

THE LOGIC OF QUANTUM MECHANICS

BY GARRETT BIRKHOFF AND JOHN VON NEUMANN

(Received April 4, 1936)

1. **Introduction.** One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system \mathfrak{S} does not in general enable one to predict with certainty the result of an experiment on \mathfrak{S} , and that in particular one can never predict with certainty both the position and the momentum of \mathfrak{S} (Heisenberg's Uncertainty Principle). It further asserts that most pairs of observations are incompatible, and cannot be made on \mathfrak{S} simultaneously (Principle of Non-commutativity of Observations).

The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sums*, and *orthogonal complements*—and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*.

In order to avoid being committed to quantum theory in its present form, we have first (in §§2–6) stated the heuristic arguments which suggest that such a calculus is the proper one in quantum mechanics, and then (in §§7–14) reconstructed this calculus from the axiomatic standpoint. In both parts an attempt has been made to clarify the discussion by continual comparison with classical mechanics and its propositional calculi. The paper ends with a few tentative conclusions which may be drawn from the material just summarized.

I. PHYSICAL BACKGROUND

2. **Observations on physical systems.** The concept of a physically observable “physical system” is present in all branches of physics, and we shall assume it.

It is clear that an “observation” of a physical system \mathfrak{S} can be described generally as a writing down of the readings from various¹ compatible measurements. Thus if the measurements are denoted by the symbols μ_1, \dots, μ_n , then

¹ If one prefers, one may regard a set of compatible measurements as a single composite “measurement”—and also admit non-numerical readings—without interfering with subsequent arguments.

Among conspicuous observables in quantum theory are position, momentum, energy, and (non-numerical) symmetry.

an observation of \mathfrak{S} amounts to specifying numbers x_1, \dots, x_n corresponding to the different μ_k .

It follows that the most general form of a prediction concerning \mathfrak{S} is that the point (x_1, \dots, x_n) determined by actually measuring μ_1, \dots, μ_n , will lie in a subset S of (x_1, \dots, x_n) -space. Hence if we call the (x_1, \dots, x_n) -spaces associated with \mathfrak{S} , its "observation-spaces," we may call the subsets of the observation-spaces associated with any physical system \mathfrak{S} , the "experimental propositions" concerning \mathfrak{S} .

3. Phase-spaces. There is one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical "phase-space."

According to this concept, any physical system \mathfrak{S} is at each instantly hypothetically associated with a "point" p in a fixed phase-space Σ ; this point is supposed to represent mathematically the "state" of \mathfrak{S} , and the "state" of \mathfrak{S} is supposed to be ascertainable by "maximal"² observations.

Furthermore, the point p_0 associated with \mathfrak{S} at a time t_0 , together with a prescribed mathematical "law of propagation," fix the point p_t associated with \mathfrak{S} at any later time t ; this assumption evidently embodies the principle of *mathematical causation*.³

Thus in classical mechanics, each point of Σ corresponds to a choice of n position and n conjugate momentum coördinates—and the law of propagation may be Newton's inverse-square law of attraction. Hence in this case Σ is a region of ordinary $2n$ -dimensional space. In electrodynamics, the points of Σ can only be specified after certain *functions*—such as the electromagnetic and electrostatic potential—are known; hence Σ is a function-space of infinitely many dimensions. Similarly, in quantum theory the points of Σ correspond to so-called "wave-functions," and hence Σ is again a function-space—usually⁴ assumed to be Hilbert space.

In electrodynamics, the law of propagation is contained in Maxwell's equations, and in quantum theory, in equations due to Schrödinger. In any case, the law of propagation may be imagined as inducing a steady fluid motion in the phase-space.

It has proved to be a fruitful observation that in many important cases of classical dynamics, this flow conserves volumes. It may be noted that in quantum mechanics, the flow conserves distances (i.e., the equations are "unitary").

² L. Pauling and E. B. Wilson, "An introduction to quantum mechanics," McGraw-Hill, 1935, p. 422. Dirac, "Quantum mechanics," Oxford, 1930, §4.

³ For the existence of mathematical causation, cf. also p. 65 of Heisenberg's "The physical principles of the quantum theory," Chicago, 1929.

⁴ Cf. J. von Neumann, "Mathematische Grundlagen der Quanten-mechanik," Berlin, 1931. p. 18.

4. **Propositions as subsets of phase-space.** Now before a phase-space can become imbued with reality, its elements and subsets must be correlated in some way with "experimental propositions" (which are subsets of different observation-spaces). Moreover, this must be so done that set-theoretical inclusion (which is the analogue of logical implication) is preserved.

There is an obvious way to do this in dynamical systems of the classical type.⁵ One can measure position and its first time-derivative velocity—and hence momentum—explicitly, and so establish a one-one correspondence which preserves inclusion between subsets of phase-space and subsets of a suitable observation-space.

In the cases of the kinetic theory of gases and of electromagnetic waves no such simple procedure is possible, but it was imagined for a long time that "demons" of small enough size could by tracing the motion of each particle, or by a dynamometer and infinitesimal point-charges and magnets, measure quantities corresponding to every coördinate of the phase-space involved.

In quantum theory not even this is imagined, and the possibility of predicting in general the readings from measurements on a physical system \mathfrak{S} from a knowledge of its "state" is denied; only statistical predictions are always possible.

This has been interpreted as a renunciation of the doctrine of pre-determination; a thoughtful analysis shows that another and more subtle idea is involved. The central idea is that physical quantities are *related*, but are not all computable from a number of *independent basic* quantities (such as position and velocity).⁶

We shall show in §12 that this situation has an exact algebraic analogue in the calculus of propositions.

5. **Propositional calculi in classical dynamics.** Thus we see that an uncritical acceptance of the ideas of classical dynamics (particularly as they involve n -body problems) leads one to identify each subset of phase-space with an experimental proposition (the proposition that the system considered has position and momentum coördinates satisfying certain conditions) and conversely.

This is easily seen to be unrealistic; for example, how absurd it would be to call an "experimental proposition," the assertion that the angular momentum (in radians per second) of the earth around the sun was at a particular instant a rational number!

Actually, at least in statistics, it seems best to assume that it is the *Lebesgue-measurable* subsets of a phase-space which correspond to experimental propositions, two subsets being identified, if their difference has *Lebesgue-measure* 0.⁷

⁵ Like systems idealizing the solar system or projectile motion.

⁶ A similar situation arises when one tries to correlate polarizations in different planes of electromagnetic waves.

⁷ Cf. J. von Neumann, "*Operatorenmethoden in der klassischen Mechanik*," *Annals of Math.* 33 (1932), 595–8. The difference of two sets S_1, S_2 is the set $(S_1 + S_2) - S_1 \cdot S_2$ of those points, which belong to one of them, but not to both.

But in either case, the set-theoretical sum and product of any two subsets, and the complement of any one subset of phase-space corresponding to experimental propositions, has the same property. That is, by definition⁸

*The experimental propositions concerning any system in classical mechanics, correspond to a "field" of subsets of its phase-space. More precisely: To the "quotient" of such a field by an ideal in it. At any rate they form a "Boolean Algebra."*⁹

In the axiomatic discussion of propositional calculi which follows, it will be shown that this is inevitable when one is dealing with exclusively compatible measurements, and also that it is logically immaterial which particular field of sets is used.

6. A propositional calculus for quantum mechanics. The question of the connection in quantum mechanics between subsets of observation-spaces (or "experimental propositions") and subsets of the phase-space of a system \mathfrak{S} , has not been touched. The present section will be devoted to defining such a connection, proving some facts about it, and obtaining from it heuristically by introducing a plausible postulate, a propositional calculus for quantum mechanics.

