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ALGEBRA OF ANALYSIS

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TABLE OF CONTENTS

INTRODUCTION

The statements of analysis can be grouped into three classes according to the depth to which the limit concept is used in their formulation and proof.

A first class consists of theorems which are entirely independent of the concept of limit, and deal with approximations. To this group belong graphical and numerical differentiation and integration as well as statements concerning the reciprocity of these two approximative operations.

A second class consists of formulae in whose proofs the concept of limit is used in a mild, so to speak, algebraic, way. This group comprises the bulk of formulae of calculus concerning elementary functions and some formulae concerning all differentiable functions: the rules for the formation of the derivatives of elementary functions, the determination of antiderivatives by substitution and by parts, etc. (Not included in this group is the theorem that each two antiderivatives of the same function differ at most by a constant).

A third group of statements is based on the assumption that in each closed interval each continuous function assumes its maximum. To this group belong the mean value theorem and its applications, of which I mention the Taylor development and its implications concerning maxima and minima, indeterminate forms, and the theorem about the antiderivatives of the same function.

In this book, we shall develop the second group of statements from a few assumptions concerning three operations (addition, multiplication, substitution) and two operators (derivation and antiderivation). A first part is devoted to the three operations. A second part deals with the Algebra of Derivation and Antiderivation. A third part contains a sketch of the theory of functions of several variables.

In developing the Algebra of Analysis, we shall make use of the notation of the operator theory. Furthermore, we shall completely avoid variables in our formulae. These principles necessitate changes of the current notation most of which result in formal as well as conceptual simplification and systematization.

In initiating students into calculus, at the present time one may find it hard to take full advantage of all these simplifications - not on account of any specific difficulties inherent to the proposed set-up or the proposed notation, but because a student of calculus must be enabled to read books on differential equations, theoretical physics, mathematical economics, etc., all of which at present are written in the classical notation. It will take a long time till these applied fields will be presented in a more modern way. In the meantime, students must acquire not only the knowledge but an operative grasp of the traditional notation with all its shortcomings from which, in fact, some applied fields, as physical chemistry, suffer more than mathematics proper. However, a gradual change of our obsolete notation probably is not only desirable but unavoidable. The first step in this direction is undoubtedly an uncompromising exposition of the new ideas for professional mathematicians and especially teachers of mathematics. It is one of the aims of this book to provide the reader with such an exposition.

Several sections of this publication may be helpful in simplifying the current presentation of calculus even when our treatment is translated into the classical notation. In this connection, we mention the Algebra of Antiderivation, the development of the entire differential calculus from a few formulae, our treatment of the exponential and tangential functions on the basis of the functional equations and the formulae

$$D \exp 0 = 1 \quad \text{and} \quad D \tan 0 = 1,$$

our introduction of the power functions, and the treatment of antiderivation by substitution.

Apart from these pedagogical aspects, our Algebra of Analysis seems to open an extended field of research. Many additional results will be published in more technical papers. Perhaps one will find the general idea of Algebra of Analysis related to that of our Algebra of Geometry whose development the author has outlined in the second lecture in the Rice Institute Pamphlets, Vol. 27, 1940, p. 41-80.

I. THE ALGEBRA OF FUNCTIONS OF ONE VARIABLE

1. The Classical Foundation of the Theory of Functions.

The classical theory starts with the assumption that a field of numbers is given. That is to say, it starts with a system N of things, called numbers, which we add and multiply according to the well-known laws of a field.

Next, the theory of functions of one variable explicitely defines a function f as the association of a number f(x) with each number x of some subset D_f of N. This set D_f is called the domain of f. The set R_f of all numbers f(x) which f associates with the numbers x of D_f, is called the range of f. A function whose range consists of exactly one number is called constant*).

The definition of the concept of functions is followed by explicit definitions of the concept of equality of functions and of three binary operations: addition, multiplication, and substitution. We define: $f = g$ if and only if $D_f = D_g$ and $f(x) = g(x)$ for each number x of $D_f = D_g$. (The concept of equality of numbers of the given field is assumed to be known). If $D_f = D_g$, we call sum of f and g [product of f and g] the function which associates the number $f(x) + g(x)$ [the number $f(x) \cdot g(x)$] with each number x of $D_f = D_g$. If R_g is a subset

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*) Analyzing the somewhat vague concept of "association" we see that f is a set of ordered pairs of numbers (d,r) such that each element of D_f occurs as the first element of exactly one pair. The second element, r, of the pair (d,r) is that element f(d) of R_f which f "associates" with d.

of D_f, we call f of g the function which associates the number f(g(x)) with each number x of D_g.

It is an odd fact that in the classical calculus no symbols for the sum of f and g, the product of f and g, and f of g, are introduced. Using the notation of the calculus of operators, we shall denote the results of the three operations by

$$f + g, \qquad f\cdot g, \qquad fg,$$

respectively, so that the numbers associated by these functions with the number x are

$$(f+g)(x), \qquad (f\cdot g)(x), \qquad (fg)(x),$$

respectively. We shall never omit the dot symbolizing multiplication in order to avoid a confusion of multiplication with substitution. In this notation the classical definitions of $f+g$, $f\cdot g$, fg read

$$(f+g)(x) = f(x) + g(x), \; (f\cdot g)(x) = f(x)\cdot g(x), \; (fg)(x) = f(g(x)).$$

From these explicit definitions of the classical theory, one deduces properties of the defined concepts. As two examples we mention the commutative law for multiplication and the associative law for substitution.

In order to prove $f\cdot g = g\cdot f$, by virtue of the equality concept, we have to show that $(f\cdot g)(x) = (g\cdot f)(x)$ for each x. From the definition of the product of functions we obtain $(f\cdot g)(x) = f(x)\cdot g(x)$ and $(g\cdot f)(x) = g(x)\cdot f(x)$. From the commutativeness of multiplication in the given field of numbers, it follows that $f(x)\cdot g(x) = g(x)\cdot f(x)$ which completes the proof.

In order to prove $f(gh) = (fg)h$, we have to prove $[f(gh)](x) = [(fg)h](x)$ for each x. Now from the definition of substitution it follows that

$$[f(gh)](x) = f[(gh)(x)] = f[g(h(x))]$$

$$[(fg)h](x) = (fg)(h(x)) = f[g(h(x))]$$

which completes the proof.

The commutative law for substitution does not generally hold, as we see, if we set $f(x) = 1 + x$ and $g(x) = x^2$. We have $f(g(x)) = 1 + x^2$ and $g(fx)) = (1 + x)^2$.

2. Algebra of Functions (Tri-Operational Algebra).

In contrast to the classical approach we do not start with a given field of numbers and do not define functions in terms of numbers. In fact, we shall not give any explicit definition of functions, and we shall abolish the dualism between numbers and functions*).

We start out with a system of things, called functions and denoted by small letters $f, g, \ldots$, and three binary operations: addition denoted by $+$, multiplication denoted by $\cdot$, and substitution denoted by juxtaposition. For these operations we postulate the following laws which in the classical theory (as we have seen in two examples) are deduced from the explicit definitions:

Operation:	Addition	Multiplication	Substitution
Commutative Law:	$f+g=g+f$	$f\cdot g=g\cdot f$	-----------
Associative Law:	$(f+g)+h=f+(g+h)$	$(f\cdot g)\cdot h=f\cdot(g\cdot h)$	$(fg)h=f(gh)$

*) At the same time we rid the foundations of analysis of set theoretical elements which are contained in the explicit definition of functions (see footnote p.4).

Distributive Law:	$(f\cdot g)h = fh\cdot gh$	$(f+g)h = fh+gh$	$(f+g)\cdot h = f\cdot h+g\cdot h$
Neutral Elements:	$f+0 = f$	$f\cdot 1 = f$	$fj = jf = f$
	$0 \neq 1 \neq j \neq 0$		$10 = 1$
Opposite Elements:	$f+(-f) = 0$		

Commutativeness of substitution is not postulated because it does not hold in the classical theory. The three distributive laws listed under

addition multiplication substitution

will be called

multiplicative-substitutive, additive-substitutive, additive-multiplicative, distributive laws, respectively, or briefly,

m.s.d. law a.s.d. law a.m.d. law.

From the commutative law of multiplication and the a.m.d. law it follows that $h\cdot(f+g) = h\cdot f+h\cdot g$. In absence of a commutative law for substitution the a.s.d. law and the m.s.d. law do not imply

$$h(f+g) = hf+hg \quad \text{and} \quad h(f\cdot g) = hf\cdot hg.$$

In fact, in the classical theory these formulae are not generally true. Each of them represents a functional equality characterizing a special class of functions h.

The neutral elements 0 and 1 in our algebra correspond to the classical functions associating with each x the numbers 0 and 1, respectively. From the commutativity of addition and multiplication in conjunction with the postulates concerning 0 and 1, it follows that $0+f = f$ and $1\cdot f = f$. For the neutral element of substitution, j, both $jf = f$ and $fj = f$ must be postulated. The element j corresponds to the classical function associating with each number x the number x. Oddly enough, the classical calculus does not introduce a symbol for this fundamental function.

In the same way as one assumes $0 \neq 1$ in defining a field, we assume that the three neutral elements 0, 1, and j are mutually different. Furthermore, we had to postulate that 0 substituted in 1 yields 1 because we shall have to make use of this assumption and, as Rev. F. L. Brown proved by an example, it is independent of the other postulates. (Under certain conditions, the postulate $10 = 1$ could be replaced by the simpler formula $10 \neq 0$).

The three neutral elements can easily be shown to be unique. For instance, if we had also $fj' = f$ for each f, then by applying this formula to $f = j$ we should obtain $jj' = j$. Now $jj' = j'$ by virtue of the neutrality of j. Hence $j = j'$.

With regard to addition we assume the existence of an opposite element to each f. We denote by $-f$ the function which, added to f, yields the sum 0. The formulae which we assumed with regard to addition and multiplication are just those valid in a commutative ring with a unit, i.e., a neutral element of multiplication*).

