A TEXT-BOOK
OF
GENERAL PHYSICS

FOR COLLEGES

MECHANICS AND HEAT

BY

J. A. CULLER, Ph.D.

PROFESSOR OF PHYSICS, MIAMI UNIVERSITY

PHILADELPHIA

J. B. LIPPINCOTT COMPANY

Printed by J. B. Lippincott Company
The Washington Square Press, Philadelphia, U.S.A.

PREFATORY NOTE

In the following treatise on mechanics and heat an effort has been made to present the subject in as clear a manner as possible for use of a college student. A knowledge of plane trigonometry is necessary before undertaking this study, and the more mathematics a student knows the better he will ordinarily succeed in physics.

Some changes have been made in the form of the usual college text and in the method of presentation. The aim of the writer has constantly been to say the words that would help the student to understand the subject. Thus it is hoped that the book will prove to be not only a treatise but also a text-book for students.

Reference matter and tables are placed in the appendix instead of being scattered through the text. This takes less room and is much more convenient for reference. A number of short lists of problems are found where they are needed to illustrate the application of principles learned. Answers to problems are given at the end of the lists, but a student should be made to understand that numerical results are not so important here as his ability to present the line of argument involved in the problem. The tables of sines, cosines, tangents, etc., are intended to make the book more desirable as a complete working text.

We acknowledge our obligation to the Ball Engine Co. for cuts of the steam engine, to D. Van Nostrand Co. for cut of the Parsons steam turbine, to the Taylor Instrument Companies for the cuts of pyrometers, and to the De Laval Steam Turbine Co. for cuts of the De Laval turbine.

CONTENTS

CHAPTER I.

KINEMATICS.

CHAPTER II.

DYNAMICS.

CHAPTER III.

SOLIDS.

CHAPTER IV.

GASES.

CHAPTER V.

LIQUIDS.

CHAPTER VI.

HEAT.

APPENDIX.

GENERAL PHYSICS

MECHANICS

CHAPTER I

KINEMATICS

1. Metrology. — Metrology is the science of weights and measures. Convenient and well-defined units of measurement are essential in any highly organized social state. Exact units are particularly necessary to the advancement of science. The scientist is constantly trying to make exact determinations of length, area, volume, and mass. He must express these quantities in fixed units. A record is thus made which can be compared with the results of other investigations and can be understood by all who are familiar with the units used. If the units employed by different investigators are not exactly defined or are carelessly used, great confusion is sure to result and progress will be checked.

In early times, when the people of a community were not so closely dependent on each other as now, the head of a family or clan might choose units of length and weight as he thought best. He would, however, choose some convenient and natural unit. For short distances the length of his foot, the breadth of his hand, or the length of his forearm would be chosen. Longer distances would be designated as so many paces, and still longer distances by the distance a man could travel in one day.

For determination of mass he would naturally use seeds, as is evidenced by terms still in use, *e.g.*, the *grain* and the *carat* (from *carob*, bean).

When people began to live together in larger and more compact communities, it became necessary for the king or some one in high authority to fix certain standards to be used in common by a great number. Thus, it is said, the English yard was first determined by the length of the king's arm.

As the various nations advanced in science, arts, and industrial pursuits, it became necessary to fix and define certain units which all would use. The units of length having most extensive use are the *yard* and the *metre*.

The **yard** was defined by the English Parliament in 1855. It is a solid square bar made of a special bronze, 38 inches in length and one square inch in cross section. Near each end a circular hole is sunk to half the depth of the bar, Fig. 1. At the bottom of each hole is a gold plug upon which is inscribed a transverse line. When the temperature of the bar is $16\frac{2}{3}°$ C., the distance between the lines is the imperial standard yard of 36 inches. This bar is carefully preserved at the standards office, Westminster. Four other bars called Parliamentary copies were made and deposited for safe keeping at other places. These, by law, must be compared with the original once every ten years, so that if the original is lost or destroyed, it may be exactly reproduced from its copies. A number of other standard yards were made of the same material and distributed to various institutions in Great Britain and to other nations. Bronze standard No. 11 was presented to the United States. It is .000088 inch shorter than the imperial standard.

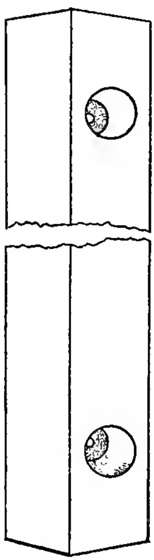


Fig. 1.

This description is sufficient to show the care which has been taken to define and preserve a unit of length.

The standard unit of **mass** in the English system is a piece of platinum marked P.-S., 1844, 1 lb. This is the *avoirdupois pound*, and $\frac{1}{7000}$ of this mass is the *grain*.

The unit of **time** is the *mean solar second*. A solar day is the interval between the passages of the sun across the meridian. These intervals are not equal, for the earth does not move with the same speed at all points of its orbit, but the mean of all the intervals in one year is the same as in another year. If there is any difference it has not yet been detected. A mean solar day is divided into 24 mean solar hours, the hour into 60 minutes, and the minute into 60 seconds, making 86,400 seconds in a mean solar day.

If it is found, as some think, that the earth is rotating on its axis more slowly than formerly, a different and less variable unit of time may be selected as a standard.

The English standards of length and mass are arbitrary,—*i.e.*, they were selected by Parliamentary enactment, and their perpetuity depends on the care with which they are preserved. As standards they are probably as good as any of this character. The chief objection to the English system is the manner in which the standards are divided. The system is not a decimal one, and the various derived units are very inconvenient.

In the latter part of the eighteenth century the subject of a rational system of weights and measures was strongly agitated in France. This was a time when the French people were making many changes and were in a mood to make this one, however radical the change might have seemed at other times. The plan was to agree on some **natural unit** of length, something that would not change while the world stands.

The French Academy of Sciences recommended the length of the earth's meridian from the equator to the pole. The measurement of this distance was made by Méchain and Delambre (1791–1798). Of course they could not measure the entire distance, but chose the distance from Dunkirk, in the northern part of France, to Barcelona, in Spain, on the shore of the Mediterranean Sea. This distance was very carefully measured by the method of triangulation, taking into account the curvature of the earth. The difference of latitude between these points was found to be 9° 40′ 45″ Knowing the length and the number of degrees, the length of 1° is easily found, and then the length of 90°, the quadrant sought.

One ten-millionth $\left(\frac{1}{10^7}\right)$ of the length of the quadrant was called 1 metre (m.).

Thus the effort was made to determine a *natural standard* of length, but it has since been found that the quadrant is more nearly 10,000,880 metres. The metre in use is practically an arbitrary standard, just as the yard is, and is defined as the distance between two transverse lines on a certain platinum-iridium bar when the temperature is 0° C. This bar is preserved with great care by the International Metric Bureau at Sèvres, near Paris.

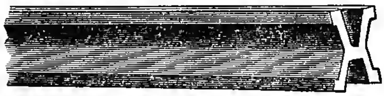


Fig. 2.

The standard of **mass** in the metric system is the *kilogram*, originally intended to be the mass of one cubic decimetre of water at its greatest density, 4° C. A mass of platinum supposed to be equal to this quantity of water was selected as the standard, but it has since been found that 1000 c.c. of pure water at 4° C. weighs about .04 g. less than the standard mass of platinum.

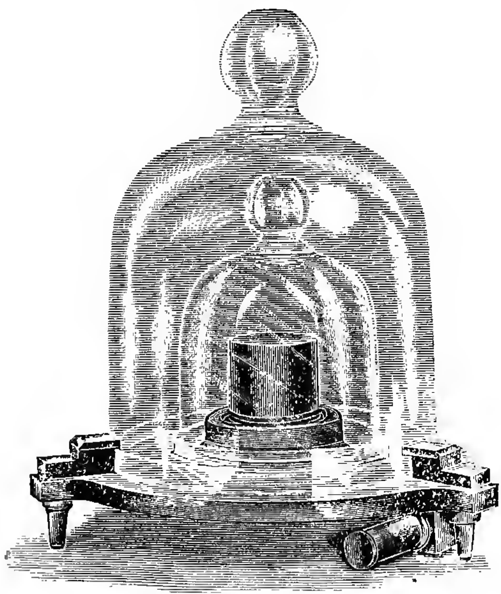


Fig. 3.

The subdivisions of the metric standards are made on a decimal basis. The convenience of the metric system has led to its universal adoption for scientific purposes, and in many countries of Europe it is used for all purposes.

The **unit of time** in the metric system is the mean solar second, as in the English system.

In the year 1866 the metric system was made lawful throughout the United States, the weights and measures in use being defined in terms of the metric units. The yard was defined as $\frac{3600}{3937}$ metre, and the pound avoirdupois as $\frac{1}{2.2046}$ kilogram. Thus the metric system is made the standard in the United States, but its use is not compulsory.

The original advocates of the metric system failed to establish a natural and invariable standard, but the arbitrary metre and kilogram as now defined are better standards than natural ones which are subject to change or for which new values are likely to be found by later and more refined processes.

In recent times Prof. Michelson has determined the length of the metre in terms of **wave lengths of light.** He found that for red light, whose wave length is .64384722μ, the length of the metre is 1,553,163.5 waves; for green light, of wave length .50858240μ, 1,966,249.7 waves; for blue light, of wave length .47999107μ, 2,083,372.1 waves.

Thus the length of the metre is fixed in terms of an invariable natural unit,—wave lengths of light,—and if for any reason the present standard metre should be destroyed, it could be exactly reproduced from the record of its length in terms of light waves.

2. Fundamental and Derived Units. — Fundamental units are those that are chosen as a basis for a system of units. The fundamental units most commonly employed are those of *length*, *mass*, and *time*. The unit of nearly all other physical magnitudes can be fixed in terms of these three.

Derived units are those whose magnitude is expressed in terms of the fundamental units. Area, for example, is a *length* squared. Volume is a *length* cubed. Velocity is a *length* divided by a *time*. Density is a *mass* divided by a *length* cubed.

A system of units fixed in this manner is called an **absolute system,** because physical magnitudes are thus determined not by reference to some other magnitude of the same kind which might have been adopted as a standard, but by reference to fundamental units which do not change.

The most common absolute system in physical investigations is that in which the unit of length is the *centimetre;* the unit of mass, the *gram;* and the unit of time, the *second.* This is the centimetre-gram-second or c.g.s. system.

3. Dimensions.—Dimensions of a derived unit are expressed by a power of the fundamental unit. Thus $[L^3]$ expresses the dimensions of volume in terms of length.

Dimensional equations are found by calling length L, mass M, time T, and then placing these in the proper relation to express the physical quantity under consideration. Velocity, for example, is a length divided by a time; hence it is expressed by $\left[\frac{L}{T}\right]$ or $[LT^{-1}]$. Momentum is a product of mass by velocity; hence it is expressed by $[M]\,[LT^{-1}]=[MLT^{-1}]$.

These expressions only show the relation between the fundamental and derived units. When the magnitudes of the fundamental units have been determined, the magnitude of the derived unit may be found from its dimensional equation. Thus

$$\text{velocity } [V]=[LT^{-1}]$$

and, if c.g.s. units are used, the velocity is expressed as $^{\text{cm}}/_{\text{sec}}$.

To express the magnitude of any physical quantity we must know not only the unit used but also the number of units. In the expression

$$x\,[LT^{-1}]$$

x represents a pure number and is called the **numeric.**

Dimensional formulæ are valuable in many ways, as will appear in later discussions. To illustrate one of their uses, suppose it is desired to convert a velocity of 12 m. per minute to cm. per second. Let x be the numeric sought,—*i.e.*, the *number* of cm. per sec. Also let L_1 and T_1 be the length and time in metres per minute and L_2 and T_2 be the same for centimetres per second. Then,

$$12\left[\frac{L_1}{T_1}\right]=x\left[\frac{L_2}{T_2}\right]$$

$$\therefore\ x=12\left[\frac{L_1 T_2}{L_2 T_1}\right]=12\ .\ 100\ .\ \frac{1}{60}=20\ ^{\text{cm}}/_{\text{sec}}.$$

4. Motion and Rest.—Motion is a change of position. The position of a body is indicated by its distance and direction from a known point of reference. If the position of A is known and B is 30 m. due north, then B may be located. This method of locating objects is in constant use. If either the distance or direction of a body in reference to the fixed point of reference is changing, the body is in motion.

Motion, then, is only a change of distance or direction in reference to another body, and hence is **relative** and **not absolute.** A and B may both be in motion in such a manner that there would be no change in distance or direction in relation to each other. They would then be said to be at rest provided one is located by reference to the other. In reference to C, however, both A and B may be in motion.

In our ordinary judgments of motion we assume that the solid earth and the fixed objects upon it are at rest. There is no rest except in a relative sense. There is no point in the universe which is *fixed* and to which all other objects may be referred. The earth, for example, rotates from west to east, so that bodies on the equator are moving about 1040 miles per hour. If a cannon were stationed on the equator and a ball projected westward with the same velocity as that of the earth eastward, observers would ordinarily consider the cannon to be at rest and the ball to be the only body in motion. More properly the force of the explosion of powder only served to bring the ball to rest while the observers and the cannon moved on with their original speed. The object aimed at would move up and *strike the ball.* Such would be the appearance to an observer who did not partake of the motion of the earth on its axis.

5. Axes of Reference.—Although all positions and motions are relative, yet it is necessary for us to assume certain points and lines as fixed and then calculate the position and motion of bodies in reference to these.

One common method is illustrated in Fig. 4. The point o is called the *origin.* The lines XoX_1 and YoY_1 are called *coordinate axes.* When the angles at o are right angles, the lines of reference are called rectangular axes. The X-axis is often called the *abscissa* and the Y-axis the *ordinate.* The distance between the fine lines in the figure is one millimetre, which may be taken to represent one metre, one foot, or any other space. Suppose it is desired to locate the point in the first quadrant.

It is seen that P_1 is 10 mm. from o as measured along the X-axis and 15 mm. from o as measured along the Y-axis. The abscissa of P_1 is then 10, and the ordinate 15. This is usually written P_1 (10, 15), the abscissa being given first. Any abscissa to the right of o is called positive (+), and to the left negative (—). The ordinates above o are positive, and below, negative. Thus the point P_2 is located by P_2 (—10, 5). The location of the point in the third quadrant is P_3 (—10, —10), and in the fourth, P_4 (10, —15).

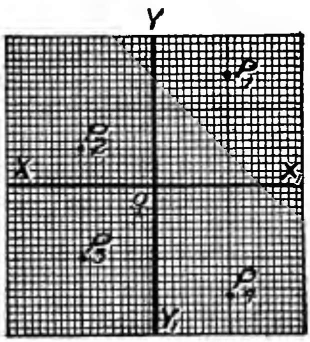


Fig. 4.

A point is also often located, as shown in Fig. 5, by a line, r, indicating its distance from the origin, o, and by an angle, θ, made by r with the *initial line*, oX. The line r and the angle θ are called the **polar coordinates** of the point P. The position of P is then expressed by $P(r, \theta)$. It is easy to pass from polar to rectangular coördinates and *vice versa* by observing the trigonometric relations shown in Fig. 5, where

$$y = r \sin \theta \quad (1)$$

and

$$x = r \cos \theta \quad (2)$$

whence

$$\frac{y}{x} = \frac{\sin \theta}{\cos \theta} = \tan \theta \quad (3)$$

or

$$\theta = \tan^{-1} \frac{y}{x} \quad (4)$$

Also

$$r^2 = x^2 + y^2 \quad (5)$$

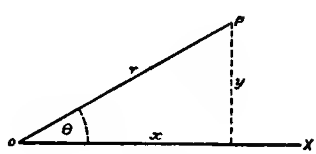


Fig. 5.

From these relations it is possible to find r and θ when x and y are known, or having r and θ to find x and y.

Problems.

1. Locate on coördinate paper as in Fig. 4, $P(7, 15)$ $P(8, -12)$ $P(-3, -14)$.

2. Construct polar coördinates for $P(12, 45°)$ $P(20, 270°)$ $P(10, 0°)$ $P(40, 360°)$.

3. Given the rectangular coördinates for a point $P(8, 6)$, find the corresponding polar coördinates.

4. Find the rectangular coördinates of $P(2, 60°)$.

1. ——.
2. ——.
3. $P(10, 36° 52')$.
4. $P(1, \sqrt{3})$

6. Translation and Rotation.—A motion is called a *translation* when a body moves in such a manner that all its particles have the same motion. If, for example, a body changes its position from A to B, Fig. 6, in such a manner that a line connecting any two of its parts remains parallel to itself, the motion

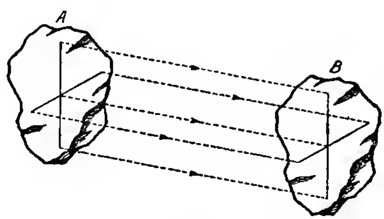


Fig. 6.

is one of translation. A locomotive running on a straight track would be an example of this kind of motion. If, however, one point or axis of a body is fixed and the other points are moving about it, the motion is called *rotation*. An example of this is a wheel or sphere rotating on a stationary axis. A body rolling along a plane has a combination of both translation and rotation.

7. Measurement of Length or Distance.—Numerous methods are employed in the measurement of displacement and length. For ordinary measurements of long distances extreme accuracy is not necessary. For distances between cities an error of several rods is in most cases of little consequence. In certain surveys, however, such as was used in finding the distance from Dunkirk to Barcelona, the length of a base line must be measured with extreme care, for it enters into all subsequent calculations.

For measurement of small lengths, such as are used in the shops and laboratory, it is often sufficient to lay a **graduated scale** on the object to be measured, with the scale divisions close to the object. If the scale is graduated in millimetres, the object may be measured to *tenths* of a millimetre with a fair degree of accuracy. The object A, Fig. 7, is 5.4 divisions long. Practice will enable one to estimate tenths with sufficient accuracy for many purposes.

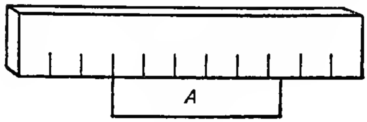


Fig. 7.

To assist in measuring tenths or other fractional part of a scale division, a **vernier** is frequently used. This is a small auxiliary sliding scale, as V in Fig. 8. Let the vernier be as long as nine of the scale divisions and divided into ten equal parts. Each of the vernier divisions will then be nine-tenths of a scale division,—*i.e.*, will be shorter by one-tenth. In Fig. 8, A, the first vernier space is shorter than the first scale space by one-tenth of the scale unit. The second mark on the vernier is two-tenths to the left of the corresponding mark on the scale; the third, three-tenths, and so on, the tenth mark on the vernier coinciding with the ninth on the scale. If now the vernier be moved to the right so that 1 on it and the scale coincide, the distance moved must be one-tenth of one of the scale divisions. When 2 coincides with 2, the distance is two-tenths of a scale division. Thus, whatever mark on the vernier coincides with any mark on the scale is the number of tenths, in this particular vernier, of one scale division. To determine the length of the line ab, Fig. 8, B, read the two whole divisions on the scale, then look along the vernier for coincident lines. In the figure it is the seventh line; hence the length of ab is 2.7 scale divisions. If the scale were in millimetres, ab would be 2.7 mm. long; if in tenths of an inch, .27 inch long.