Accordingly, let us observe that if $\alpha_1, \dots, \alpha_n$ are any compatible observations on a quantum-mechanical system \mathfrak{S} with phase-space Σ , then¹⁰ there exists a set of mutually orthogonal closed linear subspaces Ω_i of Σ (which correspond to the families of proper functions satisfying $\alpha_1 f = \lambda_{i,1} f, \dots, \alpha_n f = \lambda_{i,n} f$) such that every point (or function) $f \in \Sigma$ can be uniquely written in the form

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + \dots [f_i \in \Omega_i]$$

Hence if we state the

DEFINITION: By the "mathematical representative" of a subset S of any observation-space (determined by compatible observations $\alpha_1, \dots, \alpha_n$) for a quantum-mechanical system \mathfrak{S} , will be meant the set of all points f of the phase-space of \mathfrak{S} , which are linearly determined by proper functions f_k satisfying $\alpha_1 f_k = \lambda_1 f_k, \dots, \alpha_n f_k = \lambda_n f_k$, where $(\lambda_1, \dots, \lambda_n) \in S$.

Then it follows immediately: (1) that the mathematical representative of any experimental proposition is a closed linear subspace of Hilbert space (2) since all operators of quantum mechanics are Hermitian, that the mathematical representative of the *negative*¹¹ of any experimental proposition is the *orthogonal*

⁸ F. Hausdorff, "Mengenlehre," Berlin, 1927, p. 78.

⁹ M. H. Stone, "Boolean Algebras and their application to topology," Proc. Nat. Acad. 20 (1934), p. 197.

¹⁰ Cf. von Neumann, op. cit., pp. 121, 90, or Dirac, op. cit., 17. We disregard complications due to the possibility of a continuous spectrum. They are inessential in the present case.

¹¹ By the "negative" of an experimental proposition (or subset S of an observation-space) is meant the experimental proposition corresponding to the set-complement of S in the same observation-space.

complement of the mathematical representative of the proposition itself (3) the following three conditions on two experimental propositions P and Q concerning a given type of physical system are equivalent:

(3a) The mathematical representative of P is a subset of the mathematical representative of Q .

(3b) P implies Q —that is, whenever one can predict P with certainty, one can predict Q with certainty.

(3c) For any statistical ensemble of systems, the probability of P is at most the probability of Q .

The equivalence of (3a)–(3c) leads one to regard the aggregate of the mathematical representatives of the experimental propositions concerning any physical system \mathfrak{S} , as representing mathematically the propositional calculus for \mathfrak{S} .

We now introduce the

POSTULATE: *The set-theoretical product of any two mathematical representatives of experimental propositions concerning a quantum-mechanical system, is itself the mathematical representative of an experimental proposition.*

REMARKS: This postulate would clearly be implied by the not unnatural conjecture that all Hermitian-symmetric operators in Hilbert space (phase-space) correspond to observables;¹² it would even be implied by the conjecture that those operators which correspond to observables coincide with the Hermitian-symmetric elements of a suitable operator-ring M .¹³

Now the closed linear sum $\Omega_1 + \Omega_2$ of any two closed linear subspaces Ω_i of Hilbert space, is the orthogonal complement of the set-product $\Omega'_1 \cdot \Omega'_2$ of the orthogonal complements Ω'_i of the Ω_i ; hence if one adds the above postulate to the usual postulates of quantum theory, then one can deduce that

The set-product and closed linear sum of any two, and the orthogonal complement of any one closed linear subspace of Hilbert space representing mathematically an experimental proposition concerning a quantum-mechanical system \mathfrak{S} , itself represents an experimental proposition concerning \mathfrak{S} .

This defines the calculus of experimental propositions concerning \mathfrak{S} , as a calculus with three operations and a relation of implication, which closely resembles the systems defined in §5. We shall now turn to the analysis and comparison of all three calculi from an axiomatic-algebraic standpoint.

II. ALGEBRAIC ANALYSIS

7. Implication as partial ordering. It was suggested above that in any physical theory involving a phase-space, the experimental propositions concern-

¹² I.e., that given such an operator α , one "could" find an observable for which the proper states were the proper functions of α .

¹³ F. J. Murray and J. v. Neumann, "On rings of operators," *Annals of Math.*, 37 (1936), p. 120. It is shown on p. 141, loc. cit. (Definition 4.2.1 and Lemma 4.2.1), that the closed linear sets of a ring M —that is those, the "projection operators" of which belong to M —coincide with the closed linear sets which are invariant under a certain group of rotations of Hilbert space. And the latter property is obviously conserved when a set-theoretical intersection is formed.

ing a system \mathfrak{S} correspond to a family of subsets of its phase-space Σ , in such a way that “ x implies y ” (x and y being any two experimental propositions) means that the subset of Σ corresponding to x is contained set-theoretically in the subset corresponding to y . This hypothesis clearly is important in proportion as relationships of implication exist between experimental propositions corresponding to subsets of different observation-spaces.

The present section will be devoted to corroborating this hypothesis by identifying the algebraic-axiomatic properties of logical implication with those of set-inclusion.

It is customary to admit as relations of “implication,” only relations satisfying

S1: x implies x .

S2: If x implies y and y implies z , then x implies z .

S3: If x implies y and y implies x , then x and y are logically equivalent.

In fact, S3 need not be stated as a postulate at all, but can be regarded as a definition of logical equivalence. Pursuing this line of thought, one can interpret as a “physical quality,” the set of all experimental propositions logically equivalent to a given experimental proposition.¹⁴

Now if one regards the set S_x of propositions implying a given proposition x as a “mathematical representative” of x , then by S3 the correspondence between the x and the S_x is one-one, and x implies y if and only if $S_x \subset S_y$. While conversely, if L is any system of subsets X of a fixed class Γ , then there is an isomorphism which carries inclusion into logical implication between L and the system L^* of propositions “ x is a point of X ,” $X \in L$.

Thus we see that the properties of logical implication are indistinguishable from those of set-inclusion, and that therefore it is *algebraically* reasonable to try to correlate physical qualities with subsets of phase-space.

A system satisfying S1-S3, and in which the relation “ x implies y ” is written $x \subset y$, is usually¹⁵ called a “partially ordered system,” and thus our first postulate concerning propositional calculi is that *the physical qualities attributable to any physical system form a partially ordered system*.

It does not seem excessive to require that in addition any such calculus contain two special propositions: the proposition \square that the system considered *exists*, and the proposition \odot that it does *not exist*. Clearly

S4: $\odot \subset x \subset \square$ for any x .

\odot is, from a logical standpoint, the “identically false” or “absurd” proposition; \square is the “identically true” or “self-evident” proposition.

8. Lattices. In any calculus of propositions, it is natural to imagine that there is a weakest proposition implying, and a strongest proposition implied by,

¹⁴ Thus in §6, closed linear subspaces of Hilbert space correspond one-many to experimental propositions, but one-one to physical qualities in this sense.

¹⁵ F. Hausdorff, “*Grundzüge der Mengenlehre*,” Leipzig, 1914, Chap. VI, §1.

a given pair of propositions. In fact, investigations of partially ordered systems from different angles all indicate that the first property which they are likely to possess, is the existence of greatest lower bounds and least upper bounds to subsets of their elements. Accordingly, we state

DEFINITION: A partially ordered system L will be called a "lattice" if and only if to any pair x and y of its elements there correspond

S5: A "meet" or "greatest lower bound" $x \cap y$ such that (5a) $x \cap y \subset x$, (5b) $x \cap y \subset y$, (5c) $z \subset x$ and $z \subset y$ imply $z \subset x \cap y$.

