where $a_n = 1 + 2\bar{z}_0^n$. On the other hand let us take

$$p_n = \begin{cases} 1/3 \text{ when } n = l.m, m \text{ even}, \\ 1 \text{ when } n = l.m, m \text{ odd}, \\ O \text{ otherwise.} \end{cases}$$

The "product" is the function $\sum_{n=0}^{\infty} C_n z^n$, where

$$C_n = \begin{cases} 1 \text{ when } n = l.m, m \text{ even}, \\ -1 \text{ when } n = l.m, m \text{ odd}, \\ O \text{ otherwise.} \end{cases}$$

This function is equal to $\dfrac{1}{1 + z^l}$, and is therefore regular at $z = 1$.

Putting the coefficients p_n equal to one or to zero, we obtain as a corollary to our theorem:

"A power series having only one singular point z_0 on its closed circle of convergence is such that all its sub-series having the same circle of convergence are singular at z_0."

A METHOD OF OBTAINING NORMAL REPRESENTATIONS FOR A PROJECTIVE CONNECTION

By J. H. C. Whitehead

Department of Mathematics, Princeton University

Communicated October 3, 1930

Normal representations for a projective connection, Π, are characterized by the equations

$$P^{\alpha}_{jk} z^j z^k = 0, \tag{1.1}$$

where $P^{\alpha}_{\beta\gamma}$ are the components of Π in the normal representation $z + z^0$; (Greek letters, used as indices, will take the values $o, \ldots n$, and Roman letters the values $1, \ldots n$). In a paper in a forthcoming issue of the *Annals of Mathematics* we give a construction for obtaining such representations. In the introduction to that paper we give a brief historical account of generalized projective geometry, in which references to the literature may be found.

In this note we give an alternative method depending on solutions to the partial differential equations.

$$\left(\frac{\partial^2\theta}{\partial x^j \partial x^k} - \frac{\partial\theta}{\partial x^i}\Pi^i_{jk} - \theta\,\Pi^0_{jk}\right) X^j X^k = 0, \tag{1.2}$$

where $\Pi^\alpha_{\beta\gamma}$ are the components of the connection Π in the representation $x + x^0$, and X is an affine vector defined as follows. Let y^i be affine normal coördinates for Π^i_{jk}—treated as the components of an affine connection—the coördinate system x, and a given point q. Then X is to be the affine vector whose components in y are y^i. At any point near q, therefore, X touches the path joining that point to q.

If we put $Z = e^{x^0}\theta$, the equations (1.2) may be written

$$Z_{;\beta;\gamma}\, V^\beta V^\gamma = 0, \tag{1.3}$$

where the semicolon denotes projective differentiation with respect to Π, and V is any projective vector such that $V^i = X^i$. The identities $Z_{;\beta;0} = Z_{;0;\gamma} = 0$ arise out of the conditions $\Pi^\alpha_{\beta 0} = \delta^\alpha_\beta$. The equations (1.3) are invariant under all changes of representation, and so any solution Z is a projective scalar of index unity. We shall show that these equations admit a unique solution $Z = e^{x^0}\theta(x)$, where θ depends only on $x^1, \dots x^n$, satisfying the initial conditions

$$\left(\frac{\partial Z}{\partial x^\beta}\right)_q = a_\beta,$$

where a_β are any given constants, and q a given point, together with a value, q^0, of the factor.

We take the $(n + 1)$ solutions Z^α determined by the conditions

$$\left(\frac{\partial Z^\alpha}{\partial x^\beta}\right)_q = \delta^\alpha_\beta, \tag{1.4}$$

and show that the normal representation for q and $x + x^0$ is given by

$$\begin{cases} z^0 = x^0 - q^0 + \log\theta^0 \\ z^i = \dfrac{\theta^i}{\theta^0}. \end{cases} \tag{1.5}$$

We shall do no more than establish this result, as a geometrical discussion of these representations, with further developments of the theory, are given in the paper to which we have referred.