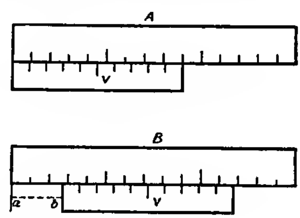


Fig. 8.

If the divisions on a scale are $\frac{1}{20}$ inch and 25 divisions on the vernier are equal to 24 on the scale, then each division on the vernier is $\frac{1}{25}$ of $\frac{1}{20}$, or $\frac{1}{500}$ inch shorter than a scale division. With this arrangement direct readings are made to within .002 of an inch.

Stating these facts in a general way, if n is the number of divisions on the vernier and $n-1$ the number on the scale, then

$$nV=(n-1)\,S \qquad (6)$$

where V and S represent the value of the divisions on the vernier and scale respectively. From equation (6)

$$V=\left(\frac{n-1}{n}\right)S=S-\frac{1}{n}S$$

$$\therefore S-V=\frac{1}{n}S \qquad (7)$$

The term $\frac{1}{n}S$ is often called the *least count.* It indicates the degree of precision for which the vernier has been constructed. In Fig. 8 the least count is one-tenth of the scale division.

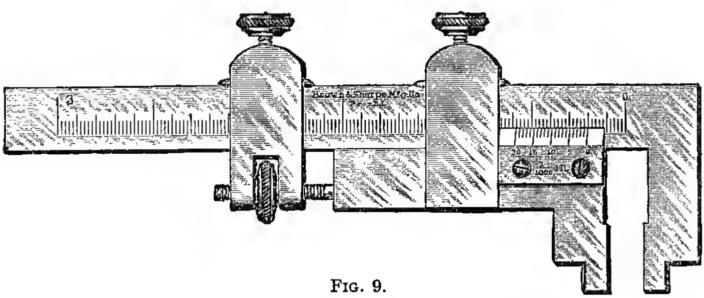


FIG. 9.

In Fig. 9 is shown the use of a vernier on a **vernier caliper.** One jaw is fastened rigidly to the scale while the other is movable and carries a sliding vernier. Many instruments of this character are provided with verniers.

Another kind of instrument, based on the use of a **micrometer screw,** is often employed in refined measurements of length. The manner of its use may be gathered from Fig. 10. A strong frame, F, carries at one end a stop, E, and at the other end a graduated arm, A. Within the arm is a fixed nut in which the screw turns. Attached to the head of the screw is a sleeve, S, which fits on the arm and is graduated on its bevelled edge.

The graduations on the arm are usually in millimetres or fractions of an inch. In a common form of this instrument the pitch of the screw is .5 mm., the arm is divided into millimetres, and the sleeve has its bevelled circumference divided into 50 equal parts. Two turns of the screw, then, would move the sleeve one millimetre on the arm. A turn of the screw through one division of the sleeve would move the screw longitudinally $\frac{1}{50}$ of $\frac{1}{2}$ or $\frac{1}{100}$ mm. The least count is $\frac{1}{100}$ mm., but by estimating in tenths measurement may be made to thousandths of a millimetre. To facilitate setting of this instrument and to secure the same pressure in different measurements, a ratchet, *R*, is provided, which will turn without turning the screw when a certain pressure has been reached.

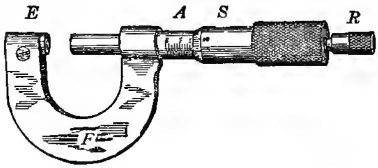


FIG. 10.

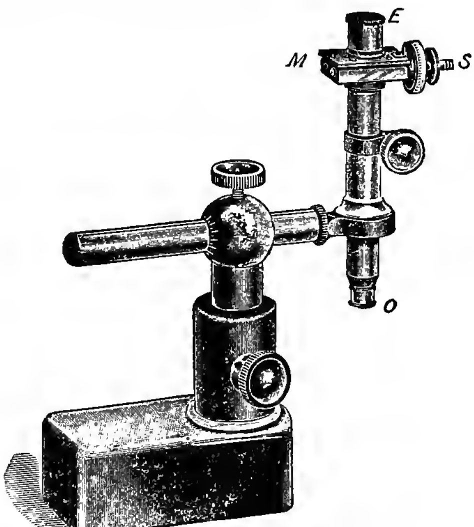


FIG. 11.

Another valuable instrument for measurement of small lengths is the **micrometer microscope.** This consists of a simple microscope with micrometer, mounted as shown in Fig. 11. The objective, *o*, of the microscope produces an enlarged image of the object within the tube in the plane *MS*. This image is viewed through an eye-piece, *E*. The micrometer, shown nearly full size in Fig. 12, is mounted in the plane of the image. The screw is attached to a light frame, across which are stretched spider lines. By turning the screw the lines are made to move across the field from one end of the image to the other. The head of the screw is a disk the circumference of which is divided into 100 equal parts. The number of

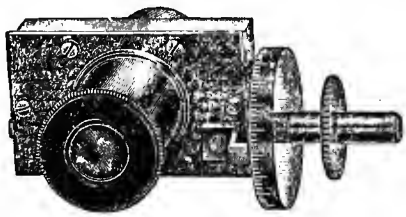


FIG. 12.

turns of the screw is a measure of the length of the image. To measure an object in millimetres it is necessary first to determine the constant of the instrument,—*i.e.*, the number of turns of the screw necessary to cause the spider lines to move over a space of one millimetre as seen through the eye-piece. Suppose this constant is found to be 9.35, then the length of an object which requires for its measurement 3.927 turns is $3.927 \div 9.35$ or .42 mm.

Numerous other devices are employed in the measurement of length, but most of them involve the principles already explained.

8. Measurement of Change in Direction.—In case the motion is a simple rotation, the displacement involves a change of direction but not of distance from the origin. Measurement is then made of the angle through which the line connecting the body with the centre of the circle moves. If a body moves from *X* to *P* along an arc whose radius is *oX*, it has changed in direction but not in distance from *o*. The displacement of the body may then be indicated

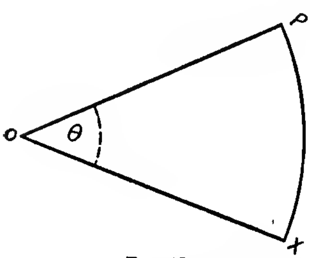


FIG. 13.

by giving the number of degrees which the line *oP* makes with *oX*.

For the construction and accurate measurement of angles, various forms of **protractors** are frequently used. In Fig. 14 is shown a protractor having a vernier, a transparent centre for accurate placing over the origin, o, and a long arm one edge of which is in a true line with the centre. With such an instrument a difference of one minute in the magnitude of an angle may be measured. The degree is the unit in this method of measurement.

Another unit, often much more convenient in the measurement of angular displacement, is the **radian.** A radian is the angle at the centre of a circle measured by an arc whose length is that of the radius of the circle. If the radius oa, Fig. 15, is laid off on the circumference, not as a chord but so as to coin-

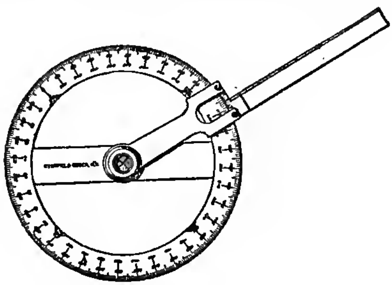


FIG. 14.

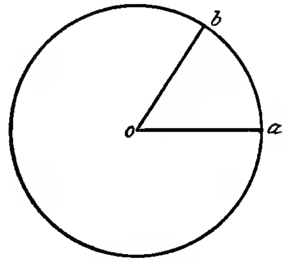


FIG. 15.

cide with the curve of the circumference, as ab, then the angle at o formed by the lines ao and bo is a radian. It is evident that the angle at o is not quite 60°, as would be the case if the radius were inscribed as a chord.

Since the circumference of a circle is $2\pi r$, the radius can be laid off on the circumference 2π or 6.2832 times,—*i.e.*, there are 6.2832 radians in 360° The value of one radian is then 57.2958°, about 57.3°.

When the angular displacement of a body is given in radians, it is easy to pass to degrees by multiplying by 57.3, or to find the linear distance passed over by multiplying radians by the length of the radius.

Problems.

1. What must be the structure of a vernier that a scale graduated in $\frac{1}{2}$ mm. may be read to $\frac{1}{1000}$ cm.?

Find value of V from (7) and substitute in (6) for n.

2. The divisions on the scale of a barometer are $\frac{1}{20}$ inch. 25 divisions on the vernier are equal in length to 24 on the scale. What is the least count in fraction of an inch?

3. In a circle whose radius is 20 feet two bodies on the circumference are 85.95° apart. Express this angular distance in radians. What is the distance in feet as measured along the circumference

4. A wheel 10 cm. in radius rotates 360 times per minute. What is its angular velocity and what is the speed of any point on the circumference?

1. 50 divisions on vernier to 49 on scale.
2. .002 inch.
3. 1.5 radians.
   30 feet.
4. 37.7 radians (nearly).
   377 cm.

9. Velocity.—Velocity is the rate of change of position. It is the change of position which would take place in a unit of time if the body continued its motion uniformly during that time. A body moving with a velocity of 500 cm. per minute may in fact be in motion for only one second of time or less, but while it was in motion its rate was such that it would traverse a distance of 500 cm. if its motion continued for one minute. A velocity of 10 radians per second simply indicates the amount of angular displacement which would occur in one second at that rate.

Since there are two kinds of motion, there are two kinds of velocity,—*linear* and *angular*. Linear velocity is that which occurs along a line, whether that line be straight or curved. It is the distance traversed per unit of time as measured along the line of motion. Angular velocity is the rate at which the angle at the centre changes when the motion is a rotation. It is the number of degrees or radians by which a body changes its direction in a unit of time.

10. Uniform and Accelerated Motion.—A motion is uniform when the same distances or angles are traversed during each successive unit of time,—*i.e.*, the motion is uniform when the velocity is constant. Such a moving body changes its position a certain number of centimetres or radians per second during each second of its motion.

Whether the motion is actually uniform or not, it is possible, when the distance and time are given, to find a **uniform rate** of motion by which the same space would be traversed in the given time. If the distance from A to B is 500 cm., and the time required for a body to move from the one point to the other is 20 seconds, then, no matter what the nature of the motion may be between A and B, the uniform rate or average velocity is $500 \div 20 = 25$ cm. per sec. In all such cases

$$v = \frac{s}{t} \qquad (8)$$

v being the average velocity, s the space traversed, and t the time in seconds.

When the velocity is increased or diminished a certain amount each second, the motion is said to be **accelerated.** The acceleration may be positive or negative,—*i.e.*, the velocity may increase or decrease. If the velocity at the beginning of a certain period is known, and to this are added the successive accelerations during the time, the result is the velocity at the end of the period. If the accelerations are negative, they must be subtracted from the initial velocity.

11. Uniformly Accelerated Motion. — When the change of velocity is the same for each successive unit of time, the motion is said to be *uniformly accelerated.* If a body starts from rest and moves with uniformly accelerated motion (U. A. M.), its velocity at the end of any period of time is found by the equation

$$v = at \qquad (9)$$

where v is velocity, a the acceleration, and t the time. The truth of this statement is apparent from the definition of a, for if a is the increase in velocity per second, then the increase per second times the number of seconds must be the final velocity.

If under similar conditions it is desired to find the total distance through which the point moves, first find the average velocity and then multiply by t. Since the acceleration is uniform, the average is one-half the sum of the first and last velocities. This may be expressed by

$$\frac{o + at}{2} = \tfrac{1}{2}at$$

assuming that the motion begins from rest. This is the uniform velocity the moving point must have to traverse the space in t seconds. The total space, s, must then be

$$s = \tfrac{1}{2}at \,.\, t = \tfrac{1}{2}at^2 \qquad (10)$$

These two equations, (9) and (10), are the fundamental ones for U. A. M., but a number of other important equations may be derived from them.

Problems.

1. Combine (9) and (10) so as to eliminate t and show that

$$v^2 = 2as \tag{11}$$

2. Make use of equation (10) to find the total distance when the time is $(t-1)$. Subtract this from the distance when the time is t. Call the distance during any one second of U. A. M. d and show that

$$d = \tfrac{1}{2}a(2t-1) \tag{12}$$

3. If a body has a uniform motion, of velocity V, and is given, in addition, a U. A. M., show that

$$d = V \pm \tfrac{1}{2}a(2t-1) \tag{13}$$

4. Show that

$$v = V \pm at \tag{14}$$

5. Show that when there is initial velocity V and U. A. M. the distance s in time t is

$$s = Vt \pm \tfrac{1}{2}at^2 \tag{15}$$

6. By eliminating t from (14) and (15) show that

$$v^2 = V^2 \pm 2as \tag{16}$$

7. Prove that the distance passed over in the first unit of time of U. A. M. is one-half the acceleration. Use (10).

12. Angular Velocity and Acceleration.—Angular velocity is the rate of change in direction. This may be either uniform or accelerated, and the acceleration may be uniform or variable, just as in linear velocity.

When a body rotates on an axis, its various particles have different linear velocities depending on their distance from the axis, but all have the same angular velocity, for there are 360° in any circle whatever its radius may be. Let the radian be the unit and let ω represent the number of radians per second. The angular distance in any given time is then expressed by

$$\omega t$$

Any particle located at a distance r from the axis has evidently a linear velocity

$$v = \omega r \tag{17}$$

for ω is the number of radians per second and r is the length of the arc which subtends one of them.

13. Vectors.—A vector is a quantity in which both *magnitude* and *direction* are considered. Velocity is a vector quantity, because it is designated by a number and a direction. Distinction is made between velocity and *speed* in that speed is designated by a number without consideration of direction. A race-horse moves with a certain speed, the direction of his motion being a matter of no interest.

Quantities are classified as *vector* and *scalar*. Examples of vectors are velocity, acceleration, momentum, currents of water, and force. Examples of scalars are speed, mass, and density.

A line may be drawn to represent any given magnitude which has also a direction. In case of velocity, the length of the line drawn to any convenient scale will represent the magnitude of the velocity while its direction is that of the motion.

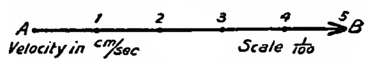


FIG. 16.

If, for example, a velocity of 500 $^{cm}/_{sec}$ due eastward is to be represented by a line, we may choose a certain scale, say 1 cm. of length to 100 cm. of velocity, and draw the line AB, Fig. 16, assuming directions as on a map. If the direction is marked by an arrow-head, the information in Fig. 16 is complete, and any one accustomed to this form of representation will at once read 500 $^{cm}/_{sec}$ due eastward.

The lines themselves may be called vectors, but only in the sense that they represent vector quantities.

14. Composition and Resolution of Velocities.—Two or more velocities may combine and produce a **resultant** velocity. The resultant is easily calculated by a consideration of the vectors which represent the component velocities. Let two component velocities be represented by the vectors oY and oX. A moving point starting at o must in one second be oY distant from oX, and oX distant from oY,—*i.e.*, it must be at R and its velocity must be oR.

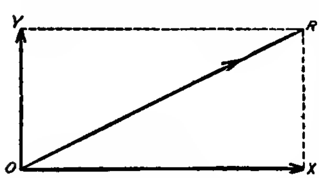


FIG. 17.

This is the **parallelogram of velocities,** in which the two components are taken as the two adjacent sides of a parallelogram and the resultant is the diagonal drawn from the common origin. This is true no matter what the angle formed by the components may be.

In case there are more than two components, as a, b, and c in Fig. 18, it is only necessary to combine a and b in the manner already shown and then combine their resultant with c.

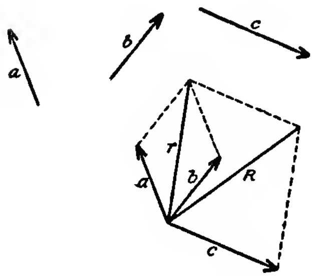


FIG. 18.

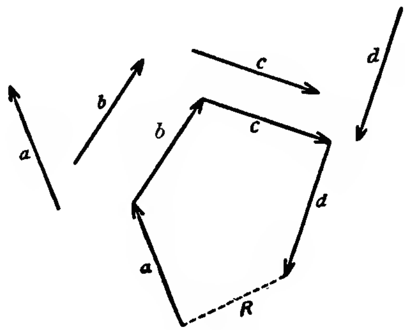


FIG. 19.

Another method, known as the **polygon of vectors,** consists in placing the vectors end to end,—*i.e.*, placing the end of one to the origin of the next. The line connecting the end of the last with the origin of the first is the vector of the resultant. Thus the resultant of a, b, c, and d, Fig. 19, is R. If the component vectors form a closed polygon, there is no resultant.

Not only can resultants thus be found when components are given, but any given vector may be **resolved** into components which would produce the same effect. Let oR be a vector and oY, oX, the rectangular coördinates drawn through the origin, o (Fig. 20). It is evident that oR represents a velocity oH along the X-axis and HR along the Y-axis. oH and HR may then be considered components of oR.

Since the coördinates may have any position, an indefinite number of rectangular components may be drawn.

The components may also be at any angle with each other, as, in Fig. 21, a and b may be taken as components of oR.

The kind of resolution which should be made depends on the nature of the problem which is to be solved. Numerous applications of these principles will be found in later pages of this work.

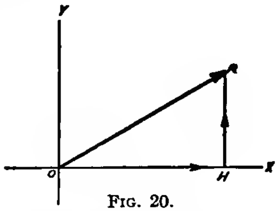


Fig. 20.

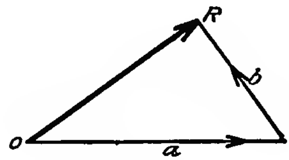


Fig. 21.

15. Methods of Calculating Resultants.—Two methods are commonly used in finding the magnitude and direction of the resultant when the components are known,—namely, the **graphical** and the **mathematical.** By the graphical method exact drawings are made in the manner just indicated, the vectors being drawn on some convenient scale and true in direction. The resultant is then measured and its value found from the scale adopted. For example, if the component vectors of velocity are drawn so that each centimetre represents one metre of velocity, and the resultant is found by measurement to be 4.5 cm. long, the resultant velocity is 4.5 m. This method is in common use in drafting rooms where problems of this character are considered.

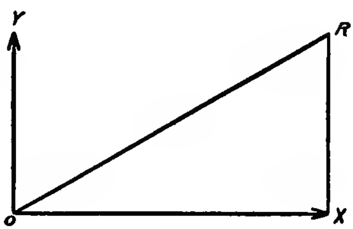


Fig. 22.

By the mathematical method the principles of geometry and trigonometry are in most cases sufficient. When two vectors are at right angles to each other, as in Fig. 22, it is sufficient to apply the well-known principle of geometry,

$$r^2 = x^2 + y^2 \qquad (18)$$

In case the vectors are not perpendicular to each other, as a and b, Fig. 23, they may be referred to the axes in such a manner that a, say, will coincide with the X-axis. Then b may be resolved into two components, $b \cos \theta$, which is along the X-axis and so can be added to a, and $b \sin \theta$, which is parallel to the Y-axis and so is at right angles to a. Now we have two vectors at right angles and can solve as above by equation (18).

