of zero. Our proof will therefore be given if we show that the equation

$$r=\frac{\partial^2 u}{\partial x^2}=\frac{K}{\left(1-\frac{x+y+u+p+q}{\rho}\right)\left(1-\frac{s+t}{\rho_1}\right)}-K$$

admits of a solution represented by a Maclaurin expansion with all coefficients positive or zero, or if we do so for any other equation where the quantity in the right-hand side would be replaced by a dominant one. Now, Goursat introduces such a dominant by writing $\frac{x}{\alpha}$ instead of x, denoting by α a positive number smaller than 1, the choice of which we shall examine presently.

For this new equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{K}{\left(1-\frac{\frac{x}{\alpha}+y+u+p+q}{\rho}\right)\left(1-\frac{s+t}{\rho_1}\right)}-K,$$

we seek to find a solution only depending on the variable

$$\sigma = x + \alpha y.$$

The function u of σ will have to satisfy the ordinary differential equation

$$u''=\frac{d^2u}{d\sigma^2}=\frac{K}{\left(1-\frac{\frac{\sigma}{\alpha}+u+(1+\alpha)u'}{\rho}\right)\left(1-\frac{(\alpha+\alpha^2)u''}{\rho_1}\right)}-K$$

or

$$\left[1-\frac{K}{\rho_1}(\alpha+\alpha^2)\right]\frac{d^2u}{d\sigma^2}-\frac{\alpha+\alpha^2}{\rho_1}\left(\frac{d^2u}{d\sigma^2}\right)^2=\frac{K}{1-\frac{\frac{\sigma}{\alpha}+u+(1+\alpha)\frac{du}{d\sigma}}{\rho}}-K.$$

If we now take α such that $1 - \frac{K}{\rho_1}(\alpha + \alpha^2) > 0$, not only will this differential equation admit (on account of Cauchy's first theorem) of a holomorphic solution vanishing with σ, but the expansion of the solution will have all its coefficients positive*. Q.E.D.

Nothing essential need be changed in the above when several independent variables $x, y, z, \ldots$ exist, the double series S merely becoming a multiple one, the quantity σ being $x + \alpha(y + z + \ldots)$ and some numerical coefficients appearing in our dominant functions.

11. The expansion (4 a) thus obtained depends on the choice of y_0 and is only valid if we assume not only that $|x| < R$, but also that it is confined within a proper interval I around y_0. On the contrary, the expansion (4) is independent of y_0. More precisely, if we give to y_0 two different values such that the two corresponding intervals I overlap each other, every function u_h will be the same for both cases in the common part, this being a consequence of the fact that the holomorphic solution of our Cauchy problem is unique.

Therefore, if our hypotheses concerning f, u_0, u_1 are satisfied throughout any segment, however large, of the y-axis, our preceding calculations will give the expansion (4) in the vicinity of that whole segment. The numbers K, ρ and ρ_1 having, as we know, the first a maximum and the two others minima all over the aforesaid segment, the corresponding limit of convergence R for $|x|$ can also, if necessary, be considered as constant.

As has already been stated, the above determination of R (even if we take it as different for different values of y_0) generally leads, for the domain of convergence of the series (4) and, a fortiori, for the domain of existence of the solution u, to limits which are so small as to be useless in practice.

12. Characteristics. The conclusions are quite different in the exceptional case, where the sign $\neq$ in (3) (p. 9) is replaced by $=$. In

* A quantity Y defined by $aY - bY^2 = X$, with $a > 0$, $b > 0$, has, in powers of X, a Maclaurin expansion with all coefficients positive, as appears, e.g. by direct resolution of the quadratic equation.

this case, *the solution of Cauchy's problem does not generally exist and, if it exists, is not unique.*

In this instance, some new features appear when the number of variables is greater than two. Let us take it as 3, and also change our notation (as we shall do from now on), by representing our equation, which we now assume to be linear, in the form*

$$\text{(E)} \qquad \sum_{i,k} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu = f,$$

A_{ik}, B_i, C and f being given functions of the x's. For the exceptional case, we must assume† $A_{mm} (= A_{33}) = 0$, so that our equation is reduced to

$$\text{(6)} \qquad 2A_{13} \frac{\partial^2 u}{\partial x_1 \partial x_3} + 2A_{23} \frac{\partial^2 u}{\partial x_2 \partial x_3} + A_{11} \frac{\partial^2 u}{\partial x_1^2} + 2A_{12} \frac{\partial^2 u}{\partial x_1 \partial x_2} + A_{22} \frac{\partial^2 u}{\partial x_2^2} + \sum_{i=1}^{3} B_i \frac{\partial u}{\partial x_i} + Cu = f,$$

with Cauchy's conditions $u = u_0(x_1, x_2)$, $\frac{\partial u}{\partial x_3} = u_1(x_1, x_2)$ for $x_3 = 0$.

Here we see that the left-hand side contains no double differentiation with respect to x_3, so that (for $x_3 = 0$) the equation does not involve the coefficient u_2 of x_3^2, but only u_0 and u_1; thus, it no longer determines any unknown, but gives us a *condition of possibility* for our Cauchy problem, viz.

$$\text{(7)} \qquad 2A_{13} \frac{\partial u_1}{\partial x_1} + 2A_{23} \frac{\partial u_1}{\partial x_2} + B_3 u_1 + H = 0,$$

with $$H = A_{11} \frac{\partial^2 u_0}{\partial x_1^2} + 2A_{12} \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + A_{22} \frac{\partial^2 u_0}{\partial x_2^2} + B_1 \frac{\partial u_0}{\partial x_1} + B_2 \frac{\partial u_0}{\partial x_2} + Cu_0 - f.$$

If u_0 and u_1 are not chosen so as to satisfy this equation, the problem has no solution.

If, for instance, u_0 is given in the first place, we ought to take for u_1 a solution of (7). We notice that this gives for u_1 a linear partial

* The notation is the usual one for quadratic forms, with $A_{ik} = A_{ki}$, so that each term of the second order with different suffixes is reckoned twice.

† $A_{mm} = 0$ may be an identity in $x_1, x_2, \ldots, x_m$ or, more generally, an identity in $x_1, \ldots, x_{m-1}$ for $x_m = 0$. For simplicity's sake, we only deal with the first case, the conclusions being the same, as may be readily seen, in the second one.

differential equation of the first order, the integration of which would lead, as we know, to the drawing, on our plane $x_3 = 0$, of the system of lines (l) defined by the differential equation*

$$(l) \qquad \frac{dx_1}{A_{13}} = \frac{dx_2}{A_{23}}.$$

Let us now suppose that condition (7) is fulfilled: we have as yet no condition to determine u_2. But such a condition arises from the following equation (which, in the general case, previously dealt with, was used to find u_3) obtained by differentiating once with respect to x_3 and then making $x_3 = 0$: this obviously gives

$$(7') \qquad 2A_{13}\frac{\partial u_2}{\partial x_1} + 2A_{23}\frac{\partial u_2}{\partial x_2} + B_3 u_2 + H_1 = 0,$$

where $H_1 = 2\frac{\partial A_{13}}{\partial x_3}\frac{\partial u_1}{\partial x_1} + 2\frac{\partial A_{23}}{\partial x_3}\frac{\partial u_1}{\partial x_2} + \frac{\partial B_3}{\partial x_3}u_1 + \frac{\partial H}{\partial x_3}$ does not depend on u_2.

We see therefore that u_2 is not entirely arbitrary, but that it can, nevertheless, be chosen in an infinity of ways: we can take for it any solution of the linear partial differential equation (7'). *This equation has the same characteristics as* (7), viz. the lines (l).

The fact that, u_0 and u_1 being given, u_2 may be chosen in more than one way can be expressed by saying that *two solutions of our equation* (corresponding to the same u_0 and u_1, but to different u_2's) *may be tangent to each other*† *in every point of* $x_3 = 0$ (or, generally, $x_m = 0$).

Further differentiations with respect to x_3 would, in the same way, give us for $u_3, u_4, \ldots$ successive linear partial differential equations of the first order, the characteristics of which would still be the same lines (l).

We shall have to return later to these lines, the geometric meaning of which will then appear. For the present, the preceding calculation

* The corresponding lines (l) for $m > 3$ would be defined by $(m-2)$ differential equations between $x_1, x_2, \ldots, x_{m-1}$.

† Two functions u, v (of one or several variables) are said to be *tangent* at a determinate point (i.e. for a certain system of values of the variables) if they and all their first derivatives assume numerical values which are equal each to each at the point in question. The contact is of *order p* if similar equalities hold not only between first derivatives, but between all derivatives up to the order p.

gives us a first presumption that if our Cauchy problem is not impossible (that is, if condition (7) is fulfilled) *it becomes indeterminate,* as each of the successive u_h's can be chosen with a certain measure of arbitrariness. This, however, is only a presumption, as we do not know, as yet, whether the choice of these u_h's can be directed so as to make the series (4) convergent: the proof of which fact we shall have a further opportunity to give, and this will show, at the same time, the degree of indetermination of u.

If we take this proof for granted, we see that, briefly speaking, Cauchy's problem behaves like the resolution of a system of n ordinary equations of the first degree in n unknowns, the determinant of which is zero.

We have dealt with a linear equation, the only interesting case for what follows; the non-linear case leads to essentially similar results with some differences at the beginning of the operation, and is even readily reduced to the first case by differentiating the given equation.

13. The above exceptional case is a most important one for our further work and for any study of partial differential equations. How is it to be defined if we set Cauchy's problem not with respect to $x_m = 0$, but to any other surface such as S (§ 3)? To see this, we only have to transform condition (3) by application of the punctual transformation (T): an elementary operation. We thus recognize* that (with the notation of § 3) *the exceptional case is defined by the condition*

$$\sum_i \frac{\partial \Phi}{\partial R_i}\left(\frac{\partial G_m}{\partial X_i}\right)^2 + \sum_{i,k} \frac{\partial \Phi}{\partial S_{ik}} \frac{\partial G_m}{\partial X_i} \frac{\partial G_m}{\partial X_k} = 0.$$

Let us again suppose that we are in the linear case, our partial differential equation having the form

$$\text{(E)} \qquad \sum_{i,k} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu = f$$

(A_{ik}, B_i, C and f being given functions of $x_1, x_2, \ldots, x_m$). The con-

* The condition can be found directly, and even the calculation in the preceding § 12 performed without the use of punctual transformations: see our *Leçons sur la propagation des ondes* (Paris, Hermann, 1903), ch. VII, §§ 278 to 288.

dition for the variety $G(x_1, \dots, x_m) = 0$ to correspond to the exceptional case will be

$$(\mathbf{A}) \qquad \Sigma A_{ik} \frac{\partial G}{\partial x_i} \frac{\partial G}{\partial x_k} = 0:$$

in other words, it is to be obtained by the following

RULE. *We consider exclusively the terms of the second order in the given equation, and, in these terms, we replace each second derivative of u by the corresponding square or product of first derivatives of G.*

If condition (**A**) is fulfilled (that is, if the above quantity is zero at every point* of the surface S), S is said to be a *characteristic*† of the equation (E). The quadratic form

$$\mathbf{A}(\gamma_1, \gamma_2, \dots, \gamma_m) = \sum_{i,k} A_{ik}\gamma_i\gamma_k$$

is called the *characteristic form.*

The fundamental property of characteristics is, on account of the preceding considerations, expressed by the fact that they are the only surfaces along which two solutions of the equation can touch each other: this contact can be of any order (as, in the operations in the preceding section, we can assume $u_0, u_1, \dots, u_{h-1}$ to be the same for two different solutions, the values of u_h changing).

This property is entirely similar to the definition of characteristics for a partial differential equation of the first order, and this is the reason why the same denomination is given to both, although the former are *surfaces* and the latter *lines* (whatever be the number of variables).

Equation (**A**) is a partial differential equation of the first order, which S must satisfy. Geometrically, as is well known, it can be interpreted by saying that, at each of its points, S must have its

* The case in which condition (**A**) would be satisfied in some points of S and not in the others, though occurring in some problems already treated, would present new difficulties which have not as yet been attacked, as they have not proved interesting in applications.

† The theory of characteristics, for two independent variables, has been known since Monge and Ampere (see Darboux's *Théorie des surfaces*, vol. II and Goursat's *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*). Its extension to $m > 2$ was first given by Bäcklund (*Math. Annalen*, vol. XIII, 1878), but was not generally known before being found by Beudon (*Bull. Soc. Math. Fr.* vol. XXV, 1897).

tangent plane tangent to a certain corresponding quadratic cone, whose *tangential* equation is $\mathbf{A}(\gamma_1, \gamma_2, \ldots, \gamma_m) = 0$. This cone is called the *characteristic cone.*

(If the A_{ik}'s are not constants, each system of values of $x_1, x_2, \ldots, x_m$ will give a different characteristic cone. We shall generally consider a characteristic cone as having the corresponding point of the m-dimensional space as its vertex.)

Characteristics have an important physical meaning; they are, in fact, what the physicist means by *waves.* That the above definition of them (and, more precisely, the part they play as surfaces of contact between two solutions) is strictly equivalent to Hugoniot's conception of waves, may be easily perceived (for proof, however, we shall refer to our *Leçons sur la propagation des ondes*). In fact, the identity of both conceptions not only will appear in each case that we shall have to deal with, but will be an a posteriori consequence of our final formulæ.

14. The result of Cauchy and Sophie Kowalewsky's analysis would therefore be that Cauchy's problem has one (and only one) solution every time the surface which bears the *data is not characteristic, nor tangent anywhere to a characteristic* *.

* We shall leave the case of *systems* of partial differential equations aside and only say a word about it here for completeness' sake. The fundamental theorem allows a well-known generalization to such systems when the number of equations is equal to the number p of unknowns and they can be solved for the derivatives of the highest order with respect to one variable x: for instance, as concerns the three equations of the second order $F_1(r, r', r'', \ldots) = 0$, $F_2(r, r', r'', \ldots) = 0$, $F_3(r, r', r'', \ldots) = 0$ with the three unknowns u, u', u'' (where we have emphasized the three second derivatives $r = \frac{\partial^2 u}{\partial x^2}$, $r' = \frac{\partial^2 u'}{\partial x^2}$, $r'' = \frac{\partial^2 u''}{\partial x^2}$) if they can be solved for r, r', r''. The exceptional case will occur when such a resolution (at least a regular one) is impossible, i.e. when the Jacobian $\frac{D(F_1, F_2, F_3)}{D(r, r', r'')}$ vanishes. Then $x = 0$ will be said to be a characteristic. We easily deduce from this, by punctual transformation (§ 3) or by a direct calculation, the condition that a surface $G = 0$ be a characteristic: which will give (see our *Leçons sur la propagation des ondes*, ch. VII, § 321), in the case of the above system, a partial differential equation of the first order and sixth degree (2pth degree, if there were p equations of the second order in p unknowns).

It has sometimes been believed that the exceptional case could always be avoided by a proper punctual transformation. This, however, is an error: in other terms, *it may happen that the above defined equation of characteristics is an identity.* Various examples of this have been formed; but one is offered by a most classic and usual problem, viz. the problem of *applicable surfaces*: three partial differential equations of the first order in x, y, z as functions of u, v which cannot be solved with respect to $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$, whatever be the choice of the independent variables u, v.

How Cauchy's statement (still remaining in the analytic hypothesis) must be modified in the most general case—without changing the variables—has, as is known, been made clear by the works of Meray and Riquier, even when the number of equations is not equal to the number of unknowns.

But the effect of a punctual transformation can itself be determined, and, therefore, a new equation for characteristics established, even when the ordinary condition for these fails by being an identity, as was pointed out, in one case, by the author (*Bull. Soc. Math. Fr.* vol. XXXIV, 1906) and generally performed, thanks to the works of Le Roux (*Bull. Soc. Math. Fr.* vol. XXXVI, p. 129, 1908), Gunther and Maurice Janet (*C. R. Ac. Sc.* 1913).

CHAPTER II

DISCUSSION OF CAUCHY'S RESULT

15. The reader will probably wonder at our systematically employing a conditional form and seeming to consider as doubtful one of the most classic and well-known demonstrations of analysis. The fact is that things are not so simple as would be suggested by the above arguments. Indeed, the circumstances which we shall meet with will appear as quite paradoxical from the purely mathematical point of view and could only be foreseen by physical hints. No question offers a more striking illustration of the ideas which Poincare developed at the first International Mathematical Congress at Zurich, 1897 (see also *La Valeur de la Science*, pp. 137—155), viz. that it is physical applications which show us the important problems we have to set, and that again Physics foreshadows the solutions.

The reasonings of Cauchy, S. Kowalewsky and Darboux, the equivalent of which has been given above, are perfectly rigorous; nevertheless, their conclusion must not be considered as an entirely general one. The reason for this lies in the hypothesis, made above, that Cauchy's data, as well as the coefficients of the equations, are expressed by *analytic* functions; and the theorem is very often likely to be false when this hypothesis is not satisfied.

We say "often" and not "always," for it may also happen that the statement of Cauchy-Kowalewsky given above should prove to be accurate for a quite general choice of data; and indeed, one of the most curious facts in this theory is that apparently very slightly different equations behave in quite opposite ways in this matter.

If, in the first place, we take such a Cauchy problem as was spoken of in § 4 [Cauchy's problem with respect to $t=0$, for equations (e_1), (e_2), (e_3)], our above conclusions are valid, as we shall see as these lectures proceed, without any need of the hypothesis of analyticity.

But the conclusions will be altogether different if, for instance, we deal with Laplace's classic *equation of potentials*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

This will be immediately realized by comparison with another classic boundary problem; I mean Dirichlet's problem. This consists, as we know, in determining a solution of Laplace's equation within a given volume $\mathcal{V}$, the value of u being given at every point of the boundary surface S of that volume. It is a known fact that this problem is correctly set: i.e. it has one (and only one) solution.

This fact immediately appears as contradictory to Cauchy-Kowalewsky's theorem: for, if the knowledge of numerical values of u at the points of S (together with the partial differential equation) is by itself sufficient to determine the unknown function within $\mathcal{V}$, we evidently have no right to impose upon u any *additional* condition, and we cannot therefore, besides values of u, choose arbitrarily those of $\frac{du}{dn}$. Indeed there is, between those two sets of values, an infinity of relations which must be satisfied in order that a corresponding harmonic function should exist. Any point a *exterior* to $\mathcal{V}$ provides such a relation, since, denoting by r the distance from a to an arbitrary point M of S, we must have the well-known identity:

$$(8) \qquad \iint_S \left(v \frac{d\frac{1}{r}}{dn} - \frac{1}{r} \frac{du}{dn} \right) = 0.$$

How is it that, on the contrary, Cauchy-Kowalewsky's conclusions would lead to the arbitrary choice at every point of S, not only of u, but also of one of its first derivatives, such as the normal derivative $\frac{du}{dn}$?

A double explanation can be given for this apparent contradiction. First, the conditions are not alike in both cases. We have previously proved the possibility of Cauchy's problem with respect to a plane, then with respect to any surface which can be deduced from a plane by punctual transformation. This is the case for any (regular) *portion* of a surface, provided it is sufficiently small, but not for *whole* closed surfaces. The whole surface of a sphere, for instance, cannot, with perfect continuity, be transformed into the surface of a plane: it has a different shape in the sense of Analysis situs.

This, however, is no decisive objection, as we can see from the remark of § 11: if we solve our Cauchy problem in the neighbourhood of each portion of S, these different elements of a solution will continue each other and one solution will be constituted, which will be valid all over S and its neighbourhood.

But, in the second place, our theorem only proves the existence of the solution of Cauchy's problem in the *neighbourhood* of the initial surface S. In Dirichlet's problem, the solution has to exist in the whole extent of $\mathcal{V}$. This is the first required explanation: for this reason, and only for this, the relations (8) are necessary. If they are not satisfied by an analytic set of values of u and $\frac{d}{dn}$, these values will correspond to a certain harmonic function u in the neighbourhood of S; but u will necessarily admit of some singularities, or even cease to be defined at some place within $\mathcal{V}$.

Even this first answer to our question is not complete and does not give the only reason why Cauchy's problem is not always possible. Taking now the geometric terms of the problem exactly in the same way as Cauchy, we can see that, if we drop the hypothesis of analytic data, no solution will exist even in the immediate neighbourhood of S, or of a portion σ, however small, of S.

This may be considered as a consequence of a well-known property of harmonic functions (i.e. solutions of $\nabla^2 u = 0$), viz. that they are analytic in every region inside the domain where they exist and are continuous together with their first derivatives, and they can only lose that character of analyticity on the boundary of this domain: a form of this property being* that if two harmonic functions, each defined on one side of a surface, have, at each point of the latter, the same value and the same normal derivative, they are the analytic extension of each other, both together constituting a single harmonic (and therefore analytic) function throughout the region lying on both sides of the surface.

This shows, indeed, that (σ being, for instance, assumed to be analytic) it is, at least, impossible that Cauchy's problem with non-analytic data should have a solution on *both* sides of σ; for such two solutions u' and u'' would, on account of the above theorem, constitute together a single analytic function in a whole domain to which σ would

* This was pointed out by Duhem (see *Hydrodynamique, Elasticité, Acoustique*, Paris, Hermann (1891), vol. I, p. 169).

be interior: which is obviously contradictory to the assumption that u_0 (common value of u' and u'') or u_1 (common normal derivative) is non-analytic over σ.

Of course, it is almost evident that, *in general*, neither u' nor u'' will exist: for there is no reason why one should exist rather than the other, if the data are taken at random.

15*a*. That no solution can exist even on *one* side of σ can be shown rigorously when σ is a portion of the plane $x=0$, by making the further assumption that u_0 (or u_1) is zero: for, if e.g. u' should exist for $x \geqslant 0$, we could define u'' for $x \leqslant 0$ by

$$u(-x, y, z) = -u(x, y, z), \tag{9}$$

u' and u'' then having, on $x=0$, the same values (viz. 0) and the same normal derivative. The case is now the same as above: therefore, the solution can *never* exist if $u_1(y, z)$ is not analytic.

(Similarly, if u_1 be zero, with any value of u_0, an eventual solution u' for $x \geqslant 0$ could be extended to $x \leqslant 0$ by $u(-x, y, z) = u(x, y, z)$, and this would lead to the same impossibility as above if u_0 should not be analytic.)

If $u_0(y, z)$ had been taken different from zero, it would obviously, by itself, have determined* u but for an analytic function of x, y, z, and therefore $u_1(y, z)$ but for an analytic function of y, z.

16. The equation of heat

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0,$$

though we shall not have to deal with it in what follows, is interesting to consider from the same point of view, as has been done by Holmgren†. Let us again take Cauchy's problem with respect to $x=0$, the

* One of these possible choices for u is $\frac{1}{2\pi}$ multiplied by the potential of a double layer of density u_0 on our plane, the corresponding u_1 being the normal derivative of this potential. The combination of this with the statement in the text gives the most general form acceptable for u_1 corresponding to a given form of u_0.

† *Arkiv för Matematik, Astronomi och Fysik* (1904), p. 324, note; see also *ibid.* vol. II (1905—1906).

first function $u_0(y)$ being again taken as 0. Just as above, assuming our solution to be defined only on one side of $x=0$, say for $x \geqslant 0$, we extend it to $x \leqslant 0$ by formula (9): by means of which u and $\frac{\partial u}{\partial x}$ remain continuous for $x - 0$.

Now, a solution of the equation of heat, which is continuous and has continuous derivatives of the first order, is not necessarily analytic in both variables, as was the case with the equation of potentials; but it can be proved* that it is analytic with respect to x. As, on the other hand (compare preceding § 15 a), it is an odd function on account of $u_0 = 0$, it can be expanded in a convergent series of the form

$$(10) \qquad u = u_1 x + \frac{u_3}{3!} x^3 + \dots + \frac{u_{2p+1}}{(2p+1)!} x^{2p+1} + \dots.$$

The first coefficient u_1 is equal to our derivative $\frac{\partial u}{\partial x}$, for $x=0$. But the differential equation gives

$$u_{2p+1} = \left(\frac{\partial^{2p+1} u}{\partial x^{2p+1}}\right)_{x=0} = \frac{\partial^p}{\partial y^p}\left(\frac{\partial u}{\partial x}\right)_{x=0} = \frac{d^p u_1}{dy^p}.$$

We therefore see, first, that u_1 admits of derivatives of every order; then that we have a limitation of their order of magnitude: for, on account of the convergence of the series (10), we have (M, ρ being two fixed positive numbers)

$$(11) \qquad \left|\frac{d^p u_1}{dy^p}\right| = |u_{2p+1}| < \frac{M(2p+1)!}{\rho^{2p+1}}.$$

The analyticity of u would require inequalities such as

$$(12) \qquad \left|\frac{d^p u_1}{dy^p}\right| < \frac{Mp!}{\rho^p},$$

so that *the system of conditions* (11) *is less restrictive than the conditions for analyticity.*

17. We see that our present considerations on the subject of partial differential equations lead to results worth noting, in the theory of functions of a real variable. Such functions have been classified in a well-known way, according to their degree of regularity: the efforts

* Serge Bernstein (*Thesis*, Paris, 1904) proved this for the most general analytic parabolic equation. Very simple proofs were then given by Gevrey (*C. R. Ac. Sc.* vol. CLII and *Thesis*, Paris).

of contemporary geometers have succeeded in pointing out many interesting and important intermediates between the conception of an arbitrary function and that of a continuous function; more restrictive than the latter is the notion of functions with a limited variation, and again of functions differentiable once, twice, ..., p times. Then would come functions differentiable to any order.

Now, functions satisfying the inequalities (11) show us the usefulness of a distinction which had not been made hitherto: they are intermediate between functions differentiable to any order and analytic functions. Goursat and Gevrey (*loc. cit.*)* have called them *functions of class* 2. Similarly, functions of class α—that is, functions $\phi(y)$ differentiable to any order and such that

$$(11') \qquad \left|\frac{d^p\phi}{dy^p}\right| < \frac{M(\alpha p)!}{\rho^{\alpha p}}$$

—would appear† if we should consider from the same point of view the equations

$$\frac{\partial^m u}{\partial x^m} = \frac{\partial^n u}{\partial y^n} \qquad (m > n)$$

treated by Henrik Block‡: α would then be equal to $\frac{m}{n}$.

That there actually exist functions which satisfy the system of conditions (11′) without being analytic—or more generally, which satisfy that system for a certain value of α, but not for smaller values—can be easily shown by the example of the trigonometrical series

$$(13) \qquad \sum_{n=0}^{\infty} c_n \cos ny,$$

where the c_n's will be, let us say, real and positive numbers. Such a

* See note, p. 27.

† It is worth noting that the class defined *amongst analytic functions* by the inequalities (11′) with α smaller than 1 is already considered in Analysis: it is in fact the class of entire functions of a finite genus.

This word "class" has already been used by Baire in his works on discontinuous functions, with a quite different meaning, but, as Gevrey observes, for this very reason no confusion is possible.

‡ Henrik Block, *Arkiv för Matematik, Astronomi och Fysik*, vol. VII (1911).

series will admit of derivatives of any order, if the series

(14) $$\Sigma c_n n^p$$

is convergent for every value of p. Let us especially take $c_n = e^{-n^{\frac{1}{\alpha}}}$, α being a fixed positive number: this satisfies the aforesaid condition; but the series (14), viz.

(14′) $$\sum_{n=0}^{\infty} n^p e^{-n^{\frac{1}{\alpha}}},$$

which for p even represents the value of the pth derivative for $y=0$ (also being the maximum value of this derivative) is, as we know, of the same order of magnitude as the integral

(14″) $$\int_0^\infty n^p e^{-n^{\frac{1}{\alpha}}} dn = \alpha\Gamma\left[(p+1)\alpha\right],$$

which shows that the corresponding series (13) belongs to class α, and not—at least in any interval containing $y=0$—to any lower class.

If, in (13), we had only given to n the values $n=b^\nu$, where b is a fixed integer and $\nu = 1, 2, \ldots, \infty$, this would not have essentially changed the order of (14′), the integral (14″) being replaced by

$$\int_0^\infty b^{p\nu} e^{-b^{\frac{\nu}{\alpha}}} d\nu = \frac{\alpha}{\log b}\Gamma(p\alpha),$$

but, for this new series

$$\sum_{\nu=1}^{\infty} e^{-b^{\frac{\nu}{\alpha}}} \cos(b^\nu y),$$

we could assert that it can be of no class lower than α, not only around $y=0$, but even in any interval whatever: for such a series only changes in its first terms by changing y into $y + \frac{2l\pi}{b^k}$ (whatever be the integers k and l) and the numbers $\frac{2l\pi}{b^k}$ can approximate as closely as is desired to any given real quantity.

Gevrey (*loc. cit.*)* has shown that functions of class $\alpha > 1$ remain so through the same general operations as analytic ones, such as multiplication, substitution of one or several functions in another one, integration of differential equations, etc. But such functions differ from analytic ones and also from the well-known generalizations given to

* See also his Memoir in vol. XXXV, série 3, of the *Ann. Scient. Ec. Norm. Supre.*

them by Borel* in lacking one of their classic properties: the extension of a function of class α from one part of its domain to a neighbouring one *is not determined.* Such a function, being given in (0, 1), may be extended to (1, 2), for instance, in an infinite number of ways, without losing its property of belonging to class α.

This is easy to see, at least for $\alpha = 2$, by introduction of the function $e^{-\frac{1}{x}}$. This function, holomorphic in every interval (a, b) with $a > 0, b > 0$, is no longer so in $(0, b)$, but belongs (at most) to class 2, as can be seen by the following direct calculation†. To calculate $\phi^{(p)}(a)$ (with $a > 0$) we take the integral $\frac{p!}{2i\pi}\int \frac{\phi(z)}{(z-a)^{p+1}}dz$ along a circumference with centre a in the complex plane. For $\phi(x) = e^{-\frac{1}{x}}$, we can take this circumference tangent to the imaginary axis (fig. 2), and the absolute value of $\phi(x)$ on it will then be constant and $= e^{-\frac{1}{2a}}$ so that

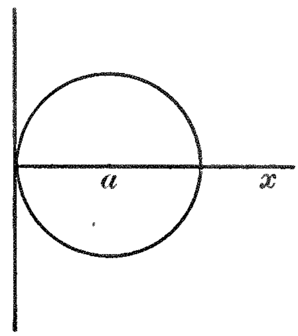

Fig. 2.

$$\left|\frac{1}{p!}\phi^{(p)}(a)\right| = \frac{1}{p!}\left|\frac{d^p}{dx^p}\left(e^{-\frac{1}{x}}\right)_{x=a}\right| \leqslant \frac{1}{a^p} e^{-\frac{1}{2a}}.$$

The maximum of this last quantity corresponds to $a = \frac{1}{2p}$, and

$$|\phi^{(p)}(a)| < e^{-p}(2p)^p p! = (\text{sensibly})\ \frac{1}{\sqrt{2}}\frac{1}{2^p}(2p)!.$$

As every derivative of $\phi(x)$ is zero (on the positive side) for $x = 0$, this shows that a function which is zero for every negative x can be extended to $x > 0$ by $e^{-\frac{1}{x}}$ and be of class 2.

From this example, a more general one can be deduced by taking the integral

$$\psi(x) = \int_0^x e^{-\frac{1}{x-z}}\chi(z)dz \qquad (x > 0),$$

where χ is an arbitrary—say continuous—function. Any derivative of ψ is to be taken by differentiating under $\int$ (no term corresponding

* See *Comptes rendus Ac. Sc.* vol. CLIV; *Acta Math.* t. XXIV.

† The very slightly different function $\frac{1}{\sqrt{x}}e^{-\frac{1}{x}}$, which is suggested by the theory of heat, can also be introduced for the same object.

to the variability of the upper limit, as the integrand is there zero) so that ψ also belongs to a class at most equal to 2; and it is obvious that it may be extended in ∞ ways beyond any value $x=a$ of the variable, as this extension depends on the arbitrary function χ.

In a recent Memoir*, Serge Bernstein extends this conclusion to functions of any class $\alpha > 1$, such a condition being still compatible with ∞ analytic prolongations of the same given function.

The question† arises whether functions $F(p)$ may exist, increasing more rapidly than $R^{-p}p!$, and, nevertheless, such that the limitation $|\phi^{(p)}(x)| < F(p)$ for the values of the successive derivatives of a function ϕ implies the fact that ϕ cannot be extended in more than one way. S. Bernstein is led to a negative answer by such an example as the functions of a class higher than 1; but shows that such a negative answer would no longer hold if the inequality $F(p) > R^{-p}p!$ were satisfied *irregularly*—that is, if there were ∞ values of p giving that inequality and ∞ giving the opposite one.

18. Returning to Cauchy's problem in general, we see that we must avoid confusion between arbitrary functions and analytic ones, and that our preceding conclusions must be examined with regard to this. This applies not only to the data, but also to the unknown function; and, therefore, even our first result, viz. that the solution of our problem, if it exists, is unique (the case of a characteristic excepted), must be looked into again: the proof having only been given that the problem admits of not more than one *holomorphic* solution.

On this point, however, our previous result subsists: Holmgren‡ has proved, at least for linear equations with analytic coefficients, that

* *Math. Ann.* vol. LXXV, pp. 440 ff.

† This was written in April, 1921. Since then, this question has been solved by the beautiful researches of Denjoy and Carleman (see *C.R. Ac. Sc.* Paris, t. 173, 174, 1921—1922): the extension is always unique when $F(p)$ is such that the series $\Sigma \frac{1}{\sqrt[p]{F(p)}}$ is divergent. (Added while correcting the proofs.)

‡ *Öfversigt af Kongl. Vetenskaps Akad. Förh.* (9 Jan. 1901), pp. 91—105. See also our *Lecons sur la propagation des ondes*, note I. The proof rests on Weierstrass' well-known theorem on approximation of continuous functions by polynomials. It would be interesting to extend it to equations with non-analytic coefficients, and (as an immediate consequence of the former) to the non-linear case: which extension would perhaps be made possible by Dunham Jackson, Serge Bernstein and de la Vallee Poussin's recent improvements on Weierstrass' theorem.

(still excluding the exceptional case) the solution of our problem, whether analytic or not, is unique*.

As we have just seen, things are much more complicated with regard to the second part of the result, viz. the existence of a solution; at a first glance, no general analytic rule seems to be assignable. Analogy with ordinary differential equations, which evidently inspired Cauchy, finally proves to have misled† us; and also the examples of the equation of sound and the equation of potentials show how little the closest and apparently most obvious analytical analogies between partial differential equations can be trusted.

But it is remarkable, on the other hand, that a sure guide is found in physical interpretation: an analytical problem always being correctly set, in our use of the phrase, when it is the translation of some mechanical or physical question; and we have seen this to be the case for Cauchy's problem in the instances quoted in the first place.

On the contrary, none of the physical problems connected with

* When stating such results, it is important, as Bôcher most rightly pointed out while lecturing at the University of Paris in 1913—1914, to state accurately what is meant by a solution. In such questions as these, u is required to admit of first and second derivatives satisfying the partial differential equation in the vicinity of $x=0$, but not necessarily on $x=0$ itself. On the contrary, the conditions that u and $\frac{\partial u}{\partial x}$ should assume given values at points of $x=0$, imply that these quantities exist and are continuous around and at these points. More exactly, the continuity of u and its different first partial derivatives will be required: this is necessary for the validity of Holmgren's proof, which (replacing the equation by a system of equations of the first order) introduces all these first derivatives as auxiliary unknowns, as classically explained in Sophie Kowalewsky's original proof of the fundamental theorem (see e.g. Goursat-Hedrick's *Differential Equations*, pp. 285—6), and applies (as we also shall do in the following Books) the usual integral transformation of Ostrogradsky or Green, for which continuity of the functions introduced is wanted.

† This analogy would have been legitimate if the second kind of methods mentioned in § 10 (successive approximations) could have been extended to the case of partial differential equations. This would contradict our above statements, and is consequently impossible, at least in the general case (attempts have been made in that direction by some authors, but, of course, have proved unsuccessful). On the contrary, methods of that kind have been applied to proper cases, account being taken of the nature of the equation and other features of the problem (especially characteristics), by Picard and his successors.

$\nabla^2 u = 0$ is formulated analytically in Cauchy's way*. All of them lead to statements such as Dirichlet's, i.e. with only one numerical datum at every point of the boundary. Such is also the case with the equation of heat. All this agrees with the fact that Cauchy's data, if not analytic, do not determine any solution of any one of these two equations.

This remarkable agreement between the two points of view appears to me as an evidence that the attitude which we adopted above—that is, making a rule not to assume analyticity of data—agrees better with the true and inner nature of things than Cauchy's and his successors' previous conception.

I have often maintained, against different geometers, the importance of this distinction. Some of them indeed argued that you may always consider any functions as analytic, as, in the contrary case, they could be approximated with any required precision by analytic ones. But, in my opinion, this objection would not apply, the question not being whether such an approximation would alter the data very little, but whether it would alter the solution very little. It is easy to see that, in the case we are dealing with, the two are not at all equivalent. Let us take the classic equation of two-dimensional potentials

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with the following data of Cauchy's†

$$(15) \qquad \begin{cases} u(0, y) = 0, \\ \dfrac{\partial u}{\partial x}(0, y) = u_1(y) = A_n \sin(ny), \end{cases}$$

n being a very large number, but A_n a function of n assumed to be very small as n grows very large $\left(\text{for instance } A_n = \dfrac{1}{n^p}\right)$. These data

* One might be tempted to assimilate the above results concerning $\nabla^2 u = 0$ to those which we previously found in the case when the variety which bears the data is a characteristic. This however would be unjustified, as in the latter case the problem, when not impossible, becomes indeterminate, which can never happen for $\nabla^2 u = 0$, on account of Holmgren's theorem. Cauchy's problem for $\nabla^2 u = 0$, in the general case, is to be compared with an algebraic problem implying *more conditions* than unknowns.

† I gave this example for the first time at a meeting of the Swiss Mathematical Society at Zurich (1917).

differ from zero as little as can be wished. Nevertheless, such a Cauchy problem has for its solution

$$u = \frac{A_n}{n} \sin(ny) \operatorname{Sh}(nx),$$

which, if $A_n = \frac{1}{n}, \frac{1}{n^p}, e^{-\sqrt{n}}$, is very large for any determinate value of x different from zero on account of the mode of growth of e^{nx} and consequently $\operatorname{Sh}(nx)$.

In this case, the presence of the factor $\sin ny$ produces a "fluting" of the surface, and we see that this fluting, however imperceptible in the immediate neighbourhood of the y-axis, becomes enormous at any given distance of it however small, provided the fluting be taken sufficiently thin by taking n sufficiently great.

19. Continuity with respect to given functions. Let us compare this case with the solution of Cauchy's problem for equation (e_1) (the equation of vibrating strings).

The general integral of the latter being

$$(16) \qquad u(x, t) = \phi(x + \omega t) + \psi(x - \omega t),$$

Cauchy's data

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x)$$

easily give us, as we know,

$$(16') \quad \phi(\xi) = \frac{1}{2}\left[u_0(\xi) + \frac{1}{\omega}\int u_1(\xi)\, d\xi\right],$$

$$\psi(\xi) = \frac{1}{2}\left[u_0(\xi) - \frac{1}{\omega}\int u_1(\xi)\, d\xi\right],$$

the constant of integration being here immaterial provided it be the same in both formulæ: which, substituted in (16), affords the solution of the problem. Now, let us assume that, along a certain interval of amplitude A, the functions u_0, u_1 be modified, but everywhere very slightly: that is, be replaced by $u_0 + \delta u_0$, $u_1 + \delta u_1$, the quantities $|\delta u_0|$, $|\delta u_1|$ being, for every value of x, less than a small constant ϵ. Then, by (16′), the corresponding alteration for ϕ, ψ will everywhere be less than $\frac{\epsilon}{2}\left(1 + \frac{A}{\omega}\right)$ and, for u, less than $\epsilon\left(1 + \frac{A}{\omega}\right)$, i.e. arbitrarily small with ϵ.

We shall say that the values of u *depend continuously* on those of u_0, u_1: a phrase suggested by an obvious analogy with ordinary continuity.

On the contrary, the example of the preceding section shows us that the solution of Cauchy's problem for the equation of potentials *does not* depend continuously on the data.

20. Various orders of neighbourhood and continuity. The above definition requires, however, to be made more precise by considerations now classic in the Calculus of Variations* and its recent generalization, the Functional Calculus.

The inequalities $|\delta u_0| < \epsilon$, $|\delta u_1| < \epsilon$ are, for certain problems, sufficient in order that $u_0 + \delta u_0$, $u_1 + \delta u_1$ be considered as very near neighbours to u_0, u_1; but, for other applications, such is not the case.

For instance, $y = g(x) = -\sin(nx)$, for large n represents a curve every point of which is very near to a corresponding point of the x-axis. Nevertheless, it cannot be approximately replaced by $y = 0$ if, e.g., length is concerned: its length, between $x = 0$ and $x = \pi$, does not† approach π for $n = \infty$.

This is due to the fact that, in this instance, $g(x)$ tends (uniformly with respect to x) towards zero with $\frac{1}{n}$, *but* $g'(x) = \cos nx$ *does not.* Such a function is said to have with zero a *neighbourhood of order zero.*

The function $\frac{g(x)}{n} = \frac{1}{n^2} \sin nx$ has, with zero, a *neighbourhood of order* 1 for very large n, i.e. $\left|\frac{g(x)}{n}\right|$ and $\left|\frac{g'(x)}{n}\right|$ are very small, but $\frac{g''(x)}{n} = -\sin nx$ would not be so.

Generally speaking, $g(x)$ and $h(x)$ are said to have, in the interval (a, b), a neighbourhood of order p if the $p+1$ differences $|g(x) - h(x)|$ $|g'(x) - h'(x)|$, ..., $\left|\frac{d^p g}{dx^p} - \frac{d^p h}{dx^p}\right|$ are very small, say smaller than the number ϵ, all over (a, b), the neighbourhood being the closer the smaller ϵ is.

* It was given by Zermelo (*Untersuchungen zur Variationsrechnung: Diss.*, Berlin, Mayer and Müller, 1894).

† It remains constant if n assumes integral values.

If the functions considered and their derivatives up to the order p are continuous—which will always be the case for the applications—it can also be said that two functions g, h have with each other a neighbourhood of order p if a correspondence between x and x'' can be established such that $|x''-x| < \epsilon$ and the same be true for each of the differences

$$(\delta) \quad |g(x')-h(x'')|, \ |g'(x')-h'(x'')|, \ \ldots, \ \left|\frac{d^p g(x')}{dx'^p} - \frac{d^p h(x'')}{dx''^p}\right|.$$

This condition is equivalent to the preceding one (ϵ being simply replaced by another quantity which becomes infinitesimal with the first one) on account of the fact that $|x''-x'| < \epsilon$ implies

$$|h(x')-h(x'')| < \eta, \quad \left|\frac{d^q h(x')}{dx'^q} - \frac{d^q h(x'')}{dx''^q}\right| < \eta$$

(for every q between 1 and p), η being infinitesimal at the same time as ϵ.

Geometrically speaking, two plane curves shall be said to have with each other a vicinity of order p if a punctual correspondence can be found between them such that the distance between corresponding points be very small as well as the differences (δ) (this implying—e.g., for $q-1$—that the angle of corresponding tangents must be very small). When this occurs, it follows from the above remarks that the choice of the correspondence is widely arbitrary and immaterial: especially, if there is no tangent parallel to $x = 0$, we can take as corresponding points those with the same abscissa and, at any rate, points such that the uniting segment cuts both curves at a finite angle.

All this is obviously similar to the classic theory of contact; and, indeed, *the latter is a sub-case of our present considerations*: it may be expressed by saying that two curves have a contact of order p at a common point A if their arcs around A have a vicinity of order p, arbitrarily close when the arcs are sufficiently small. The same, of course, applies to functions having a contact of order p for a determinate value of the variable.

The extension of all this to functions of several variables is obvious and we need not even formulate it. For instance, when two surfaces have a contact of order p at a point A, their portions around A have a neighbourhood of order p, which can be taken as close as required if the portions are taken sufficiently small.

If $G(x_1, x_2, \ldots, x_m) = 0$ is the equation of one surface, it being assumed that G has continuous derivatives up to the order p and that the first derivatives nowhere vanish simultaneously, the equation of a second surface which has with it a neighbourhood of order p will be $G + \delta G = 0$, δG being very small as well as its derivatives of the first p orders.

20 *a*. The conception of various orders of neighbourhood provides the definition of various *orders of continuity*. A quantity u being assumed to depend on the values of $g(x)$ in (a, b), this dependence will be *continuous of order p* if u is very slightly altered each time $g(x)$ is replaced by another function $h(x)$ having with it (in (a, b)) a (sufficiently close) neighbourhood of order p. Thus, the solution of Cauchy's problem for the equation of vibrating strings, as presented above, is continuous *of order zero* in u_0, u_1. The length of an arc of curve $(=\int\sqrt{1+y'^2}dx)$ is not continuous of order zero, but it is continuous of order 1.

It is to be noted that neighbourhood of order p means more than neighbourhood of order zero, and therefore continuity of order p means *less* than continuity of order zero.

The solution of Cauchy's problem for $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (§ 18) is *not continuous in* u_0, u_1 *of any order whatever.* For $A_n \sin ny$ has with zero, if $A_n = \frac{1}{n^p}$, a neighbourhood of order $p - 1$, i.e. arbitrarily great; and even, for $A_n = e^{-\sqrt{n}}$, the neighbourhood is of *infinite* order: that is, every derivative of that quantity approaches zero when $\frac{1}{n}$ does; notwithstanding which, the corresponding value of u does not approach zero.

The solution of the problem cannot be expressed by formulæ analogous to (16), (16′), as we have just seen that such expressions imply continuity of order zero.

We shall, in the following Chapters, meet with other formulæ more or less similar to (16′), except that their right-hand sides may contain (under $\int$ or not) derivatives of u_0, u_1 up to a certain order. No formula of this kind, either, can by any means represent the solution of Cauchy's problem for the equation of potentials, as this would imply continuity of order p.

21. Another paradoxical consequence furthermore appears if we consider things from the concrete point of view.

Strictly, mathematically speaking, we have seen (this is Holmgren's theorem) that one set of Cauchy's data u_0, u_1 corresponds (at most) to one solution u of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so that, if these quantities u_0, u_1 were "known," u would be determined without any possible ambiguity.

But, in any concrete application, "known," of course, signifies "known with a certain approximation," all kinds of errors being possible, provided their magnitude remains smaller than a certain quantity; and, on the other hand, we have seen that the mere replacing of the value zero for u_1 by the (however small) value (15) changes the solution not by very small but by very great quantities. Everything takes place, physically speaking, as if the knowledge of Cauchy's data would *not* determine the unknown function.

This shows how very differently things behave in this case and in those which correspond to physical questions. If a physical phenomenon were to be dependent on such an analytical problem as Cauchy's for $\nabla^2 u = 0$, it would appear to us as being governed by pure chance (which, since Poincare, has been known to consist precisely in such a discontinuity in determinism) and not obeying any law whatever.

After having been led by physical interpretation to the need of the above distinctions, we must now try to formulate them analytically. This is subordinate to the classification of linear partial differential equations of the second order into different types.

22. The three types of linear partial differential equations. These types are distinguished by the algebraic nature of the characteristic form $\mathbf{A}(\gamma_1, \gamma_2, \ldots, \gamma_m)$.

If this form contains m distinct squares, all of the same sign (in other words, if it is a *definite form*), the equation is said to belong to the **elliptic** type: the characteristics are imaginary.

If it contains less than m distinct squares (semi-definite form, if the squares are of the same sign, as is the case in all known applications), the equation belongs to the **parabolic** type.

If the characteristic form contains m distinct squares, not of the same sign (*indefinite* form), so that there are real characteristics, we have the **hyperbolic** type.

Moreover, there is a distinction to be made, in this last type, when the number m is greater than 3, as the signs could be variously distributed among the m squares. The only case which occurs in physical applications is the one in which all squares but one have the same sign: we call it the *normal hyperbolic type.* Every one of the above quoted equations (e_1), (e_2), (e_3) belongs to the normal hyperbolic type.

Geometrically speaking, as was remarked by Coulon*, the normal hyperbolic type is distinguished among all others by the following characters. Let the characteristic form (by a proper linear transformation) be resolved into squares, so that

$$\mathbf{A}(\gamma_1, \gamma_2, \ldots, \gamma_m) = A_m\gamma_m^2 - A_1\gamma_1^2 - A_2\gamma_2^2 - \ldots - A_{m-1}\gamma_{m-1}^2.$$

$\mathbf{A} = 0$ being the tangential equation of the characteristic cone, the corresponding punctual equation will be of a quite similar form

$$(17)\quad \mathbf{H}(X_1, X_2, \ldots, X_m) = \frac{X_m^2}{A_m} - \frac{X_1^2}{A_1} - \ldots - \frac{X_{m-1}^2}{A_{m-1}} = 0.$$

Such a cone consists of two sheets, and divides the m-dimensional space into three regions, the inside of the cone (i.e. $\mathbf{H} > 0$) consisting of two separate parts ($X_m > 0$ and $X_m < 0$) between which no real passage is possible otherwise than by the outside of the cone or through the vertex itself, as $X_m = 0$ is incompatible with $\mathbf{H} > 0$.

On the contrary, such a cone as

$$(17')\qquad \frac{X_1^2}{A_1} + \frac{X_2^2}{A_2} + \ldots - \frac{X_{m-1}^2}{A_{m-1}} - \frac{X_m^2}{A_m} = 0$$

(the left-hand side containing at least two positive and two negative squares) consists of one sheet and divides the m-dimensional space into two regions only. For $m = 4$, this can be interpreted in ordinary space by considering the X's as homogeneous coordinates, and $X_m = X_4 = 0$ as the plane of infinity: such an equation as (17) will then represent a hyperboloid of two sheets, and equation (17′) a hyperboloid of one sheet.

The normal hyperbolic type is the only one known in which Cauchy's problem can be correctly set. Moreover, non-normal hyperbolic types

* *Thesis*, Paris (1902), p. 30.

(which are connected with no physical application) do not lead to any problem of any kind which is known to comply with that condition*.

As to elliptic equations, they never lead to correctly set Cauchy problems. For (see Book III) the solutions of such an equation (if its coefficients are assumed to be analytic) possess, like those of $\nabla^2 u = 0$, the properties of analyticity mentioned and used in § 15, within their domain of existence; and, therefore, the arguments in that section can be applied.

The data (borne by an analytic surface S) not being analytic, if Cauchy's problem has a solution, this can only exist on one side of the initial surface S (beyond which the function thus defined cannot be extended).

23. But even for normal hyperbolic equations, physical applications do not always lead to Cauchy problems. The latter only occur when dealing with motions in completely *indefinite* media. Things would change if any limitation of these were considered, as would be the case in the classic problem of vibrating strings: this is analytically expressed by the integration of equation (e_1) with the conditions

$$(18) \qquad u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x)$$

and

$$(18') \qquad u(0, t) = 0, \quad u(l, t) = 0,$$

l being the length of the string, and $x = 0$ one of its extremities. The motion has to be calculated for $t \geqslant 0$, $0 \leqslant x \leqslant l$: i.e., graphically speaking, in the part R of the xt plane which is shaded on the diagram of fig. 3. Conditions (18) are to be fulfilled for $0 \leqslant x \leqslant l$; conditions (18′) for $t \geqslant 0$.

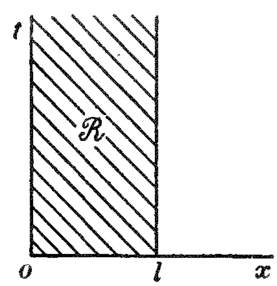

Fig. 3.

Now, it is apparent that the former are of Cauchy's type, but not

* Hamel (*Diss.*, Gottingen, 1901), who was led to the non-normal equation $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial z \partial t}$ by geometric considerations in the Calculus of Variations, has determined an unknown u by that equation and boundary conditions, but has to assume the latter to be analytic (not in all the variables, however); Coulon (*Thesis*), dealing with Cauchy's problem, also considers the case of non-normal equations; but it then appears from his very calculations that an infinity of conditions of possibility is necessary.

so the latter, so that we have here to deal not with Cauchy's problem, but with what I have called a *mixed* problem.

Indeed the two sets of conditions obviously play quite different parts, from the mechanical point of view, and are often and rightly called by different names. We hitherto, according to the geometric formulation of the question, have used the term "boundary conditions" indiscriminately; but if we now think of the mechanical meaning of the questions, we shall be led to give to conditions (18) the name "initial conditions," the name "boundary conditions" being kept only for conditions (18′) which correspond to the extremities of our string.

Now, *initial conditions* are always found to be expressed in Cauchy's form, but the opposite takes place for *boundary conditions* properly so called; these indeed rather resemble those we met with in the case of Dirichlet's problem, so that mixed problems appear every time boundaries are concerned.

24. Let us take another example by considering again a homogeneous conducting cable, but assuming it now to be indefinite only in one direction—say in the positive x direction. In the other direction, it will have to be considered as ending at a point where it will communicate, through a metallic contact, with a source maintained at every moment at a given (constant or variable) potential. Again we give the initial state of the cable (potential and intensities for $t=0$): then u, the potential, will have to satisfy the telegraphist's equation, together with the conditions (if we assume that the position of the contact is taken as the origin of the x's)

(19) $$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x),$$

(19′) $$u(0, t) = \mathbf{u}_0(t),$$

which again corresponds to what we call a mixed problem.

The two kinds of data are borne respectively by the x-axis (which, on our diagram (fig. 3), represents the cable for $t=0$) and the t-axis (representing the origin of coordinates successively considered at every positive value of time).

It is understood, of course, that (19) and (19′) must not be contradictory for $x=t=0$, so that

(20) $$u_0(0) = \mathbf{u}_0(0), \quad u_1(0) = \mathbf{u}_0'(0).$$

24*a*. In this second case, the diagram could be varied in an infinite number of ways. We can, indeed, imagine that our metallic contact, instead of being fixed at $x = 0$, be a sliding one (fig. 3*a*) and move along the cable according to a certain given law of motion $x = \xi(t)$, so that now two functions of $t \geqslant 0$ are given, one $\xi(t)$ expressing, at every positive instant, the position of the sliding contact on the cable, the other one $\mathbf{u}(t)$ giving at the same moment the value of the potential at this contact. The problem of determining the electric state for $t > 0$, when it is given for $t = 0$, will consist in finding a solution $u(x, t)$ of the telegraphist's equation satisfying the conditions

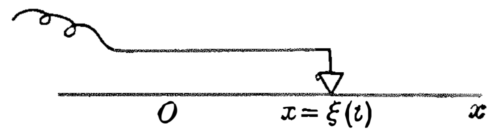

Fig. 3*a*.

$$\left.\begin{aligned} u(x, 0) &= u_0(x) \\ \frac{\partial u}{\partial t}(x, 0) &= u_1(x) \end{aligned}\right\} \quad (\text{for } x \geqslant \xi(t))$$

and $$u[\xi(t), t] = \mathbf{u}(t).$$

The line bearing the data would then be represented in the xt plane, as shown on fig. 3 *b*. Such a problem is a possible and determinate one (as is physically evident and is also seen by analytical means*). Consequently, it would not be allowable to give arbitrarily Cauchy's data for such a line.

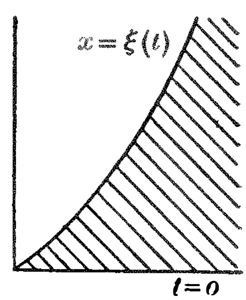

Fig. 3*b*.

25. We have an instance of analogous circumstances for problems with three independent variables, in the study of the transverse vibrations of a plane membrane, fastened at all points of its outline σ. The "indefinite" partial differential equation will be the equation (e_2) of cylindrical waves (§ 4*a*). The initial conditions will be

$$(21) \qquad u(x, y, 0) = u_0(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = u_1(x, y),$$

* See Picard, in Darboux's *Lecons sur la théorie des surfaces*, vol. IV (note I), and our Memoir in *Bull. Soc. Math. Fr.* vol. XXXI (1903).

u_0 (initial normal displacement) and u_1 (initial normal velocity) being given functions of x, y in the area S covered by the membrane. The boundary conditions are

$$u = 0 \tag{21'}$$

at every point of σ and for every value of t, which we shall however exclusively consider as positive, the motion having only to be determined after the initial instant $t = 0$. In our graphic representation (§ 5), u has to be calculated within a half cylinder having the S portion of the xy plane for its base, the lateral surface S_1 corresponding to the various positive values of t. We again have to deal with a mixed problem, (21′) being of Dirichlet's type.

The motion of any limited two-dimensional or three-dimensional medium will give occasion for similar remarks.

26. We must point out that the same still applies, whatever be the nature of the limitations. If, for instance, we take, with Duhem*, the case of a pulsating solid sphere immersed in air, the latter filling the whole space *outside* the sphere, the small motions of the gas will depend, not on Cauchy's problem, but on a mixed one, non-Cauchy-like data† corresponding to every point of the surface of the solid sphere.

27. In all the above examples, the geometric shape of the varieties which bear the data evidently presents a notable difference from the cases which depended on Cauchy's problem.

It is obvious that the simultaneous intervention of the two kinds of data is here connected with the angles or edges of our varieties bearing the data. But, for $m > 2$, we can say more. In the equation of cylindrical waves (or of vibrating membranes) the characteristic cone (referred to axes passing through its vertex) has the equation

$$x^2 + y^2 - \omega^2 t^2 = 0.$$

* *Hydrodynamique, Elasticité, Acoustique*, vol. I, chap. XII, pp. 235—237.

† They are not exactly of Dirichlet's kind, but of the so-called "Neumann's" or "hydrodynamical" kind, but they are similar to Dirichlet's data inasmuch as one quantity, and not two—the value of $\frac{du}{dn}$ alone—is given at every point of the spherical surface.

The plane which bears Cauchy's data is $t = \text{const.}$: *such a plane cuts only one sheet of the characteristic cone, and cuts it along an ellipsoid* (in this case, a sphere).

We shall say that a plane is *duly inclined* with respect to our equation (the latter being assumed to be a normal hyperbolic one) if it cuts only one sheet of the characteristic cone, the edge* of intersection being closed; or, which comes to the same, if a parallel plane drawn through the vertex of the cone has no other common point with it than the vertex itself.

The lateral cylindrical surface S_1, in our configuration for the problem of the vibrating membrane, is nowhere duly inclined: each of its tangent planes cuts a characteristic cone along a hyperbola. This is quite a general fact: as Volterra has remarked†, on a surface S which consists of several parts (whether separated from each other by edges or not), some of which are duly inclined and some are not, the correct data on the latter are Dirichlet-like ones.

We never meet with a correctly set Cauchy problem with respect to varieties (even presenting no edges) which are not duly inclined. For instance, we could not take arbitrary Cauchy's data for equation (e_2) or (e_3) with respect to $x = 0$‡. To see this, let us choose the data in question independent of t: then u itself also ought|| to be independent of t and therefore to satisfy $\nabla^2 u = 0$, which we have seen to be generally impossible if u and $\frac{\partial u}{\partial x}$, for $x = 0$, are arbitrarily chosen. What conditions ought to be imposed on u_0 and $u_1 = \frac{\partial u}{\partial x}$, in order that the problem should have a solution, is again a subject which might prove interesting from the point of view of the theory of functions: we shall speak of it briefly again in Book IV.

* See note, p. 5.

† *Intern. Congress*, Rome (1908), vol. II, p. 90.

‡ For the present, no system of data is known borne by $x=0$ and suited to determine correctly a solution of (e_2) or (e_3).

|| See below, § 29.

BOOK II

THE FUNDAMENTAL FORMULA AND THE ELEMENTARY SOLUTION

CHAPTER I

CLASSIC CASES AND RESULTS

28. Let us now come to our proper subject, that is, the solution of Cauchy's problem, which we shall, as a rule, assume to be correctly set.

This solution is very simple and was found long ago for some particular cases, the simplest of which is the case of equation (e_1) with data relative to $t = 0$ (small motions of an indefinite aerial pipe or vibrations of a string considered as of indefinite length in both directions) referred to in § 19.

Well known is also Poisson's solution of Cauchy's problem relating to the equation of sound*. This can be expressed in the following synthetic way:

Let (x_0, y_0, z_0, t_0) be any given universe-point, at which we want to calculate the value of our function u defined by conditions (C_3) (§ 4). Let us denote, in a general way, by $M_r(\phi)$ the average value of any function ϕ (x, y, z) on the surface of the sphere of radius r described in ordinary space with (x_0, y_0, z_0) for its centre. We shall have

$$(1) \qquad u(x_0, y_0, z_0, t_0) = \frac{d}{dt_0}[t_0 M_{\omega t_0}(u_0)] + t_0 M_{\omega t_0}(u_1).$$

28*a*. The proof of that formula consists in verifying directly that the right-hand side fulfils every required condition; on account of Holmgren's theorem, it is the only one which will fulfil them all.

Such a verification is given in the just quoted works and other classic books. We shall simply refer to those for what concerns the initial condition (C_3). As to the verification of the partial differential equation itself, it is generally done by means of a transformation of surface integrals into triple ones, introducing the values of u_0 or u_1 and

* Poisson, *Mémoire sur l'intégration de quelques équations aux différences partielles et particulièrement de l'équation générale du mouvement des fluides élastiques* (read at the Ac. Sc. in Paris, the 19th of July, 1919). See also Rayleigh's *Theory of Sound*, vol. II, p. 88; Poincaré's *Leçons sur la Théorie Mathématique de la Lumière*, chap. III (pp. 76—98); etc.

their derivatives not only on the surface of the aforesaid sphere but in its whole volume. Eventually, it will be convenient for us to avoid the consideration of that volume, which can be done, for instance, in the following way.

We have to show that, u being any differentiable function of x, y, z, equation (e_3) is satisfied by

$$u(x_0, y_0, z_0, t_0) = t_0 M_{\omega t_0}(u_1)$$
$$= \frac{t_0}{4\pi}\iint u_1(x_0+\omega t_0 \sin\theta\cos\phi,\, y_0+\omega t_0 \sin\theta\sin\phi,\, z_0+\omega t_0\cos\theta)\sin\theta\, d\theta\, d\phi$$
$$= t_0 I$$

(wherefrom the same conclusion will follow at once for the derivative of the right-hand side with respect to t_0, therefore also for the first term of (1)).

In the first place, derivatives with respect to x_0, y_0, z_0 are immediately obtained by differentiation under $\iint$ in I, so that

$$\omega^2\nabla^2 u = \frac{\omega^2 t_0}{4\pi}\iint \nabla^2 u_1 \sin\theta\, d\theta\, d\phi.$$

The second derivative with respect to t_0, viz. $t_0\dfrac{\partial^2 I}{\partial t_0^2} + 2\dfrac{\partial I}{\partial t_0}$, will also be determined by differentiation under $\iint$, viz.

$$\frac{\partial^2 u}{\partial t_0^2} = \frac{1}{4\pi}\iint\left(\omega^2 t_0\frac{d^2u_1}{dn^2} + 2\omega\frac{du_1}{dn}\right)\sin\theta\, d\theta\, d\psi,$$

$\dfrac{du_1}{dn}$ and $\dfrac{d^2u_1}{dn^2}$ being (exterior) normal first and second derivatives. But we have the identity*

$$\frac{d^2u_1}{dn^2} + \frac{2}{\omega t_0}\frac{du_1}{dn} = \nabla^2 u_1 - \Delta_2 u_1,$$

Δ_2 being "Beltrami's differential parameter" on the sphere

$$\frac{1}{\omega^2 t_0^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial u_1}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u_1}{\partial\phi^2}\right].$$

Also, on account of the classical integral identity† for Δ_2 and the assumed regularity of u_1, the integral of the last term in the right-hand side over the surface of our sphere is zero: which reduces the value of $\dfrac{\partial^2 u}{\partial t_0^2}$ to the above value of $\omega^2\nabla^2 u_0$. Q. E. D.

* See our *Lecons sur la propagation des ondes*, chap. I, § 34, p. 50.

† See Darboux's *Lecons sur la theorie des surfaces*, vol. III, § 674, form. (18), and our *Leçons sur la propagation des ondes*, § 35.

29. The method of descent. We shall see that many curious difficulties already arise when, instead of the equation of spherical waves, we attack equation (e_2) (equation of cylindrical waves). But nevertheless, for the present, we can immediately deduce the solution of the problem corresponding to (e_2) from the similar solution for (e_3).

Indeed we have previously seen that the former is only a special case of the latter. In order to solve it, we only need, in formula (1), to suppose that the functions u_0 and u_1 are independent of z.

We thus have a first example of what I shall call a "method of descent." Creating a phrase for an idea which is merely childish and has been used since the very first steps of the theory* is, I must confess, rather ambitious; but we shall come across it rather frequently, so that it will be convenient to have a word to denote it. It consists in noticing that he who can do more can do less: if we can integrate equations with m variables, we can do the same for equations with $(m-1)$ variables. Here, in order to solve equation (e_2) we have only to note that every solution of it is a solution of (e_3) independent of z, and conversely.

Thus, our Cauchy problem for the equation of cylindrical waves

$$\text{(A)} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - \dfrac{1}{\omega^2}\dfrac{\partial^2 u}{\partial t^2} = 0; \\ \left.\begin{aligned} u &= u_0(x, y) \\ \dfrac{\partial u}{\partial t} &= u_1(x, y) \end{aligned}\right\} \text{ for } t = 0 \end{cases}$$

is equivalent to the same problem for the equation of spherical waves

$$\text{(B)} \qquad \left|\begin{aligned} &\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} - \dfrac{1}{\omega^2}\dfrac{\partial^2 u}{\partial t^2} = 0; \\ &\left.\begin{aligned} u &= u_0(x, y) \\ \dfrac{\partial u}{\partial t} &= u_1(x, y) \end{aligned}\right\}, \end{aligned}\right.$$

* Parseval, in Lacroix's *Traité des différences et des séries*, 1st edition, p. 515; Poisson's above quoted Memoir, art. 8. See Duhem's *Hydrodynamique, Elasticite, Acoustique*, vol. II, chap. VIII.

which is the one which we have already considered (equations (e_3), (C_3)), except that the right-hand sides of the initial conditions do not contain z. The perfect equivalence of the two problems can be analytically proved with complete rigour. That any solution of (A) satisfies (B) is evident by itself. Conversely, a solution of (B) must be independent of z: for if it were not—say $u = \phi(x, y, z, t)$—then $u = \phi(x, y, z + h, t)$, meaning by h any arbitrary constant, would be a second solution of the same problem—which would contradict Holmgren's theorem. Our solution of (B) must therefore be a solution of (A).

30. This being understood, we only have to suppose in formula (1) that the functions u_0 and u_1 depend on x and y exclusively. In order to see what becomes of them in this case, we have simply to remember that an average value on a sphere—or what comes to the same, here, on a hemisphere, the limiting plane of which is parallel to x, y—is expressed by a double integral

$$M_r(\phi) = \frac{1}{2\pi r^2}\iint \phi \, d\Sigma,$$

the integral being extended over the surface of the hemisphere. Taking x and y for independent variables, we see that, ϕ being independent of z, the symbol M is to be expressed by

$$(2) \quad M_r[\phi(x, y)] = \frac{1}{2\pi r}\mu_r(\phi), \quad \mu_r(\phi) = \iint \frac{\phi \, dx\, dy}{\sqrt{r^2 - (x - x_0)^2 - (y - y_0)^2}}$$

(the integral being extended over the circle $(x - x_0)^2 + (y - y_0)^2 \leqslant r^2$) and our formula (1) becomes

$$(1') \qquad u(x_0, y_0, t_0) = \frac{1}{2\pi\omega}\left[\frac{d}{dt}\mu_{\omega t}(u_0) + \mu_{\omega t}(u_1)\right].$$

31. The intervention of waves. Let us note that these formulæ (1) and (1′) agree with what we know about the propagation of sound or light waves*. In any one of them, we recognize that, in order to calculate the value of u in (x_0, y_0, z_0) at the instant t_0, it is

* A priori reasons for that, resulting from Hugoniot's conceptions, will be seen in Duhem's *Hydrodynamique, Elasticite, Acoustique*, vol. I, or our *Lecons sur la propagation des ondes*, especially chap. IV, § 165 and chap. VII, § 290.

not necessary to know the values of our functions u_0 and u_1 (Cauchy's data for $t=0$) in the whole of space, but only in the inside and on the surface of a sphere having $O(x_0, y_0, z_0)$ for its centre and ωt_0 for its radius. The disturbances produced at the origin of t at points distant from O by more than ωt_0 are unable to act on O before that instant t_0.

In order to see the identity of this notion of waves with that of characteristics, we use our graphic representation. For convenience's sake, let us consider the equation (e_2), so that we may have directly a complete diagram in three dimensions. Construct our $t=0$ plane and our universe-point (x_0, y_0, t_0). Any disturbance produced at a point (x, y) and a certain instant t could act on that universe-point only if

$$(x-x_0)^2+(y-y_0)^2 \leqslant \omega^2 (t-t_0)^2.$$

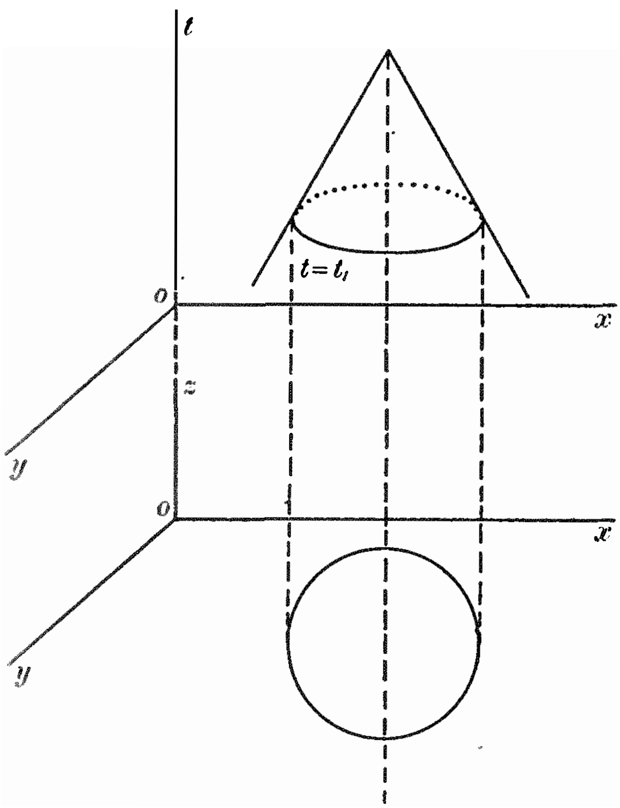

Fig. 4.

If we take the equality sign, this represents the surface of a right circular cone with (x_0, y_0, t_0) for its apex and a parallel to the t-axis for its axis; or, more exactly (since t must at present be essentially assumed to be less than t_0), the lower sheet of this cone. As to the inequality, it means that the point (x, y, t) has to be inside the conic sheet thus represented. The circle over which integral (1′) is to be extended is the trace of such a cone on the xy initial plane.

The surface of such a cone satisfies the condition

$$(3) \qquad \left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2 - \frac{1}{\omega^2}\left(\frac{\partial G}{\partial t}\right)^2 = 0,$$

which defines the characteristics of our equation of cylindrical waves, according to the rule given in § 13.

Nothing essential need be changed as yet (except the introduction of four-dimensional space) if we are to deal with the equation of spherical waves. Our cone would have to be replaced by the "hypercone"

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = \text{ or } \leqslant \omega^2 (t-t_0)^2,$$

the trace of which on the hyperplane $t=t_1<t_0$ is the sphere with centre (x_0, y_0, z_0) and radius $\omega(t_0-t_1)$. That hypercone* satisfies the partial differential equation of the first order

$$\text{(4)} \qquad \left(\frac{\partial G}{\partial x}\right)^2+\left(\frac{\partial G}{\partial y}\right)^2+\left(\frac{\partial G}{\partial z}\right)^2-\frac{1}{\omega^2}\left(\frac{\partial G}{\partial t}\right)^2=0,$$

defining the characteristics of equation (e_3).

More generally, if we express analytically the known physical rule that the normal velocity of propagation of the waves is equal to ω, we find ($G(x, y, z, t)=0$ being the wave front) condition (4).

This connection between the solutions of our problem and the waves is a general one, as will appear more evidently in the following Books.

32. Retrograde waves. We note that here, as we most frequently shall have to do in what follows, we consider waves in a manner slightly different from the usual one, viz. in the retrograde way, ascending the course of time. Instead of starting from a universe-point (x, y', z', t') or (x', y', t') and considering what successive points are reached at instants after t' by waves issuing at t' from $O'(x', y', z')$ or (x, y), we give ourselves the later universe-point (x_0, y_0, z_0, t_0) or (x_0, y_0, t_0) and inquire how the earlier one must be chosen in order that they be "just within wave," that is, that the wave issuing from this earlier universe-point reaches precisely $O(x_0, y_0, z_0)$ or (x_0, y_0) at the instant t_0: the locus of such earlier universe-points being an anti-wave quite analogous to an ordinary wave but for the fact that its propagation takes place with the decreasing values of t, i.e. by reversing the course of time.

The necessary and sufficient condition for the (ordinary) wave from (x', y', z', t') just to pass through (x_0, y_0, z_0, t_0) is that the anti-wave from the latter should just pass through the former: a fact which we shall recognize analytically to be quite a general one.

Such circumstances, in evident analogy with the principle of

* In our "descriptive geometry" configuration, it would have to be represented as in the accompanying diagram (an ordinary cone in the (x, y, t) space and, in (x, y, z), a sphere which is the base of the hypercone).

"inverse return of luminous rays," will be met with throughout the course of these lectures. We can even go a little further by considering the case in which our two universe-points (x', y', t') and (x_0, y_0, t_0) in the three-dimensional universe, for instance, are "well within wave" with respect to each other; that is, in which the wave from O' starting at the instant t' reaches O *before* the instant t_0. This means, geometrically, that (x_0, y_0, t_0) lies inside the "direct sheet" of the characteristic cone with vertex (x , y', t'), that is, inside the sheet turned towards positive t. It is interesting to note that, as is now obvious, a necessary and sufficient condition for this is that, conversely, (x', y', t') lies inside the "retrograde" or "inverse" sheet of the characteristic cone having (x_0, y_0, t_0) for its vertex. This fact will again appear a most general one: we shall recognize that the aforesaid fact is expressed by an inequality the left-hand side of which is symmetrical with respect to both points contained in it.

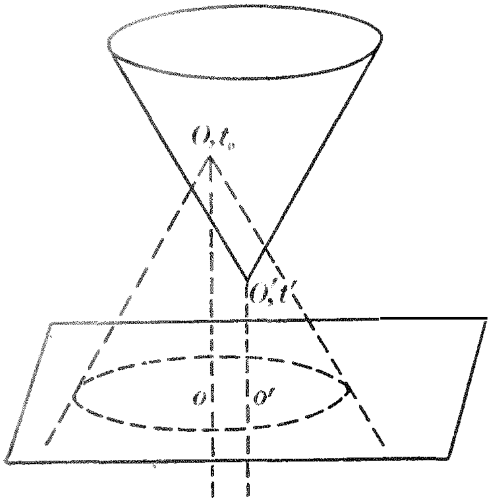

Fig. 5.

33. The question of Huygens' Principle. But, however simple be the preceding formulæ and statements, they have, nevertheless, opened somewhat important and lengthy scientific discussions, of which we have now to speak and which refer to what is called *Huygens Principle.*

As a matter of fact, it happened, as is often the case, that the question under discussion was badly set. Huygens' principle can be taken in several different senses, and these were not sufficiently distinguished.

It is known that, in his famous fundamental Memoir on light, the great Dutch scholar had to study the action of a luminous disturbance, produced initially $(t = 0)$ at a given point O, on another point a. Instead of following strictly his presentation, we shall, for our discussion's sake, put it in the form of a syllogism.

(A) (major premise). "The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the mediation of every intermediate instant $t = t'$, i.e. (assuming

$0 < t' < t_0$), in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and, from the latter, the required state at $t = t_0$."

(B) (minor premise). "If, at the instant $t = 0$—or more exactly throughout a short interval $-\epsilon \leqslant t \leqslant 0$—we produce a luminous disturbance localized in the immediate neighbourhood of O, the effect of it will be, for $t = t'$, localized in the immediate neighbourhood of the surface of the sphere with centre O and radius $\omega t'$: that is, will be localized in a very thin spherical shell with centre O including the aforesaid sphere."

(C) (conclusion). "In order to calculate the effect of our initial luminous phenomenon produced at O at $t = 0$, we may replace it by a proper system of disturbances taking place at $t = t'$ and distributed over the surface of the sphere with centre O and radius $\omega t'$."

Now it happens that, by "Huygens' Principle," different authors have meant indiscriminately any one of the above three propositions: whereas we shall, in what follows, see that our opinion concerning each of them must be quite a different one.

Proposition (A) is what philosophers (if I do not misuse their language) call one of the "laws of thought": that is, an unavoidable law of our reason, which we could by no means conceive as not existing and without which we could not think. If to-day we discover Assyrian inscriptions, we cannot dream of supposing that, at any instant between the time when they were made and the time of their discovery, those inscriptions could have ceased to exist and all trace of them have disappeared. (A) must therefore be considered as a truism, which does not mean that it cannot interest us; for the geometer does not dislike truisms. The above proposition, in particular, corresponds to the fact that the integration of partial differential equations defines certain groups of functional operations; and this, for instance, leads to quite remarkable identities concerning hypergeometric and Bessel's functions.

Proposition (C), of which we shall first speak, though not so immediately evident, will prove to be a general property of the equations we now come to.

But such is not at all the case for proposition (B). We shall perceive further on that it is quite a special property of certain special equations: indeed we do not know as yet whether our equation of spherical waves and others practically equivalent to it are not the only ones to possess it.

We shall eventually speak of (A) and (B) as "Huygens' major premise" and "Huygens' minor premise," thus distinguishing them immediately from proposition (C).

34. The aforesaid proposition (C) has been the object and result of the fundamental works of two authors: Kirchhoff dealt with it for spherical waves in his classic Memoir *Zur Theorie des Lichtstrahlen**, and in his *Lectures* on Optics; then Volterra proved it for cylindrical ones, especially in the *Acta Mathematica*, vol. XVIII, and has more recently returned to the subject in his Stockholm Lectures†.

The manner in which these two authors set the question is the following, which, for the convenience of graphic representation, we shall explain for the case of equation (e_2). Let us suppose that, initially, our xy plane is completely at rest and that, later, some impulses are communicated to it *inside a certain closed curve* σ. These will afterwards be propagated to the outside of σ and, before that, will influence the points of σ itself. We note the values of u and one of its derivatives,—for instance the normal derivative $\frac{}{nd}$—thus produced at the various points of σ at all successive instants, remembering that, in our mode of representation, those successive states of σ will be represented by the successive cross-sections of a right cylinder S having σ for its base, so that we shall consider this cylinder as bearing the aforesaid values of u and $\frac{du}{d}$. Conditions remain entirely similar for the spherical waves, except for the introduction of four-dimensional space, the curve σ being replaced by a surface (and, therefore, the cylinder by a hypercylinder).

Now Kirchhoff, for the latter problem, and Volterra, for the former, obtain the expression of u at any positive instant, and at any point outside σ, in terms of the aforesaid values of u and $\frac{}{dn}$ along our cylinder

* *Sitzungsber. der K. Ak. der Wiss.* (1882), pp. 641 ff.; see also Beltrami, *Rendic. Istituto Lombardo*, 2nd series, vol. XVII; Duhem's *Hydrodynamique, Elasticité, Acoustique*, vol. I, pp. 145—161, etc.

† *Lecons sur l'intégration des équations différentielles aux dérivées partielles*, Upsal, 1906, and Hermann, 1912, Paris.

or hypercylinder (in common language, along the curve or surface σ considered at successive instants): and these expressions are given by definite integrals taken over S, which can be physically interpreted, in any one of the aforesaid cases, by saying that the motion of the medium outside σ may be considered as resulting from properly chosen impulses issuing at various instants from different points of σ, i.e. proposition (C)*.

Analytically speaking, this problem of Kirchhoff and Volterra is no other than Cauchy's problem for the portion of universe (that is of $xyzt$, or xyt-space) lying outside S and above $t=0$ (that is, the points of which must satisfy $t \geqslant 0$), the variety which bears the data being constituted by the upper portion of S (that is, the portion of S which corresponds to $t \geqslant 0$), and a portion of plane or hyperplane $t=0$ (the portion outside σ) on which latter the data are zero.

In itself, this Cauchy problem does not belong to the class which properly interests us, for it is *not* what we call "correctly set": its possibility, as appears from the works of the aforesaid authors themselves, is subject to an infinite number of necessary conditions. Indeed, for a boundary with such a shape, the correctly set problem would be what we called a "mixed" one, consisting in giving u and $\frac{\partial u}{\partial t}$ $\left(\text{for instance } u = \frac{\partial u}{\partial t} = 0\right)$ for $t=0$ and u *alone* $\left(\text{or } \frac{du}{dn} \text{ alone}\right)$ on the aforesaid part of the variety S.

But besides their physical interest in proving form (C) of Huygens' principle, and though, strictly speaking, Volterra's formulæ could be deduced from Kirchhoff's by "descent," the methods of Kirchhoff and Volterra are directly applicable to the general case of Cauchy's problem for the corresponding two equations, and by themselves give the complete solution of it for any form of the variety bearing the data. Moreover the solution is obtained by a regular analytic method (instead of the synthetic way which we pointed out as leading to Poisson's formula), so that we may attempt to generalize such methods for other types of equations.

* From the point of view of this physical interpretation, Kirchhoff's integrals needed some transformation, which was made by B. Brunhes, *Travaux et Memoires des Facultés de Lille*, vol. IV, 16th Memoir (1895).

35. Riemann's method. Kirchhoff's and Volterra's results were not, as a matter of fact, the first of that kind thus obtained. Long before the publication of Kirchhoff's Memoir and the time of Volterra's researches, a first general solution of Cauchy's problem for an extensive class of hyperbolic equations had been given: it is Riemann's celebrated method, contained in that great geometer's paper *On the Propagation of Aerial Waves of finite Amplitude**. Though first given by Riemann for quite special equations, the method covers, in reality, any hyperbolic linear partial differential equation in two variables.

But Riemann's work remained for a while unnoticed; it was only after the publication of Kirchhoff's paper that attention was called to it by du Bois Reymond†, and it was finally brought to the knowledge of every mathematician, in its most general form, by the classic *Leçons sur la theorie des surfaces*‡ of Darboux.

Since that time, Riemann's method may be considered as universally known, and we should not need to expound it; but its principles are so closely connected with our subject that we shall necessarily come across the main steps of it in the following Chapters.