2. We shall need the following lemma.

Let $A_{jk} = A_{kj}$ *be* $\dfrac{n(n+1)}{2}$ *analytic functions of* $x^1 \dots x^n$, *which can be expanded about* $x = 0$ *in power series, convergent for* $|x^i| \le \delta$, *and l.t* $a_0 \dots a_\alpha$ *be arbitrary constants. Then the equation*

$$\left(\frac{\partial^2 y}{\partial x^j \partial x^k} - y\, A_{jk}\right) x^j x^k = 0 \tag{2.1}$$

will admit a solution $y(x)$, regular near $x = 0$, which is uniquely determined by the initial conditions

$$y(0) = a_0, \left(\frac{\partial y}{\partial x^i}\right)_{x=0} = a_i\,.$$

Suppose such a solution $y(x)$ exists, and let $y_{k_1 \ldots k_r}$ denote

$$\frac{\partial^r y}{\partial x^{k_1} \ldots \partial x^{k_r}}.$$

Then

$$x^{l_p} \frac{\partial}{\partial x^{l_p}} \ldots x^{l_1} \frac{\partial}{\partial x^{l_1}} (y_{jk} - y\, A_{jk}) x^j x^k = 0,$$

where p is any positive integer. From these identities we have

$$\left\{ y_{jkl_1} \ldots {}_{l_p} - \frac{\partial^p}{\partial x^{l_1} \ldots \partial x^{l_p}} (y A_{jk}) \right\} x^j \ldots x^{l_p} = 0,$$

and so

$$\left\{ y_{jkl_1} \ldots {}_{l_p} - [y A_{ijk}],\, l_1 \ldots l_p) \right\}_{x=0} = 0, \tag{2.2}$$

where the comma denotes partial differentiation, and $p!\, B_{(k_1 \ldots k_p)}$ stands for the sum of the quantities obtained by permuting the indices in all possible ways.

Let each set of numbers, except $a_1 \ldots a_n$ which are arbitrary,

$$a_0, a_j, \ldots, a_{jkl_1 \ldots l_p}, \ldots$$

be derived from those preceding it by the equations (2.2). If the series

$$y = a_0 + \sum_{p=1}^{\infty} \frac{1}{p!} a_{k_1 \ldots k_p} x^{k_1} \ldots x^{k_p} \tag{2.3}$$

converges for some value of x^i other than zero, it will represent a solution of the equations (2.1), which is regular near $x = 0$, and which will be uniquely determined by $a_0 \ldots a_n$. The lemma will, therefore, be established if only we can show that the series (2.3) converges.

Expanding A_{jk} as a power series about $x = 0$, we have

$$A_{jk} = A^0_{jk} + \sum_{p=1}^{\infty} \frac{1}{p!} A^0_{jkl_1 \ldots l_p} x^{l_1} \ldots x^{l_p}.$$

There exists, therefore, a number M such that

$$\left| \frac{1}{p!} A^0_{jkl_1 \ldots l_p} \right| < \frac{M}{\delta^p} \tag{2.4}$$

for all positive integral values of p. In the expansion of

$$\frac{\partial^r uv}{\partial x^{p_1} \ldots \partial x^{k_r}}$$

there will be $\dfrac{r!}{p!\,(r-p)!}$ terms in which u and v are differentiated, respectively, p and $(r-p)$ times, i.e., terms of the form

$$\frac{\partial^p u}{\partial x^{k_{a_1}} \ldots \partial x^{k_{a_p}}} \cdot \frac{\partial^{r-p} v}{\partial x^{k_{a_{p+1}}} \ldots \partial x^{k_{a_r}}},$$

where $a_1 \ldots a_r$ are the integers $1, \ldots r$. If we write λ_m for the greatest of the moduli $|a_{k_1 \ldots k_m}|$, where $a_{k_1 \ldots k_m}$ are defined by (2.2), we have, from (2.2) and (2.4)

$$\lambda^{m+2} < M\left\{\frac{m!}{\delta^m}\lambda_0 + \frac{m!}{1!\,(m-1)!} \cdot \frac{(m-1)!}{\delta^{m-1}}\lambda_1 + \ldots + \frac{m!}{p!(m-p)!} \cdot \frac{(m-p)!}{\delta^{m-p}}\lambda_p + \ldots + \lambda_m\right\}$$

$$= \frac{M\,m!}{\delta^m}\left(\lambda_0 + \lambda_1\delta + \frac{\lambda_2\delta^2}{2!} + \ldots + \frac{\lambda_m\delta^m}{m!}\right).$$