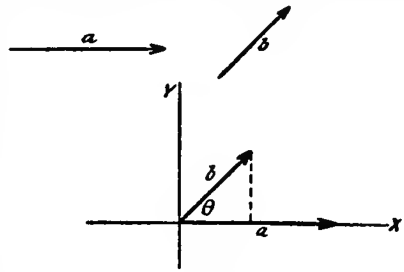


FIG. 23.

A general equation for calculating the magnitude of the resultant may be deduced as follows: Referring to Fig. 23, where a and b are the two components whose resultant is to be found, the vector value along the X-axis is seen to be $a + b \cos \theta$, and along the Y-axis it is $b \sin \theta$. Hence the square of the resultant is equal to the sum of the squares of these two rectangular components,—that is,

$$\begin{aligned} r^2 &= (a + b \cos \theta)^2 + (b \sin \theta)^2 \\ &= a^2 + 2ab \cos \theta + b^2 \cos^2 \theta + b^2 \sin^2 \theta \\ &= a^2 + b^2(\sin^2 \theta + \cos^2 \theta) + 2ab \cos \theta \\ &= a^2 + b^2 + 2ab \cos \theta \qquad (\text{since } \sin^2 \theta + \cos^2 \theta = 1) \\ \therefore r &= \sqrt{a^2 + b^2 + 2ab \cos \theta} \qquad (19) \end{aligned}$$

This is the expression for what is usually called the **addition of vectors.** Vectors may also be **subtracted** by reversing the direction of one of them and then finding the resultant in the usual manner. In Fig. 24 let a and b be the two vectors which are to be subtracted. The resultant is r,—a magnitude which is less in this particular case than that from addition. The operation must be made algebraically, and consequently the result of the subtraction may be a larger number than that from addition.

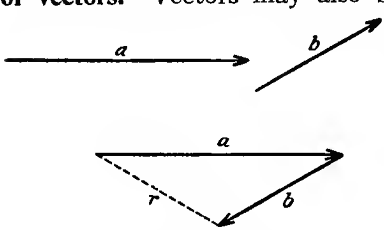


FIG. 24.

The operation of finding the resultant in case of subtraction is similar to that for addition, but the component of b along the X-axis is now $-b\cos\theta$,—*i.e.*, its direction is opposite to that of a, Fig. 25, and must be subtracted. The equation then is

$$r^2=(a-b\cos\theta)^2+(b\sin\theta)^2$$
$$\therefore r=\sqrt{a^2+b^2-2ab\cos\theta} \qquad (20)$$

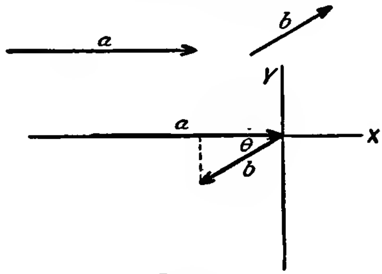


Fig. 25.

To determine the **angle** which the resultant makes with one of the components, let a and b, Fig. 26, make an angle θ with each other. Let φ be the angle made by r with a, and ψ the angle

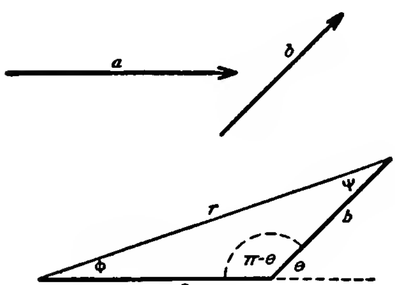
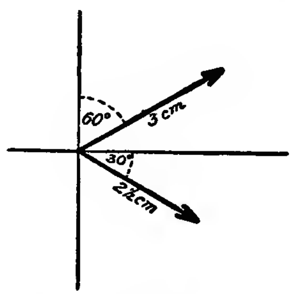


Fig. 26.



Scale, 1 cm.=20 m/sec.

Fig. 27.

with b. The supplement of θ is $\pi-\theta$. Since the sides of triangles are proportional to the sines of their opposite angles,

$$\frac{r}{b}=\frac{\sin(\pi-\theta)}{\sin\varphi}=\frac{\sin\theta}{\sin\varphi}$$
$$\therefore \sin\varphi=\frac{b}{r}\sin\theta \qquad (21)$$

When φ is known, ψ may at once be determined.

Problems.

1. Find the magnitude of the resultant of two vectors representing velocities 5 and 7 miles per hour, the angle between them being 67°.

2. If the velocity of a moving point is 48 m/sec N. 30° E., what is its velocity due northward?

3. A boat is propelled directly across a stream at the rate of 12 miles per hour, while the stream runs 4 miles per hour. Use the graphical method and find the resultant velocity by measurement.

4. Find the velocity along the Y-axis when conditions are as represented in Fig. 27.

5. If two vectors whose values are 3 and 5 form an angle of 45°, what angle does the resultant make with the 5 cm. vector?

6. Find the resultant of two velocities, one 10 m/sec E. 10° N. and the other 20 m/sec N. 30° W.

1. 10.06 miles per hour.
2. 41.569 m/sec.
3. 12.65 miles/hr.
4. 5 m/sec, positive.
5. $\phi=16°\ 35'$, approximately.
6. 19 m/sec, approximately.

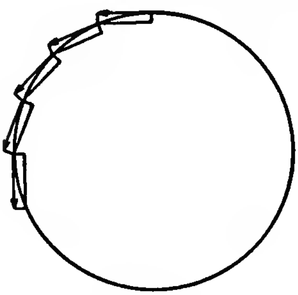


Fig. 28.

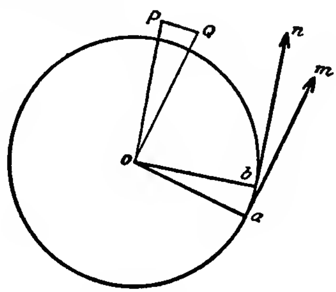


Fig. 29.

16. Uniform Circular Motion. — When a point, P, Fig. 28, moves in a curved path, any very small portion of the curve may be considered the resultant of two vectors, one of which is tangent to the curve at the point considered, and the other at right angles to the tangent. The arc which is considered the resultant must be taken infinitely small, for the direction of the motion continuously changes. The speed does not change, for it is the resultant of the same vectors at every point of the curve.

Let ab, Fig. 29, be an arc so small that it and its chord may be considered as coincident. Let am be the velocity vector when the point is at a, and bn when at b. These two lines are equal in length, since they represent equal magnitudes, the speed being uniform. The direction of the motion, however, has changed in passing from *a* to *b*. The magnitude of this change is found by taking the difference in the velocities at points *a* and *b*,—*i.e.*, by subtracting the vectors. Draw *oQ* to represent the magnitude and direction of the velocity at *a*, and *oP* to represent the same at *b*. The vectors *oQ* and *oP* are equal in length, for the motion is uniform. The line *QP* will represent the change of velocity, for *oP* is the resultant obtained by adding *oQ* and *QP*; hence *QP* is the change of velocity while a particle moves from *a* to *b*,—*i.e.*, *QP* is the acceleration. Since the unit of time is taken very small, the arc *ab* and its chord may be considered equal and the triangle *oQP* is similar to *oab*. Hence

$$\frac{QP}{Qo}=\frac{ab}{ao}$$

But *Qo* is the velocity of the particle at *a*, and *ab* is the distance traversed in unit time,—*i.e.*, velocity *v*. The radius *ao* is *r*. Hence

$$\frac{QP}{v}=\frac{v}{r}$$

$$\text{or} \qquad QP=\frac{v^2}{r} \tag{22}$$

At the limit, time $t=0$, *QP* is perpendicular to *Qo* and hence parallel to *ao*, the direction of motion changing at every instant of time. The acceleration, then, is equal in magnitude to $\frac{v^2}{r}$ and is directed toward the centre of the circle.

It has already been shown that angular velocity, ω, is equal in radians to linear velocity divided by the radius. This was expressed by

$$\omega=\frac{v}{r}$$

$$\text{whence} \qquad v^2=\omega^2r^2$$

Substituting this value of v^2 in the equation for acceleration toward the centre, (22),

$$QP=\frac{v^2}{r}=\frac{\omega^2r^2}{r}=\omega^2r \tag{23}$$

Thus, suppose a body revolves uniformly twice a second in a circle whose radius is 10 cm. In one revolution there are 2π radians, hence the magnitude of the velocity is 4π or 12.5664 radians per second. The value of QP, equation (23), is then 1579.14 $^{cm}/_{sec^2}$.

Again, let P be the period,—*i.e.*, the time required for one complete revolution. The circumference of a circle is $2\pi r$; hence the magnitude of the velocity v is

$$v = \frac{2\pi r}{P}$$

or

$$v^2 = \frac{4\pi^2 r^2}{P^2}$$

$$\therefore QP = \frac{v^2}{r} = \frac{\frac{4\pi^2 r^2}{P^2}}{r} = \frac{4\pi^2 r}{P^2} \tag{24}$$

This gives the acceleration in terms of the period and radius. Using for illustration the same problem as above,

$$QP = \frac{4\pi^2 r}{P^2} = \frac{4 \times 9.8696 \times 10}{(\frac{1}{2})^2} = 1579.14 \ ^{cm}/_{sec^2}$$

It is sometimes convenient to have this equation in still another form. The number of revolutions, n, varies inversely as the period; hence

$$n = \frac{1}{P}$$

and

$$QP = 4\pi^2 r n^2 \tag{25}$$

This gives the acceleration in terms of the radius and the number of revolutions per second.

Using the illustrative problem in this equation also,

$$QP = 4 \times 9.8696 \times 10 \times 2^2 = 1579.14 \ ^{cm}/_{sec^2}$$

Problems.

1. Assuming that the moon revolves uniformly about the earth, the period being 27 days, 7 hours, and 43 minutes (2,360,580 seconds), and that the radius of the orbit is $3.844(10)^{10}$ cm., find the acceleration.

2. How many times per minute must a body revolve in a circle whose radius is 10 cm., in order that the acceleration may be 39,478.4 $^{cm}/_{sec^2}$?

3. What is the linear velocity of a body which moves with an angular velocity of 183.36° per second in a circle of 20 cm. radius?

1. .2722 $^{cm}/_{sec^2}$.
2. 600 $^{revolutions}/_{min}$.
3. 64 $^{cm}/_{sec}$.

17. The Motion of a Projectile.—When a body, under the influence of gravity, is projected so that its line of motion makes any angle, except 90°, with a horizontal plane, it will move in a curved path called a *parabola*. This curve, as shown in Fig. 30, is the locus of all the points which are equally distant from the focus F and a line AD called the directrix. Thus $FP = PD''$, $FP' = P'D'$, and so on for all points on the curve.

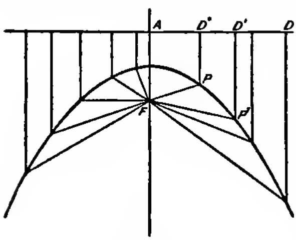


FIG. 30.

Let a body be projected from o with a velocity V in the direction oH, making an angle θ with a horizontal plane. Draw the rectangular coördinates oY and oX. The velocity V may be resolved into two components, one parallel to the X-axis, v_x, and the other parallel to the Y-axis, v_y. From trigonometrical relations

$$v_x = V \cos \theta \tag{26}$$

and

$$v_y = V \sin \theta - gt \tag{27}$$

The velocity parallel to the X-axis is uniform, for gravity acts in a direction perpendicular to this motion. But in the direction of the Y-axis the velocity is retarded by as much as gt in time t, where g is the acceleration due to gravity. Hence gt must be subtracted from the velocity which the body would have in the vertical direction without gravity. The distance the projectile moves in the direction of these axes is therefore

$$x = V \cos \theta \cdot t \tag{28}$$

and

$$y = V \sin \theta \cdot t - \tfrac{1}{2}gt^2 \tag{29}$$

In equation (28) the distance is simply the uniform velocity multiplied by the time. But in (29) the initial velocity alone would in time t raise the body a distance NH, Fig. 31. During this same time the body falls $\frac{1}{2}gt^2$,—*i.e.*, to P.

By use of these four equations it is possible to determine (1) the *time of flight* from o to R, (2) the *range* or distance from o to R, (3) the *elevation* for *maximum range*, (4) the *time of rise*, and (5) the *height* of rise, when the initial velocity and angle of elevation are known.

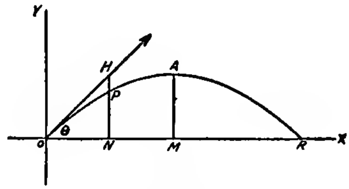


FIG. 31.

To find the **time of flight,** it is observed that when the projectile has reached R the value of y has become zero. Hence equation (29) may be written

$$V \sin\theta \, . \, t - \tfrac{1}{2}gt^2 = o$$

whence
$$t_{(\text{flight})} = \frac{2V \sin\theta}{g} \qquad (30)$$

Thus, when V and θ are known, the time of flight can easily be calculated, g in all cases being approximately 980 $^{cm}/_{sec^2}$ or 32.2 $^{feet}/_{sec^2}$.

The **range** is found from (28) when the time of flight is known, by substituting in (28) the value of t from (30). Thus,

$$x = V \cos\theta \, . \, t$$
$$= V \cos\theta \, . \, \frac{2V \sin\theta}{g}$$
$$= \frac{V^2}{g} \, 2 \sin\theta \cos\theta$$
$$\therefore \; x = \frac{V^2}{g} \sin 2\theta = \text{range} \qquad (31)$$

for $2 \sin\theta \cos\theta = \sin 2\theta$.

It is plain from (31) that the range will be greatest for any given velocity when $\sin 2\theta$ is greatest. The sine is greatest where the angle is 90°; hence, that 2θ may be 90°, θ must be 45°. Consequently the value of x, or the range, will be **maximum** when θ is 45°.

To find the **time of rise,** it is observed that when the projectile reaches its highest point the velocity in the vertical direction becomes zero. Hence (27) may be written

$$V \sin \theta - gt = 0$$

whence $$t_{(\text{rise})} = \frac{V \sin \theta}{g} \qquad (32)$$

Comparing (32) and (30) it is observed that the time of flight is twice as great as the time of rise, as would be expected, for the time of ascent is equal to the time of descent.

The **greatest height** to which a projectile will rise may be found by substituting in (29) the value of t from (32). Thus,

$$y = V \sin \theta \,.\, \frac{V \sin \theta}{g} - \tfrac{1}{2} g \frac{V^2 \sin^2 \theta}{g^2}$$

whence $$y = \frac{V^2 \sin^2 \theta}{g} - \frac{V^2 \sin^2 \theta}{2g} = \frac{V^2 \sin^2 \theta}{2g} \qquad (33)$$

By use of (33) the vertical height to which a projectile will rise may be calculated.

In case the projection is vertically upward, θ becomes 90° and its sine is 1. Under this condition equation (31) shows that the range is zero,—*i.e.*, a body thrown vertically upward will return in the same path. Under the same conditions it is observed from (32) that, since sin 90° equals 1, $V = gt$,—*i.e.*, the velocity which a body will have on its return is equal to that with which it was thrown vertically upward.

The influence of the atmosphere upon projectiles has not been considered in these discussions. When a bullet is thrown at the rate of 2000 $^{\text{feet}}/_{\text{sec}}$ through still air, the effect is the same as if the bullet were still and the air blowing 2000 $^{\text{feet}}/_{\text{sec}}$ against it.

The elevation for maximum range in vacuum is 45°, but in air a greater range will be obtained when the angle is a little less than 45°, about 44°.

Problems.

1. What is the difference in range of two projectiles thrown with the same velocity, one at an angle of 30° and the other 60°?

2. A projectile is thrown at an angle 90° to the horizontal, with a velocity of 2000 $^{\text{feet}}/_{\text{sec.}}$ What will be the range and time of flight?

3. How high will a body ascend when it is projected with a velocity of 500 $^{\text{m}}/_{\text{sec}}$ and the angle of elevation is 30°?

4. If a point is moving with a velocity V in a direction inclined $\theta°$ to the Y-axis, what is its velocity parallel to the X-axis?

5. If a projectile thrown vertically upward rises to a height of 500 m., with what velocity was it thrown? (Use (33).)

6. What must be the angle of elevation of a cannon in order that a projectile thrown with a velocity of 1600 feet/sec may strike a target at a horizontal distance of two miles?

1. No difference.
2. Range = 0. Time = 2 min. 4.2 sec.
3. Height = 3188.77 m.
4. $v_x = V\cos\left(\frac{\pi}{2} - \theta\right)$.
5. $V = 98.1$ m.
6. $\theta = 3° \, 34'$.

18. Simple Harmonic Motion.—When any series of changes recur again and again in equal intervals of time, the operation is said to be *periodic*. The time required for the completion of the series is called a period. The motion of the earth around the sun is periodic, the period being one year. Each succeeding year is a repetition of the events of the previous year. Any recurring movement of this character is periodic, and numerous examples may be cited.

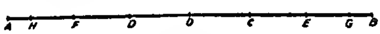


FIG. 32.

If the periodic motion is forward and backward in the same path, it is called a *vibration* or *oscillation*. Thus, a pendulum or tuning-fork is said to vibrate.

One very important kind of periodic motion is *simple harmonic motion*, which may be defined as the movement of a point to and fro along a line in such a manner that its acceleration is proportional to its displacement.

Suppose a material point is located at o, Fig. 32, and let it be set in motion so that it vibrates between A and B. Let the distance of the point on either side of o be represented by s. Then the motion will be simple harmonic when

$$\text{acceleration} \propto -s$$

The negative sign shows that the rate of retardation is proportional to the displacement of the point, the velocity being greatest at o but less and less as the point approaches A or B.

The vibrations of strings or of any body giving out a musical tone are instances of S. H. M. The name originated in the fact that harmonic sounds are produced by this kind of vibration.

A good experimental illustration of S. H. M. may be observed by suspending a heavy ball by a long string and causing the ball to move in a circle in a horizontal plane, after the manner of a conical pendulum. By observing this motion from a distance with the eyes nearly closed so that the motion may be seen indistinctly, the ball will appear to move from side to side at right angles to the line from the observer to the pendulum. The ball is moving in a circle, but the apparent motion along a

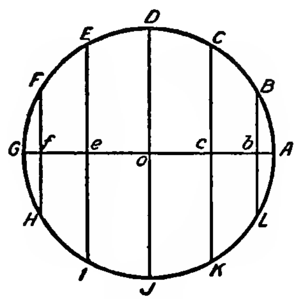


FIG. 33.

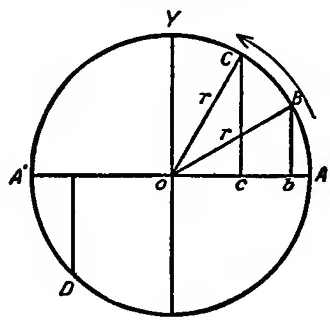


FIG. 34.

diameter of the circle is a S. H. M. In accordance with this idea a S. H. M. may be defined as the projection, upon one of the diameters of a circle, of a point which moves with uniform speed along the circumference of that circle.