S6: A "join" or "least upper bound" $x \cup y$ satisfying (6a) $x \cup y \supset x$, (6b) $x \cup y \supset y$, (6c) $w \supset x$ and $w \supset y$ imply $w \supset x \cup y$.

The relation between meets and joins and abstract inclusion can be summarized as follows,¹⁶

(8.1) In any lattice L , the following formal identities are true,

L1: $a \cap a = a$ and $a \cup a = a$.

L2: $a \cap b = b \cap a$ and $a \cup b = b \cup a$.

L3: $a \cap (b \cap c) = (a \cap b) \cap c$ and $a \cup (b \cup c) = (a \cup b) \cup c$.

L4: $a \cup (a \cap b) = a$ and $a \cap (a \cup b) = a$.

Moreover, the relations $a \supset b$, $a \cap b = b$, and $a \cup b = a$ are equivalent—each implies both of the others.

(8.2) Conversely, in any set of elements satisfying L2–L4 (L1 is redundant), $a \cap b = b$ and $a \cup b = a$ are equivalent. And if one defines them to mean $a \supset b$, then one reveals L as a lattice.

Clearly L1–L4 are well-known formal properties of *and* and *or* in ordinary logic. This gives an algebraic reason for admitting as a *postulate* (if necessary) the statement that a given calculus of propositions is a lattice. There are other reasons¹⁷ which impel one to admit as a postulate the stronger statement that the set-product of any two subsets of a phase-space which correspond to physical qualities, itself represents a physical quality—this is, of course, the Postulate of §6.

It is worth remarking that in classical mechanics, one can easily define the meet or join of any two experimental propositions as an *experimental proposition*—simply by having independent observers read off the measurements which either proposition involves, and combining the results logically. This is true in quantum mechanics only exceptionally—only when all the measurements involved commute (are compatible); in general, one can only express the join or

¹⁶ The final result was found independently by O. Öre, "The foundations of abstract algebra. I.," *Annals of Math.* 36 (1935), 406–37, and by H. MacNeille in his Harvard Doctoral Thesis, 1935.

¹⁷ The first reason is that this implies no restriction on the abstract nature of a lattice—any lattice can be realized as a system of its own subsets, in such a way that $a \cap b$ is the set-product of a and b . The second reason is that if one regards a subset S of the phase-space of a system \mathfrak{S} as corresponding to the *certainty* of observing \mathfrak{S} in S , then it is natural to assume that the combined certainty of observing \mathfrak{S} in S and T is the certainty of observing \mathfrak{S} in $S \cdot T = S \cap T$,—and assumes quantum theory.

meet of two given experimental propositions as a class of logically equivalent experimental propositions—i.e., as a *physical quality*.¹⁸

9. Complemented lattices. Besides the (binary) operations of meet- and join-formation, there is a third (unary) operation which may be defined in partially ordered systems. This is the operation of *complementation*.

In the case of lattices isomorphic with “fields” of sets, complementation corresponds to passage to the set-complement. In the case of closed linear subspaces of Hilbert space (or of Cartesian n -space), it corresponds to passage to the orthogonal complement. In either case, denoting the “complement” of an element a by a' , one has the formal identities,

$$\text{L71: } (a')' = a.$$

$$\text{L72: } a \cap a' = \odot \text{ and } a \cup a' = \square.$$

$$\text{L73: } a \subset b \text{ implies } a' \supset b'.$$

By definition, L71 and L73 amount to asserting that complementation is a “dual automorphism” of period two. It is an immediate corollary of this and the duality between the definitions (in terms of inclusion) of meet and join, that

$$\text{L74: } (a \cap b)' = a' \cup b' \text{ and } (a \cup b)' = a' \cap b'$$

and another corollary that the second half of L72 is redundant. [*Proof:* by L71 and the first half of L74, $(a \cup a') = (a'' \cup a') = (a' \cap a)' = \odot'$, while under inversion of inclusion \odot evidently becomes \square .] This permits one to deduce L72 from the even weaker assumption that $a \subset a'$ implies $a = \odot$. *Proof:* for any x , $(x \cap x')' = (x' \cup x'') = x' \cup x \supset x \cap x'$.

Hence if one admits as a postulate the assertion that *passage from an experimental proposition a to its complement a' is a dual automorphism of period two, and a implies a' is absurd*, one has in effect admitted L71–L74.

This postulate is independently suggested (and L71 proved) by the fact the “complement” of the proposition that the readings x_1, \dots, x_n from a series of compatible observations μ_1, \dots, μ_n lie in a subset S of (x_1, \dots, x_n) -space, is by definition the proposition that the readings lie in the set-complement of S .

10. The distributive identity. Up to now, we have only discussed formal features of logical structure which seem to be common to classical dynamics and the quantum theory. We now turn to the central difference between them—the *distributive identity* of the propositional calculus:

$$\text{L6: } a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \text{ and } a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

which is a law in classical, but not in quantum mechanics.

¹⁸ The following point should be mentioned in order to avoid misunderstanding: If a, b are two physical qualities, then $a \cup b, a \cap b$ and a' (cf. below) are physical qualities too (and so are \odot and \square). But $a \subset b$ is not a physical quality; it is a relation between physical qualities.

From an axiomatic viewpoint, each half of L6 implies the other.¹⁹ Further, either half of L6, taken with L72, implies L71 and L73, and to assume L6 and L72 amounts to assuming the usual definition of a Boolean algebra.²⁰

From a deeper mathematical viewpoint, L6 is the characteristic property of set-combination. More precisely, every “field” of sets is isomorphic with a Boolean algebra, and conversely.²¹ This throws new light on the well-known fact that the propositional calculi of classical mechanics are Boolean algebras.

It is interesting that L6 is also a logical consequence of the compatibility of the observables occurring in a , b , and c . That is, if observations are made by independent observers, and combined according to the usual rules of logic, one can prove L1–L4, L6, and L71–74.

These facts suggest that the distributive law *may* break down in quantum mechanics. That it *does* break down is shown by the fact that if a denotes the experimental observation of a wave-packet ψ on one side of a plane in ordinary space, a' correspondingly the observation of ψ on the other side, and b the observation of ψ in a state symmetric about the plane, then (as one can readily check):

$$\begin{aligned} b \cap (a \cup a') &= b \cap \square = b > \odot = (b \cap a) = (b \cap a') \\ &= (b \cap a) \cup (b \cap a') \end{aligned}$$

REMARK: In connection with this, it is a salient fact that the *generalized* distributive law of logic:

$$\text{L6*}: \prod_{i=1}^m \left(\sum_{j=1}^n a_{i,j} \right) = \sum_{j(i)} \left(\prod_{i=1}^m a_{i,j(i)} \right)$$

breaks down in the quotient algebra of the field of Lebesgue measurable sets by the ideal of sets of Lebesgue measure 0, which is so fundamental in statistics and the formulation of the ergodic principle.²²

11. The modular identity. Although closed linear subspaces of Hilbert space and Cartesian n -space need not satisfy L6 relative to set-products and closed linear sums, the formal properties of these operations are not confined to L1–L4 and L71–L73.

In particular, set-products and straight linear sums are known²³ to satisfy the so-called “modular identity.”

¹⁹ R. Dedekind, “*Werke*,” Braunschweig, 1931, vol. 2, p. 110.

²⁰ G. Birkhoff, “*On the combination of subalgebras*,” Proc. Camb. Phil. Soc. 29 (1933), 441–64, §§23–4. Also, in any lattice satisfying L6, isomorphism with respect to inclusion implies isomorphism with respect to complementation; this need not be true if L6 is not assumed, as the lattice of linear subspaces through the origin of Cartesian n -space shows.

²¹ M. H. Stone, “*Boolean algebras and their application to topology*,” Proc. Nat. Acad. 20 (1934), 197–202.