In the classical theory the three operations are not universal (i.e., applicable to each pair of functions). E.g., we can form fg only if R_g is part of D_f. Hence we shall not

*) It should be noted that among the formulae concerning addition and substitution, one which would be valid in a non-commutative ring has not been postulated, namely, $h(f+g) = hf + hg$.

postulate in our algebra that the three operations are universal. We shall interpret our postulates as <u>formulae which are valid if all terms are meaningful</u>. The situation is the same as in a grupoid in which $f+(g+h) = (f+g)+h$ is true provided that $g+h$, $f+g$, $f+(g+h)$, and $(f+g)+h$ exist.

However, it is worth mentioning that our postulates are consistent even in presence of the additional assumption that the three operations are universal. A model satisfying all these assumptions is the system of all polynomials $p = c_0 + c_1 \cdot j + c_2 \cdot j^2 + \cdots + c_m \cdot j^m$ with coefficients c_k belonging to a given ring (where j^k is an abbreviation for a product of k factors j) if sum, product, and substitution are defined in the ordinary way. That is, if

$q = d_0 + d_1 \cdot j + d_2 \cdot j + \cdots + d_n \cdot j^n$, then

$p + q = (c_0 + d_0) + (c_1 + d_1) \cdot j + (c_2 + d_2) \cdot j^2 + \cdots$

$p \cdot q = c_0 d_0 + (c_0 \cdot d_1 + c_1 \cdot d_0) \cdot j + (c_0 \cdot d_2 + c_1 \cdot d_1 + c_2 \cdot d_0) \cdot j^2 + \cdots$

$pq = c_0 + c_1 \cdot q + c_2 \cdot q^2 + \cdots + c_m \cdot q^m$.

A simple model consisting of four functions 0,1,j,f, is obtained if the three operations are defined by the following tables:

+	0	1	j	f
0	0	1	j	f
1	1	0	f	j
j	j	f	0	1
f	f	j	1	0

•	0	1	j	f
0	0	0	0	0
1	0	1	j	f
j	0	j	j	0
f	0	f	0	f

	0	1	j	f
0	0	0	0	0
1	1	1	1	1
j	0	1	j	f
f	1	0	f	j

In the classical theory we obtain the above system*) by considering the field consisting of the numbers 0 and 1 (modulo 2) and by calling 0,1,j,f the functions associating with the numbers 0 and 1 the numbers 0,0; 1,1; 0,1; 1,0, respectively. As Rev. F. L. Brown has recently shown, this system is the simplest one satisfying all the postulates since no system consisting of 0,1, and j only satisfies all postulates. However, Father Brown did find a system consisting of the three elements 0,1,j, satisfying all the postulates except the one concerning the existence of a negative element.

3. The Theory of Constant Functions.

We shall now single out a class of functions which we will call constant functions or, briefly, constants. The definition will be in terms of the fundamental operations. From the postulates concerning these operations, it will follow that our constants enjoy the main properties of the constant functions of the classical theory.

We call a function f constant if $f = f0$. If we know of a function that it is constant, then we shall usually denote it by letters c,d,...

From the postulates it follows that $0f = (0+0)f = 0f + 0f$. Adding $-(0f)$ to this equality we obtain the formula

$$0f = 0.$$

In particular $00 = 0$. Hence 0 is a constant. That 1 is a constant, is the content of the postulate $10 = 1$. Since $j0 = 0 \neq j$, we see that j is not a constant. We shall prove

*) It can also be described as the system of all polynomials modulo $j + j^2$ with coefficients belonging to the field 0,1 modulo 2.

the following theorem:

<u>The constants form a ring*) which is closed with respect to substitution</u>. If c_1 and c_2 are constants, then

$$c_1 + c_2 = c_1 0 + c_2 0 = (c_1 + c_2)0.$$

Similarly, $c_1 \cdot c_2 = (c_1 \cdot c_2)0$. Thus the sum and the product of two constants are constants. From the fact that 0 is a constant, it readily follows that the negative of a constant is a constant. Thus the constants form a ring. Now let c be a constant, and f any function. We have

$$fc = f(c0) = (fc)0.$$

Thus fc is a constant. Using the formula $0f = 0$ we obtain

$$cf = (c0)f = c(0f) = c0 = c.$$

Thus cf is a constant, more specifically, cf is the constant c. This completes the proof of our theorem.

The last formula can also be expressed by saying that if c is a constant, then not only $c0 = c$ but $cf = c$ for each f.

If for a constant c there exists a function c' such that $c \cdot c' = 1$, then this "reciprocal" c' is a constant. For

$$c'0 = 1 \cdot c'0 = (c' \cdot c) \cdot c'0 = c' \cdot (c \cdot c'0) = c' \cdot (c0 \cdot c'0) =$$
$$c' \cdot (c \cdot c')0 = c' \cdot 10 = c' \cdot 1 = c'.$$

We see: if for each constant $c \neq 0$ there exists a reciprocal, then the constants form a field which is closed with respect to substitution. However, the roots of an algebraic

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*) Quite accurately, we should say: For the constants all formulae postulated in a commutative ring are valid if they are meaningful. Under additional assumptions and with a sharper definition of constants, we could prove them to form a ring. We should have to call constant a function c such that c0 is defined and = 0. We should have to assume that if f0 and g0 are defined, then (-f)0, (f + g)0, and (f·g)0 are defined.

equation with constant coefficients need not be constants. For instance, each of the four functions $0, 1, j, 1+j$, studied at the end of the last section, satisfies the algebraic equation with constant coefficients $f + f^2 = 0$. The functions j and $1+j$ are not constant.

The definition of equality of two functions in the classical theory is reflected in the following fundamental proposition of our algebra: If $fc = gc$ for each constant c, then $f = g$. If this proposition holds, then we shall speak of a tri-operational algebra with a base of constants.

Clearly, in such an algebra we have $f = g$ if and only if $fc = gc$ for each constant c. Moreover, in order that f be a constant it is necessary and sufficient that $fc = f0$ for each constant c. For from $fc = f0$ it follows that $fc = f0 = f(0c) = (f0)c$. Applying the equality criterion to f and $f0$ we see that f is equal to the constant function $f0$.

A consequence of this last remark is the following first theorem: If for two constants c_0 and c_1 we have $f(c_0 + j) = c_1$, then $f = c_1$. In fact, for each constant c the assumption implies that

$$fc = f(c_0 + (c - c_0)) = f(c_0 + j)(c - c_0) = c_1(c - c_0) = c_1.$$

Another consequence is the following translation theorem: If $f(j + c) = f$ for each constant c, then f is constant. For from the assumption it follows that

$$fc = f(0 + c) = f(j0 + c0) = f[(j + c)0] = [f(j + c)]0 = f0.$$

An Algebra of Functions admits a representation by functions in the classical sense. With each function of our algebra we can associate a function in the classical sense whose domain and whose range are two sets of constants. With the function f of our algebra we can associate the function f^* in the classical sense whose domain consists of those constants which admit substitution into f, and which associates with each such constant c the constant fc. This association of functions in the classical sense with functions of our algebra is readily seen to be a homomorphism. That is to say, we have

$$(f+g)^* = f^* + g^*, \quad (f,g)^* = f^* \cdot g^*, \quad (fg)^* = f^* g^*$$

where addition, multiplication, and substitution of the classical functions on the right sides of these equalities are to be performed in the classical sense. The postulate of a base of constants implies that the above homomorphism is an isomorphism, that is to say, that $f \neq g$ implies $f^* \neq g^*$.

It is to be noted that even in an algebra in which the three fundamental operations are universal and the constants form a field, the postulate of the base of constants need not be satisfied. We obtain an example of the independence of this postulate by considering all polynomials

$$c_0 + c_1 \cdot j + c_2 \cdot j^2 + \cdots + c_m \cdot j^m$$

with coefficients 0 and 1 if addition, multiplication, and substitution are defined in the ordinary sense but modulo 2. There are infinitely many such polynomials but only two constants, viz., 0 and 1. For each polynomial the substitution of 0 and 1 yields either 0 and 0, or 0 and 1, or 1 and 0, or 1 and 1. If we write a polynomial in the form

$$p = c_o + p^{k_1} + p^{k_2} + \dots + p^{k_n}$$

where c_o is 0 or 1, then clearly p belongs to one of the following four classes:

Either $c_o = 0$ and n is even. Then $p0 = 0$, $p1 = 0$,

Or, $c_o = 0$ and n is odd. Then $p0 = 0$, $p1 = 1$.

Or, $c_o = 1$ and n is odd. Then $p0 = 1$, $p1 = 0$.

Or, $c_o = 1$ and n is even. Then $p0 = 1$, $p1 = 1$.

If p_1 and p_2 are two of the infinitely many polynomials belonging to the same class, then for each constant (that is, for $c = 0$ and $c = 1$) we have $p_1c = p_2c$ and yet $p_1 \neq p_2$. The homomorphic representation of the functions of our algebra by functions in the classical sense which we described above, would lead to the four functions defined in the field modulo 2 mentioned at the end of the preceding section. Each function of the first class would be mapped onto the function representing 0, each function of the second class on the function representing j, each function of the third class on the function representing $1 + j$, and each function of the fourth class on the function representing 1.

The following finite example for the same situation may be omitted in a first reading. We consider the polynomials of the preceding example modulo $j + j^4$. That is to say, we set $j + j^4 = 0$. We thus retain a model consisting of 16 polynomials $c_o + c_1 \cdot j + c_2 \cdot j^2 + c_3 \cdot j^3$ with coefficients $c_k = 0,1$. There are only two constants, 0 and 1, and hence only four possibilities for p0 and p1, as before. Each possibility is realized for a class of four polynomials. E.g., we have $p0 = 0$ and $p1 = 0$ for $p_1 = 0$, $p_2 = j + j^2$, $p_3 = j + j^3$, $p_4 = j + j^4$.