The circle whose diameter is the path of the S. H. M. is called the **circle of reference.** Let a point start at A, Fig. 33, and move around in the positive direction,—*i.e.*, counter-clockwise. Draw the coördinate axes through the centre, o. Let the circumference be divided into a number of equal arcs, over each of which the point will move in equal times, since the speed is uniform. In successive intervals of time the point will pass B, C, D, E, F, G, H, and so on back to A. The component along the X-axis is found by projecting the moving point upon that axis. Likewise for the Y-axis. Thus the projection of B on the X-axis is b, and of C, c, and so on.

In a uniform circular motion the component which is parallel to the X- or Y-axis is a S. H. M.

In considering the motion of a point around a circle, let a radius be drawn from any position of the point to the centre of the circle, making an angle θ with the X-axis. When the point has moved from A to B, it has moved along the X-axis from A to b. By trigonometry

$$Bb = r \sin \theta$$

In like manner,

$$Cc = r \sin \theta$$

θ being any angle made by a radius with the X-axis. The general equation for any component along the Y-axis is

$$y = r \sin \theta \tag{34}$$

By assuming a value for r and a number of successive values for θ, as 30°, 60°, 90°, 120°, and so on to 360°, and from these finding by (34) the corresponding values of y, it is possible to construct what is called the **harmonic curve,** Fig. 35. This is

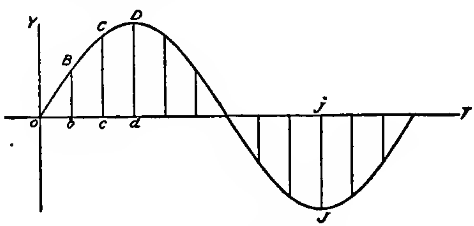


FIG. 35.

done by using the values of y as ordinates and dividing the abscissa into equal spaces to represent equal intervals of time or an interval corresponding to values chosen for θ.

Since the motion of the point is supposed to begin at A, the motion along the Y-axis will begin at o. Let r, for example, have the value 10, then, as we have assumed that the point will move over 30° in the first unit of time,

$$y = 10 \sin 30° = 5$$

At b, Fig. 35, erect an ordinate 5 units long. At the end of the next unit of time

$$y = 10 \sin 60° = 8.66$$

At c erect an ordinate 8.66 units long. When the angle is 90°,

$$y = r \sin 90° = 10$$

Erect at d an ordinate 10 units long. No ordinate can be greater than r, for the sine of an angle cannot be greater than unity.

The ordinates will all be positive as long as the angle is less than π (180°). In the third and fourth quadrants the sines are negative and therefore are dropped below the abscissa.

If now a line is drawn connecting the ends of all the ordinates, we have what is called the **harmonic** or **sine curve.** Much smaller time intervals may be chosen and consequently more ordinates constructed.

This curve is only a graphical representation of S. H. M. and is useful in a discussion of many physical problems.

The angular velocity of the point on the circle of reference is $\frac{2\pi}{P}$, where P is the period. This is the same as saying that the angular velocity is equal to 360° divided by the time of one revolution. Thus, if the point moves $\frac{1}{3}$ of the distance around the circle in one second, the period is 3 seconds. The angular velocity or distance per second is, then, $\frac{360}{3}$, or 120°, or $\frac{2\pi}{3}$ radians. The angle described in t seconds is therefore

$$\frac{2\pi t}{P}$$

and equation (34) may now be written

$$y = r \sin \frac{2\pi t}{P}$$

or

$$y = r \sin 2\pi n t \qquad (35)$$

since $\frac{1}{P}$ is equal to the number (n) of revolutions or vibrations in unit of time. For example, if the period is $\frac{1}{2}$ sec., $\frac{1}{\frac{1}{2}}$, or 2, is the number of complete vibrations in one second.

Let E, Fig. 36, be a fixed point from which the periodic time is counted. The angle EoA is then called the **epoch** e. This angle e is constant, and, since the time is counted from the moment the point P passed through E, the angle e must be added to the angle made by oP with oE to obtain the angle PoA. Hence the equation for the sine curve under this condition is

$$y = r \sin (2\pi nt + e) \qquad (36)$$

The **amplitude** of the vibration in S. H. M. is the maximum displacement of the vibrating particle,—for example, Dd or Jj in Fig. 35. This is obviously the radius of the circle of reference.

It is plain, from an inspection of Fig. 34, that the velocity along the Y-axis will be maximum when the moving particle is at A or A'. For let ω be the angular velocity in the circle of reference, then ωr is the linear velocity on the circumference of

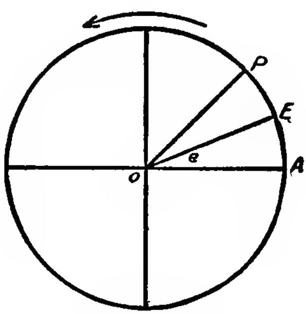


FIG. 36.

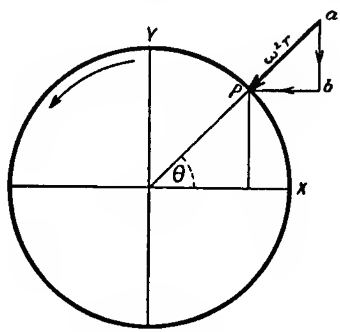


FIG. 37.

the circle. This is the velocity at all points on the circumference, since the motion is uniform; but at A and A' this motion is parallel to the Y-axis, while at all other points only a component of the velocity, or none at all, is in that direction. Hence the maximum velocity in S. H. M. is the angular velocity in the circle of reference multiplied by the amplitude, or ωr.

It has been shown in equation (23) that when a point is moving uniformly in a circle it has an acceleration which is directed toward the centre, and the magnitude of the acceleration is $\omega^2 r$, where ω is the angular velocity and r is the radius of the circle. In Fig. 37 let the point P be moving around the circle in a positive direction, and let its acceleration be represented by the vector aP, the value of which is $\omega^2 r$. This can be resolved into components parallel to the X and Y diameters.

Consider here only the motion of P along the X diameter. The acceleration in this direction is bP. Hence

$$bP = \omega^2 r \cos \theta \tag{37}$$

But the motion along the X-axis is S. H. M., and x, the displacement of P, is

$$x = r \cos \theta \tag{38}$$

Comparing equations (37) and (38),

$$bP = \omega^2 x \tag{39}$$

Stated in words, the acceleration of a particle vibrating in S. H. M. is equal to the square of the angular velocity in the circle of reference multiplied by the displacement of the particle.

The angular velocity is

$$\omega = \frac{2\pi}{P}$$

and, substituting this value in equation (39),

$$bP = \frac{4\pi^2 x}{P^2} \tag{40}$$

that is, the **acceleration in S. H. M.** is $4\pi^2$ times the displacement, x, divided by the square of the period, or acceleration is proportional to displacement.

The **phase** of vibration is the number of degrees in an arc of the circle of reference, from the position of the moving point to the point from which the angle is reckoned. In Fig. 36 the phase of the point P is PoA. Phase is also often indicated by the ratio of the angle to the circumference. For example, if the angle PoA is 60°, the phase is $\frac{60}{360}$, or $\frac{1}{6}$. If the angle is 90°, the phase is $\frac{\frac{1}{2}\pi}{2\pi} = \frac{1}{4}$, called often the quarter; 180°, the half; 120°, the third.

If two points are vibrating at the same time, their difference of phase is the difference of these angles.

By use of equation (36) it is possible to find the position of any vibrating particle when the amplitude, the number of vibrations, the time, and the epoch are known.

Suppose a particle makes 10 complete vibrations per second, and it is desired to find its position at the end of $\frac{1}{8}$ sec., the amplitude being 3 cm. and the epoch 30°. We may substitute these values as follows:

$$2\pi nt = 2\pi \,.\, 10 \,.\, \tfrac{1}{3} = 20.944 \text{ radians} = 1200°$$

$$2\pi nt + e = 1200 + 30 = 1230°$$

This shows that the point has made three complete vibrations and 150° on the next, hence

$$y = 3 \sin 150° = 3 \sin (180° - 150°) = \tfrac{3}{2}$$

i.e., the displacement of the particle is $1\frac{1}{2}$ cm.

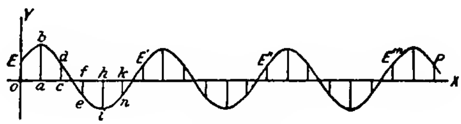


FIG. 38.

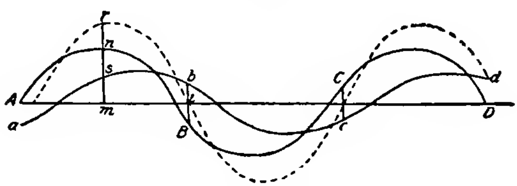


FIG. 39.

To plot the harmonic curve according to the conditions of this problem, let the abscissa, Fig. 38, be divided into a number of equal parts representing successive equal intervals of time. In this case, since the angular velocity is great, let each space represent $\frac{1}{60}$ sec. Now find, by use of equation (36), the values of y for each $\frac{1}{60}$ sec. up to $\frac{1}{3}$ sec.

When $\quad t = 0,\ \frac{1}{60},\ \frac{2}{60},\ \frac{3}{60},\ \frac{4}{60},\ \frac{5}{60},\ \frac{6}{60}$

$\qquad\quad y = 1\frac{1}{2},\ 3,\ 1\frac{1}{2},\ -1\frac{1}{2},\ -3,\ -1\frac{1}{2},\ 1\frac{1}{2}$

Hence the curve begins on the ordinate, $1\frac{1}{2}$ cm. above o. At the end of the first unit of time y is 3; so an ordinate is erected at a, 3 cm. long. This is the amplitude of the vibration. At c the ordinate is $1\frac{1}{2}$ again. At f it is $-1\frac{1}{2}$, and so is drawn below the abscissa, as are also hi and kn. At the end of $\frac{6}{60}$ sec. one cycle is completed, and the point is at E', for the period is $\frac{1}{10}$ sec. This cycle is twice repeated, bringing the particle to E'''. This makes $\frac{18}{60}$ sec., but the time given is $\frac{1}{3}$ or $\frac{20}{60}$ sec., hence the point will be at P where y is $1\frac{1}{2}$ cm. This curve shows the whole movement of the vibrating particle for $\frac{1}{3}$ sec.

Let the student take the intervals of time as $\frac{1}{120}$ sec., find corresponding values of y, and insert them in Fig. 38.

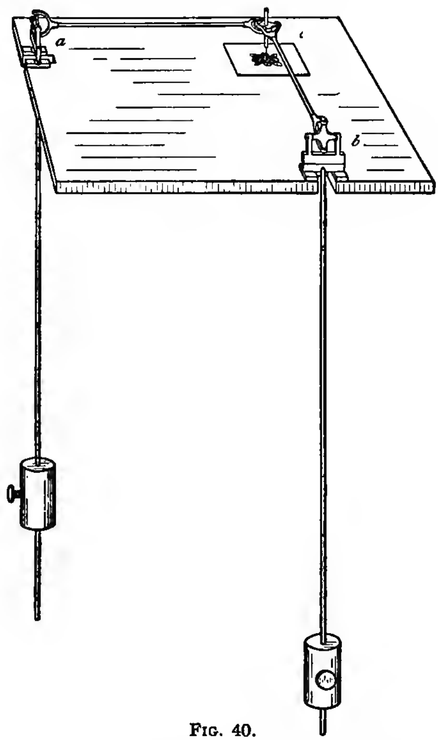


Fig. 40.

If two harmonic curves are referred to the same axis, as *abcd* and *ABCD*, Fig. 39, their **resultant** may be found by adding the ordinates. The resultant in the figure is the dotted line. The line *mr* is the sum of *mn* and *ms*. The line *bB* is composed of the positive line *ib* and the equal negative line *iB*; hence their sum is zero and the resultant curve here crosses the axis. In a similar manner the resultant curve may be found at any point.

When two S. H. M.'s at right angles are **compounded,** a large variety of resultants may be obtained, depending on the period, the phase, and the amplitude of the components. An experimental illustration may be made by suspending two pendulums so that they will swing in planes perpendicular to each other,

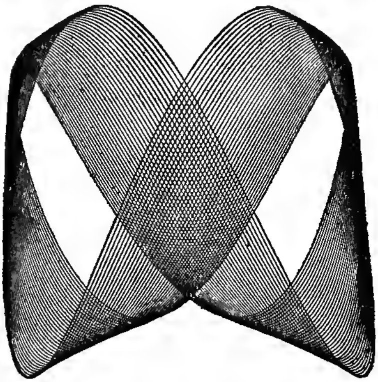


FIG. 41. ($\frac{1}{2}$)

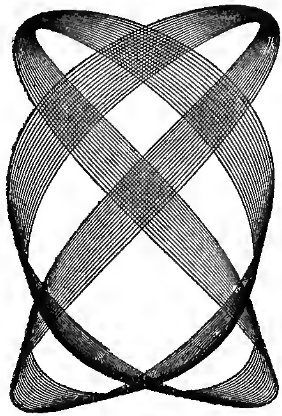


FIG. 42. ($\frac{2}{3}$)

as shown in Fig. 40. One pendulum should, preferably, beat seconds. The other can be made any length by sliding a heavy bob up or down. Both are suspended from knife edges, that

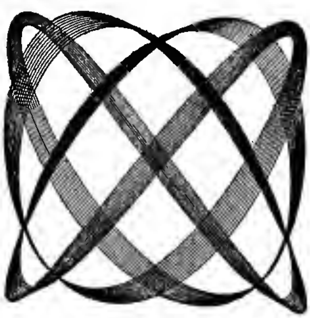


FIG. 43. ($\frac{3}{4}$)

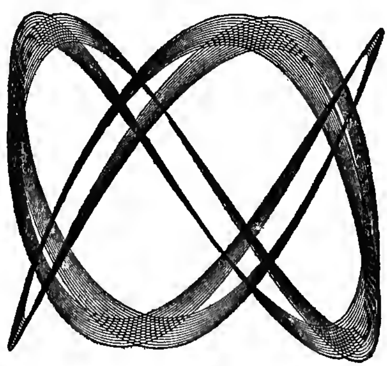


FIG. 44. ($\frac{3}{5}$)

the friction may be as little as possible. The rods extend a few centimetres above the knife edges, and to the tops of these, light rods, *ac* and *bc*, are attached by universal joints. These horizontal rods are connected at *c* by a hinge, from which a needle projects downward. The needle point must then trace the resultant motion of both pendulums. By placing a smoked glass beneath *c*, this motion may be recorded. In Fig. 41 the period of one pendulum was twice that of the other. A number of periods are required to make a figure such as this. The period of each pendulum is constant, but, because of friction and a very slight variation in the ratio of the periods, the trac-

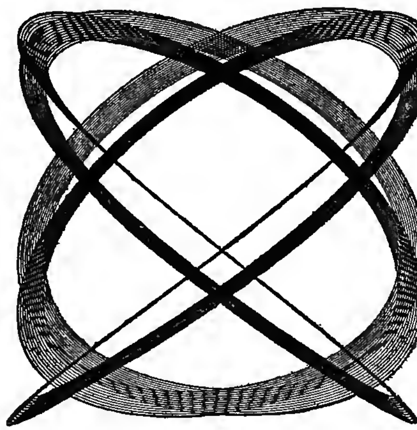


FIG. 45. ($\frac{4}{5}$)

ings are separated from one another. In Fig. 42 the periods are as 2 to 3, as indicated by the fraction beneath the figure. In Fig. 43 the pendulums are made of such length that while one is making three vibrations the other makes four. The ratios in the other figures are as indicated. The ratio in figures such as these may be determined by beginning at one of the sharp points of the tracing and counting the number of times across in a horizontal and then in a vertical direction to a corresponding symmetrical point on the other side of the figure.

CHAPTER II

DYNAMICS

19. Definition of Terms. — The preceding chapter has been devoted to a discussion of *motion*, without consideration of the *cause* of the motion or the *mass* of the moving body. The problem there was to find the path of a moving point under certain given conditions. The chapter was entitled **Kinematics** because κίνημα (kinema) means motion.

In this chapter attention is directed mainly to a consideration of *force* and its effect in changing motion or causing strain in masses of matter. Force as a cause of motion is often treated under the title **kinetics,**—that which causes motion,—while the subject of equilibrium under stress is often called **statics.** Kinetics and statics are subdivisions of dynamics.

20. Newton's Laws of Motion. — Sir Isaac Newton, in the latter part of the seventeenth century, announced three important laws which contain the fundamental principles of dynamics. These laws were written in Latin, and their interpretation is about as follows:

1. Every body of matter persists in its state of rest or motion.
2. The effect of an impulse in changing the momentum of a mass of matter is independent of other impulses which may be applied at the same time and of the momentum which the mass may already have.
3. The application of a force is always accompanied by an equal resistance in the opposite direction, and the energy expended by any body acting as agent is equal to the energy received by another body which resists the agent.

21. Inertia. — Inertia is that property by virtue of which matter persists in whatever state of rest or motion it may have. This is a general property of all matter. Newton's first law is simply a statement of the principles of inertia. In accordance with this property, a body resting at any point in space will remain there forever, and a body in motion will continue its motion forever in a straight line, provided it receives no impulse from an external force. In other words, a mass of matter separated from all other masses cannot change its state of rest or motion.

It is impossible to supply the conditions of a moving body completely unaffected by outside influences, but we assume that the law is true in reference to moving bodies, and on this assumption we solve problems in mechanics the results of which conform with experimental tests. For example, if a body is projected vertically upward with a certain velocity, we assume that it will continue its velocity uniformly. We also know the effect which gravity will have on the body at the same time. By subtracting the opposing gravity effects we obtain results which are in accordance with the facts of experience. Hence the assumption is regarded as correct.

22. Force. — The word "force" is derived from the word *fortis*, meaning strong, and the primary idea of force was probably related to muscular ability. The muscular effort required to lift a weight, for example, would be a measure of force or strength. Then a push or pull, by muscular effort against any resistance, would be called a force. Then inanimate bodies which by their motion or otherwise would do what might be done by muscular effort, would by a kind of personification be said to exert force. As applied to inanimate objects, force is a consequence of the laws and properties of material bodies. Thus, expanding gas or steam may move adjacent bodies only to secure more room for itself. A steel spring in a strained condition may push or pull other bodies to restore its own molecular equilibrium. One body may collide with another, and the motion of both will be changed as a result of the inertia of the bodies. Numerous examples of this kind may be cited where force is said to be exerted by one mass on another.

The common effect of force, as the term is used in mechanics, is *motion* or *strain* in bodies of matter. Hence force may be defined as *that which produces or tends to produce motion.*

23. Units of Force.—A force is measured by the effect which it produces. Any force, however small, will set in motion any mass, however large, provided no resistance is offered except that of the inertia of the mass. A constant force—*i.e.*, one which continues to be of the same magnitude—will produce in a mass a uniformly accelerated motion. If the mass is small, the acceleration may be large, and if the mass is large, the acceleration may be small. In any case, for a given constant force, the product of the *mass* and *acceleration* is a constant quantity. This is usually expressed by

$$F = ma \tag{41}$$

By this equation it is seen that for any given mass, m, the force, F, varies directly as the acceleration, a. Therefore the acceleration which is produced may be employed as a measure of the force applied. There are two common units based on this principle. They are the **dyne** and the **poundal.**

The dyne is a force which will cause a mass of one gram to have an acceleration of one centimetre per second every second.