²² A detailed explanation will be omitted, for brevity; one could refer to work of G. D. Birkhoff, J. von Neumann, and A. Tarski.

²³ G. Birkhoff, op. cit., §28. The proof is easy. One first notes that since $a \subset (a \cup b) \cap c$ if $a \subset c$, and $b \cap c \subset (a \cup b) \cap c$ in any case, $a \cup (b \cap c) \subset (a \cup b) \cap c$. Then one notes

L5: If $a \subset c$, then $a \cup (b \cap c) = (a \cup b) \cap c$.

Therefore (since the linear sum of any two finite-dimensional linear subspaces of Hilbert space is itself finite-dimensional and consequently closed) set-products and *closed* linear sums of the *finite dimensional* subspaces of any topological linear space such as Cartesian n -space or Hilbert space satisfy L5, too.

One can interpret L5 directly in various ways. First, it is evidently a restricted associative law on mixed joins and meets. It can equally well be regarded as a weakened distributive law, since if $a \subset c$, then $a \cup (b \cap c) = (a \cap c) \cup (b \cap c)$ and $(a \cup b) \cap c = (a \cup b) \cap (a \cup c)$. And it is self-dual: replacing \subset, \cap, \cup by \supset, \cup, \cap merely replaces a, b, c , by c, b, a .

Also, speaking graphically, the assumption that a lattice L is "modular" (i.e., satisfies L5) is equivalent to²⁴ saying that L contains no sublattice isomorphic with the lattice graphed in fig. 1:

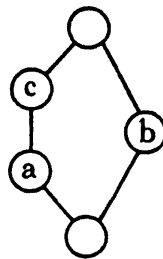


FIG. 1

Thus in Hilbert space, one can find a counterexample to L5 of this type. Denote by $\xi_1, \xi_2, \xi_3, \dots$ a basis of orthonormal vectors of the space, and by a, b , and c respectively the closed linear subspaces generated by the vectors $(\xi_{2n} + 10^{-n}\xi_1 + 10^{-2n}\xi_{2n+1})$, by the vectors ξ_{2n} , and by a and the vector ξ_1 . Then a, b , and c generate the lattice of Fig. 1.

Finally, the modular identity can be proved to be a consequence of the assumption that there exists a numerical dimension-function $d(a)$, with the properties

D1: If $a > b$, then $d(a) > d(b)$.

D2: $d(a) + d(b) = d(a \cap b) + d(a \cup b)$.

This theorem has a converse under the restriction to lattices in which there is a finite upper bound to the length n of chains²⁵ $\odot < a_1 < a_2 < \dots < a_n < \square$ of elements.

Since conditions D1–D2 partially describe the formal properties of probability, the presence of condition L5 is closely related to the existence of an

that any vector in $(a \cup b) \cap c$ can be written $\xi = \alpha + \beta$ [$\alpha \in a, \beta \in b, \xi \in c$]. But $\beta = \xi - \alpha$ is in c (since $\xi \in c$ and $\alpha \in a \subset c$); hence $\xi = \alpha + \beta \in a \cup (b \cap c)$, and $a \cup (b \cap c) \supset (a \cup b) \cap c$, completing the proof.

²⁴ R. Dedekind, "Werke," vol. 2, p. 255.

²⁵ The statements of this paragraph are corollaries of Theorem 10.2 of G. Birkhoff, *op. cit.*

“a priori thermo-dynamic weight of states.” But it would be desirable to interpret L5 by simpler phenomenological properties of quantum physics.

12. Relation to abstract projective geometries. We shall next investigate how the assumption of postulates asserting that the physical qualities attributable to any quantum-mechanical system \mathfrak{S} are a lattice satisfying L5 and L71–L73 characterizes the resulting propositional calculus. This question is evidently purely algebraic.

We believe that the best way to find this out is to introduce an assumption limiting the length of chains of elements (assumption of finite dimensions) of the lattice, admitting frankly that the assumption is purely heuristic.

It is known²⁶ that any lattice of finite dimensions satisfying L5 and L72 is the direct product of a finite number of abstract projective geometries (in the sense of Veblen and Young), and a finite Boolean algebra, and conversely.

REMARK: It is a corollary that a lattice satisfying L5 and L71–L73 possesses independent basic elements of which any element is a union, if and only if it is a Boolean algebra.

Again, such a lattice is a single projective geometry if and only if it is *irreducible*—that is, if and only if it contains no “neutral” elements.²⁷ $x \neq \odot, \square$ such that $a = (a \cap x) \cup (a \cap x')$ for all a . In actual quantum mechanics such an element would have a projection-operator, which commutes with all projection-operators of observables, and so with all operators of observables in general. This would violate the requirement of “irreducibility” in quantum mechanics.²⁸ Hence we conclude that the *propositional calculus of quantum mechanics has the same structure as an abstract projective geometry*.

Moreover, this conclusion has been obtained purely by analyzing internal properties of the calculus, in a way which involves Hilbert space only indirectly.

13. Abstract projective geometries and skew-fields. We shall now try to get a fresh picture of the propositional calculus of quantum mechanics, by recalling the well-known two-way correspondence between abstract projective geometries and (not necessarily commutative) fields.

Namely, let F be any such field, and consider the following definitions and constructions: n elements x_1, \dots, x_n of F , not all = 0, form a right-ratio $[x_1 : \dots : x_n]_r$, two right-ratios $[x_1 : \dots : x_n]_r$, and $[\xi_1 : \dots : \xi_n]_r$ being called “equal,” if and only if a $z \in F$ with $\xi_i = x_i z, i = 1, \dots, n$, exists. Similarly, n elements y_1, \dots, y_n of F , not all = 0, form a left-ratio $[y_1 : \dots : y_n]_l$, two left-ratios $[y_1 : \dots : y_n]_l$ and $[\eta_1 : \dots : \eta_n]_l$ being called “equal,” if and only if a z in F with $\eta_i = z y_i, i = 1, \dots, n$, exists.

²⁶ G. Birkhoff “Combinatorial relations in projective geometries,” *Annals of Math.* 36 (1935), 743–8.

²⁷ O. Öre, *op. cit.*, p. 419.

²⁸ Using the terminology of footnote¹³ and of *loc. cit.* there: The ring MM' should contain no other projection-operators than 0, 1, or: the ring M must be a “factor.” Cf. *loc. cit.*¹³, p. 120.

Now define an $n - 1$ -dimensional projective geometry $P_{n-1}(F)$ as follows: The "points" of $P_{n-1}(F)$ are all right-ratios $[x_1 : \dots : x_n]_r$. The "linear subspaces" of $P_{n-1}(F)$ are those sets of points, which are defined by systems of equations

$$\alpha_{k1}x_1 + \dots + \alpha_{kn}x_n = 0, \quad k = 1, \dots, m.$$

($m = 1, 2, \dots$, the α_{ki} are fixed, but arbitrary elements of F). The proof, that this is an abstract projective geometry, amounts simply to restating the basic properties of linear dependence.²⁹

The same considerations show, that the ($n - 2$ -dimensional) hyperplanes in $P_{n-1}(F)$ correspond to $m = 1$, not all $\alpha_i = 0$. Put $\alpha_{1i} = y_i$, then we have

$$(*) \quad y_1x_1 + \dots + y_nx_n = 0, \quad \text{not all } y_i = 0.$$

This proves, that the ($n - 2$ -dimensional) hyperplanes in $P_{n-1}(F)$ are in a one-to-one correspondence with the left-ratios $[y_1 : \dots : y_n]_l$.

So we can identify them with the left-ratios, as points are already identical with the right-ratios, and (*) becomes the definition of "incidence" (point \subset hyperplane).