* *Gött. Abhandl.* vol. VIII (1860). 8th Memoir of the 2nd German edition by Weber and Dedekind.

† Leipzig, 1864, and Tubingen, 1883. See Darboux's *Leçons*, vol. II, no 358. (Cf. next footnote.)

‡ Vol. II, book IV, nos. 357—359, pp. 71—81 of the 2nd edition. See also Dini, *Rendic. Accad. Lincei*, vol. V (1896) and vol. VI (1897).

CHAPTER II

THE FUNDAMENTAL FORMULA

36. To generalize Kirchhoff's, Riemann's and especially Volterra's method to any (normal) hyperbolic linear equation with any number m of independent variables, is the object of the present lectures.

Let us see how the three quoted authors proceed.

They may all be considered as starting from the same formula. Indeed we can say that there is only one formula (which we shall call the "fundamental formula") in the whole theory of linear partial differential equations, no matter to which type they belong. We shall begin by writing it down.

This formula is well known in the potential theory: it is the classic formula

$$\iiint (v\nabla^2 u - u\nabla^2 v)\, dx\,dy\,dz = -\iint \left(v\frac{du}{dn} - u\frac{dv}{dn}\right) dS.$$

It is well known that this has its origin in the identity

$$v\nabla^2 u - u\nabla^2 v = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z};$$

$$P = v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}, \quad Q = v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}, \quad R = v\frac{\partial u}{\partial z} - u\frac{\partial v}{\partial z}.$$

The starting-point of Riemann's method, for instance, is a quite similar one, viz. the identity

$$v\mathscr{F}(u) - u\,\mathscr{G}(v) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

which gives, by integration,

$$(\mathrm{F}_1) \qquad \iint [v\mathscr{F}(u) - u\,\mathscr{G}(v)]\, dx\,dy = \int (P\,dy - Q\,dx),$$

the simple integral in the right-hand side being taken in the direct sense on the boundary of the area of integration on the left-hand side, and the functions u and v being arbitrary but for the condition of being regular.

In these two formulæ, $\mathcal{F}(u)$ means, symbolically, any given linear polynomial in u and its derivatives with respect to x, y, and xy,

$$\mathcal{F}(u) = \frac{\partial^2 u}{\partial x \partial y} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu;$$

P and Q are, for instance*,

$$P = \frac{1}{2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + Auv, \quad Q = \frac{1}{2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + Buv,$$

while $\mathcal{G}(v)$ denotes the following determinate polynomial*, the "adjoint polynomial" of $\mathcal{F}$,

$$\mathcal{G}(v) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(Av) - \frac{\partial}{\partial y}(Bv) + Cv.$$

The relation between two adjoint polynomials is a reciprocal one: that is, if we should operate on $\mathcal{G}(v)$ as we just did on $\mathcal{F}(u)$, in order to obtain its adjoint polynomial, we should find this equal to $\mathcal{F}(u)$. This, of course, is verified immediately, but is also a consequence of the fact that the adjoint may be considered as defined by identity (F_1)†, which does not change when we exchange $\mathcal{F}$ with $\mathcal{G}$ if we at the same time change the signs of P and Q.

37. Now, we can write such an identity for any linear differential polynomial with any number m of independent variables,

$$\mathcal{F}(u) = \sum_{i,k} A_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu.$$

If we multiply this by v, easy integrations by parts will show us that we can again (with the help of $A_{ik} = A_{ki}$) write the identity

$$(5) \qquad v\mathcal{F}(u) - u\mathcal{G}(v) = \frac{\partial \mathcal{P}_1}{\partial x_1} + \frac{\partial \mathcal{P}_2}{\partial x_2} + \dots + \frac{\partial \mathcal{P}_m}{\partial x_m},$$

where we may choose, for $\mathcal{P}_2, \dots, \mathcal{P}_m$,

$$(5') \qquad \mathcal{P}_i = \sum_k vA_{ik}\frac{\partial u}{\partial x_k} - \sum_i uA_{ik}\frac{\partial v}{\partial x_k} - uv\left(\sum_k \frac{\partial A_{ik}}{\partial x_k} - B_i\right),$$

* See following footnote.

† It is easy to see that changes could be made in P and Q without altering the right-hand side of (F_1) but, $\mathcal{F}$ being given, there is only one polynomial $\mathcal{G}$ which can give our identity (F_1), as results from the fundamental Lemma of the Calculus of Variations: see Darboux, *Lecons sur la theorie des surfaces*, vol. II, book IV, chap. V, p. 114 of the 2nd edition; Goursat, *Cours d'Analyse*, vol. II, § 404, 2nd edition.

and $\mathcal{G}(v)$ stands for the following polynomial†

$$\mathcal{G}(v) = \sum_{i,k} \frac{\partial^2}{\partial x_i \partial x_k}(A_{ik}v) - \sum_i \frac{\partial}{\partial x_i}(B_i v) + Cv,$$

which is called the "adjoint" of $\mathcal{F}$.

The relation between $\mathcal{F}$ and $\mathcal{G}$ is a reciprocal one, as it was in the former case (and for the same reasons).

38. From (5), we must go on to an integral formula similar to (F_1). This requires some geometrical notations and definitions concerning m-dimensional space. If, in such a space, we have any hypersurface S defined by giving $x_1, x_2, \ldots, x_m$ as functions of $m-1$ parameters (curvilinear coordinates) $\lambda_1, \ldots, \lambda_{m-1}$, the cosines of the normal to S at any point will be, by definition, proportional to the quantities

$$D_1 = \pm \frac{D(x_2, \ldots\ldots x_m)}{D(\lambda_1, \ldots\ldots \lambda_{m-1})},$$

$$D_2 = \pm \frac{D(x_1, x_3, \ldots x_m)}{D(\lambda_1, \ldots\ldots \lambda_{m-1})},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_i = \pm \frac{D(x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_m)}{D(\lambda_1, \ldots\ldots\ldots\ldots\ldots\ldots \lambda_{m-1})},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

in which the right-hand sides are functional determinants and their signs are chosen in such a way that they are proportional to the corresponding minors of the determinant

$$\begin{vmatrix} \frac{\partial x_1}{\partial \lambda_1} & \frac{\partial x_2}{\partial \lambda_1} & \cdots & \frac{\partial x_m}{\partial \lambda_1} \\ \frac{\partial x_1}{\partial \lambda_2} & \frac{\partial x_2}{\partial \lambda_2} & \cdots & \frac{\partial x_m}{\partial \lambda_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial \lambda_{m-1}} & \frac{\partial x_2}{\partial \lambda_{m-1}} & \cdots & \frac{\partial x_m}{\partial \lambda_{m-1}} \\ * & * & \cdots & * \end{vmatrix}$$

with respect to the elements of the last row (which convention still leaves one sign arbitrary). In order to have the exact analogue of the

† See preceding footnote (p. 59).

direction cosines in ordinary space, we should have to choose these "cosines" such that the sum of their squares be 1, i.e.

$$\pi_1 = \cos(n, x_1) = \frac{\pm D_1}{\sqrt{D_1^2 + D_2^2 + \ldots + D_m^2}},$$

$$\pi_2 = \cos(n, x_2) = \frac{\pm D_2}{\sqrt{D_1^2 + D_2^2 + \ldots + D_m^2}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\pi_m = \cos(n, x_m) = \frac{\pm D_m}{\sqrt{D_1^2 + D_2^2 + \ldots + D_m^2}}.$$

The "element of surface" dS around the same point of S will be by definition

$$dS = \sqrt{D_1^2 + D_2^2 + \ldots + D_m^2}\, d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1},$$

so that

$$(6) \quad \pi_1 dS = D_1 d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}, \quad \pi_2 dS = D_2 d\lambda_1 \ldots d\lambda_{m-1}, \quad \ldots, \quad \pi_m dS = D_m d\lambda_1 \ldots d\lambda_{m-1};$$

in which formulæ one sign ought to remain arbitrary, this corresponding to the two possible directions on the normal. But nevertheless we have cancelled the + as we shall be able to produce any wanted inversion of sign by inverting the order of the curvilinear coordinates.

39. We have defined our cosines and our element of surface in order to keep the analogy with the corresponding notions of ordinary geometry. The fact is that we shall have to reckon with the left-hand sides of formulæ (6) as a whole, so that multiplying all the cosines by any common factor will be immaterial if, at the same time, we divide dS by the same factor. Especially, another form of the quantities (6) will frequently be of use to us: it corresponds to the case in which S is given by its equation $G(x_1, x_2, \ldots, x_m) = 0$. Then the cosines of the normal are proportional to the quantities $\frac{\partial G}{\partial x_i}$, the proportionality factor being the normal derivative $\frac{dG}{dn}$. We shall denote by dS_G the quantity

$$dS_G = dS : \left(\frac{dG}{dn}\right),$$

and our quantities (6) will be equal (except for sign) to

This superficial element dS_G is such that $dS_G \,.\, dG$ represents the (cylindrical) element of volume between the element dS of the surface $G=0$ and a corresponding element of the neighbouring surface $G=dG$ (where dG is an infinitesimal constant): so that we can speak of that element dS_G as the "quotient" $\frac{dx_1 dx_2 \dots dx_m}{dG}$ of the space element over dG.

40. By means of these definitions and phrases, the well-known identity ("Green's formula") between multiple integrals will be written, for m-dimensional space,

$$\text{(g)} \quad \mathbf{SSS}\left(\frac{\partial \mathscr{P}_1}{\partial x_1} + \dots + \frac{\partial \mathscr{P}_m}{\partial x_m}\right) dT$$
$$= - \mathbf{SS}\,(\pi_1 \mathscr{P}_1 + \pi_2 \mathscr{P}_2 + \dots + \pi_m \mathscr{P}_m)\, dS$$

(dT standing for the space element $dx_1 dx_2 \dots dx_m$).

Sacrificing accuracy to clearness, in this formula and future ones, I represent by the triple summation **SSS** what I ought to denote by m integral signs, viz. an m-fold integral extended over a certain portion of our space; by a double **SS**, an $(m-1)$-fold integral extended over a hypersurface in that space; by **S**, if necessary, an $(m-2)$-fold integral relative to an edge: in other words, the notation will be the same as if we had $m=3$. An integral relative to an edge will be distinguished from an integral along a line by the fact that the latter will be written with an ordinary $\int$ sign.

In formula (g), the **SSS** is extended over a certain (limited) portion T of m-dimensional space, the integral on the right-hand side over the limiting surface S of T. n denotes the *inner* sense of the normal to S (the signs in formulæ (6) being chosen accordingly). As has been seen above, we can replace $\cos(n, x_1), \cos(n, x_2), \dots, \cos(n, x_m)$ by $\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \dots, \frac{\partial G}{\partial x_m}$ respectively, if we replace dS by dS_G (with the hypothesis that G is increasing towards the inside of T).

Let us apply this to the identity (5): with the expressions (5′) of the $\mathscr{P}$'s, the factor of dS under **SS** on the right-hand side of (g) will become

$$v \sum_{i,k} A_{ik} \pi_i \frac{\partial u}{\partial x_k} - u \sum_{i,k} A_{ik} \pi_i \frac{\partial v}{\partial x_k} + Luv$$
$$= v \sum \frac{1}{2} \frac{\partial u}{\partial x_k} \frac{\partial \mathbf{A}}{\partial \pi_k} \quad u \sum \frac{1}{2} \frac{\partial v}{\partial x_k} \frac{\partial \mathbf{A}}{\partial \pi_k} + Luv,$$

A being the *characteristic form* defined above, and the π's denoting indifferently the cosines of the (inner) normal to S or the proportional quantities $\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_m}$, if, in the second case, dS be replaced by dS_G.

Here we shall introduce, as was done by d'Adhémar*, a particular direction depending on the tangent plane of S at any considered point M: we shall set down

$$\frac{dx_1}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \pi_1}} = \frac{dx_2}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \pi_2}} = \ldots = \frac{dx_m}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \pi_m}} = d\nu,$$

and the denominators (which cannot be zero simultaneously, as the discriminant of **A** is supposed to be different from zero) will be proportional to the direction cosines of a certain direction which we shall call the *transversal** to S in M. This definition was given a very simple geometrical interpretation by Coulon†, connecting it with the characteristic cone (§ 13) in M, i.e. the cone which has M for its vertex and $\mathbf{A}(\gamma_1, \gamma_2, \ldots, \gamma_m) = 0$ for its tangential equation. The transversal, then, is the conjugate diameter of the tangent plane of S with respect to this characteristic cone, two directions being generally said to be transversal to each other if conjugate with respect to the cone.

For the equation of potentials $\nabla^2 u = 0$, the characteristic cone is the isotropic cone, so that "transversal" is synonymous with normal.

We immediately note that the transversal direction lies in the tangent plane when and only when the latter is characteristic. (It is on the same side of the tangent plane as the corresponding normal when and only when $\mathbf{A}(\pi_1, \pi_2, \ldots, \pi_m) > 0$.)

By means of this new definition, we get to the final form of the *fundamental formula*

$$\text{(F)} \quad \mathbf{SSS}\,[v\,\mathcal{F}(u) - u\,\mathcal{G}(v)]\,dT = -\,\mathbf{SS}\left(v\frac{du}{d\nu} - u\frac{dv}{d\nu} + Luv\right)dS,$$

* *C. R. Ac. Sc.* February 11, 1901. We use the phrase in the text (instead of the word "conormal," which we previously used in accordance with d'Adhémar) as it occurs, with the same meaning and construction, in questions of Calculus of Variations closely connected with the present ones.

† *Thesis*, Paris (1902), p. 34. It would be of interest to complete Coulon's interpretation by defining geometrically, not only the direction, but also the magnitude of the small segment $d\nu$.

L denoting

(7) $$L = \sum_i \pi_i \left(B_i - \sum_k \frac{\partial A_{ik}}{\partial x_k}\right).$$

In the case of two variables (§ 36),

$$\left(v \frac{du}{d\nu} - u \frac{dv}{d\nu} + Luv\right) ds$$

will similarly be the value of the integrand in (F_1), § 36, that is, of

$$\int P dy - Q dx = \int \frac{1}{2}\left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}\right) dy$$
$$- \frac{1}{2}\left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}\right) dx + uv\,(A dy - B dx),$$

ν being such that $\frac{dx}{d\nu} = -\frac{dx}{ds}$, $\frac{dy}{d\nu} = \frac{dy}{ds}$; such a direction ν will be symmetrical with the tangent to the line of integration with respect to parallels to the axes (which is in agreement with our general construction of the transversal, as the characteristic conoid reduces to the straight lines $x = \text{const.}$; $y = \text{const.}$).

41. This formula is, as we said, the basis of any research concerning linear partial differential equations of the second order and especially of the above quoted investigations*. u habitually denotes the unknown function of the problem; v is an auxiliary arbitrary function, precisely in the choice of which the whole skill of the operator lies. It will be most generally chosen so as to satisfy the "adjoint equation†"

($\mathscr{E}$) $$\mathscr{G}(v) = 0.$$

In the ordinary theory of potential, v is simply the elementary potential

$$\frac{1}{r} = \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

if $m - 3$,—or $\log \frac{1}{r}$ if $m = 2$,—or also one of the quantities (*Green's functions*) deduced from the elementary potential by addition of

* Kirchhoff does not form directly the quadruple integral corresponding to the left-hand side of (5) in the spherical wave problem; but the succession of a triple and a simple integration which he performs is equivalent to it.

† The adjoint equation will always be—except when contrary indication is specified—taken as homogeneous, even if, in the primitive equation, the right-hand side is given $\neq 0$.

certain terms which remain regular for $r = 0$. This elementary potential,—whose introduction commends itself by the most evident analytical and even physical reasons,—owes the part it plays in the theory (as is immediately seen by inspection of the formulæ) essentially to the nature of its singularity for $r = 0$, for which it becomes infinite. Let us remark immediately that this singularity occurs not only at a single real point ($x = x_0$, $y = y_0$, $z = z_0$), but also along a whole imaginary surface, viz., the isotropic cone having this point (x_0, y_0, z_0) for its vertex, which (in accordance with a general theorem which we shall soon restate) coincides with the corresponding characteristic cone as defined above.

42. Riemann's Method. If we now come to Riemann's method for the integration of the hyperbolic equation in two variables

$$(\epsilon) \qquad \mathcal{F}(u) = \frac{\partial^2 u}{\partial x \partial y} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu = f,$$

it seems, at first, that the quantity introduced by Riemann in the fundamental formula is of a quite different character from the elementary potential. Though the method is explained in Darboux's *Leçons** and several other treatises†, we sum it up briefly.

The question is to determine the value of u at a given point $a(x_0, y_0)$. Cauchy's data being borne by an arc of a plane curve S, which we shall assume to intersect any characteristic (i.e. parallel to the x- or to the y-axis) at one point only, Riemann applies the fundamental formula

$$(\mathrm{F}_1) \qquad \iint [v\,\mathcal{F}(u) - u\,\mathcal{G}(v)]\,dx\,dy = \int P\,dy - Q\,dx$$

within a triangular domain T (fig. 6 or 6a) enclosed between an arc $\alpha\beta$ of S and the segments $a\alpha$, $a\beta$ of the two characteristics drawn through a (which cut S at α and β), $a\alpha$ being parallel to the x-axis. u means the unknown function of the problem.

Now, for v, we take, with Riemann, a quantity‡,—which is a function of x, y, but depends also on the position of a, so that it has

* Vol. II, book IV, chap. IV, pp. 71—81 of the 2nd edition.

† See the author's *Lecons sur la propagation des ondes*, chap. IV, pp. 163—166, and Goursat's *Cours d'Analyse*, vol. III, chap. XXVI, pp. 146—152 of the 2nd edition.

‡ As to the existence of that quantity, see Darboux, *loc. cit.* §§ 364, 365, pp. 96—106 and our Book V.

to be written $\mathcal{V}(x, y; x_0, y_0)$—satisfying (in x, y for x_0, y_0 constant) the adjoint equation* $\mathcal{G}(v) = 0$ and defined, moreover, by the supplementary conditions that

$$\mathcal{V} = e^{\int_{x_0}^{x} A dy}, \text{ for } x = x_0,$$

$$\mathcal{V} = e^{\int_{y_0}^{y} B dx}, \text{ for } y = y_0,$$

(which especially give $\mathcal{V} = 1$ for $x = x_0, y = y_0$). It is immediately seen that the two integrals $\int P dy - Q dx$ along the two rectilinear segments $a\alpha$, $a\beta$, viz. (if, on S, y is a decreasing function† of x)

$$-\int_{x_0}^{x_1} Q(x, y_0)\, dx = -\int_{x_0}^{x_1} \left[\frac{1}{2}\left(\mathcal{V}\frac{\partial u}{\partial x} - u\frac{\partial \mathcal{V}}{\partial x}\right) + Bu\mathcal{V}\right] dx,$$

$$\int_{y_2}^{y_0} P(x_0, y)\, dy = \int_{y_2}^{y_0} \left[\frac{1}{2}\left(\mathcal{V}\frac{\partial u}{\partial y} - u\frac{\partial \mathcal{V}}{\partial y}\right) + Au\mathcal{V}\right] dy,$$

(where x_1, y_0 are the coordinates of α and x_0, y_2 the coordinates of β) respectively reduce to $\frac{1}{2}(u\mathcal{V})_\alpha - \frac{1}{2}u_a$ and $\frac{1}{2}(u\mathcal{V})_\beta - \frac{1}{2}u_a$, so that, $\mathcal{G}(\mathcal{V})$ vanishing and $\mathcal{F}(u)$ being equal to f, we find

$$u_a = \tfrac{1}{2}(u\mathcal{V})_\alpha + \tfrac{1}{2}(u\mathcal{V})_\beta + \int_{\alpha\beta} (P dy - Q dx) - \iint f\, dx\, dy.$$

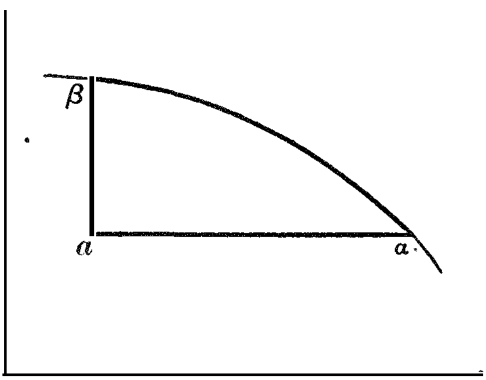

Fig. 6.

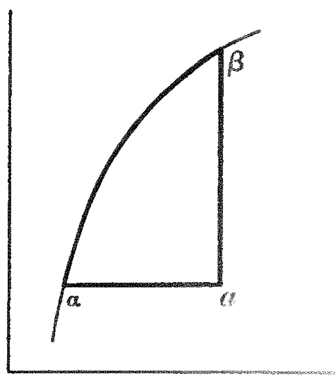

Fig. 6 *a*.

This, as required, gives the value of u in terms of quantities which are assumed to be known, viz. Cauchy's data on S: especially ν is the transversal direction, located as said at the end of § 40.

* See footnote to § 41.

† When y is, on S, a decreasing function of x (fig. 6), the two segments $a\alpha$, $a\beta$ are both in the positive directions (fig. 6) or both in the negative directions of the axes and $\alpha\beta$ is the *direct* sense on the outline of T, as necessary in writing (F_1); if, on the contrary, x and y increase simultaneously on S, the sense is the retrograde one and signs in (F_1) have to be reversed. A unique formula covering all cases could be given by the use of Méray's notation for multiple integrals.

If the line is an ascending one (i.e. y an increasing function of x) the sign of the double integral is to be reversed*.

The function v of Riemann is, as we see, like the elementary potential, a function of the coordinates of two points. The property of symmetry is also generalized to the present subject by the following *interchange property*†: *the quantity* $\mathcal{V}$ *does not change by the simultaneous interchange of* x, y *with* x_0, y_0 *and of the polynomial* $\mathcal{F}$ *with its adjoint* $\mathcal{G}$ (this giving a symmetry in (x, y) and (x_0, y_0) if $\mathcal{F}$ is identical with its adjoint, as is the case for $\nabla^2 u$).

But it immediately appears that the quantity $\mathcal{V}$ thus introduced into the operations has a priori no singularity at a: in fact, it is a perfectly *regular*, holomorphic function of the variables on which it depends, when the coefficients of the equation themselves are such. For instance, for the equation $\frac{\partial^2 u}{\partial x \partial y} + \lambda u = 0$ (λ constant) (see below, § 69 and Book IV) $\mathcal{V}$ is equal to $J_0\left[\sqrt{\lambda (x - x_0)(y - y_0)}\right]$, in which J_0 is Bessel's well-known integral transcendental function‡.

Nevertheless, we shall soon see that Riemann's function is derived most directly from the quantity which corresponds to the elementary potential.

43. The case is again different with the expressions introduced by Kirchhoff and Volterra. The former uses

$$\frac{1}{r} F(r - \omega t), \tag{8}$$

where again $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ and F is an arbitrary function of one variable. Such a quantity is singular for $r = 0$, i.e., $x = x_0$, $y = y_0$, $z = z_0$: that is, in the common language of ordinary space, at one point, but in our present conception of the "universe," along a whole *line*, as t is liable to take any arbitrary value.

Similarly the quantity used by Volterra (at least for the problem with which we are occupied at present) is

$$\mathbf{v} = \log \frac{(t - t_0) + \sqrt{(t - t_0)^2 - r^2}}{r} \tag{8'}$$

* See preceding footnote (p. 66).

† Darboux, *loc. cit.* no. 359, p. 81; Goursat, *loc. cit.*

‡ The discontinuity arises nevertheless from the fact that integration is extended over the domain limited by the two characteristics issuing from (x_0, y_0): which comes to the same as making v equal to 0 outside that domain and, therefore, discontinuous along its limiting lines.

which has two kinds of singularities in the real domain: (1) the surface $(t-t_0)^2-r^2=0$, that is, the characteristic cone with vertex (x_0, y_0, t_0), this being entirely analogous to the case of the elementary potential; (2) the line $r=0$ (that is, $x=x_0$, $y=y_0$, t arbitrary).

In consequence of the presence of this linear singularity, both Kirchhoff's and Volterra's methods do not give directly the value of the unknown u at the point chosen, but only the integral of u along a certain segment of the line* $r=0$: from which the value of u itself is then easily deduced (by differentiation, for instance, as in Volterra's Memoir).

The same applies to the extension given by Tedone† to the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\dots+\frac{\partial^2 u}{\partial x^2_{m-1}}-\frac{\partial^2 u}{\partial x^2_m}=0,$$

which behaves quite similarly to the preceding ones (e_1) and (e_3) or to (e_2) according as m is even or odd. Tedone again does not get directly to the value of u itself, but to an integral such as

$$\int_{t_1}^{t_0}(t_0-t)^{m-3}u(t)\,dt,$$

which he has to differentiate $(m-2)$ times with respect to t_0.

44. This indirect character of the method would be only a secondary disadvantage, but it implies a much more serious one, which is that the origin—at least the analytical one—of the expressions (8) and (8′) does not appear. Kirchhoff's function is suggested by physical considerations; but Volterra's has to be formed a priori; and this was precisely one of the greatest difficulties overcome by the great Italian geometer.

This difficulty will occur in a much higher degree if we try to generalize the aforesaid methods in order to apply them when other equations than (e_2) or (e_3) are concerned. Here we shall have no

* Of course, in our three-dimensional space (for (e_2)) or four-dimensional space (for (e_3)), the point $r=0$ describes a straight line, on account of the variability of t. When applying the fundamental formula, this whole line, and not merely one point of it, has to be abstracted from the field of integration by a cylinder (or hypercylinder) having it for its axis. The simple integral mentioned in the text arises from the SS extended over the surface of that cylinder, by letting its radius approach zero.

† *Annali di Matematica*, 3rd series, vol. I (1898), p. 1.

guide, at least no sure one, in the construction of expressions corresponding to (8) and (8′).

This was precisely the case with the two geometers, Coulon and d'Adhemar, who undertook the extension of the method to such equations as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} + Ku = 0,$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} + Ku = 0,$$

(equations of damped spherical or cylindrical waves) or to equations with variable coefficients. The formulæ obtained by them were not at all the equivalent of Kirchhoff's or Volterra's; in order to obtain the required value of u, they had to be followed, not by simple differentiations, but by the resolution of more or less complicated integral equations, requiring quite a new calculus of successive approximations. The reason of this is evidently that the auxiliary quantities introduced by them were not the ones truly analogous to (8) and (8′). (How it is that these analogues exist, but are not to be discovered without the proper clue, will appear from our further considerations.)

45. The reason why these methods of Kirchhoff and Volterra do not prove suitable for generalization will appear even better if we inquire in what measure the aforesaid singular line $r = 0$ is connected with the equation itself. That this connection is a rather loose one, would evidently have been seen by these two authors if, at the time their works were composed, science had possessed our present ideas on Relativity. We now know that speaking of a fixed point of space considered at successive instants has no definite meaning (or, if preferred, has an infinity of meanings), that there exist (as was known even before) an infinity of linear transformations on x, y, z, t (or x, y, t)—forming "Lorenz's group"—which leave our partial differential equation invariant, and that such transformations leave the characteristic cone unchanged, but may change the straight line $r = 0$ into any other straight line drawn through the vertex of the characteristic cone and inside it. We then know that this line $r = 0$ has no essential and particular part to play in our operations.

In what follows, we shall find that *every result of the theory can be and has to be deduced from the consideration of the elementary solution only.*

CHAPTER III

THE ELEMENTARY SOLUTION

1. General Remarks

46. Of course, we have now to define what the elementary solution is, and to construct it. The first extension of the elementary potential to other equations than Laplace's is due to Picard*. He considers the equation with two independent variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+Cu=0,$$

C being a given function of x and y, and proves that (x_0, y_0) being any given point, which we can call the *pole*, and $r=\sqrt{(x-x)^2+(y-y)^2}$, this admits of a solution of the form

$$\mathcal{U}\log\frac{1}{r}+w, \tag{9}$$

$\mathcal{U}$ and w being properly chosen functions of x, y (and also x_0, y_0) which are regular in the neighbourhood of $x=x_0$, $y=y_0$. Of course, w is, to a certain extent, arbitrary, as any regular solution of the given equation can be added to it.

This result was extended a little later by Hilbert and Hedrick†, and by the author‡, independently, to the more general equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}+Cu=0. \tag{10}$$

The method used in this case, however, implies that the coefficients A, B, C, functions of x, y, be analytic, which was not necessary in Picard's proof.

* *Comptes Rendus Ac. Sc.* April 6, 1891, and June 5, 1900. An equivalent result has been obtained by Sommerfeld, *Encycl. der Math. Wiss.* II A, 7 c, 1900.

† Hilbert, Lectures at Gottingen, 1901 (unpublished); Hedrick, *Uber den Analytischen Character der Lösungen von Differentialgleichungen* (Diss., Göttingen, 1901).

‡ *Second International Congress of Mathematicians*, Paris, 1900; *Notice scientifique*, Paris, Hermann, 1901.

But this method enables us to answer a question which previously arose, viz. the relation between Riemann's function and the elementary solution.

It is clear, in fact, that, if we remain in the analytic case, there is no essential distinction between (10) and Laplace's hyperbolic equation

$$(\epsilon) \qquad \mathcal{F}(u) = \frac{\partial^2 u}{\partial x \partial y} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu = 0,$$

(A, B, C again being functions of x and y), which form obviously arises from the former one by the changing of $x+iy$, $x-iy$ into x, y. As this changes $r^2 = (x-x_0)^2 + (y-y_0)^2$ into $(x-x_0)(y-y_0)$, we have to find for (ϵ) a solution of the form

$$\mathcal{U} \log[(x-x_0)(y-y_0)] + w.$$

It is sufficient that by substituting the first term in (ϵ), the result will be made a regular function, say

$$\mathcal{F}[\mathcal{U} \log(x-x_0)(y-y_0)] = \mathcal{M},$$

for, if so, we shall only have to take, for w, any regular solution of the equation of

$$\mathcal{F}(w) = -\mathcal{M}.$$

Now, we have

$$\mathcal{F}\{\mathcal{U} \log[(x-x_0)(y-y_0)]\} = \mathcal{F}(\mathcal{U}) \log[(x-x_0)(y-y_0)] + \frac{1}{x-x_0}\left(\frac{\partial \mathcal{U}}{\partial y} + A\mathcal{U}\right) + \frac{1}{y-y_0}\left(\frac{\partial \mathcal{U}}{\partial x} + B\mathcal{U}\right).$$

This will be a regular function of x, y near each of the lines $x = x_0$, $y = y_0$ if*:

(1) The logarithmic term vanishes, so that $\mathcal{U}$ itself is a solution of (ϵ);

(2) The numerators of the two fractions vanish at the same time as the denominators, so that

$$\frac{\partial \mathcal{U}}{\partial y} + A\mathcal{U} = 0 \qquad \text{(for } x = x_0\text{)},$$

$$\frac{\partial \mathcal{U}}{\partial x} + B\mathcal{U} = 0 \qquad \text{(for } y = y_0\text{)}.$$

* It is evident that these conditions are not only sufficient, but necessary: see our *Leçons sur la propagation des ondes*, ch. VII, § 344.

But these conditions (together with $\mathcal{U}=1$, for $x=x_0$, $y=y_0$) are precisely those which determine the function of Riemann (except that we have written $\mathcal{G}$ instead* of its adjoint polynomial $\mathcal{G}$).

Thus we see that *Riemann s function coincides with the coefficient of the logarithmic term in the elementary solution of the equation* so that, though regular, it is in a direct connection with the logarithmic quantity (9), a particular case of general relations which our further analysis will give us.

47. Extensions of the elementary solution to $m > 2$ were successively given in a fundamental memoir of Fredholm's† for equations of any order analogous to (e_2) and by Holmgren‡. But even in the latter's works, the extension is not complete, as the coefficients of the terms of the second order are assumed to be constants, which can be obtained by a point transformation if $m=2$, but generally cannot‖ if $m > 2$.

We shall construct the elementary solution for the most general (analytic and non-parabolic) linear equation of the second order.

48. The characteristic conoid. In the case of the elementary potential, it already appears that our elementary solution must be singular not only at one point—the pole—but along a certain surface (real or imaginary).

What that surface must be, appears from an important theorem of Le Roux and Delassus¶, viz. *any singular surface of a solution of a linear differential equation*** (the coefficients being regular) *must*

* Such a permutation is equivalent to that of x, y with x_0, y_0, on account of the interchange property (see above, § 44).

† *Acta Mathematica*, vol. XXIII. See also Le Roux, *C.R. Ac. Sc.*, vol. CXXXVII, p. 1230. Zeilon, *Nov. Act. Soc. Sc. Upsaliensis*, series 4, vol. V.

‡ *Arkiv for Matematik, Astronomi och Fysik*, vol. I.

‖ The possibility of such a reduction depends, as we shall see, on the possibility of the conformal representation of a certain linear element on the euclidian ds^2, so that the conditions for it are given by Cotton's researches (*Thesis*, Paris, 1899, ch. II, nos. 15—17).

¶ Le Roux, *Thesis*, Paris, 1895, Part II; Delassus, *Ann. Scient. Ec. Norm. Supre*, 3rd series, vol. XIII (1896), p. 35.

** An assumption is made on the nature of the singularity, viz. that the principal part of the solution u is $UF(G)$, U being regular and F such that *be characteristic.* Such a singular surface must therefore satisfy the differential equation of the first order

$$(\mathbf{A}) \qquad \mathbf{A}\left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_m}; x_1, x_2, \ldots x_m\right) = 0,$$

of § 13. Among the solutions of this equation, one which was especially considered by Darboux* will play the chief part in our present considerations: it is the one which has a given point

$$a\,(a_1, a_2, \ldots a_m)$$

as a conic point (its tangent cone being the characteristic cone above defined), which we shall call *the characteristic conoid.* It coincides with the characteristic cone itself when the coefficients of the equation (or at least the coefficients of the terms of the second order) are constants; in the general case. it is a kind of cone with curved generatrices, the construction of which, well known since Darboux's Memoir, will be given below, and even in a somewhat more precise form, as we shall write down its equation.

2. Solutions with an Algebroid Singularity

49. In the first place, let us examine the case of a surface without a singular point (the result of which examination can also be applied to the characteristic conoid outside the neighbourhood of the vertex). We shall prove not only Le Roux and Delassus' theorem under the conditions which concern us, but also its converse, which is important for us, by constructing, for our equation, a solution of the form

$$(11) \qquad u = UG^p + w,$$

where $G = 0$ is the equation of such a given regular surface, p a given constant index, U and w regular functions. Of course, as in § 46, we have only to contrive that the result of the substitution of the first term in the left-hand side of our equation be regular.

$\dfrac{F''(G)}{F'(G)}$ and $\dfrac{F'(G)}{F(G)}$ are infinite for $G=0$, a condition which is satisfied for all practical cases, especially for $F(G) = G^p$ and $F(G) = \log G$ (the only ones which we shall use). Le Roux (*Journ. de Math.* series 5, vol. IV, 1898, p. 402) gives another proof, not wanting the above assumption.

* *Memoire sur les solutions singulieres des equations aux dérivées partielles du premier ordre*, § 2, p. 34 (*Mémoires des Savants étrangers*, vol. XXVII, 1880).

We start from the equation (E), which we take as homogeneous (i.e. $f=0$), viz.

$$\text{(E)} \qquad \mathcal{F}(u) = \sum_{i,k} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu = 0.$$

Let us replace u by $U.F(G)$. Writing π_i for $\frac{\partial G}{\partial x_i}$, we have

$$\frac{\partial u}{\partial x_i} = U\pi_i F'(G) + \frac{\partial U}{\partial x_i} F(G),$$

$$\frac{\partial^2 u}{\partial x_i \partial x_k} = U\pi_i\pi_k F''(G) + \left(\pi_i \frac{\partial U}{\partial x_k} + \pi_k \frac{\partial U}{\partial x_i} + U \frac{\partial^2 G}{\partial x_i \partial x_k}\right) F'(G) + \frac{\partial^2 U}{\partial x_i \partial x_k} F(G).$$

We have to multiply the first line (for every i) by B_i, the second (for every i, k) by A_{ik}, and add* to CUF. In this combination we see that:

(1) the coefficient of $F''(G)$ is $\mathbf{A}(\pi_1, \dots \pi_m)$;

(2) in the coefficient of $F'(G)$, the terms in $\frac{\partial U}{\partial x_i}$ are†

$$\frac{\partial U}{\partial x_i} \cdot \sum_k 2A_{ik}\pi_k,$$

that is

$$\frac{\partial U}{\partial x_i} \frac{\partial \mathbf{A}}{\partial \pi_i}.$$

Therefore, we have

$$UF''(G)\mathbf{A}(\pi_1, \pi_2, \dots \pi_m) + F'(G)\left(\sum_i \frac{\partial U}{\partial x_i}\frac{\partial \mathbf{A}}{\partial \pi_i} + MU\right) + F(G)\mathcal{F}(U) = 0,$$

where M stands for

$$\text{(12)} \qquad M = \mathcal{F}(G) - CG.$$

Especially, for $F(G) = G^p$, we get

$$\text{(13)} \quad p(p-1)G^{p-2}U.\mathbf{A}(\pi_1, \dots \pi_m) + pG^{p-1}\left(\sum_i \frac{\partial U}{\partial x_i}\frac{\partial \mathbf{A}}{\partial \pi_i} + MU\right) + G^p\mathcal{F}(U).$$

* In such an addition, the suffixes i and k may be permuted, as $A_{ik} = A_{ki}$.

† See preceding footnote.

If the cases of $p=0$ and $p=1$ are excluded, this cannot vanish identically or even be a regular function (the first term evidently being of a greater order of magnitude than the following ones), if the coefficient $\mathbf{A}(\pi_1, \ldots \pi_m)$ is not zero: that is, if $G=0$ is not a characteristic. The equation $\mathbf{A}\left(\frac{\partial G}{\partial x_1}, \ldots \frac{\partial G}{\partial x_m}\right)=0$ must be either an identity or a consequence of $G=0$, so that we have in every case

$$(13a) \qquad \mathbf{A}\left(\frac{\partial G}{\partial x_1}, \ldots \frac{\partial G}{\partial x_m}\right) = \mathbf{A}_1 G,$$

$\mathbf{A}_1$ being regular even for $G=0$. Thus Delassus' theorem is proved. We now assume that this condition is satisfied, so that G^{p-2} disappears from (13). Let us express the condition that the terms in G^{p-1} also vanish: we have to write that, on the surface $G=0$,

$$(14) \qquad \Sigma \frac{\partial U}{\partial x_i}\frac{\partial \mathbf{A}}{\partial \pi_i} + [M + (p-1)\mathbf{A}_1]\, U = 0.$$

This is a linear partial differential equation of the first order in U, the integration of which would lead to the introduction of the lines defined by the ordinary differential equations

$$(\mathrm{L}_1) \qquad \frac{dx_1}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \pi_1}} = \frac{dx_2}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \pi_2}} = \ldots = \frac{dx_m}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \pi_m}} = ds.$$

In the denominator, we find the direction cosines of the transversal to $G=0$; but this is, in the present instance, tangent to that surface (as the latter is a characteristic; the transversal is the direction of the generatrix of contact between the plane $(\pi_1, \ldots \pi_m)$ and the characteristic cone), so that a line satisfying (L_1) and issuing from a point of $G=0$ *is entirely situated on that surface.* These lines are in fact *the characteristics of equation* ($\mathbf{A}$), as defined in the general theory of partial differential equations of the first order. We shall call them the *bicharacteristics**. If $\mathbf{A}_1 = 0$, that is, if the function G identically satisfies $\mathbf{A}=0$, the aforesaid theory of the partial differential

* For their physical meaning as sound or light rays, see our *Leçons sur la propagation des ondes*, ch. VII, §§ 309, 319 and 351.

equations of the first order* shows that besides (L_1), these lines also satisfy

$$(L_2) \qquad ds = \frac{d\pi_1}{-\frac{1}{2}\frac{\partial \mathbf{A}}{\partial x_1}} = \frac{d\pi_2}{-\frac{1}{2}\frac{\partial \mathbf{A}}{\partial x_2}} = \dots = \frac{d\pi_m}{-\frac{1}{2}\frac{\partial \mathbf{A}}{\partial x_m}}$$

so that they can be determined a priori (that is, without knowing the equation $G = 0$) by the integration of the system of ordinary differential equations (L_1) and (L_2).

Bicharacteristics have already appeared in our former operations. They are indeed the lines which, in § 12, Book I, we consider as defined on $x_1 = 0$ by the differential equation (l) in order to determine our successive $u_1, u_2, \dots$: $x_3 = 0$ being tangent to the characteristic cone at each of its points, the generatrix of contact has its direction cosines proportional to A_{31}, A_{32}.

50. The same lines and the fact that they are preserved in any punctual transformation (which is obvious from their analytical or geometric signification) will in the first place be used by us to simplify our equation. Our given characteristic surface being assumed to be a regular one, we can change the variables so as to give its equation the form $x_m = 0$, and this, moreover, in such a way that every $x_m =$ const. is a characteristic: this will be expressed by A_{mm} being identically zero.

We shall assume, further on, that the edge† of intersection of $x_m = 0$ and $x_{m-1} = 0$ is nowhere tangent to a bicharacteristic direction; by virtue of which we can take our new variables so that the bicharacteristics situated on $x_m =$const. shall be $x_1 =$ const., $x_2 =$ const., ..., $x_{m-2} =$ const. This will be expressed by

$$A_{im} = 0. \qquad (i \neq m-1)$$

We shall divide by‡ $A_{m,m-1}$, and we can also make B_m vanish by changing u into $ue^{\int B_m dx_{m-1}}$. Replacing the letters x_m and x_{m-1} by x and y respectively, we see that we can write our equation

$$(15) \qquad \frac{\partial^2 u}{\partial x \partial y} - \mathscr{F}_1(u) = 0,$$

* See Goursat-Hedrick, *loc. cit.* § 87.

† See note *, p. 5.

‡ $A_{m,m-1}$ must be different from zero, or else $\gamma_1 = \gamma_2 = \dots = \gamma_{m-1} = 0$ would make all the derivatives of $\mathbf{A}(\gamma_1, \dots \gamma_m)$ vanish, which is excluded, as we are not in the parabolic case.

where the new differential polynomial $\mathcal{F}_1(u)$ contains no differentiation with respect to x.

51. $p=0$. *Beudon's result.* The cases of $p=0$ and $p=1$ were excluded above. They correspond to no singularity of u, and bring us back to the considerations of our first Book*. But as we are taking the case of a characteristic surface, they will, for that very reason, interest us as leading to the answer of the question which was set in § 12; i.e. the nature of the indetermination of Cauchy's problem in that case.

This question is solved by the following result:

In the present case of $x=0$ characteristic†, *we can determine a solution of our differential equation by knowing its value $u(x, \ldots, x, 0, y)$*

$$u(x_1, \ldots x_{m-2}, 0, y) = u_0(x_1, \ldots x_{m-2}, y);$$
$$u(x_1, \ldots x_{m-2}, x, 0) = \mathbf{u}(x_1, \ldots x_{m-2}, x)$$

on each of the surfaces $x=0$, $y=0$‡, *which values can be chosen arbitrarily, but for the condition that they imply no contradiction along the edge of intersection,* i.e.

$$(16)\quad u_0(x_1, \ldots x_{m-2}, 0) = \mathbf{u}(x_1, \ldots x_{m-2}, 0), \text{ say } = \mathbf{u}_0(x_1, \ldots x_{m-2}).$$

This theorem contains, as a particular case, the proof of existence of Riemann's function (§ 42). It was first given, for that purpose, by Darboux‖ for $m-2$, so that $x=0$, $y=0$ were two lines, the data being analytic, and *both* lines being assumed as characteristics. It was extended by Goursat‖ to non-linear equations, assuming only that the initial tangents at their points of intersection have characteristic directions.

* The case of p integral and >1 may be considered as included in our present as well as in our former considerations b: $p=2$, for instance (with $U_1=0$), so that $u=Ux^2$ would correspond to a Cauchy problem with $u_0=u_1=0$, for which we already know that no solution different from zero can be obtained if $x=0$ is not a characteristic.

† It is even sufficient to assume that $x_m=0$ is tangent to a characteristic at every point in common with $x_{m-1}=0$.

‡ The hypothesis that the edge of intersection is nowhere tangent to a bicharacteristic is again implied.

‖ Darboux, *Leçons sur la theorie des surfaces*, vol. II, pp. 91—94. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre* (see below, 2nd footnote, p. 78).

Beudon (*loc. cit.*)* generalized Darboux-Goursat's result to $m > 2$, and, after Goursat, used it to prove the indetermination of Cauchy's problem for a characteristic.

But, as Picard† (admitting even non-analytic data) and then Goursat† (for the non-linear case) have shown for the case of $m = 2$, the hypothesis that $y = 0$ is characteristic is not necessary. We shall now prove Beudon's theorem with the same improvement.

We assume, at least for the present, all the data to be analytic, so that the coefficients in $\mathcal{F}_1$ shall be holomorphic functions of the independent variables‡, and we again substitute for u an expansion in powers of x

$$u = u_0 + u_1 x + \ldots + u_h x^h + \ldots. \tag{17}$$

By equating the coefficients of similar powers of x, we obtain the successive conditions

$$\text{(18)} \quad \left\{ \begin{aligned} &\frac{\partial u_1}{\partial y} = \mathcal{F}_1(u_0), \\ &2\frac{\partial u_2}{\partial y} = \mathcal{F}_1(u_1) + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &h\frac{\partial u_h}{\partial y} = \mathcal{F}_1(u_{h-1}) + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \right.$$

the system of which is conversely equivalent to (15).

The first of them will give us $\frac{\partial u_1}{\partial y}$ (the right-hand side containing no other term of (17) than u_0); the second, $\frac{\partial u_2}{\partial y}$ etc., the right-hand side of the equation for $\frac{\partial u_h}{\partial y}$ depending only on u_0||, $u_1, \ldots u_{h-1}$, and the coefficients of $\mathcal{F}_1$. Hence we see:

* See note †, p. 20.

† Picard, in Darboux, *loc. cit.* vol. IV, pp. 355—359 (Note I). Goursat, *Equations aux dérivées partielles du second ordre*, vol. II, pp. 303—308.

‡ x generally appears explicitly among the coefficients of $\mathcal{F}_1$: the terms due to this circumstance are those which we have replaced by dots, on the right-hand sides of equations (18) (in the first term, on the other hand, we must make $x = 0$).

|| That the $u_{h'}$'s with suffixes h' less than $h - 1$ also appear in $\frac{\partial u_h}{\partial y}$, results from the foregoing footnote.

(1) that u_0 remains arbitrary;

(2) that we can also choose arbitrarily the values of each of the following u's for $y=0$; after which, we shall have ($\mathbf{u}_h$ being what u_h becomes for $y=0$)

(19) $$u_h = \mathbf{u}_h + \int_0^y \mathcal{F}_1(u_{h-1})\,dy.$$

The latter fact is equivalent to saying that we can take arbitrarily the value of expansion (15) for $y=0$, i.e. the function

(20) $$\mathbf{u}(x_1, x_2, \ldots x_{m-2}, x) = \sum_h \mathbf{u}_h x^h$$

except its first term $\mathbf{u}_0(x_1, x_2, \ldots x_{m-2})$, which must be equal to the value of u_0 for $y=0$. This is nothing but another form of condition (16). The condition for $\frac{\partial u_1}{\partial y}$ is no other than the condition of possibility for Cauchy's problem written in § 12, Book I. We see that, when it is fulfilled, the expansion (17) remains indeterminate, the arbitrary elements in it being the successive coefficients $\mathbf{u}_h$ in the expansion (20).

52. We shall have the right to speak of a solution u presenting the same degree of indetermination when we shall have proved the convergence of (17). We shall do this* under the hypothesis that all our data are holomorphic around a given point on our edge $x=y=0$ (which we shall choose for the origin of the coordinates): granting which, we shall give the required proof for the multiple Maclaurin series which expands u in powers of $x, y, x_1, x_2, \ldots x_{m-2}$.

In the first place, we see that each of the calculations (19) implies only differentiations, integrations (from the lower limit 0), multiplications and additions, so that every coefficient in (17) will be expressed in terms of previously calculated ones (that is, with a smaller h and a not greater total degree in all the variables) and

* This will mean the construction of a solution satisfying our conditions for sufficiently small x's *and a small domain D of values of* $x_1, \ldots, x_{m-2}, y$. For the contrary case in which D is sufficiently extensive, we simply point out, as it concerns non-linear equations which we are leaving aside, a remarkable result of Goursat's (*Ann. Fac. Sc. Toulouse*, 2nd series, vol. VIII, 1906), viz. that it may happen that even the (seemingly indeterminate) problem of Cauchy relating to $x=0$ admits of no solution which is valid throughout D and regular; but this can not occur in our linear case.

coefficients in the expansions of the A_{ik}, B_{ik}, C which enter in $\mathcal{F}_1$, by a polynomial with only positive terms. Therefore, we only have to show the existence of our solution by replacing the expansions $\mathcal{F}_1$, u_0, u by properly chosen dominant ones.

We may also again assume the functions u_0, u to be identically zero, as, in the contrary case, we should only have to introduce, instead of u, the new unknown

$$u - u_0 - \mathbf{u} + \mathbf{u}_0$$

and we may suppose this to have been done beforehand.

Such zero values of u_0 and u_1 will be, as in Book I, § 10, dominated by any expansions with positive coefficients.

A dominant expansion for $\mathcal{F}_1$ will be

$$(21) \qquad K \frac{u + \Sigma' p_i + \Sigma' r_{ik}}{1 - \dfrac{x + y + x_1 + \ldots + x_{m-2}}{\rho}} \qquad (K > 0,\ \rho > 0, \text{ constants})$$

(where p_i stands for $\dfrac{\partial u}{\partial x_i}$, r_{ik} for $\dfrac{\partial^2 u}{\partial x_i \partial x_k}$ and the Σ' relates to all values of i, or i and k, from 1 to $m-1$), so that we have again only to show that the equation obtained by equating $\dfrac{\partial^2 u}{\partial x \partial y}$ to this quantity or to any dominant one admits of a solution represented by a Maclaurin expansion with positive coefficients.

Again using Goursat's device*, we write a dominant of (21) by changing, in the denominator, x into $\dfrac{x}{\alpha}$ with $\alpha < 1$, so that we start from equation

$$\frac{\partial^2 u}{\partial x \partial y} = K \frac{u + \Sigma' p_i + \Sigma' r_{ik}}{1 - \dfrac{\dfrac{x}{\alpha} + y + x_1 + \ldots + x_{m-2}}{\rho}},$$

for which we have to find a solution in the form of a Maclaurin

* This was not necessary in Beudon's original note, because Beudon assumed $y=0$ to be also a characteristic, i.e. $A_{m-1,m-1}$ to be also zero: so that the extension of the theorem to the case where $x=0$ alone is a characteristic is connected with Goursat's introduction of the parameter α.

expansion, with coefficients positive or zero. Let us take for u a function of the variables

$$(22) \qquad X = x + \alpha y, \quad Y = x_1 + \dots + x_{m-2}.$$

For such a form of u, our equation becomes (writing C for the numerical coefficient $\frac{(m-1)(m-2)}{2}$)

$$\alpha \frac{\partial^2 u}{\partial X^2} = K \frac{u + \alpha \frac{\partial u}{\partial X} + (m-2)\frac{\partial u}{\partial Y} + \alpha^2 \frac{\partial^2 u}{\partial X^2} + \alpha(m-2)\frac{\partial^2 u}{\partial X \partial Y} + C\frac{\partial^2 u}{\partial Y^2}}{1 - \frac{1}{\rho}\left(\frac{X}{\alpha} + Y\right)},$$

or, solving with respect to $\frac{\partial^2 u}{\partial X^2}$,

$$(23) \qquad \frac{\partial^2 u}{\partial X^2} = \frac{K}{\alpha} \frac{u + \alpha \frac{\partial u}{\partial X} + (m-2)\frac{\partial u}{\partial Y} + \alpha(m-2)\frac{\partial^2 u}{\partial X \partial Y} + C\frac{\partial^2 u}{\partial Y^2}}{L - \frac{1}{\rho}\left(\frac{X}{\alpha} + Y\right)},$$

where $L = 1 - K\alpha$.

Let us take α such that $L > 0$. Then we see that, expanding the fraction on the right-hand side of (23) in powers of X and Y, every coefficient in it will be positive.

Now, on account of Cauchy-Kowalewsky's theorem, equation (23) will admit of a solution vanishing, together with $\frac{\partial u}{\partial X}$, for $X = 0$, the expansion of which, as appears in § 10, has only positive coefficients.

Substituting for X and Y the values (22), the required dominant expansion is obtained, which gives the proof of our theorem.

53. *p arbitrary.* We shall now establish a corresponding conclusion for any value of the constant p, except a negative integer*.

Taking $u = Ux^p$, we have for U the equation

$$(15') \qquad x\frac{\partial^2 U}{\partial x \partial y} + p\frac{\partial U}{\partial y} = x\mathcal{F}_1(u).$$

Replacing U by

$$(24) \qquad U = U_0 + U_1 x + \dots + U_h x^h + \dots$$

* Another method has been proposed by Le Roux (*Thesis*, No. 32).

and equating the coefficients of similar powers of x, we have (dots on the right-hand side having the same meaning as in § 51)

$$(18')\qquad \begin{cases} \dfrac{\partial U_0}{\partial y} = 0, \\ (p+1)\dfrac{\partial U_1}{\partial y} = \mathcal{F}_1(U_0) + \dots, \\ \dots\dots\dots\dots\dots\dots \\ (p+h)\dfrac{\partial U_h}{\partial y} = \mathcal{F}_1(U_{h-1}) + \dots, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

Conversely, the system of (24) (when convergent) and (18′) is equivalent to (15′).

If p is a negative integer, this system generally admits of no solution*: the impossibility of satisfying it appears in that one of the equations (18′), e.g. for $p = -1$, the second of these equations, no longer contains U_1 and becomes

$$\mathcal{F}_1(U_0) = 0,$$

which ought to admit of a solution independent of y.

The hypothesis of p a negative integer being laid aside, equations (18′) make known to us the successive U_h's. In each one of these functions, however, an additive constant remains indeterminate. Their values can therefore be chosen arbitrarily for one determinate value of y, e.g. $y = 0$.

To prove the convergence of the expansion (24) thus obtained, it is sufficient to observe (as is done in the classic theory of the differential equation $x\dfrac{dy}{dx} = ax + by + \dots$) that the ratio $\dfrac{p+h}{h}$, which is never zero under the hypothesis which we have adopted and whose limit is 1 when h increases without limit, is always greater in numerical value than a fixed positive number q. We therefore shall obtain dominants for the successive U if we dominate $\mathcal{F}_1$ and replace at the same time, in the equation that gives $\dfrac{\partial U_h}{\partial y}$, the coefficient $(p+h)$ by qh.

* See *Leçons sur la propagation des ondes*, p. 339, § 356.

Now, this is equivalent* to making $p=0$ in equation (15′), after multiplying the right-hand side by $\frac{1}{q}$. It is then possible to divide right through by x, and we are brought back to Beudon's problem, which we treated above. Since, in this latter problem, *we can choose arbitrarily the values of the unknown for* $y=0$, that is, *on any surface which intersects the first without being tangent to one of its bicharacteristics*, the same is true in the problem now set.

The conclusion aimed at is thus established only upon the hypothesis that we are in the analytic case: a restriction, however, which is no immaterial one, as we have already seen.

3. The case of the characteristic conoid. The elementary solution.

54. The characteristic conoid with any point $a\,(a_1, a_2, \ldots a_m)$ as its vertex has that point for a singular point, so that the preceding calculation ceases to be valid: and in fact we shall see that p cannot, as above, be taken arbitrarily.

To treat this new case, we must first form the equation of the aforesaid characteristic conoid. This is, as we know, the locus of all bicharacteristics issuing from a. Analytically speaking, we have to take any set of quantities $p_1, \ldots p_m$ fulfilling the condition (**A**) and, with the initial conditions $p_i = p_{0i}$, $x_i = a_i$ for $s=0$, integrate the above written differential equations

$$\text{(4)} \qquad \frac{dx_i}{ds} = \frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_i}, \quad \frac{dp_i}{ds} = -\frac{1}{2}\frac{\partial \mathbf{A}}{\partial x_i}.$$

As the quantities $p_{01}, \ldots p_{0m}$ (or more exactly their mutual ratios) under condition (**A**) depend on $m-2$ parameters, the locus of the line thus generated is a surface. We must give a precise form for the equation of this surface: which form was suggested by Coulon†.

* Strictly speaking, the first equation (18′), viz. $\frac{\partial U_0}{\partial y}=0$, would disappear for $p=0$, as its left-hand side was originally multiplied by p. Of course, we have to preserve it in the present argument. Knowing the values of U for $y=0$, it determines them for $x=0$, thus giving all the data required by Beudon's theorem.

† *Thesis*, p. 22.

55. Introduction of geodesics. For that purpose, we shall construct every line issuing from $(a_1, a_2, \ldots a_m)$ and satisfy the differential system (L) *whether the initial values* $p_{01}, \ldots p_{0m}$ *of the variables* p_i *satisfy* (**A**) or not. Such lines are in fact the geodesics of a properly chosen linear element.

*Results in the general theory of geodesics**. Let

$$\mathbf{H}(dx_1, dx_2, \ldots dx_m; x_1, x_2, \ldots x_m) = \Sigma H_{ik} dx_i dx_k$$

be any quadratic form (except that its discriminant will be supposed $\neq 0$) in $dx_1, \ldots dx_m$, the coefficients H_{ik} being given functions of $x_1, \ldots x_m$. If $dx_1, \ldots dx_m$ be considered as differentials of $x_1, \ldots x_m$, $\mathbf{H}$ can be taken as a linear element in an m-dimensional variety. The integral

$$L = \int \sqrt{\mathbf{H}(dx_1, \ldots dx_m)} = \int \sqrt{\mathbf{H}(x_1', \ldots x_m')}\, dt$$

$\left(\text{where, in the last member, } x_i' \text{ stands for } \dfrac{dx_i}{dt}\right)$ will thus represent the length of an arc of. curve in that variety, and the corresponding *geodesics* are the lines which make the variation of L vanish. Their differential equations are

$$(25) \qquad \frac{d}{dt}\left(\frac{\partial \sqrt{\mathbf{H}}}{\partial x_i'}\right) - \frac{\partial \sqrt{\mathbf{H}}}{\partial x_i} = 0. \qquad (i = 1, 2, \ldots m)$$

Classic dynamical principles also lead to writing these differential equations in a different form, viz.

$$(25') \qquad \frac{d}{ds}\left(\frac{\partial \mathbf{H}}{\partial x_i'}\right) - \frac{\partial \mathbf{H}}{\partial x_i} = 0, \qquad (i = 1, 2, \ldots m)$$

this governing the motion of a system the vis viva of which would be $\mathbf{H}(x', x)$, and on which no forces would act.

These two forms (corresponding to two forms of the principle of least action) are not exactly equivalent, but are so conditionally. The first one determines the required lines but not t, the latter remaining

* See Darboux's *Leçons*, vol. II.

Many of the following principles, such as geodesics for an indefinite linear element, geodesics of zero length, differential parameters, etc., will be familiar to many readers, as they are now of constant use in the recent theory of Relativity. We, however, shall not assume this theory to be known.

an arbitrary parameter the choice of which is immaterial. Equations (25) remain unchanged by the change of the independent variable t to $\phi(t)$, ϕ being any function.

The second form (25′) defines not only a line, but a motion on that line, and this motion is no longer an arbitrary one in time: it must satisfy the integral of vis viva

$$\mathbf{H} = \text{const.}, \tag{26}$$

so that the representative point $(x_1, \ldots x_m)$ must move on the curve with constant kinetic energy. But if we take account of this latter equation, the two systems (25) and (25′) are (in general) equivalent*.

We shall start from system (25′) and reduce it to Hamilton's form by introducing the quantities

$$p_i = \frac{1}{2}\frac{\partial \mathbf{H}}{\partial x_i'}. \tag{27}$$

By eliminating the x_i''s, $\mathbf{H}$ becomes a quadratic form $\mathbf{A}$ in the p's,—viz. the adjoint form of $\mathbf{H}$ divided by the discriminant D of $\mathbf{H}$—and, as we know, the m equations (25′) of the 2nd order are replaced by the $2m$ Hamilton equations of the 1st order

$$\frac{dx_i}{ds} = \frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_i}, \tag{L_1}$$

(equivalent to (27)) and

$$\frac{dp_i}{ds} = -\frac{1}{2}\frac{\partial \mathbf{A}}{\partial x_i}. \tag{L_2}$$

These equations again admit of the integral $\mathbf{A} = \text{const.}$ equivalent to (26).

* If, in (25), we suppose the arbitrary parameter t to be chosen so that $\mathbf{H} = \text{const.}$, then the denominator $2\sqrt{\mathbf{H}}$ in $\frac{\partial\sqrt{\mathbf{H}}}{\partial x_i'} = \frac{1}{2\sqrt{\mathbf{H}}}\frac{\partial \mathbf{H}}{\partial x_i'}$ can come out from under the sign $\frac{d}{dt}$, and we find (25′).

Conversely, if we intend to write (25′) so that the independent variable t may become arbitrary, we have only to note that, as a function of such a quantity t, s can be easily calculated by (26), viz. $ds = \sqrt{\mathbf{H}}\,dt$. Replacing ds by this value and accordingly x_i' by $\frac{x_i'}{\sqrt{\mathbf{H}}}$, we find (25). (See Darboux's *Leçons*, vol. II, § 571 of the second edition.)

All this fails for the special geodesics (bicharacteristics) such that $\mathbf{A}(=\mathbf{H})=0$; then the system (25) ceases to have any meaning, (25′) remaining valid.

56. We shall use the preceding calculations in such a way that the quadratic form **A** be the one which we above denoted by that letter, viz. the characteristic form of our equation. How we must choose **H** for this purpose, is well known: the relation between **A** and **H** is a reciprocal one, so that we have to take **H** equal to the adjoint form of **A** divided by the discriminant Δ of **A**. The two discriminants D and Δ of **A** and **H** are reciprocal of each other. The variables p_i in **A** are connected with the variables x_i' in **H** by any one of the two (equivalent) systems (27), (L_1), the variables x_i being, of course, the same in both forms.

It is useful to note that the use we make of this conception of geodesics is slightly different from the usual one in the sense that **A** (or **H**) may be—and will be actually, in the hyperbolic case, the one which will concern us especially—an *indefinite* form. This, of course, will not matter for most of the analytical properties of geodesics; L may become imaginary, but not its square, which is precisely the quantity which we shall have to introduce*.

57. The above defined geodesics will now be treated by a method essentially equivalent to the well-known one due to Lipschitz†.

As we have noticed, equations (25′) do not admit of an arbitrary change in the independent variable; but they still admit of any linear one, in which s is replaced by $\alpha s + \beta$, α and β being any constants: indeed, such a change leaves equation (26) unaltered but for a change of constant on the right-hand side.

The corresponding property of equations (L) is that they are not altered when we change s into αs and p_i into — (α being constant), without altering the x's; and this even leaves every integral curve of (L) unchanged (only changing its parametric representation).

Let us now exclusively consider the geodesics issuing from a special given point $a\,(a_1, a_2, \ldots a_m)$ of the m-dimensional space, s being zero

* The only important difference introduced into operations by the possibility of **A** being indefinite is that we cannot, as is frequently done, choose on each geodesic the variable s so that the constant **A** becomes equal to 1, as the case of $\mathbf{A}=0$ happens particularly to concern us.

† *Bull. des Sc. Math.* 1st series, vol. IV, pp. 99—110. See Darboux's *Leçons*, vol. II, Book V, § 518.

at this point. One of them will be determined if we give the initial values (values for $s=0$) $p_{01}, \dots p_{0m}$ of $p_1, \dots p_m$. Moreover, the same one will be obtained, as we have just seen, if we replace s by αs and $p_1, \dots p_m$ by $\frac{p_1}{\alpha}, \dots \frac{p_m}{\alpha}$, the p_{0i}'s having of course to be also changed into $\frac{p_{0i}}{\alpha}$. This can be expressed by saying that the $2m+1$ quantities

$$p_1, \dots p_m, \quad p_{01}, \dots p_{0m}, \quad s$$

only occur in the $2m$ combinations

$$P_i = sp_i, \quad q_i = sp_{0i}. \qquad (i = 1, 2, \dots m) \tag{28}$$

Thus, the integrating formulæ of (L) must be of the form

$$\begin{cases} x_i = \phi_i(q_1, \dots q_m;\ a_1, \dots a_m), \\ P_i = \psi_i(q_1, \dots q_m;\ a_1, \dots a_m), \end{cases} \tag{29}$$

and we may immediately notice that those formulæ do not change by permutation of the x_i with the corresponding a_i and, at the same time, of P_i with $-q_i$ (as the differential equations (L_1), (L_2) show by changing s into $-s$).

Let us now consider the first series of equations (29) as defining a punctual transformation between the x's and the q's, the point corresponding to $q_1 = \dots = q_m = 0$; as $\left(\frac{dx_i}{ds}\right)_0 = \frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_{0i}}$, we see that the expansion of $x_i - a_i$ has $\frac{1}{2}\frac{\partial \mathbf{A}}{\partial q_i}$ for its term of the first degree. The Jacobian

$$J = \frac{D(x_1, \dots x_m)}{D(q_1, \dots q_m)} \tag{30}$$

has therefore, at a, the value Δ, and is $\neq 0$, so that the q's can certainly be expressed as functions of the x's in the neighbourhood of a.

The variables q are very simply connected with the *normal variables* of Lipschitz*: those being, by definition, the quantities:

$$\xi_i = s\left(\frac{dx_i}{ds}\right)_0,$$

so that we have, between them and the q's, the linear substitution with constant coefficients

$$q_i = \frac{1}{2}\frac{\partial \mathbf{H}_0}{\partial \xi_i},$$

* *Loc. cit.* See footnote †, preceding page.

or, in another equivalent form,

$$\xi_i = \frac{1}{2} \frac{\partial \mathbf{A}_0}{\partial q_i},$$

$\mathbf{A}_0$, $\mathbf{H}_0$ being the forms $\mathbf{A}$, $\mathbf{H}$ considered at a, viz.

$$\mathbf{A}_0 (q_1, \ldots q_m) = \mathbf{A} (q_1, \ldots q_m;\ a_1, \ldots a_m),$$
$$\mathbf{H}_0 (\xi_1, \ldots \xi_m) = \mathbf{H} (\xi_1, \ldots \xi_m;\ a_1, \ldots a_m).$$

If we use the special coordinates q_i (or ξ_i) the geodesic lines issuing from a will be represented by straight lines, the coordinates being proportional to s.

57 *a*. Nothing of the above requires the assumption that the coefficients A_{ik} be analytic: they only need to be regular, inasmuch as this is necessary for the application of general theorems concerning the existence of the integrals of differential equations and their differentiability with respect to initial conditions. (See Additional Note at the end of the present Book.)

If the A_{ik}'s are holomorphic functions of the x s, then the x s will be, in the vicinity of a, holomorphic functions of the q's, and, conversely, the q s will be holomorphic functions of the x s.

All our present considerations—and, consequently, those of the following § 58—will continue to be valid as long as the solution of the first series of equations (29) with respect to the q's is possible: in other words, so long as the problem of joining the point $x\,(x_1, \ldots x_m)$ to a by a geodesic line may be considered as a determinate one. The region $\mathscr{R}$ in which such a validity persists may be defined by considering, for instance, a one-parameter family of surfaces, containing a inside them and enveloping each other as the parameter increases (such as spheres with centre a): the inside of such a surface will belong* to $\mathscr{R}$

* See our *Lecons sur le Calcul des Variations*, final note A. A region $\mathscr{R}$ is most frequently obtained if, on every geodesic line going out from our point a, we determine the arc issuing from a and around which the required property (i.e. the fact that any point is to be joined to the initial one by a uniquely determined and continuous geodesic line) does not cease to exist: which arc is limited by the point defined by J (formula (30)) $=0$, the so-called *conjugate focus* of a (see *Leçons sur le Calcul des Variations*, Book II, ch. III) at which the geodesic may be touched by the envelope of a properly chosen one-parameter family of other geodesics issuing from a; but in some cases such a definition of $\mathscr{R}$ may prove to be erroneous.

so long as each of them will cut an arc of geodesic issuing from a at one point P only, and, moreover, the Jacobian (30) does not vanish on the arc aP.

Instead of considering the Jacobian (30), we could take any geodesic from a as a function of $m-1$ parameters $\lambda_1, \dots \lambda_{m-1}$ defining its initial direction, each point on one of these geodesics being thus defined by a system of values of $\lambda_1, \dots \lambda_{m-1}, s$. The Jacobian

$$(30\,a) \qquad \mathbf{J} = \frac{D(x_1, \dots x_m)}{D(\lambda_1, \dots \lambda_{m-1}, s)}$$

would play the same part as the Jacobian (30).

58. Equation of the characteristic conoid. Having thus defined the auxiliary quantities P and q, we form the expression

$$\Gamma = \mathbf{A}(P_1, \dots P_m;\ x_1, \dots x_m) = \mathbf{A}(q_1, q_2, \dots q_m;\ a_1, a_2, \dots a_m).$$

This is a quadratic form in the q's, with constant coefficients, and a holomorphic function of the x's; its expansion in powers of the $(x_i - a_i)$'s begins with terms of the second degree, viz.

$$(31) \qquad \Gamma = \mathbf{H}_0(x_i - a_1, \dots x_m - a_m) + \dots.$$

Γ is in fact *the square of the geodesic distance from point* $(x_1, x_2, \dots x_m)$ *to point* $(a_1, a_2, \dots a_m)$, this distance being calculated by means of the linear element $\mathbf{H}$ as defined in § 56.

This enables us to evaluate the partial derivatives of Γ: for those of $\sqrt{\Gamma}$ are given by the classic equation in Calculus of Variations*

$$\delta(\sqrt{\Gamma}) = \sum_i \frac{\partial\sqrt{\mathbf{H}}}{\partial x_i'}\,\delta x_i = \frac{1}{\sqrt{\mathbf{H}(x')}}\,(p_1\delta x_1 + \dots + p_m \delta x_m).$$

We thus see, on account of $\Gamma = \mathbf{A}(P) = s^2\mathbf{A}(p) = s^2\mathbf{H}(x')$, that the partial derivative of Γ with respect to x_i is no other than† $2sp_i = 2P_i$ and that the function Γ *is a solution of the partial differential equation of the first order*

$$(32) \qquad \mathbf{A}\left(\frac{\partial\Gamma}{\partial x_i};\ x_i\right) = 4\Gamma.$$

* See Bolza, *Lectures on the Calculus of Variations* (Decennial publications of the University of Chicago), formula 15 *b* (p. 123) and two last formulæ of p. 154; or our *Lecons sur le Calcul des Variations*, book II, ch. III, p. 142.

† Therefore the tangent plane to any surface Γ = const. is transversal (§ 40) to the corresponding geodesic: a general fact, besides, in the Calculus of Variations (see our *Leçons sur le Calcul des Variations*, § 137).

The expression Γ is symmetrical with respect to the two points $(x_1, x_2, \ldots x_m)$ and $(a_1, a_2, \ldots a_m)$ on which it depends.

$\Gamma = 0$ *is the equation of the characteristic conoid.*

If the normal variables ξ are taken as Cartesian coordinates, the characteristic conoid is an ordinary quadratic cone (or rather hypercone) which is real for a hyperbolic equation. As stated in Book I, *when, moreover, the equation is normal,* it consists of two sheets and divides space into three regions, two of which are interior and one exterior.

These qualitative properties also hold in the primitive space where the coordinates are $x_1, x_2, \ldots x_m$, as the punctual transformation between the x's and the ξ's is a regular one. We can speak, therefore, of the two sheets of the characteristic conoid, or, as we shall often say more briefly, of the two *half conoids* with any given vertex a.

We generally write the equation in such a way that $\Gamma > 0$ corresponds to the interior regions, i.e. that the characteristic form consists of one positive and $m-1$ negative squares.

59. Lamé-Beltrami's differential parameters for Γ. The above equation can be written in Lame's notation of differential parameters*, viz.

$$\Delta_1 \Gamma = 4\Gamma,$$

the left-hand side being the differential parameter of the first order, with respect to the linear element $\mathbf{H}$; a result which, besides, may also be considered as a mere consequence of the well-known equation†

$$(32') \qquad \Delta_1 (\sqrt{\Gamma}) = 1,$$

satisfied by the geodesic distance. We can get a second useful formula by using the m relations

$$(33) \qquad \frac{\partial \Gamma}{\partial x_i} = 2sp_i,$$

together with the differential equations (L_1) (§ 55). If we multiply these by the corresponding derivatives $\frac{\partial U}{\partial x_i}$ of an arbitrary function U and add the m products thus obtained, we find

$$\frac{dU}{ds} = \sum_i \frac{1}{2} \frac{\partial U}{\partial x_i} \frac{\partial \mathbf{A}}{\partial p_i},$$

* See Darboux, *Leçons sur la théorie des surfaces*, vol. III, book VII, ch. I.

† Darboux's *Leçons*, vol. II, book V, ch. V.

which, in Lamé's notation, (33) gives

$$(34) \qquad \Delta_1(\Gamma, U) = 2s\frac{dU}{ds}.$$

Finally, this last result enables us to find the value of the parameter of the second order $\Delta_2\Gamma$. We know indeed that the latter can be defined, after Beltrami* by the existence of the integral identity

$$(35) \quad \mathsf{SSS}\,\Delta_1(\Gamma, U)\,dx_1 dx_2 \ldots dx_m + \mathsf{SSS}\,U\Delta_2\Gamma\,dx_1 dx_2 \ldots dx_m = \mathsf{SS} \ldots,$$

where the m-uple integrals on the left-hand side are extended over a portion of the m-dimensional space; the $(m-1)$-uple one on the right-hand side over its boundary, the quantity replaced by dots under SS being† the product of U by a linear combination of first derivatives of Γ.

We shall use this by transforming the first SSS by tne introduction of the previously defined coordinates $s, \lambda_1, \lambda_2, \ldots \lambda_{m-1}$. Denoting again by J the Jacobian of $x_1, x_2, \ldots x_m$ with respect to these parameters, this first integral becomes

$$\mathsf{SSS}\,2s\frac{\partial U}{\partial s}\,|J|\,d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}\,ds,$$

which can be directly transformed into

$$\begin{aligned}(36) \quad & \pm\,\mathsf{SS}\,2U|J|\,s\,d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1} \\ & \qquad - \mathsf{SSS}\,2U\frac{\partial(|J|s)}{\partial s}\,d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}ds \\ & = \pm\,\mathsf{SS}\,2U|J|\,s\,d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1} \\ & \qquad - \mathsf{SSS}\,2U.\frac{1}{J}\frac{d(Js)}{ds}\,dx_1 dx_2 \ldots dx_m.\end{aligned}$$

The coefficient of U in the m-uple integral of this final expression is necessarily the required value of $\Delta_2 J$

$$(37) \qquad \Delta_2\Gamma = 2\left(1 + s\frac{d\log J}{ds}\right).$$

* See Darboux, *loc. cit.* vol. III, § 674. Strictly speaking, we ought to define the symbol Δ_2 a little differently, by taking as a factor, in the element of each of the SSS in (35), not $dx_1 dx_2 \ldots dx_m$, but an element of volume, equal to $dx_1 dx_2 \ldots dx_m$ multiplied by $\sqrt{|D|} = \frac{1}{\sqrt{|\Delta|}}$. In our case, this would have been an unnecessary complication. It would lead to writing $\log\frac{J}{\sqrt{\Delta}}$ instead of $\log J$ in (37).

† This quantity can be easily seen, by (36), to be $\pm 2JsU$; but this is a useless verification, as, on account of the fundamental lemma of the Calculus of Variations, no two transformations of the form (35) for the same quantity can exist (and be valid with U arbitrary) without coinciding term by term.

60. It may be useful to note that equation (32) characterizes our present form of the equation of the characteristic conoid; that is, any function holomorphic* around a vanishing on the conoid and satisfying (32) is no other than Γ itself.

For such a quantity ought to be of the form

$$\Gamma\Pi$$

(Π again holomorphic). Substituting in (32), we have

$$4\Pi\Gamma = \Pi^2\Delta_1\Gamma + 2\Pi\Gamma\Delta_1(\Pi, \Gamma) + \Gamma^2\Delta_1\Pi,$$

or (on account of (34) and noticing that the equation is satisfied for $\Pi = 1$)

$$\Pi + s\frac{d\Pi}{ds} + \frac{\Gamma}{4\Pi}\Delta_1\Pi - 1 = \frac{d}{ds}[s(\Pi - 1)] + \frac{\Gamma}{4\Pi}\Delta_1\Pi = 0.$$

This shows us that $\Pi = 1$ all over the conoid: whence $\Pi = 1 + \Gamma^l R$, l being a positive index and the new holomorphic function R not vanishing on all the surface of the conoid. But this would imply a contradiction, as, substituting $\Pi - 1 = \Gamma^l R$ in the above equation, we should find that, on our conoid,

$$s\frac{dR}{ds} + (2l+1)R = 0,$$

which admits of no other *regular* solution than zero. Q.E.D.

61. Construction of the elementary solution. This being understood, let us come to the problem which we have in view, and seek to find, for the given equation, a solution of the form

$$a = U\Gamma^p, \tag{38}$$

Γ being the function which we have just formed, in which the pole a, with coordinates $a_1, a_2, \ldots a_m$, will be considered as given, and the point x $(x_1, x_2, \ldots x_m)$ as variable. We shall for the present take only the analytic case, so that the coefficients are assumed to be holomorphic in the x's.

We again join x to a by a geodesic line, on which we have

$$\frac{dx_1}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_1}} = \frac{dx_2}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_2}} = \cdots = \frac{dx_m}{\frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_m}} - ds. \tag{L$_1$}$$

Let us write down, under these conditions, equation (13): $\mathbf{A}_1$, as defined by (13 a), is identically equal to 4, from equation (32). As for the quantity

$$M = \Sigma A_{ik}\frac{\partial^2\Gamma}{\partial x_i \partial x_k} + \Sigma B_i\frac{\partial \Gamma}{\partial x_i},$$

* On the other hand, non-holomorphic solutions exist in infinite number, viz. the square of the geodesic distance (still calculated with respect to $\mathbf{H}$) from $(x_1, \ldots x_m)$ to any surface inscribed in the conoid.

formula (31) gives us its value at the origin; we have

$$\frac{\partial^2 \Gamma}{\partial x_i \partial x_k} = 2H_{ik} + \ldots,$$

and, in consequence (the derivatives $\frac{\partial \Gamma}{\partial x_i}$ being initially zero),

$$M = 2 \sum_{i,k} A_{i,k} H_{i,k} + \ldots = 2m + \ldots.$$

The π_i's in (13), i.e., the derivatives of Γ, have to be replaced by $2P_i = 2sp_i$.

Therefore (dividing (13) by $p\Gamma^{p-1}$ after having replaced G by Γ), we have

$$(39)\quad 2\sum \frac{\partial U}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} + (M + 4p - 4)\,U + \frac{\Gamma}{p}\mathcal{F}(U)$$

$$= 2\sum \frac{\partial U}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} + (2m + 4p - 4 + \ldots)\,U + \frac{\Gamma}{p}\mathcal{F}(U) = 0,$$

and, in consequence, for $\Gamma = 0$, regard being had for (34),

$$(39')\quad 2s\frac{dU}{ds} + \left(\frac{M}{2} + 2p - 2\right)U = 2s\frac{dU}{ds} + (m + 2p - 2 + \ldots)\,U = 0.$$

Since U must be a regular function of s, *this equation is possible only if we have*

$$(40)\qquad p = -\frac{m-2}{2} - p_1,$$

p_1 being a positive integer or zero; U is then, for s in the neighbourhood of zero, of the order of s^{p_1}. For $p_1 = 0$, and therefore

$$(40')\qquad p = -\frac{m-2}{2},$$

U will have, at the point a, a value other than zero. We shall take this as equal to $\frac{1}{+\sqrt{|\Delta_a|}}$, the reciprocal of the square root of the discriminant of $\mathbf{A}$ at the point a.

The corresponding solution u is the only one that we shall need to consider, because the others, deduced from $p_1 > 0$, may easily be reduced to the first. For u, when once obtained, will be a function, not only of the x's, but also of the a's, and it is an analytic function of those quantities*. The quantities

$$\frac{\partial u}{\partial a_1}, \frac{\partial u}{\partial a_2}, \dots \frac{\partial u}{\partial a_m}$$

are solutions of the given equation, and it is immediately evident that they possess, at the point a, the singularity corresponding to $p_1 = 1$. We should obtain in the same way† the solution corresponding to values of p_1 following this, by differentiating again with respect to the a's.

But, from what we have seen before, there is a set of values of m for which none of the above solutions exist (generally, at least), and for which, consequently, *the problem is generally impossible*: these are the even values, the number p then becoming a negative integer. We shall, of course, meet this impossibility again in the course of the process that will determine the solution.

62. To carry this out, we should note that equation (39′) gives us the values of U on our conoid. We have (since U is equal to $\frac{1}{+\sqrt{|\Delta_a|}}$ at the vertex)

$$U = \frac{1}{\sqrt{|\Delta_a|}} e^{-\int_0^s \frac{1}{2s}\left(\frac{M}{2} + 2p - 2\right) ds}. \tag{41}$$

Let us determine a function U_0 that shall be equal *throughout our space* (or the portion of space where Γ is defined) to the above expression; in other words, that fulfils, through the whole of this space, equation (39′). U_0 will be a holomorphic function of the x's, as is immediately evident if the a's are taken as variables (compare equation (45a) below). We shall obviously get

$$U = U_0 + \Gamma U_1,$$

U_1 being a regular function.

* This property of u may be considered as almost evident under our present hypothesis that A_{ik}, B_i, C are holomorphic; it will follow quite strictly, at least for m odd, from the fact that each term of the series (43) (see below) satisfies this requirement, and that, on the other hand, this series is uniformly convergent. The same will hold for m even, if proper precision is given to the definition of u.

† Picard's solutions with simultaneously polar and logarithmic singularity for $m=2$ (*Comptes rendus Ac. Sc.* vol. CXXXVI, p. 1293, June, 1903) also result from the elementary solution, such as found in § 46, by the operation in the text.