The poundal is a force which will cause a mass of one pound to have an acceleration of one foot per second every second.

These two units are **absolute,** because they are not affected by surrounding conditions. The mass of a body is the same whether it is located in the vicinity of the earth, the moon, or is alone in space. Such units are of great value in science.

There are also other units of force, such as the *pound*, the *gram*, the *kilogram*, and in fact any of the units by which the quantity of mass is ordinarily determined may be used as units of force.

The pound or gram of force is that force which will support a mass of one pound or one gram. This gives the primary idea of what is meant by one pound or one gram of force,—*i.e.*, the unit is thus determined. The force may then be exerted in any direction, whether gravity is considered or not, and it will be such a force that if it *were* exerted against gravity it would lift one pound or one gram as the case may be. Such are known as **gravitational units.** They are not absolute, because gravity changes with change of distance from the centre of the earth and with change of location on the surface.

The relation between the **dyne** and the **gram** as units of force is found from a consideration that if 1 dyne gives 1g (mass) an acceleration of 1 $^{cm}/_{sec^2}$, and 1g (force) gives to 1g (mass) an acceleration of 980 $^{cm}/_{sec^2}$, as determined by the acceleration of falling bodies, then 1g (force) must be equal to 980 dynes. In the same manner it may be shown that 1 lb. (force) is equal to

32.2 poundals. These numbers are not exact except for certain locations on the earth's surface. At points near the equator the acceleration due to gravity is 978 $^{cm}/_{sec^2}$; near the poles, about 983; at Cincinnati, 980. (See page 298.)

The value of the poundal in terms of dynes, found by multiplying the number of centimetres per foot (30.48) by the number of grams per pound (453.593), is 13,825.5. The same result is found by use of the dimensional equations, as follows: Since $F=ma$, the dimension of force is the product of the dimensions of m and a,

$$[M]\,[LT^{-2}]=[MLT^{-2}]$$

Let the dimensions when the unit is the poundal be $[M_1L_1T_1^{-2}]$, and when the unit is the dyne be $[M_2L_2T_2^{-2}]$. The numeric x will be the number of dynes in the poundal. Hence

$$1\,[M_1L_1T_1^{-2}]=x\,[M_2L_2T_2^{-2}]$$

$$x=\left[\frac{M_1L_1T_2^2}{M_2L_2T_1^2}\right]$$

$$\frac{M_1}{M_2}=453.593 \qquad \frac{L_1}{L_2}=30.48 \qquad \frac{T_2^2}{T_1^2}=1$$

$$\therefore\ x=453.593\ \cdot\ 30.48\ \cdot\ 1=13825.5$$

$$\therefore\ 1\text{ poundal}=13{,}825.5\text{ dynes}$$

The purpose of this kind of solution of so simple a problem is not to obtain a result, but to give exercise in the use of dimensional equations.

24. Impulse and Momentum.—Impulse involves two factors, —a *force* and the *time* during which the force acts. It is a matter of common experience that the velocity of a freely moving body will be increased by increasing either the force which causes the motion or the time during which the force is applied. The product of these two factors is called *impulse*. Thus,

$$\text{impulse}=Ft$$

The effect of the impulse is a quantity of motion in a mass of matter which is free to move. The amount of motion in any moving body depends on two factors,—the *mass* of the body and its *velocity*. The product of these is

$$\text{momentum}=mv$$

The momentum is therefore proportional to the impulse, and we may write

$$Ft = mv \tag{42}$$

This equation may also be derived from a consideration of the fact that

$$F = ma$$

and $$a = \frac{v}{t}$$

$$\therefore F = \frac{mv}{t}$$

or $$Ft = mv$$

There are no names for the units of impulse and momentum, but the symbols of the units may be found from the dimensional formulæ. Impulse is the product of force by time, hence its dimensions are

$$[MLT^{-2}]\,[T] = [MLT^{-1}]$$

Momentum is the product of mass by velocity, hence the dimensions are

$$[M]\,[LT^{-1}] = [MLT^{-1}]$$

Hence the symbol of the unit for either impulse or momentum in c.g.s. units is $\frac{1\text{g } 1\text{cm}}{1 \text{ sec}}$,—*i.e.*, the unit momentum is the amount of motion in 1 g. moving with a velocity of 1 cm. per sec., and unit impulse is that which will produce this effect.

To use equation (42), F must be in dynes or poundals, depending on the units employed.

By Newton's second law, any impulse will produce a certain change of momentum independent of other impulses applied at the same time, and independent of the momentum which the body may already have. Momentum involves not only a quantity of motion (mv), but also a direction. It is therefore a vector quantity. Change of momentum is always in the direction in which the force is applied.

25. Stress and Strain.—*Stress* is a mutual action between two forces or between a force and that which resists it.

Strain is the deformation resulting from a stress.

For illustration, if two forces are applied in opposite directions at the ends of a wire, the wire is subjected to a stress called *tension*. The effect is the same when one end of the wire is fastened to a rigid support, the resistance of the support being equal to the active force at the other end of the wire. The strain in this case is the change in length of the wire, and its measure is the ratio of the increase in length to the original length.

If a body of gas is enclosed in a cylinder and a piston is pressed down upon it, there is the mutual action of the force upon the piston and the reaction of the gas within the cylinder. This stress is called a *pressure*. The strain in this case is the decrease in volume, and it is measured by the ratio of the diminution of volume to the original volume.

If an elastic rod is rigidly fastened at one end and the other end is twisted through a few degrees, there is the mutual relation of force and resistance called **shearing stress.** The strain in this case is measured in a manner which is described later under elasticity.

In all cases, whenever a force is applied, an equal and opposing force or resistance is offered to it. Without resistance there could be no such thing as force. If a force is applied to a body which does not move, the effect is a deformation resulting from a stress. If the body is free to move, the resistance due to the inertia of the body is equal to the force which causes the motion. The force need not be a "little greater" than the resistance to produce motion.

26. Graphical Representation of Forces. — Since force is a vector quantity, it may be represented in magnitude and direction by a line whose length, drawn to proper scale, represents the magnitude and whose direction is that in which the force is applied. The composition and resolution of forces are effected in a manner which has already been described for velocities. The parallelogram and polygon of forces, the methods of calculating resultants and their direction, and the resolution of forces are the same as in the case of velocities.

27. Resultant and Equilibrant. — A resultant force is one which may be substituted for two or more other forces and which will produce the same effect as the others combined. An equilibrant is equal and opposite to the resultant.

The resultant of two parallel forces in the same direction is their sum; when parallel but opposite in direction, the resultant is their difference; when the two forces are perpendicular to each other, the resultant is the hypotenuse of a right-angled triangle of which the two forces are the legs. When the angle between the forces is acute or obtuse, the resultant is found by use of the well-known equation

$$r^2 = a^2 + b^2 + 2ab \cos \theta \qquad \text{(compare equation 19)}$$

The graphical method of solution may also be used in any of these cases. An illustration is here given. Let *oa*, *ob*, *oc*, and *od* represent forces acting as indicated in Fig. 46. Through *o* draw rectangular coördinates and resolve each force into its components along the X- and Y-axes. The components *oe* and *of* are positive, while *oh* and *og* are negative. Also, *ea* and *hd* are positive and *fb* and *gc* are negative. By measurement and addition, all the forces may thus be composed into two at right angles. The resultant can then be easily found.

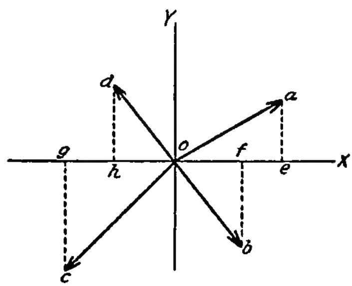


Fig. 46.

When the angles made by the vectors with the axes are known, the values of the components can most easily be calculated from the simple trigonometrical relations.

28. Resolution of Forces.—The principles and methods of resolution of vector quantities have already been discussed. These principles have many important applications, and additional illustrations will here be given. Suppose a force is applied in the direction BC, Fig. 47, on a rope attached to the top of a pillar AB, and it is desired to know the component of this force which is effective in pulling the pillar over. BC can be resolved into the two components BD and DC. An infinite number of components may be drawn, but these are the two which are desired in this case, for DC has no effect in pulling the pillar over, while BD is acting at the greatest advantage for the purpose in view. Hence the effective part of the force is $BC \sin \theta$. The larger θ becomes the greater the sine will be, and at 90°

the sine is unity. Hence the longer the rope is the more effective the force will be.

Another illustration of resolution may here be given, in case of wind or water directed against a turbine wheel. Let *TT'*, Fig. 48, represent one blade of the wheel. Let the plane of rotation be as indicated by the arrow *D* and let the force of the wind be represented by the vector *wo*. Resolve *wo* into *wp*,

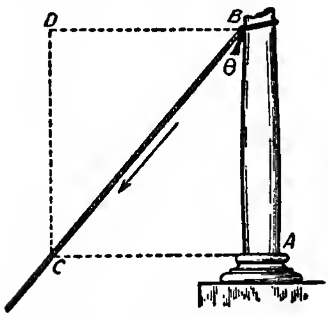


FIG. 47.

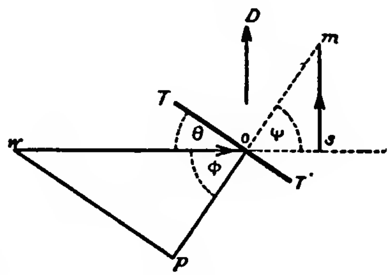


FIG. 48.

parallel to the blade, and *po*, perpendicular. It is evident that *po* is the only effective component, but it is not in the direction of the rotation. Let *po*, for convenience, be extended to *m*, making *om* equal to *po*. Resolve *om* into *sm* and *os*. Then *sm* is the component whose value is to be found. Inspection of the figure shows that, since the plane of rotation is kept perpendicular to the direction of the wind,

$$sm = om \sin \psi$$
$$= po \sin \varphi$$
$$= po \cos \theta$$

But $$po = wo \cos \varphi$$
$$= wo \sin \theta$$
$$\therefore sm = wo \sin \theta \cos \theta \qquad (43)$$

Suppose, for example, that the blades of a wind turbine are set at an angle of 45° to the plane of rotation and the wind blows with a force of 20 oz. to the square inch. Then

$$sm = 20 \,.\, \tfrac{1}{2}\sqrt{2} \,.\, \tfrac{1}{2}\sqrt{2} = 10$$

i.e., one-half the force is expended in producing rotation.

Problems.

1. What acceleration per second will be produced by a constant force of 10 g. acting on a mass of 5 g.?

2. What force will be required to lift a mass of 10 lbs. vertically with an acceleration of 20 $^{ft}/_{sec^2}$?

3. What change of momentum will be produced by a force of 25 g. acting 10 sec.?

4. What is the resultant of two forces whose magnitudes are 20 and 30 dynes respectively, the angle between them being 24° 10′?

5. If a force of 500 lbs. is required to push a car on a straight track, what force would be needed if applied in a line making an angle of 30° with the track?

6. What force is required to move a mass of 75 g. a distance of 200 cm. in 10 sec.?

7. When wind blows at right angles to the plane of rotation of a turbine wheel, show by inspection of the table of sines and cosines that the maximum rotating effect will be obtained when the blades are set at an angle of 45° to the direction of the wind.

8. A bullet is shot horizontally from a rest 120 feet high with a velocity of 2000 feet per second. When and where will it strike the ground?

9. By use of a dimensional equation change 986 feet per minute per minute to centimetres per second per second?

1. 1960 $^{cm}/_{sec^2}$.
2. 522 poundals.
3. 245,000 $^{g.\,cm}/_{sec}$.
4. 48.9 dynes.
5. 577.35 lbs.
6. 300 dynes.
7. ——.
8. 5460 ft., 2.73 sec.
9. 8.348 $^{cm}/_{sec^2}$.

29. Moment of Force.—The moment of a force is the product of that force by the perpendicular distance from the axis of rotation to the line of direction of the force. Thus, let *o* be an axis about which a body *oa* may rotate. The moment of the force *F* applied at *f*, perpendicular to *oa*, is *F* . *of*. It is observed that the importance of *F* increases as the distance from *o* increases.

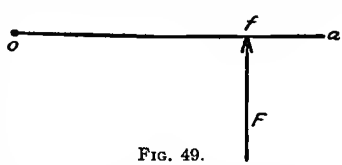


FIG. 49.

It is a common experience that the longer the arm of a lever the greater will be the effect of any given force in producing rotation about the fulcrum—the axis of rotation.

Ordinarily the effect of a force or several forces is to produce both rotation and translation accompanied by a deformation of the body upon which the forces act. For our present purpose only rotation is considered, and *oa* is taken to represent a rigid body fastened at *o*.

Only that component of a force perpendicular to the line connecting *o* and *f* is effective in causing rotation. Let *bf*, Fig. 50, represent a force acting at an angle θ to *oa*. Resolve this force into *bc* parallel to *oa*, and *cf* perpendicular. The component *bc* plainly causes only a strained condition of the body and has no effect in producing rotation about *o*. The moment in this case is, then, $of \,.\, F \sin \theta$.

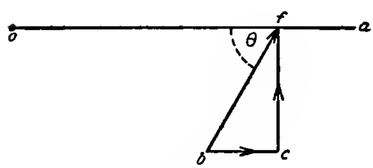


Fig. 50.

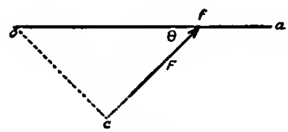


Fig. 51.

Instead of resolving the force, the same result is obtained by drawing *oc*, Fig. 51, perpendicular to the direction of the force, and the moment is $F \,.\, oc$, which is equal to $F \,.\, of \sin \theta$, the same as before.

30. Equilibrium of Moments.—A moment is positive when it produces rotation counter-clockwise, and negative when clockwise. When the sum of all the moments is zero, there is equilibrium as far as rotation is concerned.

Let *ab*, Fig. 52, be a body whose axis is at *o*, and let forces 5, 7, 6, and 3 dynes at a distance of 1 cm. from one another act at right angles to *ab*, the force of 6 dynes being applied at *o*. The positive moments are $7 \times 1 + 3 \times 1 = 10$, while the negative moments are $5 \times 2 = 10$. Their sum is zero. Hence no rotation is produced by this arrangement. It is observed, however, that there are 14 dynes acting in one direction while only 7 dynes act in the opposite direction. To secure equilibrium both in respect to rotation and translation, it is necessary that an additional 7 dynes be applied at *o* in a direction opposite to the force of 6 dynes.

To make this principle clear, another illustrative problem is here given. Let *ab*, Fig. 53, be a line through any body; let forces 10, 20, 8, and 6 dynes act at right angles to *ab*. It is required to find a resultant force and its point of application such that there will be neither motion of translation nor rotation. Assume the forces to be 2 cm. apart. The magnitude of the resultant is 28 dynes and it is directed to the right. The equilibrant is, then, 28 dynes directed to the left. This would prevent motion of translation, but, to prevent rotation also, the equilibrant must

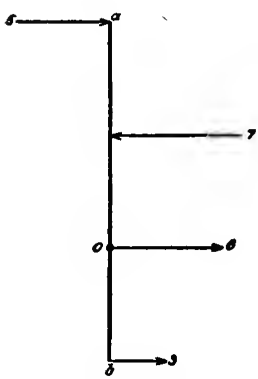


FIG. 52.

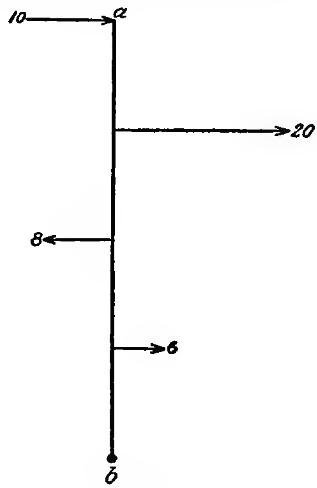


FIG. 53.

be applied at such a distance from any axis that the moment of the equilibrant in reference to that axis will be equal in magnitude to the sum of the moments of the components in reference to the same axis. Any point may be selected as the axis. Take *b*, 2 cm. from the force of 6 dynes. The negative moments are 10×8, 20×6, and 6×2. Their sum is -212. The positive moment is $8\times4=+32$. The sum of these is -180. Consequently, the 28 dynes must be applied at such a point that its moment will be $+180$. The distance from *b* is therefore $6\frac{3}{7}$ cm.

In case of two parallel forces acting in the same direction, the resultant is their sum applied at a point between the two and at distances from the lines of the forces inversely as those forces. An example of this is a beam with weights suspended from its ends, the beam being supported at the point through which the resultant passes.

31. The Couple.—When two equal and oppositely directed forces, whose lines are parallel but not coincident, are applied to a body, they constitute what is called a mechanical *couple*.

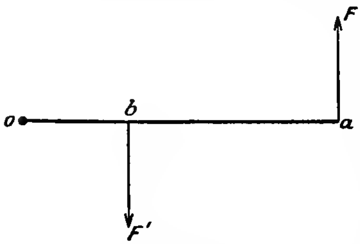


FIG. 54.

In this case it is evident that the resultant is zero, and so there can be no motion of translation. There will, however, be rotation. In Fig. 54 let F be equal, parallel, and opposite to F', o being the axis of rotation. The moment of F is $+F\ .\ oa$ while that for F' is $-F'\ .\ ob$. The sum of these is $F(oa—ob)$ or $F\ .\ ab$, since $F=F'$. Hence the moment of a couple is the product of either force by the perpendicular distance between them.

An approximate experimental illustration of the action of a couple may be made by supporting a bar magnet on a cork on the surface of water. Under the influence of the earth's magnetic field, which may be assumed to be uniform in the region of the experiment, the magnet will turn and will take a position in the magnetic meridian, but will have no motion of translation.

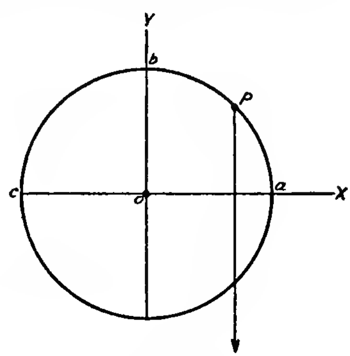


FIG. 55.

Problems.

1. Two forces, 20 and 50 lbs., act in opposite directions, their lines of direction being 5 ft. apart. Find the magnitude and position of the resultant.

2. A wheel is free to rotate on its axle. A force of 100 g. is applied at P midway between a and b and directed parallel to the Y-axis. The radius of the wheel is 3 cm. What force at c will produce equilibrium?