This is another example in which each function p, each constant function as well as each of the 14 non-constant functions, satisfies an algebraic equation with constant coefficients, namely, $p + p^4 = 0$, as we see by substituting p into the equality $j + j^4 = 0$.

4. The Lytic Operations.

While we did not postulate universality of the three fundamental operations, we saw that a postulate to this effect would be compatible with our assumptions. Now we are going to introduce a function of a special kind whose very nature, in presence of the other postulates, is incompatible with universality of substitution. We shall call this function rec (an abbreviation for reciprocal) and define it by the equality

$$\text{rec}\cdot j = 1.$$

If we substitute 0 in this equality we obtain $(\text{rec}\cdot j)0 = 10$ from which, in view of $0\cdot f = 0$, it follows that

$$0 = \text{rec } 0\cdot 0 = \text{rec } 0\cdot j0 = (\text{rec}\cdot j)0 = 10 = 1.$$

This contradicts the assumption $0 \neq 1$. We see that in presence of the definition of rec we must give up some of our postulates or abandon the universality of substitution by forbidding the substitution of the function 0 into the function rec. We shall follow the latter course.

If the constants form a field, then 0 is the only constant which cannot be substituted into rec. Into rec f we cannot substitute those constants c for which $fc = 0$. For instance, 1 cannot be substituted into $\text{rec}(j - 1)$.

It goes without saying that, in the classical notation, rec is the function associating with each number $x \neq 0$ the number $\frac{1}{x}$. From the point of view of domains, the function rec·j is not identical with the function 1. The latter is an extension of the former. For the domain of 1 comprises all numbers; that of rec, and hence of rec·j, all numbers $\neq 0$.

We shall disregard this difference and thus from now on be compelled to interpret our postulates as formulae which are valid whenever their terms are meaningful, and we shall have to interpret each result as a formula admitting those substitutions which are admissible in all terms involved in the derivation of the result from the postulates.

To make the analogy between rec f and -f more conspicuous we shall frequently write neg f instead of -f. This notation is justified since there exists a function neg such that we obtain -f by substituting f into the function neg. This function neg is -j or -1·j. In the classical notation it is the function associating with each number x the number -x.

Instead of postulating the existence of -f for each f it would be sufficient to postulate the existence of a function neg such that j + neg = 0. In view of 0f = 0, jf = f, and the a.s.d. law this postulate implies f + neg f = 0 for each f. We tabulate some analogous facts of the algebra of the functions neg and rec which we shall call <u>the lytic functions</u> with regard to addition and multiplication, respectively.

j + neg = 0	j·rec = j
f + neg f = 0	f·rec f = j
neg (f + g) = neg f + neg g	rec(f·g) = rec f·rec g
neg neg = j	rec rec = j

rec neg = neg rec.

In fact, we have

rec(f•g) = rec(f•g)•1 = rec(f•g)(f•rec f•g•rec g) =

rec(f•g)•(f•g)•(rec f•rec g) = 1• rec f•rec g = rec f•rec g.

Using the formula rec•rec rec = 1 obtained by substituting rec into j•rec = 1 we see

rec rec = (j•rec)•rec rec = j•(rec•rec rec) = j•1 = j.

The proof of neg neg = j is similar.

From neg f = -1•f it follows that neg f•neg g = f•g. Using this formula we obtain

rec neg = rec neg•(j•rec) = rec neg•(neg j•neg rec) =

rec neg•(neg•neg rec)=(rec neg•neg)•neg rec=1•neg rec = neg rec.

We define: f is <u>even</u> if and only if f neg = neg f, and f is <u>odd</u> if and only if f neg = f. The last of the tabulated formulae can be expressed by saying that rec is odd. Clearly, the product of two even or of two odd functions is even, the product of an odd and even function is odd.

Concerning the lysis of substitution, we mention that if for two functions f and g we have fg = j, then we shall call g the right inverse of f, and f the left inverse of g. For instance, the function j is its own right and left inverse since we have jj = j. If c_0 and c_1 belong to the ring of constant functions, and c_1 has a reciprocal c_1', then the function $c_1 \cdot j + c_0$ has the function $c_1' \cdot (j - c_0)$ as inverse on either side. In the classical notation, in view of j(x) = x, the definition of a pair of inverse functions reads

$$f(g(x)) = x.$$

In other words, f and g are pairs of inverse functions in the classical sense, as log and exp or arctan and tan.

We shall postulate the existence of inverse functions only for special functions f. While there are functions neg and rec satisfying the equations j + neg = 0 and j·rec = 1 and such that we obtain the negative and the reciprocal of f by substituting f into neg and rec, respectively, there certainly does not exist a function inv satisfying the equation j inv = j and such that we could obtain the inverse of f by substituting f into inv. For by virtue of the definition of j we should have j inv = inv, so that from j inv = j it would follow that inv = j. But by substituting f into j we obtain f which in general is not the inverse of f. Or we can say: By substituting f into the equality j inv = j we obtain j(inv f) = f rather than f(inv f) = j.

A constant function c clearly does not have inverse functions on either side. For, whatever function f may be, cf and fc are constants, thus $\neq j$ since j is not a constant.

If g is a right inverse of f and has itself at least one right inverse, h, then g has only one right inverse, namely, f, and only one left inverse, namely, f. And f has only one right and one left inverse, namely, g. In fact, from fg = j and gh = j it follows that h = jh = (fg)h = f(gh) = fj = f, a situation familiar from the axiomatics of group theory.

5. Exponential Functions.

We call the function f an exponential function if and only if

$$f(g+h) = fg \cdot fh \quad \text{and} \quad f \neq 0.$$

We shall denote exponential functions by exp. Thus exp(g + h) = exp g·exp h. Substituting a constant c_0 for g, and j for h we see that

$\exp(c_o + j) = \exp c_o \cdot \exp j = \exp c_o \cdot \exp$. If $\exp c_o = 0$, then it follows that $\exp(c_o + j) = 0$. If our algebra has a base of constants, then the last formula, by virtue of the first theorem of Section 3, implies that $\exp = 0$ in contradiction to the assumption $\exp \neq 0$. We thus see that $\exp c_o \neq 0$ for each constant c_o. In further consequence, $\exp f \neq 0$ for each f. For if we had $\exp f = 0$, then we should obtain

$$\exp(fc_o) = (\exp f)c_o = 0c_o = 0$$

which, in view of the fact that fc_o is a constant, contradicts the preceding remark.

If $\exp c_1 = 1$, then $\exp(f + c_1) = \exp c_1 \cdot \exp f = \exp f$ for each f. Conversely, if the constants form a field, from $\exp(c + c_1) = \exp c$ in view of $\exp c \neq 0$ it follows that $\exp c_1 = 1$. Now $\exp(c + 0) = \exp c$. Thus $\exp 0 = 1$. Consequently, $1 = \exp(j + \text{neg}) = \exp \cdot \exp \text{neg}$ and hence $\exp \text{neg} = \text{rec} \exp$.

Obviously, in each ring the function 1 is an exponential function. From $\exp 0 = 1$ it follows that 1 is the only constant exponential function. From now on, when talking about exponential functions we shall always mean exponential functions $\neq 1$.

If the constants form a finite field, then no exponential function exists. Let indeed $p \neq 0$ be the characteristic of the field of constants. Since $p - 1$ is the sum of $p - 1$ summands 1, for an exponential function we should have

$$\exp(p - 1) = (\exp 1)^{p-1}.$$

Now in a field of characteristic p we have $c^{p-1} = 1$ for each c. Hence $\exp(p - 1) = 1$. From $(p - 1) + 1 = 0$ it would follow that

$\exp(p-1)\cdot\exp 1 = \exp 0$. Since $\exp 0 = \exp(p-1) = 1$ we should have $\exp 1 = 1$. But then $\exp 2 = \exp 1\cdot\exp 1 = 1$, $\exp 3 = \exp 2\cdot\exp 1 = 1$, etc., hence $\exp c = 1$ for each c.

However, exponential functions do exist in finite rings. For instance, one readily verifies that in the ring of residues modulo 9 the function which under substitution

of 0,1,2,3,4,5,6,7,8

yields 1,4,7,1,4,7,1,4,7, respectively,

is an exponential function. In the infinite ring without divisors of 0 consisting of the numbers $m + n\cdot i$ where m and n are integers and $i^2 = -1$, the function which under substitution of $m + n\cdot i$ yields i^{m+n} is an exponential function. If the constants form the ring of all integers or the field of all rational numbers, then no exponential functions defined for all constants, exist.

6. The Logarithmic Functions.

We shall now take a step towards the algebra of real functions by assuming, in addition to a base of constants, three postulates about the ring of constants. For the sake of brevity we shall call a constant c a square if there exists a constant $c_1 \neq 0$ such that $c = c_1^2$. Now we postulate for each constant c which is not a divisor of 0:

1. If c is a square, then $-c$ is not a square.
2. If c is not a square, then $-c$ is a square.
3. There exists a constant 1/2 such that $1/2 + 1/2 = 1$ (and consequently for each c a constant c/2 such that $c/2 + c/2 = c$, namely, $c\cdot 1/2$).

Clearly, the product of two squares, as well as the product of two negative squares, is a square. The product of a square and a negative square is not a square. It follows that if a square has a reciprocal, the reciprocal is a square.

Postulate 3 is satisfied in each field of characteristic $\neq 2$. Postulate 1 can be expressed by saying that $c_1^2 + c_2^2 = 0$ implies $c_1 = c_2 = 0$, a weaker form of the postulate for a real field. Postulates 1 and 2 are sufficient to establish in the ring what may be called a multiplicative order: If we call each square "positive", then for each element c of the ring either c is a divisor of 0, or c is positive, or -c is positive, and the product of two positive elements is positive. However, even if the ring is a field, postulates 1 and 2 are not sufficient to order the field (i.e., to guarantee that also the sum of two positive elements is positive) as the field of residues modulo 7 shows if we call 1,2,4 positive. Neither does each ordered field satisfy postulate 2 as the field of all rational numbers shows.

