$$P^{\alpha}_{\beta}(t_1,t_2) = K^{\alpha}_{\gamma}(t_1)K^{\gamma}_{\beta}(t_2) \qquad (11)$$

where the Greek indices take on exactly the same values as in (10). This is the crux of the proof of the theorem below.

THEOREM. *The one-parameter family of projective functional transformations* (2) *generated by a regular infinitesimal projective functional transformation* (1) *is a one-parameter continuous group.*

[1] A general theory of linear functional equations on a composite range with application to projective functional transformations including a fuller account of the work of this note is to be published elsewhere. These developments are embodied in a California Institute thesis. I am indebted to Prof. A. D. Michal for suggesting these topics and for invaluable suggestions and criticisms.

[2] L. L. Dines, *Trans. Am. Math. Soc.*, **20**, 45 (1919), has given in different notation the inversion and group properties for transformation of type (2) and has shown the existence of the one-parameter family satisfying (4) and (5). G. Kowalewski, *Wien. Ber.*, **120**, 1435, has given the name "regular infinitesimal projective functional transformation" to (1).

[3] Loc. cit., p. 59; see also, I. A. Barnett, *Bull. Am. Math. Soc.*, **36**, 273 (1930)

A SPECIAL TYPE OF UPPER SEMI-CONTINUOUS COLLECTION[1]

BY HARRY MERRILL GEHMAN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BUFFALO

Communicated July 26, 1930

1. *Introduction.*—The object of this paper is to show, in the final section, an application of the special type of upper semi-continuous collection of continua[2] which is discussed in § 3. Before doing so, we shall prove certain theorems concerning upper semi-continuous collections in general.

2. *G-Maps on a Cactoid.*—R. L. Moore has shown that an upper semi-continuous collection of continua which fills up a sphere is topologically equivalent to a cactoid.[3] Since a plane is topologically equivalent to a sphere minus a point, this theorem can be extended to the case where the collection fills up a plane, in which case the collection is topologically equivalent to a cactoid minus a non-cut point.

If then G is an upper semi-continuous collection of continua which fills up a sphere (or plane) S, a given correspondence T between the elements of G and the points of a cactoid (or cactoid minus a non-cut point) Σ, affects a kind of "mapping" of the points of S upon the points of Σ. To define this "mapping" more precisely:

Let F be any subset of S, and let G_F be the collection of elements of G

which have points in common with F. If Φ is the set of points of Σ, which correspond to the elements of G_F under the correspondence T, then we shall call the set Φ the *G-map of F on* Σ. It should be noted that the G-map of a set F depends upon the correspondence T as well as upon the collection G.

If the set F has a point in common with an element X of the collection G, then it is obvious that the sets F and $F + X$ will have on Σ the same G-map Φ. In general, we have:

THEOREM 1. *If H_F is the set of all points belonging to the elements of G_F, and if K is any subset of H_F, then the sets F and $F + K$ will have on Σ the same G-map.*

THEOREM 2. *The G-map of a continuum is a continuum in Σ.*

For if F is a continuum, then H_F is a continuum,[4] and hence the G-map of F on Σ is a continuum in Σ—that is, is connected and closed in Σ.

THEOREM 3. *The G-map of a continuous curve is a continuous curve in Σ.*

This theorem follows at once from Theorem 2 and the properties of continuous curves and of upper semi-continuous collections.

As before, let H_F denote the set of all points belonging to the elements of G_F, where G_F is the collection of elements of G which have points in common with F. Let H_A denote the set of all points of H_F which lie in elements of G_F consisting of a single point, and let $H_B = H_F - H_A$. By definition, H_A is a subset of F.

THEOREM 4. *If the G-map on Σ of a continuum F is a continuous curve, then F is connected im kleinen at every point of H_A which is not a limit point of H_B.*

Suppose on the contrary that a continuum F is not connected im kleinen at some point P of H_A which is not a limit point of H_B, and that Φ, the G-map of F on Σ, is a continuous curve. Then within some neighborhood R of P that contains no points of H_B, we have the state of affairs described in § 3 of R. L. Moore's "Report on Continuous Curves."[5] Since there is a continuous (1–1) correspondence between the points of F in the neighborhood R, and the points of some subset of Φ, it follows that Φ fails to be connected im kleinen at some point, which is contrary to hypothesis.