3. The sides of an equilateral triangle are 10 cm. long. A force of 500 dynes acts along one side. What is the moment about the opposite vertex?

4. How high above the base of a tall pole should a rope be fastened, the rope being 20 ft. long, in order that a force applied at the other end may be most effective in pulling the pole over?

1. 30 lbs. $8\frac{1}{3}$ feet from force of 20 lbs.
2. 70.71 g.
3. 4330.
4. 14.142 feet.

32. Moment of Inertia. — *Force* in its relation to *linear acceleration* has already been discussed, the relation being expressed by $F=ma$; *i.e.*, force is measured by the acceleration it will give to a mass of matter.

We will now consider the relation of *moment of force* to *angular acceleration.* Let m, Fig. 56, be a mass at a distance r from an axis of rotation o. Let a force F act in a direction perpendicular to r. It will cause a linear acceleration in m in accordance with

$$F=ma$$

But angular acceleration is the linear acceleration divided by the radius; hence

$$A=\frac{a}{r}$$

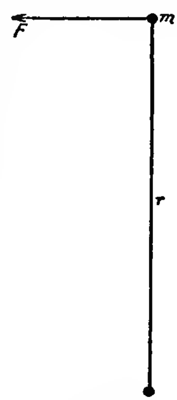


Fig. 56.

where A is the angular acceleration. Hence

$$F=mrA$$

$$\text{or} \qquad Fr=mr^2A \qquad (44)$$

Comparing this equation with

$$F=ma$$

it is observed that force in one corresponds to moment of force in the other; linear acceleration in one, to angular acceleration in the other; and mass in one, to the quantity mr^2 in the other.

It is this quantity, mr^2, which measures the inertia of matter in case of rotation.

We have considered here only a single particle m at a distance r from the axis. If each of the infinite number of particles that compose a body be multiplied by the square of its distance from the axis of rotation, the sum of all these products is called the *moment of inertia* of the body. Sometimes also called the inertia of rotation.

The moment of inertia depends not only on the mass of a body but also on the distribution of the mass. Two wheels may have the same mass, but the one which has its mass farther from the axle will have the greater moment of inertia.

Moment of inertia may be defined as that property of a body by virtue of which it resists the moment of force that tends to produce angular acceleration.

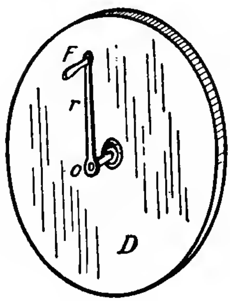


FIG. 57.

The dimensions of this quantity are $[ML^2]$, for it is a mass times the square of a distance. The c. g. s. unit is then 1 *g.* 1 *cm.*^{2}; *i.e.*, a mass of 1 g. placed at a distance of 1 cm. from the axis of rotation has unit moment of inertia.

The symbol ordinarily used for moment of inertia is I.

For regular and homogeneous bodies the value of I may be calculated by the methods of the integral calculus. This operation consists in the addition of the quantities mr^2 for each of the particles of which a body is composed. This may be expressed by

$$I = \Sigma mr^2$$

In this way it can be shown that for a cylinder rotating on its own axis and having a mass M and a radius R,

$$I = \tfrac{1}{2} MR^2 \qquad \text{(Appendix 7)}$$

and for a sphere rotating on an axis through its centre,

$$I = \tfrac{2}{5} MR^2 \qquad \text{(Appendix 9)}$$

When a body is irregular or is not homogeneous, the value of I cannot ordinarily be calculated, but must be determined by experiment. In such a case it may be determined by the moment of force required to produce unit acceleration. Thus, if force F. Fig. 57, is one dyne and r is one centimetre, and if Fr causes the body D to rotate with a change of velocity of one radian per second each second, then D is said to possess unit moment of inertia. The value of Fr required to produce unit change of velocity in any other body is a measure of I of that body, or, using the same value of Fr, the value of I will be inversely as the angular acceleration produced. This relation is expressed by

$$Fr = IA \tag{45}$$

or

$$I = \frac{Fr}{A}$$

where A is the angular acceleration.

The total mass of a rotating body may be supposed to be concentrated at a point whose distance from the axis is such that the resistance to angular acceleration would be the same as that actually observed. Let k be this distance, then, since the total mass is now by supposition at a distance k from the axis,

$$I = Mk^2$$

or

$$k^2 = \frac{I}{M} \tag{46}$$

This distance k is called the **radius of gyration.** The value of k^2 can be found from any equation for I by omitting the factor M.

It may be shown that the moment of inertia (I) about any axis is equal to the moment of inertia (I_o) about a parallel axis through the centre of gravity, increased by the product of the mass and the square of the distance between the axes. (See page 275.) This may be expressed by

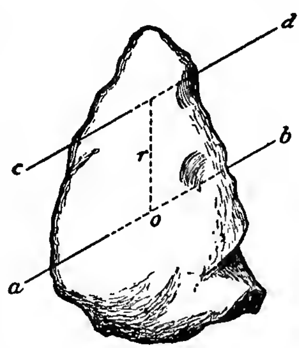


Fig. 58.

$$I = I_o + mr^2 \tag{47}$$

or, since

$$k^2 = \frac{I}{M},$$

$$k^2 = k^2_o + r^2 \tag{48}$$

In Fig. 58, the axis ab passes through the centre of gravity o, while cd is parallel to ab and at a distance r from it.

33. Centripetal Force.—A body in motion will continue its motion uniformly in a straight line except in so far as it receives an impulse from without. Consequently, when a mass is moving in a circular path, there must be a force which changes the direction of its motion.

It has already been shown, under "uniform circular motion," that the acceleration is directed toward the centre of the circle and that its value is $\frac{v^2}{r}$.

Since $F=ma$, and a in this case is $\frac{v^2}{r}$, the force directed toward the centre is

$$\frac{mv^2}{r}$$

Representing this force by F_c, we may write

$$F_c=\frac{mv^2}{r} \tag{49}$$

It has also been shown that acceleration may be expressed by $\frac{4\pi^2 r}{P^2}$ or $4\pi^2 rn^2$ (see (24) and (25)); hence equation (49) may be written

$$F_c=\frac{4\pi^2 mr}{P^2} \tag{50}$$

or

$$F_c=4\pi^2 mrn^2 \tag{51}$$

where n is the number of revolutions per second.

It should be noted that the mass moving in a circular path has no tendency to move away from the centre in the direction of the line connecting it with the centre. If the force F_c should cease, the body would continue its motion in a direction tangent to the circle.

The force which produces acceleration toward the centre is called *centripetal force*. There is no such thing as "centrifugal force" unless by that term is meant the inertia of the mass which causes it to continue in a straight line except in so far as centripetal force changes its motion.

By inspection of equation (49) it is seen that if the radius r decreases, the velocity being unchanged, the force F_c must increase, for the smaller the circle is, the more rapid is the change of direction. To round a sharp curve requires greater effort toward the centre. If F_c decreases, m and v remaining the same, the mass meets with less interference in its tendency to move in a straight line. If velocity alone increases, the force must increase as the square of the velocity.

Instances and applications of centripetal force are found on every hand. The planets move in elliptic orbits under the influence of the sun. Bodies on the surface of the earth move in a circular path under the influence of gravity. "Centrifugal" governors are employed to regulate the admission of steam to an engine. In laundries clothes are dried by rapid rotation in perforated cylinders. An indefinite number of such examples may be given.

34. Stability of a Rotating Body.—A body in rapid rotation will change its position so as to rotate on its shortest axis. This is a natural consequence of the inertia of matter. When the

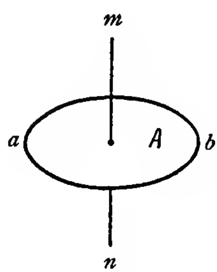


Fig. 59 *A*.

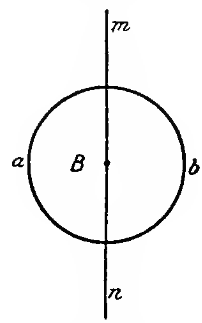


Fig. 59 *B*.

rotation is around the shortest axis, a greater number of particles of which the body is composed are thus permitted to move in larger circles,—*i.e.*, more nearly in a straight line. In Fig. 59 *B*, a circular disk is represented as rotating on an axis *mn*, while in *A* the axis is perpendicular to the disk. Only the points *a* and *b* in *B* rotate in a circle as large as the circumference of the disk, while in *A* all the particles in the circumference rotate in the largest circle. The same may be shown for any corresponding concentric circles. If the disk *B* is free to change its axis it may, provided its axis is one of perfect symmetry, continue its rotation on *mn*, but a slight disturbance will cause it to take the position *A*, which is a position of stability for a rotating body.

This may be shown experimentally by suspending a disk or ring or a piece of board by a string attached to a turning table. The string is fastened to one edge of the object, and thus will have the position *B* in the figure. By a rapid rotation it will take a horizontal position, thus rotating on its shortest axis as in *A*. The phenomena of a rolling hoop illustrate this principle. Another example is the permanence of the earth's rotation on its polar axis.

35. The Conical Pendulum. — A conical pendulum is one where a mass revolves in a horizontal plane about an axis from which it is suspended. In Fig. 60 let *m* be a heavy mass suspended from *o* by a connecting rod *l*. Let the vertical rod *os*, which is in the position of the axis, be rotated. The mass *m* will swing out farther and farther as the speed of rotation is increased, for the larger the circle the more nearly *m* will approximate a straight line. Let the period be such that *m* moves in a circle of radius *r*. The distance *h* in the figure is from *o* to the plane of revolution of *m*.

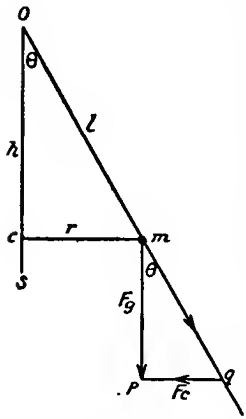


Fig. 60.

That *m* may move in a circle and not in a straight line, there must be a force directed toward *c*, the centre of the circle. This is furnished by a component of the force of gravity on the mass *m*. Let F_g, the total force of gravity, be resolved into the components *mq* and *qp*. The former causes only a tension of the rod *l* and need not here be considered, but the component *qp* is the force F_c which causes the mass *m* to move in the circle of radius *r*.

Let θ be the angle which *l* makes with *h*. Then

$$F_c = F_g \tan\theta = F_g \frac{r}{h}$$

But $$F_g = mg$$

where *g* is the acceleration due to gravity, hence

$$F_c = mg\frac{r}{h}$$

Also $$F_c = \frac{4\pi^2 mr}{P^2}$$ (see equation 50)

Hence, equating these values of F_o,

$$\frac{4\pi^2 mr}{P^2} = mg\frac{r}{h}$$

or

$$P = 2\pi\sqrt{\frac{h}{g}} \qquad (52)$$

It appears from this equation that the period of a conical pendulum varies directly as the square root of h and inversely as the square root of g. Since l does not appear in the equation, the period is independent of the length of the rod or string from which the mass is suspended. A number of masses suspended from o by strings of different length will revolve in the same horizontal plane.

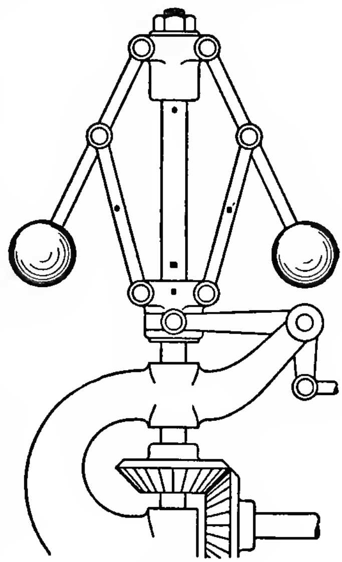


FIG. 61.

The principle of the conical pendulum has been extensively used in the regulation of the speed of the steam engine. In the Watt governor, Fig. 61, connections are made by rods a, a', from the pendulums to the ring s which slides on the rod om. The whole may be operated by a belt from the axis of the fly-wheel or by a shaft and cogs as shown in cut. The ring s is attached by proper mechanism to a valve in the steam pipe. Any increase in the speed of the engine will partly close the valve, while a decrease in speed will cause the valve to open. In this manner a fair degree of uniformity in speed may be obtained.

Other and more sensitive forms, such as the "parabolic" and "centrifugal" governors, have in large measure superseded this one, but in all of them the principle is that stated in Newton's first law of motion.

36. Effect of Rotation of the Earth on the Weight of Masses.— In Fig. 62 let the axis of the earth be represented by NS, the equator being EE'. A mass at E is acted on by a force, called gravity, which is directed toward the centre of the earth. The greater part of this force gives the body what is called *weight.* The remaining part keeps the body from moving on in a direction tangent to the equator,—*i.e.*, gives the mass an acceleration toward the centre and thus causes motion in a circle. Hence the equation for this latter part of the force of gravity is

$$F_c = \frac{4\pi^2 mr}{P^2} \qquad \text{(see equation 50)}$$

and the acceleration is therefore

$$\frac{4\pi^2 r}{P^2} \qquad \text{(see equation 24)}$$

The period of the earth's rotation is 86,400 seconds and the radius of the earth is about $6.4(10)^8$ cm. Hence the acceleration is $3.385\ ^{\text{cm}}/_{\text{sec}^2}$. The total force of gravity will cause an acceleration of about $980\ ^{\text{cm}}/_{\text{sec}^2}$, which is very nearly 289 times

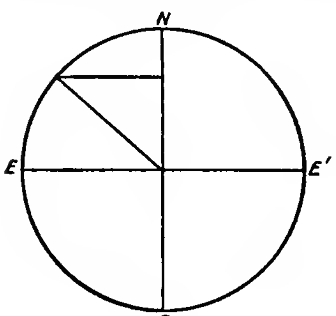


Fig. 62.

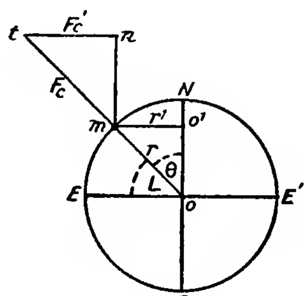


Fig. 63.

greater than 3.385,—*i.e.*, 289 times greater than the force necessary to keep the mass in a circular path at the equator. A mass at the equator will therefore weigh $\frac{1}{289}$ less than it would if the earth were not rotating.

$$\text{Since} \qquad F_c \propto \frac{1}{P^2} \qquad \text{(see equation 50)}$$

then if the period becomes 17 times shorter,—*i.e.*, if the earth should rotate 17 times faster,—the value of F_c would be 289 times greater. The weight would then be decreased $\frac{289}{289}$,—*i.e.*, all the value of F_c would be expended in keeping the mass in a circular path, consequently objects at the equator would have no weight.

A body located north or south of the equator moves in a circle of less radius than that of the earth. Let a mass m, Fig. 63, be located in latitude L. The radius of the circle in which m moves is r'. But

$$r' = r \sin \theta = r \cos L$$

Let F_c' be the component of the force of gravity directed toward o' Then

$$F_c' = \frac{4\pi^2 mr \cos L}{P^2} \tag{53}$$

By resolution of F_c as shown in the figure,

$$F_c' = F_c \cos L$$

Hence the force necessary to keep the mass m in a circle of radius r' will produce an acceleration 3.385 cos L $^{cm}/_{sec^2}$, while the total force toward o' will produce an acceleration 980 cos L $^{cm}/_{sec^2}$; consequently the ratio of the component of gravity acting toward o' to that portion needed to keep the mass in a circle is the same as at the equator. Hence, if the earth should rotate 17 times faster, the total force F_c' would be needed to produce the necessary acceleration toward o'. The force F_c has been resolved into two components,—F_c' directed toward o', and nm directed toward the equator. This resolution may be made for any latitude in the northern or southern hemisphere. As a consequence of the component nm, the shape of the earth is a spheroid having a polar diameter about 26 miles less than the equatorial.

37. The Law of Gravitation.—In the latter part of the seventeenth century Sir Isaac Newton announced a law known as the *universal law of gravitation.* The law may be stated as follows:

Every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product of the masses and inversely as the square of the distance between them.

This law may be expressed by the formula

$$F \propto \frac{mm'}{r^2} \tag{54}$$

where m and m' represent the masses of two bodies and r is the distance between their centres of mass.

This formula simply states a proportionality. The value of F in any of the adopted units of force cannot be determined until the value of a **constant of gravitation** is first found by experiment and introduced in the formula. The attractive force between two masses, each one gram, the distance between their centres being one centimetre, may be chosen as this constant. It is usually denoted by G, and its value as found by use of a torsion balance (§ 83) is $6.6579(10)^{-8}$ dynes. Consequently we may introduce this factor in (54) and write

$$F = G\frac{mm'}{r^2} \tag{55}$$

and when c.g.s. units are used the value of F is thus found in dynes.

With this knowledge it is possible to calculate the **mass of the earth** as Cavendish first did. Consider the attraction between two masses,—the earth and a unit mass of 1 g. at the surface of the earth. It is known that the force between these two masses is about 980 dynes, for gravity will give to the gram an acceleration of 980 $^{\text{cm}}/_{\text{sec}^2}$. Hence

$$980 = G\frac{mm'}{r^2}$$

where m' is 1 g. and m is the mass of the earth. Then

$$m = \frac{980r^2}{G}$$

Taking the value of r as $6.4(10)^8$ cm. and G as $6.6579(10)^{-8}$ dynes,

$$m = 6.0275(10)^{27} \text{ grams}$$
$$= 6.64(10)^{21} \text{ tons (English)}$$

Knowing the mass and volume of the earth, the mass per unit volume—density—may easily be calculated. Thus, it can be shown that the mean **density of the earth** is about 5.527 $^{\text{g}}/_{\text{cc}}$, —*i.e.*, the density of the earth as a whole is more than 5.5 times greater than that of water. Most substances on the surface of the earth are much lighter than this; consequently much of the matter in the interior must be much denser.

The law of universal gravitation applies not only to the earth and bodies on its surface, but to all bodies in the universe. The mutual influence of the sun and planets causes the latter to move in their orbital paths. The satellites, the stars, and the sun itself move in paths which are determined by this universal law.

Newton showed, for example, that the law of gravitation satisfactorily accounted for the **motion of the moon** in its orbit. Let the radius of the moon's orbit be r_o and the acceleration toward the centre a. Then

$$a=\frac{v^2}{r_o}=\frac{4\pi^2 r_o}{P^2} \qquad \text{(see equations 22 and 24)}$$

and the value of a at a distance r_o from the earth has been shown to be about .272 $^{cm}/_{sec^2}$. (See prob. 1, page 25.)

Let this result be compared with that obtained by the universal law expressed in equation (55), where the force varies inversely as the square of the distance. Since a force varies directly as the acceleration which it produces, the acceleration varies inversely as the square of the distance; hence

$$\frac{a}{g}=\frac{r^2}{r_o^2} \tag{56}$$

or

$$a=\frac{gr^2}{r_o^2} \tag{57}$$

where r = radius of the earth, r_o = radius of moon's orbit, g = acceleration at surface of earth, and a = acceleration at distance of moon. Taking $r_o=3.844(10)^{10}$ cm. and $r=6.4(10)^8$ cm.,

$$a=\frac{980\times 6.4(10)^8}{3.844(10)^{10}}=.270\ ^{cm}/_{sec^2}.$$

Considering probable inaccuracy in data, this result agrees very well with that given above.