Writing these inequalities

$$u_{m+2} < \frac{M\,\delta^2}{m+2} \cdot \frac{u_0 + \ldots + u_m}{m+1}$$

where $u_q = \dfrac{\lambda_q\,\delta^q}{q!}$, we have

$$u_{m+2} < \frac{M\,\delta^2}{m+2}\,\bar{u}_m$$

where $\bar{u}_m$ is the greatest of $u_0, \ldots, u_m$. If we chose an integer m_0 so that

$\dfrac{M\,\delta^2}{m_0 + 2} < |$, we have

$$\begin{cases} u_{m_0+-1} < \bar{u}_{m_0-1} \leqslant \bar{u}_{m_0} \\ u_{m_0+2} < \bar{u}_{m_0}. \end{cases}$$

Hence $\bar{u}_{m_0} = \bar{u}_{m_0+1} = \ldots = \bar{u}_{m_0+q} = \ldots,$ and so

$$u_N < K \tag{2.5}$$

for a suitable choice of K, and every positive integer N. From the definition of u_m and from (2.5), it follows that

$$\frac{\lambda_m}{m!} < \frac{K}{\delta_m},$$

or that the power series expansion of

$$y(x) = \frac{K}{1 - \dfrac{x' + \ldots + x^n}{\delta}}$$

dominates the series (2.3), which converges, therefore, for $|x^i| < \frac{\delta}{n}$. Hence the function $y(x)$, given by (2.3), satisfies the conditions of the lemma.

3. We shall now return to our problem. We are given a representation $x + x^0$ and are setting out to prove the existence of a representation, $z + z^0$, which is uniquely defined by the conditions

$$\begin{cases} P^{\alpha}_{jk}\, z^j z^k = 0 \\ \left(\dfrac{\partial z^{\alpha}}{\partial x^{\beta}}\right)_{x^{\alpha} = q^{\alpha}} = \delta^{\alpha}_{\beta} \\ z^{\alpha} = 0 \text{ for } x^{\alpha} = q^{\alpha}, \end{cases} \tag{3.1}$$

where q^i are the coördinates of a point, q^0 a value of the factor, and $P^{\alpha}_{\beta\gamma}$ are the components of the connection, Π, in $z + z^0$.

In case Π were flat, the differential equations

$$\frac{\partial^2 Z}{\partial x^{\beta} \partial x^{\gamma}} - \frac{\partial Z}{\partial x^{\alpha}} \Pi^{\alpha}_{\beta\gamma} = 0 \tag{3.2}$$

would be completely integrable, and would yield solutions, $Z^0, \ldots Z^n$, such that

$$\left(\frac{\partial Z^{\alpha}}{\partial x^{\beta}}\right)_q = \delta^{\alpha}_{\beta} \tag{3.3}$$

for $x^{\alpha} = q^{\alpha}$. From the condition $\Pi^{\alpha}_{\beta 0} = \delta^{\alpha}_{\beta}$ we have

$$Z^{\alpha} = e^{x^0 - q^0}\, \theta^{\alpha}(\dot{x}),$$

were $\theta^{\alpha}(x)$ depend only on $x^1, \ldots x^n$. Let $P^{\alpha}_{\beta\gamma}$ be the components of π in the representation given by

$$\begin{cases} z^0 = \log Z^0 \\ \quad = x^0 - q^0 + \log \theta^0 \\ z^i = \dfrac{Z^i}{Z^0} \\ \quad = \dfrac{\theta^i}{\theta^0}. \end{cases} \tag{3.4}$$