We shall now assume that an exponential function which admits the substitution of each constant, has an inverse on both sides which admits the substitution of each square. We shall call such a function a logarithmic function and denote it by log.

For each constant d, from $\exp d = \exp(d/2 + d/2) = \exp(d/2)\cdot\exp(d/2)$ it follows that each constant exp d is a square. This fact implies that log c admits only the substitution of squares. For if log c is defined, then

$$c = jc = (\exp \log)c = \exp(\log c)$$

and exp d is a square for each d. The same reasoning, in view of exp d $\neq$ 0, shows that log does not admit the substitution of 0. Consequently, the function log (j·j) admits the substitution of each constant $\neq$ 0.

Moreover, we have log 1 = log (exp 0) = (log exp)0 = j0 = 0.

Now let log c_1 and log c_2 be defined. That is, c_1 and c_2 are squares which implies that also $c_1 \cdot c_2$ is a square and log($c_1 \cdot c_2$) is defined. We have

$$\log c_1 + \log c_2 = j(\log c_1 + \log c_2) = \log \exp(\log c_1 + \log c_2)$$
$$= \log(\exp \log c_1 \cdot \log \exp c_2) = \log(c_1 \cdot c_2).$$

It follows that

$$0 = \log 1 = \log(j \cdot j \cdot rec \cdot rec) = \log(j \cdot j) + \log(rec \cdot rec)$$

and hence log(rec·rec) = neg log(j·j), formulae which admit the substitution of each constant $\neq$ 0.

Similarly,

$$0 = \log 1 = \log(j \cdot rec) = \log j + \log rec.$$

However, the last equality admits only the substitution of squares (and of all squares since if c is a square, rec c is a square). Thus the same holds for

$$\log rec = neg \log.$$

7. The Absolute and the Signum.

Under the assumptions of the preceding section one can introduce a function which we shall call the absolute value or, briefly, the absolute, and which we shall denote by abs. We define

$$abs = \exp[\tfrac{1}{2} \cdot \log(j \cdot j)] \text{ and } abs\ 0 = 0.$$

In the classical theory the function corresponding with abs associates with each x the number $|x|$. The function abs admits the substitution of each constant and is readily seen to enjoy the following properties:

1. $\text{abs } c_1 \cdot \text{abs } c_2 = \text{abs}(c_1 \cdot c_2)$
2. $\text{abs}^2 = j^2$
3. abs neg = abs
4. $\text{abs} \neq 0$.

It is easily seen that abs rec = rec abs.

We further definr a signum function, denoted by sgn, in the following way:

$$\text{sgn} = \text{abs} \cdot \text{rec} \quad \text{and} \quad \text{sgn } 0 = 0.$$

In the classical theory the function corresponding with sgn associates 0 with 0, 1 with each positive, -1 with each negative number.

The function sgn has the following properties:

1. $\text{sgn } c_1 \cdot \text{sgn } c_2 = \text{sgn}(c_1 \cdot c_2)$
2. $\text{sgn}^3 = \text{sgn}$
3. sgn neg = neg sgn
4. $\text{sgn} \neq 0$.

One readily verifies that sgn yields 1 or -1 according to whether a square or the negative of a square is substituted. On this fact one can base another introduction of the assumptions of the preceding section, an introduction which is more in line with the Algebra of Functions than the postulates 1 and 2 concerning squares. We can postulate the existence of a function abs or a function sgn with the four properties mentioned above and define: a constant c which is not divisor of 0, is positive or negative according to whether

$$\text{abs } c = c \quad \text{or} \quad \text{abs } c = -c \quad (\text{sgn } c = 1 \text{ or sgn } c = -1).$$

We remark that the four postulates for sgn are independent. In the field of residues modulo 3 the function s which admits the substitution of all three constants 0,1,-1 and (like the function -j) yields s0 = 0, s1 = -1, s(-1) = 1 satisfies all postulates except the first. In the field of residues modulo 5 the function s which admits the substitution of 0, $\pm 1, \pm 2$ and (like the function j) yields sc = c for each c, satisfies all postulates except the second. In the same field the function s which (like j^2) yields s0 = 0, s1 = s(-1), s2 = s(-2) = -1 satisfies all postulates except the third. In each field the function 0 satisfies all postulates except the fourth.

We have

$$\text{abs exp} = \exp\ [\tfrac{1}{2}\cdot\log(j\cdot j)]\exp = \exp[\tfrac{1}{2}\cdot\log(\exp\cdot\exp)]$$
$$= \exp[\tfrac{1}{2}\cdot 2\cdot\log\ \exp] = \exp\ j = \exp.$$

For the function log abs, on account of its importance, we shall introduce a special symbol. We shall denote it by logabs. We have

$$\text{logabs} = \log\ \exp[\tfrac{1}{2}\cdot\log(j\cdot j)] = \tfrac{1}{2}\cdot\log(j\cdot j).$$

The function logabs admits the substitution of each constant $\neq 0$. It corresponds to the function associating $\log|x|$ with each $x \neq 0$ in the classical theory. We have

$$\text{logabs exp} = (\log\ \text{abs})\exp = \log(\text{abs exp}) = \log\ \exp = j,$$
$$\exp\ \text{logabs} = \exp\ \log\ \text{abs} = j\ \text{abs} = \text{abs},$$
$$\exp(\text{logabs}\ f + \text{logabs}\ g) = \exp\ \text{logabs}\ f\cdot\exp\ \text{logabs}\ g = \text{abs}\ f\cdot\text{abs}\ g.$$

By virtue of $j = \text{sgn}\cdot\text{abs}$ and $\text{sgn}(c_1\cdot c_2) = \text{sgn}\ c_1\cdot\text{sgn}\ c_2$ it follows that

$$f\cdot g = \text{sgn}\ f\cdot\text{sgn}\ g\cdot\exp(\text{logabs}\ f + \text{logabs}\ g).$$

The last formula could be used as a definition of multiplication in terms of addition and substitution, in conjunction with the exponential and the signum functions. Algebra of Functions might be developed from postulates about two operations and two particular functions, possibly one particular function.

8. The Power Functions.

We shall now for each constant c introduce a function called the c-th power and denoted by c-po. We define c-po in the same way in which it is defined in the theory of complex functions:

$$c - po = \exp(c \cdot \log).$$

From this definition it follows that c - po admits the substitution of all squares and only of squares. More accurately, we should call the above function the c-th power based on the function exp. However, in some cases we shall see that, for algebraic reasons, power functions are independent of the particular choice of the exponential function used in defining them. For instance

$0 - po = \exp(0 \cdot \log) = \exp 0 = 1$ and $1 - po = \exp(1 \cdot \log) = \exp \log = j$.

If exp' is another exponential function, log' the inverse of exp', and if we define

$$c - po' = \exp'(c \cdot \log'),$$

then, as before, we have

$0\text{-}po' = \exp'(0 \cdot \log') = \exp' 0 = 1$ and $1\text{-}po' = \exp'(1 \cdot \log') = \exp' \log' = j$.

Moreover, we obtain the following three functional equations for the power functions

$$c_1 - po \cdot c_2 - po = (c_1 + c_2) - po$$

$$c_1 - po \; c_2 - po = (c_1 \cdot c_2) - po$$

$$c - po \; f \cdot c - po \; g = c - po \; (f \cdot g).$$

Proof:

$$c_1\text{-po}\cdot c_2\text{-po} = \exp(c_1\cdot\log)\cdot\exp(c_2\cdot\log) = \exp(c_1\cdot\log + c_2\cdot\log)$$
$$= \exp[(c_1 + c_2)\cdot\log] = (c_1 + c_2) - \text{po},$$

$$c_1\text{-po}\ c_2\text{-po} = \exp(c_1\cdot\log)\ \exp(c_2\cdot\log) = \exp[c_1\cdot\log\ \exp(c_2\cdot\log)]$$
$$= \exp[c_1\cdot(c_2\cdot\log)] = \exp[(c_1\cdot c_2)\log] = (c_1\cdot c_2) - \text{po},$$

$$c\text{-po}\ f\cdot c\text{-po}\ g = \exp(c\cdot\log f)\cdot\exp(c\cdot\log g) = \exp(c\cdot\log f + c\cdot\log g)$$
$$= \exp[c\cdot(\log f + \log g)] = \exp[c\cdot\log(f\cdot g)] = c - \text{po}(f\cdot g).$$

From the first of these functional equations it follows that

$$1 = 0 - \text{po} = (c + (-c)) - \text{po} = c - \text{po}\cdot(-c) - \text{po}, \text{ and hence}$$
$$(-c) - \text{po} = \text{rec}\ c - \text{po}, \text{ in particular, } (-1) - \text{po} = \text{rec}.$$

Moreover, we have $2 - \text{po} = 1 - \text{po}\cdot 1 - \text{po} = j\cdot j$ and, by induction, we see that for each positive integer n, the n-th power is the product of n factors j. This statement as well as the functional equations for c - po are independent of the choice of the exponential function exp used in defining the power functions.

From the second functional equation it follows that

$$c - \text{po}\ \frac{1}{c} - \text{po} = \exp(c\cdot\log)\exp(\tfrac{1}{c}\cdot\log) = \exp[c\cdot\log\ \exp(\tfrac{1}{c}\cdot\log)]$$
$$= \exp[c\cdot j(\tfrac{1}{c}\cdot\log)] = \exp(c\cdot\tfrac{1}{c}\cdot\log) = \exp(1\cdot\log) = \exp\ \log = j.$$

Thus, c - po and $\frac{1}{c}$ - po are inverse functions.