38. Gravity.—The attractive force of the earth for bodies on its surface is called *gravity*. The force of gravity is the result of the universal law of gravitation as applied to the relation of the mass of the earth and adjacent masses.

The force of gravity, like other forces, is measured by the acceleration it produces. This is about 980 $^{cm}/_{sec^2}$, or 32.2 $^{ft}/_{sec^2}$. The force is not the same at all points on the earth's surface, being greatest at the poles and least at the equator, the standard of reference being 980.6 $^{cm}/_{sec^2}$, which is the value of g at sea level in latitude 45°.

Force of gravity, and consequently acceleration, varies inversely as the square of the distance from the centre of the earth. At twice the distance the acceleration is one-fourth as

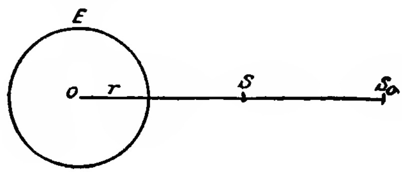


FIG. 64.

great; at three times the distance, one-ninth as great, and so on. In Fig. 64 let E represent the earth, r its radius, and s a distance from o greater than r. Let g be the acceleration on the surface and a the acceleration at the distance s. Then

$$\frac{g}{a}=\frac{s^2}{r^2} \tag{58}$$

or

$$a=\frac{gr^2}{s^2} \tag{59}$$

Thus, when r and s are known, the acceleration at any point above the surface of the earth may be calculated. The value of a is evidently not uniform, but increases each instant as s decreases, or *vice versa*.

A body falling from s_o to s would acquire a velocity which may be calculated from the equation

$$v^2=2gr^2\left(\frac{1}{s}-\frac{1}{s_o}\right) \tag{60}$$

where s and s_o represent distances from the centre of the earth. Suppose, for example, it is desired to know what velocity a body would acquire if it fell from an infinite distance to the surface of the earth. Here

$$\frac{1}{s_0}=\frac{1}{\infty}=0$$

$$\frac{1}{s}=\frac{1}{r}$$

$$\therefore\ v^2=2gr$$

$$g=32.2\ {}^{\text{feet}}/_{\text{sec}^2}=.00609\ {}^{\text{mile}}/_{\text{sec}^2}$$

$$r=4000\text{ miles}$$

$$\therefore\ v=6.98\ {}^{\text{miles}}/_{\text{sec}}$$

For a distance of a few hundred feet above the surface of the earth the change in acceleration is very slight. Even at a height of five miles the difference in acceleration as compared with that at the surface is only about 3 ${}^{\text{cm}}/_{\text{sec}^2}$. Hence no serious error will be introduced if falling bodies near the surface be regarded as acted upon by a constant force,—*i.e.*, their motion may be considered as uniformly accelerated. With this assumption the equations for falling bodies are the same as those deduced for U.A.M. Let g take the place of a and we may write

$$v=gt \tag{61}$$

$$s=\tfrac{1}{2}gt^2 \tag{62}$$

$$v^2=2gs \tag{63}$$

and so on for all equations that are true for that kind of motion.

39. Equilibrium in Orbital Motion.—If a planet, such as the earth, were moving in its orbit about the sun with uniform circular motion, the force necessary to keep it in a circular path is

$$F_c=\frac{mv^2}{r}$$

It has just been shown that the force which keeps a planet in its orbit is

$$F_g=G\frac{mm'}{r^2}$$

which shows that, while F_c varies inversely as r, F_g varies inversely as r^2. Under this condition, any change in the value of r, however little, would change F_g much more than F_c. If r should increase, the planet would leave its orbit and never return; if r should decrease, the planet would be drawn to the sun, for F_g would be greater than the force necessary to hold it in the circle. Such an orbital path is plainly an unstable one.

The actual orbit of a planet is an ellipse. This is true of any mass whose motion is periodic and in a curved path about a force which varies inversely as the square of the distance. In Fig. 65 let the sun S be at one of the foci of an ellipse, and let a planet move from aphelion A to perihelion P in direction indicated by arrows. A line drawn from S to any point on the ellipse is called a *radius vector.* During the whole trip from A to P the radius vector makes an acute angle with the direction of the planet's motion; hence a component of the sun's attraction is directed along the path of motion, increasing the velocity until the tendency to continue in a straight line carries the planet on in its orbit notwithstanding its approach to the sun. After the point P is passed, the radius vector makes an obtuse angle with the direction of motion at any point between P and A,—*i.e.*, there is a component of the sun's attraction directed opposite to the path of the motion. The velocity of the planet will thus be retarded and its tendency to continue in a straight line will be lessened to such an extent that the sun is able to deflect it from its course and cause it to round the curve at aphelion. Such a relation of planet and sun may be said to be stable, for only an enormous external force would be able to cause the planet to leave an elliptic orbit and move on in a parabola or hyperbola.

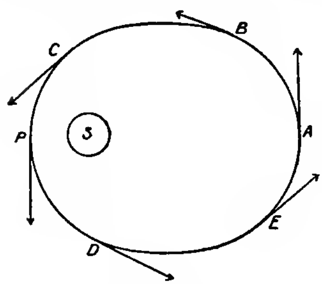


FIG. 65.

40. Gravity beneath the Surface of the Earth.—If the earth were uniform in density, acceleration of gravity at any point beneath the surface would vary directly as the distance from the centre,—*i.e.*, a body 1000 miles beneath the surface would weigh three-fourths as much as at the surface. To understand the reasons for this, let E be a shell composed of the outer crust of the earth. Let m be a mass placed anywhere within this shell. Draw through m two lines making a very small angle with each other and cutting the shell at m' and m''. The triangles thus formed are to be considered the vertical sections of cones with their vertices at m, their bases at m' and m'', and having altitudes r' and r''. The masses of the shell in the bases of the cones may be regarded as proportional to their areas. Let these masses be m' and m''. The areas of the bases of the cones vary directly as the squares of the distances from the vertices. Hence the proportion

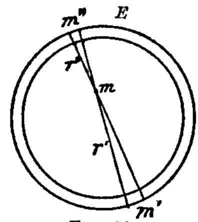


FIG. 66.

$$\frac{m'}{m''}=\frac{r'^2}{r''^2}$$

Let g' and g'' be the acceleration of m due to the masses m' and m'' respectively. Then, by the law of gravitation,

$$\frac{g''}{g'}=\frac{r'^2}{r''^2}$$

A comparison of these proportions shows that

$$\frac{m'}{m''}=\frac{g''}{g'}$$

or $$m'g'=m''g''$$

whence $$F'=F''$$

where F' and F'' are the forces measured by $m'g'$ and $m''g''$ respectively. Since F' and F'' are equal and oppositely directed, the mass m will have no acceleration and no weight, as far as the shell is concerned, for m is at any point within the shell. There will be a stress, but no motion. Suppose a mass M, Fig. 67, is located at a distance oM from the centre of the earth. It will be unaffected by the mass of a shell whose thickness is MA,—*i.e.*, as far as motion or weight is concerned. But it will be attracted by an unbalanced force due to the mass of a sphere of radius oM. Since the volumes of spheres and consequently their masses, if uniform, vary directly as the cubes of their radii, the force of attraction due to the mass

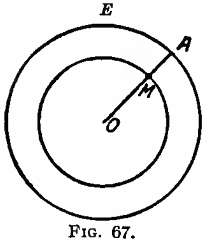


FIG. 67.

of the central sphere varies as $\overline{oM}^3$ At the same time the force varies inversely as the square of the distance from the centre; hence

$$F \propto = \frac{\overline{oM}^3}{\overline{oM}^2} \text{ or } oM \tag{64}$$

It has already been shown that the density of the earth is much greater at the interior. Consequently, as a body moves from the surface toward the centre its weight will for a time increase, because a position nearer the dense interior more than compensates for the decrease of mass due to the fact that the shell is not effective in causing weight. At a point still nearer the centre the weight will again be the same as on the surface, and from that point will decrease until at the centre the body will be subject to balanced forces and so will have no weight.

41. Weight.—The weight of a body is the force of gravity exerted upon it. It is the force which must be exerted to hold a body free from other support, or the resistance which a support must offer to equal the force of gravity. In other words, weight is the resultant of all the forces of gravity which act upon the material particles of which a body is composed. Weight is not inherent in a mass of matter, but is dependent on surrounding conditions. A mass located alone in space would have no weight. If surrounded by other masses so that attractions are equal in all directions, again there would be no weight.

Weight and **mass** should be clearly distinguished. Mass is the amount of matter in a body. The body may or may not have weight, but, no matter what changes in surrounding conditions may be made, the mass does not change.

The weight of a body is found by measuring the resultant force of gravity. One method of doing this is to suspend the body from the end of a coiled spring and note the distance the spring is stretched. **Hooke's law** in regard to elastic bodies is that, within the limits of elasticity, the force of restitution is proportional to the displacement,—*i.e.*, the effort of the spring to regain its original shape is proportional to the strain. The strain, then, may be taken as a measure of the force which causes it. A familiar application of this principle is found in the spring scales. Weight may also be determined by balancing an object against certain standard masses, in accordance with the principles of moments of force. Knowing the force of gravity on the standard, the weight of the object may be readily found.

42. Centre of Gravity.—The centre of gravity of a body may be defined as that point through which the resultant of all the forces of gravity exerted upon the particles of which a body is composed must pass, no matter what position a body may have. In other words, it is that point about which the sum of the moments of force due to the action of gravity upon the particles of the body is zero. Another form of statement is that the centre of gravity is a point at which the total weight of a body may be supposed to be concentrated, so that the resultant of the forces of gravity acting through this point may produce motion of translation but not of rotation.

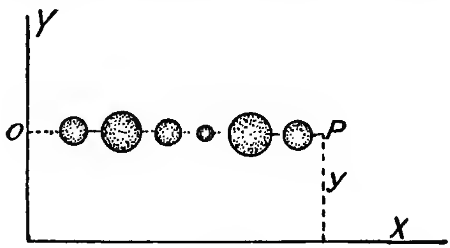


Fig. 68.

The position of the centre of gravity may be found by considering the relation of the particles of which a body is composed to certain axes or planes of reference. In Fig. 68 let a number of particles be in a straight line oP, parallel to the X-axis and distant y from it. Let the number of particles be n, their masses m_1, m_2, m_3, and so on to m_n, and their distances from the Y-axis, x_1, x_2, x_3, ... x_n, respectively. Then the moment of each particle about o is found by multiplying its weight by its distance from o. The sum of all these moments divided by the sum of the weights of all the masses—magnitude of the resultant—gives the distance from o to the point where the resultant must be applied to produce the same moment. This may be expressed by

$$\frac{m_1x_1 + m_2x_2 + m_3x_3 \; . \; . \; . \; . + m_nx_n}{m_1 + m_2 + m_3 \; . \; . \; . \; . + m_n} = \bar{x}$$

where $\bar{x}$ is the distance from o to the point of application of the resultant. The common expression for this is

$$\frac{\Sigma mx}{\Sigma m} = \bar{x} \tag{65}$$

which may be read, the sum of the moments of all the particles of which a body is composed divided by the weight of the body equals the location of the centre of gravity in reference to that axis about which the moments are taken.

In the same way the distance of the centre of gravity from the X-axis is found by

$$\frac{\Sigma my}{\Sigma m} = \bar{y}$$

In the particular case illustrated in Fig. 68, the value of y is the same for all the particles; hence $\bar{y} = y$.

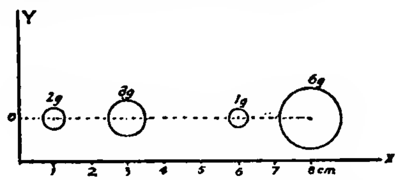


FIG. 69.

To illustrate this principle by use of a simple problem, let four bodies whose weights are 2 g., 3 g., 1 g., and 6 g. be placed in a straight horizontal line, as shown in Fig. 69. Then

$$\frac{\Sigma mx}{\Sigma m} = \frac{2\times1+3\times3+1\times6+6\times8}{2+3+1+6} = 5\tfrac{5}{12} \text{ cm. from } o$$

The point of reference o may be taken at any distance from the system of bodies; as here taken, the centre of gravity of the system is $5\frac{5}{12}$ cm. from o, or $4\frac{5}{12}$ cm. from the mass of 2 g.

When the particles are not in a straight horizontal line, but in a vertical plane, as represented in Fig. 70, the distance of the centre of gravity from both the X and Y-axis must be found. Thus $\frac{\Sigma mx}{\Sigma m}$ will locate the centre of gravity somewhere in the line a parallel to the Y-axis, and $\frac{\Sigma my}{\Sigma m}$ will locate it in line b parallel to the X-axis. It is therefore at their point of intersection.

In case the particles are so arranged that three dimensions must be considered, the centre of gravity is located by finding its distance from three planes of reference, XZ, XY, and YZ, Fig. 71. Thus $\frac{\Sigma mx}{\Sigma m}$ will determine its distance from the plane YZ; $\frac{\Sigma my}{\Sigma m}$, its distance from XZ; and $\frac{\Sigma mz}{\Sigma m}$, its distance from XY. The centre of gravity is therefore at the point of intersection of the three planes.

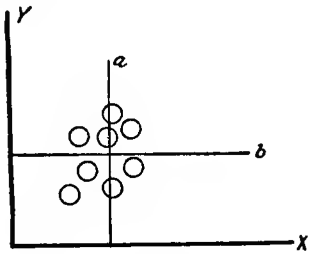


FIG. 70.

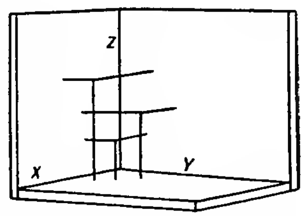


FIG. 71.

43. Centre of Mass.—The centre of gravity and *centre of mass* coincide, but their definitions are different. The centre of mass is a point whose distance from the three planes of reference is equal to the mean distance of the particles, supposed equal, from the same planes. Centre of mass may be found by the same methods as have just been described for centre of gravity. When a body is regular in shape, as a sphere or cube, and is uniform in density, the centre of mass or centre of gravity is at the centre of figure.

44. Stable Equilibrium.—In reference to the action of gravity on masses of matter, a body is said to be in *stable equilibrium* when it is so situated in reference to any axis that a moment of force tending to produce rotation on that axis causes a rise in the centre of gravity. Let A, Fig. 72, be a body suspended from s, the centre of gravity being at c, in a line drawn from s toward the centre of the earth. This line is the direction of the resultant of all the forces of gravity acting on the body, and consequently there is here no moment tending to cause rotation on the axis s. Let the body be now changed to the position B. There will then be a moment equal to the product of the resultant by the distance *sa*, tending to restore the body to its position at *A*. Any disturbance of the body that will cause rotation about *s* will cause a rise in the centre of gravity. Consequently the position at *A* is said to be stable.

If a body is placed as in Fig. 73, where the centre of gravity is directly above the support, the condition is one of **unstable equilibrium,** for the least disturbance will result in a moment that will bring *c* below *s*, and the condition will then be stable again.

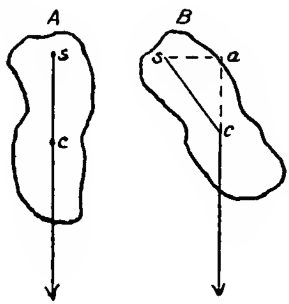


Fig. 72.

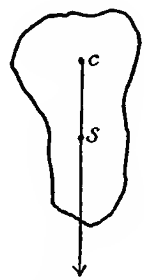


Fig. 73.

When the axis of rotation passes through the centre of gravity, the body is said to be in **neutral equilibrium,** for rotation can neither raise nor lower the centre of gravity.

In general, whether gravity is concerned or not, a system is said to be stable when it tends to restore its configuration after certain disturbing forces have been removed.

45. Determination of Mass.—One method of determining the quantity of matter or mass of a body is to apply to it a certain force and note the acceleration produced. Then, from the relation $F=ma$, m can be found. Thus, if a force of 500 dynes produces an acceleration of 5 $^{cm}/_{sec^2}$, the mass must be 100 g. This method has the advantage of being independent of the force of gravity, but it is impractical in ordinary operations.

The practical method is to determine the force of gravity, or weight, and consider the mass as proportional to the weight. That this proportionality exists has been shown experimentally by dropping masses of different size and kind from a height and noting that they reach the ground in the same time. To produce the same acceleration in a large mass as in a small one, the force must be proportional to the mass, for $F = ma$. Newton performed an experiment in which he used a pendulum with a hollow bob. By filling the bob at different times with various kinds of matter and counting the vibrations through a considerable length of time, he was not able to detect any difference in period. From this he concluded that the force of gravity is directly proportional to the mass and independent of the kind of matter.

If the force of gravity exerted on 1 g. of matter is g,—980 dynes,—then the force on m grams is mg, which is the weight. At any point on the earth g may be considered a constant quantity; hence any change in the weight, mg, must be a change in mass.

In instruments commonly used for determination of mass, two principles are employed. First, the principle stated in Hooke's law, where the amount of displacement caused by a mass suspended from the end of an elastic coil or rod is taken as a measure of the mass. All spring scales are based on this principle. Second, that there is equilibrium when the sum of the moments tending to produce positive and negative rotation about an axis is zero, and that, when the ratio of the distances from the axis to the line of direction of the forces is known, the ratio of the weights will be inversely as these distances. Thus, in Fig. 74, let a and b be distances from the axis o to the line of direction of the weights w and w' respectively. When the system is balanced,

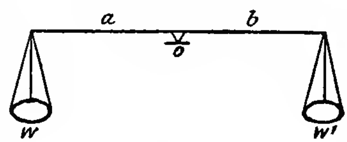


Fig. 74.

$$wa = w'b \tag{66}$$

or

$$\frac{w}{w'} = \frac{b}{a}$$

When the ratio $\frac{b}{a}$ is known, that of $\frac{w}{w'}$ is also known, and when the mass in one scale pan is known, as is usually the case, the value of the unknown mass is easily determined. If a and b are now made equal in length, w will equal w' whenever there is equilibrium. Balances for accurate weighing are constructed in this manner.

In Fig. 75 is a diagram of one style of beam used in physical and chemical balances. Its shape permits considerable stress with very little strain. At the centre is a knife-edge o made of agate and resting on a plate of agate, P. The knife may be two or three centimetres long and is here shown only in cross section. Near the end of the beam are other agate knife-edges o' and o''. Upon these rest agate plates from which scale pans are suspended. A long pointer is rigidly fastened to the beam and sweeps over the scale s when there is any rotation on o. The beam must be so constructed that it will be *stable*,—*i.e.*, it will when disturbed rotate from side to side on o and finally come to rest in its original position. The distances oo' and oo'' must be equal, to make the balance *true*. The *same results* must be obtained when equal masses are weighed at *different times*. A small difference of mass in the pans should cause a movement of the pointer,—*i.e.*, the balance should be *sensitive*. The movement of the pointer should be sufficiently *rapid*, so that too much time may not be consumed in the operation of weighing.