Replacing U by that value in equation (39), we shall see that U_1 ought to satisfy the equation

$$(42)\quad 2(p+1)\Sigma_i \frac{\partial U_1}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} + (p+1)(M+4p)U_1 + \Gamma\mathcal{F}(U_1) + \mathcal{F}(U_0)$$

$$= (p+1)\left[4s\frac{dU_1}{ds} + (M+4p)U_1\right] + \mathcal{F}(U_0) + \Gamma\mathcal{F}(U_1) = 0.$$

We shall determine a function U_1 by the equation

$$(42')\qquad 4s\frac{dU_1}{ds} + (M+4p)U_1 + \frac{1}{p+1}\mathcal{F}(U_0) = 0,$$

assumed to be satisfied throughout space. This equation (though a differential one) admits of one regular solution and one only; for it is written (with due regard for the fact that U_0 is a solution of (39′)),

$$\frac{d}{ds}\frac{sU_1}{U_0} = -\frac{1}{4(p+1)U_0}\mathcal{F}(U_0),$$

and, if U_1 must remain finite for $s=0$, this necessarily gives

$$U_1 = -\frac{U_0}{s}\int_0^s \frac{1}{4(p+1)}\frac{\mathcal{F}(U_0)}{U_0}ds;$$

U_1 is like U_0, a holomorphic function of the q's and therefore of the x's.

The rest of the working is now obvious. We shall set down

$$(43)\qquad U = U_0 + \Gamma U_1 + \ldots + \Gamma^h U_h + \ldots,$$

and the expansion thus written will give a solution of the problem if the U_h's are given by the successive equations [where each left-hand member is the coefficient of $(p+h)\Gamma^{p+h-1}$ in $\mathcal{F}(U\Gamma^h)$]*

$$(44)\quad 4s\frac{dU_h}{ds} + [M+4(p+h-1)]U_h + \frac{1}{p+h}\mathcal{F}(U_{h-1}) = 0,$$

whence

$$(44')\qquad U_h = -\frac{U_0}{4(p+h)s^h}\int_0^s \frac{s^{h-1}}{U_0}\mathcal{F}(U_{h-1})\,ds.$$

* We see that our method would allow us to construct the (unique) solution of the form $u - U\Gamma^{p'+1}$ for any given partial differential equation such as $\mathcal{F}(u) = W\Gamma^{p'}$ (with W holomorphic) provided p' be equal to none of the numbers $-\frac{m-2}{2}-1, -\frac{m-2}{2}-2, \ldots -\frac{m-2}{2}-p_1, \ldots.$ On the contrary, when p' assumes one of the last-mentioned values, the equation admits of no solution of the form $U\Gamma^{p'+1}$, as appears from the text (nor, as can also be seen, of any algebroid solution whatever).

If m is odd, and, consequently, p not an integer, all the $(p+h)$'s will be equal to integers increased by $\frac{1}{2}$: therefore, all the expressions (44′) will exist. They will be holomorphic functions: if we assume that the a's (or Lipschitz's normal variables, which comes to the same thing) have been taken as independent variables and that the quantity $\frac{1}{U_0}\mathcal{F}(U_{h-1})$ has the expansion

$$(45) \qquad \frac{1}{U_0}\mathcal{F}(U_{h-1}) = Q_0 + Q_1 + Q_2 + \dots + Q_k + \dots,$$

where Q_0, Q_1, ... are homogeneous polynomials with respect to the variables thus chosen (their degrees being denoted by their suffixes), we shall have

$$(45a) \qquad \frac{U_h}{U_0} = -\frac{1}{4(p+h)h}Q_0 - \frac{1}{4(p+h)(h+1)}Q_1 - \dots - \frac{1}{4(p+h)(h+k)}Q_k - \dots,$$

a similar expression,—in which h has to be replaced by zero and the right-hand side of (45) by $\left(\frac{M}{2} - m\right)$—applying also to $\log U_0$, on account of (41) (and with the addition of the constant term—$1/2 . \log|\Delta_a|$).

If the coefficients be merely regular (§ 9), we shall nevertheless be able to say that the U_h's exist, with the above definition and properties, up to a certain value of h.

63. Assuming the coefficients to be holomorphic, we have now to prove the convergence of the power series (43).

For this object, this time, we shall find directly (and no longer by comparison) dominant functions for the coefficients.

Let us still suppose that, for the x s, we have taken normal variables (the q's or Lipschitz's normal variables) relating to our given pole a. Moreover, we shall bring in a new simplification in our calculation by a change of unknown. Having determined U_0 as said above, we shall instead of (E) introduce the new equation

$$(E_1) \qquad \mathcal{F}_1(u) = \frac{1}{U_0}\mathcal{F}(U_0 u) = 0.$$

It is clear that any solution of (E_1) is deduced from a corresponding solution of (E) by division by U_0, and also that the elementary solutions of the two equations are connected in the same way; and it is easily verified (we do the calculation in detail further on, in Book IV) that this applies to every term in the expansions of the two numerators. Therefore, for (E_1), the new value of U_0 will be the constant

$$U_0 = \frac{1}{\sqrt{|\Delta_a|}}.$$

Let σ be the sum of the absolute values of these variables. Every coefficient of the given equation (E_1) will admit (if the positive constants α, r be properly chosen) of the dominant

$$\frac{\alpha}{1-\frac{\sigma}{r}},$$

so that, if we have ($\ll$ being, as usual, a sign for dominated functions)

$$v \ll \frac{K}{\left(1-\frac{\sigma}{r}\right)^n}, \tag{46}$$

we shall also have

$$\mathcal{F}(v) \ll \frac{\alpha' K n(n+1)}{\left(1-\frac{\sigma}{r}\right)^{n+3}}. \qquad \alpha' = \alpha\left[\left(1+\frac{m}{r}+\frac{m^2}{r^2}\right)\right] \tag{47}$$

Under these conditions, let us show that we can write

$$U_h \ll \frac{K_h}{\left(1-\frac{\sigma}{r}\right)^{2h}}, \tag{48}$$

where the K's are positive numbers which we shall calculate presently.

Let us assume that (48) is satisfied for a certain value of h, and try to prove it by changing h into $h+1$. We have (remembering that U_0 is a constant)

$$\mathcal{F}(U_h) \ll \frac{\alpha' K_h \,.\, 2h(2h+1)}{\left(1-\frac{\sigma}{r}\right)^{2h+3}},$$

whence, for U_{h+1},

$$U_{h+1} \ll \frac{\alpha' K_h \,.\, 2h(2h+1)}{4(p+h+1)} \cdot \frac{1}{s^{h+1}} \int_0^s \frac{s^h ds}{\left(1-\frac{\sigma}{r}\right)^{2h+3}}.$$

The factor

$$\frac{1}{s^{h+1}}\int_0^s \frac{s^h ds}{\left(1-\frac{\sigma}{r}\right)^{2h+3}}$$

can be written (as σ is proportional to s on every geodesic)

$$\frac{1}{\sigma^{h+1}}\int_0^\sigma \frac{\sigma^h d\sigma}{\left(1-\frac{\sigma}{r}\right)^{2h+3}}$$

and, as may easily be seen*, is dominated, as $2h+3>h+2$, by

$$\frac{1}{h+1}\frac{1}{\left(1-\frac{\sigma}{r}\right)^{2h+2}}.$$

Therefore

$$U_{h+1} \ll \frac{\alpha' K_h\, 2h(2h+1)}{4(h+1)(p+h+1)}\frac{1}{\left(1-\frac{\sigma}{r}\right)^{2h+2}}.$$

This is of the required form

$$U_{h+1} \ll \frac{K_{h+1}}{\left(1-\frac{\sigma}{r}\right)^{2(h+1)}},$$

with

$$K_{h+1}=K_h\,.\,\alpha'\frac{2h(2h+1)}{4(h+1)(p+h+1)}.$$

The ratio $\frac{K_{h+1}}{K_h}$ approaching, for $h=\infty$, the finite limit α', the series (43) will converge for $|\Gamma|<\frac{1}{\alpha'}\left(1-\frac{\sigma}{r}\right)^2$.

The existence of u is therefore completely proved.

We may add that, if we let the point a vary within any region (strictly interior to $\mathscr{R}$), the numbers r, α will have, the first a minimum, the other a maximum, so that the convergence of (43) will be a uniform one.

* For $\frac{1}{h+1}\left[\frac{\sigma^{h+1}}{\left(1-\frac{\sigma}{r}\right)^{l-1}}\right]'=\frac{\sigma^h}{\left(1-\frac{\sigma}{r}\right)^l}\left[1-\frac{\sigma}{r}+\frac{l-1}{h+1}\frac{\sigma}{r}\right]\gg\frac{\sigma^h}{\left(1-\frac{\sigma}{r}\right)^l}$ (if $l\geqslant h+2$)

64. We may immediately notice that the above analysis applies without any modification to the determination of the holomorphic solution of (E) assuming given values on the characteristic conoid, provided we already know any holomorphic function $\mathcal{U}_0$ which assumes these same values for $\Gamma = 0$. Writing

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1\Gamma + \ldots + \mathcal{U}_k\Gamma^k + \ldots, \tag{49}$$

the equations for the successive $\mathcal{U}_k$ will be those above in which we take $p = 0$, viz.

$$4s\frac{d\mathcal{U}_1}{ds} + M\mathcal{U}_1 + \mathcal{F}(\mathcal{U}_0) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$4s\frac{d\mathcal{U}_k}{ds} + (M + 4k - 4)\mathcal{U}_k + \frac{1}{k}\mathcal{F}(\mathcal{U}_{k-1}) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

or, again integrating with the help of U_0,

$$\mathcal{U}_1 = -\frac{U_0}{4s^{1-p}}\int_0^s \frac{s^{-p}}{U_0}\mathcal{F}(\mathcal{U}_0)\,ds, \tag{49'}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathcal{U}_k = -\frac{U_0}{4k\,.\,s^{k-p}}\int_0^s \frac{s^{k-p-1}}{U_0}\mathcal{F}(\mathcal{U}_{k-1})\,ds.$$

Such a problem admits, therefore, of one (and only one) holomorphic solution.

65. Let us now assume m to be even: for instance, $m = 4$, whence $p = -1$. U_0 still assuming the value (41), equation (42) becomes an impossibility if we have not

$$\mathcal{F}(U_0) = 0$$

along the whole characteristic conoid.

It is clear that this condition will not be satisfied in general. If, for instance, all the coefficients of the equation were given with the exception of C, it would make known to us the values of C on the whole conoid having a for its vertex (as the expression of U_0 is independent of C and different from zero*).

* The conditions for this to take place (and consequently for an elementary solution of the form (38) to exist) for every situation of a, would require a much more difficult investigation: an investigation, however, which would be especially interesting, as we shall see further on. We may add that an important part would be played in this by the value (37) obtained above (§ 59) of $\Delta_2\Gamma$.

A similar conclusion will evidently follow for any other even m, the impossibility arising from the equation (44) which corresponds to $h = -p = \frac{m-2}{2}$, viz.,

$$(50) \qquad \mathcal{F}(U_{-p-1}) = 0.$$

Picard's previous results (see § 46) then lead us to complete expression (37) by the addition of a logarithmic term, setting down

$$(51) \qquad u = U\Gamma^p - \mathcal{U} \log \Gamma.$$

If we substitute this new value of u, we find

$$\mathcal{F}(U\Gamma^p) - \left[2\Sigma \frac{\partial \mathcal{U}}{\partial x_i} \frac{\partial \mathbf{A}}{\partial P_i} + (M-4)\,\mathcal{U}\right] \frac{1}{\Gamma} - \log \Gamma \,.\, \mathcal{F}(\mathcal{U}) = 0,$$

where the first term has already been expanded in powers of Γ.

Again, the logarithmic terms only disappear if $\mathcal{U}$ itself is a solution* of (E).

Further, taking account of the expansion previously written for $\mathcal{F}(u)$, we see that the equations (44) corresponding to $h < -p$ are not modified. But, for $h = -p$, that is, to make the coefficient of $\frac{1}{\Gamma}$ vanish, we have to write, instead of (44),

$$(52) \qquad 4s \frac{d\mathcal{U}}{ds} + (M-4)\,\mathcal{U} - \mathcal{F}(U_{-p-1}) = 0,$$

an equation which we shall immediately notice to be of the same form as the formulæ (44′), only differing from these by the omission of the denominator (which ought to have been zero) and a change of sign.

Let $\mathcal{U}_0$ be the function defined, throughout our region $\mathcal{R}$, by the differential equation (52). $\mathcal{U}$ must be the solution of (E) which assumes on the conoid the same value as $\mathcal{U}_0$.

We have just seen how this function $\mathcal{U}$ is to be determined. The coefficients of its expansion (49) in powers of Γ depend on the equations (49′). We again see that these are exactly the same for the U_h's, with $h = p + k$.

* This proves that we do not have to look for a solution of the form

$$U\Gamma^p + \mathcal{U}\Gamma^q \log \Gamma$$

with q not zero, as the solution $\mathcal{U}\Gamma^q$ cannot exist. Algebroid and logarithmic infinitudes do not multiply each other in the present problem.

In other words, *we are led to processes exactly identical with the previous ones* (except for a numerical coefficient, the change of which is introduced by the calculation of $\mathscr{U}_0$), U_{-p+k} for $k = 0, 1, 2, \ldots$, being now denoted by $\mathscr{U}_k$.

Now, conversely, if we calculate the U_h's for $h = 0, 1, 2, \ldots, (-p-1)$ and $\mathscr{U}$ as explained above, and if we substitute in the given equation the expression

$$-\mathscr{U}\log\Gamma + \Gamma^p \sum_{h=0}^{-p-1} U_h \Gamma^h, \tag{53}$$

it follows from what we have said that the terms in $\Gamma^{p-1}, \ldots, \frac{1}{\Gamma}$ and $\log\Gamma$ will disappear. The result of substitution, denoted by $\mathscr{M}$, will therefore be a holomorphic function, and all that remains to be done is to add to expression (53) any holomorphic solution w of the equation

$$\mathscr{F}(w) = -\mathscr{M}$$

(the existence of which follows from Cauchy's fundamental theorem) to obtain a solution

$$u = -\mathscr{U}\log\Gamma + U\Gamma^p, \quad \left(U = w\,\Gamma^{-p} + \sum_{h=1}^{-p-1} U_h \Gamma^h\right)$$

of the proposed equation*.

In this case, contrary to what happened for m odd, there is a great degree of indetermination in our result, as w can be modified by the addition of any regular solution of (E).

* We could just as well find w by writing it

$$w = W_{-p} + W_{-p+1}\Gamma + \ldots + W_{-p+k}\Gamma^k + \ldots,$$

substituting the total value of u in the equation and equating to 0 the coefficients of the powers of Γ superior to $-p-1$, which have not been considered as yet: this gives the successive W_{-p+k} by

$$4s\frac{dW_{-p+k}}{ds} + (M + 4k - 4)W_{-p+k} + \frac{1}{k}\left[\mathscr{F}(W_{-p+k-1}) - 4s\frac{d\mathscr{U}_k}{ds} - (M + 8k - 4)\mathscr{U}_k\right] = 0$$

from W_{-p+1} on, while W_{-p} remains arbitrary, affording the indetermination referred to in the text.

The result will depend analytically on the coordinates of the pole if we are careful to choose this arbitrary element (for each situation of a) according to a determinate analytic law: for instance, if we agree to take $U_{-p} = 0$.

We have thus succeeded in calculating our elementary solution (for any non-parabolic equation) upon the hypothesis that the coefficients are analytic. How the same results can be attained without the help of this hypothesis, will be seen further on.

66. *Application to the elliptic case.* Still keeping to the analytic hypothesis, all the above applies to the elliptic and hyperbolic cases. In the future, we shall deal exclusively with the latter; but let us note that the existence of the elementary solution is the basis on which we can establish the theory of elliptic equations with analytic coefficients, extending to it the main properties met with for $\nabla^2 u = 0$. We can enunciate at once, for our general case, the properties obtained for two variables by Sommerfeld*, such as:

An elliptic equation with analytic coefficients has none but analytic solutions (inside their domain of existence, boundary excluded);

If two solutions of such an equation are tangent to each other† *along a surface, they are the analytic extension of each other,*

for the proof of which we only have to replace $\log \frac{1}{r}$ or $\frac{1}{r^{m-2}}$ by the elementary solution, in the classic argument‡; further, by the consideration of functions analogous to Green's||,

For an area such that the problem of determining therein a solution of the adjoint equation by its boundary values is always possible, this problem is determinate for the given equation;

etc....

67. The parabolic case remains outside the above analysis. The part of the elementary solutions is played in that case by a quantity whose value

$$(54) \qquad \frac{1}{\sqrt{y}} e^{-\frac{x^2}{4y}} \quad \text{or} \quad \frac{1}{\sqrt{t}} e^{-\frac{x^2+y^2+z^2}{4t}}$$

* *Encyclopädie der Math. Wissensch.* II A, 7 c. † See note †, p. 18.

‡ These follow from the fact that the elementary solution is an analytic function not only of the x's but also of the a's.

|| Here, we may notice that in the elliptic case, quantities deduced from the elementary solution by the addition of regular solutions of (E), such as Green's functions, may be considered as indeterminate (as long as boundary conditions are not used) as well for m odd as for m even; which will not be the case in our further operations concerning hyperbolic equations

is well known for the classic equation of heat $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$ or $\nabla^2 u = \frac{\partial u}{\partial t}$. Its extension to the more general equation containing the same terms of the second order with terms of the first has been obtained by Gevrey* and ourselves. As the parabolic equation has also been treated in a masterly way by Volterra in his Stockholm *Lectures*, we shall not go into further details about it. We simply refer to the works just quoted, and also note that it would be possible to obtain the elementary solution for the parabolic equation—even for the most general one—by a limiting process which would deduce it from an elliptic or hyperbolic one in which coefficients would vary so that one square in the characteristic form should approach zero. For instance, we easily obtain thus the first expression (54) as a limiting value of Riemann's function.

Let us take the simplest case of the equation of heat

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0. \tag{55}$$

We consider it as a limiting case (for $k = 0$) of

$$\frac{\partial^2 u}{\partial x^2} + k\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} = 0, \tag{55'}$$

(k being a constant) which we can refer to its characteristic by introducing the new variables

$$y = Y, \quad x - \frac{y}{k} = X.$$

It thus takes the form (practically equivalent to what is called "telegraphist's equation")

$$k\frac{\partial^2 u}{\partial X \partial Y} + \frac{1}{k}\frac{\partial u}{\partial X} - \frac{\partial u}{\partial Y} = 0,$$

for which Riemann's function—x_0, y_0 being, for simplicity's sake, taken equal to 0—is†

$$e^{\frac{X}{k} - \frac{Y}{k}} J_0\left(2\sqrt{\frac{XY}{k^3}}\right),$$

* See *Comptes Rendus Ac. Sc.* vol. CLII, 1911, and Gevrey's *Thesis*, Paris, 1913, chap. V.

† See below, § 69.

J_0 being Bessel's integral transcendental function

$$J_0(\xi) = 1 - \frac{\xi^2}{2^2 . 1} + \frac{\xi^4}{2^4 . (2!)^2} - \ldots + (-1)^h \frac{\xi^{2h}}{2^{2h}(h!)^2} + \ldots.$$

This gives in the former variables

$$(56) \qquad e^{\frac{x}{k} - \frac{2y}{k^2}} J_0\left(\frac{2}{k^2}\sqrt{y(kx-y)}\right).$$

Now, when k approaches zero, the argument in J_0 becomes infinite, a case for which Bessel's function admits of a well-known asymptotic evaluation, viz.

$$J_0(i\eta) \sim \frac{e^\eta}{\sqrt{2\pi\eta}}.$$

If, on the other hand, we expand $\sqrt{y(kx-y)}$ up to terms in k^2 $\left[\text{viz. } \frac{i}{2}\left(2y - kx - \frac{k^2x^2}{4y}\right) + \ldots\right]$, we see that (56) practically reduces to

$$\frac{k}{\sqrt{4\pi}} \frac{1}{\sqrt{y}} e^{-\frac{x^2}{4y}}$$

and this is precisely the fundamental quantity used in the theory of equation (55) but for the presence of the factor $\frac{k}{\sqrt{4\pi}}$ (which is removed by a corresponding denominator when substituted in our formula of § 42).

68. General conclusions. Summing up what we found for the non-parabolic case, we see that:

A non-parabolic (analytic) linear partial differential equation of the second order, with m independent variables, admits of an elementary solution, with an arbitrary point of the m dimensional space as its pole.

$\Gamma = 0$ *being the equation of the characteristic conoid with vertex a, this elementary solution is of the form* $\frac{U}{\Gamma^{\frac{m-2}{2}}}$ *(denoting by U a holomorphic function which assumes, at the pole a itself, the value* $\frac{1}{\sqrt{\Delta}}$*) for m odd; it is of the form* $\frac{U}{\Gamma^{\frac{m-2}{2}}} - \mathcal{V} \log \Gamma$ *(denoting by* $\mathcal{V}$ *another holomorphic function which may be zero) for m even.*

In the first case, its expression is a quite determinate one.

69. Some familiar examples. The elementary solution of $\nabla^2 u = 0$, for $m > 2$ variables, is

$$\frac{1}{r^{m-2}} = \frac{1}{\left[\sum_{i=1}^{m} (x_i - a_i)^2\right]^{\frac{m-2}{2}}}.$$

This (the distinction between real and imaginary not as yet arising) gives us at once the corresponding result for (e_2), (e_3'), ... and in general for any equation (equations $\Delta^{p,q} u = 0$ of Coulon) of the form

$$\Delta^{p,q} u = \left(\sum_{h=1}^{p} \frac{\partial^2}{\partial x_h^2} - \sum_{k=1}^{q} \frac{\partial^2}{\partial y_k^2}\right) u = 0,$$

viz.
$$u = \frac{1}{(\sqrt{\pm[\sum_h (x_h - a_h)^2 - \sum_k (y_k - b_k)^2]})^{m-2}}$$

for $\Delta^{p,q} = 0$ and, e.g., for (e_2)

$$u = \frac{1}{\sqrt{\omega^2 (t - t_0)^2 - (x - x_0)^2 - (y - y_0)^2}}.$$

For the slightly more general equation of *damped waves**,

$$(57) \qquad \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x^2_{m-1}} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} + Ku = 0,$$

the corresponding result can be easily reached by a simple generalization of this calculation. Setting down first $\omega t \sqrt{-1} = x_m$, $\omega t_0 \sqrt{-1} = a_m$, so that the equation becomes

$$\frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x^2_{m-1}} + \frac{\partial^2 u}{\partial x_m^2} + Ku = 0$$

and
$$\rho^2 = \sum_1^m (x_h - a_h)^2,$$

* We remind the reader that any (non-parabolic) linear equation of the second order with constant coefficients can be brought to the form in the text, because (*a*) a linear transformation on the variables brings it to the form

$$\sum_i \frac{\partial^2 u}{\partial x_i^2} + \Sigma a_i \frac{\partial u}{\partial x_i} + \beta u = 0,$$

and (*b*) the coefficients a_i are reduced to zero by a change of the unknown, viz., we take for u a function of ρ, for which we have the ordinary differential equation of Bessel

$$(58) \qquad \frac{d^2u}{d\rho^2} + \frac{m-1}{\rho}\frac{du}{d\rho} + Ku = 0.$$

This equation possesses the property that, knowing a solution u of it for a special value $m = m_0$ of m, we have a solution u_1 of the equation corresponding to $m = m_0 + 2$, by

$$(59) \qquad u_1 = \frac{1}{\rho}\frac{du}{d\rho}$$

so that we have only to integrate it for $m = 1, 2$.

For $m = 1$, we have $u = \begin{matrix}\cos\\ \sin\end{matrix}\Big\}(\sqrt{K}\rho)$ which gives the solution for $m = 3, 5, \ldots$, viz. for $m = 3$, $u = \frac{1}{\rho}\cdot\begin{matrix}\sin\\ \cos\end{matrix}\Big\}(\sqrt{K}\rho)$ in which—as well as for the following odd values of m—the symbol cos must be taken in order to obtain the required elementary solution. If, according to what we did in § 58 and most frequently shall have to do in the following Books, we write down our results as if every sign were changed in (57) so as to introduce $[\omega^2(t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2]$ in the place of Γ, we thus find, for the equation of damped cylindrical waves

$$(\mathrm{E}_2) \qquad \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - Ku = 0,$$

the elementary solution

$$(60) \qquad u = \omega\frac{\mathrm{Ch}\sqrt{K[\omega^2(t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2]}}{\sqrt{\pm[\omega^2(t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2]}}$$

(where a hyperbolic cosine appears on account of the aforesaid change of sign in Γ).

For $m = 2$, (58) has a holomorphic solution and a logarithmic one, the latter giving our elementary solution. Both are expressed by Bessel's function

$$J_0(\xi) = 1 - \frac{\xi^2}{2^2} + \frac{\xi^4}{2^4.(2!)^2} - \ldots + (-1)^h\frac{\xi^{2h}}{2^{2h}(h!)^2} + \ldots$$

and, in particular, the logarithmic solution of (58) is

$$(61) \qquad J_0(\rho\sqrt{-K})\log\rho + w \qquad [\rho^2 = \omega^2(t-t_0)^2 - (x-x_0)^2]$$

(w holomorphic), so that (with the same remark as to sign) the elementary solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} + Ku = 0$$

is $\quad J_0\left(\sqrt{-K[\omega^2(t-t_0)^2-(x-x_0)^2]}\right)\log[\omega^2(t-t_0)^2-(x-x_0)^2] + w.$

Therefore, for $m = 4$, a solution of (58) will be derived from the preceding one, viz. (61), by the operator (59), which gives

$$\frac{1}{\rho^2} J_0(\rho\sqrt{-K}) + \frac{\sqrt{-K}}{\rho} J_0'(\rho\sqrt{-K}) \log\rho + \text{holomorphic function}$$

$$= \frac{1}{\rho^2}\, j\left(\frac{K\rho^2}{4}\right) + \frac{K}{2}\, j'\left(\frac{K\rho^2}{4}\right)\log\rho + \text{holomorphic function},$$

where $\quad \rho^2 = \omega^2(t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2 - (z-z_0)^2 = \Gamma$

and the integral function j is

$$J_0(2\sqrt{-\lambda}) = j(\lambda) = 1 + \frac{\lambda}{1^2} + \frac{\lambda^2}{(2!)^2} + \dots + \frac{\lambda^n}{(n!)^2} + \dots,$$

with $$j'(\lambda) = \frac{dj}{d\lambda}.$$

This gives us the elementary solution of the equation of *damped spherical waves*

$$(\mathrm{E}_3) \qquad \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - Ku = 0,$$

viz., as the factor of $\frac{1}{\Gamma}$ must be initially $\frac{1}{\sqrt{|\Delta|}} = \omega$,

$$(61\,a) \qquad \frac{\omega j\left(\frac{K}{4}\Gamma\right)}{\Gamma} + \frac{\omega K}{4}\, j'\left(\frac{K}{4}\Gamma\right)\log\Gamma + \text{holomorphic function},$$

which quantity admits of the required singularity for

$$\omega^2(t-t_0)^2 - (x-x_0)^2 - (y-y_0)^2 - (z-z_0)^2 = 0.$$

The numerator in the first term may be simply replaced by the constant ω, as the corresponding alteration in the fraction consists of a holomorphic function. But this would no longer be allowed if we considered the formation of the elementary solution of (57) for $m = 6$, 8, ..., by successive applications of the operator (59) to the above expression.

70. The effect of descent. It will be of interest to see what becomes of those calculations if we apply a "method of descent" such as has been spoken of previously (§ 29).

In other words, simultaneously with equation $\mathcal{F}(u) = 0$ containing m independent variables, we consider the new equation

$$\mathcal{F}_1(u) = \mathcal{F}(u) - \frac{\partial^2 u}{\partial z^2} = 0,$$

where z is a supplementary independent variable which is not contained in any of the coefficients. For this new equation, we shall have to consider again the characteristic conoid $\Gamma = 0$ with vertex $(a_1, a_2 \ldots a_m, c)$, if we denote by c a determinate value of the $(m+1)$th coordinate z. The new characteristic form being

$$\mathbf{A}'(P_1, \ldots P_m, R) = \mathbf{A}(P_1, \ldots P_m) - R^2,$$

the equations (L) have to be completed by the addition of

$$ds = \frac{dz}{-4r} \quad \frac{dr}{0}$$

(r, R being supplementary variables respectively analogous to $p_1, \ldots p_m$; $P_1, \ldots P_m$), which give $r = \text{const.}$, $z - c = sr = R$. Therefore

$$\Gamma' = s^2[\mathbf{A}(p_1, \ldots p_m) - r^2] = \mathbf{A}(P_1, \ldots P_m) - R^2 = \Gamma - (z-c)^2.$$

M', the new value of M (formula (12)), is obviously equal to $M + 2$.

Then, if (c being taken $= 0$) we want to form the function

$$U' = U_0' + U_1'\Gamma' + \ldots = U_0' + U_1'(\Gamma - z^2) + \ldots + U_h'(\Gamma - z^2)^h + \ldots$$

analogous to U, this will compel us to write down the successive equations

$$\left\{\begin{aligned}
&2\sum_{i=1}^{m} \frac{\partial U_0'}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} - 4R\frac{\partial U_0'}{\partial z} + [M' + 4(p - \tfrac{1}{2}) - 4]\, U_0' \\
&= 2\Sigma \frac{\partial U_0'}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} - 4R\frac{\partial U_0'}{\partial z} + (M + 4p - 4)\, U_0' = 0, \\
&2\sum_{i=1}^{m} \frac{\partial U_1'}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} - 4R\frac{\partial U_1'}{\partial z} + [M' + 4p - 2]\, U_1' + \frac{1}{p + \frac{1}{2}}\mathcal{F}_1(U_0') \\
&= 2\Sigma \frac{\partial U_1'}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} - 4R\frac{\partial U_1'}{\partial z} + (M + 4p)\, U_1' + \frac{1}{p + \frac{1}{2}}\mathcal{F}_1(U_0') = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&2\sum_{i=1}^{m} \frac{\partial U_h'}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} - 4R\frac{\partial U_h'}{\partial z} + [M' + 4(p - \tfrac{3}{2} + h)]\, U_h' + \frac{1}{p + h - \frac{1}{2}}\mathcal{F}_1(U'_{h-1}) \\
&= 2\Sigma \frac{\partial U_h'}{\partial x_i}\frac{\partial \mathbf{A}}{\partial P_i} - 4R\frac{\partial U_h'}{\partial z} + [M + 4(p + h - 1)]\, U_h' + \frac{1}{p + h - \frac{1}{2}}\mathcal{F}_1(U'_{h-1}) = 0. \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right.$$

The first of these only differs from equation (39′), which is satisfied by U_0, by the term in $\frac{\partial U_0'}{\partial z}$: therefore, it also is satisfied by U_0; and, as we know that this equation, with the condition that U_0 assumes the value $\frac{1}{\sqrt{|\Delta|}}$ at a, entirely determines it, we see that U_0' is not distinct* from U_0.

In the same way, the second equation, defining U_1' (in which $\mathcal{F}_1(U_0') = \mathcal{F}_1(U_0) = \mathcal{F}(U_0)$), is satisfied by $U_1' = \frac{p+1}{.1} U_1$; and, as its regular solution around a is unique, U_1' must have precisely that value*.

Each successive equation will behave similarly, and we see that *all the U''s are independent of z*, and only differ from the corresponding U's by numerical factors: we have

$$(62)\quad \begin{cases} U_0' = U_0, \quad U_1' = \dfrac{p+1}{p+\frac{1}{2}} U_1 = \dfrac{-\frac{m}{2}+2}{-\frac{m}{2}+\frac{3}{2}} U_1 = \dfrac{\frac{m}{2}-2}{\frac{m}{2}-\frac{3}{2}} U_1; \ \ldots \\[2ex] U_h' = \dfrac{(p+1)(p+2)\ldots(p+h)}{(p+\frac{1}{2})(p+\frac{3}{2})\ldots(p+h-\frac{1}{2})} U_h \\[2ex] \qquad = \dfrac{\left(\frac{m}{2}-2\right)\left(\frac{m}{2}-3\right)\ldots\left(\frac{m}{2}-h-1\right)}{\left(\frac{m}{2}-\frac{3}{2}\right)\left(\frac{m}{2}-\frac{5}{2}\right)\ldots\left(\frac{m}{2}-h-\frac{1}{2}\right)} U_h, \ldots \end{cases}$$

these relations holding until the denominator or numerator (according to the parity of m) becomes zero on account of $p+h=0$ or $p+h-\frac{1}{2}=0$; after that they will hold, only the value of the numerical factor being changed, viz.†

$$(62a)\qquad \mathcal{U}'_{p-\frac{1}{2}+h} = U_h' = \frac{(m_1-\frac{5}{2})(m_1-\frac{7}{2})\ldots(\frac{1}{2})}{(m-2)(m-3)\ldots 1} \cdot \frac{1}{2} \cdot \frac{\frac{3}{2}\cdot\frac{5}{2}\ldots(h-m_1+\frac{3}{2})}{1.2\ldots(h-m_1+1)} U_h,$$

* The same conclusion easily follows if we consider the successive U_h as determined by (45a).

† The change merely consists, as seen above, in replacing the factor 0 by −1 when it occurs in the numerator or denominator in (62), every other factor remaining unchanged.

from $h = \frac{1}{2} - p = m_1 - 1$ on, for m odd $= 2m_1 - 1$; and

$$(6^{\circ}b) \quad U_h' = \frac{(m_1-2)(m_1-3)\dots 1}{(m_1-\frac{3}{2})(m_1-\frac{5}{2})\dots(\frac{5}{2}).(\frac{3}{2})} \cdot \frac{1}{(\frac{1}{2})} \cdot \frac{1.2\dots(h-m_1+1)}{\frac{1}{2}\dots(h-m_1+\frac{1}{2})} U_h$$

(with $U_h = \mathscr{U}_{p+h}$),

from $h = -p - m_1 - 1$ on, for m even $= 2m_1$.

We shall, in the following Books, find this again under a simpler and more instructive form, showing the relations which exist not only between the coefficients U_h, U_h in the expansions of the elementary solutions, but between these elementary solutions themselves.

ADDITIONAL NOTE

ON THE EQUATIONS OF GEODESICS

We have considered above the geodesics, which satisfy Hamilton's equations

$$\text{(L)} \qquad \frac{dx_i}{ds} = \frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_i}, \qquad \frac{dp_i}{ds} = -\frac{1}{2}\frac{\partial \mathbf{A}}{\partial x_i}$$

and especially those which issue from a determinate point $a(a_1, a_2 \dots a_m)$, each of them being individuated by the values of $m-1$ parameters $\lambda_1, \dots \lambda_{m-1}$, so that the coordinates $x_1, x_2, \dots x_m$ are functions of these parameters and s.

We had also to consider the derivatives of these functions not only with respect to s, but with respect to any of the λ's. General theorems, now classic*, show that such partial derivatives

$$\bar{x}_1, \bar{x}_2, \dots \bar{x}_m, \quad \bar{p}_1, \bar{p}_2, \dots \bar{p}_m$$

exist† and, on any determinate geodesic—in other words, for any determinate system of values of the λ's—satisfy the *linear* differential system, "variational equations" in Poincare's terminology (Darboux's "auxiliary system")

$$(\bar{\text{L}}) \qquad \frac{d\bar{x}_i}{ds} = \frac{1}{2}\frac{\partial \bar{\mathbf{A}}}{\partial \bar{p}_i}, \qquad \frac{d\bar{p}_i}{ds} = -\frac{1}{2}\frac{\partial \bar{\mathbf{A}}}{\partial \bar{x}_i} \qquad (i = 1, 2, \dots m),$$

$\bar{\mathbf{A}}$ being a quadratic form in the $\bar{x}$'s, $\bar{p}$'s (viz., the quadratic part of the Taylor expansion of

$$\mathbf{A}(x_1 + \bar{x}_1, x_2 + \bar{x}_2, \dots x_m + \bar{x}_m, \; p_1 + \bar{p}_1, p_2 + \bar{p}_2, \dots p_m + \bar{p}_m)$$

in powers of $\bar{x}_1, \bar{x}_2, \dots \bar{x}_m, \bar{p}_1, \bar{p}_2, \dots \bar{p}_m$). They are indeed the solutions of that system which are such that the $\bar{x}_i$'s are initially 0, as we limit ourselves to geodesics with the common origin a.

The same can be said of higher partial derivatives with respect to the λ's. If x_i, $\bar{p}_i$ denote no longer derivatives of the first order, but derivatives of order h,—say $x_i = \frac{\partial^h x_i}{\partial\lambda_1^{h_1}\partial\lambda_2^{h_2}\dots}$, $\bar{p}_i = \frac{\partial^h p_i}{\partial\lambda_1^{h_1}\partial\lambda_2^{h_2}\dots}$,—such

* See, e.g. Goursat's *Cours d'Analyse*, 2nd edition, vol. III (1913), chap. XXIII, especially § 462; or the author's *Lecons sur le Calcul des Variations*, §§ 20—22.

† The choice of the λ's is assumed to be such that the p_{0i}'s depend regularly on them.

quantities satisfy a linear system (only differing from ($\overline{\mathrm{L}}$) in that they are non-homogeneous)

$$(\overline{\mathrm{L}}') \qquad \frac{d\bar{x}_i}{ds} = \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \bar{p}_i} + X_i, \qquad \frac{d\bar{p}_i}{ds} = -\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \bar{x}_i} + P_i,$$

where the X's, P's depend on previously known derivatives of the x's, p's, that is on derivatives of order less than h, and containing also the coefficients A_{ik} and their partial derivatives up to the $(h+1)$th order. The x_i's are again 0 for $s=0$ and, therefore, at least of the first order in s.

This not only shows that we can speak of the derivatives in question, but also enables us—which will be eventually of use—to obtain upper limits for their absolute values, if we know: (1) their initial values (or at least, corresponding upper limits); (2) upper limits for the absolute values of the A_{ik} and their derivatives up to the order $(h+1)$. As to the first variational system (L), this is a consequence of the known methods used for proving the fundamental theorem in the theory of differential equations*; as to the following systems (L), it can be proved in the same way or, more simply, results from the known integration of a non-homogeneous linear system after the corresponding homogeneous one is integrated.

We also deduce from the above remarks that *the punctual transformation* (§ 57), *which introduces the normal variables instead of the x's, is regular* (up to the same order but one as the A_{ik}'s) *throughout the whole region $\mathscr{R}$* where it is defined (§ 57 a).

We may examine the expressions of our solutions x, p from another point of view: for they depend not only on the corresponding initial values, but also on the functions $A_{ik}(x_1, x_2, \ldots, x_m)$ which represent the coefficients of our partial differential equation (coefficients of the terms of the second order). We may want to know their *order of continuity* (Book I, § 20 a) with respect to these A_{ik}'s.

* Picard's proof for the fundamental theorem gives the following result: "If, in the linear homogeneous system,

$$\frac{dy_i}{ds} = a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{iN}y_N, \qquad (i=1, 2, \ldots N)$$

the a_{ik}'s have absolute values everywhere less than K, and the initial values (values for $s=0$) of the y's are all less than M, we have, for every s, $|y_i| < Me^{NKs}$."

It follows from the above that this order is 1 for the x, p themselves, 2 for their first derivatives with respect to the λ's, ..., $(h+1)$ for the derivatives of order h. If we have constructed a determinate geodesic from a for a given equation $\mathcal{F}(u)=0$, the coefficients of which (for the second order) are A_{ik}, the equations of that line being valid throughout the interval $0 \leqslant |s| \leqslant s_0$, and if, given any positive number s_0' less than s_0 and any positive number η, however small, we consider the most general altered values $A_{ik}+\bar{A}_{ik}$ such that the increments $\bar{A}_{ik}$ and their first partial derivatives have absolute values everywhere less than ϵ, the quantity ϵ can be chosen small enough so that, for every such alteration* of the A_{ik}'s: (1) a geodesic of the new kind issues from a, with the same p_{0i} as the former one, the equations of which are valid for $0 \leqslant |s| \leqslant s_0'$; (2) the values of the x's, p's for that new geodesic differ from the corresponding ones for the original geodesic only by increments smaller than η. This follows immediately from the general proof of the fundamental theorem; and the corresponding conclusion similarly holds for the above considered derivatives of the x's and p's.

* It must be understood, of course, that Lipschitz's condition (as assumed for the fundamental theorem) is satisfied by the first derivatives (if the x's and p's themselves are concerned) of the new as well as of the original A_{ik}'s.

BOOK III

THE EQUATIONS WITH AN ODD NUMBER OF INDEPENDENT VARIABLES

CHAPTER I

INTRODUCTION OF A NEW KIND OF IMPROPER INTEGRAL

1. Discussion of preceding results

71. We shall now see what use can be made of the elementary solution and what relation it has to the functions previously employed.

For the equation of cylindrical waves (e_2) with $w = 1$ (which may be assumed with a proper choice of units), the elementary solution is

$$(1) \qquad \frac{1}{\sqrt{(t_0 - t)^2 - (x - x_0)^2 - (y - y_0)^2}} = \frac{1}{\sqrt{(t_0 - t)^2 - r^2}}.$$

As we said, Volterra did not use this quantity, but the following one

$$(2) \qquad \mathbf{v} = \log \frac{t_0 - t + \sqrt{(t_0 - t)^2 - r^2}}{r}.$$

These two expressions are simply related to each other; (2) can be deduced from (1) by a mere integration with respect to t_0, viz.

$$\mathbf{v} = \int \frac{dt_0}{\sqrt{(t_0 - t)^2 - r^2}}:$$

geometrically speaking, by letting the vertex of the characteristic cone vary on the line $x = x_0$, $y = y_0$ and integrating between proper limits* with respect to that variation. No wonder at all that the introduction of such a quantity in our fundamental formula gives an expression of the integral $\int u(t_0)\,dt_0$ along this same line.

As Volterra remarks†, such a proceeding exactly corresponds to what one would find, for $\nabla^2 u = 0$, by integrating and immediately re-differentiating, with respect to z_0, the classic formula

$$u(x_0, y_0, z_0) = \frac{1}{4\pi} \iint_S \left(u \frac{d\frac{1}{r}}{dn} - \frac{1}{r}\frac{du}{dn} \right) dS - \frac{1}{4\pi} \iiint \frac{f}{r}\, dx\,dy\,dz$$
$$(r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}),$$

* See below, § 73.

† Stockholm Lectures, French edition (Paris, Hermann, 1912), p. 45.

this giving

$$u(x_0, y_0, z_0) = \frac{\partial}{\partial z_0}\left[\frac{1}{4\pi}\int_S\left(u\frac{d\phi}{dn} - \phi\frac{du}{dn}\right)dS - \frac{1}{4\pi}\iiint \phi f\, dx\, dy\, dz\right],$$

where $\phi = \log\left[\frac{z - z_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} + \sqrt{1 + \frac{(z - z_0)^2}{(x - x_0)^2 + (y - y_0)^2}}\right].$

We have, in other terms, integrated at first and made the inverse differentiation at the end, and the same remark would apply to Tedone's extension (§ 43).

Of course Volterra had an imperative reason for operating thus. If he had directly introduced the elementary solution $v = \frac{1}{\sqrt{(t_0 - t)^2 - r^2}}$ in his fundamental formula, he would have found meaningless integrals, the quantities under the signs of integration becoming infinite in an unpermissible manner on the surface of the characteristic cone. This immediately appears on performing such a calculation. We also shall recognize it if we perform the (equivalent) operation which consists in actually doing the differentiation in the above formula (for $\omega = 1$),

$$(1') \qquad u(x_0, y_0, t_0) = \frac{1}{2\pi}\left[\mu_{t_0}(u_1) + \frac{d}{dt_0}\mu_{t_0}(u_0)\right],$$

$$\mu_{t_0}(f) = \iint \frac{f(x, y)\, dx\, dy}{\sqrt{t_0^2 - (x - x_0)^2 - (y - y_0)^2}}.$$

The usual method for that would consist in differentiating with respect to t_0 under the sign $\iint$, which would affect only the denominator; and on the other hand, taking account of the fact that the boundary is variable with t_0, which ought to give place to a supplementary boundary term, a simple circumference integral. But it appears immediately that both double and simple integrals are meaningless: the former on account of the presence of an infinity of order 3/2 along the boundary, the latter because every element of it is infinite. Of course, simple devices would allow us to perform the differentiation so as to avoid such difficulties*: but they would not be of interest

* We could, for instance, refer the inside of the circle to polar coordinates r, ϕ with (x_0, y_0) as their pole and, in the place of the first of them, introduce an auxiliary variable λ defined by $r = \lambda t_0$. The integration with respect to λ and ϕ now taking place within fixed limits $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi$, differentiation with respect to t_0 would offer no special difficulty.

to us, as—paradoxical as it seems—our proposed method will consist in not avoiding them.

72. Let us first note that we could strictly imitate both Volterra's and Tedone's procedure. For $m=3$, for instance, let us consider our equation

$$\mathcal{F}(u)=\sum_{i=1}^{3}\sum_{k=1}^{3} A_{ik}\frac{\partial^2 u}{\partial x_i\,\partial x_k}+\sum_{i=1}^{3} B_i\frac{\partial u}{\partial x_i}+Cu=f,$$

the adjoint equation

$$\mathcal{G}(v)=\Sigma\frac{\partial^2}{\partial x_i\,\partial x_k}(A_{ik}v)-\Sigma\frac{\partial}{\partial x_i}(B_i v)+Cv=0,$$

and its elementary solution

$$v=\frac{V}{\sqrt{\Gamma}}.$$

Let us suppose that the point (a_1, a_2, a_3) describes an *arbitrary* given segment of line $\mathcal{L}$ (straight or curved), only such that it lies entirely inside the characteristic conoid having any of its points for vertex. We shall consider

$$\mathbf{v}(x)=\int_{\mathcal{L}} v(x\,|\,a)\,\chi(t)\,dt, \tag{3}$$

t being the parameter which defines the position of a point (a_1, a_2, a_3) on $\mathcal{L}$ and $\chi(t)$ an arbitrarily chosen function. This will lead to (2) when we start from (1), the line $\mathcal{L}$ being a parallel to the t-axis and $\chi(t)$ being simply taken equal to 1. For other purposes (such as the solution of Cauchy's problem for systems), Volterra has introduced other similar expressions which can be deduced from (3) by the choice of other forms* of the function $\chi(t)$.

We shall recognize that such an expression has a logarithmic singularity like that of (2). The calculation being sometimes of use, we now say something about it. In the equation of the characteristic conoid

$$\Gamma(x_1, \ldots x_m;\ a_1, \ldots a_m)=0,$$

let us suppose that one of the two points $(x_1, \ldots x_m)$ and $(a_1, \ldots a_m)$ lies near the line

$$\mathcal{L}[a_1=a_1(t), \ldots a_m=a_m(t)],$$

while the other one describes that very line.

* The solution Φ_1, which Volterra forms in his Memoir of the *Acta Mathematica*, vol. XVIII, p. 169, and uses for the extension of the theory to systems (such as occur in Elasticity), corresponds to $\chi(t)=t_0-t$.

If we suppose all our functions to be developable by Taylor's formula (at least up to terms of a certain degree) around one point

$$a^0 [a_1^0 = a_1(t_0), \ldots a_m^0 = a_m(t_0)],$$

which corresponds to a certain value t_0 of the parameter, and if we recollect the form (31) (§ 58), of the first terms of the expansion of Γ when the two points x and a are very near each other, we see that, in the neighbourhood of a_0, the expansion of $v[x; a(t)]$ according to powers of $(x_i - a_i^0)$, $(t - t_0)$ begins with terms of the second degree, the coefficient of $(t - t_0)^2$, viz.

$$N_0 = \mathbf{H}[a_1'(t_0), a_2'(t_0), \ldots a_m'(t_0)],$$

being different from 0 as $\mathscr{L}$ is not tangent to the characteristic cone*. Then, by a proper application of Weierstrass's and Poincare's (or rather Cauchy's)† "Theorem of factorization" for functions of several variables, we can write

$$\Gamma[x; a(t)] = N(x, t)[(t-\beta)^2 - \gamma], \quad N = N_0 + \ldots$$

* The tangent to $\mathscr{L}$ being interior to the characteristic cone, N_0 will be positive if we write our equation (as said previously) so that $\mathbf{H} > 0$ corresponds to the *inside* of the characteristic cone.

† *Bull. de Férussac*, 1831. *Exercices d'Analyse*, vol. II, and various other places. See Lindelof's *Leçons sur la Theorie des residus*, note of p. 27, and Osgood, *Madison colloquium*, Lecture IV, § 1, where, however, a distinction is made between two forms of the theorem which we consider as equivalent in the above text. The use of that theorem of factorization can be avoided or, at least, restricted to its quite elementary case concerning the first degree, i.e. to the fact that

$$c_1\tau + c_2\tau^2 + \ldots - T = 0,$$

with $c_1 \neq 0$, can be "inverted," giving

$$\tau - \left(\frac{T}{c_1} + C_2 T^2 + \ldots\right) = 0,$$

with the obvious consequence that the quotient of the two left-hand sides is a power series in τ, T, with a constant term c_1. To see this, we first solve the equation $\frac{\partial}{\partial t}\Gamma[x; a(t)] = 0$ with respect to t, which can be done regularly (as the coefficient of $(t - t_0)$ is $2N_0$) and gives $t = \vartheta$, where ϑ is a power series in the $(x_i - a_i^0)$'s. Setting down $t - \vartheta = \tau$, we find

$$\Gamma[x; a(t)] = -K + N_0\tau^2 + \ldots;$$

K (which is the minimum of Γ when x remains fixed and a describes $\mathscr{L}$) is again a power series in the $(x_i - a_i^0)$'s, beginning with quadratic terms, and the dots

β and γ being again expansions in x $(x_1-a_1^0)$, $(x_2-a_2^0)$, ... $(x_m-a_m^0)$, the former generally beginning with linear terms, the latter with quadratic terms. Now, in integral (3), where, as yet, we take $m=3$, whence $v=\frac{V}{\sqrt{\Gamma}}$, we can suppose $\frac{V[x; a(t)]\chi^{(t)}}{\sqrt{N(x,t)}}$ to be expanded in powers of $(t-\beta)$, so that

$$\frac{V}{\sqrt{N}}=P_0+P_1(t-\beta)+\dots, \qquad \frac{V}{\sqrt{\Gamma}}=\frac{P_0+P_1(t-\beta)+\dots}{\sqrt{(t-\beta)^2-\gamma}},$$

the P's being regular functions of the x's.

Every odd term in $(t-\beta)$ gives, by integration, a positive power of $\sqrt{(t-\beta)^2-\gamma}$. Then, in the even terms, we can introduce the variable $(t-\beta)^2-\gamma$ instead of $(t-\beta)^2$. The expansion in integral powers of that new variable being

$$Q_0+Q_1[(t-\beta)^2-\gamma]+Q_2[(t-\beta)^2-\gamma]^2+\dots,$$

it appears that every term after the first one gives quantities which are finite and even infinitesimal in the neighbourhood of $\mathscr{L}$; the first term, on the other hand, has the indefinite integral

$$(4) \quad Q_0\int\frac{dt}{\sqrt{(t-\beta)^2-\gamma}}=Q_0\log\left[\frac{t-\beta+\sqrt{(t-\beta)^2-\gamma}}{\sqrt{\gamma}}\right],$$

an expression entirely similar to (2) from our point of view, $\sqrt{\gamma}$ corresponding to the quantity $\sqrt{(x-x_0)^2+(y-y_0)^2}$ of Volterra*.

represent terms in τ^3, τ^4, etc. The square root of the sum of terms other than $-K$ can be extracted, giving

$$\Gamma[x; a(t)]=-K+N_0(\tau+\dots)^2=[-\sqrt{K}+\sqrt{N_0}(\tau+\dots)][+\sqrt{K}+\sqrt{N_0}(\tau+\dots)].$$

If we now apply to each factor the aforesaid inversion principle, the first one, for instance, will be found to be proportional to such a series as

$$\tau-C_1\sqrt{\frac{K}{N_0}}-C_2\frac{K}{N_0}-C_3\left(\frac{K}{N_0}\right)^{\frac{3}{2}}-\dots=\tau-\mu\sqrt{K}-\nu$$

(μ, ν, power series in K); and the product

$$(\tau-\mu\sqrt{K}-\nu)(\tau+\mu\sqrt{K}-\nu)=(\tau-\nu)^2-\mu^2K$$

will have the required form, with $\vartheta+\nu=\beta$, $\mu^2K=\gamma$.

The result in the text and the way in which we use it are due to Poincare, in his Memoir *Sur les proprietes du potentiel et sur les fonctions abeliennes* in *Acta Mathematica*, vol. XXII, 1899, pp. 114 ff.

* γ is, but for a factor different from zero, the minimum of Γ when a describes $\mathscr{L}$, the point x remaining fixed.

73. We have only considered the indefinite integral of

$$v(x;\, a)\, \chi(t)\, dt.$$

But an essential remark for our object concerns the choice of the limits of integration.

If we take them constant, whatever the values of these constants be, so that the segment of integration on $\mathscr{L}$ is completely independent of the location of the point x, the integral (3) thus obtained will certainly satisfy the given equation, for the same reason as that classically known to apply to the theory of potential (viz., that every differentiation with respect to the xs can be carried out under the integral sign, treating t as a constant).

Darboux* was, so far as I know, the first to discover a general reason for a noteworthy fact, special cases of which had already occurred in some anterior formulæ; viz. that the same property of **v** holds good when integrating between properly chosen variable limits. This remark of Darboux may be considered as implicitly containing our main further consideration. His argument is a remarkably simple one and can be, in our terminology, expressed as follows.

Integrating, at first, along a fixed arc of $\mathscr{L}$, it may obviously happen that Γ is liable to change sign along that arc: that, indeed, will be the case if at least one sheet of the characteristic conoid from x intersects $\mathscr{L}$ inside the arc in question. We suppose, for instance (fig. 7), that only one sheet, the "direct" one, does so, the segment of integration—which will correspond, e.g., to $t_1 \leqslant t \leqslant t_0$—being thus divided by the point of intersection ω into two parts, one lying outside the

* See *Leçons sur la théorie des surfaces*, vol. II, p. 67. Darboux deals with

$$\int \Phi(u)\,(u-x)^{\mu}\,(y-u)^{\mu'}\,du,$$

which, if taken between constant limits, is most easily found to satisfy (as a function of x and y) "Euler-Poisson's" partial differential equation, and assumes the constant limits to be taken so as to include both x and y: which would correspond, in our terminology, to the case in which *both* sheets of the characteristic conoid having the point x for its vertex would intersect $\mathscr{L}$ inside the primitive (fixed) segment of integration, the useful part of the integral relating to the portion of $\mathscr{L}$ which is *exterior* to that conoid. As to the constant indexes μ, μ', Darboux observes that the argument in the text applies whenever their values are fractional numbers of the form $\frac{2p+1}{[illegible]}$ (where p and q are integers).

conoid, the other inside: let us suppose that the latter corresponds to greater values of t, i.e. contains the upper limit t_0. Now, if θ denotes the value of t for the dividing point ω, our integral (in which V and χ are, of course, real quantities) will consist of two parts, one imaginary $\mathbf{v}_1$ and one real $\mathbf{v}_2$, and it is clear, therefore, that *each of them must be separately a solution of our given equation.*

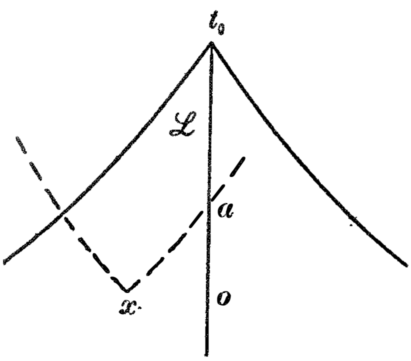

Fig. 7.

This is Darboux's general conclusion. We see that, instead of integrating from 0 to t_0, we can take one limit equal to θ, which quantity, in the notation of the preceding section, has the value

$$\beta + \sqrt{\gamma}.$$

These, for instance (θ being equal to $t + r$), are the limits between which we have to integrate (1) in order to obtain the quantity (2) of Volterra; t being, on the other hand, replaced by t_0, the indefinite integral calculated in the preceding section will be also the definite one and give the value of (3) if the latter is defined in the above way. Of course, this expression of a solution of our differential equation is only valid when θ is assumed to be smaller than t_0, i.e. when the point x lies inside the inverse half conoid having $t = t_0$ for its vertex.

This solution, as we have just proved, admits of $\mathscr{L}$ as a logarithmic singularity. It is to be foreseen that, when substituted in the fundamental formula, it will behave exactly like (2) in Volterra's method, the small cylinder of Volterra* having to be replaced by a tubular surface around $\mathscr{L}$, and give a value of the integral

$$\int_{\mathscr{L}} \chi(t)\, u\,[a(t)]\, dt.$$

74. It may be convenient, however, to introduce, in such cases, a special system of curvilinear coordinates for our point x. One of them will be the above quantity θ. For every given value of it, the locus of the point x will be a characteristic half conoid (having the point ω corresponding, on $\mathscr{L}$, to $t = \theta$ for its vertex), on which conoid the position of that point will be completely determined if we give:

* *Acta Math.* vol. XVIII, p. 174.

(1) a parameter* λ defining the initial direction of one of the bicharacteristics which generate the conoid;

(2) a value of s, defining one point on that bicharacteristic.

The value of s would contain (see Book II) an arbitrary factor of proportionality (α, in the notation of § 57): we can choose this factor in a determinate way for each value of λ, and assume this to have been done in such a way that:

(a) s be positive on our useful (that is inverse) half conoid from ω;

(b) the initial values (values at the vertex) of $\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}$ on each bicharacteristic (λ) be regular functions of θ and λ;

(c) these three quantities do not vanish simultaneously, and, for instance, the sum of their squares be always greater than a fixed positive number, whatever θ and λ may be.

Under such assumptions, we can take θ, λ and s as curvilinear coordinates; x_1, x_2, x_3 will be regular (or even holomorphic) functions of θ, λ, s, and the reverse will be true whenever we are not in the neighbourhood of $\mathscr{L}$.

75. a being a point of $\mathscr{L}$ corresponding to a value of t greater than θ, the quantity $\Gamma(x; a)$ will be of the form

$$\Gamma = (t-\theta)\, w\,(\theta, \lambda, s, t), \tag{5}$$

$w > 0$ being regular and not zero when the point x approaches a point of the conoid other than the vertex a (the sign of the first factor having to be reversed if the useful half-conoid from w be the one which contains the direction of increasing t's on $\mathscr{L}$).

In the neighbourhood of the vertex, this expression holds no longer, but the following remains still valid: denoting by τ the difference

$$\tau = t - \theta,$$

we can write

$$\Gamma = 2Ms\tau + N\tau^2,$$

M and N being two regular functions, assuming, when x coincides with a, the values

$$M_0 = \sum_i \tfrac{1}{2}\xi_i \frac{\partial H}{\partial \alpha_i}, \quad N_0 = H(\alpha_1, \alpha_2, \alpha_3),$$

where the α, ξ's stand for

$$\alpha_i = \frac{da_i}{dt}, \quad \xi_i = \left(\frac{\partial x_i}{\partial s}\right)_{s=0}, \qquad (i = 1, 2, 3)$$

M_0, N_0 are not only different from zero, but positive (N_0, as said above, on account of our assumption concerning the sign of $\mathbf{H}$; M_0, because of our assumption (a) concerning the sign of s on our half conoid)†.

* We treat the case $m=3$. But analogous calculations would hold for every odd m.

† For x must be inside the conoid from a—therefore Γ positive—when s and τ are both positive (the latter sufficiently small).

We immediately obtain the above formula if we observe that the second term may represent $\Gamma(\omega; a)$ and the first the difference $\Gamma(x; a) - \Gamma(\omega; a)$. We even see that we can assume N to be independent of s and λ. Such is not the case, of course, with M; but we can assume its initial value M_0 to be independent of λ (being a function of θ only), by a proper choice of the factor of proportionality in s (without contradicting our previous assumptions (*a*), (*b*), (*c*)). As, for small values of s, we have sensibly $x_i = a_i + s\frac{dx_i}{ds} = a_i + s\xi_i$, this means geometrically that, for such values of s, the directions of the tangents to the two coordinate lines which respectively correspond to θ as well as s being taken as constants and λ as well as s being taken as constants are transversal to each other, i.e. conjugate with respect to the characteristic conoid, so that if that small value of s is kept constant as well as θ, λ varying alone, the corresponding point will sensibly describe a small ellipse the plane of which is transversal to $\mathscr{L}$.

With such an evaluation of Γ, we have

$$\mathbf{v} = \int_0^{t_0-\theta} \frac{\chi(\theta+\tau)\, V(\theta, \lambda, s, \theta+\tau)}{\sqrt{2Ms\tau + N\tau^2}}\, d\tau;$$

V is a holomorphic function, equal to $\frac{1}{\sqrt{\Delta}}$ for $s=\tau=0$. As for $\chi(t)$, let us suppose, not only that it is regular, but that it does not change sign—say $\chi(t) > 0$—in the neighbourhood of a determinate point $A\ (t=\theta_0)$ of $\mathscr{L}$, which we are going to consider.

We try, x being taken near such a point A, to find asymptotic values for $\mathbf{v}$ and the derivative $\frac{d\mathbf{v}}{ds}$.

Beginning with the latter, we have

$$(6) \qquad \frac{d\mathbf{v}}{ds} = -\int_0^{t_0-\theta} \frac{\chi V\left(M + s\frac{dM}{ds}\right)\tau\, d\tau}{(2Ms\tau + N\tau^2)^{\frac{3}{2}}} + \int_0^{t_0-\theta} \frac{\chi \frac{\partial V}{\partial s}\, d\tau}{\sqrt{2Ms\tau + N\tau^2}}.$$

Let us begin with the latter quantity, the first term of which obviously gives the principal part. Around A as centre, we can describe a small sphere such that if the points x and a are taken in its inside, the quantities M, N, χ, V can be replaced, with an arbitrarily small error, by the values M_0, N_0, χ, (θ_0), $\frac{1}{\sqrt{\Delta_A}}$, which they have at A itself. If we denote by τ_1 the positive value of τ corresponding to an intersection of that spherical surface with $\mathscr{L}$, the integral from τ_1 to $t_0 - \theta$ remains finite and continuous when s approaches zero. The integral from zero to τ_1, in which every element is positive, can be, with a very small relative error, represented by

$$\frac{M_0}{\sqrt{\Delta}} \wedge (\theta) \int_0^{\tau_1} \frac{\tau\, d\tau}{(2M_0 s\tau + N_0\tau^2)^{\frac{3}{2}}} = -\frac{\chi}{\sqrt{\Delta}\, s}\sqrt{\frac{\tau_1}{2M_0 s + N_0\tau_1}}.$$

As τ_1 is chosen once for all, this gives, when s approaches zero,

$$\text{(6a)} \qquad \frac{\partial \mathbf{v}}{\partial s} \sim -\frac{\chi}{s\sqrt{N_0 \Delta}},$$

the sign $\sim$ signifying asymptotic equality.

An analogous method would give the approximate value of $\mathbf{v}$ itself; but we find it immediately by integrating the above asymptotic equality, viz.:

$$\text{(6b)} \qquad \mathbf{v} \sim \frac{-\chi}{\sqrt{N_0 \Delta}} \log s,$$

which we could easily see agrees with the result of § 72; and we should obviously have a similar evaluation for the second term of (6).

We also could get in the same way asymptotic expressions of the other derivatives.

As to the other singularity of $\mathbf{v}$, which is the half conoid with vertex $t=t_0$, it is easy to find which is the form of $\mathbf{v}$ when the point x approaches any determinate point of it (other than the vertex): for, as we have

$$\Gamma = (t-\theta)\, w$$

where w is holomorphic (and not zero) even when θ and t are nearly equal, this gives

$$\mathbf{v} = \int_\theta^{t_0} \chi(t)\, v(x;a)\, dt = \int_\theta^{t_0} \frac{W dt}{\sqrt{t-\theta}},$$

$W = \frac{V\chi(t)}{\sqrt{w}}$ being again holomorphic, and such an expression is sensibly equal to

$$2\, W_0 \sqrt{t_0 - \theta},$$

W_0 being the value of W at the limiting point of x. Derivatives of $\mathbf{v}$ with respect to λ or s would evidently be of a quite similar form.

76. The analogy of $\mathbf{v}$ with Volterra's quantity (2) is thus evident; let us see the consequences in our problem of integration.

S being, in our three-dimensional space, a surface at every point of which Cauchy's data are given, and which is, moreover, everywhere duly inclined (§ 27) with respect to characteristic conoids, let a be a determinate point at which we want to find the value of the solution u of the given equation

$$\text{(E)} \qquad \mathcal{F}(u) = f,$$

which corresponds to the above-mentioned data on S. From a as vertex, we construct a half conoid Γ, which we assume to enclose with S a limited volume T (fig. 8). From the same point to a point α of S, inside Γ, we also draw an arbitrary line $\mathcal{L}$ (only subject, as previously, to be interior to any conoid having one of its points for its vertex), the given point a corresponding to the value t_0 of the parameter. By means of v and an arbitrary regular function $\chi(t)$, we construct the function $\mathbf{v}(x)$, a solution of $\mathcal{G}(v)=0$; and we substitute $\mathbf{v}$ with the unknown function u in the fundamental formula

$$\text{(F)} \qquad \iiint [\mathbf{v}\mathcal{F}(u) - u\mathcal{G}(\mathbf{v})]\, dx_1\, dx_2\, dx_3 = \iint \left(u\frac{d\mathbf{v}}{d\nu} - \mathbf{v}\frac{du}{d\nu} - Lu\mathbf{v}\right) dS.$$

This can not be done at once in the whole domain T: we have to exclude the singularities of $\mathbf{v}$, which are $\mathscr{L}$ and, strictly speaking, the conoid Γ. But it is easy to see, in the first place, that the latter has no influence. Let us indeed replace it by a neighbouring conoid Γ' the vertex of which shall be the point $t=t_0'$ on $\mathscr{L}$. On Γ' we know that Γ is of the order of (t_0-t_0'); and so will be as well $\frac{d\Gamma}{d\nu}$, because we know that the transversal direction to Γ' is the bicharacteristic one, so that $\frac{d}{d\nu}$ is a derivative with respect to s: therefore, letting t_0' approach t_0, not only can we limit our domain of integration by Γ, but, just as in Volterra's method (which we are strictly imitating), no corresponding surface term need be considered.

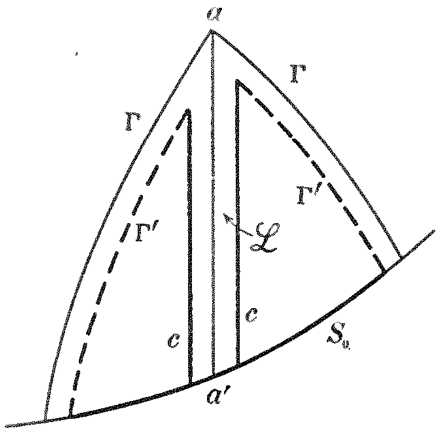

Fig. 8.

77. Let us now consider the singularity $\mathscr{L}$, which, at first, we have to exclude from T. We do this by a small surface G (corresponding to Volterra's cylinder), which we obtain by equating the curvilinear coordinate s to a very small positive constant: on which surface we have to take the double integral on the right-hand side of (F).

Terms containing $\mathbf{v}$ only as a factor can be neglected: for that quantity only becomes infinite like log s, while the surface element is of the order of s. Let us now express $\frac{d\mathbf{v}}{d\nu}$. The π's, on C, are given by

$$\pi_1 dS=\left(\frac{\partial x_2}{\partial\lambda}\frac{\partial x_3}{\partial\theta}-\frac{\partial x_3}{\partial\lambda}\frac{\partial x_2}{\partial\theta}\right)d\lambda\, d\theta, \qquad \pi_2 dS=\ldots,$$

except for sign (or, what comes to the same, for a suitable permutation between the x's). We shall obtain the correct sign in these formulæ by remarking that the direction of increasing s on each bicharacteristic is directed towards the inside of our domain of integration, so that we have to write the above quantities in such a way that $\pi_1\frac{\partial x_1}{\partial s}+\pi_2\frac{\partial x_2}{\partial s}+\pi_3\frac{\partial x_3}{\partial s}$, i.e. the determinant (Jacobian of x_1, x_2, x_3 with respect to s, λ, θ)

$$D=\begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial\lambda} & \frac{\partial x_1}{\partial\theta} \\ \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial\lambda} & \frac{\partial x_2}{\partial\theta} \\ \frac{\partial x_3}{\partial s} & \frac{\partial x_3}{\partial\lambda} & \frac{\partial x_3}{\partial\theta} \end{vmatrix}$$

be positive (geometrically speaking, that the trihedral of the positive directions s, λ, θ and the trihedral of coordinates be disposed in the same way). Let us assume (by means of a permutation between the x's or a change of the sign of λ, if necessary) that this is so: then, denoting by ϕ_1, ϕ_2, ϕ_3 the partial derivatives of any function ϕ, the transversal derivative of ϕ along C will be (§§ 38–40) given by

$$\frac{d\phi}{d\nu}ds=\sum_i \tfrac{1}{2}\pi_i ds\frac{\partial \mathbf{A}}{\partial \phi_i}=\begin{vmatrix} \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_1} & \frac{\partial x_1}{\partial \lambda} & \frac{\partial x_1}{\partial \theta} \\ \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_2} & \frac{\partial x_2}{\partial \lambda} & \frac{\partial x_2}{\partial \theta} \\ \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_3} & \frac{\partial x_3}{\partial \lambda} & \frac{\partial x_3}{\partial \theta} \end{vmatrix} d\lambda\, d\theta$$

$$=(\text{sensibly})\ s\begin{vmatrix} \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_1} & \xi_1' & \alpha_1 \\ \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_2} & \xi_2' & \alpha_2 \\ \frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_3} & \xi_3' & \alpha_3 \end{vmatrix} d\lambda\, d\theta$$

$\left(\text{with } \xi_i'=\frac{\partial \xi_i}{\partial \lambda}\right)$, on account of $x_i=a_i+s\xi_i$. For the determinant thus obtained as the factor of $d\lambda\, d\theta$, we can use the abbreviative notation

$$s\left|\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_i} \quad \xi_i' \quad \alpha_i\right|. \tag{7}$$

We already see that the coefficients of $\frac{\partial \mathbf{A}}{\partial \phi_i}$ are holomorphic functions of θ, λ, s, which contain s as a factor. A simple way of finding the value of (7) is to multiply it by the discriminant

$$\frac{1}{\Delta}=|H_{i1} \quad H_{i2} \quad H_{i3}|$$

of $\mathbf{H}$, which $\left(\text{remembering that the relations } \psi_i=\frac{1}{2}\frac{\partial \mathbf{A}}{\partial \phi_i} \text{ are equivalent to } \phi_i=\frac{1}{2}\frac{\partial \mathbf{H}}{\partial \psi_i}\right)$

gives
$$s\left|\phi_i \quad \frac{1}{2}\frac{\partial \mathbf{H}}{\partial \xi_i'} \quad \frac{1}{2}\frac{\partial \mathbf{H}}{\partial \alpha_i}\right|;$$

then by the above written Jacobian D, which gives

$$s\begin{vmatrix} \frac{\partial \phi}{\partial s} & \frac{\partial \phi}{\partial \lambda} & \frac{\partial \phi}{\partial \theta} \\ 0 & s\mathbf{H}(\xi_1', \xi_2', \xi_3') & * \\ M_0 & * & N_0 \end{vmatrix}.$$

The elements replaced by * are sensibly equal to sM_0', i.e., at any rate, of the order of s, but become of the order of s^2 under our assumption † that M_0 is inde-

† When this holds, the direction λ is sensibly transversal to the plane constructed through the directions θ and s.

pendent of λ. The above determinant thus contains s as a factor and (7) is sensibly equal, when s is small, to

$$s^2 \frac{\Delta}{D} \mathbf{H}(\xi_1', \xi_2', \xi_3') \left(N_0 \frac{\partial \phi}{\partial s} - M_0 \frac{\partial \phi}{\partial \theta} \right).$$

As to D (which is positive under our assumptions), it is sensibly equal to

$$D \sim s \,|\, \xi_i \quad \xi_i' \quad \alpha_i \,|,$$

so that

$$\iint_c u \frac{d\mathbf{v}}{d\nu} ds = \iint_c u \frac{s\Delta \mathbf{H}(\xi_1', \xi_2', \xi_3')}{\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1' & \xi_2' & \xi_3' \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}} \left[(N_0 + \ldots) \frac{\partial \mathbf{v}}{\partial s} - (M_0 + \ldots) \frac{\partial \mathbf{v}}{\partial \theta} + (\ldots) \frac{\partial \mathbf{v}}{\partial \lambda} \right] d\lambda\, d\theta$$

(dots standing for terms of higher order in s).

$\frac{\partial \mathbf{v}}{\partial \theta}$ $\left(\text{and similarly } \frac{\partial \mathbf{v}}{\partial \lambda}\right)$ can be eliminated when integrating the corresponding term $\left(\text{say } \iint P \frac{\partial \mathbf{v}}{\partial \theta} d\lambda\, d\theta\right)$ with respect to θ, an integration by parts giving a simple integral ($\int P\mathbf{v}\, d\lambda$ for $\theta = t_1, t_0$) and a double integral in $\mathbf{v}$, viz. $\iint \mathbf{v} \frac{\partial P}{\partial \theta} d\lambda\, d\theta$, which are infinitesimal with s, as before. Finally taking account of ($6a$), we have only to integrate, with respect to θ, the product of the integral

$$(8) \qquad \int \frac{\mathbf{H}(\xi_1', \xi_2', \xi_3')}{D} d\lambda = \int \frac{\mathbf{H}(d\xi_1, d\xi_2, d\xi_3)}{\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \xi_1 & \xi_2 & \xi_3 \\ d\xi_1 & d\xi_2 & d\xi_3 \end{vmatrix}}$$

by the quantity

$$(9) \qquad -u\chi \sqrt{N_0 \Delta}.$$

Such an integral as (8) taken along the conic

$$\mathbf{H}(\xi_1, \xi_2, \xi_3) = 0,$$

described by the point the homogeneous coordinates of which are ξ_1, ξ_2, ξ_3, is, of course, finite when the point $(\alpha_1, \alpha_2, \alpha_3)$ lies inside that conic (as is the case here). Its value is found by observing that the integrand does not change when we multiply ξ_1, ξ_2, ξ_3 by any common factor (whether a constant or a function of λ) and that, moreover, it is multiplied by $\mathcal{D}$ when the variables ξ (and therefore also α) are subjected to a linear substitution with determinant $\mathcal{D}$. As, by the latter operation, we can reduce our form $\mathbf{H}$ to $\xi_3^2 - \bar{\xi}_1^2 - \xi_2^2$ (denoting by $\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3$ the new variables) and, therefore, simply take $\bar{\xi}_1 = \xi_3 \cos\lambda$, $\bar{\xi}_2 = \bar{\xi}_3 \sin\lambda$ (the determinant $\mathcal{D}$ being then equal to $\sqrt{\Delta}$), the above integral is

$$(8') \qquad \pm \frac{1}{\mathcal{D}} \int_0^{2\pi} \frac{d\lambda}{\alpha_3 - \alpha_1 \cos\lambda - \bar{\alpha}_2 \sin\lambda} = \pm \frac{1}{\mathcal{D}} \frac{1}{\sqrt{\bar{\alpha}_3^2 - \bar{\alpha}_1^2 - \bar{\alpha}_2^2}}$$

$$= \pm 2\pi \frac{1}{\sqrt{\Delta \mathbf{H}(\alpha_1, \alpha_2, \alpha_3)}} = \pm 2\pi \sqrt{\frac{1}{\Delta N_0}}.$$

The sign depends on the direction in which the conic is described. In the present case, as we have seen, this direction must be such that the determinant in the denominator be positive. As the point (ξ_1', ξ_2', ξ_3') belongs to the region outside the conoid (i.e. such that $\mathbf{H}(\xi_1', \xi_2', \xi_3') < 0$), the right sign is $-$. The factor $\sqrt{N_0}\Delta$ in the expression (9) being removed by (8′), the integration with respect to θ finally gives

$$\lim \iint_c u \frac{d\mathbf{v}}{d\nu}\, dS = 2\pi \int_{\mathscr{L}} \chi(t)\, u[a(t)]\, dt,$$

so that our fundamental formula becomes (as $\mathscr{F}(u) = f$ and $\mathscr{G}(\mathbf{v}) = 0$)

(10)

$$2\pi \int_{\mathscr{L}} \chi(t)\, u[a(t)]\, dt = \iiint_T \mathbf{v} f\, dx_1 dx_2 dx_3 + \iint_{S_0} \left(\mathbf{v}\frac{du}{d\nu} - u\frac{d\mathbf{v}}{d\nu} + L u \mathbf{v}\right) dS.$$

This is the result which exactly corresponds to Volterra's, being, of course, subject to the same foregoing observations.

78. Greater number of variables. We have said that the theory of (e_2) and (e_3) has been extended by Tedone to the analogous equations in m independent variables. As for (e_2) and (e_3), the formulæ given by Tedone (*Annali di Mat.*, 3rd series, vol. I, 1898) for the solution of the equation

$$(e_{m-1}) \qquad \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_{m-1}^2}\right) = 0$$

do not express immediately the value of u_a, but an integral of the form

$$(10') \qquad \int_{t_1}^{t_0} u(t)(t_0 - t)^{m-3}\, dt,$$

from which u_a has to be deduced by $(m-2)$-fold differentiation.

It may be foreseen, therefore, that the expressions, analogous to Volterra's quantity (2), introduced by Tedone in his operations for m odd, are to be deduced from the elementary solution by integrating several times along a line such as $\mathscr{L}$ (which is, in fact, parallel to the t-axis). We shall see that such is indeed the case; but the fact is that this will require the use of generalized integrals, such as we are going presently to define.

But even before this, we must note a most peculiar feature of these solutions of Tedone. If we consider Poisson's formula for spherical waves (formula (1) of Book II), we immediately see that the given values of the function u_0 are to be differentiated, at least for a radial displacement of the point (x, y, z), so that this function is required to admit of first derivatives for such a displacement, i.e. in any direction, as the point (x_0, y_0, z_0) is arbitrary, and we shall very soon see that the existence of such derivatives is also implicitly presupposed in formula (1′), Book II, concerning cylindrical waves. Of course, we may consider this hypothesis as a natural one, as the non-existence of first or even second derivatives could be said to make the differential equation itself meaningless (though problems implying such an apparent discrepancy are usually studied by analysts*).

But if we now turn to Tedone's solutions for higher values of m, we see that they imply higher derivatives of the data, the order of derivation being $\frac{m}{2} - 1$ for m even and $\frac{m}{2} - \frac{3}{2}$ for m odd: that is, of any order, however great, if the number of independent variables is sufficiently great.

What would happen if the functions u_0 and u_1 did not admit of derivatives up to such an order? We have to expect then that *no solution of Cauchy's problem can exist.*

To prove this with perfect rigour, we shall not, at first, start from the final formulæ (above alluded to) which give the solution u itself, but from the preparatory formulæ which (as explained above) give the value of the quantity (10′).

That this quantity admits of at least $(m-2)$ derivatives, the $(m-2)$th one being, but for a numerical factor, equal to $u(t_0)$, is well known, it being sufficient for the validity of this that u be finite and continuous: so that we are certain that all the differentiations which Tedone performs on the right-hand sides of the formulæ in question (formulæ (11), (12) of his Memoir) in order to obtain the following ones (formulæ (13) to (24)) must be possible.

* Such is the case with Dirichlet's problem, which refers to a differential equation of the second order and which analysts nevertheless try to solve without supposing the existence of even a first derivative for the data at the boundary. Of course, the differential equation becomes meaningless on that boundary itself, but is assumed to be satisfied at any neighbourhood of it, however close.

It must be observed that in the above we have already assumed first derivatives to exist and be continuous, as such derivatives are involved in the quantities (5′) (p. 59), the continuity of which is implied by the use of Green's formula. Discontinuities must even be considered when using Poisson's and Kirchhoff's formulæ (see Love's paper, *Proceedings of the London Math. Society*, series 2, vol. I, §§ 822, 823).

Let us simply take for S the hyperplane* $t=0$ (so that $t_1=0$ in (10)), and only take account of the first given function u_0, the second one u_1 being assumed to vanish. Let us also choose the case m even† (so as to avoid the difficulties which we are to meet presently). Then, if we denote by r the distance $r=\sqrt{(x_1-a_1)^2+\ldots+(x_{m-1}-a_{m-1})^2}$ between a point $(x_1, \ldots x_{m-1})$ of our $t=0$ variety and the point $(a_1, \ldots a_{m-1})$, the quantity which we have to differentiate consists (for $\omega=1$), in our notation‡, of the product of a numerical factor by the integral

$$(11) \qquad \int_0^{t_0} (t^2-r^2)^{\frac{m}{2}-2}\, rM_r\, dr,$$

where M_r stands for the average value of u_0 along the hyperspherical edge of radius r in the hyperplane S. The first $\frac{m}{2}-2$ of the aforesaid differentiations can be carried out under $\int$ at any rate; but, the result thus obtained being of the form

$$(11') \qquad \int_0^{t_0} rX(t_0, r)\, M_r\, dr,$$

where $X(t_0, r)$ is a homogeneous polynomial|| of degree $\frac{m}{2}-2$ in t_0 and r such that $\xi(t)=tX(t, t)$ does not vanish (except for $t=0$), the following $\frac{m}{2}$ derivatives of (11) *can not* exist if the first $\frac{m}{2}-1$ derivatives M_r do not¶.

* Tedone himself treats any form of S.

† The corresponding conclusion for m odd results therefrom, by descent (see Book IV).

‡ See Tedone's formula (22), p. 13. Tedone calls m what we call $m-1$, and ϕ what we call u_0; his number p is equal, in our notation, to $\frac{m}{2}-1$ (for m even).

|| $X(t, 1)$ differs from Legendre's polynomial of order $\frac{m}{2}-2$ by a numerical factor only.

¶ The first derivative of (11′) is immediately found to be equal to

$$M_{t_0}\xi(t_0)+\int_0^{t_0} M_r \,.\, r\frac{\partial X}{\partial t_0}\, dr.$$

The second term of this quantity can surely be again differentiated, so that this can *not* be the case for the total expression unless the same is true of M_{t_0}.

Assuming then M_r' to exist, the second derivative of (11′) will include the term $\xi(t_0)M_{t_0}'$, a term in M_{t_0} and an integral term. If we now try to differentiate a third

2. The finite part of an infinite simple integral

79. The above considerations, at least for $m=3$, fully extend Volterra's solution to the most general (normal) hyperbolic equation. Only we recognise the indirect character of the solution which lies in the introduction of the arbitrary line $\mathscr{L}$, which is of course finally to be fully eliminated.