In the equation $2 - \text{po} = j\cdot j$, the right side admits the substitution of any constant, while the left side admits only the substitution of squares. However, we may consistently extend the definition of c - po by the following stipulations:

1. $c - \text{po}\ 0 = 0$

2. If c is a rational number n_1/n_2 with an odd denominator n_2 and a numerator n_1 which is relatively prime to n_2, then the function c - po is even if the number n_1 is even, and odd if n_1 is odd.

We do not permit the substitution of negative squares into c - po in the remaining cases, that is, if c is a rational number with an even denominator or not rational. We remark that in case that c is a rational number n_1/n_2 whose denominator is even, we have not only

$$n_2\text{- po } c\text{-po} = n_1\text{-po} \quad \text{but also} \quad n_2\text{-po neg } c\text{-po} = n_1\text{-po}.$$

After the above extension our definition includes all the cases covered by the classical theory of power functions.

In case that c is a positive integer we readily see that the extended function c - po is identical with the product of c factors j. For $c = \frac{2m}{2n+1}$ or $= \frac{2m+1}{2n+1}$ (m and n integers) it is easily seen that the extended c-th powers can be written

$$\frac{2m}{2n+1}\text{ - po} = \exp[\frac{2m}{2n+1}\cdot\text{logabs}] = \exp[\frac{m}{2n+1}\cdot\log(j\cdot j)]$$

$$\frac{2m+1}{2n+1}\text{ - po} = \text{sgn}\cdot\exp[\frac{2m+1}{2n+1}\cdot\text{logabs}].$$

In operating with integers we have omitted and shall omit the multiplication dot. 2m+1 stands, of course, for 2·m+1.

The reader can easily check to which extent our supplementary stipulations concerning the definition of c - po are compatible with the three functional equations for power functions. For instance, the equation $c_1\text{-po } c_2\text{-po} = (c_1\cdot c_2)\text{-po}$ can not be upheld after the extension; in other words, it does not permit the substitution of negative constants. As an example, we mention

$$\frac{1}{2}\text{ - po } 2\text{-po} = \exp[\frac{1}{2}\cdot\log\ \exp(2\cdot\text{logabs})] = \exp\ \text{logabs} = \text{abs}.$$

Thus the extended function 2 - po and the function $\frac{1}{2}$ - po are not inverse. The function $\frac{1}{2}$ - po 2 - po is not = j but = abs which, in fact, is Cauchy's representation of the function abs.

9. The Trigonometric Functions.

We call f a tangential function if and only if

$$f(g+h) = \frac{fg+fh}{1-fg\cdot fh} \quad \text{and} \quad f \neq 0.$$

We shall denote a tangential function by tan. In the second chapter we shall single out among the tangential functions the ordinary tangent function.

From the definition it follows: If tan g = 0, then tan(f + g) = tan f. Moreover,

$$\tan 0 = \tan(0+0) = \frac{2\cdot\tan 0}{1-\tan 0\cdot\tan 0} .$$

Consequently, if the constant tan 0 is to be real, we have tan 0 = 0. Furthermore, it readily follows from the definition that tan is an odd function.

In this section we shall assume that the constants form the field of real numbers. Moreover, we shall postulate the existence of a smallest constant c > 0 such that tan c = 0. Then tan does not admit the substitution of the constants c/2 and -c/2. For if, say, tan (c/2) were defined, then we should have

$$0 = \tan c = \frac{2\cdot\tan(c/2)}{1-\tan(c/2)\cdot\tan(c/2)} .$$

This equality would imply tan (c/2) = 0 in contradiction to our assumption that c is the smallest number 0 for which tan c = 0. It further follows that tan (c/4) = 1 or -1 since, by virtue of the definition of tan, every other value of tan (c/4) would entail a value for tan (c/2). We shall only admit the substitution into tan of constants between $-c/2$ and $c/2$.

Relative to each tangential function we define a sine and a cosine function in the following way:

$$\sin(2\cdot j) = \frac{2\cdot \tan}{1+\tan^2} \quad \text{and} \quad \cos(2\cdot j) = \frac{1-\tan^2}{1+\tan^2}$$

We obviously have

$$\frac{\sin(2\cdot j)}{\cos(2\cdot j)} = \frac{2\cdot \tan}{1-\tan^2} = \tan(2\cdot j).$$

Substituting $\frac{1}{2}\cdot j$ into the equality, we see that $\tan = \sin/\cos$.

Other useful identities are

$$\sin^2 + \cos^2 = 1 \quad \text{and} \quad 1+\tan^2 = \text{rec}\ \cos^2.$$

We postulate an inverse of tan on both sides and call it arctan.

II. THE ALGEBRA OF CALCULUS

1. The Algebra of Derivatives.

We shall now introduce an operator D associating with a function f a function Df, called the derivative of f. We shall not attempt to formulate criteria as to which functions form the domain of the operator D or as to which constants, if any, may be substituted into Df. In our Algebra of Derivatives, we shall adopt the same point of view as in our Algebra of Functions: We shall derive formulae which are valid in classical calculus provided that the terms involved in the derivation of the formulae are meaningful. In classical calculus, for a given function f and a given constant c, the symbol (Df)c is meaningful if the function $\frac{f - fc}{j - jc}$ has a limit for c. We, too, might define (Df)c in terms of a limit operator, L, and derive the fundamental properties of D from postulates concerning L. But in the present exposition we start out with an undefined operator D subject to a few assumptions connecting D with the Algebra of Functions.

Since D is not a function, the postulates of the Algebra of Functions can not be applied to D. Especially the associative law for substitution does not hold for D. Thus the symbol Dfg is ambiguous. It may mean D(fg) or (Df)g. In order to save parentheses we shall make the convention that the symbol D refers only to the immediately following function or group of functions combined in parentheses. Thus we shall briefly write Dfg for (Df)g and reserve parentheses for the case D(fg).

Three postulates will connect D with the three fundamental operations of the Algebra of Functions:

$$\text{I.}\quad D(f+g) = Df + Dg$$

$$\text{II.}\quad D(f\cdot g) = f\cdot Dg + g\cdot Df$$

$$\text{III.}\quad D(fg) = Dfg\cdot Dg$$

Postulate III replaces the associative law for substitution with respect to D. It states that D(fg) and (Dg)f differ by the factor Dg.

By postulate I we have

$$D0 = D(0+0) = D0 + D0.$$

Thus D0 = 0. This formula has two important consequences.

By means of it we first see that

$$0 = D0 = D(f + \text{neg } f) = Df + D \text{ neg } f,$$

and thus

$$D \text{ neg } f = \text{neg } Df.$$

Secondly, if c is a constant, that is to say, if c = c0, we obtain

$$Dc = Dc0 = D(c0) = Dc0\cdot D0 = Dc0\cdot 0 = 0.$$

Postulate II now yields

$$D(c\cdot f) = c\cdot Df + f\cdot Dc = c\cdot Df + 0 = c\cdot Df.$$

We shall call this result the Constant Factor Rule.

In view of fj = f postulate III yields

$$Df = D(fj) = Dfj\cdot Dj = Df\cdot Dj.$$

Hence Dj = 1 unless Df = 0 for each f which we shall later exclude. Anticipating this development, we shall from now on assume that Dj = 1. A frequently used consequence of Dj = 1 and Dc = 0 is the formula

$$D(j + c) = 1.$$

Applying the formula D neg f = neg Df to f = j we obtain

$$D\ neg = -1.$$

If f is even, that is, if f = f neg, then

$$Df = D(f\ neg) = Df\ neg \cdot D\ neg = Df\ neg \cdot -1 = neg\ Df\ neg$$

from which it follows that neg Df = Df neg, or in other words, that Df is odd. Similarly, one can prove that if f is odd, then Df is even. Using this fact, we see that

$$0 = D1 = D(j \cdot rec) = j \cdot D\ rec + rec \cdot Dj = j \cdot D\ rec + rec.$$

It follows that j·D rec = neg rec and D rec = neg (-2) - po. By virtue of postulate III we conclude further that

$$D(rec\ g) = neg\ (-2)\text{-}po\ g \cdot Dg.$$

By means of postulate II we obtain

$$D(f \cdot rec\ g) = f \cdot D(rec\ g) + rec\ g \cdot Df = f \cdot neg(-2)\text{-}po\ g \cdot Dg + rec\ g \cdot Df,$$

that is, the Quotient Rule

$$D(f \cdot rec\ g) = (g \cdot Df - f \cdot Df) \cdot (-2)\text{-}po\ g.$$

Let g be a right inverse of f. From fg = j it follows by virtue of postulate II that

$$Dfg \cdot Dg = Dj = 1, \quad \text{and thus} \quad Dg = rec\ Dfg.$$

If h is a left inverse of f, then hf = j implies that

$$Dhf \cdot Df = Dj = 1, \quad \text{and thus} \quad Dhf = rec\ Df.$$

If h is also a right inverse, then substitution of h into the last formula yields the preceding formula for the derivation of a right inverse. For

$$Dh = Dhj = Dhfh = rec\ Dfh.$$

By induction we obtain from the three postulates

$$D(f_1 + f_2 + \cdots + f_n) = Df_1 + Df_2 + \cdots + Df_n$$

$$D(f_1 \cdot f_2 \cdot \ldots \cdot f_n) = p_1 \cdot Df_1 + p_2 \cdot Df_2 + \cdots + p_n \cdot Df_n$$

where p_k denotes the product of the n factors $f_1, f_2, \ldots, f_n$ with the exception of f_k.

$$D(f_1 f_2 \ldots f_n) = Df_1 f_2 \ldots f_n \cdot Df_2 f_3 \ldots f_n \cdot \ldots \cdot Df_{n-1} f_n \cdot Df_n.$$

The second of these rules, for equal factors, yields the formula

$$Df^n = n \cdot f^{n-1} \cdot Df,$$

in particular

$$Dj^n = n \cdot j^{n-1}.$$

This formula in conjunction with postulate I and the Constant Factor Rule enables us to derive each polynomial

$$D(c_0 + c_1 \cdot j + c_2 \cdot j^2 + \cdots + c_m \cdot j^m) = c_1 + 2 \cdot c_2 \cdot j + \cdots + m \cdot c_m \cdot j^{m-1}.$$