Reciprocally, any abstract $n - 1$ -dimensional projective geometry Q_{n-1} with $n = 4, 5, \dots$ belongs in this way to some (not necessarily commutative field $F(Q_{n-1})$, and Q_{n-1} is isomorphic with $P_{n-1}(F(Q_{n-1}))$).³⁰

14. Relation of abstract complementarity to involutory anti-isomorphisms in skew-fields. We have seen that the family of irreducible lattices satisfying L5 and L72 is precisely the family of projective geometries, provided we exclude the two-dimensional case. But what about L71 and L73? In other words, for which $P_{n-1}(F)$ can one define complements possessing all the known formal properties of orthogonal complements? The present section will be spent in answering this question.^{30a}

²⁹ Cf. §§103–105 of B. L. Van der Waerden's "*Moderne Algebra*," Berlin, 1931, Vol. 2.

³⁰ $n = 4, 5; \dots$ means of course $n - 1 \geq 3$, that is, that Q_{n-1} is necessarily a "Desarguesian" geometry. (Cf. O. Veblen and J. W. Young, "*Projective Geometry*," New York, 1910, Vol. 1, page 41). Then $F = F(Q_{n-1})$ can be constructed in the classical way. (Cf. Veblen and Young, Vol. 1, pages 141–150). The proof of the isomorphism between Q_{n-1} and the $P_{n-1}(F)$ as constructed above, amounts to this: Introducing (not necessarily commutative) homogeneous coördinates x_1, \dots, x_n from F in Q_{n-1} , and expressing the equations of hyperplanes with their help. This can be done in the manner which is familiar in projective geometry, although most books consider the commutative ("Pascalian") case only. D. Hilbert, "*Grundlagen der Geometrie*," 7th edition, 1930, pages 96–103, considers the non-commutative case, but for affine geometry, and $n - 1 = 2, 3$ only.

Considering the lengthy although elementary character of the complete proof, we propose to publish it elsewhere.

^{30a} R. Brauer, "*A characterization of null systems in projective space*," Bull. Am. Math. Soc. 42 (1936), 247–54, treats the analogous question in the opposite case that $X \cap X' \neq \circ$ is postulated.

First, we shall show that it is *sufficient* that F admit an *involutory antimorphism* $W: \bar{x} = W(x)$, that is:

- Q1. $w(w(u)) = u$,
- Q2. $w(u + v) = w(u) + w(v)$,
- Q3. $w(uv) = w(v) w(u)$,

with a *definite diagonal Hermitian form* $w(x_1)\gamma_1\xi_1 + \dots + w(x_n)\gamma_n\xi_n$, where

Q4. $w(x_1)\gamma_1x_1 + \dots + w(x_n)\gamma_nx_n = 0$ implies $x_1 = \dots = x_n = 0$,

the γ_i being fixed elements of F , satisfying $w(\gamma_i) = \gamma_i$.

Proof: Consider ennuples (not right- or left-ratios!) $x:(x_1, \dots, x_n)$, $\xi:(\xi_1, \dots, \xi_n)$ of elements of F . Define for them the vector-operations

$$xz: (x_1z, \dots, x_nz) \quad (z \text{ in } F),$$

$$x + \xi: (x_1 + \xi_1, \dots, x_n + \xi_n),$$

and an "inner product"

$$(\xi_1x) = w(\xi_1)\gamma_1x_1 + \dots + w(\xi_n)\gamma_nx_n.$$

Then the following formulas are corollaries of Q1–Q4.

- IP1 $(x, \xi) = w((\xi, x))$,
- IP2 $(\xi, xu) = (\xi, x)u$, $(\xi u, x) = w(u)(\xi, x)$,
- IP3 $(\xi, x' + x'') = (\xi, x') + (\xi, x'')$, $(\xi' + \xi'', x) = (\xi', x) + (\xi'', x)$,
- IP4 $(x, x) = w((x, x)) = [x]$ is $\neq 0$ if $x \neq 0$ (that is, if any $x_i \neq 0$).

We can define $x \perp \xi$ (in words: " x is orthogonal to ξ ") to mean that $(\xi, x) = 0$. This is evidently symmetric in x, ξ , and depends on the right-ratios $[x_1: \dots : x_n]_r$, $[\xi_1: \dots : \xi_n]_r$ only so it establishes the relation of "polarity," $a \perp b$, between the points

$$a: [x_1: \dots : x_n]_r, \quad b: [\xi_1: \dots : \xi_n]_r \text{ of } P_{n-1}(F).$$

The polars to any point $b: [\xi_1: \dots : \xi_n]_r$ of $P_{n-1}(F)$ constitute a linear subspace of points of $P_{n-1}(F)$, which by Q4 does not contain b itself, and yet with b generates whole projective space $P_{n-1}(F)$, since for any ennuple $x:(x_1, \dots, x_n)$

$$x = x' + \xi \cdot [\xi]^{-1}(\xi, x)$$

where by Q4, $[\xi] \neq 0$, and by IP $(\xi, x') = 0$. This linear subspace is, therefore, an $n-2$ -dimensional hyperplane.

Hence if c is any k -dimensional element of $P_{n-1}(F)_1$ one can set up inductively k mutually polar points $b^{(1)}, \dots, b^{(k)}$ in c . Then it is easy to show that the set c' of points polar to every $b^{(1)}, \dots, b^{(k)}$ —or equivalently to every point in c —constitute an $n-k-1$ -dimensional element, satisfying $c \cap c' = \odot$ and $c \cup c' = \square$. Moreover, by symmetry $(c')' \supset c$, whence by dimensional considerations $c'' = c$. Finally, $c \supset d$ implies $c' \subset d'$, and so the correspondence $c \rightarrow c'$ defines an involutory dual automorphism of $P_{n-1}(F)$ completing the proof.

In the Appendix it will be shown that this condition is also necessary. Thus the above class of systems is exactly the class of irreducible lattices of finite dimensions > 3 satisfying L5 and L71–L73.

III. CONCLUSIONS

15. Mathematical models for propositional calculi. One conclusion which can be drawn from the preceding algebraic considerations, is that one can construct many different models for a propositional calculus in quantum mechanics, which cannot be differentiated by known criteria. More precisely, one can take any field F having an involutory anti-isomorphism satisfying Q4 (such fields include the real, complex, and quaternion number systems³¹), introduce suitable notions of linear dependence and complementarity, and then construct for every dimension-number n a model $P_n(F)$, having all of the properties of the propositional calculus suggested by quantum-mechanics.

One can also construct infinite-dimensional models $P_\infty(F)$ whose elements consist of all closed linear subspaces of normed infinite-dimensional spaces. But philosophically, Hankel's principle of the "perseverance of formal laws" (which leads one to try to preserve L5)³² and mathematically, technical analysis of spectral theory in Hilbert space, lead one to prefer a continuous-dimensional model $P_c(F)$, which will be described by one of us in another paper.³³

$P_c(F)$ is very analogous with the model furnished by the measurable subsets of phase-space in classical dynamics.³⁴

16. The logical coherence of quantum mechanics. The above heuristic considerations suggest in particular that the physically significant statements in quantum mechanics actually constitute a sort of projective geometry, while the physically significant statements concerning a given system in classical dynamics constitute a Boolean algebra.

They suggest even more strongly that whereas in classical mechanics any propositional calculus involving more than two propositions can be decomposed into independent constituents (direct sums in the sense of modern algebra), quantum theory involves irreducible propositional calculi of unbounded complexity. This indicates that quantum mechanics has a *greater logical coherence*

³¹ In the real case, $w(x) = x$; in the complex case, $w(x + iy) = x - iy$; in the quaternionic case, $w(u + ix + jy + kz) = u - ix - jy - kz$; in all cases, the λ_i are 1. Conversely, A. Kolmogoroff, "Zur Begründung der projektiven Geometrie," Annals of Math. 33 (1932), 175–6 has shown that any projective geometry whose k -dimensional elements have a *locally compact topology* relative to which the lattice operations are continuous, must be over the real, the complex, or the quaternion field.

