The attraction of a finite cylinder on any external point can be expressed to a precision of 1 part in 100,000 by seven terms of a zonal harmonic series.

The observations during 1926 gave five values of the constant of gravitation as follows:

	6.661
	6.661
	6.667
	6.667
	6.664
mean	6.664×10^{-8}
Average departure from mean	0.002

No previous investigator has obtained results agreeing to more than 2 significant figures.

It is planned to carry out an additional series of observations during 1927, by which it is hoped to obtain six or seven more values of this important constant.

* Published by permission of the Director of the National Bureau of Standards of the U. S. Department of Commerce.

[1] C. V. Boys, *Phil. Trans. Roy. Soc. A.*, **1895**, Part 1, p. 1.

[2] Carl Braun, *Denkschriften k. Akad. Wissens.* (*math. und naturwiss. Classe*), **64**, 1897, p. 187.

[3] Article "Gravitation," *Encyclopedia Britannica*, XI Edition.

CONTACT TRANSFORMATIONS OF THREE-SPACE WHICH CONVERT A SYSTEM OF PATHS INTO A SYSTEM OF PATHS[1]

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Introduction.—A contact transformation of the surface elements of three-dimensional space[3] converts every point, curve or surface into a point, curve or surface, but not necessarily a point into a point, a curve into a curve or a surface into a surface.

An important special contact transformation is the duality. This is characterized among proper contact transformations, i.e., those which do not reduce to point transformations by the property of converting each of the straight lines of space again into a straight line.

The ∞^4 straight lines of space are an example of a system of paths. By this we mean a system of analytic curves analytically distributed[4] over a region of space so that there is one and only one curve joining any two given points of the region, and one and only one through any given point in any given direction.

These considerations suggest the problem: to determine all proper contact transformations of space which convert the curves of some (unspecified) system of paths again into curves. This problem is solved by the following theorem, which it is the purpose of the present paper to establish.

THEOREM.—*If a proper contact transformation* Γ *of space converts the curves of a system of paths F into curves, then* Γ *must have the form* PDP_1 *where D is a duality and P and* P_1 *are point transformations; and F must be linear, i.e., convertible by a point transformation (P) into the straight lines.*

Proof.—Let Γ proceed from the space S containing F to the space S' containing the system of paths F' into which F is transformed by Γ.

There are three possibilities to consider: Γ converts an arbitrary point of S either into a point, a curve or a surface.

The first of these is ruled out by the hypothesis that Γ is a proper contact transformation.

If Γ converts an arbitrary point p of S into a curve C_p in S', then as p describes any curve of F, C'_p must move so that its envelope shall be a curve of F'. But this can only be when C'_p remains fixed as p describes the curve of F. It follows that C'_p must be the same for all points p of S; therefore, every union in S is carried by Γ into C'_p—a degenerate case which we may dismiss.

There remains only the case where an arbitrary point p of S is converted into a surface Σ'_p in S'. The surfaces Σ' form a triply infinite system corresponding to the ∞^3 points of S.

Imagine that p describes a curve C of F. Then, by hypothesis, the corresponding ∞^1 surfaces Σ' must intersect in the same curve C' of F', in the manner of a pencil of planes. Thus the system of paths F' consists of the mutual intersections of the ∞^3 surfaces Σ'.

Now there is a theorem of Felix Klein[5] which applies in this situation, being essentially to the effect that a system of paths F' in space which is identifiable with the mutual intersections of ∞^3 surfaces must be a linear system; i.e., there must exist a point transformation P' which converts F' into the straight lines of a projective space T'.

By applying the above reasoning to Γ^{-1}, it follows that the system of paths F can be converted by a point transformation P into the straight lines of a projective space T.

Consider the transformation $P^{-1}\Gamma P'$. Being a composition of contact transformations, it is itself a contact transformation, which proceeds from the space T to the space T'. By following the three component transformations from T to S to S' to T', we see that $P^{-1}\Gamma P'$ converts the straight lines of T into the straight lines of T'. Hence $P^{-1}\Gamma P'$ must be either a collineation C or a duality D. It cannot be a collineation, for then Γ would be PCP'^{-1}, or a point transformation, contrary to hypothesis. Therefore, we must have

$$P^{-1}\Gamma P' = D,$$

whence

$$\Gamma \overset{\circ}{=} PDP'^{-1} = PDP_1,$$

where P_1 as well as P is a point transformation. This completes what was to be proved.

[1] Presented to the American Mathematical Society, May 7, 1927.

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[3] Hereafter the word "space" will mean three-dimensional space.

[4] This means that in the finite equations of the paths, $x^i = f^i\ (t, a_1, a_2, a_3, a_4), i = 1, 2, 3$, the functions f^i are analytic in the parameters a which vary from curve to curve as well as in the parameter t which varies along each curve.

[5] *Über einen Satz aus der Analysis Situs, Gesammelte Mathematische Abhandlungen, erster Band*, pp. 306–310; also *Über die sogenannte nicht-euklidische Geometrie*, in the same volume, pp. 331–343, especially pp. 333–343.

MOTION AND COLLINEATIONS IN GENERAL SPACE

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1. By a general metric space we shall understand an n-dimensional continuum in which the element of arc of a curve is given by

$$\left(\frac{ds}{dt}\right)^2 = F(x, \dot{x}), \tag{1.1}$$

where $\dot{x}$ stands for dx/dt and F is homogeneous of the second degree in $\dot{x}$.

The geometry of spaces of this sort has been studied by Finsler,[1] Synge,[2] Taylor[3] and Berwald[4] and it is the tensor analysis as developed by Berwald that we employ in the present paper. We shall denote the covariant derivative of a tensor by means of a subscript preceded by a comma, while a subscript preceded by a period will denote partial differentiation with respect to $\dot{x}$. It is understood that the summation convention of a repeated index is used.

As most of the functions with which we shall have to deal are homogeneous of degree zero in $\dot{x}$ and since we cannot equate coefficients of $\dot{x}$ as we do in Riemannian geometry or in the corresponding geometry of paths, we shall make use of the following lemma:

If $f_{i_1 i_2 \ldots i_\gamma}(x, \dot{x})\dot{x}^{i_1}\dot{x}^{i_2}\ldots\dot{x}^{i_\gamma} = 0$, *the functions* f *being homogeneous of degree*