We call f an algebraic function, more specifically, an algebraic function belonging to the polynomials $p_0, p_1, \ldots, p_n$, if

$$p_0 + p_1 \cdot f + p_2 \cdot f^2 + \cdots + p_n \cdot f^n = 0.$$

By virtue of the formulae derived in this section we obtain

$$Df = \operatorname{neg}\left(\sum_{k=0}^{n} f^k \cdot Dp_k\right) \cdot \operatorname{rec}\left(\sum_{k=1}^{n} k \cdot f^{k-1} \cdot p_k\right).$$

2. The Derivation of Exponential Functions.

Let exp be an exponential function. We apply the formula $\exp(f+g) = \exp f \cdot \exp g$ to $f = j$ and $g = c$. We obtain $D[\exp(j+c)] = D\exp(j+c) \cdot D(j+c) = D\exp(j+c) \cdot 1 = D\exp(j+c)$. On the other hand

$$D[\exp(j+c)] = D(\exp j \cdot \exp c) = D(\exp \cdot \exp c) = \exp c \cdot D\exp.$$

Thus, $D\exp(j+c) = \exp c \cdot \exp$. Substituting 0 in this equality we obtain

on the left side: $D\exp(j+c)0 = D\exp(0+c) = D\exp c$

on the right side: $\exp c0 \cdot D\exp 0 = \exp c \cdot D\exp 0$.

Thus $D\exp c = \exp c \cdot D\exp 0$ for each constant c. If we have a base of constants it follows that D exp = exp·D exp 0. We see that the derivative of an exponential function is a constant multiple of the function.

We shall postulate the existence of an exponential function for which D exp 0 = 1. From now on we shall restrict the symbol exp to this exponential function defined by the two postulates

1. exp(f + g) = exp f·exp g
2. D exp 0 = 1.

Postulate 2 makes the previous stipulations exp $\neq$ 0 and $\neq$ 1 superfluous since D0 = D1 = 0 and thus D00 = D10 = 0 $\neq$ 1. In the classical theory, the only differentiable (and even the only continuous) function satisfying the postulates 1 and 2 is the function associating e^x with each x.

From the two postulates we have derived that D exp = exp.

In Chapter I we saw in the algebra of the exponential functions that exp c $\neq$ 0 for each c. Hence D exp c $\neq$ 0 for each c. Thus our postulate 2 concerning the exponential function implies the existence of a function which, in an algebra with a base of constants, justifies our conclusion Dj = 1 in the preceding section, in the sense that Djc = 1 for each constant c. We merely have to apply our previous reasoning to f = exp. From exp j = exp it follows that

D exp = D(exp j) = D exp j·Dj = D exp·Dj

hence D exp c = D exp c·Djc for each constant c. Since D exp c $\neq$ 0, we may multiply both sides of this equality by rec D exp c and thus obtain Djc = 1 for each constant c. Hence Dj = 1 if we have a base of constants.

Applying postulate 3 to the formula D exp = exp, we obtain

$$D(\exp f) = D \exp f \cdot Df = \exp f \cdot Df.$$

3. The Derivation of Logarithmic Functions.

By log we shall from now on denote the inverse of the function exp for which D exp 0 = 1 and D exp = exp.

From exp log = j by virtue of postulate III it follows that

$$1 = Dj = D(\exp \log) = D \exp \log \cdot D \log = \exp \log \cdot D \log = j \cdot D \log.$$

Thus, D log = rec.

The function rec on the right side admits the substitution of each constant ≠ 0, the function log on the left side the substitution of squares only. Instead of log we shall study the function logabs = log abs which, like rec, admits the substitution of each constant ≠ 0.

$$D \text{ logabs} = D[\tfrac{1}{2} \cdot \log(j \cdot j)] = \tfrac{1}{2} \cdot D[\log(j \cdot j)] = \tfrac{1}{2} \cdot D \log(j \cdot j) \cdot D(j \cdot j)$$
$$= \tfrac{1}{2} \cdot \text{rec}(j \cdot j) \cdot 2 \cdot j = \text{rec}(j \cdot j) \cdot j = (\text{rec } j \cdot \text{rec } j) \cdot j = (\text{rec} \cdot \text{rec}) \cdot j$$
$$= \text{rec} \cdot (\text{rec} \cdot j) = \text{rec} \cdot 1 = \text{rec}.$$

Next we compute D abs. We have

$$D \text{ abs} = D(\exp \log \text{abs}) = D(\exp \text{logabs}) = D \exp \text{logabs} \cdot D \text{ logabs}$$
$$= \exp \text{logabs} \cdot \text{rec} = \text{abs} \cdot \text{rec} = \text{sgn}.$$

We remark that the formulae D log = rec and D abs = sgn by virtue of postulate III entail the formula D logabs = rec.

Applying the last formula and postulate III we obtain

$$D(\text{logabs } f) = D \text{ logabs } f \cdot Df = \text{rec } f \cdot Df.$$

4. Logarithmic and Exponential Derivation.

The formulae at the end of the two preceding sections can also be written as follows:

Df = f·D(logabs f) and Df = rec exp f·D(exp f).

Replacing f in the former formula by a particular function f is called logarithmic derivation (or differentiation) of f. Similarly, replacing f in the latter formula by a particular function f might be called exponential derivation of f.

We apply the former method with benefit whenever logabs f is simpler than f. As an example of logarithmic derivation, we treat the power functions. From c - po = exp(c·log) it follows that log c - po = c·log which is indeed simpler than c - po. We have D(log c - po) = c·D log = c·rec. Hence by the formula of logarithmic derivation

D c - po = c - po·D(log c - po) = c - po·c·rec = c·(c - 1) - po.

We mention that this formula holds also for the extended c-th powers in case that c is a rational number with an odd denominator. For in these cases we obtain

$$D\,\frac{2m}{2n+1} - po = \frac{2m}{2n+1} - po\cdot D\left\{\log\exp\left[\frac{2m}{2n+1}\cdot \text{logabs}\right]\right\}$$
$$= \frac{2m}{2n+1} - po\cdot D\left[\frac{2m}{2n+1}\cdot\text{logabs}\right] = \frac{2m}{2n+1}\cdot\frac{2m}{2n+1} - po\cdot rec$$
$$= \frac{2m}{2n+1}\cdot\frac{2(m-n)-1}{2n+1} - po.$$
$$D\,\frac{2m+1}{2n+1} - po = \frac{2m+1}{2n+1} - po\cdot D\left\{\text{logabs}\left[\text{sgn}\cdot\exp\left(\frac{2m+1}{2n+1}\cdot\text{logabs}\right)\right]\right\}$$
$$= \frac{2m+1}{2n+1} - po\cdot D\left\{\text{logabs}\exp\left(\frac{2m+1}{2n+1}\cdot\text{logabs}\right)\right\}$$
$$= \frac{2m+1}{2n+1} - po\cdot D\left(\frac{2m+1}{2n+1}\cdot\text{logabs}\right) = \frac{2m+1}{2n+1}\cdot\frac{2m+1}{2n+1} - po\cdot rec$$
$$= \frac{2m+1}{2n+1}\cdot\frac{2(m-n)}{2n+1} - po.$$

We see that, in accordance with the general rule, the derivative of the even function $\frac{2m}{2n+1}$ - po is odd, and the derivative of the odd function $\frac{2m+1}{2n+1}$ - po is even.

As another example we apply logarithmic derivation to the function f = exp(j·logabs) in the classical theory denoted by x^x. We have logabs f = j·logabs, thus

$$D(\text{logabs } f) = \text{logabs} + j\cdot\text{rec} = \text{logabs} + 1.$$

Hence, Df = f·D(logabs f) = exp(j·logabs)·(logabs + 1).

In general, for functions starting with the symbol exp the function logabs f is simpler than f, and hence logarithmic derivation is convenient. The same is true for functions f which are products $f_1 \cdot f_2 \cdot \ldots \cdot f_n$ provided that we can find D(logabs f_i) for i = 1,2,...,n. For D(logabs f) is the sum of these n functions.

Exponential derivation is convenient whenever exp f is simpler than f. This is the case for functions starting with the symbol log or logabs. As an example, we treat the function f = logabs(j + logabs). Now, D(exp f) = D(j + logabs) = 1 + rec. Hence,

$$Df = \text{rec exp } f\cdot D(\exp f) = \text{rec}(j + \text{logabs})\cdot(1 + \text{rec}).$$

5. <u>The Derivation of the Trigonometric Functions.</u>

Let tan be a tangential function, c a constant. From the definition of tan it follows that

$$\tan(j + c) = \frac{\tan j + \tan c}{1 - \tan j\cdot\tan c} = \frac{\tan + \tan c}{1 - \tan\cdot\tan c}$$

By virtue of the quotient rule we obtain

$$D \tan(j + c) = D[\tan(j + c)]$$

$$= \frac{(1 - \tan\cdot\tan c)\cdot D \tan - (\tan + \tan c)\cdot - \tan c\cdot D \tan}{(1 - \tan\cdot\tan c)^2}$$

$$= D \tan\cdot(1 + \tan c\cdot\tan c)\cdot\text{rec}(1 - \tan\cdot\tan c)^2.$$

Substituting 0 we obtain

$$D \tan c = D \tan 0\cdot(1 + \tan c\cdot\tan c)\cdot\text{rec}(1 - \tan 0\cdot\tan c)^2.$$

Since tan 0 = 0 we have

$$D \tan c = D \tan 0 \cdot (1 + \tan c \cdot \tan c) \text{ for each constant } c.$$

If we have a base of constants, then

$$D \tan = D \tan 0 \cdot (1 + \tan^2).$$

We shall now postulate that there is a tangential function tan for which D tan 0 = 1. From now on we shall reserve the symbol tan for this function given by the postulates

1. $\tan(f + g) = \dfrac{\tan f + \tan g}{1 - \tan f \cdot \tan g}$

2. $D \tan 0 = 1.$

For this function we have

$$D \tan = 1 + \tan^2 = \text{rec} \cos^2.$$

In the classical analysis, for each constant a the function tan (a·x) satisfies postulate 1. The function associating tan x with x is the only one which satisfies postulates 1 and 2. In a paper "e and π in Elementary Calculus" (to appear in the near future) we describe how the postulates D tan 0 = 1 and D exp 0 = 1 in conjunction with the functional equations for the tangential and exponential functions yield an intuitive introduction of π and e, as well as a simple development of the "natural" tangential and exponential functions e^x and tan x (x measured in radians).