If, for instance, we should wish to integrate the equation of *damped cylindrical waves*

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x_2^{\,2}}-Ku=0,$$

we should have to take (as results from the expression of the elementary solution, found in Book II)

$$\text{(12)} \qquad \mathbf{v}=\int\frac{\mathrm{Ch}\sqrt{K[(t-t_0)^2-r^2]}}{\sqrt{(t-t_0)^2-r^2}}\,dt_0.$$

This is the expression which Coulon, in the above quoted *Thesis*, ought to have introduced, in order to imitate strictly Volterra's procedure. Its complexity explains to us perfectly why he was not able to discover it, not possessing the general way of attaining it. At the same time, we see that this complexity is entirely due to a quadrature, the effect of which must finally disappear.

Is it possible to obtain the required result without having recourse to this finally useless mediation?

I thought it worth while to attain this, though we cannot do so without introducing a rather paradoxical notion which I shall now speak of.

time, this is already proved to be possible for every term except $\xi(t_0)M_{t_0}'$, so that the existence of M_{t_0}'' is necessary; and so on for all following derivatives, so that the conclusion in the text is proved.

As, under $\int$, the degree of the coefficient of M_r decreases by one unit from each operation to the following one, the $\left(\frac{m}{2}-1\right)$th derivative contains no such integral term and is therefore represented by an $\mathbf{S}$ extended over the spherical edge $r=t_0$, as happens for the case of (e_3). We shall come back to that fundamental difference between even and odd values of m.

80. We again take the differentiation we had to deal with, or, even more simply, we take the corresponding questions for simple integrals. Let us start from the integral

$$\int_a^b \frac{A(x)}{\sqrt{b-x}}\,dx. \tag{13}$$

If we tried to differentiate it with respect to b, we have already remarked that we should have to use, in the common conception, a suitable device (easily to be found), a direct differentiation appearing as impossible: indeed the latter would consist in writing the absurd expression

$$-\frac{1}{2}\int_a^b \frac{A(x)\,dx}{(b-x)^{\frac{3}{2}}} + \left[\frac{A(x)}{\sqrt{b-x}}\right]_{x=b}, \tag{13'}$$

a sum of two terms, the first of which has no meaning as containing an infinity of order $\frac{3}{2}$ under $\int$ and the second being evidently meaningless.

There is nevertheless an immediate means of performing directly (i.e., without any transformation) this differentiation: it would consist in replacing the real integral (13) by the half of the complex integral taken along a circuit constituted by two lines along ab, connected by a small circle around b (fig. 9): for such a circuit, differentiation presents no difficulty*.

Fig. 9.

80*a*. Of course we must have somehow a means of doing this without introducing complex quantities. Indeed we have only to note that (replacing b by x in the upper limit), not the integral

$$\int_a^x \frac{A(x)}{(b-x)^{\frac{3}{2}}}\,dx, \tag{14}$$

but the algebraic sum

$$\int_a^x \frac{A(x)}{(b-x)^{\frac{3}{2}}}\,dx - 2\,\frac{A(x)}{\sqrt{b-x}},$$

approaches a perfectly definite limit when x approaches b. Moreover, the same takes place for

$$\int_a^x \frac{A(x)}{(b-x)^{\frac{3}{2}}}\,dx + \frac{B(x)}{\sqrt{b-x}} \tag{15}$$

* We here suppose A to be analytic: a hypothesis which is easy to get rid of if only A is supposed to have a derivative.

if B be any function of x, provided it is differentiable (or at least satisfies Lipschitz's condition $|B(x_2) - B(x_1)| < K|x_2 - x_1|$), and such that $B(b) = -2A(b)$.

Furthermore, *the result obtained is independent of the choice of this function* B, under the above conditions: this being due to the fact that the denominator is of a fractional order, while a change of the function B (under our hypothesis) would alter it by terms containing at least $(b-x)$ in the first power as factor, so that the corresponding terms in the fraction would necessarily vanish for $x = b$. Therefore, in order to calculate the limit of (15), we do not even need to indicate what special function B we choose. We denote that limit by "the finite part" of the integral in (13′), and write it

(16) $$\left|\overline{\int_a^b \frac{A(x)}{(b-x)^{\frac{3}{2}}} dx},\right.$$

the sign $|\overline{\quad}$ being read "finite part of".

To conform with what precedes, this expression shall be taken to mean the limit of the sum of integral (14) and an additive fractional term in $(b-x)$ of the form $\frac{B(x)}{\sqrt{b-x}}$, taking for B any function such that:

it can be differentiated at least once (or at least satisfies Lipschitz's condition);

the sum in question does have a limit (the value of this limit being independent of the choice of the additional term, provided B fulfils the aforesaid conditions).

But the above definition supposes A itself to satisfy Lipschitz's condition.

If A is analytic, the expression (16) can just as well be defined as half of the corresponding integral taken along the aforesaid circuit.

81. No difficulty arises in defining the same symbol for higher orders of infinity, provided they always are fractional. The integral

$$\int_a^b \frac{A(x)}{(b-x)^{p+\frac{1}{2}}} dx$$

is meaningless (p being any integer), but we can define the quantity

(16′) $$\left|\overline{\int_a^b \frac{A(x)}{(b-x)^{p+\frac{1}{2}}} dx}\right.$$

(the "finite part" of the integral in question):

(1) If A is analytic, as half of the corresponding integral taken along the above-mentioned circuit;

(2) A being supposed only to admit of p derivatives in the vicinity of b, as the limit for $x=b$, of the sum

$$(15') \qquad \int_a^x \frac{A(x)}{(b-x)^{p+\frac{1}{2}}} dx + \frac{B(x)}{(b-x)^{p-\frac{1}{2}}},$$

$B(x)$ being again any function bound by the conditions:

(a) that the limit in question must exist;

(b) that B must admit of p derivatives, at least in the vicinity of $x=b$.

That both definitions agree is immediately verified by performing the calculation (see below).

Again, the arbitrary choice of B has no influence on the value of the limit obtained; for condition (a) determines the values of the $(p-1)$ first derivatives of B in b, so that what remains arbitrary in the numerator of our additive term is at least an infinitesimal of the order of $(b-x)^p$.

We may say briefly—the sense of which we hope to have made clear by the above explanations—that we give a meaning to our integral by *removing fractional infinities* at b.

We must not forget, however, that A itself is supposed to admit of the corresponding derivatives at b.

82. Of course, we could also introduce the above conception for the integral

$$\left| \overline{\int_a^b \frac{A(x)\,dx}{(b-x)^{p+\mu}}} \right.,$$

μ being no longer necessarily equal to $\frac{1}{2}$, but still being necessarily contained between 0 and 1, limits excluded: which quantity can be defined upon the same hypotheses as (16) or (16').

It can also, as in the above cases, be expressed by means of a complex integral taken along the circuit of fig. 9, an integral which ought, this time, to be divided by $1-e^{2i\pi\mu}$. It can also, therefore, be considered as obtained by differentiation of

$$(12') \qquad \int_a^b \frac{A(x)\,dx}{(b-x)^{\mu}}.$$

The integral

$$\int_a^b \frac{A(x)\,dx}{[(x-a)(b-x)]^{p+\frac{1}{2}}},$$

where the integrand is infinite at both limits, is the half of the corresponding complex integral along the circuit of fig. 9*a*. For the analogous one

$$\int_a^b \frac{A(x)\,dx}{[(x-a)(b-x)]^{p+\mu}},$$

we ought to give the circuit such a form as is represented in fig. 9*b*, in order that the integrand should come back with a final value equal to its initial one.

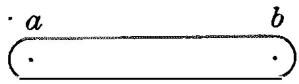

Fig. 9*a*.

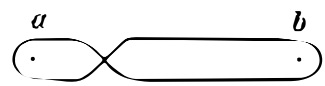

Fig. 9*b*.

It is also clear that other functions than powers of $(b-x)$ could be introduced and treated in the same way; for instance,

$$\frac{1}{(b-x)^{p+\mu}} \log (b-x).$$

83. Such considerations would even hold good to a certain extent for

$$(17) \qquad \int_a^b \frac{A(x)}{(b-x)^p}\,dx,$$

with p integral.

This could be reduced to a finite value by adding the terms

$$(17a) \qquad \frac{B(x)}{(b-x)^{p-1}} + B_1(x) \log (b-x).$$

But, for $p>1$, we could, by adding to $B(x)$ terms in $(b-x)^{p-1}$, modify the result in an arbitrary manner. This result, then, is not determined when we merely know the integral (17), but requires that the additive terms (17*a*) be given as well.

The same does not apply to $p=1$. But, on the other hand, the result obtained is not invariant when the variable is changed, as is seen below in the case of μ fractional. Some operations of this kind, but of course with an explicit specification of the additive terms (17*a*), have nevertheless been used in Calculus: such is the case for

Cauchy's "principal value," and also for some forms of the second derivatives of a Newtonian space potential (such as will be used further on, § 115a).

84. Actual calculation. A simple way of obtaining the quantity (16) consists in finding first

$$(18) \qquad \overline{\int_a^b \frac{dx}{(b-x)^{\frac{3}{2}}}} = -\frac{2}{(b-a)^{\frac{1}{2}}},$$

then replacing $A(x)$ by $[A(x) - A(b)] + A(b)$, so that our expression is resolved into

$$\int_a^b \frac{A(x) - A(b)}{(b-x)^{\frac{3}{2}}} dx,$$

an ordinary improper integral (as A is assumed to satisfy Lipschitz's condition) and

$$\frac{2A(b)}{(b-a)^{\frac{1}{2}}}.$$

Similarly, to calculate $\overline{\int_a^b \frac{A(x)\,dx}{(b-x)^{p+\frac{1}{2}}}}$ we shall subtract from $A(x)$ its expansion in powers of $(b-x)$ by Taylor's formula up to the term in $(b-x)^{p-1}$, by which our expression becomes an ordinary integral; then we must integrate (according to our meaning) such terms as $\overline{\int_a^b \frac{dx}{(b-x)^{q+\frac{1}{2}}}}$, the value of which is $\frac{1}{(q-\frac{1}{2})}\frac{1}{(b-a)^{q-\frac{1}{2}}}$, so that

$$(16a) \qquad \overline{\int_a^b \frac{A(x)\,dx}{(b-x)^{p+\frac{1}{2}}}} = -\frac{A(b)}{(p-\frac{1}{2})(b-a)^{p-\frac{1}{2}}} + \dots$$
$$-\frac{(-1)^{p-1}}{(p-1)!}\frac{A^{(p-1)}(b)}{\frac{1}{2}(b-a)^{\frac{1}{2}}} + \int_a^b \frac{A_1(x)\,dx}{(b-x)^{p+\frac{1}{2}}},$$

$$A_1(x) = A(x) - [A(b) - A'(b)(b-x) + \dots + (-1)^{p-1}A^{(p-1)}(b)(b-x)^{p-1}].$$

This is equivalent to using our former definition and taking, for $B(x)$,

$$B(x) = \frac{A(b)}{(p-\frac{1}{2})(b-x)^{p-\frac{1}{2}}} - \frac{A'(b)}{(p-\frac{3}{2})(b-x)^{p-\frac{3}{2}}} + \dots + \frac{(-1)^{p-1}}{(p-1)!}\frac{A^{(p-1)}(b)}{\frac{1}{2}(b-x)^{\frac{1}{2}}}.$$

If we take as understood that $B(x)$ is chosen in that way, we see that what we may call the "remainder" of our improper integral,—i.e. the difference between (15′) and its limit—is $\int_x^b \frac{A_1(x)}{(b-x)^{p+\frac{1}{2}}}\,dx$. We, therefore, shall have an upper limit of it—viz. $M\,|\,b-x\,|$—if we have one for the absolute value of $\frac{A_1(x)}{(b-x)^p}$ or, which comes to the same, an upper limit M for the pth derivative of A (divided by $p!$) in the neighbourhood of $x=b$.

If A is a function not only of x, but of several parameters $\alpha, \beta, \ldots$ (on which b may also depend), but M is independent of $\alpha, \beta, \ldots$, the remainder can be evaluated in terms of $|b-x|$ independently of $\alpha, \beta, \ldots$: we may say that (16′) converges *uniformly*.

85. Principal properties. The rules of calculation concerning such a symbol as (16′) are generally identical to rules relating to ordinary integrals as concerns *equalities*, such as $\int_a^b = \int_a^c + \int_c^b$ and so on. Especially *a changing of the variable* is allowed, provided it be regular in b; that is, one variable has with respect to the other a derivative, finite and different from zero, so that the order of infinitesimals around b is not changed.

But any property implying *inequality* requires once more due precautions. First, we cannot conclude anything as to the sign of the expression $\overline{\int \frac{A(x)\,dx}{(b-x)^{p+\frac{1}{2}}}}$ from the knowledge of the sign of the function A in our interval of integration, as the example of (18) immediately shows.

Limitation of the values of our improper integrals.—This applies in particular to the finding of upper limits for the values of such expressions as (16′). For this object, it is not sufficient, as it would be for ordinary integrals, to have upper limits of the integrand and of the interval of integration.

Calculating

$$I = \overline{\int_a^b \frac{A(x)\,dx}{(b-x)^{p+\frac{1}{2}}}} \qquad (16')$$

as explained in § 84, we immediately (on account of the well-known expression of the remainder of Taylor's series) find that

$$(19) \qquad I \leqslant \frac{|A(b)|}{(p-\frac{1}{2})(b-a)^{p-\frac{1}{2}}} + \frac{|A'(b)|}{(p-\frac{3}{2})(b-a)^{p-\frac{3}{2}}} + \cdots$$
$$\cdots + \frac{|A^{(p-1)}(b)|}{(p-1)!\,.\,\frac{1}{2}\,.\,(b-a)^{\frac{1}{2}}} + \frac{2\,(A_p)}{p!}\sqrt{b-a},$$

where (A_p) is an upper limit for the modulus of the pth derivative of A in (a, b).

Therefore, we can limit the absolute value of I if we have:

1°. An upper *and a lower* limit of the interval of integration;

2°. Upper limits of the absolute values of the function A and its first p derivatives* (of A itself and the first $(p-1)$ derivatives at b itself, of the pth throughout the interval); or, at least [as our interval can be resolved into $(a, b-\epsilon)$ and $(b-\epsilon, b)$] of the absolute value of A throughout (a, b) (as usual), of the $(p-1)$ first derivatives for $x=b$ and of the pth in a certain partial interval $(b-\epsilon, b)$ adjacent to b, the *reciprocal* of the amplitude of which also enters into the limitations, so that, if a approached b, our improper integral would not approach zero, but, generally, infinity.

86. Continuity. Replacing the function A by another one $\bar{A}$, whereby I is changed into $\bar{I}$, and applying the above limitation to the difference $(I-\bar{I})$, we see that *the value of our symbol* (16′) *is continuous of order p*, but not, of course, of order zero, *with respect to the function A*.

* It is easy to give instances of expressions such as (16′) that assume values as great as can be desired although A remain finite. We only need to take the following

$$I = \left| \overline{\int_0^a \frac{f(Nx)\,dx}{x^{p+\frac{1}{2}}}} \right.,$$

N being a very large positive number and f a finite function for any value, however great, of x. By effecting the change of variable $Nx=z$, it appears immediately that we shall get the asymptotic equality

$$I = N^{p-\frac{1}{2}}\, I_1,$$
$$I_1 = \left| \overline{\int_0^\infty \frac{f(z)\,dz}{z^{p+\frac{1}{2}}}} \right. .$$

If, then, I_1 differs from zero, I will increase indefinitely with N.

87. Differentiation. From the first conception by which we obtained our new symbol, we immediately see that *it admits directly of differentiation with respect to b,* which is to be performed by differentiating under $\int$ and not writing any term corresponding to the upper limit, the latter terms being included in the fractional infinite terms which are meant to be added in order to make the integral have a limit.

It follows from this that *any (linear) differential equation which would be satisfied by our integral* (considered as a function of b) *if taken between constant limits a, c, is so as well when one limit is b itself.* This is, as we said in § 73, Darboux's fundamental remark.

3. THE CASE OF MULTIPLE INTEGRALS

88. The above notion will be extended to multiple integrals by the usual reduction to simple ones. Let us take (in ordinary space, for instance) such an integral as

$$\iiint_T \frac{A(x, y, z)}{[G(x, y, z)]^{p+\frac{1}{2}}}\, dx\, dy\, dz, \tag{20}$$

one part of the boundary of the domain of integration T being constituted by the surface $G = 0$, with the essential hypothesis that this part of the boundary contains no singular point, i.e. that at no point of it the first partial derivatives of G simultaneously disappear. Then, for any neighbouring point, the distance to the boundary (or rather to the aforesaid part of it) is exactly of the same infinitesimal order as the value of G.

Let us assume, in the first place, that $\frac{\partial G}{\partial z}$ especially is everywhere $\neq 0$, and even that any parallel to the z-axis cuts the surface in question in not more than one point $z = z_1$ and *at a finite angle,* so that we can write $G = (z - z_1)\, G_1$. Let us assume, moreover, for the present, that every part of the boundary adjacent to $G = 0$ consists of a cylindrical surface parallel to the z-axis (fig. 10). Then we shall write, by definition,

$$\overline{\iiint_T \frac{A}{G^{p+\frac{1}{2}}}\, dx\, dy\, dz} = \iint dx\, dy \left|\overline{\int_{z_1}^{Z} \frac{A}{G^{p+\frac{1}{2}}}\, dz}\right. \tag{21}$$

$$= \iint dx\, dy \left|\overline{\int_{z_1}^{Z} \frac{A\, dz}{G_1^{p+\frac{1}{2}} (z - z_1)^{p+\frac{1}{2}}}}\right.$$

(if, for instance, the segment (z_1, Z) intercepted by T on any parallel to the z-axis has z_1 for its lower limit and if the upper one Z corresponds to no singularity of the integrand).

If ζ be a function of x, y and a small parameter ϵ, infinitesimal together with ϵ and developable in powers of ϵ at least to the pth order, the term in ϵ itself being always different from zero, this means that we take the limit of

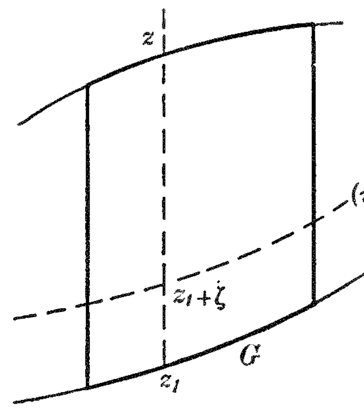

Fig. 10.

$$(22) \qquad \iint dx\,dy \int_{z_1+\zeta}^{Z} \frac{A}{G^{p+\frac{1}{2}}}\,dz$$

after we have subtracted suitable terms, infinite of fractional order in ϵ (viz. of the form

$$\frac{\mathscr{B}(x, y, \epsilon)}{\epsilon^{p-\frac{1}{2}}} = \frac{\mathscr{B}_0(x, y) + \epsilon\mathscr{B}_1 + \ldots + \epsilon^{p-1}\mathscr{B}_{p-1}}{\epsilon^{p-\frac{1}{2}}})$$

under the assumption that the convergence of $\left|\int_{z_1}^{Z} \frac{A}{G^{p+\frac{1}{2}}}\,dz\right|$ is uniform (§ 84) when x and y vary, in order that exchange of our limiting process and integration with respect to x, y be allowed, which will be the case if $\dfrac{\partial^p\left(\dfrac{A}{G_1^{p+\frac{1}{2}}}\right)}{\partial z_1^p}$ be contained between finite limits all over $G = 0$ and in its neighbourhood. On the other hand, G is supposed to admit of derivatives up to the pth order with respect to x, y, z, so that such is also the case, on the surface considered, for z as a function of x and y.

This definition, in its turn, is obviously equivalent to the following one:

Let the neighbourhood of $G = 0$ be separated from our domain T by a surface (τ) such as

$$(\tau) \qquad G = \gamma(x, y, z, \epsilon)$$

in which γ denotes a quantity having with zero a neighbourhood of the pth order (§ 20),—that is, very small together with its partial derivatives up to the pth order, when ϵ approaches zero. For instance, let γ be equal to $D\epsilon$, denoting by D a differentiable expression independent of ϵ. Then, if it is first extended to the domain T_1 deduced from T by cutting out the neighbourhood T_2 of our surface, our integral will not approach a limit when ϵ approaches zero; but it will do so if we subtract a properly chosen expression of the form

$$\frac{B(\epsilon)}{\epsilon^{p-\frac{1}{2}}} = \frac{B_0 + B_1\epsilon + \ldots + B_{p-1}\epsilon^{p-1}}{\epsilon^{p-\frac{1}{2}}} \tag{23}$$

(which, of course, completely determines the coefficients B). This limit is equal to (21)—for (τ) can obviously be written in the form $z = z_1(x, y) + \zeta$;—and, therefore, is utterly independent of the choice of the function D or even γ, under the above specified conditions.

89. But this new form of the definition is also independent of our previous restrictions concerning the location of our domain T with respect to the axes of coordinates. The calculation itself can be made independent of them (under conditions of regularity which we shall presently specify) by the use of a punctual transformation.

Assuming now any number m of dimensions, let us refer the neighbourhood of $G=0$ to a system of curvilinear coordinates, one of which shall be G, the others being denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$, such that they admit of derivatives up to the pth order with respect to the Cartesian coordinates and that the Jacobian K never vanishes, the element of volume being

$$dT = dx_1 dx_2 \ldots dx_m = K d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}\, dG$$

(in which $K d\lambda_1\, d\lambda_2 \ldots d\lambda_{m-1}$ is precisely what we previously called dT_G or the quotient $\frac{dT}{dG}$). If so, the lines

$$\lambda_1 = \text{const.}, \ \lambda_2 = \text{const.}, \ \ldots \ \lambda_{m-1} = \text{const.},$$

which we shall denote by l and which shall now replace the parallels to the z-axis, must cut $G = 0$ at finite angles.

Let us assume, in the first place, that every part of our boundary adjacent to $G=0$ is a locus of coordinate lines l. We must immediately note that this hypothesis, together with those which we have already made, implies:

(1) that the surface $G = 0$ is regular (§ 9): more precisely that one of its Cartesian coordinates, considered as a function of the others, has continuous partial derivatives up to the pth order;

(2) that the same property belongs to every part of the boundary adjacent to $G=0$;

(3) that such boundaries cut $G=0$ at an angle which never vanishes (nor becomes equal to π).

These conditions are necessary for the validity of the definition we are about to give. Conversely, if they are satisfied, we can, in ∞ ways, find such a system of curvilinear coordinates as is assumed above. Then, we may take

$$\mathbf{SSS}_T \frac{A}{G^{p+\frac{1}{2}}} dx_1 dx_2 \ldots dx_m = \mathbf{SSS}_T \frac{A}{G^{p+\frac{1}{2}}} K d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1} dG$$
$$= \mathbf{SS} d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1} \int \frac{AK}{G^{p+\frac{1}{2}}} dG,$$

this third form of the definition being again obviously equivalent to the second one (and therefore independent of the particular punctual transformation used, under the above assumption)*.

We obtain the finite part of the simple integral l by taking it, not from $G=0$, but from $G=\gamma$, then adding a certain complementary term

$$\frac{\mathscr{B}}{\gamma^{p-\frac{1}{2}}},$$

where $\mathscr{B}$ is a regular function of $\lambda_1, \lambda_2, \ldots \lambda_{m-1}, \gamma$ (and, therefore, also of the x's). Integrating with respect to the λ's, we shall obviously have an $(m-1)$tuple integral along (τ), viz.

$$\mathbf{SS} \frac{\mathscr{B}}{\gamma^{p-\frac{1}{2}}} d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}, \tag{24}$$

and the value of (21′) will be obtained by adding $\mathbf{SSS}_{T_1}$ and (24), then letting ϵ (and therefore γ) approach zero.

For $p=1$, the complementary term (24) can be written so that its independence of the choice of our punctual transformation is thrown into evidence, viz. (on account of our previous remark concerning $K d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}$)

$$-2\mathbf{SS} \frac{A}{\sqrt{\gamma}} d\tau_G.$$

* It may often happen that the use of our curvilinear coordinates will only be possible in the neighbourhood of the singular surface G. It will then be expedient to divide T into two parts, a central one T' where the integral is to be calculated in the ordinary fashion, and another T'' including the whole neighbourhood of G, where we have to use the form in the text.

90. It will be useful for what follows to notice that one of our preceding geometric restrictions can be dropped. If, for instance, in (20), we intend to begin by integrating with respect to z, we can do so even if some parts H of the boundary of T, adjacent to G, are inclined with respect to the z-axis.

To see this*, we again cut T (the two parts thus separated being again called T_1 and T_2) by (τ), which surface cuts H along a certain edge (in ordinary space, a line) Λ. If, through every point of Λ, we draw a parallel to the z-axis, till it meets G, we thus generate a cylinder $\mathscr{C}$, the region inside which will be the region filled by the parallels to the z-axis which meet (τ) inside T. Let $\mathcal{T}$ be the part common to that interior region and to T; J the integral

$$J = \overline{\left| \iiint \frac{A}{G^{p+\frac{1}{2}}}\, dx\,dy\,dz \right.}$$

extended to $\mathcal{T}$, while I_1 will be the (ordinary) integral extended to T_1.

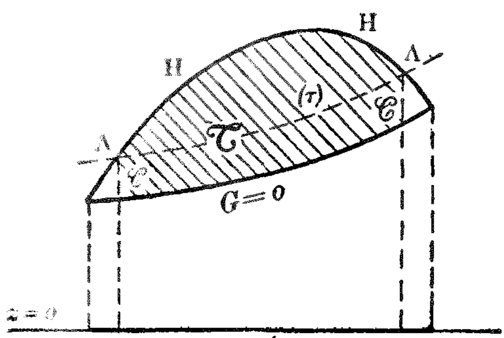

Fig. 11.

On account of the cylindrical boundary of $\mathcal{T}$, the integral J is to be expressed as explained above, viz.

$$J = \iint_{s'} dx\,dy\, \overline{\left| \int_{z_1}^{Z} \frac{A}{G^{p+\frac{1}{2}}}\, dz \right.},$$

z_1 still being the ordinate of G and the double integration being extended to the base s' of the cylinder $\mathscr{C}$ on the xy plane. If $z_1 + \zeta$ again denotes the corresponding ordinate of (τ), and ζ depends on ϵ as above specified, we have, by the definition of our symbol,

$$j(x, y) = \overline{\left| \int_{z_1}^{Z} \right.} = \int_{z_1+\zeta}^{Z} + \frac{\mathscr{B}(x, y, \epsilon)}{\epsilon^{p-\frac{1}{2}}} + \eta(x, y),$$

* We suppose, for convenience in treatment, that we are in the case (which is the one that interests us) where the useful parts of H—and even the whole domain T—project on to the xy plane inside the area s of integration for I. The diagrammatic figure 11 represents a section of T, $\mathcal{T}$ (the latter shaded on the diagram), etc., by a plane parallel to the z-axis.

$\mathscr{B}(x, y, \epsilon)$ being a function which admits of Maclaurin's expansion in powers of ϵ at least up to the pth order and $\eta = \eta(x, y, \epsilon)$ an infinitesimal. This gives, by integrating over s',

$$\iint_{s'} j(x, y)\, dx\, dy = J = I_1 + \iint \eta\, dx\, dy + \frac{1}{\epsilon^{p-\frac{1}{2}}} \iint_{s'} \mathscr{B}(x, y, \epsilon)\, dx\, dy.$$

But the double integral in η is again infinitesimal when ϵ is*; and, in the last term, because of the assumed regularity of (τ) and H, the second factor (in which account must be taken of the simultaneous variation of the integrand and the area of integration) admits of successive derivatives† in ϵ and is therefore itself a Maclaurin expansion (at least up to the $(p-1)$th order), so that

$$(25) \qquad J - I_1 = \epsilon_1 + \frac{B(\epsilon)}{\epsilon^{p-\frac{1}{2}}},$$

where ϵ_1 means the infinitesimal quantity $\iint \eta\, dx dy$ and $B(\epsilon)$ is again regular in ϵ.

If we now denote by I the integral (21) which we intend to calculate, the difference $I - J$ will, in general, be infinite (this being the case for the simple integrals along segments of generatrices of the cylinder $\mathscr{C}$ when their length approaches zero); but this infinity will be a fractional one of the form (23) as both $J - I_1$ (as has just been proved) and $I - I_1$ (by definition) are of that form. This is equivalent to saying that

$$(26) \qquad I = \left|\overline{\iint_s j(x, y)\, dx\, dy}\right. = \left|\overline{\iint_s dx\, dy \left|\overline{\int_{z_1}^{Z} \frac{A}{G^{p+\frac{1}{2}}}\, dz}\right.}\right.,$$

so that the calculation of the finite part of our triple integral is reduced to two "finite parts," one of a simple, the other of a double integral.

All this obviously holds for integrals with any number of dimen-

* We admit that η tends towards zero uniformly with respect to x, y, which is legitimate (§ 84) when the pth derivative $\frac{\partial^p}{\partial z_1^{\,p}}\left(\frac{A}{G_1^{\,p+\frac{1}{2}}}\right)$ is finite in the whole neighbourhood of G.

† However, we must note that this assumes the existence of derivatives of B up to the order $(p-1)$ with respect to x, y, z, so that, as B contains $(p-1)$ derivatives of A and G_1 (with respect to z), derivatives of the integrand up to the order $2p-2$ are postulated.

sions, and also when parallels to the z-axis are replaced by lines l such as considered above.

Such lines are not, therefore, bound to be drawn on the non-singular part H of the boundary.

As to the condition that the angle between one of them and the singular part G nowhere becomes infinitesimal, it remains essential, as we shall see by examples, when using the present mode of evaluation.

91. The same applies to our first hypothesis concerning the regularity of our surface.

If the boundary surface $G = 0$ possess a singular point a (which will precisely be the case in the application we shall meet with), we must proceed just as for ordinary improper integrals, that is, first cut out the neighbourhood of this singular point by a small portion of surface Σ; then pass to the limit. Only Σ must have with the singular point a neighbourhood of the pth order, i.e., by a natural extension of § 20, not only must the radius vector* from the singular point to any point of Σ be very small, but also its derivatives up to the pth order with respect to the direction cosines of its direction, which condition, however, is quite generally fulfilled (for instance if Σ is derived from a regular fixed surface by homothecy with respect to the singular point, or by translation, etc.). Whether the limit exists (though general sufficient conditions for this could probably be formed without great difficulty) shall be investigated in each case.

92. The above considerations, nevertheless, still apply when G is the product of two factors $G = G'G''$, in such a way that $G = 0$ is composed of two parts $G' = 0$, $G'' = 0$, which intersect: this is what would happen, for instance, if T were a rectangle, G denoting the product of the four sides.

* Such radii vectores may be meant to be rectilinear ones, but as well (which is equivalent, on account of the classic rules of Differential Calculus) may be taken along any regular set of curves, i.e. any set of curves through a, depending on $m-1$ parameters (for instance, $m-1$ direction cosines of their tangents at a) and such that continuous derivatives of the coordinates up to the pth order with respect to these parameters and the arc s exist, the Jacobian never vanishing. We shall have to operate in this way, using the geodesics (§§ 55 ff.) from a, on which occasion we shall come back to the subject (see § 106).

Let us suppose, for instance, that $G = xy$, the domain of integration lying in the region $x \geqslant 0$, $y \geqslant 0$. Then, to define the integral

$$\iint_T \frac{A(x, y)\,dxdy}{x^{p+\frac{1}{2}}\,y^{p+\frac{1}{2}}},$$

we shall begin by limiting the integration to $x \geqslant \epsilon'$, $y \geqslant \epsilon''$—denoting by ϵ', ϵ'' two small positive numbers—under which conditions [as appears if we expand $A(x, y)$ in powers of x, y and calculate

$$\iint \frac{dx\,dy}{x^{q+\frac{1}{2}}\,y^{r+\frac{1}{2}}}, \qquad (q, r = 0, 1, 2, \ldots, p)]$$

the integral can be made finite by subtracting complementary terms in

$$(27) \qquad \frac{B}{\epsilon'^{p-\frac{1}{2}}}, \quad \frac{B}{\epsilon''^{p-\frac{1}{2}}}, \quad \frac{B}{\epsilon'^{p-\frac{1}{2}}\,\epsilon''^{p-\frac{1}{2}}},$$

B denoting quantities regular in ϵ', ϵ''. The general case of $G = G'G''$ being reduced to the former by punctual transformation (in which G' and G'' are to be taken as new variables x and y*), the same result will be then obtained by complementary terms of the above form (if T_1 is separated from T_2 by the surfaces $G' = \epsilon'$, $G'' = \epsilon''$), with the same meaning for B. As the third class of terms possesses, like the two first, the property of never remaining finite for all infinitesimal values of ϵ', ϵ'' without vanishing, the preceding theory is again applicable: i.e., there will be an infinity of ways of choosing the terms (27) so as to obtain a finite limit, but the latter will have the same value in every case.

Nothing similar will occur in general for other kinds of singularities of $G = 0$. We shall have to take the precautions alluded to in the foregoing section, and shall find that they actually change the result.

93. We can repeat for multiple integrals all what we have said concerning equalities and inequalities. Especially, *changing the variables* is allowed if mutual derivatives up to the pth order exist and if the Jacobian vanishes at no point on the surface of singularity $G = 0$.

* We simplify the argument in the text by assuming that we are given not only the two surfaces $G' = 0$, $G'' = 0$, but the left-hand sides G', G'' of their equations (which assumption was not wanted in the above); this condition will be satisfied in the application which we shall have to make of the present section, so that we need not take the separating surfaces (τ) under the more general form of § 88.

Differentiation, when the parameter influences the shape of the singular part of the boundary $G=0$, is to be performed without writing any term corresponding to this effect (since such is the case for the simple integral along each line l).

An *upper limit* is obtained by integrating along the l's* and applying to each simple integral formula (19) (§ 85). For this purpose, of course, we must again know upper limits for the integrand and its derivatives up to the pth order, and conversely this will be sufficient if we know upper and lower limits for the lengths of the arcs of lines l included in T.

94. Briefly speaking, our new symbol consists in giving our integral a value by subtraction of fractional infinities. In the calculation, therefore, it happens that we have to cancel such fractional infinities: if two different integrals of the above kind extended to the same domain T are such that, when extended to T', they differ by a quantity which we know to be necessarily of the form (23), their finite parts must be equal and, therefore, no account is to be taken of the difference in question.

For instance, in Green's formula (g), let us assume that the integrand on the left-hand side is of the form $\frac{A}{G^{p+\frac{1}{2}}}$ and the P_i's of the form $\frac{B_i}{G^{p-\frac{1}{2}}}$, where the B's are again regular functions of the x's. If the boundary of the domain of integration T is entirely constituted by a surface $G=0$ (on which we still make the same general assumptions), *the finite part of the corresponding multiple integral will be zero.* For, if it is first taken within the surface

$$(\tau) \qquad G=\gamma(x_1, x_2, \ldots, x_n, \epsilon),$$

this integral will (on account of the identity (g) in question) be reduced to a fractional infinity in ϵ: which is equivalent to the above conclusion.

If some parts S' only of the boundary belong to $G=0$, only the

* If we operated as in § 90, we ought to know upper limits for the derivatives up to the order $2p-2$; on the other hand it would no longer be necessary to know *lower* limits for the lengths of the arcs of lines l included in T (but only of the angles between these lines and $G=0$).

SS relating to its remaining portions S''—or rather the finite parts of these **SS**—will have to be written down: formula (g) is to be written

$$\left|\overline{\mathbf{SSS}}_T = -\right|\overline{\mathbf{SS}}_{S''}.$$

The same becomes obvious on using complex variables as in §§ 80, 81. In order to simplify the geometric interpretation, let us limit ourselves to a double integral $\iint \frac{A(x, y)\, dx dy}{x^{p+\frac{1}{2}}}$ extended over a rectangle having one side along $x = 0$. Leaving y real, we shall replace x by $x + ix'$ and consider x' as a third coordinate in a three-dimensional space. Then I is equal to half the integral over a double sheet which should be folded around $x = 0$ and cover our rectangle twice (or if preferred, over an infinitely flattened elliptic half cylinder having a focal line along $x = 0$ and the opposite side for its axis).

If we should deal with $\iint \frac{A(x, y)\, dx dy}{[x(a - x)]^{p+\frac{1}{2}}}$, the rectangle of integration having $x = 0$ and $x = a$ for its opposite sides, we should consider a segment of a whole elliptic cylinder having these for focal lines.

In all these cases, our relation with a curvilinear integral would appear immediately with the help of the above transformation.

4. Some important examples

95. We shall have to consider, from the above point of view, the integral

$$\int \frac{dz}{(z^2 - \alpha)^{n+\frac{1}{2}}}.$$

Let us take it first between $+\sqrt{\alpha}$ and the fixed number $z_1 > \sqrt{\alpha}$. For $n = 0$, we have

$$\int_{\sqrt{\alpha}}^{z_1} \frac{dz}{\sqrt{z^2 - \alpha}} = \log \frac{z_1 + \sqrt{z_1^2 - \alpha}}{\sqrt{\alpha}} = -\tfrac{1}{2} \log \alpha + P(\alpha),$$

where

$$P(\alpha) = \log (z_1 + \sqrt{z_1^2 - \alpha}) = \log z_1 + \log \left(1 + \sqrt{1 - \frac{\alpha}{z_1^2}}\right)$$

is a series in integral and positive powers of α.

We deduce from this, by differentiation,

$$(28)\qquad \int_{\sqrt{a}}^{z_1} \frac{dz}{(z^2-\alpha)^{n+\frac{1}{2}}} = \frac{(-1)^n}{2nC_n}\frac{1}{\alpha^n} + \frac{\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\frac{d^n}{d\alpha^n}P(\alpha),$$

C_n denoting the numerical coefficient*

$$(29)\qquad C_n = \frac{\frac{1}{2}\cdot\frac{3}{2}\ldots(n-\frac{1}{2})}{1\;2\ldots n} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}$$

$$= \frac{1}{\pi}B(n+\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{nB(n,\frac{1}{2})}.$$

For $z_1=\infty$, the last term on the right-hand side of (28) disappears (as $P(\alpha)$ is in powers of $\frac{\alpha}{z_1^2}$, and therefore every term but the first contains z_1 in its denominator): whence†

$$(30)\qquad \int_{\sqrt{a}}^{\infty} \frac{dz}{(z^2-\alpha)^{n+\frac{1}{2}}} = \frac{(-1)^n}{2nC_n\,.\,\alpha^n}\qquad (n \text{ an integer} > 0).$$

We must equally note the value of the integral for $n<0$, i.e.

$$(28')\qquad \int_{+\sqrt{\alpha}}^{z_1} dz\,(z^2-\alpha)^{n'-\frac{1}{2}} = \frac{(-1)^{n'-1}}{2} C_{n'}\alpha^{n'}\log\alpha + P_1(\alpha)$$

$$(n' \text{ an integer} \geqslant 0),$$

C_n denoting the same numerical coefficient as before. This is again deduced from the case $n'=0$ by integration with respect to α,—the value of the integral being evident for $\alpha=0$,—or more simply, by expanding the integrand‡ in powers of $\frac{\alpha}{z^2}$.

* Especially, $C_0=1$, $C_1=\frac{1}{2}$.

† The value of (30) could also be found from another point of view, giving a good example of our first definition in § 80. If we replace the real segment $(\sqrt{\alpha}, \infty)$ by a circuit around it, the corresponding integral—which is twice (30)—can be transformed, by Cauchy's theorem, into an integral along the imaginary axis, equal to $e^{-ni\pi}\int_{-\infty}^{+\infty}\frac{dz_1}{(\alpha+z_1^2)^{n+\frac{1}{2}}}$; the latter quantity is immediately and classically reduced to an Eulerian integral B.

‡ The term in α^n is the only one which contains z to the powers -1 and, therefore, leads to a logarithmic result.

96. But the formulæ relating to the cases where the limits are $-\sqrt{\alpha}$, $+\sqrt{\alpha}$ are especially important for what follows.

The integral

$$(31)\qquad \left|\int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} \frac{dz}{(\alpha - z^2)^{n+\frac{1}{2}}}\right| \qquad (n \text{ an integer} > 0)$$

is zero.

The integral

$$\int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} (z^2 - \alpha)^{n'-1} \sqrt{\alpha - z^2}\, dz \qquad (n' \text{ an integer} \geqslant 0)$$

is equal to $2\pi A$, *A denoting the coefficient of* $\log \alpha$ *in expression* (28′), i.e.

$$(31')\qquad \int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} (z^2 - \alpha)^{n'-1} \sqrt{\alpha - z^2}\, dz = (-1)^{n'-1} \pi C_{n'} \alpha^{n'},$$

so that

$$(31a)\qquad \int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} (\alpha - z^2)^{n'-\frac{1}{2}}\, dz = \pi C_{n'} \alpha^{n'}.$$

This form of the result, implying a relation between (28′) and (31′) (which relation may also be considered as holding between (28) and (31)), will be of special interest for us later on. It can be established by considering (31′) as a period of (28′). If we let α, starting from a determinate positive value, come back to it by a direct circuit around the origin in the complex plane, the point $\sqrt{\alpha}$ performs, as we know, a half circuit, going from $+\sqrt{\alpha}$ to $-\sqrt{\alpha}$. As, simultaneously, z_1 remaining fixed (at a finite or infinite distance according to the case), (28′) increases by $2i\pi A$, our relation is proved.

Of course, the same formulæ (31′), or rather (31a), can be obtained directly, and precisely in terms of Eulerian integrals of the first kind: for, by the change of variable $z = \sqrt{\alpha t}$, integral (31a), which is also immediately (setting $z = \sqrt{\alpha} \cos \phi$) reducible to Wallis' integral, becomes

$$\int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} dz\, (\alpha - z^2)^{n'-\frac{1}{2}} = \alpha^{n'} \int_0^1 (1-t)^{n'-\frac{1}{2}} t^{-\frac{1}{2}}\, dt = \alpha^{n'} B(n' + \tfrac{1}{2}, \tfrac{1}{2}).$$

If we differentiate this with respect to α, which, as we have seen, may be accomplished by mere differentiation under $\int$ and without writing any terms for limits, this gives, by a sufficient number of differentiations, the value 0 for (31).

Similarly, q being any positive integer, we have, for $n' \geqslant 0$ (the left-hand side being again reducible to Eulerian integrals as well as to Wallis'),

(32) $$\int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} z^{q-1}(\alpha - z^2)^{n'-\frac{1}{2}}\,dz$$
$$= \alpha^{n'+\frac{q-1}{2}} \int_0^1 t^{\frac{q-2}{2}}(1-t)^{n'-\frac{1}{2}}\,dt = \alpha^{n'+\frac{q-1}{2}} B\left(n'+\tfrac{1}{2}, \frac{q}{2}\right).$$

Again starting from $n' = 0$, we shall obtain finite parts of integrals containing $(1-t)^{n+\frac{1}{2}}$ or $(\alpha - z^2)^{n+\frac{1}{2}}$ in the denominator by differentiating with respect to α (or by a classic integration by parts, with respect to t, applied to the second form of the integral). We thus see:

(1) that

$$\left|\overline{\int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} \frac{z^{q-1}}{(\alpha - z^2)^{n+\frac{1}{2}}}\,dz}\right. \quad \text{or} \quad \left|\overline{\int_0^1 \frac{t^{\frac{q-2}{2}}}{(1-t)^{n+\frac{1}{2}}}\,dt}\right.$$

is zero, when q is odd and $n > \frac{q-1}{2}$;

(2) that otherwise

$$\left|\overline{\int_{-\sqrt{\alpha}}^{+\sqrt{\alpha}} \frac{z^{q-1}}{(\alpha - z^2)^{n+\frac{1}{2}}}\,dz}\right. = \alpha^{\frac{q-1}{2}-n} \frac{(q-\frac{1}{2})(q-\frac{3}{2})\dots\left(\frac{q+1}{2}-n\right)}{(-\frac{1}{2})(-\frac{3}{2})\dots(-n+\frac{1}{2})} B\left(\tfrac{1}{2}, \frac{q}{2}\right)$$
$$= \alpha^{\frac{q-1}{2}-n} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q-1}{2}-n\right)} \cdot \frac{\Gamma(\frac{1}{2}-n)}{\Gamma(\frac{1}{2})} B\left(\tfrac{1}{2}, \frac{q}{2}\right).$$

It is almost evident—and is immediately verified—that, expressing the symbol B in terms of Γ-functions, the numerical factor will be found to be the same as in (32), except for the change of n' into $-n$, i.e. *equal to* $B\left(\frac{1}{2}-n, \frac{q}{2}\right)$.

Especially

(33) $$\left|\overline{\int_{-1}^{+1} \frac{z^{q-1}}{(1-z^2)^{n+\frac{1}{2}}}\,dz}\right. = \left|\overline{\int_0^1 \frac{t^{\frac{q}{2}-1}}{(1-t)^{n+\frac{1}{2}}}\,dt}\right. = B\left(\tfrac{1}{2}-n, \frac{q}{2}\right),$$

so that, in this case, we get formulæ exactly similar to (32) except for the introduction of our symbol $\left|\overline{}\right.$.

97. The volume of m-dimensional hyperboloids.—Let us consider the m-dimensional quadric

$$(H_1) \qquad x_1^2 + x_2^2 + \ldots + x_{m-1}^2 - x_m^2 = 1,$$

analogous to the hyperboloid of one sheet, and the (m-dimensional) volume between this quadric and the "asymptotic cone"

$$(C) \qquad x_1^2 + x_2^2 + \ldots + x_{m-1}^2 - x_m^2 = 0.$$

To calculate this, we can evaluate $x_1, x_2, \ldots, x_{m-1}$ in terms of

$$r = \sqrt{x_1^2 + x_2^2 + \ldots + x_{m-1}^2},$$

and of angular parameters $\phi_1, \ldots \phi_{m-2}$, defining a direction from the origin in the $(m-1)$-dimensional space $(x_1, x_2, \ldots x_{m-1})$: by which

$$dx_1 dx_2 \ldots dx_{m-1} = r^{m-2}\, dr\, d\Omega_{m-2},$$

where $d\Omega_{m-2}$ (corresponding to variations of the ϕ's) is an element of surface of the sphere of radius 1 in the $(m-1)$-dimensional space. Summing, in the first place, for all possible directions in the latter, we express our volume by the double integral

$$(34) \qquad \Omega_{m-2} \iint r^{m-2}\, dr\, dx_m.$$

Ω_{m-2} denotes the surface of the hypersphere with radius 1 in the $(m-1)$-dimensional space. It is equal to $\dfrac{2\,[\Gamma(\frac{1}{2})]^{m-1}}{\Gamma\left(\frac{m-1}{2}\right)}$, i.e. to $2\dfrac{\pi^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)}$ (the volume of the same sphere is $\dfrac{1}{m-1}\Omega_{m-2}$, and we may observe that this allows us to speak even of Ω_0, the value of which is $\int_{-1}^{+1} dx = 2$).

We ought to extend (34) between one branch of the hyperbola $r^2 - x_m^2 = 1$ and its asymptotes. Now, if we set down

$$x_m = rz,$$

$z =$ constant will represent, in the r, x_m plane, the radius vector from the origin and, in our original m-dimensional space, a hypercone. The volume enclosed between two such consecutive hypercones

$(z,\ z+dz)$ and our quadric (easily calculated by expressing the variables x_m and r in (34) in terms of z and $\rho=\sqrt{r^2-x_m^2}$, and letting ρ vary from zero to 1) will be

$$\frac{\Omega_{m-2}}{m}\frac{dz}{\sqrt{1-z^{2}}^{m}}.$$

If we should integrate between $-z$ and $+z$, with $0<z<1$, this would give us the volume between the two corresponding cones and our quadric.

If m is odd, there will be a finite part for such an integral, when z approaches 1. It will be said, by definition, to be "the finite part of the volume between the quadric and its asymptotic cone."

From the conclusions of the preceding section, we see that *this finite part is zero.*

Let us take, on the other hand, the quadric

$$(\mathrm{H}_2)\qquad x_1^2+x_2^2+\dots+x^2_{m-1}-x_m^2=-1,$$

which, for $m=3$, is the *hyperboloid of two sheets.* Let us again consider the volume between the sheet corresponding to $x_m>0$ and the asymptotic cone (C).

This volume will be represented by the integral

$$\frac{\Omega_{m-2}}{m}\int\frac{dz}{\sqrt{z^2-1}^{m}},$$

which we must take between 1 and $+\infty$. *If m is odd* and equal to $2m_1+1$, the expression thus obtained has a finite part, which will be called the "finite part" of the aforesaid volume z. *The value of this finite part will be*

$$(35)\qquad \frac{\Omega_{2m_1-1}\cdot(-1)^{m_1}}{(2m_1+1)\,2m_1C_{m_1}}.$$

For instance, as concerns the ordinary hyperboloid of two sheets, we have

$$\frac{2\pi}{3}\left|\int_0^{\infty}\frac{dz}{(z^2-1)^{\frac{3}{2}}}\right.=-\frac{2\pi}{3}.$$

If our hyperquadric be given under the more general form

$$\mathbf{H}(x_1,\ x_2,\dots x_m)=1$$

(**H** being any quadratic form with one positive and $m-1=2m_1$ negative squares), our result is obviously to be divided by $+\sqrt{|D|}$, D being the discriminant of **H**.

The sign of the finite part obtained is, as seen from (35), variable according to what was said in § 85: it depends on the parity of $m_1=\frac{m-1}{2}$.

98. A similar treatment would apply to the hyperquadric

$$\text{(H}') \qquad x^2+x_2^2+\ldots+x_p^2-y^2-y_2^2-\ldots-y_q^2=1,$$

which occurs in connection with the equations $\Delta^{p,\,q}u=0$ of Coulon (see § 69). Introducing the auxiliary variables

$$r=\sqrt{x_1^2+x_2^2+\ldots+x_p^2}, \quad r'=\sqrt{y_1^2+y_2^2+\ldots+y_q^2},$$

the calculation of our volume would, as above, be brought back to the double integral

$$\Omega_{p-1}\,\Omega_{q-1}\iint r^{p-1}\,r'^{q-1}\,dr\,dr',$$

which, by $r'=rz$, would lead to a simple integral

$$(36) \qquad \frac{\Omega_{p-1}\,\Omega_{q-1}}{p+q}\int\frac{z^{q-1}\,dz}{\sqrt{1-z^{2}}^{\,p+q}}.$$

If we extend this integral between $z=-1$ and $z=+1$, the finite part of it will have a meaning for $p+q$ odd. *For q odd it will be zero* (see § 96).

Only in the case of q even, p odd, will the finite part of the volume between the quadric (H′) and its asymptotic cone $\sum_1^p x^2-\sum_1^q y^2=0$ exist and differ from zero: its value will be $B\left(\frac{p+q}{2}-1,\frac{q}{2}\right)\frac{\Omega_{p-1}\Omega_{q-1}}{p+q}$.

99. The numbers Ω have very simple relations with the coefficients C_n introduced above: we have

$$\Omega_{m-1}=\Omega_{m-2}\cdot\frac{m}{m-1}\,\pi\,C_{\frac{m}{2}}=\Omega_{m-2}\,\pi\,C_{\frac{m}{2}-1}$$

(and, more generally, relations between Ω_{p-1}, Ω_{q-1} and Ω_{p+q-1}).

This is immediately verified, but is hardly distinct from the argument in the preceding section: the classic determination of Ω_{m-1} (as a subcase of Dirichlet's integral) is entirely similar to the above calculation; the latter applied to the volume of the m-dimensional sphere with radius 1 $\left(\text{equal to } \frac{1}{m}\Omega_{m-1}\right)$ reduces it to $\Omega_{m-2}\iint r^{m-2}\,dr\,dx$, the latter double integral being therefore extended to $r^2+x^2 \leqslant 1$ and immediately reducible to (31′) as above.

100. If we divide the volume of one of our hyperboloids of $m=2m_1+1$ dimensions by a central plane (which is assumed to cut the surface) the latter can always be considered as a diametral plane and therefore the two partial volumes thus separated must be equivalent to each other: so that the finite part of the volume of the half hyperboloid of one sheet is zero, and the finite part of the volume of the half hyperboloid of two sheets is equal to half the value obtained in § 97.

A consequence is that any such two planes, the intersection of which lies outside the asymptotic cone (if the hyperboloid of two sheets is concerned), include between them in the portion of space enclosed between the surface and its asymptotic cone an infinite volume the finite part of which is again zero.

101. Our notion of improper integral, as developed in the above, allows us to find the relation between Tedone's expressions and the elementary solution. It is to be foreseen (by the first example of $m=3$) that the latter, in order to admit of ordinary Calculus, must be first integrated several times—at least m_1 times—with respect to t_0.

Now, such successive (or rather superimposed) integrations of a function $F(t)$ from a common lower limit T to the upper one t' can again be replaced by a single one with the help of a factor $(t'-t)^{n-1}$, so that, for the study of the generalized equation of cylindrical waves (e_{m-1}) for $m=2m_1+1$, starting from the elementary solution

$$\frac{1}{[(t_0-t)^2-r^2]^{m_1-\frac{1}{2}}},$$

we have to deduce from it the definite integral

$$\int_T^{t'}\frac{(t'-t_0)^{n-1}\,dt_0}{[(t_0-t)^2-r^2]^{m_1-\frac{1}{2}}},$$

in which, as we did in § 73, we shall take t' independent of t, but the lower limit T equal to $t+r$.

Tedone's expressions correspond* to a number of integrations much greater than strictly necessary, viz. $n=m-2=2m_1-1$ (the advantage being to obtain rather simple expressions which only depend on the ratio $(t'-t):r$). His function† is, but for a numerical factor, equal to a definite (improper) integral‡

$$\mathbf{v}_1 = \int_{t+r}^{t'} \frac{(t'-t_0)^{2m_1-2}}{[(t_0-t)^2-r^2]^{m_1-\frac{1}{2}}}\, dt_0.$$

That applying the fundamental formula to such a quantity together with the unknown function u, leads to the value of (10′) (§ 78), whence u follows by differentiations, is now evident *a priori*. We see that the (m_1-1)th derivative of u could have been used for the same purpose.

* In Tedone's notation, our number m is denoted by $m+1$ and (for odd m's) our number m_1 is called p.

† ϕ_2, in his notation.

‡ $\mathbf{v}_1$, considered as a function of t and the x's, satisfies (e_{m-1}), as results from our principles; this and the fact of being a mere function of $\frac{t'-t}{r}$ (Tedone's variable θ) together with its logarithmic singularity (which is easily shown by the principles of the theory of functions) prove its identity with Tedone's expression. A direct calculation, giving $\mathbf{v}_1$ under Tedone's form, is obtained by setting down

$$\frac{t_0-t-r}{t_0-t+r}=\tau,$$

and, similarly,

$$\frac{t'-t-r}{t'-t+r}=\tau',$$

which gives

$$\mathbf{v}_1 = \left| \int_0^{\tau'} \frac{1}{(1-\tau')^{2m_1-2}} \frac{(\tau'-\tau)^{2m_1-2}}{\tau^{m_1-\frac{1}{2}}(1-\tau)} d\tau \right.$$

$$= \int_0^{\tau'} \frac{d\tau}{(1-\tau)\sqrt{\tau}} + \frac{1}{(1-\tau')^{2m_1-2}} \left| \int_0^{\tau'} \frac{1}{\tau^{m_1-\frac{1}{2}}} \frac{(\tau'-\tau)^{2m_1-2}-\tau^{m_1-1}(\tau'-1)^{2m_1-2}}{1-\tau} d\tau. \right.$$

The first term is the logarithmic one; and, as

$$\frac{(\tau'-\tau)^{2m_1-2}-\tau^{m_1-1}(\tau'-1)^{2m_1-2}}{1-\tau}$$

$$=[\tau(\tau'-1)+\tau'(\tau'-\tau)]\,\Sigma\,(\tau'-\tau)^{2(m_1-h-1)}\tau^{h-1}(1-\tau')^{2h-2},$$

the remaining (improper) integral can be expanded in powers of

$$\frac{4\tau'}{(1-\tau')^2}=\frac{(t'-t)^2-r^2}{r^2},$$

with the common factor

$$1+\frac{2\tau'}{1-\tau'}=\frac{t'-t}{r^2},$$

the coefficients being Eulerian functions B (see § 96).

CHAPTER II

THE INTEGRATION FOR AN ODD NUMBER OF INDEPENDENT VARIABLES

102. With these principles in mind, we can come to Cauchy's problem for the equation

$$\text{(E)} \qquad \mathscr{F}(u) = \sum_{i,k} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu = f.$$

From our previous considerations, it is to be foreseen that we shall have to distinguish between the cases of m even and m odd. We shall begin with the latter. The elementary solution is then unique. It contains no logarithmic term, but has an irrational denominator.

We shall have to assume that solution to have been constructed not for the given equation, but for the adjoint one*

$$(\mathscr{E}) \qquad \mathscr{G}(v) = 0,$$

the elementary solution of which will be of the same form, viz.

$$v = v(x;\ a) = \frac{V}{\Gamma^{m_1 - \frac{1}{2}}},$$

for $m = 2m_1 + 1$; in which Γ is still the square of the geodesic distance between the two points $x(x_1, x_2, \dots x_m)$ and $a(a_1, a_2, \dots a_m)$, while V is a holomorphic function of the $2m$ coordinates of these two points, taking on, when they coincide, the value $\dfrac{1}{\sqrt{|\Delta|}}$.

For the present, we shall suppose that our equation belongs to the hyperbolic type, and even to the normal one, so that the characteristic form

$$\mathbf{A} = \Sigma A_{ik} \gamma_i \gamma_k$$

consists of squares all of which but one have the same sign. Then, the characteristic conoid consists of two different sheets and divides the space into three regions, two of which are interior, viz. one inside

* The adjoint equation will always be taken homogeneous (i.e. $\mathscr{G}(v) = 0$) even if the proposed equation is non-homogeneous ($\mathscr{F}(u) = f \neq 0$).

each sheet. We assume the equation to be written in such a way that Γ is positive in these interior regions (physically speaking, positive when the two points x and a are well within wave with respect to each other in the meaning of § 32). This is the case when the form $\mathbf{A}$ has one positive and $m-1$ negative squares*.

102 *a*. Cauchy's problem consists in finding a solution u of (E) when we know the values of u and one of its first derivatives at every point of a certain surface S. As the knowledge of any first derivative (u being assumed to be known all over S) is equivalent to the knowledge of any other one (provided the corresponding direction is *not tangent to* S), we shall assume that the derivative in question is the *transversal* one† (§ 41).

We intend to calculate the values of u in a certain region $\mathscr{R}$ of the m-dimensional space; and it will be assumed that if, from any point a of $\mathscr{R}$ as vertex, we draw the characteristic conoid, one of its sheets will cut out a certain portion S_0 (finite in every direction) of S and, together with S_0, be the boundary of a portion T of our space. This geometric condition is expressed by saying that we have to deal with the *interior* problem (we thus see that no interior problem exists for non-normal hyperbolic equations).

We also know that it is essential to say what we precisely mean by a solution; and, here, we shall begin by understanding this in a rather restrictive manner, viz., by admitting that u has to admit of partial derivatives up to the order m_1,—or, at least, to the order (m_1-1) satisfying Lipschitz's condition—a restriction which would seem a very artificial one if it were not for the remarks of § 78, but which will now appear as justified.

This being admitted, we shall see that the above data allow us to calculate u and, further on, that, conversely, upon one geometric hypothesis more (viz., that S is everywhere duly inclined (§ 27)) and some hypotheses of regularity concerning the data (which correspond to the preceding ones on u itself), the function u thus determined

* This would lead us to change the sign in equation (e_2), instead of writing it as in § 4 *a*.

† If S is not characteristic, the transversal direction will not be tangent to S. For the contrary case, see below, § 113.

satisfies all required conditions. (On the contrary, no *exterior* Cauchy problem admits of a solution for arbitrary regular data.)

103. Accordingly, a $(a_1, a_2, \dots a_m)$ being any point of $\mathscr{R}$ at which we intend to calculate the value of u, we draw one half characteristic conoid Γ with vertex a, the one which cuts S along a closed edge and which we shall call (§ 32) the *inverse* or *retrograde* sheet. Let T be the portion of $\mathscr{R}$ thus limited, i.e., which is both interior to Γ and on the same side of S as a (fig. 12).

We shall apply the fundamental formula* in the domain T to the unknown u and to the elementary solution $v(x;\ a)$ of the adjoint equation which is singular in a.

The quantity $vf = \dfrac{fV}{\Gamma^{m_1 - \frac{1}{2}}}$ is infinite on one part of the boundary, viz. the conoid Γ: it is an infinite quantity of the fractional order $m_1 - \frac{1}{2}$. The m-tuple integral bearing on this quantity is subject, then, to the consideration explained above (these becoming unnecessary only for $m_1 = 1$, i.e. $m = 3$).

But there is an exception for the neighbourhood of the point a, where Γ is infinitely small of the order 2, not of the order 1. We shall therefore have to proceed as is done in the case of ordinary multiple integrals, and abstract from the domain of integration all the neighbourhood of the point a, by means of a small surface Σ (fig. 12) surrounding this point, it being understood that the infinitely close vicinity between Σ and the point a is of the m_1th order (§ 91), such being the case, for instance, if we take a small sphere with centre a.

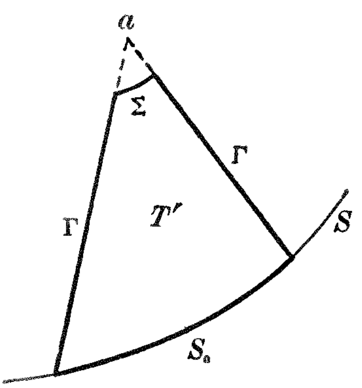

Fig. 12.

Let T' stand for what remains of T after all the part which lies inside Σ has been removed.

The fundamental formula

$$\text{(F)}\quad \mathbf{SSS}\, vf\, dx_1\, dx_2 \dots dx_m + \mathbf{SS}\left(v\frac{du}{d\nu} - u\frac{dv}{d\nu} + Luv\right) dS = 0$$

* Fig. 12 is supposed to be, as previously, a diagrammatic one, obtained by cutting the one which we have to consider (and which has m dimensions) by a two-dimensional plane drawn through the point a.

will be applied in the domain T', with the convention that we take the finite part of the left-hand side, by omitting the infinite quantities fractional on Γ. We must therefore

(1) Take the finite part of the first integral in (F) (m-tuple integral with respect to T');

(2) Take likewise the finite part of the $(m-1)$-tuple integral with respect to the given multiplicity S;

(3) *Cancel the integral relating to the boundary* Γ, this integral being an infinite quantity of a fractional order. This same integral vanishes in Kirchhoff's method, because it integrates exactly, and in Volterra's because it is identically zero. The same thing happens here, as we see, by a different mechanism, which we have explained in § 94. The integral in question being of a fractional order, and the complementary boundary terms implicitly understood in (1), (2), and (4) (see below) also being such*, the very fact that the sum of the integrals (F) vanishes implies that the sum of these four fractional infinite quantities also cancels out separately;

104. (4) If the shape of T were such that a is exterior to it, as happens in fig. 12 a or 12 b, we should have to apply merely the fundamental formula, thus interpreted; and this would give

$$\text{(F}'\text{)} \quad \left|\overline{\mathbf{SSS}\, vf\, dx_1\, dx_2 \ldots dx_m} + \left|\overline{\mathbf{SS}\left(u\frac{dv}{d\nu} - v\frac{du}{d\nu} - Luv\right) dS} = 0,\right.\right.$$

which corresponds to the classic formula in the theory of potential (formula (8) in Book I, § 15) for the case when the origin of the radii vectores lies outside the surface of integration.

But in the present case, we must take account of the integral relating to Σ, of which we must take (as on S) the finite part. We have to see what this quantity becomes when Σ approaches the point a.

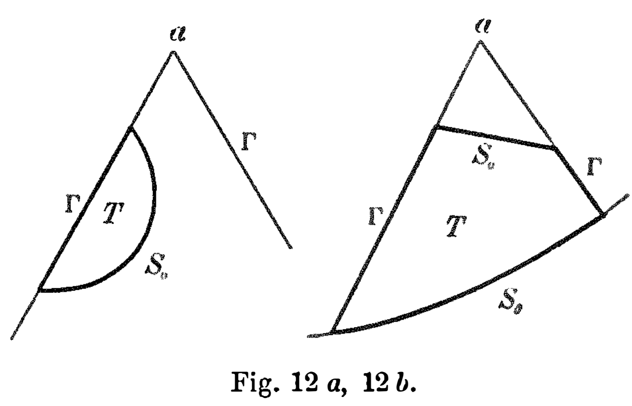

Fig. 12 *a*, 12 *b*.

* In (1), this complementary term would be extended to Γ or rather to a surface coinciding finally with Γ; in (2) and (4), the limiting positions for the domains of integration corresponding to the complementary terms would be the edges of intersection of Γ with S and Σ.

Through this point, let us draw geodesics (defined by the differential equations (L)), inside T, and depending on $m-1$ parameters $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$, while, on each one of them, any point will be defined by the variable s of § 55. s will have, at each point of Σ, a determinate value (a function of $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$), which will be very small, together with its derivatives* up to the order m_1. The integral

$$(37) \qquad \left|\overline{\mathbf{SS}\left(v\frac{du}{d\nu}+Luv\right)dS}\right. - \left|\overline{\mathbf{SS}\frac{\left(V\frac{du}{d\nu}+Luv\right)}{\Gamma^{\frac{m-2}{2}}}dS}\right.$$

will approach zero. For the quantities

$$\pi_i dS = \pm \frac{D(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_m)}{D(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})} d\lambda_1, d\lambda_2, \ldots, d\lambda_{m-1},$$

which appear in the numerator under $\mathbf{SS}$, are of the order s^{m-1}, whereas the denominator

$$\Gamma^{\frac{m-2}{2}} = s^{m-2}\,\mathbf{H}^{\frac{m-2}{2}}(x_1', x_2', \ldots, x_m'),$$

where $x_i' = \dfrac{dx_i}{ds}$, contains as a factor only s^{m-2}. The coefficient of

$$\frac{1}{\mathbf{H}^{\frac{m-2}{2}}(x_1', x_2', \ldots, x_m')}$$

is therefore of the order of s, and the same is true of its derivatives of the various orders with respect to $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$. Under these conditions, the evaluations of §§ 85 and 93 show that the integral (37) is also of the order of s.

This applies again, in the term

$$-\left|\overline{\mathbf{SS}\, u\frac{dv}{d\nu}dS},\right.$$

to the part

$$\left|\overline{\mathbf{SS}\frac{u\frac{dV}{d\nu}}{\Gamma^{\frac{m-2}{2}}}dS},\right.$$

in which Γ is not differentiated.

* This results from the principles in the Additional Note to Book II. See below, § 106.

105. Lastly, let us take the remaining part

$$(37') \quad +\frac{m-2}{2}\,\Big|\mathbf{SS}\frac{uV\dfrac{d\Gamma}{d\nu}}{\Gamma^{\frac{m}{2}}}dS = +\frac{m-2}{2}\,\Big|\mathbf{SS}\frac{\frac{1}{2}uV.\Sigma\dfrac{\partial\Gamma}{\partial x_i}\dfrac{\partial\mathbf{A}}{\partial\pi_i}.dS}{\Gamma^{\frac{m}{2}}}.$$

We have (still with the notations of §§ 55–58)

$$\frac{\partial\Gamma}{\partial x_i} = 2sp_i$$

and consequently

$$\sum_i\frac{\partial\Gamma}{\partial x_i}\frac{\partial\mathbf{A}}{\partial\pi_i} = 2s\sum_i p_i\frac{\partial\mathbf{A}}{\partial\pi_i} = 2s\sum_i\pi_i\frac{\partial\mathbf{A}}{\partial p_i} = 4s\sum\pi_i\frac{dx_i}{ds}.$$

But, if we refer back to the expression of $\pi_i dS$ written as above, we shall see that the coefficient of $4s$ (multiplied by dS) is in fact a development of the determinant

$$\begin{vmatrix} \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \dots, & \dfrac{dx_m}{ds} \\ \dfrac{\partial x_1}{\partial\lambda_1}, & \dfrac{\partial x_2}{\partial\lambda_1}, & \dots, & \dfrac{\partial x_m}{\partial\lambda_1} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial x_1}{\partial\lambda_{m-1}}, & \dfrac{\partial x_2}{\partial\lambda_{m-1}}, & \dots, & \dfrac{\partial x_m}{\partial\lambda_{m-1}} \end{vmatrix} d\lambda_1\, d\lambda_2 \dots d\lambda_{m-1},$$

the λ's being taken in such an order as to make the determinant positive, since the direction of s increasing is away from a and entering into the domain T'.

All the rows of the above determinant, except the first, contain s as a factor*, so that the integrand (for determinate values of the λ's) remains finite when s approaches zero.

On the other hand, we have, practically, on Σ,

$$uV = \frac{1}{\sqrt{\Delta_a}}u(a_1, a_2, \dots, a_m) = \frac{1}{\sqrt{\Delta_a}}u_a,$$

the word "practically" being used, of course, with a slightly different meaning from that which it usually has in such a case, and signifying

* It is true that each element contains, besides a part proportional to s, a part containing the derivatives of that quantity with respect to the λ's: but upon the assumption in the text concerning Σ, this will not change the order of magnitude of our determinant.

that the neglected quantities, *together with their derivatives with respect to the* λ's (up to the order m_1), are very small.

Still from the same point of view, we can replace the $\frac{dx_i}{ds}$'s by the values x_i' which they have at the origin, and the derivatives $\frac{dx_i}{d\lambda}$ by $s\frac{\partial x_i'}{\partial \lambda}$: thus we get the expression

$$(38) \qquad \frac{(m-2)\,u_a}{\sqrt{\Delta}} \mathbf{SS} \frac{\begin{vmatrix} x_1' & x_2' & \cdots & x_m' \\ \frac{\partial x_1'}{\partial \lambda_1} & \frac{\partial x_2'}{\partial \lambda_1} & \cdots & \frac{\partial x_m'}{\partial \lambda_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1'}{\partial \lambda_{m-1}} & \cdots & \cdots & \frac{\partial x_m'}{\partial \lambda_{m-1}} \end{vmatrix} d\lambda_1\, d\lambda_2 \ldots d\lambda_{m-1}}{\mathbf{H}^{\frac{m}{2}}(x_1', x_2', \ldots, x_m')}.$$

The integral which is multiplied by $(m-2)\,u_a$ is easy to reduce to the result of § 97. Generally speaking, if through the origin we draw a variable straight line

$$\frac{x_1}{\alpha_1} = \frac{x_2}{\alpha_2} = \ldots = \frac{x_m}{\alpha_m}$$

whose direction depends on $m-1$ parameters $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$, and choose on it one point

$$P\,(x_1 = \alpha_1 s,\ x_2 = \alpha_2 s,\ \ldots,\ x_m = \alpha_m s)$$

(denoting by s another function of the λ's), the integral

$$\frac{1}{m}\mathbf{SS} \begin{vmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_m \\ \frac{\partial \alpha_1}{\partial \lambda_1}, & \frac{\partial \alpha_2}{\partial \lambda_1}, & \ldots, & \frac{\partial \alpha_m}{\partial \lambda_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \alpha_1}{\partial \lambda_{m-1}}, & \frac{\partial \alpha_2}{\partial \lambda_{m-1}}, & \ldots, & \frac{\partial \alpha_m}{\partial \lambda_{m-1}} \end{vmatrix} s^m d\lambda_1\, d\lambda_2 \ldots d\lambda_{m-1}$$

will represent* the volume between the surface S described by P and

* For, if dS be an element of the surface S and $\pi_1, \pi_2, \ldots, \pi_m$ the direction-

the cone with the outline of S for its base and the origin for its vertex.

The **SS** of formula (38) is therefore equal to $m=2m_1+1$ times the finite part of the volume of the hyperboloid of two sheets, as is calculated in § 97, and found equal to

$$\frac{1}{2m_1+1}\,\frac{(-1)^{m_1}\,\Omega_{2m_1-1}}{2m_1 C_{m_1}}\cdot\frac{1}{\sqrt{D}},$$

D being the discriminant of **H**. The integral along Σ has therefore (on account of the equality $D\Delta=1$) the limiting value

$$\frac{(-1)^{m_1}\,\Omega_{2m_1-1}}{C_{m_1}}\cdot\frac{2m_1-1}{2m_1}\,u_a=(-1)^{m_1}\frac{\Omega_{2m_1-1}}{C_{m_1-1}}\cdot u_a$$
$$=(\S\ 99)\ (-1)^{m_1}\,\pi\,\Omega_{2\ \ 1-2}\cdot u_a$$

and the required value of u_a is given by the formula

$$(39)\quad (-1)^{m_1}\pi\,\Omega_{2m_1-2}\cdot u_a=(-1)^{m_1}\frac{\Omega_{2m_1-1}}{C_{m_1-1}}\cdot u_a$$
$$=-\left|\overline{\mathbf{SSS}_T\, vf\, dx_1,\, dx_2,\, \ldots,\, dx_m}\right.$$
$$+\left|\overline{\mathbf{SS}_{S_0}\left(u\frac{dv}{d\nu}-v\frac{du}{d\nu}-Luv\right)dS}\right.$$
$$=-\left|\overline{\mathbf{SSS}_T\, vf\, dx_1\ldots\ldots dx_m}+\right|\overline{\mathbf{SS}_{S_0}\left(u_0\frac{dv}{d\nu}-u_1 v-Lu_0 v\right)dS}.$$

cosines of its outer normal, we have, account being taken of our values of $\pi_1 dS$, $\pi_2 dS$, ...,

$$V=\mathbf{SS}\,\frac{1}{m}\,(\pi_1 x_1+\ldots+\pi_m x_m)\,dS$$

$$=\frac{1}{m}\,\mathbf{SS}\begin{vmatrix} x_1 & x_2 & \ldots & x_m\\ \dfrac{\partial x_1}{\partial\lambda_1} & \dfrac{\partial x_2}{\partial\lambda_1} & \ldots & \dfrac{\partial x_m}{\partial\lambda_1}\\ \ldots & \ldots & \ldots & \ldots\\ \dfrac{\partial x_1}{\partial\lambda_{m-1}} & \dfrac{\partial x_2}{\partial\lambda_{m-1}} & \ldots & \dfrac{\partial x_m}{\partial\lambda_{m-1}}\end{vmatrix}d\lambda_1\,d\lambda_2\ldots d\lambda_{m-1}.$$

If we now replace x_i by α_i's, and, therefore, $\dfrac{\partial x_i}{\partial\lambda_k}$ by $s\,\dfrac{\partial\alpha_i}{\partial\lambda_k}+\alpha_i\dfrac{\partial s}{\partial\lambda_k}$, the second terms in this latter expression for $i=1, 2, \ldots, m$ can be cancelled as being proportional to the x's which constitute the first row of the determinant, and this gives the result in the text.

For $m = 3\ (m_1 = 1)$, the coefficient of u_a is

$$-\Omega_1 \frac{1}{2C_1} = -2\pi\ (= -\pi\Omega_0).$$

106. A consequence of the above is that the first **SSS** in (39) exists when extended to T, at least if we still let Σ approach a in such a way that the vicinity be of the m_1th order. We can also see this directly. Calculating it with the system of curvilinear coordinates $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, s$, we shall get

$$\left|\mathbf{SSS}_T\, vf\, dx_1\, dx_2 \ldots dx_m\right.$$

$$= \left|\mathbf{SSS} \frac{Vf}{s^{m-2}\mathbf{H}^{\frac{m-2}{2}}} \frac{D(x_1, \ldots, x_m)}{D(\lambda_1, \ldots, \lambda_{m-1}, s)} d\lambda_1\, d\lambda_2 \ldots d\lambda_{m-1}\, ds.\right.$$

Now, the functional determinant $\dfrac{D(x_1, x_2, \ldots, x_m)}{D(\lambda_1, \lambda_2, \ldots, s)}$ contains s^{m-1} as a factor, and therefore the quantity under **SSS** thus only contains the fractional infinity

$$\frac{1}{\mathbf{H}^{\frac{m-2}{2}}\left(\frac{x_1 - a_1}{s}, \frac{x_2 - a_2}{s}, \ldots, \frac{x_m - a_m}{s}\right)}.$$

Under this form or the preceding one we see that the error committed by substituting T' for T, the difference between the values of **SSS** extended over T' and over T, is limited in terms of:

The partial derivatives of f up to the order $m_1 - 1$ and of the coefficients of the equation up to the order m_1;

The corresponding derivatives of V;

The partial derivatives, up to the order m_1, with respect to the λ's, of the coordinates of the point where each geodesic issuing from a intersects Σ or, which is equivalent, of the corresponding value of s: these derivatives being themselves limited* by the corresponding derivatives of the x's with respect to s on the geodesic, the correspond-

* The choice of the parameters λ and the proportionality factor for s (see § 57) on each geodesic are assumed to be such as to satisfy usual conditions of regularity (regularity of the initial values of the x's with respect to the λ's).

ing partial derivatives of the left-hand side $\Sigma(x_1, \ldots, x_m)$ of the equation of Σ and by the reciprocal of $\frac{d\Sigma}{ds}$ [as is seen by expressing the x's and thereby Σ in terms of $(\lambda_1, \ldots, \lambda_{m-1}, s)$—say

$$\Sigma(x_1, \ldots, x_m) = \Phi(\lambda_1, \ldots, \lambda_{m-1}, s)$$

—and differentiating the implicit equation

$$\Phi(\lambda_1, \ldots, \lambda_{m-1}, s) = 0].$$

This proves to us, what will be useful in several circumstances, that the **SSS** in (39) *converges uniformly**: that is, in a region where the coefficients of the equation are holomorphic (therefore, V holomorphic in the $2m$ variables which it contains), f regular (derivatives continuous up to the order $m_1 - 1$), we do not need to know the position of a in order to indicate a (very small) upper limit for the above-mentioned error if we only know (very small) upper limits for the distance between Σ and the point a,—say ϵ—and for the partial derivatives (up to the order m_1) of the x's with respect to the λ's. The latter limit will exist and be very small with ϵ (a lower limit of $\frac{d\Sigma}{ds}$ inside the conoid being known), if the derivatives of any order $k \leqslant m_1$ of Σ with respect to the x's remain finite, or even if their products by ϵ^{k-1} remain, in absolute value, below a fixed limit† as would take place, for instance, if Σ were a sphere of radius ϵ around a.

* It must be emphasized that the meaning of this word is distinct from what it was in § 84.

† This is equivalent to saying that the absolute values of the corresponding derivatives on Σ_1 all lie below a fixed upper limit, Σ_1 being deduced from Σ by a homothecy whose pole is a and ratio $\frac{1}{\epsilon}$.

Such a homothecy would alter the derivatives of the x's with respect to $\lambda_1, \ldots, \lambda_{m-1}, s$, but not the order of magnitude of their quotients by s (which, as to derivatives with respect to s, would even be diminished, as is easy to see).

Such an alteration would be avoided if, before submitting the diagram to homothecy, we could transform it by introducing "normal" coordinates (Book II, § 57): a punctual transformation which, as we know, would not change the order of magnitude of our derivatives if the derivatives of the A_{ik} are finite up to the same order of differentiation, augmented by one (see Additional Note to Book II).

107. The above formula has a meaning only if the surface S is regular and the quantities u_0, $u_1 - \frac{du}{d\nu}$ differentiable (with respect to regular coordinates of S, or, e.g. to $(m-1)$ of the Cartesian coordinates). As a corresponding assumption for u has been made from the beginning (§ 102), the above argument would not be sufficient to prove that these conditions are necessary for the existence of u, and that our method did not omit some solution, if the latter were not differentiable a sufficient number of times.

But an answer to this is given by the example of § 78, concerning the case of (e_{m-1}), and where the non-existence (in general) of the solution appears by a direct calculation.

The same method (imitating Tedone's) could be used for the general equation (E): in other words, we could, by repeated integration along a line $\mathscr{L}$, deduce from v another solution $\mathbf{v}$ of the adjoint equation admitting of a logarithmic singularity along $\mathscr{L}$, but becoming zero (and not infinite) along Γ; then obtain, by its help, the value of such an integral as (10′) and, finally, find u itself by $(m-2)$-fold differentiation. If this differentiation be possible, the result is unique, so that:

The solution will not generally exist when our formula (39) is meaningless;

In the contrary case, no other solution will exist than the one which is given by that formula.

108. When $m=3$, it is clear that the above results must be equivalent to those which can be deduced from the operations in §§ 72—77, by differentiation of formula (10) with respect to t_0. It can be shown, indeed, that such is the case, and we can even carry out the differentiation by elementary methods and obtain the required value of u_a as the limit of a sum of a double integral and a curvilinear one, getting an explicit expression of the latter complementary term. The points of S_0 being referred to the curvilinear coordinates θ and λ (in terms of which s will, therefore, be considered as expressed), let us divide S_0 into two parts S_1 and S_2, the latter containing the neighbourhood of the line γ of intersection of S and Γ: the boundary (τ') between S_1 and S_2 will thus correspond to $\theta = t_0 - \tau'$, denoting by τ' a small quantity, constant or variable with λ (but, in the latter case, such that its derivative with respect to λ be also small, and of the same order as τ itself); finally, we shall let τ and, therefore, S_2 approach zero, so that we can neglect any term which becomes infinitesimal with τ'.

The integral over S_1 can be differentiated by differentiation under the integral sign without any difficulty*; this being the same as replacing $\mathbf{v}$ by its derivative with respect to t_0, i.e. by $\chi(t_0) \,.\, v$, we evidently obtain the corresponding term to our quantity (38), with the only difference that integration is limited to S_1, instead of being carried out over the whole of S_0.

This precaution is even unnecessary as regards the terms which only contain $\mathbf{v}$ as a factor, for, the latter quantity being proportional to $\sqrt{t_0 - \theta}$, its derivative will contain no other infinity than $\frac{1}{\sqrt{t_0 - \theta}}$, the integration of which is allowed all over S_0: or, which comes to the same, the corresponding terms extended over S_2 would (if we operate in the same way as we shall presently) give derivatives which would vanish with τ'.

Therefore, there remains only to deal with the term in $\frac{d\mathbf{v}}{d\nu}$. We have seen that

$$\mathbf{v} = 2\left(\frac{\chi V}{\sqrt{w}} + \dots\right)\sqrt{t_0 - \theta}.$$

The terms which we have replaced by dots contain higher powers† of $(t_0 - \theta)$. Therefore, ν denoting the transversal to S, we have

$$\frac{d\mathbf{v}}{d\nu} = -\left(\frac{\chi V}{\sqrt{w}} + \dots\right)\frac{1}{\sqrt{t_0 - \theta}}\frac{d\theta}{d\nu}$$

which can be written (still replacing by dots terms of higher order in $t_0 - \theta$)

$$\text{(40)} \qquad \frac{d\mathbf{v}}{d\nu} = \left(\frac{\chi}{w}\frac{V}{\sqrt{w}} + \dots\right)\frac{1}{\sqrt{t_0 - \theta}}\frac{d\Gamma}{d\nu}$$

as $\Gamma = w(t_0 - \theta) \dots$, where the first factor is not zero.