The following example shows the necessity of assuming that the point P is not a limit point of the set H_B.

Example.—In a Euclidean 3-space, let S be the sphere $x^2 + y^2 + z^2 = 1$. Let the elements of G be the circles in which the planes parallel to the XY-plane intersect the sphere. The collection G is topologically equivalent to the straight line from (0, 0, 1) to (0, 0, -1). Since every subcontinuum of a straight line is a point or a straight line, it follows that the G-map of *every* subcontinuum of S is a continuous curve. It should be noted, however, that for any subcontinuum F of S, the only points of S which could possibly be points of H_A are the points (0, 0, 1) and (0, 0, -1), and if F contains one of these points, then that point is a limit point of H_B.

3. *A Special Type of G-Map.*—Let L be a bounded continuum lying on a sphere S. The collection of continua G_L consisting of L and of the points of S-L is an upper semi-continuous collection and is topologically equivalent to the cactoid Σ consisting of a contracting sequence of spheres, each one of which is tangent to each one of the other spheres at a point Λ. Such a set has been called by R. L. Moore a simple aspiculate cactoid.[6]

In case the continuum L lies in a plane S, the surface Σ may be defined as the cactoid just described, with some point (different from Λ) omitted from one of the spheres, or it may be defined as a surface consisting of a plane Π and a contracting sequence of spheres, each one of which is tangent to the plane Π at the same point Λ of the plane. In the discussion of § 4, we shall consider Σ as the latter type of set.

THEOREM 5. *A necessary and sufficient condition that the G_L-map on Σ of a continuum F be a continuous curve, is that F be connected im kleinen at every point of S-L.*

The condition is necessary by Theorem 4, since $H_A = F(S-L)$ and $H_B = FL$ is closed. It is sufficient, because if Φ, the G_L-map of F, is not a continuous curve, it fails to be connected im kleinen at some point of $\Phi - \Lambda$, and then by an argument similar to that used in the proof of Theorem 4, it follows that F fails to be connected im kleinen at some point which is not a point of L.

THEOREM 6. *If L_1 and L_2 are bounded subcontinua of S and $L_1L_2 = 0$, a necessary and sufficient condition that a continuum F be a continuous curve is that the G_{L_1}-map of F on Σ_1 and the G_{L_2}-map of F on Σ_2 be continuous curves.*

The condition is necessary by Theorem 3. It is sufficient by Theorem 5, since under our hypotheses F is connected im kleinen at every point of $(S-L_1) + (S-L_2) = S$.

4. *An Application to Theorems on Accessibility.*—The above considerations can be used to show that the generalization of the concept of a *point's* being accessible by arcs (or by continua) to the concept of a *continuum's* being accessible by arcs (or by continua) which was recently announced by G. T. Whyburn[7] is a generalization which is more apparent than real.

As in § 3, let L be a bounded continuum lying in a plane S, and let G_L be the upper semi-continuous collection and Σ the surface described there. According to Whyburn's definition, L is said to be *accessible by arcs* from a point set D, provided that if A is any point of D, then G_L contains a simple continuous arc of elements from A to L, every point of which is contained in $D + L$. This is equivalent to saying that L is *accessible by arcs* from D, provided that if A is any point of Δ, the G_L-

map of D on Σ, then Σ contains a simple continuous arc from A to Λ, every point of which is contained in $\Delta + \Lambda$. That is, Whyburn's "generalized accessibility" in S is equivalent to "ordinary accessibility" in Σ.

Consideration of the G_L-map of D rather than D itself has the advantage that we replace an arc of elements, which when considered as a point set in S may not be an arc or even a continuous curve, by an ordinary arc of points in Σ. If then throughout the discussions in Whyburn's paper we replace each subset of S by its G_L-map on Σ, this replacing has the effect (1) of reducing his "generalization" to the ordinary notion of accessibility by arcs, and (2) of replacing the plane S by the more complicated space Σ. We shall next show that the difficulties involved in (2) are not sufficient to cause ordinary accessibility in Σ to be considered as a generalization of ordinary accessibility in S.