From tan arctan = j we obtain

$$1 = D(\tan \arctan) = D \tan \arctan \cdot D \arctan$$
$$= (1 + \tan^2)\arctan \cdot D \arctan = (1 + j^2) \cdot D \arctan.$$

It follows that

$$D \arctan = \text{rec}(1 + j^2).$$

From the definition of the sine function we conclude by virtue of the Quotient Rule

$$2\cdot D\ \sin(2\cdot j) = D[\sin(2\cdot j)]$$

$$= 2\cdot[(1+\tan^2)\cdot D\ \tan - 2\cdot\tan\cdot D\ \tan\cdot\tan]\cdot \mathrm{rec}(1+\tan^2)^2$$

$$= 2\cdot(1-\tan^2)\cdot D\ \tan\cdot\mathrm{rec}(1+\tan^2)^2 = 2\cdot(1-\tan^2)\cdot\mathrm{rec}(1+\tan^2)$$

$$= 2\cdot\cos(2\cdot j).$$

It follows that $D[\sin(2\cdot j)] = \cos(2\cdot j)$. Substituting $\frac{1}{2}\cdot j$ we obtain

$$D\ \sin = \cos.$$

Similarly, we arrive at D cos = neg sin. (It goes without saying that the symbols sin and cos are reserved for the functions defined in terms of the tangential function for which D tan 0 = 1).

6. The Foundation of the Algebra of Antiderivatives.

The Algebra of Antiderivatives is based on an equivalence relation which we shall symbolize by $\sim$, and a right inverse of the operator D which we shall symbolize by S. We shall read the symbol Sf "an antiderivative of f" or "an integral of f" indicating by this expression the multi-valuedness of the operator S in contrast to the uni-valuedness of D. The latter is expressed in the implication

If $f = g$, then $Df = Dg$

which will be of basic importance for the Algebra of Antiderivatives.

Sf is what in the classical analysis is denoted by $\int f(x)dx$ while $f\sim g$ expresses the relation $f'(x) = g'(x)$ for which the classical theory does not introduce a special symbol. Only to some extent $f\sim g$ corresponds to what in classical integral calculus is denoted by $f(x) = g(x) + \text{const.}$ As we shall see in this section, $f = g + c$ implies $f\sim g$. However, our Algebra of Antiderivation neither infers nor postulates that conversely $f \sim g$ implies $f = g + c$. In classical analysis the proof of the fact that functions with equal derivatives differ by a constant, requires deeper logical methods than the proof of any theorem corresponding to our Algebra of Analysis (see Introduction).

In view of the connection of our antiderivation with the classical calculus of indefinite integrals, we shall call f the integrand of Sf.

The two fundamental concepts of the Algebra of Antiderivation are introduced by the postulates:

A. $f \sim g$ if and only if $Df = Dg$

B. $D(Sf) = f$.

No ambiguity will arise if we write postulate B in the form $DSf = f$ since we shall leave DS undefined. We might, of course, express postulate B in the form $DS = j$. At the beginning of this section, in calling S a right inverse of D, we adopted this point of view. But we shall refrain from elaborating on this idea (as in the Algebra of Antiderivates we refrained from briefly writing D neg = neg D instead of D neg f = neg Df) since its consistent extension would necessitate the use of functions of more variables.

From the definition A it follows that the equivalence relation is reflexive, symmetric, and transitive. Since $D0 = 0$ and $D1 = 0$, we have $0 \sim 1$. In fact, for each constant c we have $c \sim 0$. More generally, from the Algebra of Derivatives it follows that $f + c \sim f$.

Next we consider two fundamental consequences of postulate B. If $Sf \sim g$, then $DSf = Dg$, thus by postulate B, $f = Dg$.

Conversely, if $f = Dg$, then from B it follows that $DSf = Dg$ and hence $Sf \sim g$. We thus see

C. $Sf \sim g$ if and only if $f = Dg$.

Secondly, we see: If $SDf \sim g$, then $DSDf = Dg$ and from B it follows that $Df = Dg$. Hence $f \sim g$ and $g \sim f$. We thus obtain the result

D. $SDf \sim f$.

Obviously, this Algebra of Antiderivation solves all the difficulties connected with the multi-valuedness of the operator S. In our formula, Sf stands for any function g for which $Dg = f$. The formulae concerning antiderivatives resulting from our two postulates of the Algebra of Antiderivation express only the equivalence (never the equality) of antiderivatives with functions or other antiderivatives. For instance, from $D0 = Dc = 0$ it follows that $S0 \sim c$. Clearly, also the classical integral calculus lacks formulae expressing the equality of any antiderivation and any other function.

7. Formulae of the Algebra of Derivation in the Notation of Antiderivation.

The formulae A. - D. of the preceding section enable us to translate each formula of the Algebra of Derivation into a formula about antiderivation. We start translating the postulates I - III of the Algebra of Derivation:

$$Sf + Sg \sim SD(Sf + Sg) \sim S(DSf + DSg) \sim S(f + g).$$

$$f \cdot g \sim SD(f \cdot g) \sim S(f \cdot Dg + g \cdot Df) \sim S(f \cdot Dg) + S(g \cdot Df).$$

$$fg \sim SD(fg) \sim S(Dfg \cdot Dg).$$

We thus obtain

$$\text{I}' \quad S(f + g) \sim Sf + Sg$$

$$\text{II}' \quad f\cdot g \sim S(f\cdot Dg) + S(g\cdot Df)$$

$$\text{III}' \quad fg \sim S(Dfg\cdot Dg).$$

Important is the special case of II' for $f = c$ and $g \sim Sh$. We obtain the Constant Factor Rule

$$c\cdot Sh \sim S(c\cdot h).$$

Translating the formulae

$$D\ \exp = \exp, \qquad D\ \text{logabs} = \text{rec}, \qquad D\ \tan = \text{rec}\ \cos^2$$

we obtain

$$S\ \exp \sim \exp, \qquad S\ \text{rec} \sim \text{logabs}, \qquad S\ \text{rec}\ \cos^2 \sim \tan.$$

From $D\ c - \text{po} = c\cdot(c-1) - \text{po}$, it follows that

$$S[c\cdot(c-1) - \text{po}] \sim c - \text{po}.$$

Applying the Constant Factor Rule for $\frac{1}{c}$ (if $c \neq 0$) and replacing $c+1$ by c, we obtain

$$S\,c - \text{po} = \frac{1}{c+1}\cdot(c+1) - \text{po} \qquad \text{if } c \neq 0.$$

8. <u>The Three Methods of Antiderivation.</u>

If in formula III' we replace f by Sh we obtain

$$\text{III}^* \quad Shg \sim S(hg\cdot Dg).$$

The formula III^* is the source of two methods for the computation of antiderivatives.

The first of these methods consists in applying formula III^* read from the right to the left, that is, in the form

$$S(hg\cdot Dg) \sim Shg.$$

In words: If the integrand of an antiderivative which we wish to find, can be represented as the product of what results from a function h by substitution of a function g times the derivative of this function g, then we obtain the antiderivative we are looking for, by substituting g into the antiderivative of h. The problem of finding the antiderivative of hg·Dg is thus reduced to the problem of finding the antiderivative of h.

Examples:

$$S(\text{rec } g \cdot Dg) \sim \text{logabs } g$$

$$S(\tan g \cdot Dg) \sim \text{rec } \cos^2 g$$

$$S(\exp g \cdot Dg) \sim \exp g, \quad \text{etc.}$$

The second method, called antiderivation by substitution, consists in substituting into formula III*, read from the left to the right, the right inverse of g which we shall denote by g^*. We obtain

$$Sh\, g\, g^* \sim S(hg \cdot Dg)g^*$$

thus

$$\text{E.} \quad Sh \sim S(hg \cdot Dg)g^*.$$

In words: We find the antiderivative of h by substituting into h any function g, multiplying the result by Dg, finding the antiderivative of the product, and substituting into this antiderivative the right inverse of g.

While formula E is correct for each h and g, it is of practical use for given h only if we can find a function g with a right inverse such that S(hg·Dg) is simpler than Sh.

Example:

$$Sh \sim S(h \tan \cdot D \tan)\text{arctan}.$$

The formula is useful if $S(h \tan \cdot \text{rec } \cos^2)$ is simpler than Sh. For instance, this is the case if $h = (-\frac{3}{2}) - po(1 + 2 - po)$, in classical notation, $h(x) = (1 + x^2)^{-3/2}$. We have

$$h \tan = (-\tfrac{3}{2})\text{-po}\,(1+\tan^2) = \cos^3,$$

$$h \tan\cdot\text{rec}\,\cos^2 = \cos,$$

$$S[(-\tfrac{3}{2})\text{-po}\,(1+2\text{-po})] \sim S \cos \arctan \sim \sin \arctan.$$

The third method, called antiderivation by parts, consists in an application of formula II', written in the following form

F. $S(f\cdot Dg) \sim f\cdot g - S(g\cdot Df)$.

While this formula holds for each f and g, it is of practical use for the computation of an antiderivative Sh only if we succeed in representing h as the product of two functions f and f_1 such that

1) Sf_1 can be found,

2) $S(Df\cdot Sf_1)$ can be found.