We have to express $\frac{d\Gamma}{d\nu}$ or, more exactly, $\frac{d\Gamma}{d\nu} dS$. In doing this, we must observe that two kinds of derivatives of the x's with respect to θ and λ may occur: viz. we may consider every x as a function of three independent variables θ, λ, s, or only of the first two of them, s being a function of θ and λ defined by the equation of S. The symbol ∂ shall be kept for derivatives of the first kind, and those which correspond to the second hypothesis shall be denoted by ordinary d's: they are connected with the former by the relations

$$\text{(41)} \qquad \frac{dx_i}{d\theta} = \frac{\partial x_i}{\partial \theta} + \frac{ds}{d\theta}\frac{\partial x_i}{\partial s},$$

* The same holds for the space integral, which we shall assume to be zero, as (for $m = 3$) its treatment requires no special precaution.

† Our language relates to the hypothesis of analytic data: the working would be easily, by proper devices such as integrations by parts, extended to the case in which these data would be simply assumed to be regular.

and similarly for $\frac{dx_i}{d\lambda}$. This being understood, we have (a suitable order being chosen between the x's in connection with the direction of the normal)

$$(40') \qquad \frac{d\Gamma}{d\nu}dS = \Sigma\left[\pi_i dS \cdot \frac{\partial \mathbf{A}}{\partial P_i}\right] = \pm \begin{vmatrix} \frac{dx_1}{d\theta} & \frac{dx_2}{d\theta} & \frac{dx_3}{d\theta} \\ \frac{dx_1}{d\lambda} & \frac{dx_2}{d\lambda} & \frac{dx_3}{d\lambda} \\ \frac{\partial \mathbf{A}}{\partial P_1} & \frac{\partial \mathbf{A}}{\partial P_2} & \frac{\partial \mathbf{A}}{\partial P_3} \end{vmatrix} d\theta\, d\lambda.$$

The quantities $\frac{\partial \mathbf{A}}{\partial P_i}$ are, as we know, respectively equal to $2s\frac{\partial x_i}{\partial s}$. This shows us that derivatives d in the above formula can be indifferently replaced by derivatives ∂, as follows from (41).

Another expression of $\frac{d\Gamma}{d\nu}dS$ would be obtained by starting from the equation $G(x_1, x_2, x_3)=0$ of S and writing $\pi_i dS = \frac{\partial G}{\partial x_i}dS_G$, which, on account of the above values of $\frac{\partial \mathbf{A}}{\partial P_i}$, gives $2s\frac{\partial G}{\partial s}dS_G$. We shall not introduce this expression in our further operations; but it supplies an easy answer to the question of sign* in the above formula (40′): for we know that G must be written so as to be positive in our domain of integration, i.e. on the side of S where a lies; then $s\frac{\partial G}{\partial s}$ will be evidently negative, and we see that we have to take the sign $-$ before the right-hand side of (40′), the determinant being taken in absolute value.

Taking account of that formula (40′), we find an element of integration $u\frac{d\mathbf{v}}{d\nu}dS$ which is of the form

$$\frac{Q}{\sqrt{t_0-\theta}}\, d\theta\, d\lambda,$$

Q being expanded in powers of $(t_0-\theta)$, viz. $Q=Q_0+\ldots$. Leaving the factor $d\lambda$ aside till the end of operations, the result of integration with respect to θ from $\theta=t_0-\tau'$ to $\theta=t_0$ will be

$$2(Q_0+\ldots)\sqrt{\tau'}$$

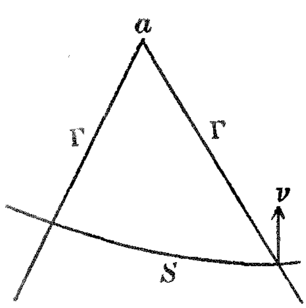

Fig. 13.

* Geometrically, we could observe that the transversal ν is directed towards the same side of S as the corresponding normal (since $\mathbf{A}(\pi_1, \pi_2, \pi_3)>0$). As this transversal direction is, on the other hand, interior to the characteristic cone, it is clear (see the accompanying diagram) that it is directed towards the *outside* of Γ.

(the coefficient of $2\sqrt{\tau'}$ being now expanded in powers of τ', with the constant term Q_0).

This is what we have to differentiate with respect to t_0. Such a differentiation is to be carried out as well on $\sqrt{\tau'}$ as on the coefficients of the expansion Q; but the only useful term—that is, the only one which does not become infinitesimal with τ'—is obviously $\frac{Q_0}{\sqrt{\tau'}}$ or, after division by χ and taking account of the value of Q_0 as defined by (40) and (40′),

$$-\frac{u\Gamma}{w\sqrt{w}\sqrt{\tau'}}\left|\frac{dx_i}{d\theta}\,\frac{dx_i}{d\lambda}\,\frac{\partial \mathbf{A}}{\partial P_i}\right| = -uv \cdot \frac{1}{w}\left|\begin{array}{ccc} \frac{dx_1}{d\theta} & \frac{dx_2}{d\theta} & \frac{dx_3}{d\theta} \\ \frac{dx_1}{d\lambda} & \frac{dx_2}{d\lambda} & \frac{dx_3}{d\lambda} \\ \frac{\partial \mathbf{A}}{\partial P_1} & \frac{\partial \mathbf{A}}{\partial P_2} & \frac{\partial \mathbf{A}}{\partial P_3} \end{array}\right| = -uvK,$$

the determinant being still taken in absolute value and every factor other than v receiving the value which it assumes on Γ itself.

The initial value of w is the same as that of $\left|\frac{\partial \Gamma}{\partial \theta}\right|$, and this allows us to write the factor of $-uv$ in another form. For, on account of the relations

$$P_1\frac{dx_1}{d\lambda}+P_2\frac{dx_2}{d\lambda}+P_3\frac{dx_3}{d\lambda}=0,$$

and (as Γ is a characteristic)

$$2\mathbf{A}(P_1, P_2, P_3)=P_1\frac{\partial \mathbf{A}}{\partial P_1}+P_2\frac{\partial \mathbf{A}}{\partial P_2}+P_3\frac{\partial \mathbf{A}}{\partial P_3}=0,$$

the quantities

$$\frac{dx_2}{d\lambda}\frac{\partial \mathbf{A}}{\partial P_3}-\frac{dx_3}{d\lambda}\frac{\partial \mathbf{A}}{\partial P_2},\quad \frac{dx_3}{d\lambda}\frac{\partial \mathbf{A}}{\partial P_1}-\frac{dx_1}{d\lambda}\frac{\partial \mathbf{A}}{\partial P_3},\quad \frac{dx_1}{d\lambda}\frac{\partial \mathbf{A}}{\partial P_2}-\frac{dx_2}{d\lambda}\frac{\partial \mathbf{A}}{\partial P_1}$$

are proportional to $2P_1$, $2P_2$, $2P_3$, and, as

$$\frac{\partial \Gamma}{\partial \theta}=2\left(P_1\frac{\partial x_1}{\partial \theta}+P_2\frac{\partial x_2}{\partial \theta}+P_3\frac{\partial x_3}{\partial \theta}\right),$$

the proportionality factor is precisely the value of K. Introducing (as is of use in the theory of Abelian functions and was also done by Fredholm in his Memoir of *Acta Math.* vol. XXIII) three arbitrary (and really immaterial) quantities k_1, k_2, k_3, we see that after integration with respect to λ, the result will be* (f being

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* $\frac{dx_i}{d\lambda}d\lambda$ is the same thing as dx_i, on γ', if τ' is a constant; if τ' is variable, this is no longer true, but the corresponding relative error is infinitesimal $\left(\text{on account of our assumption concerning } \frac{d\tau'}{d\lambda}\right)$ and will not alter the final limit in the text.

still assumed to be 0)

(42)
$$2\pi u_a = \lim_{\tau'=0} \left\{ \iint_{S_1} \left(v\frac{du}{d\nu} - u\frac{dv}{d\nu} + Luv \right) dS + \int_{(\tau')} \overline{\left| \frac{\begin{vmatrix} k_1 & k_2 & k_3 \\ dx_1 & dx_2 & dx_3 \\ \partial P_1 & \partial P_2 & \partial P_3 \end{vmatrix}}{2(k_1P_1 + k_2P_2 + k_3P_3)} \right|} uv \right\},$$

where no influence of the function χ or the choice of the line $\mathscr{L}$ remains any longer.

109. A consequence of the presence of our symbol $\overline{|}\ \cdot$ is that, though expressed by a definite integral containing the values of u_0, u_1, f under **SS** or **SSS**, the value of u is not continuous of order zero with respect to these quantities. The continuity is of order m_1 (§ 20 *a*) in u_0, $m_1 - 1$ in u and f.

u is also continuous of order m_1 with respect to the shape of S. This follows from the fact that, S being cut in a point M (at a finite angle) by any geodesic issuing from an arbitrarily given point a—which geodesic depends on $m-1$ parameters $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$—the derivatives, to any order p, of the coordinates of the point of intersection with respect to the λ's are functions of the coordinates themselves, of the derivatives (up to the same order) along the geodesic and of the derivatives along S. Now, a new surface S very near to S will be cut by the same geodesic in a point M very near* to M; and,

* This fact is hardly different from the classic theorem on continuity of implicit functions, and is proved by the same argument. If, through a point M of S, we draw a geodesic such that $\left(\frac{\ }{ds}\right) \neq 0$—where $G=0$ is the equation of S; s, the (ordinary) length of an arc of geodesic reckoned from M, and the derivative is taken at M itself ($s=0$)—there will be, on each side of M, an arc of it along which $\frac{dG}{ds}$ will keep its sign. Let us denote by s' the length of such an arc if smaller than ϵ, and take $s'=\epsilon$ in the contrary case: then, for $s=\pm s'$, the function G will assume two values G' and $-G''$ respectively positive and negative. If now, through every point of M of S (or of a limited portion of it) we draw all the geodesics for which $\left(\frac{\ }{ds}\right)_0$ is greater than a fixed number $a>0$, as these depend continuously on the $2m$ parameters included in the general equation of geodesics, it follows therefrom that the quantities G', G'' corresponding to them will have a minimum G_1.

We shall be certain that a second surface S must cut each of the above if the neighbourhood is of the order p, the above-mentioned derivatives will be but slightly altered by the changing of S into S: this provides the same conclusion as to our improper integral.

Similarly, any first derivative of u_a will be continuous of order $m_1 + 1$ with respect to u_0, of order m with respect to u_1 and f, and of order $m_1 + 1$ with respect to the shape of S.

110. Consequences concerning waves and their diffusion. Classic results immediately follow, on the other hand, from the shape of the area of integration S_0 in our formula (38). It is, indeed, obvious that it illustrates the intervention of characteristics, with the physical signification of waves, just as the formulæ in the beginning of Book II already did for the most usual special equations. We see that not all the data on S enter in the value of u_a, but only those which relate to points of S_0, that is, points lying inside the retrograde half conoid from a. Conversely (cf. Book II, § 32), the values of u_0 and u_1 at a determinate point x' (fig. 14) taken on S have no influence on the values of u at points which lie outside the *direct* half conoid from x'. Physically speaking, this means, as previously (Book II), that no initial impulse at x' can react on a distant point before the time when the corresponding wave reaches that point.

If the initial impulse starts not only from one point but from a certain region $\mathcal{S}$ of S, the portion of space (or rather universe) on which the effect of such an impulse may be sensible is constituted by the insides of all the half conoids the vertices of which are within $\mathcal{S}$: such a region is limited by the envelope of the half conoid in question when its vertex x describes the boundary Λ of $\mathcal{S}$. This envelope (according to known principles concerning partial differential equations of the first order) again satisfies (**A**): it is again a characteristic or, in other words, a wave*.

geodesics at a distance from S less than ϵ if it lies in a sufficiently close neighbourhood (even of order 0) of S, i.e. if its equation be of the form $G=\delta$ with $|\delta| < G_1$.

The corresponding conclusion concerning the derivatives will follow from this and the principles in the Additional Note to Book II.

* The envelope in question consists of two sheets, an exterior one (corresponding to a propagation of the waves towards the outside of ϖ) and an interior one (waves propagating inside $\mathcal{S}$): the former generally (see below) limits the region mentioned in the text.

Such circumstances also show, as was already observed for spherical or cylindrical waves*, that the solutions of our hyperbolic equation need not be analytic†: for (if the data u_0 and u_1 are not themselves analytic) there is obviously no relation between the values of u in the respective neighbourhoods of two points a and a' when the traces of the characteristic conoids from them on S are exterior to each other and, consequently, no analytic continuation from one of these sets of values to the other. It may be added, besides, that a discontinuity of the Nth derivatives of u_0 or u_1 would produce a corresponding discontinuity at any point a situated on the same conoid; and if two different sets of values of u_0 and u_1 have with each other a contact of an arbitrary order N along an edge, the respectively corresponding functions u will have a contact of the same order all along the above-mentioned wave issuing from that edge.

111. *The diffusion of waves.* We have already said that distinctions must be made concerning such propagations by waves and especially Huygens' principle in its special meaning, what we called proposition (B), or Huygens' minor premise.

A mere inspection of formulæ (1) and (1′) (Book II) shows that spherical waves and cylindrical waves behave quite differently from that point of view. Formula (1) gives the value of the solution by means of a double integral—which we ought to denote, in our system of notation, by a single **S**—over the surface of a sphere—in our language, over the *edge* of intersection of the characteristic cone with the initial plane. A point x of the latter can act on the universe-point which is represented by the vertex of the cone, when, and only when, it is just in wave (Book II, § 32) with it. If u_0 and u_1 are zero everywhere except within a small region around a determinate point x' (initial impulse localized in the immediate neighbourhood of x'), the value of u representing the ulterior effect of that impulse will be zero everywhere except in the immediate neighbourhood of the direct half conoid from x: which, at any given point (x_0, y_0, z_0) of

* See Duhem's *Hydrodynamique*, *Elasticité*, *Acoustique*, Vol. II, Book II, p. 168.

† The contrary conclusion would be incompatible with the very existence (which we shall prove a little further on) of the solution of Cauchy's problem, as results from the arguments in Book I, § 15.

ordinary three-dimensional space, corresponds to a small interval of time, after which everything will come back to rest. This is precisely proposition (B).

Such is by no means the case for cylindrical waves. Volterra's formulæ, or, confining ourselves to the simplest case of the problem relating to $t=0$, formula (1′), express the solution of (e_2) in terms of u_0 and u_1 by double integrals, corresponding to the sign **SS** in our notation, being extended (in the plane $t=0$) all over the *inside* of the trace of the characteristic conoid. They show, therefore, that, for such a kind of wave, a point of the initial plane $t=0$ is likely to act on the universe-point (x_0, y_0, t_0), that is on the point (x_0, y_0) at the instant t_0, not only if just within wave, but also if well within wave with each other. In other words, the action of an initial impulse over our two-dimensional medium will propagate with the constant velocity ω and will begin to be perceptible at (x_0, y_0) when the wave thus generated just reaches that point; but it also *continues* so *after* that instant. There will exist what we shall call a *residual integral*, corresponding to this effect of a distant impulse continuing after the time when the wave is past. If we initially ($t=0$) suppose that the impulse is localized within a certain region $\mathcal{S}$ of our plane, the functions u_0 and u_1 being identically zero outside that region, the quantity $u(x_0, y_0, t_0)$ will be zero if the circle to which **SS** in (39) is extended is entirely outside $\mathcal{S}$ (this means physically that no wave issuing from the initial impulse will have had time enough to reach that point). It will of course be different from zero if the circumference of this circle cuts $\mathcal{S}$ (cases at which some waves issuing from the initial impulse precisely reach our point at the time t_0). It will remain different from zero—and will be what we call a *residual integral*—if the aforesaid circle includes $\mathcal{S}$ entirely inside it*. This means that Huygens' minor premise—the proposition which we previously denoted by (B)—will not be true in

* The domain of influence of an initial impulse localized in the region of $t=0$ is bounded, as explained above (see footnote * p. 174), by the exterior wave issuing from the edge which limits ϖ. For waves without diffusion, such as spherical waves, this domain would consist of the annular space between the two (viz. exterior and interior) sheets of the characteristic issuing from that edge (see accompanying diagram).

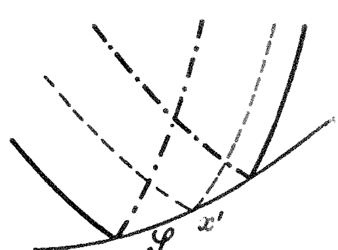

Fig. 14.

this case. After any given time t, the effect of a wave issuing from a given point O is, for our present problem, localized not only on the circumference of a circle with centre O and radius at, but also in the whole inside of that circle.

This is also often expressed by saying that cylindrical waves—unlike to (ordinary, not damped) spherical ones—*diffuse.*

If we now take any other equation with an odd number of independent variables, it is clear from our formula (39) that we shall always reach the same conclusion.

Huygens' minor premise (B) *holds for no phenomenon governed by a linear partial differential equation of the second order with an odd number of independent variables**.

Any such equation with an odd number of independent variables admits of residual integrals.

112. Moreover, a rather curious fact may be noticed concerning the signs of such residual integrals. Let us suppose, for simplicity's sake, that (E) is homogeneous, i.e. $f=0$, so that we shall only have to deal with the surface integrals **SS**. It is clear that, at least for a sufficiently near to S, the most important part of any element of such an integral will be given by the highest power of Γ in the denominator, therefore by the term $u\frac{dv}{d\nu}$, or, more exactly, the part

$$-\frac{m-2}{2}\frac{uV\dfrac{d\Gamma}{d\nu}}{\Gamma^{\frac{m}{2}}}=-(m-2)\,uVs\frac{\Sigma\pi_i\dfrac{dx_i}{ds}}{\Gamma^{\frac{m}{2}}}.$$

This sign is that of u, for, the π_i's denoting the direction para-

* Duhem (*Hydrodynamique, Elasticité, Acoustique*, Vol. II, p. 139) inquires whether (e_2) could admit of solutions such as $\psi(r)\,F(r-\omega t)$ containing the arbitrary function F (where, on the other hand, ψ is supposed to be a determinate expression in r). The negative conclusion which he reached may be considered as evident a posteriori by the result in the text: for (e_3) admits of such solutions, and their existence is sufficient (see, e.g., *ibid.* Vol. I, Book VII, § 2) to prove Huygens' principle in the sense (B). The same hypothesis for (e_2) would therefore carry the same conclusion, which we now know to be false.

meters of the normal to S *inside* T, i.e. of the one which is directed towards the side where a is, the sum $\Sigma\pi_i \frac{dx_i}{ds}$ is negative.

If then we had to deal with an ordinary integral, it would have the sign of u on S (assuming this sign to be constant). But here the **SS** is modified by a complementary $(m-2)$-tuple integral, which necessarily has the opposite sign.

Now, the left-hand side must have the same sign as the values of u on S, if we continue to assume that the point u is in the neighbourhood of S.

Therefore, for even values of m_1, i.e. *for* $m = 5, 9, 13, \ldots$, *it is the* $(m-1)$*-tuple integral that gives its sign*; but for m_1 odd, i.e. *for* $m = 3, 7, \ldots$, *it is, on the contrary, the complementary term that preponderates.*

But if we take the above-mentioned case of the residual integral, *the complementary term vanishes.* Therefore, *if u is positive, the residual integral is positive for equations with* $4p+1$ *variables, but negative for equations with* $4p+3$ *variables.*

Such is, in particular, the equation of cylindrical waves.

This is true, at least, so long as the point considered is near enough to S and the given values of $\frac{dv}{d\nu}$ are not too large in comparison with those of u.

113. The case of characteristic boundaries. Some noteworthy circumstances appear when S is constituted by portions of characteristics, as already occurs in the application of Riemann's method* for $m = 2$.

Our preceding formulæ remain valid in that case, as d'Adhémar and Coulon† pointed out, provided S still possesses the geometric property of cutting any characteristic conoid Γ having a point of a certain region $\mathscr{R}$ for its vertex so as to be, together with Γ, the limit of a portion T of space.

The transversal derivative which we systematically introduced, no longer satisfies, however, in this case, the condition of relating to a

* See Darboux's *Leçons*, Vol. II (2nd edition), § 359, p. 79.

† d'Adhemar, *C.R. Ac. Sc.* Febr. 11, 1901. Coulon (*Thesis*, pp. 53 ff.).

direction exterior to S: so that this derivative $\frac{du}{d\nu}$ *may be considered as taken along lines* (the bicharacteristics) *drawn on S itself.* The knowledge of it is no longer distinct from the knowledge of u itself at every point of S.

As, on account of our formulæ, u and $\frac{du}{d\nu}$ are the only quantities which we need to know on S in order to determine our unknown function, we see that *a solution of* (E) *is determined by knowing its values on a boundary constituted by portions of characteristics* (under the above-mentioned geometrical condition).

Though only one numerical value is thus given at each point of S, such a problem has all the properties of Cauchy's problem*.

114. The interchange property. Let us take, for S, a sheet of a characteristic conoid Γ', with vertex a', and located in such a manner as to limit with Γ a domain T; for u, the solution of the given homogeneous equation

$$\mathcal{G}(u) = 0$$

analogous to v, i.e. the one which is singular in a', and which is, around that point, of the order of

$$\frac{1}{\Gamma'^{\frac{m-2}{2}}}.$$

This quantity is no longer finite in T, but infinite along Γ'; it is again, however, of a fractional order, so that if we integrate in T, the terms relating to the boundaries Γ, Γ' will again disappear. No alteration of this conclusion will be caused by the presence of the intersection of Γ and Γ', owing to what has been established in § 92. All we shall have to do, then, as before, is to isolate the points a and a', applying to them what we said in §§ 104, 105; we shall obviously get

$$u_a = v_{a'}.$$

* This is the way in which characteristic surfaces constitute the transition from duly inclined ones, for which we have to take Cauchy's problem, to non-duly inclined ones, along which only one numerical value can be chosen arbitrarily.

The elementary solution does not change its value if we simultaneously exchange the two points on which it depends and the given equation with its adjoint.

This is the *interchange relation*, entirely similar to that which exists for Riemann's function in the hyperbolic equation with two variables, or the symmetry of Green's function in the theory of potential. It holds good, as we see, thanks to the precaution which we took to divide by $\sqrt{|\Delta_a|}$ the solution singular in a.

From the above relation, we see that the function v, *considered as a function of the point* a (assuming $x_1, x_2, \ldots, x_m$ to be fixed), is a solution of the given (homogeneous) equation, $\mathcal{F} = 0$.

As the two sides of the above written relation are analytic functions, it may be noticed that *the interchange property remains valid for the elliptic case**.

* A direct proof for this would be more difficult than in the hyperbolic case, but, at the same time, would not present the same interest, the reason being that the theory of the elliptic equation does not rest on the elementary solution itself, but on Green's functions, which, although deduced from that solution by addition of regular terms, must be separately formed for each kind of boundary conditions.

CHAPTER III

SYNTHESIS OF THE SOLUTION OBTAINED

115. We have now to prove that the function u defined by our above formula (39) actually satisfies all the requirements of the problem*: by which (and by which only) we shall have proved that our problem of Cauchy admits of a solution. This proof, of course, consists of two parts: first, we shall show that the indefinite partial differential equation is satisfied; then, but under a geometric assumption,—viz. that S is everywhere duly inclined—we shall show the same for the definite conditions.

The verification of the partial differential equation itself, which otherwise is not devoid of difficulty, becomes quite simple when using our special symbol of integration. It is immediate for the homogeneous equation, i.e. when expression (39) of $(-1)^{m_1}\pi\Omega_{2m_1-2}u_a$ is reduced to its second term. For we know (§§ 87, 95) that, to differentiate this with respect to the a's, all that is necessary is to differentiate it under the symbol **SS**. Now, the quantity to be integrated only contains the a's through the factor v, which is (by the preceding section) a solution of the given equation.

115 a. Now let f be $\neq 0$. All we shall have to concern ourselves with will be the m-tuple integral

$$(43)\qquad -\left|\overline{\mathbf{SSS}_T vf\, dx_1\, dx_2 \ldots\ldots dx_m}\right. .$$

We shall apply to it methods entirely similar to those of the classic theory of potential.

To effect the first differentiation of this integral with respect to one of the coordinates a, all that is necessary is to differentiate under **SS**. For the integral thus obtained

$$(43')\qquad \left|\overline{\mathbf{SSS}\, \frac{\partial v}{\partial a_i} f\, dx_1\, dx_2 \ldots\ldots dx_m}\right.$$

has a meaning: i.e., by isolating the point a by a neighbouring surface

* This was undertaken, for the first time, by d'Adhemar (*Bull. Soc. Math. Fr.* Vol. XXIX (1901), pp. 190 ff., and *Thesis*, Paris, 1904) at least for the homogeneous equation. See also his work *Les équations aux dérivées partielles à caractéristiques réelles*, Paris, Gauthier-Villars, 1907.

(compare the diagrammatic fig. 12) one obtains an integral that approaches a determinate limit when Σ approaches a. This is seen by following the same procedure as in §§ 104, 105. But moreover, the above integral is uniformly convergent, so that the error committed by substituting the domain of integration T' (fig. 12) for T has an upper limit which can be assigned without the point a being given, as long as it is known to be near enough to Σ.

Therefore, according to a well-known argument, the integral (43′) is the derivative of (43), even when taken in the domain T.

To differentiate a second time, we shall again consider the surface Σ, which resolves the domain of integration into two parts, one T', between S and Σ, the other T'', between Σ and a.

In T', we shall differentiate directly under **SS**; in T'', we shall write

$$-\frac{\partial v}{\partial a_i} = -\left(\frac{\partial v}{\partial a_i} + \frac{\partial v}{\partial x_i}\right) + \frac{\partial v}{\partial x_i}.$$

The quantity $\left(\frac{\partial v}{\partial a_i} + \frac{\partial v}{\partial x_i}\right)$ gives an integral that can be differentiated under **SS**, the proof of this being hardly different from the argument in § 106, if we first observe that the terms of lowest degree in Γ only contain the combinations $(x_1 - a_1)$, $(x_2 - a_2)$, ..., $(x_m - a_m)$: for, on account of this, $\left(\frac{\partial \Gamma}{\partial x_i} + \frac{\partial \Gamma}{\partial a_i}\right)$, if expanded in powers of these differences, again begins with quadratic terms and

$$\frac{\partial}{\partial a_k}\left(\frac{\partial v}{\partial a_i} + \frac{\partial v}{\partial x_i}\right) = \frac{\partial}{\partial a_k}\left[\frac{1}{\Gamma^{\frac{m-2}{2}}}\left(\frac{\partial V}{\partial a_i} + \frac{\partial V}{\partial x_i}\right) - \frac{m-2}{2}\frac{V}{\Gamma^{\frac{m}{2}}}\left(\frac{\partial \Gamma}{\partial a_i} + \frac{\partial \Gamma}{\partial x_i}\right)\right] = \frac{Q_3}{\Gamma^{\frac{m}{2}+1}},$$

the numerator Q_3 beginning with cubic terms. In the coordinates s, λ of §§ 104—106, the integrand will therefore contain no power of s in its denominator, but only $\mathbf{H}^{\frac{m}{2}+1}$, as was to be proved: so that

$$(44) \qquad \frac{\partial}{\partial a_k}\,\mathbf{SSS}\left(\frac{\partial v}{\partial a_i} + \frac{\partial v}{\partial x_i}\right) dx_1 \qquad dx_m$$

$$= \mathbf{SSS}\,\frac{\partial}{\partial a_k}\left(\frac{\partial v}{\partial a_i} + \frac{\partial v}{\partial x_i}\right) dx_1\, dx_2 \qquad dx_m.$$

As for the integral

$$\overline{\mathbf{SSS}_{T''} f \frac{\partial v}{\partial x_i} dx_1\, dx_2 \ldots\ldots dx_m},$$

it will be transformed by Green's* formula into

$$-\overline{\mathbf{SSS}_{T''} v \frac{\partial f}{\partial x_i} dx_1 \ldots\ldots dx_m} + \overline{\mathbf{SS}\, vf\pi_i dS},$$

π_i being, as before, a direction-cosine of the normal to Σ directed towards the inside of T' (and therefore, towards the outside of T'').

Differentiation under **SS** no longer presents any difficulty, and we have

$$-\frac{\partial^2}{\partial a_i \partial a_k} \overline{\mathbf{SSS}_T vf\, dx_1 \ldots\ldots dx_m}$$

$$= -\overline{\mathbf{SSS}_{T'} \frac{\partial^2 v}{\partial a_i \partial a_k} f\, dx_1 \ldots\ldots dx_m} + \overline{\mathbf{SS}_\Sigma f\pi_i \frac{\partial v}{\partial a_k} dS} + R$$

(R being the m-tuple integral (44) taken in T''); or, again letting Σ approach the point a

$$-\frac{\partial^2}{\partial a_i \partial a_k} \overline{\mathbf{SSS}_T vf\, dx_1 \ldots\ldots dx_m}$$

$$\doteq \lim \left\{-\overline{\mathbf{SSS}_{T'} \frac{\partial^2 v}{\partial a_i \partial a_k} f\, dx_1 \ldots\ldots dx_m} + \overline{\mathbf{SS}_\Sigma f\pi_i \frac{\partial v}{\partial a_k} dS}\right\}.$$

The result of the substitution of expression (43) in the differential polynomial $\mathcal{F}$ is then

$$\lim \overline{\mathbf{SS}_\Sigma f \sum_{i,k} A_{ik}\pi_i \frac{\partial v}{\partial a_k} dS},$$

a limit which is entirely similar† to that of quantity (37) in § 105, to which it is easily reduced.

116. Let us come to the boundary conditions.

Here we have to make a proper geometric assumption as to the shape of S. We assume that its tangent plane is everywhere *duly inclined.* We already know from Book I that if such were not the

* We again operate as has been said in § 94.

† The two expressions only differ by the change of $\frac{\partial v}{\partial x_k}$ into $\frac{\partial v}{\partial a_k}$, and by the fact that the A_{ik}'s are taken, in one case, at the point $(x_1, x_2, \ldots x_m)$ and, in the other, at the point $(a_1, a_2, \ldots a_m)$.

case, our problem would not generally be possible. We even, for the present, shall admit that this condition is *strictly* satisfied: i.e., that no tangent plane of S shall have a characteristic direction. Such tangent planes, therefore, will make a finite angle with every direction of line which is either bicharacteristic or interior to the characteristic cone.

If so, when a approaches indefinitely any determinate point P of S, the corresponding characteristic conoid will cut out of S an infinitely small area S_0 in the immediate neighbourhood of P, the segments s of geodesics from a between that point and S_0 also being all infinitely small.

If, moreover, the surface S is a regular one, so that one of the coordinates admits, with respect to the others, of finite partial derivatives up to a certain order p, the derivatives of s with respect to $\lambda_1, \ldots, \lambda_{m-1}$ up to that same order will also be infinitesimal.

We have to show:

(1) that u approaches the given value $(u_0)_P$;

(2) that the derivative $\frac{du}{d\nu}$ in the direction of the transversal at P approaches the given value $(u_1)_P$.

The first proof is immediate. The partial **SS**'s in (39) do not differ essentially from those which we consider in (4) (§§ 104, 105) in order to find their limiting values: only here the surface of integration S remains fixed instead of moving towards a, the latter point being assumed, on the other hand, to come infinitely near S. But as nothing in our previous argument assumed a to be fixed, we again can assert that one of the **SS** s (39) approaches $(-1)^{m_1} \pi \Omega_{2m_1-2} u_P$ and the rest approach zero. For similar reasons, we can also say (by § 96) that the **SSS** is also infinitely small. Our first conclusion is thus proved.

117. The direct proof of our second conclusion, concerning $\frac{du}{d\nu}$, would be more delicate. As, on account of the presence of $\frac{dv}{d\nu}$, one term of the **SS** in the value of u is comparable to a potential of double layer, the classic difficulties which occur in the study of the normal derivative of such a potential would also appear in our proof, the intervention of our symbol | introducing a new complication.

An indirect argument will lead us rather simply to the result: it consists in using the fact that the conclusion wanted would certainly be true if S were analytic as well as the other data u, $\frac{du}{d\nu}$ and f, as, in that case, we know by Cauchy's fundamental theorem that the problem admits of a solution, the latter being necessarily given by our above formula.

But, on the other hand, we can consider an analytic surface $\bar{S}$ which would have with S, at P, a contact of a certain order q (this requiring only that S be regular up to that order). Similarly, we can consider two analytic functions u_0 and u_1 of the coordinates of an arbitrary point M of S having, with u_0 and u_1 (values of these quantities for the point M corresponding to M), a contact of order q at P, and an analytic function f having with f a contact of the same order. If we should replace S by S, f, u_0, u_1 by f, u_0, u_1, thus changing u to $\bar{u}$, the convergence of $\frac{d\bar{u}}{d\nu}$ towards $(u_1)_P$ would be certain. But on the other hand, when a approaches P, we know that our domains of integration become also confined to the immediate neighbourhood of P and that, in such regions, the surface S and the functions f, u_0, u_1 have with S, f, u_0, $\bar{u}_1$ respectively infinitely close neighbourhoods of order q. Therefore (by § 109) if q is great enough, the difference $\frac{du}{d\nu} - \frac{d\bar{u}}{d\bar{\nu}}$ approaches zero and $\frac{du}{d\nu}$ also has the limit $u_1(P)$.

118. *An analogy with ordinary potentials.*—The limiting value $u_0(P)$ is obtained by the application of our formula to the small conoidal domain T constructed by means of a, on whatever side we let a approach S (fig. 15), and this limiting value is thus the same on both sides. But it must be noticed that, when passing from one side to the other, the sense on the normal must be changed on account of the rules of Book II, § 38, and so must be as well the sense of the transversal ν. If we kept the same sense on ν in both

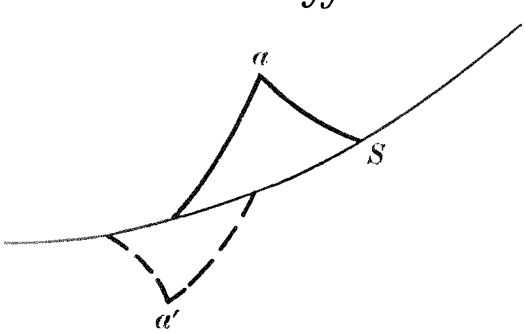

Fig. 15.

cases, the value of the **SS** would at once *change sign* when crossing S: a discontinuity which is obviously similar to that of ordinary surface potentials. This analogy, which with others of the same kind (see further on) was pointed out by Volterra (*Congress of Rome*, 1908, vol. II, p. 90), is completed by the fact that on S itself, the **SS** takes the value zero, which is the arithmetical mean between the two aforesaid opposite values.

119. The case of a characteristic boundary. It is obvious that the success of the above verifications essentially depends on our geometrical hypothesis concerning S, it being necessary for them that the intersection of the characteristic conoid with S be reduced to an infinitely small area around P, when a indefinitely approaches the latter point.

We must expect to meet with quite different circumstances when that geometrical condition is no longer satisfied, which will occur as soon as S ceases to be duly inclined.

It is remarkable—though to be foreseen by what we already know concerning the case of analytic data*—that our verification yet succeeds in a case where the area S_0 no longer becomes infinitesimal in every sense: we mean the case (intermediate between duly inclined and not duly inclined surfaces) of S being a *characteristic*.

As we have seen in § 113, the data are then reduced to the value of u alone at each point, so that (nothing being changed of course, as to the indefinite partial differential equation itself) there is only one kind of boundary conditions to verify.

This verification, however, presents some peculiar difficulties, owing to two circumstances.

One of them is that which we just mentioned, and was noticed by d'Adhémar (see *Rendic. Circ. Mat. di Palermo*, vol. xx. p. 143 (1909)); it was to be foreseen by remarking that our present case is intermediate between the considerations of §§ 116, 117 and those which we shall meet with further on. Let us suppose that we have to deal with the equation of cylindrical waves (with $\omega = 1$), so that Γ is a circular cone having a right vertex angle, and also that

* We have obtained in § 64, Book II, the construction of the solution when u is given along a characteristic conoid.

S is a plane of characteristic direction, that is making with the t-axis an angle of 45°. Then, the latter will no longer cut Γ along an ellipse, but along a parabola (fig. 16), which, when a approaches S, no longer reduces to a point but to a whole line, viz. one-half of a generatrix of Γ, the volume included between S and Γ being (for any position of a not on S) indefinitely extended in one direction. Secondly, as is seen by the above example, a *regular* characteristic S is not sufficient, by itself, to constitute, with Γ, the complete boundary of our domain Γ. In order to enclose a volume, it would be necessary *either* (as in our above instance) to introduce a second surface S', such as a second characteristic plane cutting the first (see fig. 16); *or*, to assume that S has a singular point (being itself, as in Book II, § 64, a characteristic cone, or a kind of polyhedral angle with characteristic faces, etc.). A general proof ought to take account of all those possible singularities*.

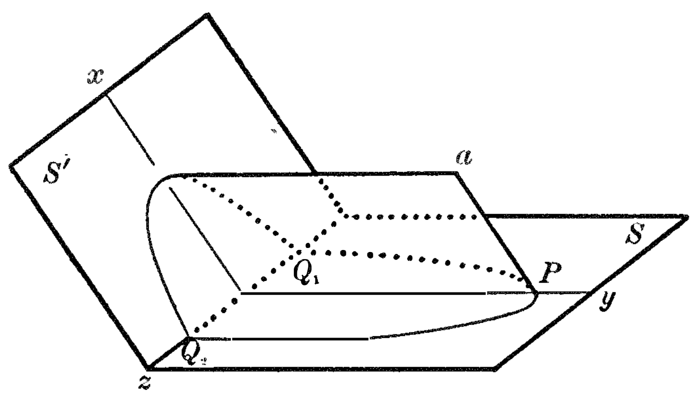

Fig. 16.

Limiting ourselves, for simplicity's sake, to $m = 3$, we shall, in the first place, suppose that our boundary consists of two intersecting regular characteristics S and S'; and, letting a approach any given point P of S, we have to prove that the quantity u given by (39) will approach the given value $(u_0)_P$. It is useful to observe that this is equivalent to proving that the problem has a solution† (as this solution cannot be other than (39)).

* The proof of d'Adhémar (*Rendic. Circ. Mat. di Palermo*, *loc. cit.*) concerns the case where S is a characteristic cone.

† The importance of this lies in the fact that we have (see below) to change variables, after which only our verification will be done. That the same verification would succeed when using the original variables, would not be evident

By means of a suitable punctual transformation, we can assume S and S' to be two coordinate planes $x = 0$ and $y = 0$, all the planes $x = \text{const.}$ becoming characteristics, and even in such a way that the corresponding bicharacteristics are parallels to the y-axis and the x-axis, the y-axis itself going through P. We also admit that the useful portions of the planes S and S'—i.e., the portions which bear the data—are the positive ones.

It is easy to ascertain that the general disposition of the diagram will be the same as in the above special example. Let us denote by x, y, z the coordinates of any point; by $x_0, y_0, 0$ the coordinates* of a, through which we draw a parallel to the y-axis which will be a bicharacteristic. In the neighbourhood of any point $(x_0, y_0, 0)$ of this line, we can expand Γ in powers of $x - x_0$, z, the coefficients being functions of y. By remembering:

(a) That $\Gamma = 0$ touches the plane $x = x_0$ along the line $z = 0$;

(b) That it is situated, with respect to this plane, on the side of decreasing x's;

(c) That $\Gamma > 0$ corresponds to the inside of the conoid and, therefore, $\Gamma < 0$ at points such that $x = x_0$, $z \neq 0$,

we see that the expansion must be of the form

$$(45) \quad \Gamma = n(x_0 - x) - Nz^2 - 2N_1(x_0 - x)z - N_2(x_0 - x)^2 + \dots$$

(where the dots stand for terms of higher order), both coefficients n and N being positive. The first of them, but not the second, vanishes at a itself and is, in general, practically proportional to $(y_0 - y)$.

Equating (45) to 0 and making $x = 0$, we evidently obtain a kind of parabola having P for its vertex and $z = 0$ for its axis, which flattens along the latter line when x_0 becomes 0. Any line $y = \text{const.}$ cuts that curve in two points $z = \alpha \pm \sqrt{\beta}$, denoting by α and β two expansions

without showing that expression (39) remains invariant (or invariant but for a proper factor) for such transformations. Such a proof is avoided by the remark in the text (the existence of a solution evidently being an invariant property), and even the invariance of (39) could, if necessary, be deduced from our arguments.

* The assumption $z_0 = 0$ does not restrict the generality, as we can take P variable with the position of a, replacing it by another point P' situated on the same parallel to the x-axis as a, and letting P' finally approach P like a. This would require a variable translation of the axes, but this is immaterial.

in powers of x_0 without constant terms. The theorem of factorization (see p. 112 and corresponding footnote, which again applies to our present case) shows us that we can write

(46) $$\Gamma = [\beta - (z - \alpha)^2]\, G\,(x, y, z, x_0, y_0),$$

where G is another expansion in $(x_0 - x)$, z having N for its constant term.

This being understood, we come to the determination of the limiting value of u_a, viz. (the $\mathbf{S}$'s being here replaced by ordinary $\int$'s)

(39′) $$u_a = -\frac{1}{2\pi}\left[-\iiint f v\,dx\,dy\,dz + \overline{\left|\iint\left(u\frac{dv}{d\nu} - v\frac{du}{d\nu} - Luv\right)dS\right.}\right].$$

By exclusively considering the case $m = 3$, we get the advantage of not meeting with any difficulty concerning the valuation of our symbol $\overline{|\ }$. This is clear, in the first place, for the space-integral in the first term, which, v being only infinite of the order $\frac{1}{2}$, has a meaning in the classic sense, and becomes infinitely small at the same time as the volume of integration.

In the double integral, an infinity of order $\frac{3}{2}$ occurs only in $\dfrac{dv}{d\nu}$, and can be eliminated with the help of an integration by parts. For, on S, the direction ν being parallel to the y-axis, we can take $d\nu = dy$ by setting down

$$dS = K\,dy\,dz,$$

K being a proper function of y, z; and similarly on S', we can take $d\nu = dx$ with

$$dS = K'dx\,dz.$$

Therefore, the double integral $\overline{\left|\iint u\dfrac{dv}{d\nu}\,dS\right.}$ relating to S will be $\left(\text{except for the factor } \dfrac{1}{2\pi}\right)$

$$\overline{\left|\iint K u\frac{\partial v}{\partial y}\,dy\,dz\right.} = \int dz\,\overline{\left|\int K u\frac{\partial v}{\partial y}\,dy\right.} = \int dz\left[(Kuv) - \int v\,\frac{\partial (Ku)}{\partial y}\,dy\right].$$

The limits of integration, for any given value of z, will be given first by the edge of our dihedron (i.e., a segment Q_1Q_2 of the z-axis) and by the characteristic conoid. But the term corresponding to the latter limit has a fractional infinity and has, therefore, to be cancelled: so that the value of the integral on S in (39′), multiplied by -2π, is

$$(47)\qquad -\int^{Q_2} Kuv\,dz + \iint\left(-K\frac{\partial u}{\partial y} - \frac{\partial (Ku)}{\partial y} - KLu\right)v\,dy\,dz,$$

the simple integral being taken along the z-axis. In this way, every infinity of order higher than $\frac{1}{2}$ has disappeared.

Moreover, the value of v is

$$v = \frac{V}{\sqrt{\Gamma}} = \frac{V_0}{\sqrt{\Gamma}} + V_1\sqrt{\Gamma} + \ldots.$$

In the first term—obviously the only one which can give anything else than zero in (47)—we shall replace Γ by the value (46), the first factor of which can also be written

$$-(z - z_1)(z - z_2),$$

$z = z_1$ and $z = z_2$ denoting the intersections of any parallel to the y-axis with the characteristic conoid with vertex a, so that z_1 and z_2 are functions of y (x being zero) and the coordinates of a.

In this way, the simple integral along Q_1Q_2 becomes

$$-\int_{z_1}^{z_2} \frac{KuV\,dz}{\sqrt{(z - z_1)(z_2 - z)}\sqrt{G}}$$

and, when a approaches P and therefore Q_1Q_2 becomes infinitesimal, such an integral is practically equal to

$$(48)\qquad -\frac{K^0u^0V^0}{\sqrt{N^0}}\int_{z_1}^{z_2}\frac{dz}{\sqrt{(z - z_1)(z_2 - z)}}$$

(K^0, u^0, V^0, N^0 denoting the values of the quantities K, u, V, N, at the origin of coordinates), the limit of which may be written immediately, the last factor being, as well known, always equal to π.

The same treatment obviously applies to the double integral; by writing it

$$\int dy\int_{z_1}^{z_2}\left(-K\frac{\partial u}{\partial y} - \frac{\partial (Ku)}{\partial y} - KLu\right)\frac{V}{\sqrt{(z - z_1)(z_2 - z)}\sqrt{G}}\,dz$$

and operating on every simple integral relative to z, (47) is thus found to approach the limit

We operate quite similarly on the integral relative to S', only with the simplification that there is no simple integral like that in (47) (the corresponding segment approaching zero): the corresponding limit will be

$$(48\,b) \qquad \pi \frac{K'^0 u^0 V^0}{\sqrt{N^0}}.$$

The question is whether the sum of (48), (48 a) and (48 b) is equal to $2\pi u_P$. The method for answering it is well known from the Calculus of Variations: we remove $\frac{\partial u}{\partial y}$ under $\int$ by means of an integration by parts, which being done, the value of our sum will be the expected one if: (a) after that transformation, the terms in u vanish at the same time under $\int$, so that no integral whatever is left*; (b) the terms in u^0 also cancel each other; (c) the coefficient of u_P is equal to 2π.

These conditions are sufficient; but—on account of the fundamental Lemma of the Calculus of Variations—it is well known that they are also *necessary*. The consequence is that, in the present case, we can assert *a priori* that they are satisfied, and need no calculation for that. For we have seen in Book II, § 64, that our problem has a solution (and, therefore, the present verification *must* succeed) whenever the data are analytic. Thus, the sum of the quantities (48) to (48 b) reduces to $2\pi u_P$ for every analytic u, and this cannot be† otherwise than by our three conditions above being satisfied.

120. The direct investigation of (48), (48 a) and (48 b) is however interesting in some respects and deserves to be undertaken. It, at first, seems to meet with an insuperable difficulty on account of the

* In other terms, we must have identically (for $x=z=0$)

$$\left(\frac{dK}{dy}+KL\right)\frac{V_0}{\sqrt{N}}=2\frac{d}{dy}\left(\frac{KV_0}{\sqrt{N}}\right).$$

† The fundamental Lemma continues, as is well known, to apply when the arbitrary function mentioned in its hypothesis is required to be analytic. The argument seemed, at first, to assume S (or S') itself to be analytic; but, in the contrary case, we could substitute instead of S, another *analytic* characteristic having a contact of an arbitrary order with it in one point of our y-axis (and consequently, on account of known properties of partial differential equations of the first order, all along that bicharacteristic), which substitution (as in § 117) would not change the results. The hypothesis of analyticity of S is therefore immaterial.

fact that the values of V along the bicharacteristic are deduced (by means of M) from those of the second derivatives of Γ, and the latter depend on the general integration of the differential equations of geodesics, or at least, of the corresponding "variational equations*." In the present case, nevertheless, the values of V,—which, on the bicharacteristic, reduces to V_0,—can be found by a quadrature. The reason for this is that, though not knowing all the geodesics in general, we implicitly assume (by our very choice of coordinates indicated above) that we know the geodesics of zero length (i.e. bicharacteristics)†.

To obtain such an expression of V_0, we shall complete the simplifications in the preceding section by means of Book II, § 50. We have seen there that, coordinates being chosen as above, the homogeneous partial differential equation can be (changing the unknown if necessary) taken in the form

$$\text{(E)} \qquad 2\frac{\partial^2 u}{\partial x \partial y} - \mathcal{F}_1(u) = 0,$$

$\mathcal{F}_1$ including no differentiation with respect to x (and the adjoint equation will be of a similar form).

This being assumed, and taking also account of our above assumptions as to the axes of x and z, the characteristic form $\mathbf{A}$ will be, denoting by α, β, γ the variables, of the form

$$\mathbf{A} = 2\alpha\beta - \lambda\gamma^2$$

(where the coefficient λ must be positive, in order that we have only one positive square), which gives for the discriminant Δ the value λ.

We now proceed to the determination of the coefficients n and N

* See Additional Note to Book II. More precisely, as we shall see there, V_0 is (on account of equation (37), § 59) connected with the Jacobian J.

† Without insisting on this point—which I perhaps shall do at another time—I simply point out that we ultimately have an application of the well-known theorem that the integral of a linear differential non-homogeneous system can be found by quadrature when the general integral of the corresponding homogeneous system is known. The linear system here considered is constituted by our variational equations, one of them being replaced by the (variational) relation deduced from the theorem of vis-viva, the right-hand side of which has to be taken as zero when bicharacteristics alone are concerned, and to an arbitrary constant for the study of geodesics in general.

in (45) (in terms of y). We, for this, only need to substitute (45) in the partial differential equation of the first order for Γ,

$$\mathbf{A}\left(\frac{\partial \Gamma}{\partial x}, \frac{\partial \Gamma}{\partial y}, \frac{\partial \Gamma}{\partial z}\right) = 4\Gamma.$$

Denoting by accents any differentiations with respect to y, we get

$$\begin{aligned} 2\,[-n + 2N_1 z + 2N_2 (x_0 - x) + \ldots] \\ \times [+n'(x_0 - x) - N' z^2 - 2N_1' (x_0 - x) z - N_2' (x_0 - x)^2 + \ldots] \\ - 4\lambda\,[Nz + N_1 (x_0 - x) + \ldots]^2 \\ = 4n (x_0 - x) - 4Nz^2 - 8N_1 (x_0 - x) z - 4N_2 (x_0 - x)^2 + \ldots, \end{aligned}$$

dots still denoting terms of higher order. We shall obtain the required result by equating the coefficients of $(x_0 - x)$ and also the coefficient of z^2, this giving

$$-2nn' = 4n,$$

$$2nN' - 4\lambda N^2 = -4N.$$

The first relation gives (n necessarily vanishing at a)

(49) $$n = 2 (y_0 - y).$$

We have then for N

(49′) $$-N'(y - y_0) + N - \lambda N^2 = 0,$$

an equation of the Bernoulli type, one solution of which only is finite at a, viz.

(50) $$N = \frac{y_0 - y}{\int_{y_0}^{y} \lambda\, dy}.$$

This first result being attained, we now can obtain the quantity M (§ 49): its expression is reduced to

$$M = 2\frac{\partial^2 \Gamma}{\partial x \partial y} - \lambda \frac{\partial^2 \Gamma}{\partial z^2} = -2n' + 2\lambda N$$

for $x = x_0$, $z = 0$ (every other term vanishing as both Γ and $\frac{\partial \Gamma}{\partial z}$ are zero along that line). But taking account of (49), (49′), this gives

$$M = 4 + 2\lambda N = 6 + \frac{2N'}{N}(y_0 - y).$$

Therefore, by its definition, Book II, § 62 (in which $s = y_0 - y$),

$$V_0 = \text{const.} \sqrt{N} = \sqrt{N},$$

the constant factor being 1, as both V_0 and $\sqrt{N}$ are equal to $\sqrt{\lambda}$ at a.

121. Let us now come to our above formulæ (48) to (48 b). If we take account of the above value of **A** and the equalities (relative to S)

$$\pi_1 dS = dy dz, \qquad \pi_2 = \pi_3 = 0,$$

the definition of ν shows that $K = 1$. As to L, it is equal to 0, as defined by formula (7) in Book II, § 40. Thus, as we had to show, no integral remains in (48 a), which reduces to

$$(51) \qquad -2\pi (u_P - u^0),$$

while (48) gives $-\pi u^0$.

The only term yet remaining to find is (48 b). But, just as above, we see that K'^0 is, like K^0, equal to 1, so that (48) and (48 b) are found to cancel the second term of (51) and the verification is performed.

***122.** Our required conclusion being thus established, this implies, of course, the original form (§ 119) of the result: it can be said, with the notations of § 119, that the limiting value of the double integral over the first characteristic (containing P) is

$$-\pi \left[2u_P - \left(\frac{KuV}{\sqrt{N}} \right)_0 \right],$$

K being such that*, for any ϕ,

$$\frac{d\phi}{d\nu} dS = K \frac{\partial \phi}{\partial y} dy dz,$$

and N being the coefficient of z^2 in the expansion of Γ. The conclusion, under this form, holds if S is no longer assumed to be a coordinate plane (the plane $x = 0$ being, however, still assumed to be tangent to S along the y-axis): the ratio of K to $\sqrt{N}$ is, of course, independent† of the choice of the second variable z (which will then be a curvilinear coordinate) on S.

* K (and so would be also K' in (48 b)) is equal to the coefficient of $2 \frac{\partial^2 u}{\partial x \partial y}$ in the equation (as is seen in the same way in § 120).

† This can be verified in a direct way: for if a new variable Z be introduced in the place of z, the variable y remaining the same at least on $z = 0$, this Z will reduce practically to $\gamma z + \beta y + \alpha$ (α, β, γ, constants) in the neighbourhood of any determinate point of our y-axis and both K and N will be divided by γ.

*123. If we now come to the case in which S, instead of consisting of a dihedron, would admit of a singular point O—which will be taken as the origin of coordinates—we shall be able to overcome the difficulties which are special to it by bringing it back to the preceding case, under suitable geometric assumptions.

We begin by observing that we can no longer admit S to be a coordinate plane (as it is singular at O), so that we are compelled to operate as said in the preceding section. But we shall admit that the characteristic which will be taken as a coordinate plane leaves the whole of S on its positive side. We further admit that S can be generated by regular lines (λ) issuing from the point O, each of which will be, in the neighbourhood of this point, directed in the sense of x increasing, so that, if the coordinates x, y, z be expressed as functions (of course regular) of the arc s, we have $\frac{dx}{ds} \geqslant 0$. Moreover, as some of these lines (λ) make infinitesimal angles with $x=0$, we shall again consider the planes $y=\text{const.}$, which will still be assumed to be characteristics, and we admit that, θ being a certain positive angle, the tangents to any of the lines (λ) (in the neighbourhood of O) make an angle greater than θ with at least one of the planes $x-0$, $y=0$. Therefore, if we cut S by a characteristic plane $y=\epsilon$, ϵ being a small quantity, the portion S_2 of S adjacent to O bounded by that plane and the half conoid from a will intercept on each line (λ) an arc less than a length σ which we can take as small as we wish by taking ϵ sufficiently small (and a sufficiently near P).

A part of S_2 will consist of a portion S_2 of our nrst characteristic (at least an angular portion to which the bicharacteristic OP will be interior); the remaining part will be denoted by S_2''.

x, y, z will be continuous in both parameters λ and s: we shall admit that they are regular in s:

$$(52) \qquad \begin{cases} x=\xi_1 s+\xi_2 s^2+\dots, \\ y=\eta_1 s+\eta_2 s^2+\dots, \\ z=\zeta_1 s+\zeta_2 s^2+\dots. \end{cases}$$

Of course, all the coefficients ξ_h, η_h, ζ_h will be continuous in λ; but their derivatives or $\frac{\partial x}{\partial \lambda}$, $\frac{\partial y}{\partial \lambda}$, $\frac{\partial z}{\partial \lambda}$ may have a finite number of discontinuities of the first kind (values of $\frac{\partial \xi_h}{\partial \lambda}$, ... existing on both sides of the discontinuity, but being different from each other). The sum $\left(\frac{d\xi_1}{d\lambda}\right)^2+\left(\frac{d\eta_1}{d\lambda}\right)^2+\left(\frac{d\zeta_1}{d\lambda}\right)^2$ will be different from zero so that the angle between two consecutive lines (λ) will be of the same order as the difference of the corresponding values of λ.

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* If the lines (λ) are bicharacteristics, the second term alone will exist. This will occur in d'Adhemar's case where S is a characteristic conoid; our operating mode in the text is necessary in order to treat other forms of S_2, such as a polyhedral angle, the part S_2' of which is a *regular* characteristic.

On account of the above, the quantities $\pi_i dS$ will be, on S_2, the products of $d\lambda\, ds$ by functional determinants such as $\frac{D(y, z)}{D(\lambda, s)}, \ldots$ which all contain s as a factor. The superficial element will be of the form $Hs\,d\lambda\, ds$, where H is finite, but also everywhere different from zero and consequently greater than a fixed positive number.

124. This being understood, let us take the double integral in (39′): (*a*) over the part S_1 of S which corresponds to $y > \epsilon$; (*b*) over S_2. The limiting value of the first integral will be (as found above)

$$-2\pi u_P + \pi\left(\frac{Kuv}{\sqrt{\Delta}}\right)_{x=0,\, z=0,\, y=\epsilon}. \tag{53}$$

On S_2, we begin by observing that the transversal direction ν is tangent and therefore, for any ϕ,

$$\frac{d\phi}{d\nu} dS = \left(\alpha\frac{\partial\phi}{\partial\lambda} + \beta\frac{\partial\phi}{\partial s}\right) d\lambda\, ds,$$

where α and β are regular functions of λ and s, the second of which again contains s as a factor*.

As to the given values of u on S_2, let us assume that they have finite first derivatives, the derivatives $\frac{\partial u}{\partial\lambda}$ vanishing with s; and also, to begin, *that u itself is zero at O* (therefore, that $|u|$ admits of an upper limit proportional to s). Then, in our integral, which is to be written

$$\iint\left[u\left(\alpha\frac{\partial v}{\partial\lambda} + \beta\frac{\partial v}{\partial s}\right) - v\left(\alpha\frac{\partial u}{\partial\lambda} + \beta\frac{\partial u}{\partial s}\right) - Luv\right] d\lambda\, ds,$$

we transform the terms in $\frac{\partial v}{\partial s}$ and $\frac{\partial v}{\partial\lambda}$ by Green's formula.

The simple integral along the intersection of S_2 with Γ is to be cancelled as above. The term along the intersection with $y=\epsilon$ cancels the corresponding

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* To express these coefficients, we can remark that on account of the assumption that s is the (ordinary) arc of (λ), the quantity

$$\frac{\partial x}{\partial s}\frac{\partial^2 x}{\partial\lambda\partial s} + \frac{\partial y}{\partial s}\frac{\partial^2 y}{\partial\lambda\partial s} + \frac{\partial z}{\partial s}\frac{\partial^2 z}{\partial\lambda\partial s}$$

is zero. We then obtain the required values if we multiply the equations

$$\frac{dx}{d\nu} = \frac{1}{2}\frac{\partial\mathbf{A}}{\partial\pi_1} = \alpha\frac{\partial x}{\partial\lambda} + \beta\frac{\partial x}{\partial s},\quad \frac{dy}{d\nu} = \ldots$$

(in which we have taken $dS = d\lambda\, ds$ and the π's accordingly) by $\frac{\partial x}{\partial\lambda}, \frac{\partial y}{\partial\lambda}, \frac{\partial z}{\partial\lambda}$ or by $\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}$. The two results being of the orders of s^2 and s respectively, we get the order of magnitude in the text.

term of (53) (as the coordinates λ and s could be as well used in the part of S_1 adjacent to S_2')*

$s=0$ should be considered, in the present case, as being a part of the boundary, but the corresponding term vanishes as u is assumed to be zero with s.

There remains, therefore, to evaluate simple integrals of the form

$$\int huv\,ds=(s)\int\frac{H\,ds}{\sqrt{\Gamma}} \tag{54}$$

(where (s) is a value s contained in the interval of integration and H a finite quantity) along lines $\lambda=\text{const.}$ (corresponding to discontinuities of $\frac{\partial x}{\partial \lambda}, \ldots$ such as edges of the polyhedral angle, and situated on S_2'') and double integrals such as

$$\iint\frac{Hu\,d\lambda\,ds}{\sqrt{\Gamma}},\quad \iint\frac{H\frac{\partial u}{\partial\lambda}\,d\lambda\,ds}{\sqrt{\Gamma}},\quad \iint\frac{H\beta\frac{\partial u}{\partial s}\,d\lambda\,ds}{\sqrt{\Gamma}} \tag{54'}$$

where H is again finite and the factor s again appears in the numerator under the integral sign on account of the presence of one of the factors u, $\frac{\partial u}{\partial \lambda}$, β.

We now remember the expansion of Γ, as written above, viz.

$$\Gamma=n(x_0-x)-Nz^2-\ldots=2(y_0-y)(x_0-x)-Nz^2-\ldots \tag{45}$$

or (as every term not explicitly written contains either (x_0-x) or z^2 as a factor)

$$\Gamma=2\hat{y}_0(x_0-x)-\hat{N}z^2,$$

where $\hat{y}_0$ and $\hat{N}$ stand for quantities which differ but infinitesimally from y or N respectively. We have, in the first place, to substitute this for Γ in the simple integrals (54), which all relate to lines belonging to S_2'' and, therefore, making a finite angle either with the plane $x=0$ (i.e. $\xi_1>\xi_1'>0$) or with the plane $z=0$ (i.e. $|\zeta_1|>\zeta_1'>0$), ξ_1' and ζ_1' being constants.

The first case will always occur if the coefficient ξ_2 in the expansion of x in powers of s is negative and algebraically smaller than $-\frac{N\zeta_1'^2}{2}$: for, if so, the coefficient ξ_1 must be greater than a fixed positive number† and so will be $\frac{dx}{ds}$ for a sufficiently small s.

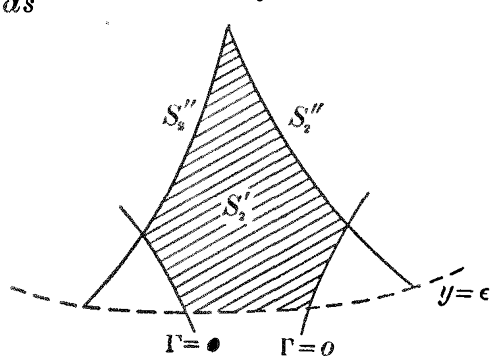

Fig. 17.

* The part of the intersection of S and $y=\epsilon$ contained in S_2 lies entirely in S_0', for x_0 sufficiently small, the relative disposition of S_2, $\Gamma=0$, $y=\epsilon$ being such as shown in the accompanying diagram.

† If lines (λ) exist such that

$$\xi_2<-\frac{N\zeta_1'^2}{2}$$

and ξ_1 approach zero, they would have (by

In this first case, denoting by s_1 the value of s which corresponds to the intersection of (λ) with the conoid, the integral will be of the form

$$\int_0^{s_1} \frac{H}{\sqrt{2\hat{y}_0}(\xi_1+\ldots)} \frac{ds}{\sqrt{s-s_1}},$$

that is, less than

$$2\sqrt{s_1} \times \max. \left| \frac{H}{\sqrt{2\hat{y}_0\hat{\xi}_1'}} \right| < H_1\sqrt{\sigma},$$

with H_1 finite.

125. In the second case, taking account of the inequality $\xi_2 > -\frac{N}{2}\zeta_1'^2$, the expansion of Γ in powers of s will contain a negative term in z^2, the coefficient being numerically greater than the fixed quantity $\frac{N}{2}\zeta_1'^2$. If we begin by cancelling the factor (s), we see, by the theorem of factorization, that the remaining factor in the integrand is the quotient of Kds (with K finite) by the square root of a quadratic polynomial in s with -1 as coefficient of s^2, the integral being taken from zero to a root of the polynomial. Such an integral (in which the indefinite integral is an arc sin) is always smaller than $K\pi$. If we take account of the factor (s), (54) will therefore be infinitesimal when ϵ and x_0 are very small.

The evaluation of double integrals such as (54′), when extended over S_2'', is immediately deduced from the above by integrating expressions like (54) with respect to λ.

On S_2, we shall operate differently and again introduce y and z, in terms of which we may admit that we have expressed* x. The element $sd\lambda ds$ only differs, as we have seen, by a finite factor from the superficial element of S_2', and therefore, from $dydz$. On the other hand, the coefficient of z^2 in the expansion of x

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Bolzano-Weierstrass' theorem) a limiting position such that $\xi_1=0$ and

$$\xi_2 \leqslant \frac{N\zeta_1'^2}{2},$$

which is contradictory to the assumption $\frac{dx}{ds} \geqslant 0$. Similarly, if, with

$$\xi_2 < \frac{N\zeta_1'^2}{2},$$

it could happen, for suitable values of λ and s, that $\frac{dx}{ds}$ could approach zero, either s would remain greater than a fixed quantity s_1, and this could be excluded by taking ϵ and x_0 suitably small; or it might approach zero—but this is impossible, as we have just seen that ξ_1 must remain greater than a fixed number.

* If S_2' belongs to a regular characteristic, we can take $x=0$; if to a characteristic conoid, x will be a function of y and z, the derivatives of which are discontinuous at O but remain finite and vanish on the y-axis.

(for any fixed y) is necessarily positive*, so that, expanding in the same way, the coefficient of z^2 is numerically greater than N. Therefore, again, the integrand is the quotient of a finite quantity K by the square root of a quadratic polynomial in z with the coefficient -1 for z^2, so that the integral relative to z is constantly less than $K\pi$. Integrating with respect to y, the result is again infinitesimal with ϵ.

The integral over S_0 being thus the sum of infinitesimal quantities and a term which disappears with the second term in (53), our conclusion is proved as far as u is admitted to be zero in O.

But the latter hypothesis does not restrict generality: for we can begin by taking a first set u' of values of u (different from zero at O), coinciding with the values of a given solution of (E), for which, therefore, the verification must succeed, as we know beforehand that the problem has a solution, and set down $u=u'+u''$, where u'' is zero at O and can be treated by the above analysis: so that our proof is complete.

For $m=5, 7, \ldots$ we must expect to meet similar calculations, with some more complications, especially on account of the intervention of the symbol | : a subject on which, however, we shall enter into no further detail.

126. A non duly inclined boundary. It is clear from the above, that the success of our synthesis in the preceding case depends on quite special circumstances: these will no longer occur if S ceases to be characteristic, so that its tangent plane at any point cuts the conoid from the same point along two distinct generatrices; and, in the first place, the area of integration S_0 will not be any longer infinitesimal even in one of its dimensions.

If, for instance, the partial differential equation being (e_2), S should, as in § 25, consist of an area in the xy plane, the lateral surface S_1 of the cylinder having this area for its cross-section, this would enable us to calculate, by (39), a value of u_a throughout the volume thus enclosed—whatever the given distribution of values of u_0 and u_1 at the various points of S is—and this quantity u would satisfy the partial differential equation. But, if we let a approach any determinate point P of S, there would be no reason why u_a should approach u_0 (P), as is seen by mere inspection of the accompanying diagram (fig. 18).

* If S is a characteristic conoid, it even increases indefinitely with $1/y$, as does the curvature of the surface.

We thus find, as we have already said in §§ 23 ff., that Cauchy's problem is in general insoluble in this case; and we indeed can immediately write an infinity of conditions of possibility (quite similar to conditions (8), § 15) by now taking a outside our cylinder (fig. 18 a). If a is so chosen, there is no longer any singular point of the conoid inside the domain of integration T and the result of the application of the fundamental formula (f being, in the above example, assumed to vanish) is, as we saw in § 104, reduced to

$$\mathbf{SS}\left(\frac{dv}{d\nu} - v\frac{du}{d\nu} + Luv\right) dS = 0\,;$$

so that no solution can exist if this equation is not satisfied for every position of a outside the cylinder.

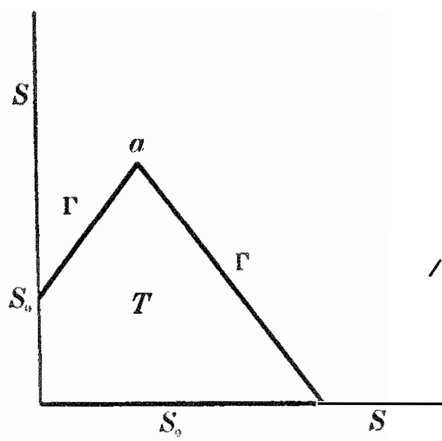

Fig. 18.

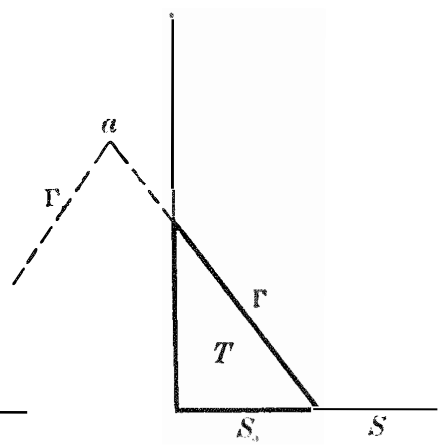

Fig. 18 a.

This case, therefore, does not give a correctly set problem. But it is nevertheless important to notice, as corresponding to Kirchhoff's and Volterra's proofs of Huygens' principle in the most general of the three senses spoken of in § 33; that is, what we call proposition (C). Let us imagine, for that purpose, that we investigate our phenomenon *outside* a certain closed curve of the xy plane,—so that the region $\mathscr{R}$ where we want to determine u is situated on the positive side of $t=0$ and outside the cylinder $\mathscr{C}$ which has σ for its base,—and that, the medium being initially at rest (so that the quantities u_0 and u_1 are zero all over the xy plane), certain disturbances are produced inside σ. As the motion thus generated satisfies (e_2), the corresponding value of u has the expression* (39), which, in the present case, is

* The explicit expression corresponding to that case is given below, § 131.

reduced to integrals extended over $\mathscr{C}$. As the region of integration on $\mathscr{C}$ is constituted (compare fig. 19) by points (just or well) within wave with a, this, as Volterra notes, comes back to representing the motion as produced by disturbing centers properly distributed over $\mathscr{C}$; and Kirchhoff's method is a quite analogous one for (e_3), assuming the latter equation to be integrated as in the next Book.

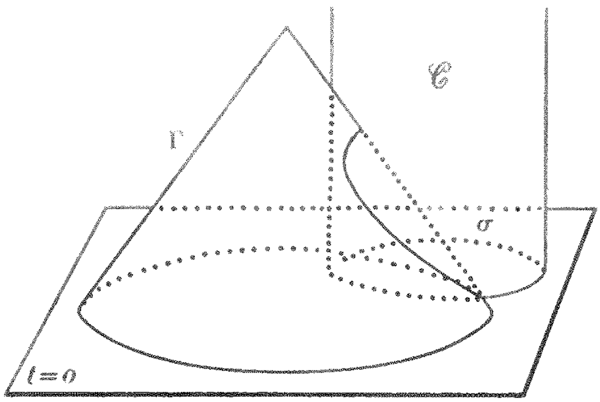

Fig. 19.

It is clear that every phenomenon governed by a hyperbolic equation with three independent variables would allow such a form of Huygens' principle to be presented.

This fully illustrates the necessity of the distinctions which we established above between various formulations of Huygens' principle: we indeed see that our formulæ may be considered as proving the *accurateness* of that principle, if we take it under the form (C),—as they actually are, for instance, in Volterra's fundamental Memoir of the *Acta Mathematica*, t. XVIII,—while we have seen, in § 111, that they prove that same principle to be *false* in the formulation (B).

It may be added, on the other hand, that a difference—though a less essential one—also exists between the proposition (C) as proved by Kirchhoff or Volterra, and Huygens' own conception: for a fundamental character of the new disturbances by which Huygens replaces the initial one issuing at $t=0$ from the point O is their distribution over a surface of a *sphere* having O for its centre, which represents the front at the instant $t=t'$, of the waves emitted by O at $t=0$: while Kirchhoff's or Volterra's fictitious disturbances are distributed over *any* closed surface surrounding the initial centre.

127. Some indications on the exterior problem. The case of a non duly inclined boundary was treated by Volterra from another point of view, introducing what the Italian geometer calls the "exterior problem."

It concerns the case where the domain of integration T_1, instead of lying inside one sheet of the characteristic conoid from a (with a boundary constituted by portions of that sheet and of S), lies *outside* the conoid and is bounded by *both* its sheets and portions of S (fig. 20): it happens then, at least in the most obvious examples, that S is nowhere duly inclined (which is not the case for the interior problem).

Such a problem behaves quite differently from the other one, this being a consequence of the conclusions of § 97. Let us operate on the domain T_1 as we did previously on T, applying the fundamental formula to the unknown function u and the elementary solution (with pole a), in which solution v we only change the sign of Γ so as to make it positive outside the conoid: everything behaves as in the

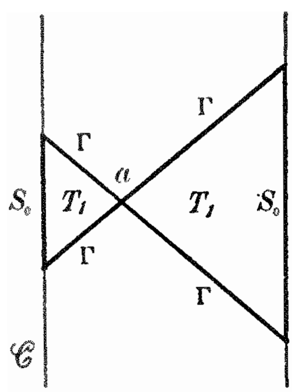

Fig. 20.

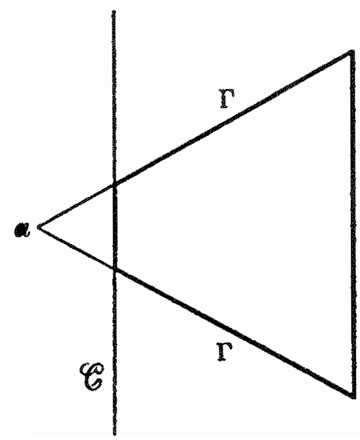

Fig. 20 *a*.

preceding operations, so that we shall have an **SSS** over T and an **SS** over the portion S_0 (which is an annular one) intercepted on S between the sheets of the conoid. But if we again construct the small surface Σ which is necessary to cut out the neighbourhood of a, the limiting value of the corresponding improper integral will no longer contain as a factor the finite part of the volume of the hyperboloid of two sheets, but the finite part of the volume of the hyperboloid of one sheet, and this is zero, as we have seen in § 97. Therefore, no term corresponding to the singularity in a is to be inscribed, and the formula reduces to

$$(55)\quad \mathbf{SSS}\, vf\, dx_1 dx_2 \ldots dx_m + \mathbf{SS}\left(u\frac{dv}{d\nu} - v\frac{du}{d\nu} - L\, uv\right) dS = 0.$$

It no longer determines the value of u_a, but, containing only the data of the problem, *represents a condition of possibility for it.*

We can thus obtain an infinite number of such necessary conditions, by taking the point a arbitrarily inside S. But, of course, we can still obtain other ones as we did in the preceding section, by taking a outside S (fig. 20 a).

Other quantities must be substituted for v in order to obtain the required singularity in a, leading to the expression of u_a. Moreover, the question of finding such a v is no longer a determinate one, precisely because the problem is no longer well set and therefore the solutions, if any exist, can be written in an infinite number of ways in terms of the data (by combination with the conditions of possibility (55)).

Volterra uses the expression

$$\int_0^\Theta \log(1-\Theta^2)\frac{d\Theta}{\sqrt{1-\Theta^2}} + \log r . \operatorname{arc} \sin \frac{t-t_0}{r}$$

with $\Theta = \frac{t-t_0}{r}$, the useful singularity of which is again a whole line, parallel to the t-axis. If we operate on it as we did on (2), i.e., differentiating it with respect to t_0, we find

$$\frac{1}{\sqrt{r^2-(t-t_0)^2}} \log\left(\frac{r^2-(t-t_0)^2}{r}\right).$$

We see that the latter still admits of the singularity $r=0$; but such a fact is by no means abnormal in the present case, on account of the aforesaid necessary indetermination of our expression for the solution.

The determination of analogous quantities for the general hyperbolic equation would depend on the general study of such kinds of singularities (algebraico-logarithmic on a characteristic conoid and logarithmic on another variety)*.

128. Another kind of generalized surface potentials. Let us come back to the interior problem, but still assuming that S is not duly inclined. The expressions

(56) $$\mathbf{SS}\, u_1 v\, dS,$$

(57) $$\mathbf{SS}\, u \frac{dv}{d\nu}\, dS,$$

* The indications in my Memoir of *Acta Math.* Vol. XXXI (p. 367) are erroneous.

again behave like surface potentials of simple or double layers, as they did in § 118, but with somewhat different characters. The domain of integration for an ordinary surface potential extends over the whole surface, independently of the location of the point a at which the potential is calculated. In the case of § 118, the area of integration S_0 becomes infinitesimal when a approaches S. In our present instance, the behaviour of S_0 is an intermediate one: if we take, for example, equation (e_2), S containing a plane parallel to the t-axis, S_0 is the inside of a branch of hyperbola, the intersection of the plane with a sheet of cone of revolution (fig. 21), and, when a comes on the plane, the hyperbola reduces to its asymptotes and S_0 to the angular space between them.

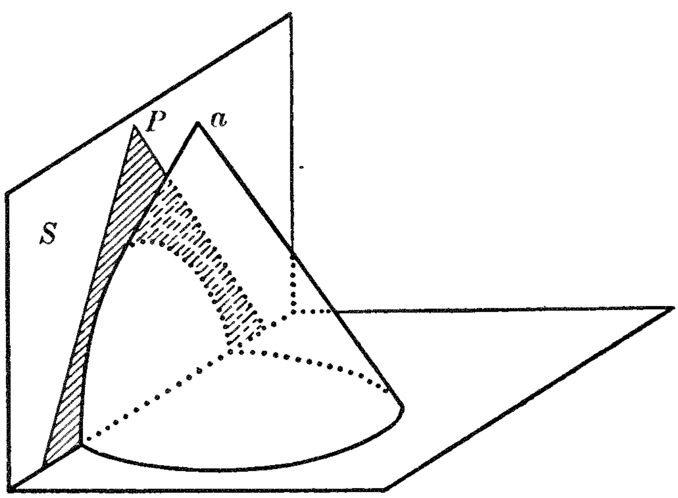

Fig. 21.

As it happens in the ordinary theory of potential, the expressions (56), (57) continue to have a meaning when a is on S. This is to be seen*, as concerns (56), by operating as in § 104, i.e., referring S to lines L from a, each of which is characterized by giving the values of $m-2$ parameters $\lambda_1, \lambda_2, \ldots \lambda_{m-2}$, a point being defined on it by† an $(m-1)$th parameter s. The factor s^{m-2} in the denominator will be

* In the present section we give the arguments briefly. The reader will complete them easily, as being combinations of the above methods with those which are known in the classic theory of potential.

† We can admit that the expressions of the x's on (λ) are tangent to the corresponding ones on the geodesic which touches (λ) in a, so that points taken respectively on both lines with the same value of s lie at a mutual distance of the order of s^2.

cancelled by a similar one in the expression of the superficial element of S, so that things behave as in § 104. Moreover, the convergence is uniform with respect to the location of a on or outside S, so that (56) remains continuous.

In (57), the denominator contains s^{m-1} as a factor; but (as for the ordinary potential of a double layer) a supplementary factor s appears in the numerator on account of the presence of $\frac{d\Gamma}{d\nu}$: for the latter quantity would be zero (§ 58) at any point x of S, if the direction ν were transversal to the geodesic ax, and this is approximately the case when x, varying on S, approaches a, as ν is transversal to the coordinate line (λ) which passes through x and makes with the geodesic an infinitesimal angle of the order of s.

When a is taken near a point P of S, but outside S, the factor $\frac{d\Gamma}{d\nu}$ is no longer infinitesimal for points x neighbouring P, so that the convergence of (57) is not a uniform one and that integral is discontinuous.

Let us examine its mode of discontinuity.

For this purpose, let us admit, in the first place, that S is a locus of geodesics from P (the initial directions of which are of course all in the same $(m-2)$-dimensional plane, so that S is regular), and consider any function u, coinciding with the given one on S, but defined and regular also outside S: let f be $\mathcal{F}(u)$. Let us associate with S, if necessary, another (duly inclined) portion of surface S' so as to enclose, together with one of our half-conoids, a portion of space T, as in fig. 18 or 18 a. To such a domain, we apply our formula with the successive hypotheses that a lies on one side of S, on the other side and at P. In the first case, the sum of integrals **SSS** and **SS** is equal to $-2\pi u_a$, in the second to zero. In the intermediate case of a at P, the integral (38) (see § 105) is extended over initial directions located between the characteristic cone and the tangent plane to S. We are therefore led to the finite part of half the volume of the hyperboloid of two sheets (§ 100), equal to $-\pi u_P$. The discontinuity of our algebraic sum of integrals is thus exactly divided into two equal parts by the value which it assumes when a is on S, as usual.

On the other hand, this discontinuity occurs only in the integral (57) relating to S: for the integrals over S' are evidently continuous, and the other ones converge uniformly.

There remains only to get rid of the hypothesis that S is a locus of geodesics. This is done by considering the geodesics tangent to S in P, which generate a second surface $\bar{S}$. The difference of the potentials (57) relating to those two surfaces and the same point a is an integral which converges uniformly* with respect to the position of a. Therefore, our conclusion is extended to any regular form of S.

* We use the remark in the preceding footnote (†, p. 204). The distance between corresponding points on S and S being of the second order and the angle between their tangent planes infinitesimal, it is easy to see, as for ordinary potentials, that the difference of the values of $\frac{d\Gamma}{d\nu}$ at them is of the order of s^2. As to the values of u, they can be assumed to be the same at corresponding points.