Methods of proof will not be materially changed if the space considered is Σ instead of S. So far as the topology of Σ is concerned, it is locally 2-dimensional, except in the neighborhood of the point Λ. Hence practically all topological theorems, particularly those concerning accessibility, hold true in this space, with appropriate modifications in some cases. If, for instance, L is a subset of a continuous curve F in S, the G_L-map of F on Σ is a continuous curve containing the point Λ, by Theorem 3. This continuous curve on Σ can be thought of as the sum of a sequence of continuous curves, one lying on the plane Π and one (which may degenerate into the point Λ only) on each of the spheres. Also if a set D is a subset of one component of S-L, then Δ the G_L-map of D on Σ lies on Π or on one of the spheres. Hence L is accessible from D if and only if the point Λ is accessible from a set Δ lying on a plane or a sphere. If D has points in common with more than one component of S-L, then L is accessible from D if and only if the point Λ is accessible from each one of a sequence of sets $\Delta_1, \Delta_2, \Delta_3, \ldots$, which lie respectively in a sequence of different spaces, one of which is a plane and the remainder are spheres.

As has just been shown, in considering questions of accessibility, it is in general no more difficult to prove theorems for the space Σ than for the space S. Hence it can be seen that the announced generalization of the idea of accessibility is in reality no generalization at all. Many of the results of Whyburn's paper[7] are merely restatements for the space Σ of theorems previously proved by Whyburn for the plane.[8]

While the above discussion has been restricted to the plane for the sake of definiteness, the same ideas can obviously be used in considering questions of accessibility in any Euclidean space of n dimensions.

[1] The major portion of this paper was presented to the American Mathematical Society at Bethlehem, Penna., December 27, 1929.

[2] For definitions, see: R. L. Moore, "Concerning Upper Semi-continuous Collections of Continua," *Trans. Amer. Math. Soc.*, 27, 416–28 (1925).

[3] R. L. Moore, "Concerning Upper Semi-continuous Collections," *Monatshefte für Mathematik und Physik*, **36**, 81–88 (1929). See especially Theorem 2.

[4] R. L. Moore, *Transactions*, loc. cit., Theorem 1.

[5] *Bull. Amer. Math. Soc.*, **29**, 289–302 (1923).

[6] R. L. Moore, *Monatshefte*, loc. cit., Definitions, pp. 81–82 and Theorem 6. The set described above does not satisfy Moore's condition (*b*), p. 81—a condition which is not essential under any circumstances.

[7] G. T. Whyburn, "A Generalized Notion of Accessibility," *Fund. Math.*, **14**, 311–326 (1929).

[8] G. T. Whyburn, "On Certain Accessible Points of Plane Continua," *Monatshefte für Mathematik und Physik*, **35**, 289–304 (1928).

ON EXPANSIONS OF ARITHMETICAL FUNCTIONS

BY R. D. CARMICHAEL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated July 24, 1930

1. The remarkable expansions of arithmetical functions obtained by Ramanujan in his notable memoir (*Collected Papers*, No. 21) on certain trigonometrical sums and their applications are contained as special cases of much more general expansions which have also other special cases of particular interest. The purpose of this paper is to present these generalizations and to draw from the expansions some conclusions of importance obtained by means of a hitherto unnoticed fundamental property of the Ramanujan sums $c_q(n)$, namely, that expressed by the relations

$$\sum_{n=1}^{q\rho} c_q(n)c_\rho(n) = 0 \text{ if } \rho \neq q, \qquad \sum_{n=1}^{q} c_q^2(n) = q\varphi(q),$$

where $\varphi(q)$ denotes Euler's φ-function of q. This has led to the notion of orthogonal arithmetical functions analogous to the notion of orthogonal functions in analysis.

2. By $\chi(a)$, $\chi_1(a)$, $\chi_2(a)$, ..., we denote any characters modulis k, k_1, k_2, ..., respectively, and by $\chi_0(a)$, $\chi_{10}(a)$, $\chi_{20}(a)$, ... we denote the principal characters for these moduli. If a symbol for a character is written with an argument which is not an integer it may have for this argument any conveniently assigned value. The character which is equal to unity for all arguments will sometimes be replaced by 1. We use $\mu(a)$ to denote the Möbius function. By $\eta_q(n)$ we denote the sum of the n^{th} powers of the q^{th} roots of unity; then $\eta_q(n)$ is q or 0 according as q is or is not a factor of n.