If we set $Sf_1 \sim g$, then formula F enables us to compute $S(f\cdot f_1)$:

F'. $S(f\cdot f_1) \sim f\cdot Sf_1 - S(Df\cdot Sf_1)$.

While it is immaterial which antiderivative of f_1 we use in the expression on the right side, it is essential that on both places the same antiderivative Sf_1 is selected.

Example:

$$S \text{ logabs} \sim S(\text{logabs}\cdot 1) \sim S(\text{logabs}\cdot Dj)$$

$$\sim j\cdot\text{logabs} - S(j\cdot\text{rec}) \sim j\cdot\text{logabs} - S1 \sim j\cdot\text{logabs} - j.$$

III. ON FUNCTIONS OF HIGHER RANK

1. The Algebra of Functions of Higher Rank.

As in the first chapter, we shall denote functions by small letters $f, g, h, \ldots$ But we shall assume now that with each function f a positive integer r, called the rank of f, is associated. The rank will correspond to the number of variables of f in the classical notation. Whenever it is necessary to indicate the rank of f we shall write $f^{(r)}$ or, where no confusion with powers can arise, briefly f^r.

Only one operation will be assumed, substitution, denoted by juxtaposition. If f is of rank r, then for each ordered r-tuple of functions $g_1, \ldots, g_r$ there is a function $f(g_1, \ldots, g_r)$. It is called the function obtained from f by substituting g_i at the index i for $i = 1, \ldots, r$. If a function f is followed by r functions in parentheses, separated by commas, it will be understood that f is of rank r. If g_i is of rank s_i, then $f(g_1, \ldots, g_r)$ is of rank $s_1 + \cdots + s_r$.

Substitution will be assumed to satisfy the following laws:

I. Associative Law.

$$[f(g_1^{s_1}, \ldots, g_r^{s_r})](h_1, \ldots, h_{s_1 + \ldots + s_r})$$

$$= f[g_1(h_1, \ldots, h_{s_1}), \ldots, g_r(h_{s_{r-1}+1}, \ldots, h_{s_r})].$$

For some purposes it is convenient to denote the s_i functions substituted into g_i by $h_{i1}, \ldots, h_{is_i}$ $(i = 1, 2, \ldots, r)$. In this notation the associative law reads:

$$[f(g_1,\ldots,g_r)](h_{11},\ldots,h_{rs_r})$$

$$= f[g_1(h_{11},\ldots,h_{1s_1}),\ldots,g_r(h_{r1},\ldots,h_{rs_r})].$$

II. Law of a Neutral Element.

$$jf = f(j,\ldots,j) = f.$$

III. Law of Depression. If for the function f of rank $r > 1$ we have

$$f = f(j,\ldots,j,g,j,\ldots,j)$$

no matter which function g we substitute at the index i, then there exists a function $f_{(i)}$ whose rank is by 1 less than that of f, for which

$$f_{(i)} = f(j,\ldots,j,g,j,\ldots,j), \text{ and thus}$$

$$f_{(i)}(g_1,\ldots,g_{r-1}) = f(g_1,\ldots,g_{i-1},g,g_i,\ldots,g_{r-1}).$$

We say of such a function f that it admits the suppression of the index i. In the classical notation, a function admitting the suppression of the index i is one which does not depend upon its i-th variable, as $f(x,y,z) = 4\cdot x + 5\cdot \log z$ does not depend upon y.

Definition: If for a function f of rank 1 we have $fg = f$ for each g, then f is called a constant.

If for a function f of rank r we have $f = f(g_1,\ldots,g_r)$ no matter which functions $g_1,\ldots,g_r$ we substitute, then we can suppress any $r-1$ of the indices and thus arrive at a constant function. We may call f a constant function of rank r. By substituting r constant functions into any function of rank r, we obtain a constant function.

If a function of rank r admits the suppression of each of its indices, then it is constant. E.g., for $r = 2$,

if $f(g,j) = f$ and $f(j,h) = f$,

then $f(g,h) = [f(j,h)](g,j) = f(g,j) = f.$

It is easy to prove that the function obtained from f by substituting a constant at the index i, admits the suppression of the index i if the rank of f is > 1, and is a constant if the rank of f is $= 1$.

IV. Law of Identification. Let R be the set of numbers $\{1,\ldots,r\}$, and $R = R_1 + \cdots + R_m$ a splitting of R into $m\,(<r)$ mutually disjoint, non-vacuous sets $R_j = \{i_{j,1},\ldots,i_{j,k_j}\}$. Then for each function f of rank r there exists a function $f_{R_1,\ldots,R_m}$ of rank m such that $f_{R_1,\ldots,R_m}(g_1,\ldots,g_m)$ is equal to the function obtained from f by substituting g_1 at the indices belonging to $R_1,\ldots,$ and g_m at the indices belonging to R_m. For instance, if $R = \{1,\ldots,6\}$, $R_1 = \{1,2,4\}$, $R_2 = \{5\}$, $R_3 = \{3,6\}$, and f is of rank 6, then there is a function f_{R_1,R_2,R_3} of rank 3 such that

$$f_{R_1,R_2,R_3}(g_1,g_2,g_3) = f(g_1,g_1,g_3,g_1,g_2,g_3).$$

Obtaining f_{R_1,R_2,R_3} from R corresponds to the formation of $f(x,x,y,x,z,y)$ from $f(x_1,\ldots,x_6)$ in the classical notation. For each function f we have $f_R g = f(g,\ldots,g)$. This is the case $m = 1$.

We remark that for each function f of rank 2, and each two functions g_1 and g_2 of rank 1, we clearly have

$$[f(g_1,g_2)]_R h = f(g_1 h, g_2 h).$$

V. Law of Permutation. If f is a function of rank r and if ρ is the permutation $i_1,\ldots,i_r$ of the numbers $1,\ldots,r$, then there is a function $f\rho$ of rank r such that for each r-tuple of constant functions $c_1,\ldots,c_r$ we have

$$f\rho(c_1,\ldots,c_r) = f(c_{i_1},\ldots,c_{i_r}).$$

For each function f^r of rank r the permutations ρ for which $f^r\rho = f^r$, form a subgroup Γf^r of Σ_r, the symmetric group of r elements. Γf^r is called the group of f^r. If $\Gamma f^r = \Sigma_r$, then f^r is called a symmetric function.

In formulating this law, we substituted into f only constant functions, since without this restriction none but constant functions f would satisfy the law. Indeed, let f be a function of rank 2, and let ρ be the permutation 2,1 of the numbers 1,2. If we had postulated the existence of a function $f\rho$ such that $f\rho(g_1,g_2) = f(g_2,g_1)$ for each pair of functions g_1,g_2 of rank 1, then by substituting the functions h_1,h_2 into the two above functions of rank 2 we should obtain

$$[f\rho(g_1,g_2)](h_1,h_2) = [f(g_2,g_1)](h_1,h_2).$$

By virtue of the associative law for substitution this equality would imply

$$f\rho(g_1h_1,g_2h_2) = f(g_2h_1,g_1h_2)$$

for each quadruple of functions g_1,g_2,h_1,h_2. Applying this formula to

$$g_1 = h_2 = 0,\ h_1 = j$$

we see that

$$f\rho(0,g_20) = f(g_2,0)$$

for each function g_2. Now, since g_20 is a constant, we see that $f\rho(0,g_20)$ is a constant. Hence, f would permit the suppression of the index 1. Similarly we could prove that f would permit the suppression of the index 2. Thus f would be a constant.

2. Sum and Product.

We call a function f of rank 2 <u>associative</u> if

$$f[f(g_1,g_2),g_3] = f[g_1,f(g_2,g_3)].$$

A constant function n is said to be <u>neutral</u> with respect to f if

$$f(n,g) = f(g,n) = g.$$

An associative, symmetric function of rank 2 may be considered as an associative, commutative binary operation. Instead of f(g,h) we may write $g \circ h$. We shall postulate the existence of two such functions s and p whose corresponding operations will be denoted by + and $\cdot$, and called addition and multiplication, respectively. We shall postulate the existence of neutral elements denoted by 0 and 1, respectively, and shall assume a distributive connection of s and p.

In order to establish the connection of these concepts with those of the Algebra of Functions developed in Part I, we remark that the sum of two functions g and h of rank 1 considered in Part I, is $[s(g,h)]_R$ rather than s(g,h). For s(g,h) is a function of rank 2 whereas the sum of two functions considered in Part I was a function of rank 1. We had $(f+g)h = fh+gh$. By virtue of the remark following the Law of Identification in the preceding section, this formula (i.e., the a.s.d. law) is indeed valid for $[s(g,h)]_R$. In the classical notation, s(g,h) corresponds to g(x) + h(y) while $[s(g,h)]_R$ corresponds to the sum g(x) + h(x) which we considered in Part I. Similarly the product $g \cdot h$ of Part I is $[p(g,h)]_R$.

3. The Algebra of Partial Derivatives.

If f is a function of rank r, we introduce r operators D_i. We call $D_i f$ the partial derivative of f for the index i. This operator is connected with substitution and identification according to the following postulates:

I. $D_{ij}[f(g_1,\ldots,g_r)] = D_i f(g_1,\ldots,g_r)\cdot D_j g_i.$

Here the symbol ij refers to the j-th index in g_i, in the same way as we could denote the $s_1 + \cdots + s_r$ functions to be substituted into the function $f(g_1,\ldots,g_r)$ by $h_{11},\ldots,h_{1s_1},\ldots,h_{r1},\ldots,h_{rs_r}$.

II. $$D_i f_{R_1,\ldots,R_m} = \sum_{j\,\mathrm{in}\,R_i} (D_j f)_{R_1,\ldots,R_m}.$$

Here $R_1 + \cdots + R_m$ is a decomposition of the set $R = \{1,\ldots,r\}$ into non-vacuous, disjoint subsets.

A detailed development of the Algebra of Partial Derivation on this foundation will be the content of another publication.