CHAPTER IV

APPLICATIONS TO FAMILIAR EQUATIONS

129. Let us take some simple instances of the calculation of our formulæ. The first one which occurs to us is the equation of *cylindrical waves* (e_2): let us take it non-homogeneous with $\omega = 1$. The elementary solution is (x_0, y_0, t_0 and x, y, t being the coordinates of a and x)

$$v = \frac{1}{\sqrt{\Gamma}}, \quad \Gamma = (t_0 - t)^2 - (x_0 - x)^2 - (y_0 - y)^2$$

and, as $L = 0$ in this case, our general formula for Cauchy's problem is

$$(58) \quad 2\pi u_a = 2\pi u(x_0, y_0, t_0)$$
$$= \iiint_T \frac{f\,dx\,dy\,dt}{\sqrt{\Gamma}} + \left|\overline{\iint \left(\frac{1}{\sqrt{\Gamma}} u_1 - u \frac{d}{dv}\frac{1}{\sqrt{\Gamma}}\right) dS.}\right.$$

The second term under $\iint$ alone has to be transformed in order not to introduce anything but the ordinary symbols of Calculus. A first general way of doing this has been given in § 108. Introducing

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

and the azimuth angle ϕ, the coordinates of any point of the half conoid from a will be

$$x = x_0 + r\cos\phi, \quad y = y_0 + r\sin\phi, \quad t = t_0 - \epsilon r,$$

ϵ denoting $+1$ if the useful (inverse) half conoid is directed towards the decreasing t's (case of $t_0 > 0$ when S is the plane $t = 0$) and -1 in the contrary case. Then,

$$P_1 = \frac{1}{2}\frac{\partial\Gamma}{\partial x} = -(x - x_0), \quad P_2 = -(y - y_0), \quad P_3 = t - t_0,$$

and the integrand, in the second term of the formula (42) of § 108, will be

$$\left|\frac{\begin{vmatrix} k_1 & k_2 & k_3 \\ dx_1 & dx_2 & dx_3 \\ \dfrac{\partial \mathbf{A}}{\partial P_1} & \dfrac{\partial \mathbf{A}}{\partial P_2} & \dfrac{\partial \mathbf{A}}{\partial P_3} \end{vmatrix}}{2(k_1P_1 + k_2P_2 + k_3P_3)}\right| uv = uvr\,d\phi,$$

and (58) will become (notation of § 108)

$$(59)\quad 2\pi u(x_0, y_0, t_0) = \iiint_T \frac{f\,dx\,dy\,dt}{\sqrt{\Gamma}}$$
$$+ \lim \left\{ \iint_{S_1} \left(\frac{u_1}{\sqrt{\Gamma}} - u \frac{d}{d\nu} \frac{1}{\sqrt{\Gamma}} \right) dS + \int_{(\gamma')} \frac{ur}{\sqrt{\Gamma}} d\phi \right\}.$$

130. This formula is a general one, for any shape of S. But a better form for practical calculation is obtained by applying the general rules for our symbol $\big|$ (see especially § 84): for this, we shall simply take the two kinds of surfaces S which have been most usually considered.

If, in the first place, S is the plane $t = 0$, we shall have

$$dS = dx\,dy = r\,dr\,d\phi.$$

The transversal ν will coincide (sense included) with the inner normal, so that $\frac{d}{d\nu} = \epsilon \frac{\partial}{\partial t}$ and

$$\frac{d}{d\nu} \frac{1}{\sqrt{\Gamma}} = \frac{|t_0|}{\Gamma^{\frac{3}{2}}}$$

whatever the sign of t_0 be. We shall write the negative second term under $\iint$ in (58)

$$|t_0| \int d\phi \left| \int_0^{|t_0|} \frac{u r\,dr}{(t_0^2 - r^2)^{\frac{3}{2}}} \right. .$$

As previously explained, we have

$$\left| \int_0^{|t_0|} \frac{u r\,dr}{(t_0^2 - r^2)^{\frac{3}{2}}} \right. = \int_0^{|t_0|} \frac{(u - \bar{u})\,r\,dr}{(t_0^2 - r^2)^{\frac{3}{2}}} - \frac{\bar{u}}{|t_0|},$$

where $\bar{u}$ stands for the value of $u = u_0$ at the extremity of the corresponding radius, that is

$$\bar{u} = u_0(x_0 + |t_0| \cos\phi, \quad y_0 + |t_0| \sin\phi);$$

finally we have

$$(60)\quad 2\pi u(x_0, y_0, t_0) = \iiint \frac{f\,dx\,dy\,dt}{\sqrt{\Gamma}}$$
$$+ \iint \left[\frac{1}{\sqrt{\Gamma}} u_1 - \frac{|t_0|}{\Gamma^{\frac{3}{2}}} (u - \bar{u}) \right] r\,dr\,d\phi + \int_0^{2\pi} \bar{u}\,d\phi.$$

The intervention of $|t_0|$, giving two different expressions according to the sign of t_0, is in agreement with our remarks of § 118 concerning the discontinuity of the **SS**'s used in our general formulæ*.

We also verify our conclusions of § 112 on the sign of the residual integral, by admitting that, in (60), u_0 is positive for r smaller than a certain value $r_1 < |t_0|$ and zero for $r > r_1$ (so that $u = 0$), and that $f = u_1 = 0$.

131. Secondly, we shall admit that S consists of a (finite or, as in fig. 19, infinite) portion S' of the plane $t = 0$ and of a cylindrical part S'' having the outline σ of S' for its cross-section. We, moreover, shall assume t_0 to be so great as to make the half conoid from a cut only S'' and not S' (fig. 22): these being, as we have said in § 126, the conditions in which Volterra operated in order to prove Huygens' principle (in its form (C)) for (e_2).

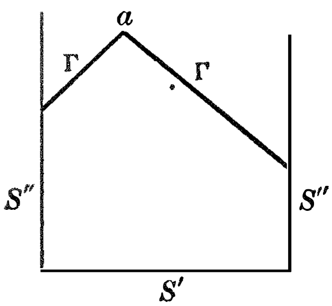

Fig. 22.

The triple integral and the double integral over S' will be

$$(61) \qquad \iiint_T \frac{f}{\sqrt{\Gamma}}\, dx\, dy\, dt + \iint_{S'} \left(\frac{u_1}{\sqrt{\Gamma}} - \frac{|t_0|}{\Gamma^{\frac{3}{2}}} u \right) dx\, dy,$$

no $\lceil^{-}$ being necessary this time. On S'', the transversal (because of $\mathbf{A}(\pi_1, \pi_2, \pi_3) < 0$) will be *opposite* to the inner normal n (which is parallel to the normal to the outline σ directed towards the inside of S'): which gives ($d\sigma$ being the element of arc of σ)

$$\iint \left(u \frac{d}{dn} \frac{1}{\sqrt{\Gamma}} - \frac{1}{\sqrt{\Gamma}} \frac{du}{dn} \right) d\sigma\, dt = \iint \left(\frac{ur \dfrac{dr}{dn}}{\Gamma^{\frac{3}{2}}} - \frac{1}{\sqrt{\Gamma}} \frac{du}{dn} \right) d\sigma dt.$$

The first term alone wants transformation. We integrate it first with respect to t, along the segment l of the corresponding generatrix of the cylinder included inside Γ, i.e. from $t = 0$ to $t = t_0 - \epsilon r$, replacing u by $(u - \bar{u}) + \bar{u}$, where

$$\bar{u} = u(x, y, t_0 - \epsilon r)$$

* This discontinuity also appears as concerns the term in u_1: as said in § 118, we must not forget that $u_1 = \epsilon \frac{\partial u}{\partial t}$ and, therefore, changes its sign at once when the point a crosses the plane $t = 0$.

is the value of u at the intersection of the generatrix with the conoidal sheet. Now (for any sign of t_0)

$$\left|\int_t \frac{dt}{\Gamma^{\frac{3}{2}}} = \right|\int_t \frac{dt}{[(t-t_0)^2-r^2]^{\frac{3}{2}}} = \left|\int_r^{|t_0|} \frac{dt'}{(t'^2-r^2)^{\frac{3}{2}}} = -\frac{1}{r^2}\frac{|t_0|}{\sqrt{t_0^2-r^2}},\right.$$

so that the value of $2\pi u_a$ is the sum of (61) and

$$(61') \quad \iint_{S_0'} \left[\frac{(u-\bar{u}')\,r\,\dfrac{dr}{dn}}{\Gamma^{\frac{3}{2}}} - \frac{1}{\sqrt{\Gamma}}\frac{du}{dn}\right] d\sigma\, dt - |t_0| \int_0 \frac{\bar{u}\, dr}{r\, dn} \frac{d\sigma}{\sqrt{t_0^2-r^2}}.$$

The simple integral must be considered as taken along the curve of intersection of S'' and Γ, although $d\sigma$ and r relate to the base σ of the cylinder.

132. A quite similar treatment will apply to the equation of *damped cylindrical waves*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - Ku = 0,$$

investigated by Coulon as said above. The elementary solution is

$$v = \frac{1}{\sqrt{\Gamma}} \operatorname{Ch} \sqrt{K\Gamma}$$

(with the same value of Γ, Ch being a hyperbolic cosine, which assumes the value 1 on the circle $r = |t_0|$) and we should have

$$2\pi u_a = \left|\iint_{S_0}\left[\frac{u_1 \operatorname{Ch}\sqrt{K\Gamma}}{\sqrt{\Gamma}} + \frac{1}{2}\frac{d\Gamma}{d\nu}\left(\frac{\operatorname{Ch}\sqrt{K\Gamma}}{\Gamma^{\frac{3}{2}}} - \frac{\sqrt{K}\operatorname{Sh}\sqrt{K\Gamma}}{\Gamma}\right)u\right] dS.\right.$$

The transformation of § 129 will give

$$2\pi u_a = \lim_{\tau'=0}\left\{\iint_{S_1}\left[\frac{u_1 \operatorname{Ch}\sqrt{K\Gamma}}{\sqrt{\Gamma}}\right.\right.$$

$$\left.\left. + \frac{1}{2}\frac{d\Gamma}{d\nu}\left(\frac{\operatorname{Ch}\sqrt{K\Gamma}}{\Gamma^{\frac{3}{2}}} - \frac{\sqrt{K}\operatorname{Sh}\sqrt{K\Gamma}}{\Gamma}\right)u\right] dS + \int_{(\tau')} \frac{u}{\sqrt{\Gamma}} r\, d\phi\right\}$$

(where again S_1 and (τ') have the meaning of § 108), the complementary simple integral being exactly the same as in (59) (even with the same value of v, as the substitution of 1 for $\operatorname{Ch}\sqrt{K\Gamma}$ in the numerator of v is immaterial for τ' infinitesimal). The transformation of § 130, taking for S the plane $t=0$, will give

$$2\pi u_a = \iint \left[\frac{\mathrm{Ch}\sqrt{K\Gamma}}{\sqrt{\Gamma}} u_1 - |t_0| \left(\frac{\mathrm{Ch}\sqrt{K\Gamma}-1}{\Gamma^{\frac{3}{2}}} - \frac{\sqrt{K}\,\mathrm{Sh}\sqrt{K\Gamma}}{\Gamma} \right) u \right] r\,dr\,d\phi$$

$$- |t_0| \iint \frac{(u-\bar{u})\,r\,dr\,d\phi}{\Gamma^{\frac{3}{2}}} + \int_0^{2\pi} \bar{u}\,d\phi.$$

The simplicity of this result, compared with the difficulties which, as we said, Coulon met with, and with the complication of expression (12) (§ 79) which he ought to have introduced in order to imitate Volterra's operations, seems to me sufficient to illustrate the importance of avoiding the roundabout method which consists implicitly in integrating and finally redifferentiating.

If the equation were non-homogeneous with the right-hand number f, a corresponding term $\iiint_T \frac{f\,\mathrm{Ch}\sqrt{K\Gamma}}{\sqrt{\Gamma}}\,dx\,dy\,dt$ would have to be added; and the case of a cylindrical S could also be easily treated in the same way as in § 131.

133. Let us again take the (ordinary) wave equation with two more variables:

$$(\mathrm{e}_4) \qquad \frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} \right) = f.$$

The elementary solution will be $\frac{1}{\Gamma^{\frac{3}{2}}} = \frac{1}{[(t_0-t)^2 - r^2]^{\frac{3}{2}}}$, r standing, of course, for $\sqrt{(x_1-a_1)^2 + \ldots + (x_4-a_4)^2}$; and, in our notation, the solution of Cauchy's problem, with respect to $t=0$, will be given by

$$4\pi^2 u_a = -\left|\overline{\mathbf{SSS}\,\frac{f}{\Gamma^{\frac{3}{2}}}\,dx_1 \ldots dx_4\,dt}\right. + \left|\overline{\mathbf{SS}_{t=0}\left(u_0 \,.\, \epsilon \frac{\partial}{\partial t} \,.\, \frac{1}{\Gamma^{\frac{3}{2}}} - \frac{u_1}{\Gamma^{\frac{3}{2}}} \right) dS}\right.$$

$$= -\left|\overline{\mathbf{SSS}\,\frac{f}{\Gamma^{\frac{3}{2}}}\,dx_1 \ldots dx_4\,dt}\right. + \left|\overline{\mathbf{SS}\left(\frac{3\,|t_0|\,u_0}{\Gamma^{\frac{5}{2}}} - \frac{u_1}{\Gamma^{\frac{3}{2}}} \right) dx_1 \ldots dx_4}\right..$$

In the space integral (i.e. 5-fold) and in the surface integral containing u_1 we have to introduce the values f, u_1 of f and u_1 at the point where a perpendicular to the axis of our characteristic cone drawn through any point $(x_1, \ldots x_4, t)$ meets the surface of this characteristic cone.

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, 0 denote direction-cosines of a direction parallel to $t = 0$, such that

$$\frac{x_1 - a_1}{\alpha_1} = \frac{x_2 - a_2}{\alpha_2} = \frac{x_3 - a_3}{\alpha_3} = \frac{x_4 - a_4}{\alpha_4} > 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1,$$

$\bar{f}$ will stand for

$$f\,[a_1 + \alpha_1 \,|\, t_0 - t \,|,\; a_2 + \alpha_2 \,|\, t_0 - t \,|,\; a_3 + \alpha_3 \,|\, t_0 - t \,|,\; a_4 + \alpha_4 \,|\, t_0 - t \,|,\; t].$$

As the integral containing u_0 introduces $\frac{1}{\Gamma^{\frac{5}{2}}} = \frac{1}{(t_0^2 - r^2)^{\frac{5}{2}}}$, we shall have to use not only $\bar{u}$, but also $\bar{u}'$, the value of the radial derivative $\bar{u}' = \frac{1}{2r}\frac{\partial u_0}{\partial r}$, at any point of the edge $r = |\,t_0\,|$ along which our characteristic cone cuts $t = 0$. If $d\Omega$ denotes an element of solid angle in the four-dimensional space (x_1, x_2, x_3, x_4), the formula will be

$$4\pi^2 u_a = -\mathbf{SSS} \frac{(f - \bar{f})}{\Gamma^{\frac{3}{2}}}\, dx_1 \ldots dx_4\, dt$$

$$+ \mathbf{SS} \left[2\,|\,t_0\,| \frac{u - \bar{u} + \bar{u}'(t_0^2 - r^2)}{\Gamma^{\frac{5}{2}}} - \frac{(u_1 - \bar{u}_1)}{\Gamma^{\frac{3}{2}}} \right] dx_1 \ldots dx_4$$

$$+ 2\left[\int |\,t_0 - t\,|\, dt\, \mathbf{S} \bar{f}\, d\Omega + \mathbf{S}\,(|\,t_0\,|\, \bar{u}_1 + 3t_0^2 \bar{u}')\, d\Omega\right] + 2\mathbf{S}\bar{u}\, d\Omega,$$

every term now having a meaning in the usual way if f and u_1 have derivatives up to the first order, u_0 up to the second.

Of course we should have no difficulty in writing the analogous formula if there were a "damping" term of the form Ku.

BOOK IV

THE EQUATIONS WITH AN EVEN NUMBER OF INDEPENDENT VARIABLES AND THE METHOD OF DESCENT

CHAPTER I

INTEGRATION OF THE EQUATION IN $2m_1$ VARIABLES

1. General formulæ

134. The first cases in which the solution of Cauchy's problem was known in Analysis do not, as we have seen, belong to the above treated class: Riemann's and Kirchhoff's methods correspond respectively to $m = 2$ and $m = 4$.

We shall presently see that, in such cases, singularities such as we met with in the preceding Book no longer occur, every improper integral being even eliminated. This explains why the above-mentioned solutions were found first.

In the general case, nevertheless, even values of m must be considered as bringing in new difficulties. The above methods are no longer valid, and this for two reasons:

First, the elementary solution is no longer *well determined* (§ 65).

Next, we can no longer introduce the *finite part* of the integrals which we shall be led to use, as the exponent

$$\left(\frac{m-2}{2} \quad \text{or} \quad \frac{m}{2}\right),$$

with which Γ will appear in the denominator of the elementary solution or its derivatives, will be an integer.

It will actually follow from the very form of the expressions which we shall find, that they could not have been obtained by mere imitation of our former method.

But, as we have already mastered the case of $2m_1 + 1$ independent variables, this will enable us to reach the same result when the number of variables is $2m_1$ by using our method of *descent* (§ 29). The solution of the equation

$$\text{(E)} \quad \mathcal{F}(u) = \sum_{i,k=1}^{m}\sum A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{m} B \cdot \frac{\partial u}{\partial x_i} + Cu = f(x_1, x_2, \ldots, x_m),$$

m being equal to $2m_1$, will be deduced from the corresponding one for the equation in $2m_1+1$ variables

$$(\text{E}') \qquad \mathcal{F}'(u)=\mathcal{F}(u)-\frac{\partial^2 u}{\partial z^2}=f(x_1, x_2, \ldots, x_m),$$

denoting by z an $(m+1)$th auxiliary variable.

If, as we still assume, the characteristic form

$$\mathbf{A}(P_1, P_2, \ldots, P_m)=\Sigma A_{ik}P_iP_k$$

of (E) contains one positive and $(m-1)$ negative squares, the corresponding form

$$\mathbf{A}'(P_1, \ldots P_m, R)=\mathbf{A}-R^2$$

relating to (E′) will consist of one positive and m negative squares. We have already seen that the quantity Γ', analogous to Γ and relating to (E′), is

$$\Gamma'=\Gamma-(z-c)^2,$$

denoting by $(x_1, x_2, \ldots x_m, z)$ and $(a_1, a_2, \ldots a_m, c)$ two points of the $(m+1)$-dimensional space. We have also found, in § 70, what relations exist between the elementary solutions of both equations: we have seen that the coefficients of the successive powers of Γ in one of them differ by numerical factors from the coefficients of the corresponding powers of Γ' in the other.

Considering the adjoint equations

$$(\mathcal{E}) \quad \mathcal{G}(v)=0, \qquad (\mathcal{E}') \quad \mathcal{G}'(v)=\mathcal{G}(v)-\frac{\partial^2 v}{\partial z^2}=0$$

of (E) and (E′), the formulæ of § 70 express that, if

$$v'=\frac{V'}{\Gamma'^{\frac{m-1}{2}}}=\frac{V'}{\Gamma'^{m_1-\frac{1}{2}}}$$

be the elementary solution of $(\mathcal{E}')$, with

$$(1) \qquad V'=\sum_0^\infty V_h'\,\Gamma'^h=(-1)^h\,V_h'\,[(z-c)^2-\Gamma]^h,$$

then the elementary solution of $(\mathcal{E})$ will be

$$(2) \qquad v=\frac{V}{\Gamma^{m_1-1}}-\mathcal{V}\log\Gamma+w$$

(w being a regular function); and, if we use the coefficients C_n of

§ 95, formulæ (62), (62 *b*) of § 70, applied to the adjoint equation, can be written

$$(3) \qquad V = (m_1 - 1)\, C_{m_1-1} \sum_{h=0}^{m_1-2} \frac{1}{(m_1 - h - 1)\, C_{m_1-h-1}} V_h' \Gamma^h,$$

$$(3') \qquad \mathfrak{V} = (m_1 - 1)\, C_{m_1-1} \sum_{h=m_1-1}^{\infty} C_{h-m_1+1} V_h' \Gamma^{h-m_1+1}$$

$$= (m_1 - 1)\, C_{m_1-1} \sum_{k=0}^{\infty} C_k V'_{k+m_1-1} \Gamma^k.$$

Now, this can be obtained directly by operating on v', or rather on the similar quantity

$$(v') = \frac{V'}{(-\Gamma')^{\frac{m-1}{2}}}$$

which relates to $\Gamma' < 0$, as said in § 73. We form the expression (again a solution of $(\mathscr{E}')$)

$$(4) \qquad \overline{\left|\int_{z+\sqrt{\Gamma}}^{c_1} (v')\, dc\right.} = \overline{\left|\int_{z+\sqrt{\Gamma}}^{c_1} \frac{V'}{[(c-z)^2 - \Gamma]^{m_1 - \frac{1}{2}}} dc\right.}$$

(c_1 constant); this (changing c into $z + c'$ under $\int$) can be written

$$(4') \qquad \overline{\left|\int_{\sqrt{\Gamma}}^{c_1 - z} (v')\, dc'\right.} = \overline{\left|\int_{\sqrt{\Gamma}}^{c_1}\right.} - w_1,$$

$w_1 = \int_{c_1 - z}^{c_1} (v')\, dc'$ being a regular function; the first term, which is independent of z, is (but for a numerical factor) the singular part* of the elementary solution v of $(\mathscr{E})$. For, by substituting (1) for V', the integral of each term will be given by the operations of § 95 (formulæ (28), (28′) in that number) and these precisely give, for the result, a quantity of the required form.

Moreover, we again obtain the values of the coefficients in (3),

* The difference (4′) is a solution of $(\mathscr{E}')$ and, the second term being holomorphic, we have

$$\mathscr{G}'\left(\overline{\left|\int_{\sqrt{\Gamma}}^{c_1}\right.}\right) = \mathscr{G}'(w_1).$$

The common value of both sides is a holomorphic function (as we see from its second form) and independent of z (as appears from the first form). Therefore, as noticed above, there exists a function w of $x_1, x_2, \ldots, x_m$ only, such that which become identical, of course, with those in Book II (§ 70), if we multiply (4′) by the constant factor

(5) $$(-1)^{m_1-1}\, 2\,(m_1-1)\, C_{m_1-1} = (-1)^{m_1-1}\mathbf{k}.$$

The case of $m=2$ is special. (4′) does not want the symbol $\sqrt{}$ and has the value

(6) $$\Sigma \int (-1)^h V_h'(c'^2-\Gamma)^{h-\frac{1}{2}}\, dc' = -\tfrac{1}{2}\log\Gamma \,.\, \Sigma\, C_h V_h' \Gamma^h + w.$$

Therefore, there exists no V, but only a $\mathcal{V}$ (equal to $\Sigma\, C_h V_h' \Gamma^h$), which is, as said previously, no other than Riemann's function $\left(\text{multiplied by the constant factor } \frac{1}{\sqrt{\Delta}}\right)$, the number $\mathbf{k}$ being equal to* 2.

135. This being understood, to obtain a solution of equation (E) complying, as to the multiplicity S, with the given conditions, we shall consider, in the $(m+1)$-dimensional space E_{m+1} defined by the coordinates $(x_1, x_2, \dots x_m, z)$, the multiplicity S' (hypercylinder) the projection of which is S (fig. 23)†, i.e., the one obtained by taking successively for $(x_1, x_2, \dots x_m)$ the coordinates of any point of S and for z all possible real values. If S is duly inclined with respect to Γ, S' will be duly inclined with respect to Γ'.

As we shall have to consider multiple integrals both in the m-dimensional space E_m and in the $(m+1)$-dimensional space E_{m+1}, the notation which we used in § 38 will be modified in the following way: the symbols **SS** and **SSS** will be kept for E_m, while a surface integral (i.e. an integral over an m-fold variety, which will always be a hypercylinder) in E_{m+1} will be denoted by **SS**$\int$, a volume integral (i.e. $(m+1)$-tuple integral) in E_{m+1} by **SSS**$\int$.

A solution u of equation (E) being defined by the double condition:

Of assuming at each point $(x_1, x_2, \dots x_m, z)$ of S' the value that u must have at the corresponding point $(x_1, x_2, \dots x_m)$ of S;

Of having for transversal derivative at the point $(x_1, x_2, \dots x_m, z)$ the given value of $\frac{du}{dv}$ at $(x_1, x_2, \dots x_m)$;

* This is not in agreement with (5): the factor (m_1-1), which ought to be zero, is replaced by 1, as it was several times in Book II.

† Fig. 23, relating to $m=2$, can be used as a diagrammatic figure for the general case.

we know (§ 29) that the solution will be unique and independent of z and will satisfy (E). Therefore such a solution will be a solution of the given problem, and conversely.

Function u will be given by formula (39) (§ 105), viz. (in our new notation)

$$(7) \quad (-1)^{m_1} \pi \Omega_{m-2}\, u_a = - \overline{\mathbf{SSS}} \int v' f \, dx_1 \dots dx_m dz + \overline{\mathbf{SS}} \int \left(u \frac{dv'}{d\nu} - v' \frac{du}{d\nu} - Luv' \right) dS',$$

where T' denotes the portion of $(m+1)$-dimensional space between S' and Γ', S_0' (fig. 23) the corresponding portion of S'; and $m = 2m_1$.

136. Strictly speaking, we can say that, in this way, we have solved the problem : but, remembering a celebrated word of Poincare*, we must acknowledge that it is very "insufficiently solved." For the above solution contains foreign elements,—the space E_{m+1}, the auxiliary variable z and all that is relative to them ;—and we evidently have to try to transform it in order to get rid of these.

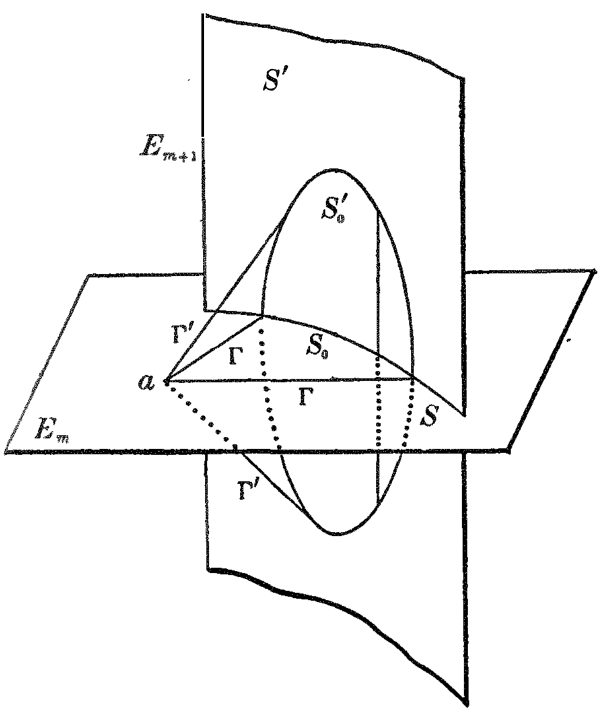

Fig. 23.

Geometrically, the relation between the diagrams in E_m and E_{m+1} is the following.

T' is projected on to the m-dimensional space E_m along the region T, included between S and Γ : that is, if the point $(x_1, x_2, \dots x_m, z)$ belongs to T', the point $(x_1, x_2, \dots x_m)$ belongs to T, and, conversely, any point of T is the common projection of points of T', i.e. of all those

* "Il n'y a plus des problèmes résolus et d'autres qui ne le sont pas ; il y a seulement des problemes *plus ou moins* resolus." Poincare's *Lecture* at the IV Mathematical Congress, Rome, 1908 ; *Atti del IV Congresso intern. dei Matematici*, Vol. I, p. 175.

whose z's are between $-\sqrt{\Gamma}$ and $+\sqrt{\Gamma}$ (supposing the $(m+1)$th co-ordinate of a to be zero).

In the same way, S_0' is projected on to E_m along S_0, each point of S_0 being the projection of an infinite number of points of S_0', having their z's between $-\sqrt{\Gamma}$ and $+\sqrt{\Gamma}$.

137. This being noticed, we now undertake the transformation of our formula (7).

Let us deal, for instance, with the first term in the right-hand member, which is an **SSS** extended to T'.

Our method will consist in integrating with respect to z first.

For $m=2$, v being infinite of order $\frac{1}{2}$ only, this raises no objection. But for greater even values of m, we have no right to do so all over our field of integration, as the ordinates projecting points of T' on T cut the singular surface Γ' at an angle which becomes infinitely small in the neighbourhood of Γ. We shall, therefore, divide our domain T into two parts T_1 and T_2—the second of which will include the vicinity of the conoid—by an auxiliary boundary τ (which will, finally, approach Γ).

In the portion T_1' of T' which is projected along T_1 (the boundary of which is constituted by Γ' and a cylinder τ with base τ) integrations with respect to z are legitimate, so that the corresponding **SSS** $\int$ will be obtained by integrating m times, over T_1, the simple integral

$$f \cdot \left|\overline{\int_{-\sqrt{\Gamma}}^{+\sqrt{\Gamma}} \frac{\sum_{h=0}^{\infty} V_h' (\Gamma - z^2)^h \, dz}{(\Gamma - z^2)^{m_1 - \frac{1}{2}}}}\right. .$$

If the factor of f is integrated term by term*, say

$$\Sigma V_h' \left|\overline{\int_{-\sqrt{\Gamma}}^{+\sqrt{\Gamma}} \frac{dz}{(\Gamma - z^2)^{m_1 - h - \frac{1}{2}}}}\right. ,$$

we see that each term depends on the operations in § 96.

These show us that

(1) All terms corresponding to h smaller than $m_1 - 1$ vanish;

* No difficulty as to convergence arises from the presence of our symbol $|$, as the latter only occurs in a finite number of terms of the sum.

(2) The following terms (h varying from $m_1 - 1$ to ∞) give a result which is in fact 2π, multiplied by the coefficient of the logarithm in the integral (4′), except for the factor $(-1)^{m_1-1}$, i.e., by $\frac{1}{\mathbf{k}}\mathcal{V}$.

Consequently, in this first term, the quantity, infinite on the conoid, which appeared under **SSS** in the formulæ relating to the case of m odd, is already replaced by the perfectly regular quantity $\mathcal{V}$.

For $m = 2$, our transformation is thus accomplished.

138. In T_2 (for $m > 2$), the same method is no longer valid, and the result will actually be found to be utterly different from what that method would suggest. Our lines of integration being subjected to the condition of meeting Γ' at a finite angle, we shall now consider a system of lines l, each joining a point of Γ (defined by coordinates $\lambda_1, \lambda_2, \ldots \lambda_{m-1}$) to a point of τ and the parallel lines in any plane $z = \text{const.}$ A point of T_2 will therefore be defined by the values of $\lambda_1, \lambda_2, \ldots \lambda_{m-1}$ and of Γ, this last quantity varying from zero to a quantity γ, very small if τ is very near Γ.

Let

(8) $$dx_1 dx_2 \ldots dx_m = K d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}\, d\Gamma = d\tau_\gamma d\Gamma$$

be the expression of an element of T_2.

If T_2' is the part of T' projected along T_2, a point of T_2' will be defined by the coordinates $\lambda_1, \lambda_2, \ldots \lambda_{m-1}, \Gamma, z$.

Let us first integrate along the lines l, that is, allowing

$$\lambda_1, \lambda_2, \ldots \lambda_{m-1}, z$$

to remain constant. Then we shall make z vary, and lastly

$$\lambda_1, \lambda_2, \ldots \lambda_{m-1}.$$

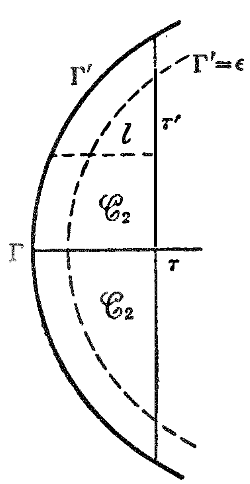

Fig. 24.

As the boundary of T_2', aside from Γ' (i.e., the cylinder), is not a locus of lines l, so that the segments of lines l included in T_2 become infinitely small in the neighbourhood of Γ', we have to apply the principles of Book III, § 90: we shall have to take the finite part of every simple integral along a line l and, integrating this with respect to z, again take the finite part of the result. This gives the double integral over the section $\mathcal{C}_2$ (fig. 24) of

T_2' by any two-dimensional plane

$$\lambda_1 = \text{const.}, \lambda_2 = \text{const.}, \ldots \lambda_{m-1} = \text{const.}:$$

an integral which remains finite (as is evident *a priori*, and as we shall verify) when we let the parameters $\lambda_1, \ldots \lambda_{m-1}$ vary, and the integration of which with respect to these parameters will be made in the classic way.

We can notice, besides, that the terms corresponding to $h \geqslant m_1 - 1$ in the expansion (1) of V' only give finite integrals in the ordinary sense, which become infinitely small when γ approaches zero. We shall therefore neglect them, so that we only have to deal with a finite number, viz. $m_1 - 1$, of terms of that expansion.

For $m = 2$, we have already noticed that no other terms exist. For $m = 4$, i.e. $m_1 = 2$, we have only one of them, viz. V_0', which we have to divide by $(\Gamma - z^2)^{\frac{3}{2}}$ and integrate with respect to the λ's, Γ and z after having multiplied it by Kf. Writing $KfV_0' = F_0$, the integral with respect to Γ is (by means of an integration by parts)

$$\left|\overline{\int_{z^2}^{\gamma} \frac{F_0 d\Gamma}{(\Gamma - z^2)^{\frac{3}{2}}}} = \left(-2 \frac{F_0}{\sqrt{\Gamma - z^2}}\right) + 2 \int_{z^2}^{\gamma} \frac{\partial F_0}{\partial \Gamma} \frac{d\Gamma}{\sqrt{\Gamma - z^2}}.\right.$$

The integral which remains on the right-hand side is an ordinary one, which approaches zero with $\gamma - z^2$. Outside of $\int$, we have a term P which is a fractional infinity in the neighbourhood of the lower limit $\gamma = z^2$: this fractional infinity is to be cancelled, and we only keep the term $\dfrac{-2F_0}{\sqrt{\gamma - z^2}}$. This is now easy to integrate with respect to z, giving $-2\pi F_0$.

Similarly, for any m_1 and $h < m_1 - 1$, let us write $KfV_h = F_h$. From the corresponding integral taken over $\mathscr{C}_2$, we shall easily eliminate the symbol $\left|\overline{}\right.$: for we have, by a classic formula, as

$$\frac{1}{(\Gamma - z^2)^{m_1 - h - \frac{1}{2}}} = \frac{(-1)^{m_1 - h - 1}}{\frac{1}{2} \cdot \frac{3}{2} \ldots (m_1 - h - \frac{3}{2})} \frac{d^{m_1 - h - 1}}{d\Gamma^{m_1 - h - 1}} \left(\frac{1}{\sqrt{\Gamma - z^2}}\right),$$

$$(9) \quad \int \frac{F_h d\Gamma}{(\Gamma - z^2)^{m_1 - h - \frac{1}{2}}}$$

$$= \frac{(-1)^{m_1 - h - 1}}{\ldots} \int \frac{d\Gamma}{\ldots} \frac{d^{m_1 - h - 1} F_h}{\ldots}$$

$$(10)\quad R_h = \frac{(-1)^{m_1-h-1}F_h}{\frac{1}{2}\cdot\frac{3}{2}\dots(m_1-h-\frac{3}{2})}\frac{d^{m_1-h-2}}{d\Gamma^{m_1-h-2}}\left(\frac{1}{\sqrt{\Gamma-z^2}}\right)-\dots$$

$$-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\dots(m_1-h-\frac{3}{2})}\frac{1}{\sqrt{\Gamma-z^2}}\frac{d^{m_1-h-2}F_h}{d\Gamma^{m_1-h-2}}$$

$$=\frac{-F_h}{m_1-h-\frac{3}{2}}\frac{1}{(\Gamma-z^2)^{m_1-h-\frac{3}{2}}}-\dots$$

$$-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\dots(m_1-h-\frac{3}{2})}\frac{1}{\sqrt{\Gamma-z^2}}\frac{d^{m_1-h-2}F_h}{d\Gamma^{m_1-h-2}}.$$

The first term on the right-hand side of (9) gives an ordinary integral, which vanishes with γ and may be neglected. On the other hand, the value R_h' of R_h in the vicinity of the boundary $\Gamma = z^2$ is a fractional infinity and shall be cancelled (being indeed the only infinity, at least for an arbitrary z, it precisely represents *the* fractional infinity which we have to remove by the definition of the notation $\overline{|\quad}$). This reduces the finite part of the simple integral (9) to

$$R_h'' = (R_h)_{\Gamma=\gamma} = -\frac{F_h}{m_1-h-\frac{3}{2}}\frac{1}{(\gamma-z^2)^{m_1-h-\frac{3}{2}}}-\dots$$

$$-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\dots(m_1-h-\frac{3}{2})}\frac{1}{\sqrt{\gamma-z^2}}\frac{d^{m_1-h-2}F_h}{d\gamma^{m_1-h-2}},$$

which we have now to integrate with respect to z, from $-\sqrt{\gamma}$ to $+\sqrt{\gamma}$, taking the finite part of the result. Now the value of the integral

$$(11)\quad \overline{\left|\int_{-\sqrt{\gamma}}^{+\sqrt{\gamma}}\frac{dz}{(\gamma-z^2)^{n+\frac{1}{2}}}\right.}$$

has been found (Book III, § 96) to be zero for every positive n. Therefore, the only term we have to consider is the last one, giving

$$-\frac{\pi}{\frac{1}{2}\cdot\frac{3}{2}\dots(m_1-h-\frac{3}{2})}\frac{d^{m_1-h-2}F_h}{d\gamma^{m_1-h-2}}.$$

We have to take the above derivative for $\Gamma=\gamma$. But finally we must let γ approach zero. We therefore take the derivative in question for $\gamma=0$ and this gives us the required limit.

There remains only, taking for h every value from zero to m_1-2, to obtain the sum of the results. We shall now see that this is in direct connection with the value of the polynomial (3): for the (m_1-h-2)th derivative of F for $\Gamma=0$ can be (on account of Leibniz's classic formula for the derivative of a product) considered as being the (m_1-2)th derivative of $F\Gamma^h$ multiplied by $\frac{(m_1-h-2)!}{(m_1-2)!}$.

Thus, again introducing the coefficients C and taking account of (3) we have*

$$(12)\qquad \lim_{\gamma=0} \left| \iint \frac{v' d\Gamma dz}{(\Gamma - z^2)^{m_1 - \frac{1}{2}}} \right.$$

$$= -\frac{\pi}{(m_1-2)!}\left(\frac{d^{m_1-2}}{d\gamma^{m_1-2}}\right)_{\gamma=0}\left(\sum_0^{m_1-2} \frac{F_h\Gamma^h}{(m_1-h-1)C_{m_1-h-1}}\right)$$

$$= -\frac{2\pi}{\mathbf{k}}\frac{1}{(m_1-2)!}\left(\frac{d^{m_1-2}KfV}{d\gamma^{m_1-2}}\right)_{\gamma=0}.$$

139. We have, in the above, applied the general result of § 90; but, in the present case, it is easy to verify directly that things actually behave as was proved at the aforesaid place. For, referring to the definition of $|$, as given in §§ 88, 89, we should have to limit T_2' by $\Gamma'=\epsilon$ (dotted line of fig. 24) and take the limit of the corresponding **SSS** after subtraction of fractional infinities in ϵ. Now, we immediately see, by (10), that the value of R_h' (for $\Gamma=z^2+\epsilon$) is such a fractional infinity; and so is the remainder of any integral (11), when limiting the segment of integration to $z=-\sqrt{\gamma}-\epsilon$ and $z=+\sqrt{\gamma}+\epsilon$, this proving—as has been done, for the general case, in § 90—that our method of procedure in two successive integrations, each time using the symbol $|$, correctly gives the value of the double integral relative to Γ and z.

140. Let us integrate, lastly, with respect to $\lambda_1, \lambda_2, \ldots \lambda_{m-1}$. On the multiplicity τ defined by the equation $\Gamma=\gamma$, where γ is any constant, the product $Kd\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}$ gives an element which, on account of (8), we previously denoted by $d\tau_\gamma$ $\left(\text{or } \frac{dT}{d\gamma}\right)$.

The integral

$$(13)\qquad I_\gamma = \mathbf{SS} fV\,d\tau_\gamma = \mathbf{SS}\,Kf d\lambda_1 d\lambda_2 \ldots d\lambda_{m-1}$$

will be a function of γ, which can be differentiated with respect to γ by differentiating KfV under **SS** and integrating with respect to the λ's.

Therefore, the quantity required will be found to be proportional to

* When f is assumed to be analytic, the same formula can be obtained by using Maclaurin expansions, as we did in the *Acta Mathematica*, Vol. XXXI.

the coefficient of γ^{m_1-2} in the expansion of I_γ, or, in other words (by (12)), it is equal* to

(14) $$\frac{2\pi}{\mathbf{k}} \cdot \frac{1}{(m_1-2)!} \frac{d^{m_1-2}}{d\gamma^{m_1-2}}_{(\gamma=0)} \cdot I_\gamma.$$

It will contain, as we see, the derivatives of f and those of V up to the order $m_1 - 2$. This function V is only partly determinate, terms containing Γ^{m_1-1} as a factor being arbitrary; but such terms have no influence on (14).

The part of the value of u_a corresponding to the term

$$-\overline{\left|\mathsf{SSS}\int v' f dx_1 \ldots dx_m dz\right.}$$

consists, then, of the m-tuple integral

(14′) $$-\frac{2\pi}{\mathbf{k}} \mathsf{SSS} f \mathcal{V} dx_1 \ldots dx_m$$

extended over the inside of Γ, and of expression (14), an $(m-1)$-tuple integral extended over the surface of Γ. These two quantities do not contain, this time, any infinite function, but only the two regular functions, $\mathcal{V}$ for the one, V for the other.

141. We must notice, however (for $m > 2$), that, in order to agree with the result of Book III, (13) ought to be calculated first by cutting T—and, therefore, τ—with a small surface Σ so as to exclude the neighbourhood of a, then by letting Σ approach a with the assumptions mentioned in § 106, this being the process obtained in §§ 105, 106, of which the present one is a mere translation. That this process will converge, and uniformly, as was said in § 106, is obvious for the same reason.

But the fact is that the precaution in question is unnecessary. We can obtain the same final value by at once extending the SS over the whole of the surface τ and applying the $m_1 - 2$ differentiations to the result thus obtained.

To prove this, we have to show that if such an integral were extended over the small part of τ which lies on the same side of Σ as the point a, its (m_1-2)th derivative with respect to γ, for $\gamma=0$, would exist and approach zero on letting Σ approach a (the restrictions of § 106 being still understood). We only need to give that proof for a suitably chosen law of variation of Σ, because we know that the final result does not depend on this law.

* We now take account of the sign − before the first term of (7).

We shall do this by introducing normal variables ξ, such as defined in § 57. This reduces Γ to a quadratic form with constant coefficients, which we can even (by a linear substitution on the ξ's) reduce to

$$\Gamma_0 = \xi_m{}^2 - \xi_1{}^2 - \xi_2{}^2 - \dots - \xi^2{}_{m-1},$$

which we shall write $$\Gamma_0 = t^2 - r^2 = t^2 - R,$$

by writing t instead of ξ_m and $R = r^2$ for $\xi_1{}^2 + \xi_2{}^2 + \dots + \xi^2{}_{m-1}$. Thus the characteristic conoid with vertex a becomes an ordinary hypercone of revolution, and $\Gamma = \gamma$ will represent a (hyper)quadric, already considered in § 97.

We can now choose Σ: we shall take for it such planes as $t = \text{const.} = \epsilon$.

The function $F = \Gamma f$ under **SSS** is assumed to be regular (see below) (and will remain so after our change of variables, which is a regular one), which being understood, we have to investigate the differentiation of the integral

(15) $$\mathbf{SS}\, F d\tau_\gamma,$$

where $d\tau_\gamma$ is such that its product by $d\gamma$ represents the element of m-dimensional volume $d\xi_1 \dots d\xi_m$. The latter will be replaced by

$$r^{m-2} d\Omega_{m-2} dr\, dt$$

($d\Omega_{m-2}$ having the same meaning as in § 97), which is equivalent to referring it to t, r and angular parameters $\phi_1, \phi_2, \dots \phi_{m-2}$; and we shall begin by integration with respect to $\phi_1, \dots \phi_{m-2}$. This will introduce the integral

$$\Phi = \mathbf{S}\, F d\Omega_{m-2},$$

a function of t and r which, moreover, is even with respect to the latter*, so that we can consider it as being regular in terms of t and $R = r^2$. By means of this introduction, the volume integral $\mathbf{SSS}\, F d\xi_1 d\xi_2, \dots d\xi_m$, relating to the m-dimensional volume enclosed between the conoid $\Gamma = 0$, the hyperquadric $\Gamma = \gamma$ and the hyperplane $t = \epsilon$, would be expressed by the double integral

(16) $$\iint \Phi \,.\, r^{m-2} dr\, dt,$$

the area of integration being bounded (fig. 25) by the three straight lines $r = 0$, $t = \epsilon$, $t = r$ and an arc of the hyperbola

$$t^2 - r^2 = \gamma.$$

(15) is the derivative† of (16) with respect to γ. As $d\tau_\gamma$ is the "quotient" of the m-dimensional space element by $d\gamma$, we have to replace $drdt$ by a corresponding element $d\Pi_\gamma$ of our hyperbola, which

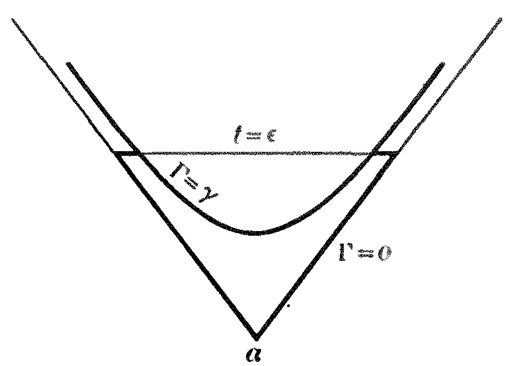

Fig. 25.

* F being expanded by Taylor's formula in powers of the ξ's and, therefore, t and r, with coefficients trigonometrical in the ϕ's, any term which is odd in r, when multiplied by $d\Omega_{m-2} dr$ and integrated with respect to the ϕ's, corresponds to an integral inside the hypersphere of radius r, which is zero, as its element is monomial in the ξ's and odd with respect to at least one of them.

† Compare below, § 147.

shall be the quotient of $d\Pi = drdt$ by $d\gamma$. This quotient (see § 38) is

$$dt : \frac{\partial \Gamma}{\partial r} = \frac{dt}{2r}.$$

The question, therefore, concerns the simple integral

$$(17) \quad \int_{\sqrt{\gamma}}^{\epsilon} r^{m-3} \Phi dt = \int_{\sqrt{\gamma}}^{\epsilon} \Phi(t, R) R^{m_1 - \frac{3}{2}} dt = \int_{\sqrt{\gamma}}^{\epsilon} \Phi(t, t^2 - \gamma)(t^2 - \gamma)^{m_1 - \frac{3}{2}} dt,$$

and is whether its $(m_1 - 2)$th derivative exists for $\gamma = 0$, and is infinitesimal with ϵ. Now, by the classic rule, the first derivative is

$$\int_{\sqrt{\gamma}}^{\epsilon} \left[\frac{\partial \Phi}{\partial R} \frac{\partial R}{\partial \gamma} (t^2 - \gamma) - (m_1 - \tfrac{3}{2}) \Phi \right] (t^2 - \gamma)^{m_1 - \frac{5}{2}} dt.$$

The equation giving R is $t^2 - R = \gamma$ and therefore $\frac{\partial R}{\partial \gamma}$ is equal to -1. On the other hand, we have written no term corresponding to the variability of the lower limit: this term is obviously zero, m_1 being assumed to be greater than 1 (and even, for the present moment, than 2). Thus, setting down

$$(18) \qquad \Phi_1(t, R) = -\left[R \frac{\partial \Phi}{\partial R} + (m_1 - \tfrac{3}{2}) \Phi \right] = -\left[\frac{r}{2} \frac{\partial \Phi}{\partial r} + (m_1 - \tfrac{3}{2}) \Phi \right],$$

we see that the derivative in question is

$$(17') \qquad \int_{\sqrt{\gamma}}^{\epsilon} \Phi_1(t, t^2 - \gamma)(t^2 - \gamma)^{m_1 - \frac{5}{2}} dt;$$

that is, analogous to the expression (17) itself, but differing from it by the fact that m_1 is changed to $m_1 - 1$.

The $(m_1 - 2)$th derivative of (17), for $\gamma = 0$, is the same thing as the $(m_1 - 3)$th derivative of (17′). In other words, if our conclusion is certain for any value of m_1, it is certain for the following one.

But, for $m_1 = 2$, we only have to examine the values of the integral itself, viz.

$$\int_{\sqrt{\gamma}}^{\epsilon} \Phi(t, t^2 - \gamma) \sqrt{t^2 - \gamma}\, dt$$

(without having to differentiate it); and, for $\gamma = 0$, this integral reduces* to

$$\int_0^{\epsilon} \Phi(t, t)\, t dt,$$

a quantity which is smaller than $H\frac{\epsilon^2}{2}$, denoting by H a maximum value of $|\Phi|$.

---

* For $m = 4$ and Φ identically equal to 1, the first form of (17) becomes $\int_{\sqrt{\gamma}}^{\epsilon} r dt$, which is immediately seen to represent half the area of the hyperbolic segment determined by the chord $t = \epsilon$ and, therefore, for $\gamma = 0$, half the area of the triangle between that chord and the asymptotes.

Our proof is, therefore, complete; and we see that the (m_1-2)th derivative of (15), for $\gamma=0$, is, for ϵ infinitesimal, an infinitesimal of the order of ϵ^2.

Instead of using the symbol Φ, we can remark that the left-hand side Φ_1 of (18) is equal to $\mathbf{S} F_1 d\Omega_{m-2}$, with

$$F_1 = -\left[\frac{r}{2}\frac{\partial F}{\partial r} + (m_1 - \tfrac{3}{2}) F\right].$$

m_1-2 similar operations (the coefficient inside the brackets decreasing each time by 1) will lead to a certain final function F_{m_1-2}; and the consequence of our above argument is that the required (m_1-2)th derivative of I_γ is

$$\text{(19)} \qquad \int dt\, \mathbf{S} F_{m_1-2} d\Omega_{m-2},$$

where, in F_{m_1-2}, we have to make $r=t$: but this can be written

$$\text{(19')} \qquad \mathbf{SS} F_{m_1-2} d\Omega_{m-2} dt,$$

an integral which is extended over Γ_0. Coming back to our original coordinates, this can be considered as an integral extended over Γ. On this latter conoid, the parameters ϕ may be considered as $(m-2)$ of those which we have called λ (each system of values of the ϕ's characterizing a generatrix of Γ_0, which corresponds to a bicharacteristic on Γ), while t, being a normal variable, can be considered as the parameter s.

142. Having thus transformed the first term in (7), an entirely similar evaluation obviously applies to the integral

$$-\left|\mathbf{SS}\int\left(v'\frac{du}{d\nu} + Luv'\right) dS' = \right|\mathbf{SS}\int v'(u_1 + Lu_0)\, dS'.$$

We shall have, for this quantity, the value

$$-\frac{2\pi}{\mathbf{k}}\left[\mathbf{SS}_{S_0}\, \mathcal{V}(u_1 + Lu_0)\, dS - \frac{1}{(m_1-2)!}\frac{d^{m_1-2}}{d\gamma^{m_1-2}}_{(\gamma=0)} \mathbf{S}_\sigma V(u_1 + Lu_0)\, d\sigma_\gamma\right],$$

where σ is the intersection of S by the surface $\Gamma=\gamma$ and $d\sigma_\gamma$, the element of σ defined by the relation

$$\text{(8 a)} \qquad d\sigma_\gamma\, d\gamma = dS.$$

143. Let us proceed, lastly, to the term

$$\text{(20)} \qquad \mathbf{SS}\int u\frac{dv'}{d\nu} dS' = \mathbf{SS}\int u\frac{dv'}{d\nu} dSdz.$$

A quite similar method can again be applied to it. For the preceding operations show us, generally speaking, that, if w' be any quantity of the form

$$w' = \frac{\Sigma\, W_h' \Gamma'^h}{\cdots}$$

with the W_h' not depending on z, and we set down $(w') = \dfrac{\Sigma\, W_h' \Gamma'^h}{(-\Gamma')^{p-\frac{1}{2}}}$; if, furthermore,

$$(21) \qquad \left|\int_{\sqrt{\Gamma}}^{c_1} (w')\, dc' = (-1)^{p-1} \left(\frac{W}{\Gamma^{p-1}} - \mathcal{W} \log \Gamma + \dots\right)\right.$$

(dots standing for terms which are regular for $\Gamma = 0$), then

(22)

$$\left|\mathbf{SS}\int w' u dS' = + 2\pi \left(\mathbf{SS}_S u \mathcal{W} dS - \frac{1}{(p-2)!} \frac{d^{p-2}}{d\gamma^{p-2}}_{(\gamma=0)} \mathbf{S}_\sigma u W d\sigma_\gamma\right).\right.$$

This will be the case if we take $w' = \mathbf{k}\dfrac{dv'}{d\nu}$ and (as seen by direct differentiation) (w') will be $-\mathbf{k}\dfrac{d(v')}{d\nu}$, the number p being $m_1 + 1$. The integral (21) will then be $-\mathbf{k}\left|\displaystyle\int_{\sqrt{\Gamma}}^{c_1} \frac{d(v')\, dc'}{d\nu}\right.$ and therefore, as we know, equal to

$$-\mathbf{k}\frac{d}{d\nu}\left|\int_{\sqrt{\Gamma}}^{c_1} (v')\, dc' = -(-1)^{m_1-1} \frac{d}{d\nu}\left(\frac{V}{\Gamma^{m_1-1}} - \mathcal{V} \log \Gamma + \dots\right)\right.$$

on account of § 134, so that

$$\mathcal{W} = \frac{d\mathcal{V}}{d\nu}, \quad W = -(m_1 - 1)\frac{V d\Gamma}{d\nu} + \frac{\Gamma dV}{d\nu} - \mathcal{V}\Gamma^{m_1-1}\frac{d\Gamma}{d\nu}.$$

These are the values which we shall have to substitute in (22), the result having to be divided by $\mathbf{k}$.

We shall immediately observe that the $(p-2)$th, i.e. (m_1-1)th, derivative of the last term of W for $\gamma = 0$ appears at once, viz.

$$(22') \qquad \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{d\gamma^{m_1-1}}_{(\gamma=0)} \mathbf{S} u \mathcal{V} \Gamma^{m_1-1} \frac{d\Gamma}{d\nu} = \mathbf{S} u \mathcal{V} \frac{d\Gamma}{d\nu}.$$

144. At first sight, the expressions thus written for the last term (20) seem to present a disadvantage not shown by those which correspond to the other terms. They appear to depend on the terms in Γ^{m_1-1} occurring in V, which terms are not determinate.

It is easy to verify that this dependence is merely an apparent one. Let us imagine that, from each point of S', a very small segment is marked off along the transversal ν, being such that the corresponding $d\nu$ is equal to a (very small) constant. Let us denote by S_ν' the locus of the points thus obtained, and give to u, at each one of them, the same value which it has at the corresponding point of S'.

Term (20) will under these conditions be the derivative, with respect to ν, of the quantity

$$(23) \qquad \left| \overline{\mathbf{SS} \int_{S_\nu'} uv' \, dS'} \right.$$

and will consequently be equal to

$$(24) \qquad \frac{2\pi}{\mathbf{k}} \frac{d}{d\nu} \mathbf{SS}_{S_\nu} \mathcal{V} u dS - \frac{2\pi}{\mathbf{k}} \frac{d}{d\nu} \frac{1}{(m_1-2)!} \frac{d^{m_1-2}}{d\gamma^{m_1-2}} \mathbf{S}_{\sigma_{\gamma\nu}} u V d\sigma_\gamma.$$

In the second term, S_ν is the base of S_ν' in the space E_m; $\sigma_{\gamma\nu}$, the intersection of S_ν by $\Gamma = \gamma$, and the differentiations $\frac{d}{d\gamma}$ and $\frac{d}{d\nu}$ can be inverted (because γ and ν are independent variables, entering into the expression of the $\mathbf{S}$ by means of $\sigma_{\gamma\nu}$). We must observe that in the original expression (23), the differentiation only concerns v'; the values of u and dS' are to be considered as independent of ν: this means that, in order to calculate them for any element at a point X of S_ν', we have to consider the corresponding element around the corresponding point $X^{(0)}$ of S', which is the "transversal projection" of X, and by means of which we have to calculate u (as being the value of the latter quantity in $X^{(0)}$) and dS', which is not the value of the element on S_ν' but the value of its transversal projection on S'.

Similar observations apply therefore to (24), so that u and $d\sigma_\gamma$ are still to be taken in transversal projection on S; but, as one shall observe, the conclusion is not that the differentiation shall only concern V: for the transversal projection, on S, of $\sigma_{\gamma\nu}$ is variable with ν. Let us suppose that between points x of σ_γ and (infinitely near) points X of $\sigma_{\gamma\nu}$ we have established a punctual correspondence (which can be done in ∞ ways), whereby another infinitesimal punctual transformation is defined, on S, between x and the transversal projection $X^{(0)}$ of X. We can imagine that the value of V at X and the value $u^{(0)}$ of u at $x^{(0)}$ are expressed in terms of the coordinates of x. As to the relation between the two elements $d\sigma_\gamma$ on σ_γ and on $\sigma_{\gamma\nu}$, we shall find it by remembering that $d\sigma_\gamma$ is the quotient of

dS by $d\gamma$, which latter quantity has the same value for x and for X. Remembering also that dS relating to X is to be taken in transversal projection on S, we see that the integral in the second term of (24) can then be written

$$(25) \qquad \mathbf{S}_{\sigma_\gamma} V u^{(0)} \frac{dS^{(0)}}{dS} d\sigma_\gamma,$$

where $\frac{dS^{(0)}}{dS}$ is the ratio of corresponding elements in the punctual transformation between x and $X^{(0)}$ and we take account also of the above agreements as to V and $u^{(0)}$.

Differentiation with respect to ν can now be carried out under S, and gives, for the derivative of (25) with respect to γ ($m_1 - 2$ times) and ν, the value

$$(26) \qquad \frac{d^{m_1-2}}{d\gamma^{m_1-2}} \frac{d}{d\nu} \mathbf{S}_{\sigma_{\gamma\nu}} u V d\sigma_\gamma$$
$$= \frac{d^{m_1-2}}{d\gamma^{m_1-2}} \mathbf{S}_\sigma \left(u_0 \frac{dV}{d\nu} + V \frac{du_0^{(0)}}{d\nu} + u_0 V \frac{d}{d\nu} \frac{dS^{(0)}}{dS} \right) d\sigma_\gamma.$$

The treatment of the first term of (24) wants no further observation and depends on the ordinary rules of Calculus. We have to differentiate first under $\mathbf{SS}$,—i.e., replace $\mathcal{V}$ by $\frac{d\mathcal{V}}{d\nu}$;—then we shall have a boundary term, as the domain S_ν,—or more exactly its transversal projection on to S,—depends on ν. This term is always a negative one if S is duly inclined, the reason for this being that (compare § 108, footnote p. 172) the transversal to S at any point of the edge of intersection with Γ is directed towards the outside of Γ and that, therefore, the transversal projection of S_ν lies everywhere inside S (fig. 26). The part thus subtracted from S consists of elements each of which is a small ($(m-1)$-dimensional) cylinder having an element of σ for its base and the small segment $xX^{(0)}$ defined above (see fig. 26) for its generatrix : the volume of such an element is, in our notation, $d\sigma_\gamma \, d\gamma$, the second factor representing the increment of Γ when passing from x to $X^{(0)}$ or, which is the same, from X to $X^{(0)}$ and being, therefore, numeri-

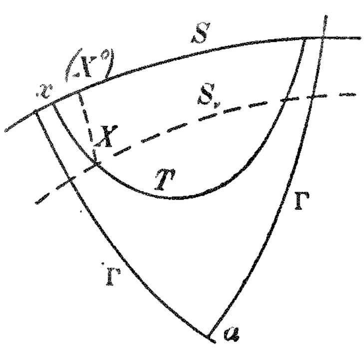

Fig. 26.

cally equal to $\left|\frac{d\Gamma}{d\nu}\right| d\nu$. Dividing by $d\nu$, we have the derivative

$$(27) \qquad \frac{d}{d\nu}\mathbf{SS}_{S_\nu}\mathcal{V}u_0 dS = \mathbf{SS}_S u_0 \frac{d\mathcal{V}}{d\nu}\, dS - \mathbf{S}_\sigma u_0 \mathcal{V}\left|\frac{d\Gamma}{d\nu}\right| d\sigma_\gamma .$$

It is well understood that this formula only concerns the case where S is duly inclined and therefore $\frac{d\Gamma}{d\nu} < 0$. The sign before the last term ought, of course, to be changed if we should replace $\left|\frac{d\Gamma}{d\nu}\right|$ by $\frac{d\Gamma}{d\nu}$, and the new formula thus obtained would be valid for any inclination of S (as $\frac{d\Gamma}{d\nu} > 0$, resulting from a non duly inclined S, would be compensated for by the fact that the corresponding parts of the transversal projection of S_ν would overlap S instead of lying inside it).

This second expression of (20), by combination of (26) and (27), is equivalent to the first one (as appears at once for the terms in $\mathcal{V}$ by comparison with (22) and (22′)); but it appears, this time, that it does not contain the terms in Γ^{m_1-1} of the expression of V.

145. The evaluation of the term (18) completes the solution of the problem. Taking account of the value (5) of $\mathbf{k}$, we obtain the following statement:

For $m = 2m_1$, let:

V, $\mathcal{V}$ be the two regular functions which appear in the fundamental solution (2) *of the adjoint equation $\mathcal{G}(v) = 0$;*

Γ, T, S_0, the domains analogous to the ones which have been defined for m odd; τ, the part of the surface $\Gamma = \gamma$ (γ being a very small positive constant) enclosed inside T; σ_γ, the intersection of this same surface with S_0, the elements $d\tau_\gamma$ and $d\sigma_\gamma$ of the varieties τ and σ being defined by the equalities (8), (8 *a*). pag 221, 228

The solution of Cauchy's problem will be given by the formula

$$(28) \quad (-1)^{m_1}(m_1-1)\, C_{m_1-1}\, \Omega_{m-2}\, u_a$$

$$= -\mathbf{SSS}_T \mathcal{V} f\, dx_1 \ldots dx_m - \mathbf{SS}_{S_0}\left[\mathcal{V}(u_1 + Lu_0) - u_0 \frac{d\mathcal{V}}{d\nu}\right] dS$$

$$+ \mathbf{S}_\sigma u_0 \mathcal{V} \frac{d\Gamma}{d\nu}\, d\sigma_\gamma$$

$$+ \frac{1}{(m_1-2)!}\frac{d^{m_1-2}}{d\gamma^{m_1-2}}_{(\gamma=0)}\left[\mathbf{SS}_\tau f V d\tau_\gamma + \mathbf{S}_\sigma V(u_1 + Lu_0)\, d\sigma_\gamma\right]$$

$$- \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}_{(\gamma=0)}\mathbf{S}_\sigma\left[\frac{\Gamma dV}{d\nu} - (m_1-1)\frac{V d\Gamma}{d\nu}\right] u_0\, d\sigma_\gamma$$

Or (in the sense explained in the preceding section)
(28 *a*)

$$= -\mathsf{SSS}_T \mathcal{V} f\, dx_1 dx_2 \dots dx_m - \mathsf{SS}_{S_0} \mathcal{V}(u_1 + Lu_0)\, dS$$
$$+ \frac{1}{(m_1-2)!} \frac{d^{m_1-2}}{d\gamma^{m_1-2}}_{(\gamma=0)} [\mathsf{SS}_{\tau_\gamma} f V d\tau_\gamma + \mathsf{S}_{\sigma_\gamma} V(u_1 + Lu_0)\, d\sigma_\gamma]$$
$$- \frac{1}{(m_1-2)!} \frac{d^{m_1-2}}{d\gamma^{m_1-2}} \frac{d}{d\nu} \mathsf{S}_{\sigma_{\gamma\nu}} V u_0 d\sigma_\gamma + \frac{d}{d\nu} \mathsf{SS}_{S_\nu} u_0 \mathcal{V} dS,$$

where the last term can be found by (26), the last but one by (27).

The coefficient of u_a in the left-hand members of both equations can be written (see § 99)

$$(29) \qquad (-1)^{m_1} \frac{m_1 - 1}{\pi} \Omega_{m-1} = \frac{2(-1)^{m_1}}{(m_1-2)!} \pi^{m_1-1}.$$

It is well understood that, in the present case, every differentiation under **SS** or **SSS** is to be done by the classic rules of Calculus; and no difficulty such as has been met with for the case of m odd arises.

146. It is clear also that several properties of the solution as written in the preceding Book immediately apply to that which we have just found, as the latter is not essentially distinct from the former.

This is the case, in the first place, for the remark at the beginning of § 104: if the shape of S is such as to enclose, together with Γ, a volume T to which a is exterior (compare fig. 12 *a*, 12 *b*) we have formulæ quite identical with (28), (28 *a*), except that the left-hand side is replaced by *zero*, this being an immediate consequence of formula (F′), § 104.

We can also already see that the same must apply to the "interchange property" (§ 114), a subject which will be examined again in the next Chapter.

The same applies to the remark which we made (§ 113) on the case where S is constituted by characteristics, in which the knowledge of u_0 suffices to write down the solution, as the knowledge of $u_1 = \frac{du}{d\nu}$ is implied in it.

Also we have no further care to take in order to verify our solution (for duly inclined boundaries): the proof that it actually satisfies every requirement of the problem must be considered as already afforded in the preceding Book, Chap. III*.

147. Another form of the formula. The terms in $d\tau_\gamma$ and $d\sigma_\gamma$ in (28), (28 a) can be written in another way by remarking that $d\tau_\gamma d\gamma$, for instance, is an element (a small cylinder, see Book II, § 38) of the volume enclosed between one surface τ and a consecutive one in which γ is changed into $\gamma + d\gamma$. It results therefrom that, by such a changing of γ, the integral $\mathbf{SSS} fV\, dx_1 \ldots dx_m$, relative to the domain which we have called T_2, will be increased by $d\gamma\, \mathbf{SS} fV\, d\tau_\gamma$, so that $\mathbf{SS} fV\, d\tau_\gamma$ is the derivative of that volume integral† with respect to γ.

Similarly, $\mathbf{S}\left(V\frac{du}{d\nu} + LuV\right) d\sigma_\gamma$ will be the derivative, with respect to γ, of $\mathbf{SS}\left(V\frac{du}{d\nu} + LuV\right) dS$, extended over the portion S_2 of S enclosed between $\Gamma = 0$ and $\Gamma = \gamma$; and the same transformation applies to the other term in $d\sigma_\gamma$. Writing the abbreviated notations $\mathbf{SSS}_2$ and $\mathbf{SS}_2$ for $\mathbf{SSS}_{T_2}$ and $\mathbf{SS}_{S_2}$, we see tnat *formula* (28 a) *is equivalent to*

$$(30)\quad 2(-1)^{m_1} \frac{1}{(m_1 - 2)!} \pi^{m_1 - 1} u_a$$
$$= - \mathbf{SSS}_T \mathcal{V} f\, dx_1 \ldots dx_m - \mathbf{SS}_{S_0} \mathcal{V}(u_1 + Lu_0)\, dS$$
$$+ \frac{1}{(m_1 - 2)!} \frac{d^{m_1 - 1}}{d\gamma^{m_1 - 1}} [\mathbf{SSS}_2 fV\, dx_1 \ldots dx_m + \mathbf{SS}_2 V(u_1 + Lu_0)\, dS]$$
$$- \frac{1}{(m_1 - 2)!} \frac{d}{d\nu} \frac{d^{m_1 - 1}}{d\gamma^{m_1 - 1}} \mathbf{SS}_2 u_0 V dS + \frac{d}{d\nu} \mathbf{SS}_{S_\nu} u_0 \mathcal{V} dS,$$

a form of the result which we shall have to use; and (28) can be transformed in a quite similar way‡.

* In the case of a characteristic S, such a verification would, in the same way, be a consequence of the corresponding one relating to m odd. The proof given in § 119 for $m=3$ would apply to $m=2$.

† The increment of the volume T_2 (volume between two consecutive positions of τ) will contain irregular parts in the vicinity of the edge σ; but they are of the second order in $d\gamma$.

‡ It will be eventually useful to remember that, the operations in this section not being essentially distinct from the preceding ones, the remarks made in § 141,

148. The expression for the required unknown differs considerably, as we see, from the one which answered the case of m odd. In the latter, the elementary solution was directly introduced. Here, the elementary solution still serves as a basis, but only in so far as it provides the two functions V and $\mathfrak{V}$, which alone appear in the operations to be performed.

On the other hand, the value of the unknown, for m even, appears in the form of a sum of two integrals, the one extended over the inside of the characteristic conoid, and the integrand of which contains as a factor the data themselves, multiplied by known functions; the other extended over the characteristic conoid itself, and containing under the same conditions the data and their derivatives up to the order $m_1 - 2$ (or even $m_1 - 1$). The integrals thus written only involve finite quantities, if the data are regular.

In the case of m odd, we had a single integral, involving the data themselves (without explicit differentiation), but possessing the paradoxical character examined formerly, and on account of it, containing, virtually, a boundary integral and certain derivatives of the functions introduced.

Such an expression should, therefore, be considered as intermediate between two expressions of the above class (the ordinary integrals met with in the case of m even) corresponding to two consecutive values of m_1.

It has also, with respect to u_1 or f for instance, an order of continuity which must be considered as intermediate* between such consecutive m_1's.

149. Application to Huygens' principle. The above formulæ enable us to answer the following fundamental question:

as to the convergence, after differentiation with respect to γ, of the term $\mathbf{SSS} fV dx_1 \ldots dx_m$ still hold under the present form. In both cases, besides, the convergence is uniform as long as derivatives of fV up to the order $(m_1 - 2)$ are numerically limited.

* This is fully verified by application of the methods peculiar to Functional Calculus, as developed in our Calculus of Variations (§§ 243–248) and beautifully improved by Frechet and Riesz. (See our Memoir in *Acta Mathematica*, t. XXXI. p. 379.)

For which equations is Huygens' principle true in its special sense, i.e., in its sense (B)?

We already know that such equations must not be looked for amongst those which we have investigated in Book III.

But, for the case of m even, we now see that the residual integral, as given by the operations in the domain which we have called T_1' and the analogous domain on S', exclusively depends on the quantity $\mathcal{V}$.

The necessary and sufficient condition for the vanishing of that residual integral is that this function $\mathcal{V}$ be identically zero, i.e. *that the elementary solution contain no logarithmic term.*

If such is not the case, the residual integral will be different from zero for arbitrary data. The question of its sign, as considered in § 112, is liable to receive any answer according to the values of the coefficients of the equation: for, at least for $m = 4$, the remark of § 65 evidently shows that we can get any sign for $\mathcal{V}$ by a proper choice of the coefficient C if the other ones are assumed to be chosen beforehand.

We have said that we give *an* answer and not *the* answer, to our question: for it is clear that we can wish it to be "plus resolu" than it has been in the above. We have enunciated the necessary and sufficient condition, but we do not know how equations satisfying it can be found, or even whether any exist except (e_{2m_1-1}) (and, of course, those which are deduced from (e_{2m_1-1}) by trivial transformations). This, and many other questions concerning the residual integral, would require further researches.

As to the more general form (C) of the principle, it may be considered as being proved by our integrating formulæ in the same sense as Kirchhoff's or Volterra's results prove it for (e_3) or (e_2).

2. Familiar examples

150. It will be useful to show some applications of the above general formulæ, and even to verify their agreement with previously known results.

(*a*) *The case of Riemann.* Let us begin with the case $m = 2$, already treated directly in Book II; which, as we have said, behaves somewhat differently from the other ones.

The characteristic conoid degenerates into a system of two characteristic lines: these will be parallels to the axes, if we take the equation in Laplace's form. We shall write the latter so as to have $|\Delta| = 1$, therefore, by multiplying it by ± 2:

$$\pm 2\left(\frac{\partial^2 u}{\partial x \partial y} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu\right) = \pm 2f.$$

We have to choose the sign according to the location of the useful angle between the characteristics (i.e. of the angle which intercepts an arc of S), the form $A(\pi_1, \pi_2)$ being bound to be positive for lines which cut both sides of that useful angle: this sign will, therefore, depend (as seen in § 42) on the sense of variation of y considered as a function of x, it happening, in this case, that we do not define by the nature of the equation what we must call a "duly inclined" line, but that, on the contrary, the equation is to be written so as to make the given line S a duly inclined one.

We have already noticed that every term in the formula (leaving aside, in the first place, the one which contains $\frac{dv'}{d\nu}$) can be transformed without distinguishing between T_1 and T_2, on account of the absence of the symbol $|\overline{\ \ }$ and of V. In any of these terms, therefore, we only have to write $\mathcal{V}$ instead of v', cancel one **S** and divide* by -2.

But the same treatment can also be applied to the term in $\frac{dv'}{d\nu}$ by using the method of § 144. $a\alpha$, $a\beta$ being, as in Book II, § 42, the two characteristics issuing from a (Γ is precisely constituted by those two lines); S the line which bears the data and which cuts the two characteristics at α and β, we have to draw a neighbouring curve S_ν such that each point of S_ν is deduced from a point of S by taking on the transversal ν to S a small segment such that $d\nu = \text{const.}$, which S_ν will cut the characteristics at α', β' (fig. 27), S_ν' being the right cylinder on the base S_ν; we have to take the derivative, with respect to ν, of the integral **SS** $\big|\, uv'dS'$ (see § 144), it being understood that the dS' of any element of S_ν' and the value of u on it relate, by definition, to the transversal projection on to S'.

* The factor 2π, given in § 137, is, of course, cancelled by the coefficient on the the left-hand side.

But $\mathbf{SS}\int uv'dS'$ is equal to* $\pi\mathbf{SS}_{S_\nu} u\mathfrak{V} ds$, a simple integral which we can now write with an ordinary $\int$ and the derivative of which will consist of the product of π by:

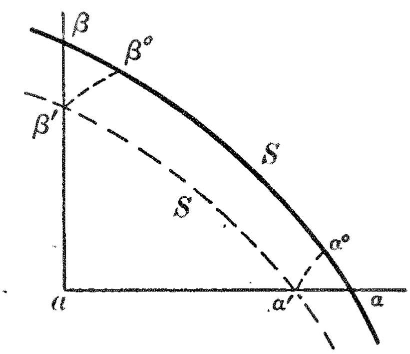

Fig 27.

(1) the term $\int u \frac{d\mathfrak{V}}{d\nu} dS$ representing the integral of the infinitesimal variation of $uv'dS$ between corresponding points of S and S_ν;

(2) two terms relating to the arcs $\alpha\alpha^{(0)}$, $\beta\beta^{(0)}$ (fig. 27) of one curve which have no corresponding points on the other, so that the limits of integration with respect to s are functions of ν. The derivatives of these two functions are equal to $+1$, as, on account of the fact that the transversal direction is symmetrical to the tangent with respect to parallels to the axes, the two triangles $\alpha\alpha'\alpha^{(0)}$, $\beta\beta'\beta^{(0)}$ are isosceles. The two terms in question will thus give $\frac{1}{2}(u\mathfrak{V})_\alpha$ or $\frac{1}{2}(u\mathfrak{V})_\beta$, evidently corresponding to the second term in (27).

Those two terms are, in that case, the only ones which we have to add to the right-hand side of formulæ (7) after having changed in them v into $\mathfrak{V}$, $\mathbf{SS}$ and $\mathbf{SSS}$ into $\int$ and $\iint$ (and cancelled, of course, the $|\quad$). This gives precisely the results which have been deduced by Riemann's method. The fact that Riemann's function $\mathfrak{V}$ is the coefficient of the logarithm in the elementary solution, as has been already found in § 46, appears, this time, as a subcase of our general considerations.

151. (*b*) **Poisson's and Kirchhoff's formulæ.** The next case is $m = 4$ (giving $m_1 = 2$, $\mathbf{k} = 1$), and the simplest equation of that type the equation (e_3) of spherical waves.

We have then, denoting by (x, y, z, t) and (x_0, y_0, z_0, t_0) our two points, and taking $\omega = 1$,

$$\Gamma = (t - t_0)^2 - (x - x_0)^2 - (y - y_0)^2 - (z - z_0)^2 = (t - t_0)^2 - r^2.$$

* We change the letter S into s, in agreement with the ordinary notation and our § 40, to denote the arc of S.

There is no logarithmic term in the elementary solution (see § 69) which is reduced to $\frac{1}{\Gamma}$, so that $\mathcal{V}=0$ and $V=1$.

For this equation (and, in general, for $m=4$, m_1-2 being zero), differentiation with respect to γ and the consideration of the auxiliary surface $\Gamma=\gamma$ will not occur except in the terms considered in § 143 (last line of formula (28 a)), and can even be avoided in these if we treat them as we do in § 144.

Let us suppose first that S is the hyperplane $t=0$, which is the case in Poisson's analysis. $\frac{d}{d\nu}$ will be $\epsilon\frac{\partial}{\partial t}$ (with $\epsilon=\pm 1$ according to the same rule as in § 129), having the value zero for V and $-2\,|\,t_0\,|$ for Γ. The element of S (i.e., element of volume in ordinary space) being $r^2\sin\theta\, dr\, d\theta\, d\phi$, the element $d\sigma_\gamma$, quotient of dS by

$$|\,d\Gamma\,| = 2r\,|\,dr\,|,$$

will be $\frac{1}{2} r \sin\theta\, d\theta\, d\phi$. Thus the term in u_1 will be (as $r=|\,t_0\,|$ for $t=\Gamma=0$ and $u_1=\epsilon\frac{\partial u}{\partial t}$),

$$\mathsf{SS}\, V u_1 d\sigma_\gamma = |\,t_0\,| \iint u_1 \sin\theta\, d\theta\, d\phi = t_0 \iint \frac{\partial u}{\partial t} \sin\theta\, d\theta\, d\phi.$$

The term in u_0 (in which $L=0$) will be (as, for $\Gamma=0$, we have $r=t_0-t$, the derivative of which, with respect to ν, is -1)

$$\frac{d}{d\nu}\left(\mathsf{SS}\, V u_0 d\sigma_\gamma\right) = -\frac{\partial}{\partial t_0}\left(t_0 \iint u_0 \sin\theta\, d\theta\, d\phi\right).$$

The sum of both, divided by 2π (which is the value of the coefficient (29)), coincides of course with the right-hand side of *Poisson's formula.*

If the partial differential equation is non-homogeneous, so that $f\neq 0$ and the new term $\mathsf{SS}\, fV d\tau_\gamma$ appears, the value of

$$d\tau_\gamma = \frac{dx\,dy\,dz\,dt}{d\gamma}$$

is immediately deduced from $d\sigma_\gamma$ by multiplying by dt, so that the supplementary term is found to be (after division by 2π)

$$\frac{1}{4\pi}\int |\,t_0 - t\,|\, dt \iint \bar{f} \sin\theta\, d\theta\, d\phi,$$

where $\bar{f}$ stands for

$$f\left[x_0+(t_0-t)\sin\theta\cos\phi,\ \ y_0+(t_0-t)\sin\theta\sin\phi,\ \ z_0+(t_0-t)\cos\theta,\ t\right].$$

152. Let us now consider Kirchhoff's hypersurface S, viz. a hypercylinder the base of which will be a closed (ordinary) surface ω (the equation being again supposed to be homogeneous). The domain of integration may be, according to the nature of the question, the inside or the outside of the cylinder. Let us also take $t_0 > 0$.

Taking the π's respectively equal to the direction cosines of the normal n to ω (directed inwards with respect to our domain, which may be inwards or outwards with respect to ω) and (for the fourth one) to zero, we have to take dS equal to the element of our hypercylinder $d\omega\, dt$; and the transversal direction ν will be opposite to n, the terms in $(x - x_0)^2, \ldots$ being negative.

Integration relating to the edge of intersection σ_0 of S with Γ, $d\sigma_\gamma$ must be such that

$$d\omega_\gamma = d\omega : \left(\frac{\partial \Gamma}{\partial t}\right) = \frac{d\omega}{2(t_0 - t)} = \frac{d\omega}{2r},$$

both sides representing the volume dS of any small cylinder (which we can suppose to be parallel to the t-axis) included on S between an element of surface of the characteristic conoid and a corresponding element of the neighbouring surface $\Gamma = \gamma$.

This is all we want for the calculation of § 142, as no differentiation with respect to γ is wanted. For the operations of § 144, we define the correspondence between x and $x^{(0)}$ so that the segment joining these points is parallel to the t-axis: in other terms, from any point x of σ_0 (which represents a point of ω associated with the value $t_0 - r$ for t), we conduct a small segment $\delta n = -\,\delta\nu$ normal to S which changes r by $\frac{dr}{dn}\delta n$ and increases t so as to let Γ take back its original value, therefore by $\delta t = -\frac{r}{t_0 - t}\frac{dr}{dn}\delta n$. This is the increment which, given to t *on the original surface* of the hypercylinder S, leads from the point x to the point $x^{(0)}$. Therefore we have

$$u^{(0)} = u + \delta t \frac{\partial u}{\partial t} = u + \frac{r}{t_0 - t}\frac{dr}{dn}\frac{\partial u}{\partial t}\delta\nu,$$

$$\frac{dS^{(0)}}{dS} = 1 + \frac{r}{(t_0 - t)^2}\frac{dr}{dn}\delta\nu.$$

Substituting these values in (26) and observing that, on our conoid, $\frac{r}{t_0 - t} = 1$, while V is identically equal to 1, we get

$$u_a = \frac{1}{4\pi}\iint \left[\frac{\bar{u}_1}{r} - \frac{1}{r}\frac{dr}{dn}\frac{\partial \bar{u}_0}{\partial t} - \bar{u}_0 \frac{d\frac{1}{r}}{dn}\right] d\omega,$$

in which (as in § 131) $d\omega$ relates to the base of the cylinder while $\bar{u}_0$, $\bar{u}_1$ are values of u_0, u_1 at the corresponding points of σ_0, viz.

$$\bar{u}_i = u_i(x, y, z, t - r). \qquad (i = 0, 1)$$

This agrees with Kirchhoff's formula (if we remember that

$$u_1 = \frac{du}{d\nu} = -\frac{du}{dn}).$$

153. If t_0 is greater than the maximum of r along ω, this formula gives the whole value of u_a. In the contrary case, and if the boundary of our domain (which can not be constituted by our hypercylinder only) is supposed to be completed by the corresponding portion of the hyperplane $t = 0$, there is a term relating to the latter surface.

This term consists, of course, of spherical integrals as in Poisson's formula, but an observation is necessary concerning the way of computing the two **S**'s corresponding to (20) in this case, as in every other case when S consists of two parts joining each other at an angle different from zero.

In Poisson's formula itself, the term which contains u_1 is

$$t_0 M_{t_0}(u_1) = \frac{t_0}{4\pi}\iint u_1 d\Omega_2,$$

$d\Omega_2 = \sin\theta\, d\theta\, d\phi$ still being an element of solid angle, and the integration being extended to the whole sphere with centre (x_0, y_0, z_0) and radius t_0. It is clear that, this time, we have to write down the same integral, with the only difference that it is limited to the portion of sphere which lies within our domain. The same obviously holds for the first term $M_{t_0}(u_0)$ of the derivative $\frac{d}{dt_0}[t_0 M_{t_0}(u_0)]$. As to the remaining part of this derivative (with abstraction of the factor $\frac{1}{4\pi} t_0$), it can be written under either of the two forms

(31) $$\frac{\partial}{\partial t_0}\iint u_0 d\Omega_2,$$

or ($\frac{d}{dr}$ denoting the radial derivative, or derivative relating to the outer normal of the sphere)

$$\text{(32)} \qquad \iint \frac{du_0}{dr} d\Omega_2,$$

which are equivalent to each other.