Then we have

$$\begin{cases} Z^i = e^{z0}\, z^i \\ Z^0 = e^{z0}. \end{cases} \tag{3.5}$$

From (3.3) and (3.5) it follows that $Z^\alpha = \delta^\alpha_0$ and $\dfrac{\partial Z^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$ for $x^\alpha = q^\alpha$. Since the expressions on the left-hand side of the equations (3.2) are the components of a projective tensor, we have

$$\frac{\partial^2 Z^\alpha}{\partial z^\beta \partial z^\gamma} - \frac{\partial Z^\alpha}{\partial Z^\sigma} P^\sigma_{\beta\gamma} = 0, \tag{3.6}$$

where Z^α are given in terms of $z + z^0$ by (3.5). It follows that

$$P^\alpha_{jk} = 0. \tag{3.7}$$

In the general case we shall establish the existence of solutions to (1.3) which satisfy (1.4), and make the same use of projective invariance to obtain the conditions (3.1). In the equations (1.3) we may change to any representation, provided we know its relation to $x + x^0$. We shall start with the representation $y + y^0$, where $y^0 = x^0 - q^0$, and y^i are affine normal coördinates for Π^i_{jk}, the point q, and the coördinate system x. For $x^\alpha = q^\alpha$ we have, therefore,

$$y^\alpha = 0, \quad \frac{\partial y^\alpha}{\partial x^\beta} = \delta^\alpha_\beta.$$

Let $\overline{\Pi}^\alpha_{\beta\gamma}$ be the components of π in $y + y^0$. Then in $y + y^0$ the components V^i are, by definition, y^i. We have, therefore.

$$\overline{\Pi}^i_{jk} V^j V^k = \overline{\Pi}^i_{jk} y^j y^k = 0,$$

and the equations (1.2) will become

$$\left(\frac{\partial^2 \theta}{\partial y^j \partial y^k} - \theta \overline{\Pi}_{jk}\right) y^j y^k = 0. \tag{3.8}$$

By our lemma we know that these equations admit a unique set of solutions $\theta^0, \ldots, \theta^n$, such that, writing $Z^\alpha = e^{y0}\theta^\alpha$,

$$\left(\frac{\partial Z^\alpha}{\partial x^\beta}\right)_q = \left(\frac{\partial Z^\alpha}{\partial y^\beta}\right)_0 = \delta^\alpha_\beta. \tag{3.9}$$

We also have $Z^\alpha_{;0;\gamma} = Z^\alpha_{;\beta;0} = 0$, and therefore Z^α satisfy the equations

$$Z^\alpha_{;\beta;\gamma} V^\beta V^\gamma = 0.$$

Let $P^\alpha_{\beta\gamma}$ and $\overline{V}_\alpha$ be the components of Π and V in the representation given by

$$\begin{cases} Z^0 = \log Z^0 \\ \quad = x^0 - q^0 + \log \theta^0 \\ z^i = \dfrac{Z^i}{Z^0} \\ \quad = \dfrac{\theta^i}{\theta^0}. \end{cases} \tag{3.10}$$

Then by the same argument as that used in the flat case

$$P^\alpha_{jk}\overline{V}^j\overline{V}^k = 0. \tag{3.11}$$

To find the equations, in z, to the paths through q, we have to solve

$$\frac{dz^i}{d\sigma} = \overline{V}^i. \tag{3.12}$$

But on any path we have

$$\left(\frac{d^2z^i}{d\sigma^2} + P^i_{kl}\frac{dz^k}{d\sigma}\frac{dz^l}{d\sigma}\right)\frac{dz^j}{d\sigma} = \left(\frac{d^2z^j}{d\sigma^2} + P^j_{kl}\frac{dz^k}{d\sigma}\frac{dz^l}{d\sigma}\right)\frac{dz^i}{d\sigma},$$

and so from (3.11) and (3.12), we have, on the paths through q,

$$d^2z^i dz^j = d^2z^j dz^i$$

On these paths, therefore, $\dfrac{dz^i}{d\sigma} = z^i f(\sigma)$, and so $\overline{V}^i$ are proportional to z^i. The conditions (3.11) may, therefore, be written

$$P^\alpha_{jk}z^jz^k = 0,$$

and from (3.9) and (3.10), we have

$$\left(\frac{\partial z^\alpha}{\partial x^\beta}\right)_q = \delta^\alpha_\beta \text{ and } z^\alpha = 0, \text{ for } x^\alpha = q^\alpha.$$

The conditions (3.1), therefore, are satisfied by $z + z^0$.

The functions θ^α are the only solutions to (3.8) which satisfy (3.9), and from (3.10) it follows that the representation $z + z^0$ is uniquely determined by the conditions (3.1).