³² L5 can also be preserved by the artifice of considering in $P_\infty(F)$ only elements which either are or have complements which are of finite dimensions.

³³ J. von Neumann, "Continuous geometries," Proc. Nat. Acad., 22 (1936), 92–100 and 101–109. These may be a more suitable frame for quantum theory, than Hilbert space.

³⁴ In quantum mechanics, dimensions but not complements are uniquely determined by the inclusion relation; in classical mechanics, the reverse is true!

than classical mechanics—a conclusion corroborated by the impossibility in general of measuring different quantities independently.

17. Relation to pure logic. The models for propositional calculi which have been considered in the preceding sections are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have had to guide logicians hitherto. Hence it is interesting to compare the modifications which they introduce into Boolean algebra, with those which logicians on “intuitionist” and related grounds have tried introducing.

The main difference seems to be that whereas logicians have usually assumed that properties L71–L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities* L6 as the weakest link in the algebra of logic. Cf. the last two paragraphs of §10.

Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that $a' \cup b = \square$ implies $a \subset b$ (or dually, the assumption that $a \cap b' = \odot$ implies $b \supset a$ —the assumption that to deduce an absurdity from the conjunction of a and not b , justifies one in inferring that a implies b).³⁵

18. Suggested questions. The same heuristic reasoning suggests the following as fruitful questions.

What experimental meaning can one attach to the meet and join of two given experimental propositions?

What simple and plausible physical motivation is there for condition L5?

APPENDIX

1. Consider a projective geometry Q_{n-1} as described in §13. F is a (not necessarily commutative, but associative) field, $n = 4, 5, \dots$, $Q_{n-1} = P_{n-1}(F)$ the projective geometry of all right-ratios $[x_1 : \dots : x_n]_r$, which are the *points* of Q_{n-1} . The ($n - 2$ -dimensional) *hyperplanes* are represented by the left-ratios $[y_1 : \dots : y_n]_l$, incidence of a point $[x_1 : \dots : x_n]_r$ and of a hyperplane $[y_1 : \dots : y_n]_l$ being defined by

$$(1) \quad \sum_{i=1}^n y_i x_i = 0$$

All linear subspaces of Q_{n-1} form the lattice L , with the elements a, b, c, \dots . Assume now that an operation a' with the properties L71–L73 in §9 exists:

$$\begin{aligned} \text{L71} \quad & (a')' = a \\ \text{L72} \quad & a \cap a' = \odot \text{ and } a \cup a' = \square, \\ \text{L73} \quad & a \subset b \text{ implies } a' \supset b'. \end{aligned}$$

³⁵ It is not difficult to show, that assuming our axioms L1–5 and 7, the distributive law L6 is equivalent to this postulate: $a' \cup b = \square$ implies $a \subset b$.

They imply (cf. §9)

$$\text{L74 } (a \cap b)' = a' \cup b' \text{ and } (a \cup b)' = a' \cap b'.$$

Observe, that the relation $a \subset b'$ is symmetric in a, b , owing to L73 and L71.

2. If $a: [x_1: \dots : x_n]_r$ is a point, then a' is an $[y_1: \dots : y_n]_l$. So we may write:

$$(2) \quad [x_1: \dots : x_n]'_r = [y_1: \dots : y_n]_l,$$

and define an operation which connects right- and left-ratios. We know from §14, that a general characterization of a' (a any element of L) is obtained, as soon as we derive an algebraic characterization of the above $[x_1: \dots : x_n]'_r$. We will now find such a characterization of $[x_1: \dots : x_n]'_r$, and show, that it justifies the description given in §14.

In order to do this, we will have to make a rather free use of *collineations* in Q_{n-1} . A collineation is, by definition, a coördinate-transformation, which replaces $[x_1: \dots : x_n]_r$ by $[\bar{x}_1: \dots : \bar{x}_n]_r$,

$$(3) \quad \bar{x}_j = \sum_{i=1}^n \omega_{ij} x_i \quad \text{for } j = 1, \dots, n.$$

Here the ω_{ij} are fixed elements of F , and such, that (3) has an inverse.

$$(4) \quad x_i = \sum_{j=1}^n \theta_{ij} \bar{x}_j, \quad \text{for } i = 1, \dots, n,$$

the θ_{ij} being fixed elements of F , too. (3), (4) clearly mean

$$\delta_{kl} = \left\{ \begin{array}{l} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l \end{array} \right\}:$$

$$(5) \quad \sum_{j=1}^n \theta_{ij} \omega_{kj} = \delta_{ik}, \quad \sum_{i=1}^n \omega_{ij} \theta_{ik} = \delta_{jk}.$$

Considering (1) and (5) they imply the contravariant coördinate-transformation for hyperplanes: $[y_1: \dots : y_n]_l$ becomes $[\bar{y}_1: \dots : \bar{y}_n]_l$, where

$$(6) \quad \bar{y}_j = \sum_{i=1}^n y_i \theta_{ij}, \quad \text{for } j = 1, \dots, n,$$

$$(7) \quad y_i = \sum_{j=1}^n \bar{y}_j \omega_{ij}, \quad \text{for } i = 1, \dots, n.$$

(Observe, that the position of the coefficients on the left side of the variables in (4), (5), and on their right side in (6), (7), is essential!)

3. We will bring about

$$(8) \quad [\delta_{i1}: \dots : \delta_{in}]'_r = [\delta_{i1}: \dots : \delta_{in}]_l \quad \text{for } i = 1, \dots, n,$$

by choosing a suitable system of coördinates, that is, by applying suitable collineations. We proceed by induction: Assume that (8) holds for $i = 1, \dots, m - 1$ ($m = 1, \dots, n$), then we shall find a collineation which makes (8) true for $i = 1, \dots, m$.

Denote the point $[\delta_{i1} : \dots : \delta_{in}]_r$ by p_i^* , and the hyperplane $[\delta_{i1} : \dots : \delta_{in}]'_l$ by h_i^* . Our assumption on (8) is: $p_i^{*'} = h_i^*$ for $i = 1, \dots, m - 1$. Consider now a point $a: [x_1 : \dots : x_n]_r$, and the hyperplane $a': [y_1 : \dots : y_n]_l$. Now $a \leq p_i^{*'} = h_i^*$ means (use (1)) $x_i = 0$, and $p_i^* \leq a'$ means (use (8)) $y_i = 0$. But these two statements are equivalent. So we see: If $i = 1, \dots, m - 1$, then $x_i = 0$ and $y_i = 0$ are equivalent.

Consider now $p_m^*: [\delta_{m1} : \dots : \delta_{mn}]_r$. Put $p_m^{*'}: [y_1^* : \dots : y_n^*]_l$. As $\delta_{mi} = 0$ for $i = 1, \dots, m - 1$, so we have $y_i^* = 0$ for $i = 1, \dots, m - 1$. Furthermore, $p_m^* \cap p_m^{*'} = 0, p_m^* \neq 0$, so p_m^* not $\leq p_m^{*'}$. By (1) this means $y_m^* \neq 0$.

Form the collineation (3), (4), (6), (7), with

$$\theta_{ii} = \omega_{ii} = 1, \quad \theta_{mi} = \omega_{im} = y_m^{*-1} y_i^* \quad \text{for } i = m + 1, \dots, n,$$

all other $\theta_{ij}, \omega_{ij} = 0$.