But in our present problem the above two forms are *no longer* equivalent. If, to take a determinate case, we assume ω to be $x = 0$, our domain lying on the positive side, and if we denote by $\mathscr{M}_0(x \ \rho)$ the average value of the function u_0 along the circumference of the circle whose centre is (x, y_0, z_0) and radius ρ, its plane being parallel to $x = 0$, viz.

$$\mathscr{M}_0 = \mathscr{M}_0(x, \rho) = \frac{1}{2\pi}\int_0^{2\pi} u_0(x, y_0 + \rho\cos\phi, z_0 + \rho\sin\phi)\, d\phi,$$

M_{t_0} will be, in our present problem, replaced by

$$\frac{1}{2}\int \mathscr{M}_0(x_0 - t_0\cos\theta, t_0\sin\theta)\sin\theta d\theta = \frac{1}{2}\int_{-1}^{\lambda_0} \mathscr{M}_0(x_0 - \lambda t_0, t_0\sqrt{1-\lambda^2})\, d\lambda,$$

where λ stands for $\cos\theta$, the upper limit (instead of $+1$, as was the case in Poisson's original formula) being

$$\lambda_0 = \frac{x_0}{t_0}.$$

(32) will be obtained if we replace, under $\int$, $\mathscr{M}_0$ by

$$\text{(33)} \qquad \frac{d\mathscr{M}_0}{dt_0} = -\lambda\frac{\partial u}{\partial x} + \sqrt{1-\lambda}\,\frac{\partial u_0}{\partial r}.$$

This is no longer equal to the derivative (31): there is lacking the term

$$\text{(34)} \qquad -\frac{1}{2}\frac{x_0}{t_0^2}\mathscr{M}_0(0, \sqrt{t_0^2 - x_0^2})$$

corresponding to the fact that λ_0 is a function of t_0.

Which of the two expressions (31) or (32) is to be introduced in our formula? The answer will, of course, appear if we remember that the term in question is due to the quantity (20) in § 143.

Now, if we suppose that S (figured by a broken line in fig. 28) consists of two parts $\bar{S}$, $\bar{\bar{S}}$, meeting at a certain angle, it is clear that the corresponding integral (20) has to be taken on each of them independently, by the methods of § 143 or § 144; and in the latter, no displacement of the boundary of S, when ν varies, has to take place on account of the presence of S, so that, everywhere else than on Γ, the boundaries of S and S must correspond to each other by transversal projection, the total S_ν being constructed as diagrammatically figured by dotted lines in fig. 28. Thus, no boundary term has to correspond to the common edge of $\bar{S}$ and S (as would also appear from the mere application of § 143).

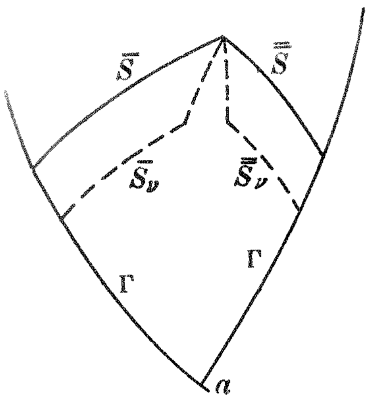

Fig. 28.

Therefore, in the particular problem above, the term (34) is *not* to be introduced. The right term in u_0 contains (32) and not (31) and the complete formula is

$$(35)\quad 4\pi u_a = \left[t_0 \iint\left(u_1 + \frac{\partial u_0}{\partial r}\right) \sin\theta\, d\theta\, d\phi + \iint u_0 \sin\theta\, d\theta\, d\phi\right]$$

$$+\left[\iint\left(\frac{\bar{u}_1}{r} - \frac{1}{r}\frac{dr}{dn}\frac{\partial \bar{u}_0}{\partial t} + \bar{u}_0 \frac{d\frac{1}{r}}{dn}\right) d\omega\right]$$

where the integrals within the first brackets are taken for $t = 0$ over a portion of sphere included in the given domain, while the integral within the second brackets relates to σ_0.

154. (*c*) *The equation of damped spherical waves.* The equation

$$(\mathrm{E}_3)\qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - Ku = 0$$

has been treated by Birkeland, Carvallo, Weber, Brillouin*, and later,

* Birkeland, *C. R. Ac. Sc.*, Vol. CXX (1895), p. 1046; Carvallo, *ibid.*, 14 Janvier, 1895; Weber, in Riemann's *Partielle Differentialgleichungen der Math. Physik*, Vol. II, edition of 1901, p. 310; Brillouin, *C. R. Ac. Sc.*, Vol. CXXXVI, 1903, p. 667.

in several papers*, by Tedone. Comparing this equation with the preceding one (e_3), we see that Γ will be the same, and we can also again take $V=1$, as remarked in Book II (§ 69), so that we have to inscribe the same terms as in the foregoing operations (§§ 151—153), but with the addition of the term in $\mathcal{V}$ of formula (28), where (the elementary solution being given by formula (61 a), § 69) we have

$$(36) \qquad \mathcal{V} = -\frac{K}{4} j'\left[\frac{(t-t_0)^2 - r^2}{4}\right],$$

$j(\lambda)$ still being $\quad 1 + \frac{\lambda}{(1\,!)^2} + \frac{\lambda^2}{(2\,!)^2} + \ldots + \frac{\lambda^h}{(h\,!)^2} + \ldots.$

If S is the hyperplane $t=0$, the formula will be (noticing that $\frac{d}{d\nu} = \frac{\partial}{\partial t}$ is, in this case, equal to $-\frac{\partial}{\partial t_0}$)

$$(37) \quad u_a = \frac{\partial}{\partial t_0}[t_0 M_{t_0}(u_0)] + t_0 M_{t_0}(u_1) + \frac{K}{2}\int_0^{t_0} r^2 M_r(u_1) j'\,dr$$
$$+ \frac{K}{2}\frac{\partial}{\partial t_0}\int_0^{t_0} r^2 M_r(u_0) j'\,dr,$$

where the symbol M has the same meaning as above, and the argument in j' is $K\frac{t_0^2 - r^2}{4}$. The identity of this result with the conclusions of the papers cited (for example, Weber's or Brillouin's) is verified without any difficulty.

When S consists of a portion of $t=0$ (i.e., of ordinary space) limited by a surface ω and of a hypercylinder on ω (see Brillouin, *loc. cit.*) the terms (35) have to be completed by the following ones (the argument in j' being now $\frac{(t_0 - t)^2 - r^2}{4}$)

$$\frac{K}{2}\iiint u_1 j'\,dx\,dy\,dz + \frac{K}{2}\frac{\partial}{\partial t_0}\iiint u_0 j'\,dx\,dy\,dz$$
$$+ \frac{K}{2}\iint u_0 \frac{dr}{d\nu}\,d\omega - \frac{K}{2}\iint d\omega\left[\frac{dr}{d\nu}\int_0^{t_0 - r} u_0 \frac{\partial j'}{\partial r}\,dt - \int_0^{t_0 - r} u_1 j'\,dt\right].$$

* *Rendic. Acc. Lincei*, 5th series, Vol. XXII (1913, 1st semester), p. 757; Vol. XXIII (1914, 1st semester), pp. 63, 120, 473.

The first two terms only differ from the corresponding integral terms in (37) (the factor 4π excepted) by the fact that

$$M_r(u_i) = \frac{1}{4\pi}\iint u_i(x_0 + r\sin\theta\cos\phi,\ y_0 + r\sin\theta\sin\phi,\ z_0 + r\cos\theta)\sin\theta\, d\theta\, d\phi$$
$$= \frac{1}{4\pi}\iint u_i d\Omega_2 \qquad (i = 0, 1)$$

is replaced by an integral extended only over a part of the corresponding sphere as in the foregoing section. The second one, for instance, can be written

$$\frac{K}{2}t_0^2\iint u_0 d\Omega_2 + \frac{K}{2}\iint_0^{t_0} r^2\frac{\partial j'}{\partial t_0}dr\iint u_0 d\Omega_2,$$

the double integrals relating to such portions of spheres.

For an arbitrary shape of S, which Tedone has also considered*, the formula is

$$2\pi u_a = \frac{K}{4}\left(\iiint_{S_0} u_1 j_0' dS - \frac{d}{d\nu}\iiint_{S_\nu} u_0 j_0' dS\right)$$
$$+ \iint_\sigma u_1 d\sigma_\gamma - \frac{d}{d\nu}\iint_{\sigma_\nu} u_0 d\sigma_\gamma;$$

the calculation of the last term must be understood as stated in § 153, in the case where S has angular edges.

155. (*d*) *Higher number of variables.* The analogous equations in 6, 8, ... variables would be treated similarly by application of formula (28) or (28 *a*).

Let us take simply the equation of (ordinary) "hyperspherical waves"

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \dots - \frac{\partial^2 u}{\partial x_{m-1}^2} = 0$$

* Tedone's results are of a quite different appearance, at first sight (a proper transformation being even necessary to find Weber's formula (37)), and are obtained by a very different method, implying the resolution of certain integral equations. This, exactly as happens in Coulon's case (see Book III, § 79), is due to the impossibility, which we now see to be even more radical than for m odd, of introducing directly the elementary solution, so that more or less indirect substitutes for it must be used. Such a necessity, on the other hand, is not devoid of advantage, as giving place to suggestive integral identities concerning Bessel's functions: a subject connected with certain consequences of what we have called proposition (A) in Book II, § 33.

and suppose that S is $t=0$, with $t_0>0$. As V is still equal to 1 and $\mathfrak{V}$ to 0, we find the same simplification as above. $d\sigma_\gamma$ is given by

$$d\sigma_\gamma = \frac{dS}{d\gamma} = \frac{r^{2m_1-2}d\Omega_{m-2}dr}{2r\,dr} = \frac{1}{2}r^{2m_1-3}d\Omega_{m-2},$$

$d\Omega_{m-2}$ denoting an element of hyperspherical surface of radius 1. As $d\gamma = -2r\,dr$, we see that the factor $(-1)^{m_1}$ is thus destroyed and*

$$(38)\quad 4\pi^{m_1-1}u_a = \left(\frac{d}{2t_0dt_0}\right)^{m_1-2}\left(t_0^{2m_1-3}\iint_{r=t_0} u_1 d\Omega_{m-2}\right) + \frac{d}{dt_0}\left(\frac{d}{2t_0dt_0}\right)^{m_1-2}\left(t_0^{2m_1-3}\iint_{r=t_0} u_0 d\Omega_{m-2}\right).$$

If $M_1(r)$ denotes the average value of u_1 on the hyperspherical surface of radius r, we can write

$$(38')\quad \frac{4\pi^{m_1-1}}{\Omega_{m-2}}u_a = \left(\frac{d}{2t_0dt_0}\right)^{m_1-2}[t_0^{2m_1-3}M_1(t_0)] + \frac{d}{dt_0}\left(\frac{d}{2t_0dt_0}\right)^{m_1-2}[t_0^{2m_1-3}M_0(t_0)].$$

The solution obtained by Tedone in 1898 (*Annali di Mat., loc. cit.*) is apparently different, the term in u_1, for instance, being

$$(39)\qquad \sum_{h=0}^{m_1-2} A_h \frac{d^{m_1-2-h}}{dr^{m_1-2-h}}[r^{m_1-1-h}M_1(r)]_{(r=t_0)},$$

where the A's are numerical constants. That

$$\left(\frac{d}{2r\,dr}\right)^{m_1-2}[r^{2m_1-3}M_1(r)]$$

is of the above form, is obvious by mere inspection: the A's may be considered as defined by the relation, identical in λ,

$$(40)\quad \Pi(\lambda) = \frac{1}{2^{m_1-2}}(\lambda+3)(\lambda+5)\dots(\lambda+2m_1-3)$$
$$= A_0(\lambda+2)(\lambda+3)\dots(\lambda+m_1-1)+\dots$$
$$+A_h(\lambda+2)(\lambda+3)\dots(\lambda+m_1-h-1)+\dots$$
$$+A_{m_1-3}(\lambda+2)+A_{m_1-2},$$

as is seen by setting down $M_1(r)=r^\lambda$ and observing, on the other hand, that such an identity in λ can be written in one way and in one way only. An easy method

* For the use of (28a), it is necessary, this time, that the differentiation

$$\frac{d}{d\nu}\left(-\frac{d}{dt_0}\right)$$

be carried out last, as it has to apply to the denominator in $\frac{d}{2t_0dt_0}$.

for the determination of the constants A_h consists in representing the left-hand side in (40) by the circuit integral

$$(41)\qquad \Pi(\lambda)=\frac{(m_1-2)!}{2i\pi}\int_C \frac{(1-z)^{-\frac{\lambda+3}{2}}}{z^{m_1-1}}\,dz$$

(integration along a closed line around the origin, as usual). If we set down

$$1-z=(1-Z)^2,\quad \text{whence } z=2Z-Z^2,$$

(41) becomes (the new path of integration C' surrounding the origin in the same way as C)

$$\Pi(\lambda)=\frac{(m_1-2)!}{2i\pi}\int_C \frac{(1-Z)^{-(\lambda+2)}dZ}{2^{m_1-2}Z^{m_1-1}\left(1-\frac{Z}{2}\right)^{m_1-1}},$$

or, finally, expanding $\left(1-\frac{Z}{2}\right)^{-(m_1-1)}$ as well as $(1-Z)^{-(\lambda+2)}$ in powers of Z and only retaining the terms in $\frac{1}{Z}$ in the integrand

$$=\Sigma\,\frac{(m_1-2+h)!}{2^{m_1-2+h}.\,h!\,(m_1-2-h)!}(\lambda+2)\ldots(\lambda+m_1-h-1),$$

so that, in (40),
$$A_h=\frac{(m_1-2+h)!}{2^{m_1-2+h}h.\,(m_1-2-h)!},$$

which is to be substituted in (39). Of course, the terms in u_0 are deduced from the above by writing $\left(\frac{d}{dr}\right)^{m_1-1-h}$ instead of $\left(\frac{d}{dr}\right)^{m_1-2-h}$; and this (taking account of the coefficient in the left-hand side of (38)) gives Tedone's form of the result (*loc. cit.*, formula (24)).

3. Application to a discussion of Cauchy's problem

156. Let us come back to formula (35), § 153. It allows us to find a solution u of (e_3), it being assumed that we know Cauchy's data both on $t=0$ and $x=0$.

But we know that such a problem is not correctly set. The one by which we have to replace it, in order to satisfy this condition is, as we have already said in Book I, a *mixed problem.*

The theory of such mixed problems is now by itself an extensive subject whith, as a whole, will not be developed here. It is classically treated for the most usual case of cylindrical domains such as spoken of in Book I, § 25, by the method of " fundamental functions " and was, indeed, the very origin of that method, as the notion of " fundamental functions " arose from Schwarz's, Picard's and Poincare's papers on the vibrations of a membrane*. Several works of some of our best known geometers† were devoted to the extension and improvement of the original ideas of the aforesaid authors, until they culminated in Fredholm's theory, and also the application of this theory to our problem‡. On the other hand, a quite different series of methods|| more nearly connected with our above considerations have been applied to the same problem in more recent times by Volterra, Goursat, Picard, etc.¶ and by ourselves**.

We shall content ourselves by treating the simple case of the plane boundary††, alluded to above, which is done very easily by a method belonging to the second class which we have just mentioned, and which is deduced from the well-known method of images in the potential theory. The device applies as well to (e_2) as to (e_3): let us develop it for the latter equation (whose theory, as we know, implies that of (e_2)). The problem is to find u defined by

$$(e_3) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

and the definite conditions

$$(C_3) \qquad u = u_0(x, y, z); \quad \frac{\partial u}{\partial t} = u_1(x, y, z) \quad (\text{for } t = 0,\ x \geqslant 0);$$

$$(\mathbf{C}_3) \qquad u = \mathbf{u}(y, z, t) \qquad (\text{for } x = 0,\ t \geqslant 0),$$

* Schwarz, *Acta Soc. Fennicae*, 1885; Picard, *C. R. Ac. Sc.*, 16 Oct. 1893; Poincaré, *American Journal*, Vol. XII, 1890; *C. R. Ac. Sc.*, Vol. CXVIII, 1894, p. 447; *Rendic. Circ. Mat. Palermo*, Vol. VIII, 1894.

† Le Roy, Stekloff, etc.

‡ See, for instance, Lauricella, *Ann. di Mat.*, series 3, Vol. XIV, 1907, pp. 143 ff.

|| On the comparison between them, see Volterra's *Lecture* at the International Congress of Mathematicians at Strasbourg, Sept. 1920.

¶ We indicate, amongst others, the works of Heaviside, Picone, those of Zaremba, some of whose results were found again later on by Rubinowicz; also recent papers of Tedone, Webster, etc.

** *Bull. Soc. Math. France*, Vol. XXXI, 1903; *Intern. Congress of Mathematicians*, Strasbourg, 1920.

†† Contrary to Cauchy's problem, the shape of the boundary in the mixed problem (we mean that part of the boundary which does not bear Cauchy's data) has a very deep influence on the nature of the problem. Thus, for the case of two independent variables, the problem of the electric cable with a sliding contact (§ 24*a*) will require quite different calculations for different laws of motion of the contact.

it being understood that the given quantities on the right-hand sides satisfy

(42) $$u_0(0, y, z) = \mathbf{u}(y, z, 0),$$

(43) $$u_1(0, y, z) = \frac{\partial \mathbf{u}}{\partial t}(y, z, 0).$$

If the retrograde half-conoid from a, limited to $t = 0$, does not meet $x = 0$ (mixed dotted cone in fig. 29), u is given by Poisson's formula.

In the contrary case, if we know the values of $\frac{du}{dn} = \frac{\partial u}{\partial x}$ along $x = 0$, we should have formula (35)

(44) $$4\pi u_a = I_1 + I_2,$$

where I_1 is the first line on the right-hand side in (35), constituted by a part of each of Poisson's integrals (viz. the part relating to that part σ of the sphere σ with radius t_0 which lies on the positive side of $x = 0$) and I_2 is Kirchhoff's integral over $x = 0$:

$$I_2 = \iint \left(\frac{\bar{\mathbf{u}}_1}{r} - \frac{1}{r}\frac{dr}{dn}\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}}\frac{d\frac{1}{r}}{dn} \right) d\omega,$$

$\bar{\mathbf{u}}_1$ being equal to $-\frac{du}{dn} = -\frac{\partial u}{\partial x}$.

There remains to eliminate these values of $\bar{\mathbf{u}}_1$ in the latter integral. We obtain this by introducing the point $a\ (-x_0, y_0, z_0)$, symmetrical

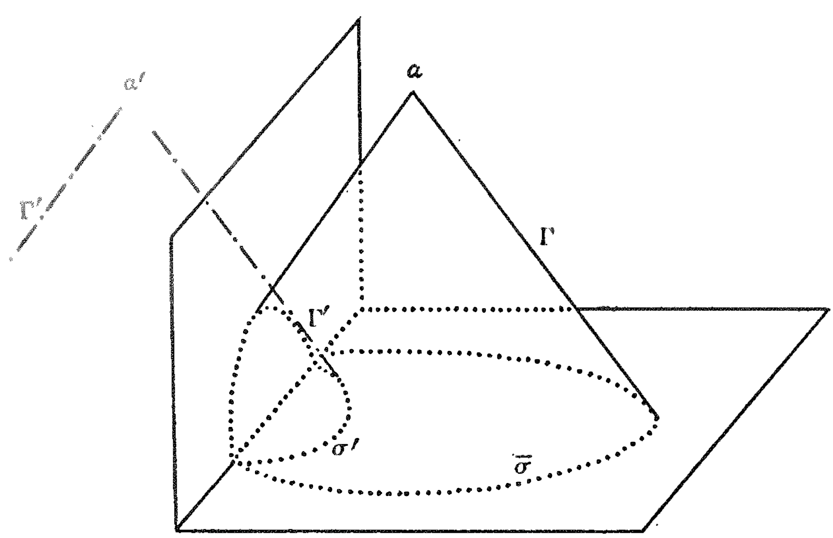

Fig. 29.

to a with respect to $x=0$. Let $\overline{\Gamma}'$ be the part of the characteristic conoid (we still mean the retrograde half-conoid with vertex a') which lies in our useful region $x \geqslant 0$, $t \geqslant 0$; σ, the trace of Γ' on $t=0$ (a portion of spherical surface in ordinary space)*: the sum

$$I_1' + I_2' = I_1' + \iint \left(\frac{\bar{\mathbf{u}}_1}{r'} - \frac{1}{r}\frac{dr'}{dn}\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \frac{d\frac{1}{r'}}{dn} \right) d\omega$$

analogous to the right-hand side of (44), but in which we start from point a' instead of a—so that we introduce

$$r' = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

instead of r—and, moreover, I_1' relates to $\overline{\sigma}'$ and no longer to $\overline{\sigma}$, *is zero,* as the corresponding domain of integration does not include the vertex a' (see § 146). Such a sum can therefore be subtracted from (44), and this combination

$$u = I_1 + I_2 - I_1' - I_2'$$

is the required one. For, the values of r and r' being equal to each other at every point of $x=0$, the terms in $\mathbf{u}_1$ disappear in the difference $I_2 - I_2$.

The other terms in $I_2 - I_2'$ duplicate each other, as the values of $\frac{d\frac{1}{r}}{dn}$ and $\frac{d\frac{1}{r'}}{dn}$ along $x=0$ are opposite: so that†

$$(45) \quad u_a = \frac{1}{4\pi}(I_1 - I_1') + \frac{1}{2\pi} J_2, \quad J_2 = \iint \left(-\frac{1}{r}\frac{dr}{dn}\frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \frac{d\frac{1}{r}}{dn} \right) d\omega.$$

* The accompanying diagram (fig. 29) (which would be a complete diagram for (e_2)) represents the projection of the true 4-dimensional diagram on to the (x, y, t)-space.

† Any "non duly inclined" plane S' can be treated in the same way (whether normal or oblique to $t=0$), because we know that (e_3) can be transformed into itself by a linear transformation on x, y, z, t such that S' becomes parallel to the new t-axis. In the non-transformed space, the cone Γ' would have the same intersection with S' as Γ, the line which joins the two vertices being transversal to S' and divided by S' into two equal parts.

We only mention briefly Volterra's important remarks (London *Proceedings*, 1904, and Stockholm Lectures) on the quite special behaviour of this method of images in the hyperbolic case.

$I_1 - I_1'$ is a double integral relating to a system of two spherical zones (of one base). It can be expressed more simply by supposing fictitiously that the values of u_0 and u_1 on the negative side of our plane $t = 0$ (i.e. relating to $t = 0$, $x < 0$) are respectively opposite to the corresponding values on the positive side: viz.

(46) $$u_i(-x, y, z) = -u_i(x, y, z) \qquad (i = 0, 1),$$

by means of which $I_1 - I_1'$ is expressed by double integrals extended over the whole surface of the sphere σ, and quite similar to the right-hand side of Poisson's formula.

We however must observe that the values of u_0 and u_1 thus introduced in the integrand are discontinuous on $x = 0$ if they do not vanish there. It may be convenient to remove this discontinuity by separating the terms which correspond to the values of u_0 and u_1 for $x = 0$, i.e. setting down

(47) $$u_0(x, y, z) = \mathbf{u}(y, z, 0) + U_0(x, y, z),$$
$$u_1(x, y, z) = \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{t=0} + U_1(x, y, z).$$

157. The verification of this solution is, as usual, necessary.

There is no difficulty as concerns the partial differential equation: for it is satisfied* by $I_1 + I_2$ and also (as the equation does not change by changing x into $-x$) by $I_1' + I_2'$.

As to Cauchy's conditions, there is no difference between the present problem and Poisson's question, the solution being simply given by Poisson's formula in the neighbourhood of $t = 0$.

Let us now suppose that the point (x_0, y_0, z_0, t_0) reaches the plane $x = 0$. Poisson-like terms $I_1 - I_1'$ disappear, as becoming exactly equal except for sign. The term $2\iint \frac{1}{r}\frac{dr}{dn}\frac{\partial u}{\partial t}\,d\omega$, in which $\frac{dr}{dn}$ is finite, behaves like the potential of a simple layer, and therefore is continuous and assumes the value zero; the remaining term

$$2\iint u \frac{d\frac{1}{r}}{dn}\,d\omega$$

* The argument in Book III, §§ 115, 116, remains valid when S is bounded by an outline and, therefore, also when it consists of two different parts, as happens in the present instance.

behaves like the potential of a double layer, and therefore (the integrand being identically zero when a assumes its limiting position P) becomes, as is well known, equal to $4\pi u_P$: which is also, therefore, the limiting value of $4\pi u_a$.

The proof is apparently complete. In reality it is not, for we have to ascertain that u and its first derivatives are continuous on the plane $x - t$ (which is the characteristic through the edge common to both parts of S): else we should have constructed a solution satisfying (C_3) and a solution satisfying ($\mathbf{C}_3$), but not *one* solution satisfying both.

In each of the regions 1 ($t < x$) and 2 ($t > x$) into which our domain is divided by the characteristic plane $x - t$, each of the terms of the solution is continuous, as well as its derivatives, if such is the case for u_0, u_1, $\mathbf{u}$ themselves. The derivatives of I_1, I_2 in 1 are expressed as stated in § 28. In 2, let us express $M_t(u_0)$ and $M_t(u_1)$ (notation of § 28) by means of the quantities

$$(48) \qquad \mathcal{M}_0(x, \rho) = \frac{1}{2\pi}\int_0^{2\pi} u_0(x, y_0 + \rho\cos\phi, z_0 + \rho\sin\phi)\, d\phi,$$

$$\mathcal{M}_1(x, \rho) = \frac{1}{2\pi}\int_0^{2\pi} u_1(\quad)\, d\phi,$$

which respectively denote the average values of u_0 and u_1 along the circumference which has its plane parallel to $x = 0$, its centre at (x, y_0, z_0) and its radius equal to ρ. These integrals give $M_{t_0}(u_0)$ and $M_{t_0}(u_1)$ as

$$\frac{1}{2}\int_{-1}^{+1} \mathcal{M}_0(x_0 - \lambda t_0, t_0\sqrt{1-\lambda^2})\, d\lambda, \qquad \frac{1}{2}\int_{-1}^{+1} \mathcal{M}_1(\quad)\, d\lambda,$$

and $M_{t_0}\left(\frac{du_0}{dr}\right)$, as
$$\frac{1}{2}\int_{-1}^{\lambda_0} \frac{d\mathcal{M}_0}{dt_0}\, d\lambda,$$

where $\frac{d\mathcal{M}_0}{dt_0}$ is given by (33), § 153.

In 2, the corresponding quantities, account being taken of our fictitious distribution for negative values of x, will be

$$(49) \qquad M_{t_0}(u_i) = \frac{1}{2}\left[\int_{-1}^{\lambda_0} \mathcal{M}_i(\quad)\, d\lambda - \int_{\lambda_0}^{1}\right], \quad (i = 0, 1) \quad \left(\lambda_0 = \frac{x_0}{t_0}\right)$$

Similarly, with the values of **u**, we shall construct the average value $\mathcal{N}(x, \rho)$ of **u** along the circumference with centre $(0, y_0, z_0)$ and radius ρ, and we shall easily get, for $\frac{1}{2\pi} J_2$, the value

$$(50) \quad \frac{1}{2\pi} J_2 = \int_{\lambda_0}^{1} \left[\mathcal{N}\left(t_0 - \frac{x_0}{\lambda}, \ \frac{x_0\sqrt{1-\lambda^2}}{\lambda}\right) + \frac{x_0}{\lambda}\frac{\partial \mathcal{N}}{\partial t} \right] d\lambda.$$

Now, each term of u, and the first derivatives of each of these terms, will be continuous in 2 as well as in 1. Moreover, when passing from 1 to 2, the continuity of u subsists, because the new terms which appear in it (viz., integrals from λ_0 to 1) begin by being infinitesimal.

But such is not the case for the derivatives with respect to* x_0 and t_0. Taking for instance the latter, it is clear that the differentiation of (49), with respect to t_0, introduces not only differentiations under the integral sign (the result of which varies continuously when $t_0 - x_0$ passes through zero) but also limiting terms corresponding to the dependence of λ_0 on t_0, and which duplicate each other, giving†

$$(51) \qquad -\frac{x_0}{t_0^2}\left[t_0 U_1(0, y_0, z_0) + u_0\right] = -\frac{1}{t_0}(t_0 u_1 + u_0).$$

The corresponding term for (49′) is zero†, and we see that $I_1 + I_1'$ *has a discontinuous derivative with respect to* t_0 (or similarly with respect to x_0), the discontinuity being given by (51).

But this discontinuity is exactly compensated by another one relating to $\frac{1}{2\pi}\frac{\partial I_2}{\partial t_0}$ which is 0 in 1 and assumes in 2 (as seen from (50)) the initial value

$$\frac{x_0}{t_0}\left[\mathcal{N}(0, 0) + x_0\left(\frac{\partial \mathcal{N}}{\partial t}\right)_0\right].$$

158. We shall use this solution of the mixed problem for a plane boundary in the study of a question which occurred to us in Book I, § 27, concerning Cauchy's problem with respect to (e_3) and to $x = 0$,

* $\frac{\partial u}{\partial y_0}$ and $\frac{\partial u}{\partial z_0}$ behave like u itself.

† For $\rho = 0$, the quantity $\mathcal{M}_i\ (x, \rho)$ is evidently equal to the corresponding value of $u_i\,(x, y_0, z_0)$ and the function u_i being assumed to be differentiable in y_0 and z_0, the derivative $\frac{}{\partial \rho}$ is zero. ($\mathcal{M}_i$ is an even function of ρ, as is seen from (48).)

that is, the problem of finding $u(x, y, z, t)$ such that (ω being still taken $=1$)

$$(e_3) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

and that

$$(52) \qquad u = u_0(y, z, t), \quad \frac{\partial u}{\partial x} = u_1(y, z, t), \text{ for } x = 0,$$

or the corresponding problem for (e_2), that is, to find $u(x, y, t)$ such that

$$(e_2) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

and that

$$(52') \qquad u = u_0(x, y, t), \quad \frac{\partial u}{\partial x} = u_1(y, t), \text{ for } x = 0.$$

We have seen that there exists no solution in general; and therefore, as in §§ 15*a*, 16 for $\nabla^2 u = 0$, or for the equation of heat, the question arises to find for what values of u_0 and u_1 the solution will exist.

A very simple *sufficient* condition has been found by Volterra*: viz. the solution will certainly exist if u_0 and u_1 be analytic in y, z (for equation (e_3)) or in y (for equation (e_2)), no matter in what (regular) manner they depend on t.

To see this, we have only to remark that our differential equations do not change by changing x into it and t into ix.

If, therefore, reasoning on (e_3), we write down the expression

$$\frac{1}{4\pi} \frac{\partial}{\partial x} \left[x \iint u_0(y + ix \sin\theta \cos\phi,\ z + ix \sin\theta \sin\phi,\ t + x\cos\theta) \sin\theta d\theta d\phi\right]$$

this will again satisfy the differential equation (the verification of this fact being still valid in this new case if we take it under the form which we gave in Book II, § 28*a*). The same is also true, of course, for

$$x \iint u_1(y + ix \sin\theta \cos\phi,\ z + ix \sin\theta \sin\phi,\ t + x\cos\theta) \sin\theta d\theta d\phi,$$

and the sum of these two terms will satisfy the boundary conditions if we take for u_0 and u_1 the data (52) of the problem.

All this, of course, assumes u_0 and u_1 to be analytic in y, z, because imaginary values of the latter are introduced.

* This is equivalent to the argument in his paper in *Rivista di Matematica*, t. IV, 1894, pp. 1—14.

A quite similar argument holds for the problem relative to (e_2) if we treat formula (1′), Book II, § 30, integrating this equation, as we just now treated Poisson's formula.

159. Can we now find a system of necessary and sufficient conditions? An answer, though a very imperfect one, is afforded by the above solution of the mixed problem, as we showed at Volterra's Seminary in Rome in 1916*.

As we have done in preceding examples, we take u_0 as arbitrarily given beforehand, and try to find the most general admissible form of u_1 by finding the most general form of the solution u. This can be done by considering u as being the solution of our mixed problem, i.e. as being defined by the conditions of § 156 in which, only, instead of the initial plane $t=0$, we take a suitably chosen parallel plane $t=\theta$ to bear Cauchy's data, so that (also reversing the notation for the data) we write

$$u=\mathbf{u}_0(x, y, z), \quad \frac{\partial u}{\partial t}=\mathbf{u}_1(x, y, z), \quad \text{for } t=\theta,\ x \geqslant 0,$$
$$u=u_0(y, z, t), \qquad\qquad\qquad \text{for } x=0,\ t\geqslant\theta.$$

The quantity θ is arbitrary: we can therefore admit that it belongs to an interval of values of t for which u_0 is regular, or even that it lies arbitrarily near to the values of t which we especially consider.

As found above, (45) gives (notation of § 157)

$$u=\frac{1}{2\pi}J_2+\bar{M}_{t_0-\theta}(\mathbf{u}_0)+(t_0-\theta)\left[\bar{M}_{t_0-\theta}\left(\frac{d\mathbf{u}_0}{dr}\right)+\bar{M}_{t_0-\theta}(\mathbf{u}_1)\right],$$

of which we have to take the derivative with respect to x for $x=0$, if we want to obtain an expression of u_1.

Again proceeding as in Book I, §§ 15*a*, 16, we observe that the first term, which is the only one depending on the given function u_0, will lead us to *one* of the possible solutions: we do not take any special care of its form and shall submit it to no further transformation.

Under the above form, however, the remaining terms are not completely independent of u, as u_0 and u_1 are subject to conditions (42).

* The principle of the calculation in § 160 was given in 1901 at Princeton (see *Princeton University Bulletin*, Vol. XIII, 1902).

(43). We avoid this inconvenience by means of the transformation (47) (§ 156), writing

$$\mathbf{u}_0(x, y, z) = u_0(y, z, 0) + U_0(x, y, z), \quad \mathbf{u}_1(x, y, z) = \frac{\partial u_0}{\partial t}(y, z, 0) + U_1(x, y, z),$$

U_0 and U_1 vanishing with x. The terms obtained when replacing $\mathbf{u}_0$ and $\mathbf{u}_1$ by $u_0(y, z, 0)$ and $\frac{\partial u_0}{\partial t}(y, z, 0)$ shall be considered as making a whole with $\frac{1}{2\pi} J_2$.

We investigate the remaining part

$$(53) \quad u' = \bar{M}_{t_0-\theta}(U_0) + (t_0 - \theta)\left[\bar{M}_{t_0-\theta}\left(\frac{dU_0}{dr}\right) + \bar{M}_{t_0-\theta}(U_1)\right],$$

in which U_0 and U_1 vanish for $x = 0$, and which represents the most general expression of a solution of (e_3) the values of which for $x = 0$ are everywhere zero. On that same hyperplane $x = 0$, we have to write down the value u_1' of the differential coefficient $\left(\frac{\partial u'}{\partial x}\right)_{x=0}$.

It is easily seen (by combining with each other the corresponding elements of the two spherical surfaces along which the M's are taken; we mean, elements which have the same projection on $x = 0$) that—taking account of the assumed conditions $U_0(0, y, z) = U_1(0, y, z) - 0$ —the derivative in question is*

$$(53') \quad u_1' = \left(\frac{\partial u'}{\partial x}\right)_{x=0}$$
$$= M'_{t_0-\theta}(U_0') + (t_0 - \theta)\left[M'_{t_0-\theta}\left(\frac{dU_0'}{dr}\right) + M'_{t_0-\theta}(U_1')\right],$$

the average values M' being now taken over the hemisphere σ (centre $(0, y, z)$, radius $t_0 - \theta$) which is situated in the region of positive x's, and U_0', U_1' denoting the derivatives of U_0, U_1 with respect to x (for any positive x). We immediately observe that these are arbitrary functions of x, y, z, any choice of U_0', for instance, giving

$$U_0 = \int_0^x U_0' \, dx.$$

* If we had operated with $\mathbf{u}_0$ and $\mathbf{u}_1$ and not with U_0 and U_1, i.e. with functions not vanishing with x, the formula would have contained complementary terms introducing the average values of these functions on the limiting circle of the hemisphere.

If, conversely, u_1' is given, the problem of finding u is reduced to the determination of U_0', U_1', that is to *solving equation* (53').

160. This can be done, at least theoretically, by the following method. In the first place, it is easy to separate the two terms on the right-hand side of (53'). For, setting down $t_0 - \theta' - t'$, one of them is evidently even*, and the other odd, in t', so that we can determine separately U_0' and U_1' by

$$(54)\quad \frac{\partial}{\partial t}[t'M'_{t'}(U_0')] = \tfrac{1}{2}[u_1'(0, y, z, \theta + t') + u_1'(0, y, z, \theta - t')],$$

which is equivalent to

$$(55)\quad t'M'_{t'}(U_0') = \frac{1}{2}\int_0^{t'} [u_1'(0, y, z, \theta + t') + u_1'(0, y, z, \theta - t')]\, dt' = W(y, z, t'),$$

and

$$(55')\quad t'M'_{t'}(U_1') = \tfrac{1}{2}[u_1'(y, z, \theta + t') - u_1'(y, z, \theta - t')] = W_1(y, z, t').$$

Each of them needs only to be considered for positive t' and is a special case of an *integral equation of the first kind.*

The unknown quantities U_0' and U_1' being required only to be continuous, we shall multiply by $t'dt'$ and integrate from 0 to t'. The result (which we, of course, only write for U_0', as the two equations (54) and (55') are entirely similar) is

$$\int_0^{t'} t'\, W dt' = \int_0^{t'} t'^2 M'_{t'}(U_0')\, dt',$$

and the right-hand side represents the integral $\frac{1}{2\pi}\iiint U_0'\, dx\, dy\, dz$ throughout the *inside* of the hemisphere.

Now, the principle of our argument will consist in observing that the latter integral admits of a derivative with respect to each of the variables y and z, without any assumption as to U_0' other than continuity. For, if the centre $(0, y_0, z_0)$ of our hemisphere be displaced parallel to the y-axis by dy, the new spherical surface will in the neighbourhood of any point M (see fig. 30, which is a plane section of the space diagram) be normally displaced by

$$dy \cos(n, y) = \frac{y - y_0}{t'}\, dy,$$

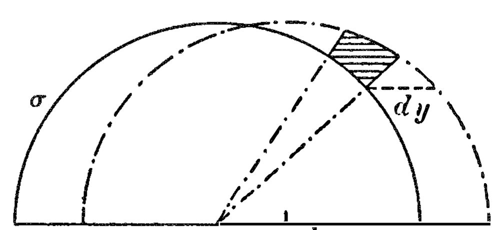

Fig 30.

* Compare formula (48), § 157.

and the new volume will differ from the original one by (positive or negative) cylindrical elements, each of which has $\frac{y-y_0}{t'}dy$ for its height and $t'^2 d\Omega$ for its base, ($d\Omega = \sin\theta d\theta d\phi$), giving a term

$$t'(y-y_0)\,U_0'\,d\Omega$$

in the integral. Thus

$$(56)\quad \frac{1}{2\pi}\frac{\partial}{\partial y_0}\iiint U_0'\,dx\,dy\,dz = \frac{\partial}{\partial y_0}\int_0^{t'} t'\,W(y_0, z_0, t')\,dt'$$
$$= \frac{t'}{2\pi}\iint (y-y_0)\,U_0'\,d\Omega = t'M'_{t'}\,[(y-y_0)\,U_0'],$$

with a similar expression for the derivative with respect to z_0.

161. Let us now define the two operators $\mathscr{C}_y$ and $\mathscr{C}_z$ by the equalities

$$\mathscr{C}_y\,\Phi = \frac{\partial}{\partial y}\int_0^{t'} t'\,\Phi(y, z, t')\,dt', \quad \mathscr{C}_z\,\Phi = \frac{\partial}{\partial z}\int_0^{t'} t'\,\Phi(y, z, t')\,dt':$$

we see that

$$\mathscr{C}_{y_0} W = t'M'_{t'}\,[(y-y_0)\,U_0'], \quad \mathscr{C}_{z_0} W = t'M'_{t'}\,[(z-z_0)\,U_0'].$$

But we can operate on (56) as we did on U_0' itself. Therefore, $\mathscr{C}^2_{y_0} W$ must exist, as well as $\mathscr{C}_{y_0}\mathscr{C}_{z_0} W$ (the latter being equal to $\mathscr{C}_{z_0}\mathscr{C}_{y_0} W$), and will give the values of

$$t'M'_{t'}\,[(y-y_0)^2 U_0'], \quad t'M'_{t'}\,[(y-y_0)(z-z_0)\,U_0'].$$

To this we can again apply our operators $\mathscr{C}_y$, $\mathscr{C}_z$, and we see that we can do so any number of times, *be the given function u_1' indefinitely differentiable or not* (the latter being, of course, the general case when U_0' only admits of derivatives to a certain order). We have

$$(57)\quad T_{hk}(y_0, z_0, t') = \mathscr{C}^h_{y_0}\mathscr{C}^k_{z_0} W = t'M'_{t'}\,[(y-y_0)^h (z-z_0)^k U_0'].$$

We can say that this enables us to obtain the value of any double integral such as $t'M'_{t'}\,[P(y, z)\,U_0']$, where P is any polynomial, as P can be expanded in powers of $(y-y_0)$, $(z-z_0)$. For instance, denoting by y_1, z_1 another system of values of y, z, the quantity

$$(y-y_0)^h (z-z_0)^k U_0'$$

can be written in the form

$$\Sigma A_{hkmn}\,(y-y_1)^m (z-z_1)^n\,U_0',$$

so that its average value on a new hemisphere σ_1 with its centre at (y_1, z_1) and its radius equal to t_1' can also be found by such an expression as

$$(58) \qquad T'_{h,k}(y_0, z_0;\ y_1, z_1, t_1') = \frac{1}{t'} \Sigma A_{hkmn} T_{mn}(y_1, z_1, t_1').$$

161 *a.* There remains to determine U_0' by means of (57) (or (58)): a kind of "problem of moments" whose solution, if existent, can be obtained by known methods. For instance, we only have to consider such an integral as

$$\frac{K^2}{\pi} \iint e^{-K^2[(y-y_0)^2+(z-z_0)^2]} U_0'(\xi, y, z)\, t'^2 d\Omega$$
$$= 2t' \Sigma (-1)^{h+k} \frac{K^{2(h+k+1)}}{h!\, k!} T_{2h,2k}(y_0, z_0, t')$$

extended to our original hemisphere σ —so that ξ denotes

$$\sqrt{t'^2 - (y-y_0)^2 - (z-z_0)^2}, —$$

which, for $K = \infty$, approaches the limit $U_0'(t', y_0, z_0)$,—or* a similar expression formed with the T'_{hk}'s and approaching

$$\sqrt{t_1'^2 - (y_0-y_1)^2 - (z_0-z_1)^2}\, U_0'[\sqrt{t_1'^2 - (y_0-y_1)^2 - (z_0-z_1)^2}, y_0, z_0].$$

We thus have a series of operations which must give us the solution U_0', if any one exists. Of course, we should have to ascertain that this is the case, i.e. that the values of U_0' found in this way actually satisfy the given equation (55): so that a system of necessary (and sufficient) conditions which u_1' must satisfy in order that U_0' should exist is the following:

(*a*) *The function $W(y, z, t')$ deduced from u_1' by* (55′) *can be subjected to the operations $\mathscr{C}_y, \mathscr{C}_z$ any number of times in any order* (*these operations being even permutable*); and the same applies to W_1;

* We also can consider the quantity

$$M_{t'}\left\{\left[\frac{(y-y_0)^2+(z-z_0)^2}{t'^2}\right]^m U_0'\right\},$$

which we again can express under the form $\Sigma B_{hk} T_{hk}$ and whose quotient by

$$M_{t'}\left\{\frac{(y-y_0)^2+(z-z_0)^2}{t'^2}\right\}^m$$

has also the limit U_0' when the index m becomes infinite.

(*b*) *The result* T_{hk} (or T'_{hk}) *must be such that*

$$2x_0 \Sigma (-1)^{h+k} \frac{K^{2(h+k+1)}}{h!\,k!} T_{2h,2k}(y_0, z_0, x_0),$$

(or a suitable analogous combination with T') *approaches a limit for* $K=\infty$;

(*c*) *This limit* $U_0'(x_0, y_0, z_0)$ *must satisfy* (55).

162. A quite similar treatment can be applied to the corresponding problem for (e_2). It is clear that, the data being the values $u_0 = 0$ and $u_1'(y, t)$ of u and $\frac{\partial u}{\partial x}$ for $x=0$, our above equation (55) would have to be replaced (notation of § 30, Book II) by

$$W = t'\mu_{t'}(U_0'),$$

that is

$$W(y_0, t') = \frac{1}{\pi}\iint_{\{x \geqslant 0,\, x^2+(y-y_0)^2 \leqslant t'^2\}} \frac{U_0'(x, y)\, dx dy}{\sqrt{t'^2 - x^2 - (y-y_0)^2}}$$

$$= \frac{1}{\pi}\int_0^{t'}\int_0^{\pi} \frac{U_0'(\rho \sin\phi, y_0 + \rho\cos\phi)\,\rho\, d\rho d\phi}{\sqrt{t'^2-\rho^2}},$$

from which again we, in the first place, deduce the value of the average

$$m_t(U_0') = \frac{1}{\pi}\int_0^{\pi} U_0'(t\sin\phi,\, y_0 + t\cos\phi)\, d\phi.$$

The only new feature characterising this case is that such a deduction is to be done by solving Abel's integral equation, viz.

$$\text{(55a)} \qquad \frac{\pi}{2} m_t(U_0') = \frac{\partial}{t\partial t}\int_0^t \frac{W(y, t')\, t' dt'}{\sqrt{t^2 - t'^2}}.$$

This being done, we should have to proceed as in the preceding case and should obtain quite similar conclusions.

163. The problem, in one case or the other, is thus solved, though "very little" in the sense of Poincare, on account of the complicated nature of our conditions and of the fact that, for instance, we cannot even say whether (*c*) is distinct from (*a*) and (*b*) or a consequence of them.

Moreover, these conditions introduce the variable t in a rôle quite different from that which is played by the other variables; and this ought not to be: for our problem is evidently invariant for Lorenz's group or, more exactly, the sub-group of it which leaves x invariant,—for (e_2) for instance, the well-known group

$$\begin{cases} y' = y \,\mathrm{Ch}\, \alpha + t \,\mathrm{Sh}\, \alpha, \\ t' = y \,\mathrm{Sh}\, \alpha + t \,\mathrm{Ch}\, \alpha. \end{cases} \quad (\alpha, \text{arbitrary parameter}).$$

It is clear that it would be desirable to write our solution in a form which would also be invariant for such a group: this would be the case if it would introduce the variations of u_1' not along parallels to the axes of the coordinate planes, but along the bicharacteristics.

Our results in Book I, § 27, show us that our integral equation (55) or (55a) has no solution when u_1' is independent of t and not analytic in the other variables; Volterra's result shows us that it always has one when u_1' is analytic in the variable or variables other than t. All this, of course, could be put in other more general forms by using the aforesaid sub-group of Lorenz's group.

CHAPTER II

OTHER APPLICATIONS OF THE PRINCIPLE OF DESCENT

1. Descent from m even to m odd

164. We have obtained the solution of our problem with an even number of independent variables by deducing it from the corresponding result in the case of m odd, i.e., from our formulæ given in Book III. Could the reverse be done? Can our present formulæ lead (by means of descent) to solutions for the case of m odd? We shall now see that this is possible, and even that the solution thus obtained is more advantageous, in some respects, than the previous one.

We apply our method of descent in the same way as before, starting from fig. 23, p. 219, and the comparative study of equations (E) and (E′), with the only difference that m will now be an odd number, which we can still write $m = 2m_1 + 1$, if, in our above formulæ, we change m_1 to $m_1 + 1$. Also the relation

$$\Gamma' = \Gamma - (z - c)^2$$

(in which we shall again assume c to be zero) will subsist. But, instead of formula (39) given in Book III, we now start from formula (28) or (28 a) of the present Book, which we shall apply to (E′): we, therefore, introduce the two functions corresponding to $\mathcal{V}$ and V. One will be a power series in Γ'

$$\mathcal{V}' = \sum_{k=0}^{\infty} V'_{m_1+k}\,\Gamma'^k = \sum V'_{m_1+k}\,(\Gamma - z^2)^k, \tag{59}$$

the other a polynomial, of degree $m_1 - 1$ (on account of our above observation concerning m_1) in the same quantity

$$V' = \sum_{0}^{m_1-1} V_h'\,\Gamma'^h = \sum V_h'\,(\Gamma - z^2)^h, \tag{60}$$

the coefficients V_h' being, in both cases, functions of the variables x.

Let us now consider a solution u of (E), which we again take as a solution of (E′) not depending on z. If it is determined by Cauchy's data on S'—or, which is the same, on S—it will be given by formula (28) or (28 a) of the preceding chapter, in which

(1) m_1 has to be changed into $m_1 + 1$;

(2) Γ has to be replaced by Γ'; V and $\mathcal{V}$ by the above expressions V' and $\mathcal{V}'$;

(3) T, S, τ, σ have to be replaced by the corresponding varieties T', S', τ', σ' relating to the problem in E_{m+1};

(4) $dx_1 dx_2 \ldots dx_m, dS, dS_\nu$ have to be multiplied by dz, while $d\tau_\gamma'$, $d\sigma_\gamma'$ are deduced from them as explained above;

(5) a $\int$ is to be written* after each **SSS** or **SS**, and will represent the integration with respect to z.

We only have to illustrate the influence of the latter operation. The treatment of the first two terms is obvious. The $\mathbf{SSS}\int$, for instance, will be written

$$(61)\quad \mathbf{SSS} f\, dx_1 dx_2 \ldots dx_m \int \mathcal{V}' dz$$

$$= \mathbf{SSS} f\, dx_1 dx_2 \ldots dx_m \int [\Sigma\, V'_{m_1+k} (\Gamma - z^2)^k]\, dz.$$

As the integration $\int$ has to be performed from $-\sqrt{\Gamma}$ to $+\sqrt{\Gamma}$, and

$$\int_{-\sqrt{\Gamma}}^{+\sqrt{\Gamma}} (\Gamma - z^2)^k\, dz = \Gamma^{k+\frac{1}{2}} B\left(k+1, \tfrac{1}{2}\right) = \frac{\Gamma^{k+\frac{1}{2}}}{(k+1)\, C_{k+1}},$$

this introduces the quantity (no longer containing z)

$$(62)\qquad \mathbf{l} v_2 = \sum_{k=0}^{\infty} \frac{1}{(k+1)\, C_{k+1}} V'_{m_1+k}\, \Gamma^{k+\frac{1}{2}},$$

where the arbitrary numerical coefficient $\mathbf{l}$ shall be disposed of presently.

Thus (61) becomes $\mathbf{l}\,\mathbf{SSS} f v_2 dx_1 dx_2 \ldots dx_m$, and similarly, in the second term of (28), we have to replace $\mathcal{V}$ by† $\mathbf{l} v_2$.

* Another symbol would, strictly speaking, be necessary for the integrals which are differentiated with respect to γ. In order not to complicate notations, we simply denote them by **SSS** and **SS** (instead of **SS** and **S**) as they will be finally expressed (see the following text) by space and surface integrals respectively.

† No special difficulty arises from the presence of the derivative $\frac{d}{d\nu}$, as the differentiation is evidently permutable with our integrations.

164 a. Let us come to the remaining terms, in which we have to differentiate with respect to γ. Before that differentiation, τ, for example, will belong to a kind of hyperboloid of two sheets, so that the point $(x_1, x_2, \ldots, x_m)$ has to vary inside the domain T_1 enclosed between $\Gamma = \gamma$ and S, a kind of hyperbola (the section of the hyperboloid in question), each position of it giving on τ two values of z

$$z = \pm \sqrt{\Gamma - \gamma}.$$

By its definition, $d\tau_\gamma'$ will be such that its product by $d\gamma$ represents the volume of the portion of the $(m+1)$-dimensional space enclosed between two consecutive surfaces $\Gamma' = \text{const.}$ in any elementary cylinder through the element in question (fig. 31). If we

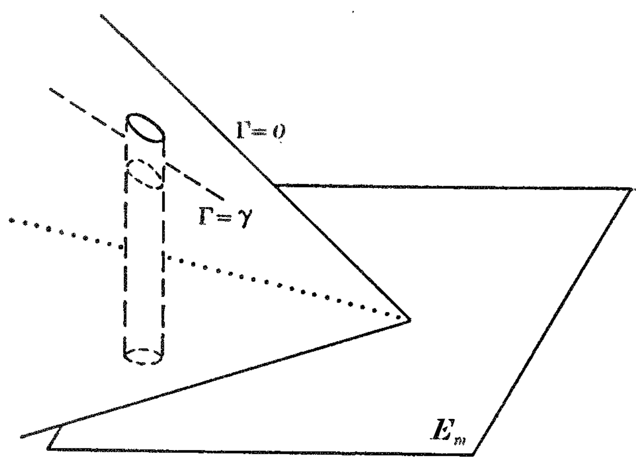

Fig. 31.

take the cylinder parallel to the z-axis, the volume will be equal to the cross-section (that is, the projection on to E_m, i.e. $dx_1\, dx_2 \ldots dx_m$) multiplied by the segment dz intercepted on the generatrix. Thus

$$d\tau_\gamma' = dx_1 \ldots dx_m : \left|\frac{\partial \Gamma'}{\partial z}\right| = \frac{dx_1 \ldots dx_m}{2\,|z|} = \frac{dx_1 \ldots dx_m}{2\sqrt{\Gamma - \gamma}},$$

which allows us to express the first integral subject to differentiation relative to γ. V' being given by (60), in which $\Gamma - z^2$ has to be replaced by γ, this integral becomes (the denominator disappearing on account of the presence of two elements $d\tau$ with the same projection on to E_m)

$$(63) \qquad \mathbf{SSS} f V' d\tau_\gamma' = \mathbf{SSS}_{T_1} f \frac{dx_1 \ldots dx_m}{\sqrt{\Gamma - \gamma}} \sum_{h=0}^{m_1 - 1} V_h' \gamma^h,$$

where T_1 is again the portion of T such that $\Gamma \geqslant \gamma$.

By a quite similar argument we have to take, in the other integrals subject to differentiation,

$$d\sigma_\gamma' = \frac{dS}{2\sqrt{\Gamma-\gamma}}.$$

We can also apply this without essential modification to the integrals in the last line of the formula if we take it in the form (28 a) (and besides note that the symbols $\frac{d}{d\gamma}$ and $\frac{d}{d\nu}$ are permutable). Finally, we have

$$(64)\quad (-1)^{m_1+1}\frac{2\pi^m}{(m_1-1)!}u_a$$
$$= -\mathbf{|}\left[\mathbf{SSS}_T f v_2\, dx_1 \dots dx_m + \mathbf{SS}_{S_0} v_2 (u_1 + Lu_0)\, dS\right]$$
$$+ \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}_{(\gamma=0)}\left[\mathbf{SSS}_T f\, dx_1 \dots dx_m \frac{\Sigma V_h'\gamma^h}{\sqrt{\Gamma-\gamma}} + \mathbf{SS}_S (u_1 + Lu_1)\, dS \frac{\Sigma V_h'\gamma^h}{\sqrt{\Gamma-\gamma}}\right]$$
$$- \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\frac{d}{d\nu}\mathbf{SS}_{S_\nu} v_0\, dS \frac{\Sigma V_h'\gamma^h}{\sqrt{\Gamma-\gamma}} + \mathbf{|}\frac{d}{d\nu}\mathbf{SS}\, u_0 v_2\, dS.$$

This new formula is, as can be seen, quite of the same appearance as the classic formula of Volterra (to which it easily reduces when applied to (e_2)), and contains no symbol which is not classic in analysis. But it combines this advantage with an entire generality, as we know that such a formula exists for any (as yet analytic) equation.

165. How this same formula could again lead to our former solution is easy to recognize. We simply have to perform the differentiation with respect to γ. As the integrand is of a fractional order in the neighbourhood of $\gamma = 0$, no boundary term is to be introduced; but, on the other hand, the use of our symbol $\mathbf{|}$ is necessary: with the help of it we can simply differentiate under $\mathbf{SSS}$ or $\mathbf{SS}$: in other words, replace $\frac{1}{\sqrt{\Gamma-\gamma}}\Sigma V_h'\gamma^h$ by its (m_1-1)th derivative*, so that we find, e.g.

$$\frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}_{(\gamma=0)}\left[\mathbf{SSS}\frac{f}{\sqrt{\Gamma-\gamma}}(\Sigma V_h'\gamma^h)\, dx_1 \dots dx_m\right] = \mathbf{|}\,\overline{\mathbf{SSS}\, v_1 f\, dx_1 \dots dx_m},$$

* This, however, is not valid in the vicinity of a, for which a limiting process is to be used as said in Book III, § 106.

v_1 being given by

$$(65)\qquad \mathbf{l}v_1 = \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\left(\frac{1}{\sqrt{\Gamma-\gamma}}\Sigma V_h'\gamma^h\right)_{(\gamma=0)}$$

$$= \Sigma\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots(m_1-h-\frac{3}{2})}{(m_1-h-1)!}\frac{V_h'}{\Gamma^{m_1-h-\frac{1}{2}}} = \sum_0^{m_1-1} C_{m_1-h-1}\frac{V_h'}{\Gamma^{m_1-h-\frac{1}{2}}}.$$

The same treatment applying to every term in (64), we evidently find the very formula (39) in Book III, § 105, with

$$(65')\qquad \mathbf{l}v = \mathbf{l}(v_1 - v_2) = \sum_{h=0}^{m_1-1} C_{m_1-h-1}\frac{V_h'}{\Gamma^{m_1-h-\frac{1}{2}}}$$

$$-\sum_{k=0}^{\infty}\frac{1}{(k+1)\,C_{k+1}}V'_{m_1+k}\Gamma^{k+\frac{1}{2}}.$$

V is thus deduced from the functions V' and $\mathfrak{V}'$ as we have already shown in Book II, § 70, the numerical ratios of corresponding coefficients in (59), (60), and in (65′) being of course in agreement* with the values found at the aforesaid place if, for the numerical coefficient $\mathbf{l}$, we take the value

$$\mathbf{l} = C_{m_1-1}.$$

But as we see the relation between v' and v can also be expressed by formulæ (62), (65) as, in the former case of m even, it was by (3), (3′).

2. Properties of the coefficients in the elementary solution

166. The importance of the elementary solution in our theory is obviously due to its essential connection with the equation itself, evidently resulting from the fact that a given linear partial differential equation of the second order with an odd number of independent variables admits of one perfectly determinate elementary solution and that, similarly, to a given equation with an even number of variables corresponds one function V and one function $\mathfrak{V}$. This property immediately shows us, for instance, how the solution in question would behave towards some simple transformations, such as changing variables.

* The verification is immediate with the help of the relation (§ 99) between Ω_{2m_1} and Ω_{2m_1-1}.

To what extent does this character also belong to the operations by which this solution u has been constructed in the above, i.e. to the single terms of its expansion? Such a question is of interest as to the value of our method for the calculation of u: the closer the connection these individual terms have with our problem itself, the more natural this method will be.

The connection is, this time, a looser one than it was as to the final value of the quantity u, although in the aforesaid expansion (taking, for instance, m odd)

$$u = \frac{1}{\Gamma^{\frac{m-2}{2}}} (U_0 + U_1 \Gamma + \ldots + U_h \Gamma^h + \ldots),$$

the coefficients U_h themselves are perfectly determinate in terms of the x, a when the expressions of the coefficients are known. We can easily foresee that the U's will, like u itself, keep their values except for a simple factor* when changing independent variables, because in such a punctual transformation our geodesics, such as introduced in Book II, § 55, and, therefore, Γ also remain unaltered. But the case would be different if we should change unknown by setting down

$$u = \mu (x_1, x_2, \ldots, x_m) u_1,$$

or even if we should simply multiply the left-hand side $\mathcal{F}(u)$ by a given quantity (viz. a given function of the x's). Either of these two operations again brings no other change in the elementary solution than multiplication by a simple factor: but the change in each individual U_h is much more complicated, as such operations, altering the characteristic form **A**, also profoundly modify the geodesics and consequently the quantity Γ.

We can only see what would happen by a suitable combination of these two operations, viz. by substituting for $\mathcal{F}(u)$ the new differential polynomial

$$\mathcal{F}_1(u) = \frac{1}{\mu} \mathcal{F}(\mu u),$$

* This factor occurs only on account of the presence of the factor $\frac{1}{\sqrt{|\Delta|}}$ in our formulæ of § 63.

the adjoint of which is (as immediately appears by considering the adjoint as defined by identity (5), § 37) $\mathcal{G}_1(v) = \mu\mathcal{G}\left(-\right)$. This preserves the values of the A_{ik}'s (and therefore Γ). As to the B's, it will be easily found, if we set down

$$\mu = e^{\nu}, \qquad \nu_i = \frac{\partial \nu}{\partial x_i},$$

that each of them is augmented by the corresponding value of $\frac{\partial A}{\partial \nu_i}$ (the explicit calculation of C not being necessary for our purpose), so that the new value of M becomes

$$M_1 = M + \Sigma \frac{\partial \mathbf{A}}{\partial \nu_i} \frac{\partial \Gamma}{\partial x_i} = M + 2\,\Sigma\, \nu_i \frac{\partial \mathbf{A}}{\partial C_i} = M + 4s\frac{d\nu}{ds}$$

(notation of Book II). Therefore U_0 will be multiplied by the quantity

$$e^{-\int_0^s \frac{d\nu}{ds} ds} = \frac{\mu_a}{\mu_x}$$

(in which we have taken account of the initial factor $\frac{1}{\sqrt{|\Delta|}}$), after which the successive equations (42′), (44′) show us that such will be the case for each U_h.

167. But another test of the intimate connection of the coefficients U_h with the question lies in the result obtained in § 114, and extended (§ 146) to even m's.

We have seen that our elementary solution admits of the interchange property, i.e. does not change its numerical value when we simultaneously exchange the two points x and a and the two mutually adjoint polynomials $\mathcal{F}(u)$ and $\mathcal{G}(v)$.

As (for m odd)

$$u = \frac{U}{\Gamma^{\frac{m-2}{2}}},$$

and Γ is symmetrical with respect to x and a, the same can be said as to the numerator U.

Can we also assert the same conclusion concerning each coefficient U_h of the expansion

$$U = U_0 + U_1\Gamma + \ldots + U_h\Gamma^h + \ldots ?$$

We certainly could if the variables $x_1, x_2, \ldots, x_m$, $a_1, a_2, \ldots, a_m$ (on which the U_h's depend) and Γ were independent. But this is not the case, so that the conclusion in question is by no means evident.

It becomes so, on the contrary, if we again use our device of "descent."

In other words, together with our given differential polynomial

$$\mathcal{F}(u) = \Sigma A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \Sigma B_i \frac{\partial u}{\partial x_i} + Cu,$$

—in which, in the first place, we shall assume the number m of independent variables to be *even*—we consider the auxiliary one

$$\mathcal{F}'(u) = \mathcal{F}(u) - \frac{\partial^2 u}{\partial z^2},$$

where z is a supplementary $(m+1)$th variable, the two adjoint polynomials respectively being

$$\mathcal{G}(v), \quad \mathcal{G}'(v) = \mathcal{G}(v) - \frac{\partial^2 u}{\partial z^2}.$$

We know that the new value of Γ relative to $\mathcal{F}'$ or $\mathcal{G}'$ will be

$$\Gamma' = \Gamma - (z-c)^2, \tag{66}$$

so that the new elementary solution will be

$$u' = \frac{U'}{[\Gamma - (z-c)^2]^{\frac{m-1}{2}}} = \frac{1}{[\Gamma - (z-c)^2]^{\frac{m-1}{2}}} \sum_h U_h' [\Gamma - (z-c)^2]^h, \tag{67}$$

the coefficients U_h' only differing from the corresponding coefficients U_h by numerical factors and especially *only depending on*

$$x_1, x_2, \ldots, x_m, \quad a_1, a_2, \ldots, a_m,$$

with the exclusion of z and c.

The number $(m+1)$ being odd, the quantity (67) and, consequently, its numerator U' admit of the interchange property, so that

$$\Sigma U_h' \Gamma'^h = \Sigma V_h' \Gamma'^h, \tag{68}$$

the successive coefficients V_h' on the right-hand side being calculated as the corresponding ones in U, but for the exchange: (1) of x with a, i.e. of x_1 with a_1, x_2 with a_2, etc. (so that, in the calculations of § 62, x must, in the first place, be considered as fixed and only the a's as variable and, e.g. the geodesic paths of integration all issue from the same x and end at various points a); (2) of the polynomial $\mathcal{F}$ (in which the independent variables are the x's) with $\mathcal{G}$ (in which the independent variables are the a's).

Now, in (68), the $(2m+1)$ variables $x_1, \ldots, x_m, a_1, \ldots, a_m, \Gamma$ *are independent*, because [by equation (66)] Γ' contains a variable $(z-c)$ which is distinct from $x_1, \ldots, x_m, a_1, \ldots, a_m$, and does not appear in the coefficients.

Therefore, (68) must be an identity with respect to Γ', and this gives the required conclusion concerning our equation $\mathcal{F}(u)=0$ at least for m even.

But a new "descent" will evidently extend it to odd values of m, as any equation $\mathcal{F}(u)=0$ with an odd number of variables can be considered as deduced from another one $\mathcal{F}(u)-\frac{\partial^2 u}{\partial z^2}=0$ (in which the number of independent variables is even) whose elementary solution has the same coefficients except for numerical factors. Our conclusion is thus completely proved.

168. Could we, in order to obtain it, replace the above indirect method by a more direct one, starting from the explicit expression of the U's?

This may be considered as a question belonging to the theory of geodesic lines. Not only, indeed, every equation (E) leads, as we have seen, to the consideration of geodesic lines, relative to the linear element **H**; but, conversely, every linear element **H** corresponds to ∞ linear partial differential equations such as (E). It will be convenient, after Cotton*, to write the most general homogeneous equation (E) corresponding to a given **H** in the form

$$\Delta_2 u + \sum_i B_i' \frac{\partial u}{\partial x_i} + C'u = 0, \tag{69}$$

where Δ_2 is again the second differential parameter of Lamé, the expression of which (account being taken of p. 91, first footnote) is

$$\Delta_2 u = \sum_i \frac{\partial}{\partial x_i}\left(\sum_k A_{ik}\frac{\partial u}{\partial x_k}\right) = \sum_{ik} A_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i \left[\frac{\partial u}{\partial x_i}\sum_k \frac{\partial A_{ik}}{\partial x_k}\right].$$

* *Ann. Sc. Ec. Norm. Sup*^{re}, Vol. XVII, 1900, pp. 211—244. See also Levi-Civita, *Atti Ist. Veneto*, Vol. LXXII, 1913, pp. 1331—1357.

It is clear that the left-hand member of (69) is of the form

$$\mathcal{F}(u) = \Sigma A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \Sigma B_i \frac{\partial u}{\partial x_i} + Cu$$

and that, conversely, every differential polynomial such as $\mathcal{F}(u)$ can be written in the form (69), with

$$B_i' = B_i - \Sigma \frac{\partial A_{ik}}{\partial x_k}.$$

Therefore, any question on an arbitrary equation (E) may be considered as a question concerning a linear element **H** and a system of $(m+1)$ functions $B_1', \ldots, B_m, C'$ of $x_1, \ldots, x_m$.

169. In the present case, the difficulty of the question obviously increases with the order of the term considered.

As concerns the first term U_0, the required conclusion can be, as we shall see, deduced from the formula obtained in § 59:

$$\Delta_2 \Gamma = 2\left(1 + s\frac{d \log J}{\cdots}\right)$$

and from properties, now classic, of geodesic lines.

Again considering the differential equations

$$\text{(L)} \qquad \begin{cases} \dfrac{dx_i}{ds} = \dfrac{1}{2}\dfrac{\partial \mathbf{A}}{\partial p_i}, \\[2ex] \dfrac{dp_i}{ds} = -\dfrac{1}{2}\dfrac{\partial \mathbf{A}}{\partial x_i}, \end{cases}$$

of § 55, we know that their general integral depends on $2m$ arbitrary constants $\mu_1, \mu_2, \ldots, \mu_{2m}$.

We also know (see Additional Note to Book II) that, if we start from any determinate solution of (L) (which corresponds to a determinate system of numerical values of the μ s) and consider the quantities

$$\text{(70)} \qquad \begin{cases} \bar{x}_i = \dfrac{\partial x_i}{\partial \mu_j}, \\[2ex] \bar{p}_i = \dfrac{\partial p_i}{\partial \mu_j}, \end{cases}$$

(j being any subscript from 1 to $2m$), these will satisfy the *variational system*

$$(\bar{\mathrm{L}}) \qquad \begin{cases} \dfrac{d\bar{x}_i}{ds} = \dfrac{1}{2}\dfrac{\partial \bar{\mathbf{A}}}{\partial \bar{p}_i}, \\ \dfrac{d\bar{p}_i}{ds} = -\dfrac{1}{2}\dfrac{\partial \bar{\mathbf{A}}}{\partial \bar{x}_i}, \end{cases}$$

which is *linear*, as $\bar{\mathbf{A}}$ is quadratic in the $\bar{x}$'s and $\bar{p}$'s.

Each value of j in (70) gives a solution of ($\bar{\mathrm{L}}$) and, as the Jacobian

$$\mathscr{D} = \frac{D(x_1, \ldots, x_m, p_1, \ldots, p_m)}{D(\mu_1, \ldots, \mu_{2m})}$$

is not zero *, the $2m$ possible values of j give us a fundamental system of solutions of ($\bar{\mathrm{L}}$).

These properties belong to any variational equations deduced from a differential system. But Hamiltonian systems like (L) and their variational systems ($\bar{\mathrm{L}}$) possess another important property†, which is that *the determinant $\mathscr{D}$ is a constant along every determinate line satisfying* (L) (in other words, $\mathscr{D}$ depends on $\mu_1, \ldots, \mu_{2m}$ only and not on s).

The geodesics themselves only depend on $2m-2$ parameters; but each solution of (L) contains two parameters more, viz. the two quantities α and β mentioned in § 56, and which therefore we have to consider as being two of the μ s (so that, having written the general equation of geodesics with $(2m-2)$ parameters, we deduce therefrom the general integral of (L) by changing s into $\alpha s + \beta$).

Coming back to our given equation in the form (69). we begin by noting that the differential polynomial $\Delta_2 u$ is identical with its adjoint polynomial ‡, as is verified directly without any difficulty, and

* The fundamental theorem of differential equations shows that the μ's can be chosen so as to give to $x_1, \ldots, x_m, p_1, \ldots, p_m$ any given values for $s=0$. The determinant $\mathscr{D}$ is what, in our *Leçons sur le Calcul des Variations*, we have called the "general determinant" of the $(2m-2)$ solutions of ($\bar{\mathrm{L}}$), while $\mathbf{J}$ is what we have called the "special determinant."

† See Poincaré's *Les methodes nouvelles de la Mécanique Celeste*, Vol. III, § 254.

‡ The integral identity which (§§ 36, 37) characterizes adjoint polynomials is no other than the one which would be deduced from that which defines Δ_2 (see formula (35), § 59) by exchanging the two functions (Γ and U in § 59) which it contains and subtracting.

as follows from the definition of the symbol Δ_2. Therefore, the adjoint polynomial of

$$\mathscr{F}(u) = \Delta_2 u + \sum_i B_i' \frac{\partial u}{\partial x_i} + C'u$$

is

$$\mathscr{G}(v) = \Delta_2 v - \sum_i B_i' \frac{\partial v}{\partial x_i} + C_1'v, \quad \left(C_1' = C' - \sum_i \frac{\partial B_i'}{\partial x_i}\right)$$

i.e., *is deduced from the former by changing the sign of each B'* and properly changing C'.

Coming then to the expression of U_0, we have, in the first place, to express the quantity M: by (37), § 56 and $\frac{\partial \Gamma}{\partial x_i} = 2P_i = 2sp_i$, we get

$$\frac{M}{2} = \frac{1}{2}\left(\Delta_2 \Gamma + \sum B_i' \frac{\partial \Gamma}{\partial x_i}\right) = 1 + \frac{sd \log \mathbf{J}}{ds} + s \sum_i B_i' p_i.$$

$\mathbf{J}$ is given by formula (30a), § 57 a:

$$\mathbf{J} = \frac{D(x_1, x_2, \ldots, x_m)}{D(\lambda_1, \ldots, \lambda_{m-1}, s)},$$

in which only the geodesics issuing from a are considered (and not all geodesics, as above) and are expressed in terms of s and $m-1$ parameters λ.

Let us introduce m new functions of $x_1, \ldots, x_m$ by writing

$$B_i' = \sum_k A_{ik} \mathscr{B}_k = \frac{1}{2} \frac{\partial \mathbf{A}}{\partial \mathscr{B}_i}:$$

the last term in the expression of $\frac{M}{2}$ thus becomes

$$s \cdot \tfrac{1}{2} \sum_{i,k} A_{ik} p_i \mathscr{B}_k = \frac{s}{2} \sum_i \mathscr{B}_i \frac{\partial \mathbf{A}}{\partial p_i} = s \sum_i \mathscr{B}_i \frac{dx_i}{ds}.$$

If we recall that

(71) $$2p - 2 = -m,$$

we see that

$$\frac{M}{2} + 2p - 2 = \frac{M}{2} - m = -(m-1) + \frac{sd \log \mathbf{J}}{ds} + s \sum_i \mathscr{B}_i \frac{dx_i}{ds},$$

and, therefore,

$$(72) \qquad U_0 = \frac{1}{\sqrt{|\Delta_a|}} e^{-\frac{1}{2}\int_0^s \left(\frac{M}{2} - m\right)\frac{ds}{s}}$$

$$= \frac{1}{\sqrt{|\Delta_a|}} \sqrt{\left(\frac{\mathbf{J}}{s^{m-1}}\right)_0 : \left(\frac{\mathbf{J}}{s^{m-1}}\right)} \, e^{-\frac{1}{2}\int_a^x \mathscr{B}_1 dx_1 + \dots + \mathscr{B}_m dx_m},$$

$\left(\frac{\mathbf{J}}{s^{m-1}}\right)_0$ denoting the limiting value of $\frac{\mathbf{J}}{s^{m}}$ when the point x approaches a along a determinate geodesic. This limit exists and is different from zero*: we easily find

$$(73) \qquad \left(\frac{\mathbf{J}}{s^{m-1}}\right)_0 = \begin{vmatrix} \frac{\partial x_1'}{\partial \lambda_1} & \frac{\partial x_1'}{\partial \lambda_2} & \cdots & \frac{\partial x_1'}{\partial \lambda_{m-1}} & x_1' \\ \frac{\partial x_2'}{\partial \lambda_1} & \frac{\partial x_2'}{\partial \lambda_2} & \cdots & \frac{\partial x_2'}{\partial \lambda_{m-1}} & x_2' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_m'}{\partial \lambda_1} & \cdots & \cdots & \frac{\partial x_m'}{\partial \lambda_{m-1}} & x_m' \end{vmatrix}_0,$$

the subscript 0 after the determinant denoting that the values of the x''s and their derivatives with respect to the λ's are taken for $s = 0$. We shall write, introducing a condensed notation,

$$\left\| \frac{\partial x_i'}{\partial \lambda_1} \quad \frac{\partial x_i'}{\partial \lambda_2} \cdots \frac{\partial x_i'}{\partial \lambda_{m-1}} \quad x_i' \right\|_0.$$

170. *The exponential factor in* (72), *viz.* $e^{-\frac{1}{2}\int_a^x \mathscr{B}_1 dx_1 + \dots + \mathscr{B}_m dx_m}$, *possesses the above enunciated interchange property.* For, as we have seen, exchanging the two adjoint polynomials $\mathscr{F}$ and $\mathscr{G}$ corresponds to a change of signs in the B''s, and, on the other hand, the permutation of a and x changes the sense of integration in the curvilinear integral

$$\int \mathscr{B}_1 dx_1 + \dots + \mathscr{B}_m dx_m$$

taken along the geodesic line.

To show the same as to the remaining factor

$$\frac{1}{\sqrt{|\Delta_a|}} \sqrt{\left(\frac{\mathbf{J}}{s^{m-1}}\right)_0 : \left(\frac{\mathbf{J}}{s^{m-1}}\right)},$$

* This fact is the very condition by which we have chosen the value of the number p in § 61.

we shall transform it with the help of known principles concerning geodesics*.

If the parameters $\mu_1, \ldots, \mu_{2m}$ be replaced by $2m$ other ones $\nu_1, \ldots, \nu_{2m}$ (the latter being functions of the former, and conversely), the determinant $\mathscr{D}$ is evidently multiplied by the Jacobian

$$\frac{D(\mu_1, \ldots, \mu_{2m})}{D(\nu_1, \ldots, \nu_{2m})}$$

(which is evidently a constant along each solution of (L)).

We shall now introduce another important determinant*

$$\mathscr{J} = \frac{D(x_1^{(0)}, \ldots, x_m^{(0)}, x_1^{(1)}, \ldots, x_m^{(1)})}{D(\mu_1, \ldots, \mu_{2m})}$$

which, as we shall presently see, is closely connected with the above-mentioned Jacobian **J**. The quantities $x_1^{(0)}, \ldots, x_m^{(0)}$ and $x_1^{(1)}, \ldots, x_m^{(1)}$ are the values assumed by the x's in two points of the same geodesic, corresponding to two different values $s^{(0)}, s^{(1)}$ of s (the latter being considered as constants in the differentiations with respect to $\mu_1, \ldots, \mu_{2m}$).

We take $s^0 = 0$, $s^{(1)}$ being the value of s which corresponds to x, so that

$$\mathscr{J} = \frac{D(a_1, \ldots, a_m, x_1, \ldots, x_m)}{D(\mu_1, \ldots, \mu_{2m})}. \tag{74}$$

$\mathscr{J}$ is multiplied by the same factor as $\mathscr{D}$ if we replace

$$\mu_1, \ldots, \mu_{2m} \text{ by } \nu_1, \ldots, \nu_{2m},$$

so that *the ratio*

$$\frac{\mathscr{J}}{\mathscr{D}}$$

does not depend on the choice of the arbitrary constant parameters in terms of which the general solution of (L) *is expressed.*

Let us accordingly suppose that m of the parameters μ are those which we previously denoted by $\lambda_1, \ldots, \lambda_{m-1}$ and α, so that the values of the x's for $s = 0$ do not depend on them and remain equal to $a_1, \ldots, a_m$ respectively as long as the other m parameters $\mu_1, \ldots, \mu_m$ remain fixed. The quantities $\frac{\partial a_i}{\partial \lambda_l}$ $(i = 1, \ldots, m;\ l = 1, \ldots (m-1))$, and $\frac{\partial a_i}{\partial \alpha}$

* See our *Leçons sur le Calcul des Variations*, § 283.

being zero, we have $\left(\text{as } \frac{\partial x_i}{\partial \alpha} = s x_i' = s \frac{\partial x_i}{\partial s}\right)$,

$$\mathcal{J} = \left\| \begin{array}{cccc:cccc} \frac{\partial a_i}{\partial \mu_1} & \frac{\partial a_i}{\partial \mu_2} & \cdots & \frac{\partial a_i}{\partial \mu_m} & \frac{\partial x_i}{\partial \mu_1} & \frac{\partial x_i}{\partial \mu_2} & \cdots\cdots & \frac{\partial x_i}{\partial \mu_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & 0 & \frac{\partial x_i}{\partial \lambda_1} & \frac{\partial x_i}{\partial \lambda_2} & \cdots \frac{\partial x_i}{\partial \lambda_{m-1}} & s x_i' \end{array} \right\|,$$

which notation must again be interpreted by understanding that each row stands for m rows obtained by successively taking

$$i = 1, 2, \ldots, m,$$

and that we have, by dotted lines, separated the m first columns from the m last ones, and the same for the rows. As m^2 elements are zero, such a determinant splits into a product of two determinants of order m, the second of which is $s\mathbf{J}$: thus we get

$$(75) \qquad \mathcal{J} = Q \,.\, s\mathbf{J}, \quad Q = \left\| \frac{\partial a_i}{\partial \mu_1} \frac{\partial a_i}{\partial \mu_2} \cdots \frac{\partial a_i}{\partial \mu_m} \right\|.$$

By the same choice of parameters, and taking $s = 0$, the constant determinant $\mathcal{D}$ becomes (same mode of notation)

$$\mathcal{D} = \left| \begin{array}{ccc:cccc} \frac{\partial a_i}{\partial \mu_1} & \cdots & \frac{\partial a_i}{\partial \mu_m} & \frac{\partial p_{i0}}{\partial \mu_1} & \cdots\cdots & & \frac{\partial p_{i0}}{\partial \mu_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots\cdots & 0 & \frac{\partial p_{i0}}{\partial \lambda_1} & \cdots & \frac{\partial p_{i0}}{\partial \lambda_{m-1}} & p_{i0} \end{array} \right|$$

$\left(\text{as } \frac{\partial p_{i0}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\alpha p_i\left(\frac{s}{\alpha}\right)\right]_{s=0} = p_{i0}\right)$: which again is a product of two factors

$$\mathcal{D} = Q \,.\, \left| \frac{\partial p_{i0}}{\partial \lambda_1} \cdots \frac{\partial p_{i0}}{\partial \lambda_{m-1}} \, p_{i0} \right|,$$

the factor Q being the same as in (75). In the second factor, we may replace each p_{i0} by the corresponding

$$\frac{1}{2}\frac{\partial \mathbf{A}}{\partial p_{i0}} = x_{i0}',$$

if we, at the same time, divide by the determinant of this linear substitution, i.e. by Δ_a. But this gives precisely the right-hand side of (73), i.e. the value of $\left(\frac{\mathbf{J}}{s^{m-1}}\right)_0$. Therefore

$$\mathscr{D} = Q \cdot \frac{1}{\Delta_a} \left(\frac{\mathbf{J}}{s^{m-1}}\right)_0,$$

and, on account of (75),

$$\frac{\mathscr{J}}{\mathscr{D}} = \Delta_a \frac{s\mathbf{J}}{\left(\frac{\mathbf{J}}{s^{m-1}}\right)_0}.$$

We thus transform the value (72) of U_0 into

$$U_0 = s^{\frac{m}{2}} \sqrt{\left|\frac{\mathscr{D}}{\mathscr{J}}\right|} \cdot e^{-\frac{1}{2}\int_a^x \Sigma \mathscr{B}_i dx_i}.$$

This completes the proof of the enunciated property as concerns U_0, as $\mathscr{D}$ is a constant which has therefore the same value at a and at x, and $|\mathscr{J}|$ obviously possesses the symmetry in question (by (74)).