One verifies immediately, that this collineation leaves the coördinates of the $p_i^*: [\delta_{i1} : \dots : \delta_{in}]_r, i = 1, \dots, n$, invariant, and similarly those of the $p_i^{*'}: [\delta_{i1} : \dots : \delta_{im}]_l, i = 1, \dots, m - 1$, while it transforms those of

$$p_m^{*'}: [y_1^* : \dots : y_n^*]_l$$

into $[\delta_{m1} : \dots : \delta_{mn}]_l$.

So after this collineation (8) holds for $i = 1, \dots, m$.

Thus we may assume, by induction over $m = 1, \dots, n$, that (8) holds for all $i = 1, \dots, n$. This we will do.

The above argument now shows, that for $a: [x_1 : \dots : x_n]_r, a': [y_1 : \dots : y_n]_l$,

$$(9) \quad x_i = 0 \text{ is equivalent to } y_i = 0, \quad \text{for } i = 1, \dots, n.$$

4. Put $a: [x_1 : \dots : x_n]_r, a': [y_1 : \dots : y_n]_l$, and $b: [\xi_1 : \dots : \xi_n]_r, b': [\eta_1 : \dots : \eta_n]_l$.

Assume first $\eta_1 = 1, \eta_2 = \eta, \eta_3 = \dots = \eta_n = 0$. Then (9) gives $\xi_1 \neq 0$, so we can normalize $\xi_1 = 1$, and $\xi_3 = \dots = \xi_n = 0$. ξ_2 can depend on $\eta_2 = \eta$ only, so $\xi_2 = f_2(\eta)$.

Assume further $x_1 = 1$. Then (9) gives $y_1 \neq 0$, so we can normalize $y_1 = 1$. Now $a \leq b'$ means by (i) $1 + \eta x_2 = 0$, and $b \leq a'$ means $1 + y_2 f_2(\eta) = 0$. These two statements must, therefore, be equivalent. So if $x_2 \neq 0$, we may put $\eta = -x_2^{-1}$, and obtain $y_2 = -(f_2(\eta))^{-1} = -(f_2(-x_2^{-1}))^{-1}$. If $x_2 = 0$, then $y_2 = 0$ by (9). Thus, x_2 determines at any rate y_2 (independently of x_3, \dots, x_n): $y_2 = \varphi_2(x_2)$. Permuting the $i = 2, \dots, n$ gives, therefore:

There exists for each $i = 2, \dots, n$ a function $\varphi_i(x)$, such that $y_i = \varphi_i(x_i)$.
Or:

$$(10) \quad \text{If } a: [1 : x_2 : \dots : x_n]_r, \quad \text{then } a': [1 : \varphi_2(x_2) : \dots : \varphi_n(x_n)]_l.$$

Applying this to $a: [1:x_2:\dots:x_n]_r$ and $c: [1:u_1:\dots:u_n]_r$ shows: As $a \leq c'$ and $c \leq a'$ are equivalent, so

$$(11) \quad \sum_{i=2}^n \varphi_i(u_i) x_i = -1 \text{ is equivalent to } \sum_{i=2}^n \varphi(x_i) u_i = -1.$$

Observe, that (9) becomes:

$$(12) \quad \varphi_i(x) = 0 \text{ if and only if } x = 0.$$

5. (11) with $x_3 = \dots = x_n = u_3 = \dots = u_n = 0$ shows: $\varphi_2(u_2)x_2 = -1$ is equivalent to $\varphi_2(x_2)u_2 = -1$. If $x_2 \neq 0$, $u_2 = (-\varphi_2(x_2))^{-1}$, then the second equation holds, and so both do.

Choose x_2, u_2 in this way, but leave $x_3, \dots, x_n, u_3, \dots, u_n$ arbitrary. Then (11) becomes:

$$(13) \quad \sum_{i=3}^n \varphi_i(u_i) x_i = 0 \text{ is equivalent to } \sum_{i=3}^n \varphi_i(x_i) u_i = 0.$$

Now put $x_5 = \dots = x_n = u_5 = \dots = u_n = 0$. Then (13) becomes:

$$\varphi_3(u_3)x_3 + \varphi_4(u_4)x_4 = 0 \text{ is equivalent to } \varphi_3(x_3)u_3 + \varphi_4(x_4)u_4 = 0,$$

that is (for $x_4, u_4 \neq 0$):

$$(a) \quad x_3x_4^{-1} = \varphi_4(u_4)^{-1} \varphi_3(u_3)$$

(14) is equivalent to

$$(b) \quad u_3u_4^{-1} = \varphi_4(x_4)^{-1} \varphi_3(x_3).$$

Let x_4, x_3 be given. Choose u_3, u_4 so as to satisfy (b). Then (a) is true, too. Now (a) remains true, if we leave u_3, u_4 unchanged, but change x_3, x_4 without changing $x_3x_4^{-1}$. So (b) remains too true under these conditions, that is, the value of $\varphi_4(x_4)^{-1} \varphi_3(x_3)$ does not change. In other words: $\varphi_4(x_4)^{-1} \varphi_3(x_3)$ depends on $x_3x_4^{-1}$ only. That is: $\varphi_4(x_4)^{-1} \varphi_3(x_3) = \varphi_{34}(x_3x_4^{-1})$. Put $x_3 = xz, x_4 = x$, then we obtain:

$$(15) \quad \varphi_3(xz) = \varphi_4(x) \psi_{34}(z).$$

This was derived for $x, z \neq 0$, but it will hold for x or $z = 0$, too, if we define $\psi_{34}(0) = 0$. (Use (12).)

(15), with $z = 1$ gives $\varphi_3(x) = \varphi_4(x)\alpha_{34}$, where $\alpha_{34} = \psi_{34}(1) \neq 0$, owing to (12) for $x \neq 0$. Permuting the $i = 2, \dots, n$ gives, therefore:

$$(16) \quad \varphi_i(x) = \varphi_i(x)\alpha_{ij}, \quad \text{where } \alpha_{ij} \neq 0.$$

(For $i = j$ put $\alpha_{ii} = 1$.)

Now (15) becomes

$$(17) \quad \begin{aligned} \varphi_2(xz) &= \varphi_2(x)w(z) \\ w(z) &= \alpha_{42}\psi_{34}(z)\alpha_{23}. \end{aligned}$$

Put $x = 1$ in (17), write x for z , and use (16) with $j = 2$:

$$(18) \quad \varphi_i(x) = \beta w(z) \gamma_i, \quad \text{where } \beta, \gamma_i \neq 0. \\ (\beta = \varphi_2(1), \gamma_i = \alpha_{i2}).$$

6. Compare (17) for $x = 1, z = u; x = u, z = v$; and $x = 1, z = vu$. Then

$$(19) \quad w(vu) = w(u)w(v)$$

results (12) and (18) give

$$(20) \quad w(u) = 0 \text{ if and only if } u = 0.$$

Now write $w(z), \gamma_i$ for $\beta w(z)\beta^{-1}, \beta\gamma_i$. Then (18), (19), (20) remain true, (18) is simplified in so far, as we have $\beta = 1$ there. So (11) becomes

$$(21) \quad \sum_{i=2}^n w(u_i) \gamma_i x = -1$$

(21) is equivalent to

$$\sum_{i=2}^n w(x_i) \gamma_i u_i = -1$$

$x_2 = x, u_2 = u$ and all other $x_i = u_i = 0$ give: $w(u)\gamma_2 x = -1$ is equivalent to $w(x)\gamma_2 u = -1$. If $x \neq 0, u = -\gamma_2^{-1} w(x)^{-1}$, then the second equation holds, and so the first one gives: $x = -\gamma_2^{-1} w(u)^{-1} = -\gamma_2^{-1} (w(-\gamma_2^{-1} w(x)^{-1}))^{-1}$. But (19), (20) imply $w(1) = 1, w(w^{-1}) = w(w)^{-1}$, so the above relation becomes:

$$x = -\gamma_2^{-1} (w(-\gamma_2^{-1} w(x)^{-1}))^{-1} = -\gamma_2^{-1} w((- \gamma_2^{-1} w(x)^{-1})^{-1}) \\ = -\gamma_2^{-1} w(w(x)(- \gamma_2)) = -\gamma_2^{-1} w(-\gamma_2)w(w(x)).$$

Put herein $x = 1$, as $w(w(1)) = w(1) = 1$, so $-\gamma_2^{-1} w(-\gamma_2) = 1, w(-\gamma_2) = -\gamma_2$ results. Thus the above equation becomes

$$(22) \quad w(w(x)) = x,$$

and $w(-\gamma_2) = -\gamma_2$ gives, if we permute the $i = 2, \dots, n$,

$$(23) \quad w(-\gamma_i) = -\gamma_i.$$

Put $u_i = -\gamma_i^{-1}$ in (21). Then considering (22) and (19)

$$(24) \quad \sum_{i=2}^n x_i = 1 \text{ is equivalent to } \sum_{i=2}^n w(x_i) = 1$$

obtains. Put $x_2 = x, x_3 = y, x_4 = 1 - x - y, x_5 = \dots = x_n = 0$. Then (24) gives $w(x) + w(y) = 1 - w(1 - x - y)$. So $w(x) + w(y)$ depends on $x + y$ only. Replacing x, y by $x + y, 0$ shows, that it is equal to $w(x + y) + w(0) = w(x + y)$ (use 20). So we have:

$$(25) \quad w(x) + w(y) = w(x + y)$$

In other words: (27) holds for $x_2 \neq 0$ too.

Permuting $i = 2, \dots, n$ (only $i = 1$ has an exceptional rôle in (27)), we see: (27) holds if $x_i \neq 0$ for $i = 2, \dots, n$. For $x_1 \neq 0$ (27) held anyhow, and for some $i = 1, \dots, n$ we must have $x_i \neq 0$. Therefore:

(27) holds for all points $a: [x_1: \dots : x_n]_r$.

8. Consider now two points $a: [x_1: \dots : x_n]_r$ and $b: [\xi_1: \dots : \xi_n]_r$. Put $a': [y_1: \dots : y_n]_l$, then $b \leq a'$ means, considering (1) and (27) (cf. the end of 7.):

$$(28) \quad \sum_{i=1}^n w(x_i)\gamma_i \xi_i = 0.$$

$a \leq a'$ can never hold ($a \cap a' = 0, a \neq 0$), so (28) can only hold for $x_i = \xi_i$, if all $x_i = 0$. Thus,

$$(29) \quad \sum_{i=1}^n w(x_i)\gamma_i x_i = 0 \text{ implies } x_1 = \dots = x_n = 0.$$

Summing up the last result of 6., and formulae (26), (29) and (28), we obtain:
There exists an involutory antisomorphism $w(x)$ of F (cf. (22), (25), (19)) and a definite diagonal Hermitian form $\sum_{i=1}^n w(x_i)\gamma_i \xi_i$ in F (cf. (26), (29)), such that for $a: [x_1: \dots : x_n]_r, b: [\xi_1: \dots : \xi_n]_r$ $b \leq a'$ is defined by polarity with respect to it:

$$(28) \quad \sum_{i=1}^n w(x_i)\gamma_i \xi_i = 0.$$

This is exactly the result of §14, which is thus justified.

THE SOCIETY OF FELLOWS, HARVARD UNIVERSITY.
 THE INSTITUTE FOR ADVANCED STUDY.

(25), (19) and (22) give together:

$w(x)$ is an involutory antisomorphism of F .

Observe, that (25) implies $w(-1) = -w(1) = -1$, and so (23) becomes

$$(26) \quad w(\gamma_i) = \gamma_i.$$

7. Consider $a: [x_1: \dots: x_n]_r$, $a': [y_1: \dots: y_n]_l$. If $x_1 \neq 0$, we may write $a: [1: x_2 x_1^{-1}: \dots: x_n x_1^{-1}]_r$, and so $a': [1: w(x_2 x_1^{-1}) \gamma_2: \dots: w(x_n x_1^{-1}) \gamma_n]_l$. But

$$w(x_i x_1^{-1}) \gamma_i = w(x_1^{-1}) w(x_i) \gamma_i = w(x_1)^{-1} w(x_i) \gamma_i,$$

and so we can write

$$a': [w(x_1): w(x_2) \gamma_2: \dots: w(x_n) \gamma_n]_l$$

too. So we have

$$(27) \quad y_i = w(x_i) \gamma_i \quad \text{for } i = 1, \dots, n,$$

where the γ_i for $i = 2, \dots, n$ are those from 6., and $\gamma_1 = 1$. And $w(1) = 1$, so (26) holds for all $i = 1, \dots, n$. So we have the representation (27) with γ_i obeying (26), if $x_i \neq 0$.

Permutation of the $i = 1, \dots, n$ shows, that a similar relation holds if $x_2 \neq 0$:

$$(27^+) \quad y_i = w^+(x_i) \gamma_i^+,$$

$$(26^+) \quad w^+(\gamma_i^+) = \gamma_i^+,$$

$w^+(x)$ being an involutory antisomorphism of F . ($w^+(x)$, γ_i^+ may differ from $w(x)$, γ_i !) Instead of $\gamma_1 = 1$ we have now $\gamma_2^+ = 1$, but we will not use this.

Put all $x_i = 1$. Then $a': [y_1: \dots: y_n]_l$ can be expressed by both formulae (27) and (27+). As $w(x)$, $w^+(x)$ are both antisomorphism, so $w(1) = w^+(1) = 1$, and therefore $[y_1: \dots: y_n]_l = [\gamma_1: \dots: \gamma_n]_l = [\gamma_1^+: \dots: \gamma_n^+]_l$ obtains. Thus $(\gamma_1^+)^{-1} \gamma_i^+ = (\gamma_1)^{-1} \gamma_i = \gamma_i$, $\gamma_i^+ = \gamma_1^+ \gamma_i$ for $i = 1, \dots, n$.

Assume now $x_2 \neq 0$ only. Then (27+) gives $y_i = w^+(x_i) \gamma_i^+$, but as we are dealing with left ratios, we may as well put

$$y_i = (\gamma_1^+)^{-1} w^+(x_i) \gamma_i^+ = (\gamma_1^+)^{-1} w^+(x) \gamma_1^+ \gamma_i.$$

Put $\beta^+ = \gamma_1^+ \neq 0$, then we have:

$$(27^{++}) \quad y_i = \beta^{+-1} w^+(x_i) \beta^+ \gamma_i.$$

Put now $x_1 = x_2 = 1$, $x_3 = x$, all other $x_i = 0$. Again $a': [y_1: \dots: y_n]_l$ can be expressed by both formulae (27) and (27^{++}), again $w(1) = w^+(1)$. Therefore

$$\begin{aligned} [y_1: y_2: y_3: y_4: \dots: y_n]_l &= [\gamma_1: \gamma_2: w(x) \gamma_3: 0: \dots: 0]_l \\ &= [\gamma_1: \gamma_2: \beta^{+-1} w^+(x) \beta^+ \gamma_3: 0: \dots: 0]_l \end{aligned}$$

obtains. This implies $w(x) = \beta^{+-1} w^+(x) \beta^+$ for all x , and so (27^{++}) coincides with (27).