The same result, as concerns the following coefficients U_1, U_2, ..., seems to be connected with more complicated properties of geodesics. It might even, for that reason, prove of interest in the theory of these lines, as depending on other principles than those which have been used as yet.

3. Treatment of non-analytic equations

171. We now return to our elementary solution as a whole. We succeeded in constructing it (at least for two points x and a sufficiently near each other) by assuming the coefficients to be analytic. It is remarkable that, in the first instance (aside from classic cases) which was given of such a construction—viz. Picard's work on $\nabla^2 u + Cu = 0$ (as quoted in our Book II)—this hypothesis was not wanted. We shall now see how we may also get rid of it.

Such a result, for the general equation of the *elliptic* type, has been obtained independently by E. Elia Levi *—one of the best and most beautifully gifted of young Italian geometers (he gave his life in the Great War),—and under a finally equivalent form by Hilbert †. The

* *Rendic. Circ. Mat. Palermo*, 1907, Vol. XXIV, pp. 275—317.

† *Grundzüge einer allg. Theorie der linearen Integralgleichungen*, 6th paper (1910). A first allusion is made at the end of the 5th paper (1906). See also Fubini, *Rendic. Ac. Lincei*, 5th series, Vol. XVIII (1909), p. 423, to which the remarks in the text also apply.

method used by both of them consists in forming a first approximation (Hilbert's "parametrix") which *does not satisfy* the given equation, but, when substituted in this equation, merely gives a result which, at the singular point, is only of the first order of infinity. Thanks to the introduction of this "parametrix," E. Elia Levi succeeds in forming the elementary solution; Hilbert's result consists in doing without the latter, i.e. making the "parametrix" play the part which usually belongs to the elementary solution itself. The two questions are really one, and E. Elia Levi's analysis is, ultimately, identical with Hilbert's. In both cases the problem is reduced to Fredholm's integral equation (as is also recognized to be the case for Picard's initial proceeding).

The method, however, wants improvement in order to be applicable to the hyperbolic case. The reason of this is that, in E. E. Levi's and Hilbert's case, there is only one real singular point, and we need not mind how our parametrix or the complementary term behaves along the imaginary singularity, which in fact is different when considered in the first approximation ("parametrix") and in the final result. In our case of the problem, on the contrary, we must immediately take account of our characteristic conoid, which is real, and our first approximation itself may not admit of any other singularity than the conoid in question.

Other difficulties seem, at first, to arise from the nature of our above solutions: for, if m is even, the elementary solution is not well-determined, and if m is odd, we have to reckon with the peculiar singularities met with in our expressions, which would require special precautions in the application of Fredholm's method. The fact is that, in what follows, we only get to the solution by dealing simultaneously with both cases, thanks to "descent."

172. The domain of validity in the analytic case. Even before coming to non-analytic equations, we have, in that line, to answer a first—and perhaps the most important—part of the question concerning analytic ones.

We must not forget—and this defect is common to all methods resting on Maclaurin's series—that our above solutions of the problem, though seemingly complete, are really, as yet, quite insufficient, the problem being "not enough" solved.

It was first solved, but "very little," in Book I, by Cauchy-Kowalewsky's argument. The problem is very little solved, not only because the expression of u is given in a very indirect and complicated form, but also because the region of validity of this calculation may be (and generally is) very small and insufficient for our wants. We know (see Book I, § 8) that this always chances to be the case (on account of the presence of imaginary singularities) when using expressions involving power series.

We shall see presently that the solutions obtained in the present Book will be of better use even from that point of view. Nevertheless again, the existence of the elementary solution (and consequently the validity of our formula), as a result of the convergence of the series (43) (Book II, § 62), have been proved only for a certain domain around the vertex of our conoid. To what extent this allows us to assert the existence of the function U, or of the functions U and $\mathcal{U}$, is what we do not know: the radius of convergence of our power series (43) may be much greater than the lower limit deduced from our previous considerations, and also the functions U, $\mathcal{U}$ may exist far beyond the range where their developments in powers of Γ are convergent.

We shall prove (the coefficients being still assumed to be analytic) not only the existence of the solution throughout the domain of regularity of the coefficients themselves, which is relatively easy (see below, § 177), but also (under the further assumption of the analyticity of f, u_0, u_1) its analyticity. The corresponding proof, for the elliptic case, has been given, for instance, by E. E. Levi in his above-cited work. We shall have, however, to modify the method used in that case in order to apply it to ours*, for reasons of the same kind as mentioned above (more precisely, on account of the fact that the domain of integration of the right-hand sides of our formulæ depends on a).

To give this proof, we take the case of m even (which does not limit the generality, thanks to descent) and we make, at least to begin, a further geometrical assumption, which, strictly speaking, we

* The method which we shall develop presently is utterly different from E. E. Levi's and, indeed, as it rests on successive extensions, could not by any means be applied to the elliptic case: a method corresponding to E. E. Levi's will be indicated further on.

ought not to make, but which is justified in every practical case. We assume that there exists a one-parameter set of surfaces which are everywhere duly inclined (and this *strictly*, i.e. it will happen nowhere that one of them is tangent to a characteristic cone, and, therefore, the angle between any of them and any direction interior to the cone from one of its points will have a positive minimum): these surfaces will be analytic (so that, by means of an analytic change of variables, the corresponding parameter t can be taken as one of the coordinates, the mth), and one of them, corresponding to $t=0$, will be* the surface S, our region $\mathcal{R}$ being on the side $t>0$. Every bicharacteristic or interior geodesic—i.e. every geodesic such that $\mathbf{H}\geqslant 0$—will, under such assumptions, have a direct and an inverse sense, the latter corresponding to $\frac{dt}{d\sigma}<0$ (where σ is the arc, according to the ordinary meaning of the word), $\left|\frac{dt}{d\sigma}\right|$ having even a positive minimum. We assume the shape of $\mathcal{R}$ to be such that any inverse geodesic issuing from a point of $\mathcal{R}$ will remain constantly inside it for $t\geqslant 0$: then such a geodesic must necessarily reach S. We suppose the choice of the variable s to have been made on each of these lines in a determinate and even analytic way: for instance, we agree that $\frac{dt}{ds}=1$ for $t=0$, so that this derivative will remain between two fixed positive limits throughout $\mathcal{R}$. x_m will be, in what follows, synonymous with t, and, similarly, a_m with c.

We shall also (though the necessity of it be not absolute, as we shall see) admit, concerning $\mathcal{R}$, our general hypothesis that any two points x and a within it can be joined to each other by a geodesic in a perfectly unique and continuous way: in other words, the first set of equations (29) of § 57 admits of a perfectly determinate solution for the q's in terms of the x and a s. We even suppose that their Jacobian never vanishes in $\mathcal{R}$. We can, therefore, define normal variables relating to a; and moreover, on these normal variables, make a linear substitution (with coefficients functions of the a s) which

* The influence of the fact that one of the surfaces $t=$const. is the given S, is merely superficial, as we shall see below, and we have used it principally in order to simplify notations (especially when resuming the matter in § 189).

we can even suppose to be a perfectly determinate one and to vary analytically* with the as, so that the quadratic form $\Gamma = \mathbf{H}(\xi)$ is reduced to its canonical form

$$\Gamma_0 = \xi_m^2 - \xi_1^2 \quad \xi_2^2 - \ldots - \xi^2_{m-1}$$

and also the surface $\xi_m = \text{const.}$ through a is tangent, at this point, to $t = \text{const.}$, with $\frac{\partial \xi_m}{\partial t} < 0$. The linear substitution on the normal variables ξ of Lipschitz will concern only the first $(m-1)$ of them if we have chosen the variables x so that the lines $x_1 = \text{const.}$, $x_2 = \text{const.}$, $\ldots, x_{m-1} = \text{const.}$ are everywhere transversal to the surfaces $t = \text{const.}$ We can suppose that our choice of the x's possesses this property, and even (by means of a suitable transformation on $x_1, x_2, \ldots, x_{m-1}$ alone) that, for $t = 0$,

$$\frac{d}{d\nu} dS = \frac{\partial}{\partial t} dx_1 \ldots dx_{m-1}$$

so that transversal derivatives are no other than derivatives with respect to t: all these transformations being analytic and regular.

ξ_m will also eventually be denoted by the synonymous letter θ.

The ξ's can be expressed in terms of θ and the ratios

$$\eta_1 = \frac{\xi_1}{\theta}, \quad \eta_2 = \frac{\xi_2}{\theta}, \quad \ldots, \quad \eta_{m-1} = \frac{\xi_{m-1}}{\theta},$$

the latter satisfying, for the inside of the conoid, the relation

$$\eta_1^2 + \eta_2^2 + \ldots + \eta^2_{m-1} \leqslant 1. \tag{76}$$

To every such system of constant values of the η's will correspond a determinate (interior or bicharacteristic) geodesic from a. Along any such geodesic, the ratio $\left|\frac{d\theta}{dt}\right|$ remains between two fixed positive limits, and so does also, therefore, the ratio $\frac{\theta}{c-t}$.

* These coefficients being arbitrary to a certain extent, the statement in the text is true under the assumption that they are calculated in a determinate way, e.g. by strictly following Lagrange's classic rule (see Serret's *Algebre superieure*, 4th ed.. Vol. I, p. 430; Bocher's *Higher Algebra*, Ch. X, § 45, p. 131). For the application of this, we shall always (under the assumption below in the text as to the choice of variables) be in the general case where the coefficients of the square terms are different from zero, as, the term in ξ_m^2 being abstracted, there remains a form in $\xi_1, \ldots, \xi_{m-1}$ which is (negative) definite.

a being any determinate point within $\mathscr{R}$, we can refer the points x which are interior altogether to $\mathscr{R}$ and to the conoid from a, to the normal variables ξ relating to a, but also to $\eta_1, \ldots, \eta_{m-1}, \theta$ or to $\eta_1, \ldots, \eta_{m-1}, t$: both latter systems will be equivalent to each other from our point of view, in the sense that each of them can be expressed in terms of the other, the expressions being holomorphic on account of our remark on $\frac{d\theta}{dt}$.

173. Things being so, we shall show that, if we have in any way constructed the elements V and $\mathfrak{V}$ of the elementary solution for all possible positions of x and a inside $\mathscr{R}$ (more exactly, for all positions of these points satisfying $\Gamma(x; a) \geqslant 0$), knowing, moreover, these quantities to be holomorphic in the x's and the as, we can assert that the solution u of Cauchy's problem (with holomorphic data) relating to $t = 0$ or to $t = t_0 > 0$, is also holomorphic.

We begin by showing this for the first term (in **SSS**) of (28 a). Generally speaking, we show that the m-tuple integral

$$(77) \qquad \mathbf{SSS}\, F(x_1, x_2, \ldots x_m)\, dx_1\, dx_2 \ldots dx_m$$

(F being holomorphic), extended over the domain included between the retrograde half conoid from a and the surface S, is holomorphic in the a's. F may even contain not only the x's, but also the as themselves and eventually other parameters: if it be holomorphic in all these quantities, so will be also (77) in the a's and the parameters. The fact may be looked at as practically evident; but its explicit deduction is very simple after the above assumptions and remarks.

It follows from these that the x's will be holomorphic functions (throughout $\mathscr{R}$) of $\eta_1, \ldots, \eta_{m-1}, t$. Let K_1 be their Jacobian (taken so as to be positive) so that

$$dx_1\, dx_2 \ldots dx_m = K_1\, d\eta_1 \ldots d\eta_{m-1}\, dt.$$

The required function will be

$$(77\,a) \quad \mathbf{SSS}\, F(x_1, x_2, \ldots, x_m)\, dx_1\, dx_2 \ldots dx_m = \mathbf{SS}\, d\eta_1\, d\eta_2 \ldots d\eta_{m-1} \int_0^c K_1 F\, dt,$$

the integration being carried out with respect to the η's over the real domain (76) and with respect to t, from the origin to the value c which corresponds to a. As this can be written (since $t = \lambda c$)

$$(77\,b)\qquad c\,\mathbf{SS}\,d\eta_1 \ldots d\eta_{m-1}\int_0^1 K_1 F d\lambda,$$

integration with respect to λ having to be carried out from 0 to 1, we only want to notice that the integrand is holomorphic, and even uniformly so*, when expressed in terms of the η's, λ and the as (and, a fortiori, in terms of the as for any values of the η's and λ). Its Taylor expansion, around any determinate position of a and any system of values of these quantities, corresponding to a determinate position of x within $\mathscr{R}$, converging uniformly with respect to the η's and λ, can therefore be integrated term by term, which gives the conclusion which we have in view: our integral is defined and holomorphic throughout $\mathscr{R}$.

If we take $F = \mathscr{V}f$, we see that the first term $\mathbf{SSS}\,\mathscr{V}f\,dT$ of (28) or (28 a) exists and is holomorphic in the region $\mathscr{R}$; and the other terms

$$\mathbf{SS}\left[\mathscr{V}(u_1 + Lu_0) - u_0\frac{d\mathscr{V}}{d\nu}\right]dS$$

in (28) relating to S_0 will evidently be treated in the same way without any difficulty (compare below, § 176).

174. No difficulty would occur, either, in the remaining terms (relating to the surface of the conoid or its edge of intersection with S), if it were not for the first of them,

$$(78)\qquad \frac{1}{(m_1-2)!}\left(\frac{d^{m_1-2}}{d\gamma^{m_1-2}}\right)_{\gamma=0}\mathbf{SS}_\tau f V d\tau_\gamma,$$

which difficulty, however, will be easily overcome with the help of § 141. We must, however, resume the considerations of that section in a somewhat more complete form, as we investigate more precisely the expansion of the integral I_γ of § 140 in powers of γ.

As in § 141, we begin by considering the x's as being expressed

* A function which is holomorphic around every point of a continuous domain (boundary included) is uniformly holomorphic, i.e. its Taylor expansion admits of a fixed dominant throughout it, as is seen by a classic argument resting on Bolzano-Weierstrass' Lemma.

in terms of the a's and the normal variables ξ defined in § 172 with respect to a. We call

$$K\, d\xi_1 \dots d\xi_m = dx_1 \dots dx_m$$

the space element (K being a Jacobian).

Instead of the variables ξ, we can (at least in the neighbourhood of $\Gamma = 0$, which alone interests us) introduce the same angular variables $\phi_1, \phi_2, \dots, \varphi_{m-2}$ as in § 141, combined with $\theta = \xi_m$ and γ: expressing in terms of the latter two, the quantity

$$\rho = \sqrt{\xi_1^2 + \dots + \xi_{m-1}^2} = \sqrt{\theta^2 - \gamma},$$

we see that (same meaning as in § 141 for $d\Omega_{m-2}$)

$$d\xi_1 \dots d\xi_m = d\xi_1 \dots d\xi_{m-1}\, d\theta = \tfrac{1}{2}\rho^{m-3}\, d\Omega_{m-2}\, d\theta\, d\gamma$$
$$= \tfrac{1}{2}(\theta^2 - \gamma)^{\frac{m-3}{2}}\, d\Omega_{m-2}\, d\theta\, d\gamma.$$

This expression of the space element gives us a corresponding expression of the element $d\tau_\gamma$ on a surface $\gamma = \text{const.}$, viz.

$$d\tau_\gamma = \tfrac{1}{2}(\theta^2 - \gamma)^{\frac{m-3}{2}}\, d\Omega_{m-2}\, d\theta.$$

We see, thus, that the value of I_γ will be obtained by integrating with respect to the φ's (after multiplication by $d\Omega_{m-2}$), the simple integral

$$(78\ a) \qquad \tfrac{1}{2}\int K f V (\theta^2 - \gamma)^{\frac{m-3}{2}}\, d\theta.$$

In this integral, we shall take for the lower limit a small positive quantity ϵ (constant or variable with the a's and ϕ's), which we begin by leaving fixed, exactly as we did in § 141. The upper limit will be the value θ' of θ corresponding to the point where our line of integration (i.e., the section of τ by $\phi_1 = \text{const.}$, $\varphi_2 = \text{const.}, \dots, \phi_{m-1} = \text{const.}$) intersects S. Now, we can replace the x's by their expressions in terms of the ξ's and consequently in terms of the ϕ's, ρ and θ, and these expressions will be uniformly holomorphic around every point within $\mathscr{R}$.

Let us begin by effecting this substitution in the integrand: $F = K f V \rho^{m-3}$ (for any determinate choice of the ϕ's) will be written

$$F(\theta, \rho) = F(\theta, \sqrt{\theta^2 - \gamma}) = \Sigma \phi_q(\theta)\, (\theta^2 - \gamma)^{\frac{q+m-3}{2}}.$$

We shall have to expand this in powers of γ, only considering the terms whose indices have the values 0 to $m_1 - 2$. Now,

$$\rho^{q'} = (\theta^2 - \gamma)^{\frac{q'}{2}} = \theta^{q'} \left[1 - \frac{q'}{2} \frac{\gamma}{\theta^2} + \dots \right.$$
$$\left. + (-1)^h \frac{\frac{q'}{2}\left(\frac{q'}{2} - 1\right) \dots \left(\frac{q'}{2} - h + 1\right)}{h!} \frac{\gamma^h}{\theta^{2h}} + \dots \right],$$

in which we see that a denominator in θ only appears for $2h > q'$. Therefore (as $q' = q + 2m_1 - 3$), our integrand, or, more exactly, the part of it which is at most of degree $m_1 - 2$ in γ, is of the form

$$\theta^{2m_1 - 3} F_0(\theta) + \gamma \theta^{2m_1 - 5} F_1(\theta) + \dots$$
$$+ \gamma^h \theta^{2m_1 - 3 - 2h} F_h(\theta) + \dots + \gamma^{m_1 - 2} \theta F_{m_1 - 2}(\theta),$$

where the F_h's (which are finite combinations of the first $(m_1 - 2)$ partial derivatives of F with respect to ρ, for $\rho = \theta$: especially, $F_0(\theta) = F(\theta, \theta)$) are again uniformly analytic around any point inside or on the retrograde half conoid from any vertex a within $\mathcal{R}$. This, in the first place, shows that, in the integrand (78 a), the coefficients of the powers of γ from 0 to $m_1 - 2$ are limited in absolute value and even infinitesimal of the order of θ and, consequently, in the integral (78 a) taken from 0 to ϵ, the totality of the corresponding terms will be at least of the order of ϵ^2: which is equivalent to the result found in § 141.

175. On account of this result, we see that we obtain the required value by taking 0 for our lower limit. If we change the variable by setting down $\theta = s\theta'$ we find

$$(79) \qquad \frac{\theta'^2}{2} \int_0^1 \left[\sum_{h=0}^{m_1 - 2} \theta'^{2(m_1 - 2 - h)} s^{2m_1 - 3 - 2h} F_h(s\theta') \gamma^h \right] ds$$

for integral (78 a), reduced to its terms of degree at most equal to $(m_1 - 2)$ in γ.

Let us also expand in powers of γ the upper limit θ', determined by an equation of the form $S(\theta, \rho) = 0$, or (operating as above)

$$(80) \quad S(\theta, \sqrt{\theta^2 - \gamma}) = S_0(\theta) + \frac{\gamma}{v} S_1(\theta) + \dots + \frac{\gamma^h}{\theta^{2h-1}} S_h(\theta) + \dots = 0,$$

the S_h's in the left-hand side being uniformly holomorphic in the variables which they contain*.

When $\gamma = 0$, this reduces to

(80 *a*) $$S_0(\theta) = S(\theta, \theta) = 0,$$

the equation which determines the intersection of S with a bicharacteristic. For the value θ_0 of θ thus obtained, we know that $S_0'(\theta)$ is different from zero and even numerically greater than a fixed positive constant ς.

Now, let us expand θ' in powers of γ and see what denominators θ_0 it contains†. In equation (80) which defines θ', let us introduce a new independent variable g and a new unknown ϑ by

(81) $$\gamma = \theta_0^2 g, \quad \theta' = \theta_0(1 + \vartheta)$$

so that the expansion of ϑ

(81′) $$\vartheta = g\vartheta_1 + g^2\vartheta_2 + \dots$$

has its constant term equal to zero. In these new variables, equation (80), taking account of equation (80 *a*) for θ_0, becomes (one factor θ_0 disappearing)

$$\vartheta S_0'(\theta_0) + \frac{\vartheta^2}{2!}\theta_0 S_0''(\theta_0) + \dots + \frac{g}{1+\vartheta} S_1[\theta_0(1+\vartheta)]$$
$$+ \frac{g^2}{(1+\vartheta)^3} S_2[\theta_0(1+\vartheta)] + \dots + \frac{g^h}{(1+\vartheta)^{2h-1}} S_h[\theta_0(1+\vartheta)] + \dots = 0.$$

But such an equation gives for ϑ an expansion in powers of g, the coefficients ϑ_h of which obviously admit of fixed dominants, on account of $|S_0'(\theta_0)| > \varsigma$. Therefore, in the expansion of θ' in powers of γ, the term of degree h may contain the denominator θ_0, but with an index

* If S is $t=0$, the left-hand side of (80) is the expression of t in terms of the ϕ's, of θ and of γ.

The argument in the text applies to any analytic regular surface S, provided it meets every interior or bicharacteristic geodesic at a finite angle. Upper limits for the coefficients of the expansion (81′) depend on upper limits for the derivatives of S with respect to the x's and a lower limit for $|S_0'(\theta_0)|$.

† This discussion of small θ_0's seems not to be absolutely necessary, as we know, by Cauchy-Kowalewsky's theorem, under our general assumption of analyticity, of the existence of a certain limit T for $|t|$ such that, below this limit, u is certainly holomorphic, so that we could limit ourselves to $|t| > T$. But the behaviour of the term (78) in the neighbourhood of S will be wanted finally (see § 196).

not higher than $2h-1$. In other words, we can say that its degree of homogeneity, with respect to γ and θ_0^2, is $\geqslant \frac{1}{2}$.

If, finally, we carry such an expression of θ' into the integrand in (79), we see that the term in γ^{m_1-2} does not contain any denominator θ_0, but, on the contrary, contains θ_0^2 as a factor, the other factor being uniformly holomorphic in all the variables which it contains, viz. the ϕ's and s: especially, for any determinate system of values of the ϕ's and s, this factor is holomorphic with respect to the a's around any position of a in $\mathscr{R}$, and this uniformly whatever that position and the (real) values of $\phi_1, \ldots, \phi_{m-2}$ (between zero and π or zero and 2π) and s (between 0 and 1) may be.

This, integrating with respect to the ϕ's and s, gives the required proof of analyticity, with the determination of the order of magnitude of the term in question when θ_0 is small (i.e. when c is small).

176. Such a proof will allow us not to insist on the treatment of the other terms (terms with the element of integration $d\sigma_\gamma$) in (28), this treatment being obviously similar to the above, but easier, inasmuch as the question corresponding to § 141 or § 174 does not occur.

The only new question concerns the expression of $d\sigma_\gamma$. A simple way of obtaining it is, in the relation

$$dS = d\sigma_\gamma \, d\gamma,$$

to replace dS by dS_G, defined by means of the set of surfaces

$$S = G = \text{arbitr. const.}$$

which we can do if we simultaneously replace π_i by $\frac{\partial S}{\partial x_i}$. As we have written

$$dx_1 \, dx_2 \ldots dx_m = K \, d\xi_1 \ldots d\xi_{m-1} \, d\theta = \frac{K}{2} \rho^{m-3} \, d\Omega_{m-2} \, d\theta \, d\gamma,$$

we have (as is seen by considering the cylinder having for its bases two elements of two surfaces S and its lateral surface constituted by small arcs of lines $\phi = \text{const.}$, $\gamma = \text{const.}$)

$$dS_G = \frac{K}{2} \rho^{m-3} \frac{d\theta'}{dG} \, d\Omega_{m-2} \, d\gamma \tag{82}$$

and

$$d\sigma_\gamma = \frac{K}{2} \rho^{m-3} \frac{d\theta'}{dG} \, d\Omega_{m-2}, \tag{82'}$$

the derivative $\frac{d\theta'}{dG}$ being taken along one of the lines in question. As, by formulæ (81), (81′), θ' is expressed in terms of γ and θ_0 (when considering the ϕ's as constants), the coefficients being taken from the expansion of the function S, such a derivative, expanded in powers of γ, will be obtained by means of $\frac{d\theta_0}{dG}$ (the latter being itself deduced from $S_0(\theta_0) = G$), viz.

$$\frac{d\theta'}{dG} = \frac{d\theta_0}{dG}\left\{1 - \frac{\gamma}{\theta_0^{\,2}}\vartheta_1 + \ldots - \frac{(2h-1)\gamma^h}{\theta_0^{\,2h}}\vartheta_h - \ldots \right.$$
$$\left. + \theta_0\left(1 + \frac{\gamma}{\theta_0^{\,2}}\frac{\partial\vartheta_1}{\partial\theta_0} + \ldots + \frac{\gamma^h}{\theta_0^{\,2h}}\frac{\partial\vartheta_h}{\partial\theta_0} + \ldots\right)\right\}$$

(the $\frac{\partial\vartheta_h}{\partial\theta_0}$ being uniformly holomorphic in a, ϕ, γ, θ_0—as the ϑ_h themselves are—and, therefore, in a, ϕ and γ), so that the factor of $d\Omega_{m-2}$ in (82′) will be uniformly holomorphic and even infinitesimal of the first order for small θ_0's.

176 *a*. The treatment of the terms containing $\frac{d}{d\nu}$ is immediate if we take the formula under the form (28 *a*). We have only to imagine that the calculation is not only made concerning S, but also concerning the auxiliary surface S_ν: the result will be an analytic function, not only of the variables hitherto mentioned, but of ν, the differentiation with respect to which therefore gives no difficulty.

Under the special assumptions made in § 172 on the choice of our variables, the auxiliary surface S_ν will be $t = \text{const.}$*

177. We have proved, so far, the existence and analyticity of the solution u within any part $\mathscr{R}'$ of $\mathscr{R}$ limited in such a way that the retrograde half conoid from any point a within it is, together with $t = 0$, the boundary of a volume interior to the region of definition of $\mathscr{V}$, e.g., interior to the region of validity of the operations in § 62.

* The condition that the parameter ν be identical with the parameter G introduced in § 176 is easily seen to be that the surface $S=0$ satisfies the partial differential equation

$$\mathbf{A}\left(\frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_m}\right) = 1.$$

This will certainly be the case, on account of the assumptions made on the variable, if we introduce the limitation $|t| < T$, denoting by T a suitably chosen positive constant: the latter can indeed be fixed,—and this once for all throughout the whole region $\mathscr{R}$, in such a way that $|t-c| \leqslant T$ (together with $\Gamma > 0$) implies the inequality

$$(83) \qquad |\Gamma| < \frac{1}{\alpha'}\left(1-\frac{\sigma}{r}\right)^2$$

of § 63, and, therefore, the convergence of the series for $\mathcal{V}$.

Now, as we are given the values of u and $\frac{\partial u}{\partial t}$ for $t=0$, and these are analytic, we are able, by the above, to calculate the values of the same quantities for any t between 0 and T, the values corresponding to $t=T$ being again holomorphic in $x_1, x_2, \ldots, x_{m-1}$ around any point* of the plane $t=T$ included in $\mathscr{R}$. But such analytic values of u and $\frac{\partial u}{\partial t}$ allow us to set a new Cauchy problem, the data of which are borne by the plane $t=T$: the solution will be, by the above, defined and holomorphic at least until $t=2T$ (we mean the part of it which lies in $\mathscr{R}$); and going on in the same way, we shall be able to reach every plane $t=$ const. containing points of $\mathscr{R}$.

178. The above result, and the method used to prove it, obviously remind us of an analogous argument in the theory of ordinary differential equations and the corresponding conclusion, viz.: *the solutions of an analytic (ordinary) linear differential equation can admit of no other singularities than those of the coefficients themselves.*

One of the proofs for the latter theorem† precisely consists in observing that the radius of convergence of the expansion of any one of the solutions in question around any point (or at least, a lower limit for this radius) can be obtained without knowing *what* solution of the equation is meant. Similarly, here, we use the fact that we can tell a priori an interval of values of t over which we can extend the definition of our solution.

Such an analogy might lead us to think that the same result could

* The useful part of the half conoid from such a point is, as we have said, assumed to lie entirely inside $\mathscr{R}$.

† See, for instance, Jordan's *Cours d'Analyse*, Vol. III, 1887, § 92, p. 108.

be reached with the help of the original methods which have been applied in general to Cauchy's problem, i.e. the Cauchy-Kowalewsky classic argument. This, however, would be an error: in other terms, the radius of convergence of the expansion of the solution of Cauchy's problem (relating to $t = \text{const.}$) with respect to t, when obtained by the calculus of limits, must depend, not only on the expansions of the coefficients, but also on the radii of convergence of the expansions (with respect to the other variables $x_1, x_2, \ldots$) of the data u_0 and u_1.

For if it were not so, the conclusion would be common to hyperbolic equations and to elliptic ones (the former being even dominant of the latter in the classic mode of calculation for the proof of Cauchy's fundamental theorem). But such is not the case, as is shown, for Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, by the simplest examples, such as

$$u = \frac{1 - x}{(1 - x)^2 + y^2} \left(\text{the real part of } \frac{1}{1 - x - iy}\right),$$

the values of which, for $x = 0$ $\left(\text{viz. } \frac{1}{1 + y^2}\right)$ as well as the values of its derivative $\frac{\partial u}{\partial x}$ $\left(\text{viz. } \frac{1 - y^2}{(1 + y^2)^2}\right)$, are holomorphic for any real y and which, nevertheless, admits of the singularity $x = 1$, $y = 0$.

179. It is evident, on the other hand, that the working in § 173 could have been replaced by the same method which we have applied in §§ 174 ff., using exclusively normal variables instead of replacing one of them by t. We then see that the assumption of S being $t = 0$ is immaterial, as it is by no means implied in the argument of § 175.

180. We have thus proved that the solution of a Cauchy problem with analytic data certainly exists and is holomorphic throughout the whole of any region $\mathscr{R}$ satisfying our above assumptions. But can we say as much of the instrument which we have to use in order to find that solution, I mean our elementary solution? Does this quantity exist and is it holomorphic as long as: (1) the coefficients of the equation are themselves holomorphic, the discriminant of $\mathbf{A}$ being constantly different from zero; (2) the equations (29) of Book II, § 57, can be solved in a unique and continuous manner, their Jacobian being different from zero, and therefore the two points a and x can be joined together by a perfectly determinate geodesic, varying continuously in terms of the coordinates of these points?

In the first place, we can observe that these assumptions are sufficient for the construction of each of the successive coefficients U_h by the operations of Book II, § 62. Moreover, there is no difficulty in showing that these functions will be holomorphic throughout the region $\mathscr{R}$ (a fact which will appear presently).

U itself is deduced from the U_h's by means of the expansion

$$U = U_0 + U_1\Gamma + \dots + U_h\Gamma^h + \dots.$$

We are going to see that this expansion converges not only around a, as we had seen in Book II, but for any point in the region (γ) such that Γ is sufficiently small, i.e. in the neighbourhood of the whole of the characteristic conoid (or more exactly, of the part of it contained within $\mathscr{R}$).

For that purpose, i.e. in order to obtain upper limits for the $|U_h|$'s, let us resume our "Calculus of limits" of § 63, except that we apply it not only to expansions around a, but to expansions around any point inside $\mathscr{R}$.

As in § 63, we take normal variables relating to a (so that geodesics from that point are represented by straight lines), the sum of the absolute values being still denoted by σ; and we change the unknown so that the first term U_0 in our series is $U_0 = \frac{1}{\sqrt{|\Delta_a|}}$.

Moreover, in order to simplify notation*, we can admit that we have taken our variables so as to let one of the axes of coordinates—say the x_m-axis—pass through the point x' around which we intend to investigate our Taylor expansions: the variable x_m will be replaced by y, the value of which at x will be denoted by y', and we have to

* It would be quite easy to repeat the argument in the text without this particular choice of axes. We should expand our functions around

$$(x_1', x_2', \dots, x_m')$$

by setting down $$x_i = x_i' + X_i.$$

The coefficients in the right-hand sides of (84) and (85) (expanded in powers of the X's) would then be functions of $x_1', \dots, x_m'$, the latter being replaced by $sx_1', \dots, sx_m'$ in the integrand of (87), and σ' would be

$$|X_1| + |X_2| + \dots + |X_m|.$$

consider expansions in powers of $x_1, x_2, \ldots, x_{m-1}$ and $y - y' = Y$. For any coefficient A of the equation, this expansion will be

$$(84) \qquad A = \sum_{k_1, k_2, \ldots, k_m} A_{k_1 \ldots k_m}(y')\, x_1^{k_1} x_2^{k_2} \ldots x_{m-1}^{k_{m-1}} Y^{k_m}.$$

Assuming all the quantities A to be holomorphic—and therefore, uniformly holomorphic—throughout $\mathscr{R}$, we suppose therefore that all the expansions (84) admit of the common dominant of § 63

$$A \ll \frac{\alpha}{1 - \frac{\sigma'}{r}},$$

which will be independent of y' (i.e., of the position of x' on the y-axis), except that σ is replaced by $\sigma = |x_1| + \ldots + |x_{m-1}| + |Y|$. We deduce therefrom, as in § 63, that if

$$\frac{K_h}{\left(1 - \frac{\sigma'}{r}\right)^{2h}}$$

be a dominant of the expansion of U_h around x' (this dominant being again assumed to be independent of y), the expansion of $\mathscr{F}(U_h)$,—say (as the coefficients are again functions of y)

$$(85) \quad \mathscr{F}(U_h) = \psi^{(h)} = \Sigma\, \psi^{(h)}_{k_1, k_2, \ldots k_m}(y') \,.\, x_1^{k_1} x_2^{k_2} \ldots x_{m-1}^{k_{m-1}} Y^{k_m},$$

will admit of the dominant

$$(86) \qquad \frac{2h(2h+1)\alpha' K_h}{\left(1 - \frac{\sigma'}{r}\right)^{2h+3}}.$$

Then, we have to construct integral (44′) (with $U_0 = \text{const.}$). The path of integration is the straight line joining the origin to the point $(x_1, x_2, \ldots, x_{m-1}, y' + Y)$ so that we can represent the coordinates of an arbitrary point of it by $sx_1, sx_2, \ldots, s(y' + Y)$, where the parameter s varies from zero to the final value 1; s is precisely the variable of integration in (44′). For every value of s, the quantity (85) will be expanded by Taylor's formula with the initial point $(0, 0, \ldots, 0, sy')$, i.e., by replacing, in (85), $x_1, x_2, \ldots x_{m-1}, y, Y$, by $sx_1, sx_2, \ldots sx_{m-1}, sy', sY$. Integrating with respect to s, we find the required expansion of U_{h+1}, viz.

(87)

$$U_{h+1} = -\frac{\displaystyle\sum_{k_1, k_2, \ldots, k_m} \int_0^1 s^{h+k_1+\ldots+k_m}\, \psi^{(h)}_{k_1 \ldots k_m}(sy')\, x_1^{k_1} x_2^{k_2} \ldots x_{m-1}^{k_{m-1}} Y^{k_m} ds}{4(p+h+1)}$$

and we obtain a dominant for it if we replace every $\psi^{(h)}$ by its value taken from (86). As the latter is independent of y and may be taken outside $\int$, we get the same final dominant as in § 63 (except that σ is still replaced by σ'), and we see that the series for U converges whenever we have

$$(83\,a) \qquad |\Gamma| < \frac{1}{\alpha'}\left(1 - \frac{\sigma'}{r}\right)^2,$$

in which σ' can be* replaced by $\sqrt{m}D$, denoting by D the distance between x and x'.

We shall take the point x' on the conoid itself and let it assume successively every position on this conoid: by which a corresponding point x such that $\sqrt{m}D < \frac{1}{2}r$ can assume every position such that Γ is smaller than a suitably chosen positive constant γ (for all real points included between the two hyperquadrics $\Gamma = \gamma$ and $\Gamma = -\gamma$ in the finite region $\mathscr{R}$ are at a distance from the conoid less than $\frac{1}{2}r$ if γ' is small enough).

Therefore, U will certainly exist and be holomorphic whenever Γ is less than γ, the smaller of the two numbers γ' and $\dfrac{1}{4\alpha'}$.

181. If, now, we combine the above result with our previous method, we shall be able to extend the definition of U to the whole part of the region $\mathscr{R}$ (the latter still satisfying the same above assumptions) which lies inside the conoid, say the direct sheet of it. More exactly, we shall reach every point x such that the plane t=const. through it (t having the same meaning as before) includes with Γ a volume entirely interior to $\mathscr{R}$.

To this end, let us denote by T a positive number such that the operations of § 63 define U whenever simultaneously $\Gamma(x;\, a) \geqslant 0$ and the difference $|t - c|$ of the t's relating to x and a is smaller than T. On the other hand, let us notice that if we draw the retrograde half conoid from any point x' interior to Γ and such that $\Gamma > \dfrac{\gamma}{2}$, and cut it by the plane $t = t' - T$, the volume thus enclosed will lie entirely

* We limit ourselves hereby to the real domain. As to the advantage of introducing D instead of σ', it lies in the possibility of changing axes, as is necessary (on account of the rotations in the text) when we let the point x' vary.

inside $\Gamma=0$ if the positive number T is sufficiently small*. Let us take it so and also smaller than the numbers denoted by the same letter in § 177.

For $0 \leqslant t \leqslant T$ (and $\Gamma \geqslant 0$, which will be implicitly understood all through the following argument), U is defined by § 63, and is holomorphic.

For $T \leqslant t \leqslant 2T$, two cases may occur. Either the retrograde half conoid from x cuts the plane $t=T$ wholly inside Γ: then u and $\frac{\partial u}{\partial t}$ are known within the whole portion of S_0 of the plane $t=T$ thus obtained and, therefore, the value of u_x is known by our general formulæ solving Cauchy's problem, and is holomorphic, as explained above†. Or the retrograde bicharacteristics from x may intersect Γ before meeting $t=T$; but, by the definition of the quantity T, this can only happen if, at x, we have $\Gamma<\frac{\gamma}{2}$ and then, the value of U_x is defined by the operations of our preceding section and is holomorphic. Moreover, the two definitions are simultaneously valid throughout a certain region $\left(\frac{\gamma}{2}<\Gamma<\gamma\right)$ and both coincide with the analytic extension of the values already found for u. Therefore, we have a single analytic function for all the domain corresponding to $0 \leqslant t \leqslant 2T$ inside Γ.

It is clear that the same operations can be applied for $2T \leqslant t \leqslant 3T$; and so on. Thus our conclusion is completely proved. U is holomorphic in the x's and (for the same reasons as above) the a's.

182. Our method, both for u and for U, has consisted in calculating these quantities for remote points or, as we can say, "events," by using intermediate ones sufficiently near to each other. We can say, therefore, that it is an illustration of what we have called Huygens' major premise.

A thorough investigation of the consequences of such a principle

* If, as allowed, we suppose Γ to be $t^2-x_1{}^2-x_2{}^2-\ldots$, the condition for T will be $T<\gamma : 4 \max. (t+\sqrt{x_1{}^2+\ldots+x^2{}_{m-1}})$.

† We have written our inequalities so that S_0 is always *strictly* interior to Γ in this first case, on account of which not only U, but u is necessarily holomorphic throughout S_0.

(which would lead however to more extensive researches which I hope to resume later on) would give us further extensions of our results. We could indeed, in that way, recognize the remarkable fact that the seemingly fundamental condition that any two points x and a within $\mathscr{R}$ can be joined to each other by a geodesic in a unique way is *not* necessary.

We, for the present, content ourselves with noticing one point. Though the region $\mathscr{R}$ be not assumed to satisfy the condition in question, let us assume, nevertheless (for an even m), that the functions V and $\mathfrak{V}$ exist and are analytic throughout it. Then, even if the point a be chosen so distant from S that, inside the domain between Γ and S, the solution of the first set of equations (29), § 57, would cease to be possible in an unique way (the Jacobian (30) in the same section vanishing, for instance, within that domain or even on S_0), *all the integrals on the right-hand sides of* (28) *or* (28 a), § 145, *could still be defined*. The only necessary condition for that is that every geodesic from a interior to the conoid or belonging to it must still cut S at a determinate point and a finite angle.

It will be sufficient, in order to define the integrals in question, to express them all with the help of the normal variables ξ corresponding to a. It is clear, in the first place, that our new assumptions do not prevent the integral (79), finally obtained in § 175, from having a meaning, the x's being still holomorphic functions of the ξ's (no matter if the converse is true or not) and from being a holomorphic function of the a s.

The same can be said of $\mathsf{SSS}\,\mathfrak{V} f\, dT$ if we express it as said in § 179, viz.

$$\mathsf{SSS}\, K\mathfrak{V} f\, d\xi_1 \dots d\xi_m = \mathsf{SS}\ \theta_1{}^m\, d\eta_1 \dots d\eta_{m-1} \int_0^1 K\mathfrak{V} f\, s^{m-1}\, ds,$$

where the η's are defined as in § 172 and θ_1, an analytic function of the a s and the η's, corresponds to the intersection of any (interior) geodesic from a with S.

The same will also be obtained for the integrals relating to S (element dS or $d\sigma_\gamma$) by writing down the values of $\pi_i dS$ as said in § 176 [formulæ (82), (82′)].

The total quantity (28 a) thus defined is a holomorphic function of the a's. In the partial region $\mathscr{R}_0$ where the ξ's are uniform functions of the x's and the a's, it is proved by our preceding operations to satisfy (E). But values of the same quantity outside $\mathscr{R}_0$ (though inside $\mathscr{R}$) are the analytic extension of values within $\mathscr{R}_0$. Therefore, *they will also satisfy* (E) and represent the solution of our problem.

183. The non-analytic case. Let us now assume our coefficients to be no longer analytic; they will, however, be assumed to be regular, i.e. to admit of derivatives, up to a certain sufficiently high order, with respect to the xs. Indeed, we know from the properties of Tedone's solutions (Book III) that such a hypothesis rests on the nature of things. As has been said in Book I, the precise determination of the order of differentiability postulated will not be undertaken: it will be sufficient for us to make sure that such an order exists for every value of m.

We again take m even $=2m_1$,

$$\text{(E)} \qquad \mathscr{F}(u) = \Sigma A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \Sigma B_i \frac{\partial u}{\partial x_i} + Cu = f$$

being again the given equation, and

$$(\mathscr{E}) \qquad \mathscr{G}(v) = 0$$

its adjoint. Simultaneously, as before, we consider the equation in $2m_1 + 1$ variables

$$\text{(E}'\text{)} \qquad \mathscr{F}'(u) = \mathscr{F}(u) - \frac{\partial^2 u}{\partial z^2} = f,$$

the adjoint of which is

$$(\mathscr{E}') \qquad \mathscr{G}'(v) = \mathscr{G}(v) - \frac{\partial^2 v}{\partial z^2} = 0.$$

Our coefficients will be assumed to admit of partial derivatives at least up to a certain order, which can be expressed by saying that up to infinitesimals of that order they resemble analytic functions. The existence of derivatives of the first few orders is obviously sufficient for us to be able to carry out the first part of our operations, that is the construction of the quantity Γ; and the existence of derivatives up to a certain order for the coefficients will imply the existence of partial derivatives up to a certain corresponding order for the quantity Γ (see Additional Note to Book II).

Let us now come to the construction of the successive quantities V_h (or V_h') as explained in our second Book. Though we cannot go on with it indefinitely, we are evidently able to calculate a certain number of these quantities V_h: the more if the existence of more derivatives of the coefficients (and consequently of Γ) is postulated.

We shall assume that this is possible up to the (m_1-1)th power of Γ or $\Gamma' = \Gamma - (z-c)^2$ (the operations concerning both cases being, as noticed in our second Book, the same but for numerical coefficients) so that we have, for equation ($\mathcal{E}'$), the expansion

$$[v'] = \frac{1}{\Gamma'^{m_1-\frac{1}{2}}} \sum_{h=0}^{m_1-1} V_h' \Gamma'^h,$$

which is identical with the expansion of v' given in Book II, § 62, but for the fact of being limited. The corresponding terms relating to ($\mathcal{E}$) will consist of:

(1) The quantity V, with respect to which our previous calculations of Book II (or this Book, § 135) want no modification, viz.

$$V = C_{m_1-1} \sum_{h=0}^{m_1-2} \frac{1}{C_{m_1-h+1}} V_h' \Gamma^h;$$

(2) The *first* term $\mathcal{V}_{(0)}$ of the expansion of the quantity which we have previously called $\mathcal{V}$.

Now, $[v]$ will *not* be a solution of the adjoint equation ($\mathcal{E}'$), but the operations in our Book II show us that we have

$$\text{(88)} \qquad \mathcal{G}'([v']) = (-1)^{m_1} \Omega_{m-2} \frac{\psi}{\sqrt{\Gamma'}},$$

ψ being a quantity* which is finite, continuous, and even (if further derivatives of our coefficients are postulated) differentiable. Moreover, this quantity ψ is independent of z. The coefficient $(-1)^{m_1} \Omega_{m-2}$, introduced in order to simplify further operations, is the same which stands on the left-hand side in (7) (§ 135), except for the factor π.

184. We shall now look at our problem from Hilbert's point of view: that is, we shall see whether we can solve our Cauchy problem if we no longer have at our disposal the true elementary solution, but only this quantity $[v']$, an incomplete elementary solution or "parametrix" in Hilbert's sense. Therefore, we shall again take our Cauchy problem for (E) and the corresponding equivalent problem for (E');

* $(-1)^{m_1} \Omega_{m-2} \psi = 2\mathcal{G}(V'_{m_1-1})$.

but, for the latter, we shall again write down our fundamental formula, in which, however, v' will be replaced by the above obtained parametrix $[v']$.

The modification which we have to make in our formula is obvious: it consists in taking account of the values (88) by the addition of a supplementary space integral

$$\mathbf{SSS}\int \frac{u\mathscr{G}([v'])}{\sqrt{\Gamma'}}\,dx_1\,dx_2\ldots dx_m\,dz$$
$$= (-1)^{m_1}\,\Omega_{m-2}\,\mathbf{SSS}\int \frac{u\psi}{\sqrt{\Gamma'}}\,dx_1\ldots dx_m\,dz$$

(the use of our symbol $|\overline{}$ not being necessary here).

Using now the same process as before to descend again to our $2m_1$-dimensional space, we obtain $\left(\text{as } \int_{-\sqrt{\Gamma}}^{+\sqrt{\Gamma}} \frac{dz}{\sqrt{\Gamma'}} = \int_{-\sqrt{\Gamma}}^{+\sqrt{\Gamma}} \frac{dz}{\sqrt{\Gamma - z^2}} = \pi\right)$

(89) $$u = H + \mathbf{SSS}_{(a)}u\psi\,dx_1\,dx_2\ldots dx_m$$

where, for brevity's sake, we have written H for the quantity which is given by

(90)
$$\frac{2(-1)^{m_1}\pi^{m_1-1}}{(m_1-2)!}H_a$$
$$= -\mathbf{SSS}_{(a)}\,\mathscr{V}_{(0)}f\,dx_1\,dx_2\ldots dx_m - \mathbf{SS}_{S_0}\,\mathscr{V}_{(0)}(u_1 + Lu_0)\,dS$$
$$+\frac{1}{(m_1-2)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}[\mathbf{SSS}_2 fV\,dx_1\,dx_2\ldots dx_m + \mathbf{SS}_2\,V(u_1+Lu_0)\,dS]$$
$$-\frac{1}{(m_1-2)!}\frac{d}{d\nu}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\mathbf{SS}_2 u_0 V\,dS + \frac{d}{d\nu}\mathbf{SS}_{S_\nu} u_0\mathscr{V}_{(0)}\,dS,$$

i.e., by formula (30) (which, this time, we prefer to (28)), except that $\mathscr{V}$ is replaced by its first term $\mathscr{V}_{(0)}$.

In both above formulæ, the points a, x are assumed to lie on the same side of S, which we shall call the "positive side"; and (a), for instance, as a suffix to the $\mathbf{SSS}$, stands for the domain which we previously denoted by T, viz., the domain enclosed, on that positive side of S, by the retrograde characteristic half conoid from a.

185. We have to determine u by means of the above equation (89). It obviously belongs to the well-known type of *integral equations of the second kind*. The *kernel*—viz. ψ—is a finite one, so that the solution is given without any difficulty by the classic methods*. Even, in this case, *we have to deal with the Volterra type,* on account of the manner in which the domain of integration depends on $(a_1, a_2, \ldots, a_m)$ and approaches zero when the latter point approaches S, and we can solve it without needing to have recourse to Fredholm's algorithm: the required quantity u will be given by the successive approximations

$$(91) \quad \begin{cases} u^{(0)} = H, \\ u_a^{(1)} = H_a + \mathbf{SSS}_{(a)} u_x^{(0)} \psi(x; a)\, dx_1\, dx_2 \ldots dx_m, \\ u_a^{(2)} = H_a + \mathbf{SSS}_{(a)} u_x^{(1)} \psi(x; a)\, dx_1\, dx_2 \ldots dx_m, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ u_a^{(n)} = H_a + \mathbf{SSS}_{(a)} u_x^{(n-1)} \psi(x; a)\, dx_1\, dx_2 \ldots dx_m, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

u being equal to $\lim_{n=\infty} u^{(n)}$.

That these approximations converge in the same way as in Volterra's case, corresponds to the way in which the shape of the domain of integration on the right-hand side depends on the position of the point $(a_1, a_2, \ldots, a_m)$. Let us take again the coordinate t by considering a one-parameter family of surfaces S_t, such that S_0 coincides with the given S and that every S_t is duly inclined with respect to the characteristic conoids and, therefore, cuts any one of them along closed edges. On any such surface, let us take for the element dS_t the quotient of the m-dimensional space element by dt, so that

$$dS_t\, dt = dT = dx_1\, dx_2 \ldots dx_m.$$

Let us, moreover, denote by K' a maximum of the ($(m-1)$-fold) integral $\mathbf{SS}\,|\psi(x; a)|\, dS_t$, extended over the section of a retrograde half conoid (having for its vertex any point a inside $\mathscr{R}$) by any surface S_t. Then if the function $\phi(x_1, x_2, \ldots)$ is given and its absolute value is, on each S_t, less than $\Phi(t)$, the absolute value of the integral

$$\mathbf{SSS}_I\, \phi(x)\, \psi(x; a)\, dx_1 \ldots dx_m$$

inside a retrograde half conoid with vertex a (limited to S) will admit of the limitation

$$(92) \quad \left| \mathbf{SSS}_T\, \phi(x)\, \psi(x; a)\, dx_1 \ldots dx_m \right| < K' \int_0^c \Phi(t)\, dt,$$

c denoting the value of t at a.

* See, e.g. Bôcher's *Introduction to the study of Integral Equations*, Cambridge University Press, 1909.

This inequality precisely yields the required convergence just as in Volterra's method and in Picard's method of successive approximations for ordinary differential equations. It shows us that if

$$|u^{(0)}| = |H| < H',$$

H' being a positive constant, then

$$(92') \qquad |u^{(n)} - u^{(n-1)}| < H' \frac{(K'c)^n}{n!}$$

which is the general term of a convergent series.

As follows from the summation of the series $u_0 + \sum_1^\infty (u^{(n)} - u^{(n-1)})$ and as is well known from the theory of integral equations, the solution u finally obtained is of the form

$$(93) \qquad u_a = H_a - \mathbf{SSS}_{(a)} \Psi(x; a) H_x dT_x,$$

where dT_x is an abbreviation for $dx_1 dx_2 \ldots dx_m$ and

$$\Psi(x; a) = \Psi(x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_m)$$

is a determinate function of the x's and a's, the so-called "reciprocal kernel"* of our integral equation, the value of which depends solely on the expression of ψ itself.

* See Bôcher, *loc. cit.*, § 6.

The calculation of Ψ with the help of ψ is given by the ordinary method in the theory of integral equations (Bocher, *loc. cit.*). Having constructed our first two approximations $u^{(0)}$ and $u^{(1)}$, we find that the third one $u^{(2)}$ is expressed in terms of $u^{(1)}$. In order to obtain it in terms of $u^{(0)} - H$, we replace $u^{(1)}$ itself by its expression. Operating as explained below in the text, we see that

$$u_a^{(2)} = H_a + \mathbf{SSS}_{(a)} [\psi_1(x; a) + \psi_2(x; a)] H_x dT_{x-1},$$

where $\psi_1 = \psi$ and ψ_2 is represented by an integral over the domain which we call $(a \between x)$ (see the text), viz.

$$\psi_2(x; a) = \mathbf{SSS}_{(a \between x)} \psi(x; a') \psi(a'; a) dT_{a'}.$$

Going on in the same way, we see that $u_a = \lim u_a^{(n)}$ is represented by (93), with

$$-\Psi(x; a) = \psi_1 + \psi_2 + \ldots + \psi_n + \ldots,$$

the terms ψ_n being the "iterated kernels," such that $\psi_1 = \psi$ and

$$\psi_n(x; a) = \mathbf{SSS}_{(a \between x)} \psi_{n-1}(x; a') \psi(a'; a) dT_{a'};$$

this series for $(-\Psi)$, corresponding to Bôcher's series (5) (*loc. cit.*, § 6), converges for reasons similar to those for the series for u (cf. Bocher, *loc. cit.*, § 6, Theorem 3, p. 23).

186. Let us now see how this solution depends, not on the form of H, but on the data themselves. Let us begin by the terms in f: in the expression of H, we have the first term

$$\text{(94)} \qquad \mathbf{SSS}\,\mathcal{V}_{(0)}(x;\,a) f(x)\, dT_x$$

in which $\mathcal{V}_{(0)}$ is a function of both the x's and the a's. Substituting that quantity in (93) and replacing x by a' as the variable of integration when necessary, we get

$$\mathbf{SSS}\,\mathcal{V}_{(0)} f(x)\, dT_x - \mathbf{SSS}\,\Psi(a';\,a)\,\mathbf{SSS}\,\mathcal{V}_{(0)}(x;\,a') f(x)\, dT_x dT_{a'}.$$

The second term is a $2m = (4m_1)$-fold integral which relates to all systems of positions of our two points a and x such that:

The point a lies between S and the retrogade half conoid with vertex a;

The point x again lies between S and the retrograde half conoid with vertex a'.

We can invert integrations, that is, we shall integrate by letting first the point x be fixed and a variable: which, f being a factor, gives, for the other factor, the quantity

$$\text{(95)} \qquad \mathcal{V}_{(1)}(x;\,a) = \mathbf{SSS}_{(a\mathfrak{X}x)}\,\Psi(a';\,a)\,\mathcal{V}_{(0)}(x;\,a')\, dT_{a'},$$

the domain of variability of a', which is denoted by $\mathfrak{X}$,—that is, the domain of integration (fig. 32) in this formula,—being included

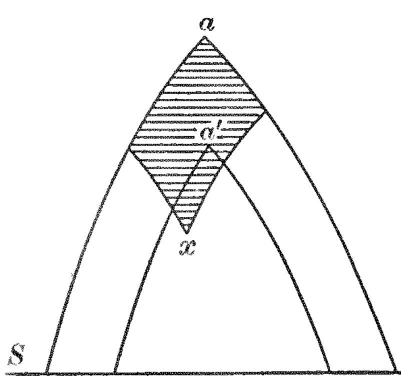

Fig. 32.

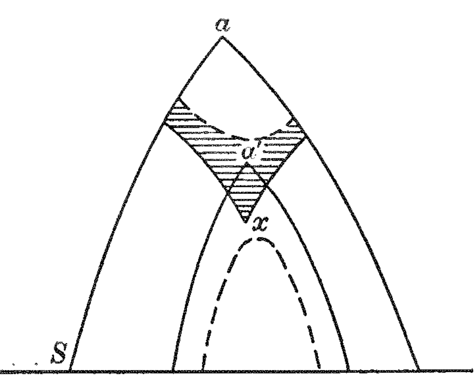

Fig. 32*a*.

between the retrograde half conoid with vertex a (which we have been considering) and the *direct* half conoid (conoidal sheet not turned towards S) with vertex x. The term in question is thus

$$\text{(94 } a\text{)} \qquad \mathbf{SSS}_{(a)}\,\mathcal{V}_{(1)}(x;\,a) f(x)\, dT_x.$$

We now take the other term containing f in the expression of H, viz.

$$(96) \qquad \frac{1}{(m_1-2)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}} \mathbf{SSS}_{(a)_2} \mathcal{V} f\, dT_x$$

the integration $\mathbf{SSS}$ being extended within the space $(a)_2$ included on the positive side of S, between the retrograde half conoid having a for its vertex and the surface $\Gamma(x; a) = \gamma$. We have to substitute this for H in the right-hand side of (93), which gives

$$(97) \quad \frac{1}{(m_1-2)!}\Bigg[\frac{d^{m_1-1}}{d\gamma^{m_1-1}} \mathbf{SSS}_{(a)_2} \mathcal{V} f\, dT_x$$
$$- \mathbf{SSS}_{(a)} \Psi(a'; a)\, dT_{a'} \frac{d^{m_1-1}}{d\gamma^{m_1-1}} \mathbf{SSS}_{(a')_2} V(x; a') f(x)\, dT_x\Bigg].$$

This, on account of the ordinary rules of differentiation under the integration sign*, may be replaced by

$$\frac{1}{(m_1-2)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}} [\mathbf{SSS}_{(a)_2} V f\, dT_x$$
$$- \mathbf{SSS}_{(a)} \Psi(a'; a)\, dT_{a'}\, \mathbf{SSS}_{(a')_2} V(x; a') f(x)\, dT_x].$$

The double $\mathbf{SSS}$, i.e. $4m_1$-fold integral, shall be transformed, as before, into

$$(97') \quad \mathbf{SSS}_{(a)} f(x)\, dT_x \cdot \mathbf{SSS}_{(a \chi x)_2} \Psi(a'; a)\, V(x; a')\, dT_{a'}.$$

In (97), the point a lies anywhere on the positive side of S inside the half conoid with vertex a, and the point x lies on the positive side of S, between the retrograde half conoid $\Gamma(x; a') = 0$ and the surface $\Gamma(x; a) = \gamma$ (a region such that $0 \leqslant \Gamma(x; a') \leqslant \gamma$; see fig. 32 a)†. Therefore, in (97'), the point x will lie anywhere in a and, for each given position of x, the field of integration for a will be bounded by

* Rigorously speaking, we ought, in the first place, as remarked above, to exclude, before differentiation, the vertices of our conoids by small surfaces Σ, e.g. by requiring x and a' to be distant from each other at least by ϵ. For such restricted domains, the operations of differentiation under $\mathbf{SSS}$ in the text would be valid. It is easy to see, by our previous considerations (including footnote to § 147), that they are also valid by making ϵ immediately 0 (the convergence of the operations in § 141 or § 174 being uniform).

† The (diagrammatic) figs. 32, 32 a are two-dimensional sections of the diagrams in the $2m_1$-dimensional space.

the retrograde half conoid with vertex a, the direct half conoid with vertex x and the surface $\Gamma(x; a') = \gamma$ (fig. 32 a). This field is what we call $(a \gamma x)_2$.

The $(m_1 - 1)$-fold differentiation of (97′) with respect to γ can be carried out under the first **SSS**, i.e., on the value of the $\mathbf{SSS}_{(a\gamma x)_2}$; and thus we see that the result of the substitution of (96) in our Volterra-like resolving formula (93) is equal to the term (96), diminished by

(94 b) $$\mathbf{SSS}_{(a)} f(x) \mathcal{V}_{(II)}(x; a)\, dT_x,$$

with

(95 a)

$$\mathcal{V}_{(II)}(x; a) = \frac{1}{(m_1 - 2)!} \frac{d^{m_1 - 1}}{d\gamma^{m_1 - 1}} \mathbf{SSS}_{(a\gamma x)_2} \Psi(a'; a)\, V(x; a')\, dT_{a'}.$$

Finally, we have now every term depending on f. If we set down

(98) $$\mathcal{V} = \mathcal{V}_{(0)} - \mathcal{V}_{(I)} + \mathcal{V}_{(II)},$$

$\mathcal{V}_{(I)}$, $\mathcal{V}_{(II)}$ being defined by (95), (95 a) respectively—the totality of these terms will be

$$-\mathbf{SSS} f(x) \mathcal{V}(x; a)\, dT_x + \frac{1}{(m_1 - 2)!} \frac{d^{m_1 - 1}}{d\gamma^{m_1 - 1}} \mathbf{SSS} f(x)\, V(x; a)\, dT_x.$$

187. The above explanations of the treatment of the terms in f will allow us to deal more briefly with the other terms (containing u_0 and u_1) as the operations will be exactly similar. The integral with respect to x will be an **SS** (instead of an **SSS**), the point x describing S; but the relation between this point and a', as well as between a and a', remaining the same for any given x on S, the fields of integration relating to a' have to be constructed as before. Similar operations can also be performed when the point x is required to describe the surface which we have called S_ν, so as to obtain a result which is to be differentiated with respect to ν. This shows, without any new difficulty, what each term of the aforesaid kind in H gives when substituted in (93): we thus find

for the term $-\mathbf{SS}_{S_0} \mathcal{V}_{(0)}(u_1 + Lu_0)\, dS$:

$$-\mathbf{SS}_{S_0} \mathcal{V}_{(0)}(u_1 + Lu_0)\, dS + \mathbf{SS}_{S_0} \mathcal{V}_{(I)}(u_1 + Lu_0)\, dS;$$

for $\frac{1}{(m_1-2)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\mathbf{SS}_2 V(u_1+Lu_0)\,dS$:

$$\frac{1}{(m_1-2)!}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\mathbf{SS}_2 V(u_1+Lu_0)\,dS - \mathbf{SS}_{S_0}\mathcal{V}_{(\mathrm{II})}(u_1+Lu_0)\,dS;$$

for $\frac{1}{(m_1-2)!}\frac{d}{d\nu}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\mathbf{SS}_2 u_0 V\,dS$:

$$-\frac{1}{(m_1-2)!}\frac{d}{d\nu}\frac{d^{m_1-1}}{d\gamma^{m_1-1}}\mathbf{SS}_2 u_0 V\,dS + \frac{d}{d\nu}\mathbf{SS}_{S_0}u_0\mathcal{V}_{(\mathrm{II})}\,dS;$$

for $\frac{d}{d\nu}\mathbf{SS}_{S_\nu}u_0\mathcal{V}_{(0)}\,dS$:

$$\frac{d}{d\nu}\mathbf{SS}_{S_\nu}u_0\mathcal{V}_{(0)}\,dS - \frac{d}{d\nu}\mathbf{SS}_{S_\nu}u_0\mathcal{V}_{(\mathrm{I})}\,dS,$$

the sum of which, added to (94), (94 *a*) with subtraction of (94 *b*), finally gives u itself under the form (30), V being calculated as said in § 183 and $\mathcal{V}$ being given by (98).

188. We have just constructed the expression of the required solution, if it exists. But we have to prove, conversely, that what we have found is a solution, satisfying the conditions of the problem.

To do this, we use the results obtained in the first place for the analytic case. In this case, we have shown that the solution exists and is analytic throughout $\mathcal{R}$: of course, this solution can not be distinct from the one which we have just obtained and which is therefore itself analytic.

Moreover, we have constructed the elementary solution and especially the function $\mathcal{V}$ (also analytic in terms of the x's and a s). Again, such a quantity *can not be distinct* from the quantity defined by (98): for, on account of the fundamental Lemma of the Calculus of Variations, the same quantity u can not admit of two distinct expressions of the form (30), equal to each other for arbitrary choices of u_0, u_1, f.

Therefore, the quantity $\mathcal{V}$ defined by (98) is, under our present assumptions, holomorphic in the z's and the a's.

189. Could we prove, in a direct way, that the operations in §§ 184—187, starting from analytic data, necessarily lead to analytic results*?

This can be done without any great difficulty as to the first part of them, the construction of u. In the first place, formula (90), which gives the value of the quantity H, is entirely similar to our previous formulæ which gave u itself, except that it contains $\mathcal{V}_{(0)}$ instead of $\mathcal{V}$. We therefore can prove the analyticity of H by the same methods which applied in §§ 173—176 a; but, this time, these methods will be valid throughout $\mathcal{R}$, as $\mathcal{V}_{(0)}$ is defined and holomorphic therein, so that the analyticity of H is proved at once in the whole of this region.

We have now to show the same for the solution of the integral equation (89). To this end, we shall resume the method (§ 173) which we have applied to the space integral (77), but with the modifications necessary to extend them to a suitably defined complex domain.

We start from our real region $\mathcal{R}$, subject to the same restrictions as above (especially, it is understood that any interior (or bicharacteristic) geodesic described in the retrograde sense from a point a in $\mathcal{R}$ remains in $\mathcal{R}$ till it reaches S).

Let x and a be two points in $\mathcal{R}$, which, therefore, as assumed, can be joined by a determinate geodesic; the Jacobian $\frac{D(x_1, x_2, \ldots, x_m)}{D(q_1, \ldots, q_m)}$ being also assumed to be always different from zero (for the corresponding values of the a's and the q's), the q's (and therefore the ξ's) *will be analytic functions*, and they will be holomorphic for any system of real or imaginary values x, a of the same variables such that

$$|\bar{x}_i - x_i| < \delta, \quad |\bar{a}_i - a_i| < \delta, \qquad (i = 1, 2, \ldots, m)$$

δ being a certain positive quantity, which, as is well known, will have a positive minimum when x and a assume all possible positions within $\mathcal{R}$. If one of the $\mathcal{R}$ points is required to describe a surface $t = \text{const.}$, while the other remains arbitrary, there will be another minimum, which will be a function of t and which we shall denote by δ_t. We thus can deduce from $\mathcal{R}$ a certain complex domain (δ_t), viz. the domain containing every point with (real or imaginary) coordinates $x_1, \ldots, x_m$, connected with, at least, one real point $(x_1, \ldots, x_m)$ in $\mathcal{R}$ by

$$(99) \qquad |\bar{x}_i - x_i| < \delta_t \qquad (i = 1, 2, \ldots, m)$$

(t still standing for x_m).

From any (in general, imaginary) point in (δ_t), we can draw geodesics in various directions. We shall especially consider those which are such that

$$|\bar{\mu} - \mu| < k\delta_t, \qquad (k, \text{ positive constant})$$

* The method given below is the one which corresponds to E. E. Levi's (*loc. cit.*); but the proof of E. E. Levi, for the elliptic case, is more complete, applying not only to u, but also to the elementary solution and even to the reciprocal kernel Ψ which generates it.

$\bar{\mu}$ being the value of any of the quantities $\frac{dx_1}{dt}, \ldots, \frac{dx_{m-1}}{dt}, \frac{ds}{dt}$ and μ the corresponding value for a (suitably chosen) interior and retrograde direction at a neighbouring real point (the neighbourhood being defined by (99)) belonging to $\mathscr{R}$. Such directions will also be called "retrograde directions in $\mathscr{R}$." It is clear, by the above remarks on $\frac{d\theta}{dt}$, that, for any such direction, the argument of $\left(-\frac{d\xi_m}{dt}\right)$ will be numerically less than k' (k' being a certain positive constant). We also see that if k be taken sufficiently great we shall always obtain a retrograde direction (at any point of (δ_t)) if we take for the differentials of the normal variables ξ real values such that

$$d\xi_1^2 + d\xi_2^2 + \ldots + d\xi^2_{m-1} \leqslant d\xi_m^2, \quad d\xi_m > 0.$$

Geodesics having a retrograde direction (whether real or imaginary) will be called "retrograde geodesics"; then, moreover, we shall agree that the independent variable* t shall always vary in such a way that the argument of dt will itself lie between $-k'$ and k': of course, this will also be the case for the argument of the difference between any two values of t on one such path, and we can find a (constant) upper limit for the ratio between the length of any arc of such a path in the plane of the complex variable t and the length of its chord, or of its projection on the real axis.

Any point which can be reached from a by a retrograde geodesic, with the above restriction for the variation of t, will be said to be *subordinate* to a.

190. It will be essential for us to modify our previous definition of (δ_t) (by suitably diminishing the values of δ_t, as allowed) in order that, *taking for* a *a point in* (δ_t), *all the points subordinate to* a *also belong to* (δ_t).

We can reach this by the help of known results connected with Cauchy's fundamental theorem on differential equations. We indeed know that, $(y_1, y_2, \ldots)$ and $(y_1, \bar{y}_2, \ldots)$ being two solutions of one and the same canonical differential system with the independent variable t and N unknown functions, if we have an upper limit ϵ_0 of the absolute values of the differences $y_i - y_i$ for $t = t_0$, we can deduce a similar upper limit ϵ_1 relating to $t = t_1$ by

$$\epsilon_1 \leqslant \epsilon_0 e^{NAT},$$

where T is the length of the path followed from t_0 to t_1 and A a positive constant, which we can find when we know a finite region where all the unknowns lie and where the right-hand sides of the differential equations are regular†.

* We take t for the independent variable in (L) by multiplying both sides of every equation by $\frac{ds}{dt} = 1 : \left(\frac{s}{2}\frac{\partial \mathbf{A}}{\partial p_m}\right)$.

† A is an upper limit of absolute values of the first partial derivatives of the right-hand sides (compare footnote to the Additional Note to Book II).

On account of this theorem, we see that the domain (δ_t) will satisfy the required condition if δ_t is chosen in terms of t so that $\delta_t e^{2mAat}$ be decreasing (denoting by a the aforesaid upper limit for the ratio between the length of a path of integration in the t-plane and its chord): for instance, δ_t' being a first choice of that function, such as considered hitherto, we denote by δ_t the minimum of $\delta'_{t'} e^{-2mAa(t-t')}$ for t' varying between zero and t.

190 *a*. A last limitation in our domain (δ_t) will be introduced by only considering values of t, the arguments of which lie between $-k'$ and $+k'$ (the definition of the domain, as concerns the other coordinates, remaining as above). A "subordinate point" of a point of the domain, with that new definition, will continue to lie in this domain if the two values of t lie on the same radius vector through the origin in the t-plane.

191. These geometric considerations are the only difficulty in the argument. Introducing any function F of the x's, or the x's and the a's, holomorphic throughout $\mathscr{R}$, we can now easily form an analytic function of the a's, also holomorphic throughout (δ_t), which will coincide with the given one for real points (i.e. within $\mathscr{R}$).

We see that such an extension will be given by the integral

$$(77b) \qquad I = c\,\mathbf{SS}\,d\eta_1 \ldots d\eta_{m-1} \int_0^1 K_1 F d\lambda$$

constructed in § 173, the η's still describing the real domain (76) and the variable λ the real segment (0, 1), so that the variable t has to go from c to the origin by the rectilinear path. By §§ 190 and 190*a*, if a lies within (δ_t), so will also every point x corresponding to such a system of values of the η's and λ's. Thus,—which is essential for us—, F being defined in the domain (δ_t), (77 *b*) is defined in the *same* domain and is holomorphic therein, for the same reasons as above.

Moreover, if we denote by $\phi(|t|)$ a maximum of the absolute value of F when t assumes all the values such that $|t|=$const., and by K' the product of $\frac{1}{m-1}\Omega_{m-2}$ by a maximum of K_1, we have

$$(92a) \qquad |I| < K' \int_0^{|c|} \phi\,|t|\,dt,$$

i.e. the inequality corresponding to (92), § 185.

192. This being obtained, there is no longer any difficulty in defining u within the whole of (δ_t), by equation (89). First, H itself, on account of the calculations in Book II, is holomorphic in the aforesaid domain (suitably restricted, if necessary). Therefore, we shall see successively that each of the integrals (91) is defined and holomorphic throughout (δ_t); further on, by the same argument as in § 185 (with the use of inequality (92*a*)), that such approximations converge.

Moreover, this convergence is uniform and, therefore, by a known theorem, the limit is again an analytic function, as we wanted to prove.

In order to afford the equivalent of E. E. Levi's proof, we ought to show, by the same method, the analyticity of $\mathfrak{V}$ itself. We shall not, however, undertake this, new difficulties (which do not exist for the elliptic case) arising obviously from the shape of the domain which we have called $(a \between x)$, if we should try to extend it by adjunction of complex points.

193. This being said, we again cease to assume that our coefficients are analytic (their regularity being, of course, still understood).

On account of the assumed regularity of the coefficients and of a known fundamental theorem of Weierstrass', we can approximate each of them as closely as we wish by a polynomial, and we may even do this in such a way that the approximation holds on differentiation up to the order for which the existence of derivatives has been postulated*.

* This, which is obtained under much more general conditions in a Memoir of Tonelli in the *Rendic. Circ. Mat. Palermo* (Vol. XXIX, 1910, pp. 1—36), results from the very methods for the proof of Weierstrass's theorem. In most of them, indeed, the approximating polynomial for a continuous function of the variables $x_1, x_2, \ldots, x_m$ is expressed by an integral of the form

$$\Pi_n = \mathbf{SSS}\, F(z_1, z_2, \ldots, z_m)\, P_n(z_1 - x_1, z_2 - x_2, \ldots, z_m - x_m)\, dz_1 dz_2 \ldots dz_m,$$

the polynomial P_n ($n = 1, 2, \ldots, \infty$) being such that: (a) for any fixed system of values of $Z_1, Z_2, \ldots, Z_m$ other than $Z_1 = Z_2 = \ldots = 0$, $P_n(Z_1, Z_2, \ldots, Z_m)$ approaches zero with $1/n$, and even uniformly as long as $Z_1^2 + Z_2^2 + \ldots$ remains greater than a fixed (arbitrarily small) positive number h; (b) the integral

$$\mathbf{SSS}\, P_n dZ_1 dZ_2 \ldots dZ_m,$$

extended over a fixed domain containing the origin in its inside (the shape of which is immaterial on account of (a)), approaches 1.

P_n may be, for instance, La Vallée Poussin's and Landau's polynomial extended by Tonelli to several variables:

$$P_n = \frac{1}{K_n}\left[1 - \lambda^2 \sum_i (z_i - x_i)^2\right]^n,$$

where λ is the inverse of the greatest dimension of the domain and

$$K_n = \frac{\Omega_{m-1}}{\lambda^m} \int_0^1 (1 - \rho^2)^n \rho^{m-1} d\rho = \frac{\Omega_{m-1}}{\lambda^m} B\left(n + 1, \frac{m}{2}\right).$$

If we now want to take any derivative of order k—say

$$D_{k_1 k_2 \ldots} = \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots}$$

—of such expressions, we differentiate P_n under $\mathbf{SSS}$ with respect to the x's,

194. We get to the conclusion which we want to prove, viz. that the solution constructed in §§ 184—187 actually satisfies the given conditions, by combining the above result with those obtained for the analytic case (§§ 173—181) and by first investigating the *mode of continuity* (Book I, §§ 19 ff.) of our expressions with respect to the functions which represent the coefficients of the equation*: a question which may be interesting in several cases and for which, exactly as in § 18, Book I, expressions in power series would give us no information whatever, while we shall be able to solve it by our calculations in §§ 184 ff.

We shall see that the quantities constructed in the aforesaid sections are continuous of a certain finite order with respect to the coefficients in question. In other words, if we replace the coefficients in question by other ones having respectively with them a neighbourhood of the order in question, and if

$$(\mathrm{E}_1) \qquad \mathcal{F}_1(u) = \Sigma A_{ik}' \frac{\partial^2 u}{\partial x_i \partial x_k} + \Sigma B_i' \frac{\partial u}{\partial x_i} + C'u = f$$

is the new equation thus obtained, the aforesaid quantities will differ very slightly, whether deduced from (E) or from (E_1). This will be, for instance, the case if we replace our coefficients by approximating polynomials constructed according to the preceding section.

The first question of that kind concerns the construction of geodesics and consequently of the quantity Γ. As to this, the answer is given by what has been said in our Additional Note to Book II. We thus know that any geodesic issuing from point a and relating to the

or—which is equivalent—to the z's with multiplication by $(-1)^k$. But, if the corresponding derivative of F exists and is continuous, an integration by parts is possible and transforms the result into

$$\mathbf{SSS} D_{k_1 k_2 \dots} F \,.\, P_n(z_1 - x_1, z_2 - x_2, \dots, z_m - x_m)\, dz_1 dz_2 \dots dz_m,$$

with addition of boundary terms.

If we finally assume (as in the case for the above written polynomial of La Vallee Poussin and Landau) that our above condition (a) is satisfied not only by P_n, but by its derivatives of any order less than k, these boundary terms approach zero, and the limit of $D_{k_1 k_2 \dots} \Pi_n$ can be obtained by operating on $D_{k_1 k_2 \dots} F$ as we did on F itself: which is the required result.

* Analogous methods have been used for elliptic equations: see Lichtenstein, *Abhandl. Ak. Berlin*, 1911 (*Anhang*).

characteristic form $\mathbf{A}_1$ of (E_1) will run very near to the corresponding geodesic relating to the form $\mathbf{A}$, calling this, for instance, the geodesic which has the same tangent in $(a_1, a_2, \ldots a_m)$. Moreover, the alteration will also be very slight as concerns the partial derivatives of the coordinates x of any point of this geodesic, with respect to the parameters previously called $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ and, consequently, as concerns the functional determinant $\mathbf{J}$. This functional determinant, being different from zero and (at least if a certain neighbourhood of a is excluded) numerically greater than a fixed positive number all over a certain region $\mathscr{R}$, when taken with respect to equation (E), will therefore remain so if we start from one of the approximate equations (E_1), as soon as the approximation is sufficiently close. This fact is a most important one for us, because we thus know that *our operations can be considered as having the same domain of validity*, whether we start from (E) or (E_1).

That the alteration in Γ and M is also slight is again obvious now.

The same conclusion extends to the successive quantities V_h $(h = 1, 2, \ldots, m_1 - 2)$ on account of their defining formulæ and similarly to $V_{m_1-1} = \mathcal{V}_{(0)}$, if, of course, the order of differentiability postulated for our coefficients is sufficiently high (the derivatives thus postulated being, as we have said, approximated by the corresponding derivatives of our approximating polynomials).

It applies also to the "kernel" ψ of our integral equation (89), as appears immediately from its expression (see footnote to § 183).

Finally, as to the solution of the same equation (89), the same fact follows from the form of the successive approximations in (91) (each of which will be but slightly altered by our substitution) and from the fact that these approximations are uniformly convergent in the circumstances we have to deal with, that is if calculations start from either of the equations (E) or (E_1).

The same can be said for the reciprocal kernel Ψ, as given by a (uniformly convergent) series of "iterated" kernels (see p. 300, footnote).

The alteration will also be very slight in the quantity which we have called H (and in its derivatives up to a certain order, related to the order of neighbourhood assumed for the coefficients). This appears from the expressions of the different terms in H, as given in

§§ 173—181, which are integrals containing the data u_0, u_1, f, functions V and $\mathcal{V}_{(0)}$ and their derivatives up to a certain finite order.

Therefore, formula (93) immediately shows that the same extends to u and—if the order of neighbourhood assumed is sufficiently high—to its first and second derivatives.

195. Our conclusion will follow now without any difficulty. Let us begin by operating on equation (E_1), the coefficients A_{ik}', B_i', C' of which will be approximating polynomials for A_{ik}, B_i, C. We shall obtain a quantity u' which will actually be the solution of the corresponding problem, i.e. satisfying

$$\mathcal{F}_1(u') = \Sigma A_{ik}' \frac{\partial^2 u'}{\partial x_i \partial x_k} + \Sigma B_i' \frac{\partial u'}{\partial x_i} + C' u = f$$

and also Cauchy's condition (C_3). But if we let the altered coefficients A_{ik}', etc. vary in such a way that their neighbourhood (of a suitable chosen order) with the corresponding coefficients of (E) becomes indefinitely close, u will approach u and $\mathcal{F}_1(u)$ approach $\mathcal{F}(u)$: the latter is therefore necessarily equal to f.

196. The same continuity proof will apply to Cauchy's conditions, as these are constantly satisfied for the approximating analytic problem.

More exactly, the first of these conditions, for instance, means that, a approaching a determinate point P which belongs to S, the quantity u_a, calculated by our method, must approach $(u_0)_P$. Now, u_a constructed with (E) may be, by the above, considered as resulting, by a limiting process, from u_a', the analogous quantity deduced from (E_1). To make sure that the limit of u_a, for a approaching P, is the same as the limiting value of u_P' when (E_1) is infinitely little different from (E),—in other words, that the two limiting processes in question may be inverted—it is sufficient, as is well known, to ascertain that the first of them (corresponding to the variation of the coefficients) is uniformly convergent (especially in the neighbourhood of S). But this appears from the expressions of the different terms, as constructed in §§ 174—176 a, it having been proved, especially, that the integrands contain integral and positive powers only of the quantity which we have called θ_0.

Things behave in the same way as to the second condition (C_3).

Our problem, thus, *is proved to have a solution*, which is given by the same formula (28) or (28 a) as in the analytic case, $\mathcal{V}$ being denned by (98).

197. To this quantity $\mathcal{V}$, the preceding considerations can be at least partially extended. We can show, as above:

(1) that $\mathcal{V}$ is the limit of the corresponding quantity relating to (E_1), at least for any pair of points x and a such that $\Gamma(x; a)$ is positive and not zero;

(2) that (at least with the same restriction) it satisfies $\mathcal{G} = 0$ (as a function of the x s) and $\mathcal{F} = 0$ (as a function of the a s).

There is no doubt that these conclusions are also valid for $\Gamma(x; a)=0$, $\mathcal{V}$ being regular even then and assuming the values $\mathcal{V}_{(0)}$ (as happens in the analytic case and was required in the above for the elementary solution); in other words, that $\mathcal{V}_{(I)}$ and $\mathcal{V}_{(II)}$ are zero with Γ. The rigorous proof of this would however present some difficulties as to $\mathcal{V}_{(II)}$, on account of geometric reasons already alluded to: for, $\Gamma(x; a)$ being very small, we should not be able to indicate a lower limit for the quantity $S_0'(\theta_0)$ (§ 175), as some bicharacteristics from x would pass very near a and others meet the characteristic conoid from a at very small angles.

198. Of course, the result obtained for m even implies the corresponding one for m odd, by means of our process of descent: V and $\mathcal{V}$—now called V' and $\mathcal{V}'$—having been constructed for $m = 2m_1 + 2$, as explained above, the value of v, for $m = 2m_1 + 1$, follows by formulæ (62), (65), (§§ 164, 165).

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