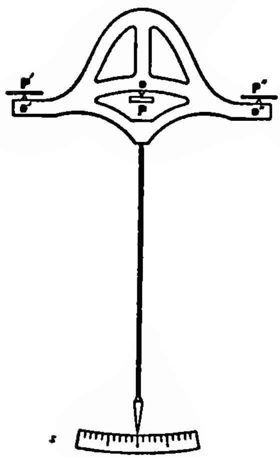


FIG. 75.

Stability is secured by constructing the beam in such a manner that its centre of gravity is a short distance below o. In the diagram, Fig. 76, let the centre of gravity be at c, directly below o when the beam is in a horizontal position. If the beam is turned through an angle θ, c will move to c', the line oc also turning through an angle θ. Let W_b be the weight of the beam, then the moment tending to restore it to a horizontal position is

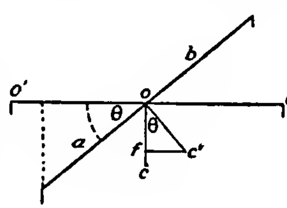


FIG. 76.

$$W_b \cdot \overline{fc'} = W_b \cdot \overline{oc'} \sin \theta$$

Let a and b be the arms of the beam, and let weights be placed in the pans such that the one suspended from o' is greater than that from o'' by a difference d. Then the moment tending to produce positive rotation is

$$d \cdot a \cos\theta$$

while that tending to produce negative rotation is

$$W_b \cdot \overline{oc'} \sin\theta = W_b \cdot x \sin\theta$$

by letting x stand for $\overline{oc}$ or $\overline{oc'}$. Hence

$$d \cdot a \cos\theta = W_b \cdot x \sin\theta \qquad (67)$$

or

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{a}{W_b x} d \qquad (68)$$

From this equation it appears that to make the balance very sensitive,—*i.e.*, to construct the beam so that a small difference d may produce as great a value for θ as possible,—a should be made long, and W_b and x as small as possible. If a, however, is made very long, the time of the swing becomes very long. If W_b or x are decreased, the moment tending to restore the beam to a horizontal position is decreased and again the time of the swing is increased. These conflicting results are all considered, and a balance is constructed best suited to the purpose for which it is intended.

Mounted at the centre of the beam is a nut which may be raised or lowered, thus changing the position of the centre of gravity. In this way different degrees of sensibility may be obtained at the will of the operator.

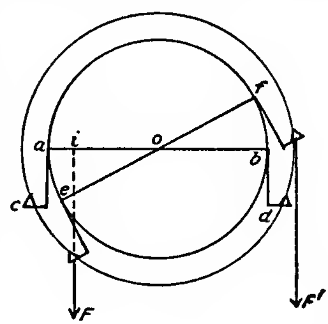


Fig. 77.

That there may be the same sensibility when the differences in the weights on the pans are not the same, it is necessary that the three knife-edges be in the same plane. In Fig. 77 let *cabd* be a rigid beam with the pans hung from c and d. Let the beam be turned to the position *ef*. It is observed that the moment of the force F has decreased, for *oa* has decreased to *oi*, but the moment due to F' has, in the position shown, actually increased. Hence the moment tending to restore the beam to a horizontal position will be increased, and therefore the pointer will not move over a number of divisions of the scale proportioned to the difference in load, as it will do when the knife-edges are in the same plane.

To obtain the true weight, the length of the arms of the beam must be equal if they are assumed to be so. If they are not

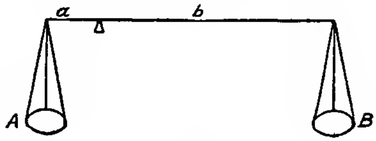


FIG. 78.

equal, the true weight may be found by double weighing. Let w' be the apparent weight as indicated by standard weights placed in the pan A, Fig. 78, while the body whose true weight is w is in B. Then

$$w'a = wb \tag{69}$$

Now let w be placed in A and standard weights w'' in B; then

$$w''b = wa \tag{70}$$

The product of these two equations is

$$w'w''ab = w^2ab \tag{71}$$

hence
$$w = \sqrt{w'w''} \tag{72}$$

The true weight is therefore equal to the square root of the product of the apparent weights. In very accurate weighing this method is often used even when the lengths of the arms are supposed to be equal.

Another method for the elimination of error due to difference in arm length is to place the object whose mass is desired in one pan and counterbalance it with any convenient mass in the other. Then remove the object and put in its place standard weights until equilibrium is again restored. The mass of the standard weights and that of the object must be the same.

Problems.

1. Calculate the moment of inertia I of a cylinder rotating on its own axis, the radius being 10 cm. and the weight 3 kg.

2. Find I of a wheel, radius 1 m., when a force of 500 dynes applied to the rim causes an acceleration of π radians per sec^2.

3. A mass of 500 g. is made to revolve 120 times per minute in a circle of radius 50 cm. Find the centripetal force.

4. What must be the speed of a conical swing that a car suspended from the top of the central pole by a rod 50 feet in length may incline at an angle of 45°?

$$\left(F_c = mg \tan\theta \qquad F_c = ma \qquad a = \frac{v^2}{r}\right)$$

5. What velocity will a body acquire in falling from a point 1000 miles above the surface of the earth to the surface, on the supposition that the body will encounter no resistance?

6. Show that, if three equal masses are placed at the corners of an equilateral triangle, their centre of gravity will be at a point one-third of the distance from the middle of the base to the opposite vertex.

7. A weight of 500 lbs. is suspended from a rope and allowed to descend from the top of a building with an acceleration of 10 $^{\text{feet}}/_{\text{sec}^2}$. What is the tension of the rope?

8. A uniform beam weighing 100 lbs. is at rest in a horizontal position on a fulcrum placed one-fourth of its length from one end. A weight of 500 lbs. hangs from the end nearest the fulcrum. What is the weight on the other end.

1. 150,000 g. cm^2.
2. 15,915.45 g. cm^2.
3. 4028.4 g.
4. 33.7 $^{\text{feet}}/_{\text{sec}}$.
5. 3.12 $^{\text{miles}}/_{\text{sec}}$.
6. ——.
7. 344.7 pounds.
8. 133.33 pounds.

46. Work and Energy. — When force applied to a body produces motion in the direction in which the force acts, work is done. The magnitude of the force times the distance through which it is applied is a measure of the work done. This is expressed by

$$W = Fs$$

where W is the work, F the force, and s the space or distance.

No work is done in the application of force unless motion results. The figure of a giant cut from stone and placed as one of the supports of a building does not do any work, however great the weight upon his shoulders.

It is not necessary that the body as a whole should be moved that work may be done. If the body is elastic, the force may compress, elongate, or twist it, and thus the point of application of the force may move through a certain distance. When this force is removed, the body may in turn do work on other bodies. By virtue of work done upon them bodies are able to do work, and are then said to possess energy. Energy is the capacity for doing work.

Work in general may be defined as the process of transferring energy from one body to another. It requires work to lift a stone to the top of a building, but the energy expended is then in the stone by virtue of its position relative to the ground. Work must be done in setting a mass of matter in motion, and the mass then contains energy by virtue of its motion. It requires work to pass a current of electricity through a storage cell, and the cell then contains the energy expended, by virtue of chemical changes which have been made.

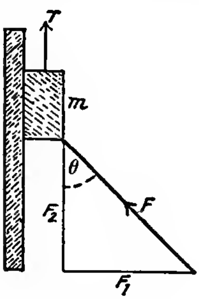


FIG. 79.

While energy exists as a distinct physical quantity, it is here assumed to have no independent existence. Wherever it is found, it is associated with matter, matter here being taken in a sense to include ether.

Whatever difference of opinion may exist in regard to the essential character of matter and energy, yet for purposes of physical investigation no false conclusions will be reached by assuming that matter has an objective existence and that energy is a relative condition of matter by virtue of which work may be done.

The chief function of matter, then, is to serve as the vehicle of energy. The mill owner cares nothing for the water in the mill-pond as a body of water, but he highly prizes the fact that the water is on a higher level than the mill-wheel. The engineer cares nothing for steam as a vaporized mass of matter, but he makes use of the rapidly moving molecules to drive the piston of his engine. The value of foods consists not so much in the material of which they are composed as in the work which has been done upon them by sunlight.

Matter in this respect may be defined as anything capable of possessing energy as a result of work done upon it.

The two factors which determine the **quantity of work** are the *force* applied and the *distance* through which it is applied.

The product of these factors is a measure of the quantity of work done. In case the direction of the force is at an angle to the direction of the motion, only that component of the force which is in line with the motion is considered. Let F, Fig. 79, be applied to raise a mass m. Let F make an angle θ with the direction of the motion. Resolve F into two components, F_1 and F_2. The component F_1 causes only pressure against the support while F_2 is effective in lifting the mass. Hence the work done is

$$W = F_2 s = F \cos \theta \, . \, s \tag{73}$$

where s is the vertical distance through which the mass is raised. The greater part of the energy expended in raising this mass against the force of gravity is now possessed by the mass. A certain quantity of energy may be said to have been transferred to it, and it as a consequence is capable of doing work.

47. Units of Work and Energy.—As in the case of force, so for work and energy there are both *absolute* and *gravitational* units of measurements.

The absolute units are the *erg* and the *foot-poundal.*

The **erg** is the work done by a force of one dyne acting through a distance of one centimetre.

The **foot-poundal** is the work done by a force of one poundal acting through a distance of one foot.

These units are absolute because the factors which enter into their definitions are invariable in time and place. They are not dependent on a variable force such as gravity.

Since work is the product of a force by a distance, the dimensions of its units are

$$[MLT^{-2}]\,[L] = [ML^2T^{-2}]$$

The symbol of the c. g. s. unit, then, is $\frac{1 \text{ g } 1 \text{ cm}^2}{1 \text{ sec}^2}$,—*i.e.*, the erg is the product of 1 cm. by a force which will give to 1 g. an acceleration of 1 $^{\text{cm}}/_{\text{sec}^2}$. The symbol for the foot-poundal is $\frac{1 \text{ lb } 1 \text{ ft}^2}{1 \text{ sec}^2}$.

The relation of these two units may be found in the usual manner by finding the numeric for dimensions in c. g. s. units when that for foot-poundals is unity. Thus,

$$1\left[\frac{ML^2}{T^2}\right]=x\left[\frac{M_1L^2_1}{T_1^2}\right]$$

hence $$x=\left[\frac{T_1^2}{T^2}\cdot\frac{M}{M_1}\cdot\frac{L^2}{L^2_1}\right]=1\times 453.59\times 30.48^2=4.214(10)^5$$

hence there are $4.214(10)^5$ ergs in a foot-poundal.

The erg is inconveniently small, being only about the amount of work required to raise one milligram to a height of one centimetre. For this reason, ten million (10^7) ergs are employed as a practical unit called the **joule.**

The gravitational units of work are the *foot-pound*, the *gram-centimetre*, the *kilogram-metre*, or any convenient product of a gravitational force by a distance.

A **foot=pound** is the work done by a force of one pound exerted through a distance of one foot. The gram-centimetre is defined in a similar manner.

In the gravitational units the force is that of gravity exerted on a pound or gram of matter. Since gravity is not the same at all points on the earth, these units will differ with location. Gravitational units are in common use by engineers, for in most structural work gravity is the chief agent against which force must be exerted, and for rough work the slight variations in the force of gravity in different localities may be neglected. It is possible, however, to select a certain locality, say latitude 45° at sea level, and consider the force of gravity per unit mass at that point as the standard. Then, if a mass m at that place is raised a distance h, the work done is mh in foot-pounds or gram-centimetres, depending on the units selected. The standard force of gravity per gram is 980.6 dynes. Then, in another locality where g is, say, 978 dynes, the work performed by the same operation is

$$W=mh\frac{978}{980.6}\text{ ft.-lbs. or g.-cm.}$$

Since weight is the product of mass, m, by the force of gravity, g, per unit mass, the work W done in raising a mass to a height h is

$$W=mh \text{ gram-centimetres or foot-pounds}$$

and $$W=mgh \text{ ergs or foot-poundals}$$

From this relation it is easy to pass from the absolute to the gravitational system or *vice versa.* One foot-pound equals approximately $1.355(10)^7$ ergs.

Work may be represented by an area, as shown in Fig. 80. Let divisions on the abscissa represent distance; and those on the ordinate, intensity of force. Suppose each millimetre on the abscissa stands for one centimetre of actual distance, and those on the ordinate represent dynes. Then 30 dynes acting through 30 cm. would do work which is here represented by the rectangle *oabc*,—900 square millimetres, each representing one erg of work. If the force is not uniform but is constantly changing, the work may be represented by the number of unit squares enclosed by the abscissa and the curve *def*. This curved line is the locus of all the points whose distances from the abscissa represent the intensity of the force at each successive small change in the distance. This graphical method of representing work will be used in later discussions.

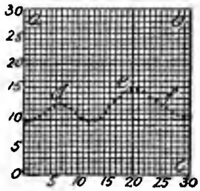


Fig. 80.

48. Power.—Power is the *rate* of doing work. If the quantity of work only is specified, there is no indication of the time within which it was performed. A small force acting for a long time may accomplish as much work as a large force acting for a short time. A small boy may do as much work as a strong man. But when the element of time enters into the consideration, then the time-rate is a measure of the *power*, or, as it is sometimes called, the *activity* of an agent. Steam engines differ in power because they differ in the rate of doing work.

A common unit of power is the **horse-power.** An engine or other agent is said to have one horse-power when it can do 33,000 foot-pounds of work in one minute or 550 ft.-lbs. in one second.

Another unit of power is the **watt,** which is the rate of doing work when one joule is performed in one second. There are approximately 746 watts in one horse-power.

49. Potential and Kinetic Energy.—Energy, as has already been defined, is the capacity for doing work, and work is an operation of transferring energy from one body of matter to another. The amount of energy lost by an agent in the performance of work is a measure of the work done. Hence the units for energy are the same as those for work.

All energy is potential in the sense that it is a capacity for doing work, but a division is commonly made into *potential* and *kinetic* energy. The energy which a body possesses by virtue of its position in relation to other objects, or by virtue of a strain which has been caused by work done, is called **potential.** Examples of this are particles of carbon in coal in relation to the oxygen in the air, the coiled spring of a watch or clock, a mass raised to a position from which it may fall.

Kinetic energy is the energy which a body has by virtue of its motion. A heavy projectile in flight possesses a great deal of energy, by virtue of the fact that the exploding powder did a great deal of work upon it.

It is desirable to have an equation by which the *kinetic energy* can be calculated when the *mass* and *velocity* of the moving body are known. To do this we assume that any moving body has the energy which was given to it by a force that started it from rest and accelerated the motion until the velocity was as now observed. While the velocity is uniform, no energy is gained or lost, as may be inferred from Newton's first law. The force F which caused the acceleration a in mass m, is

$$F = ma$$

The distance, s, through which the force acted to give the mass the observed velocity, v, is

$$s = \frac{v^2}{2a} \quad \text{(from } v^2 = 2as\text{)}$$

The formula for energy or work is

$$W \text{ or } E = Fs \tag{74}$$

hence

$$E_{\text{kin}} = ma\frac{v^2}{2a} = \tfrac{1}{2}mv^2 \tag{75}$$

From (75) the kinetic energy in ergs or foot-poundals may be found when velocity and mass are known.

50. Energy of a Rotating Body.—When the mass and angular velocity of a rotating body are given, the energy of each particle of the mass is found by (75),—namely, $E_{\text{kin}} = \frac{1}{2}mv^2$. In Fig.

81 let a solid disk be represented as rotating on axis o. Consider the energy of a particle m_1 which is moving with angular velocity ω at distance r from the axis. Since ω is the number of radians per second, ωr is the linear velocity of m_1, hence the kinetic energy of the particle m_1 is

$$E_{\text{kin}} = \tfrac{1}{2}\omega^2 m_1 r^2 \tag{76}$$

The total energy is the sum of the energies of all the particles, hence

$$E_{\text{kin}} = \tfrac{1}{2}\omega^2 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 \ldots + m_n r_n^2)$$

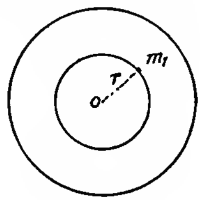


FIG. 81.

The sum of the quantities within the parenthesis is the moment of inertia of the rotating body, hence

$$E_{\text{kin}} = \tfrac{1}{2}I\omega^2 \tag{77}$$

Since $I = Mk^2$ where M is the total mass and k is the radius of gyration,

$$E_{\text{kin}} = \tfrac{1}{2}M\omega^2 k^2 \tag{78}$$

Equation (78) simply states that the kinetic energy of a rotating body is equal to one-half the mass times the square of the linear velocity of that point where the total mass is supposed to be concentrated. If c.g.s. units are used and the value of ω is taken in radians, the result will be in ergs.

51. Conservation of Energy. — The total quantity of the energy in the universe is constant. The operations of life and the activities in nature show a constant change of energy from place to place and from body to body, but what is lost at one point is gained at another, and the total quantity remains unchanged. This is the doctrine of the conservation of energy.

The material changes which are observed in the world or universe are the transferences of energy from body to body, or the result of such transferences. The potential energy, for example, which exists in coal by virtue of the relation between carbon and oxygen may, by burning the coal beneath a boiler, be converted to kinetic energy of countless numbers of particles of water, and these may in turn transmit their energy to an engine, causing mechanical motion. The engine then passes the energy on to various machines whch it operates. The sum of all the work done in this instance, counting the energy lost by friction or otherwise, is exactly equal to the energy which was originally in the coal.

Another illustration may be given in the case of a falling body. Let a mass m be located at a height s. The body is then said to possess potential energy, the quantity of which in gravitational units is ms, and in absolute units mgs. This may be expressed by

$$E_p = mgs \tag{79}$$

where E_p is potential energy. If now the body falls through the distance s, it will acquire a velocity v, and its kinetic energy will be

$$E_{\text{kin}} = \tfrac{1}{2}mv^2$$

But $v^2 = 2gs$, hence

$$E_{\text{kin}} = \tfrac{1}{2}m \,.\, 2gs = mgs = E_p$$

Hence the quantity of energy is the same whether it is kinetic or potential. When the body has fallen part of the distance, the energy is part kinetic and part potential. At one-third the distance from the top, for example, two-thirds of the energy is potential and one-third kinetic.

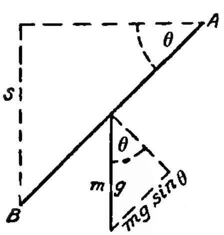


Fig. 82.

The same is true in case the body descends along an inclined path AB, Fig. 82, for the length of the path is $\frac{s}{\sin\theta}$, where s is the vertical height, and the value of F along the incline is $mg \sin\theta$, hence

$$E = \frac{s}{\sin\theta} \quad mg \sin\theta = mgs$$

52. Available Energy.—While it is an established fact of science that all the energy expended in the performance of work is conserved, yet it is not all available for the performance of other work. For example, when a heavy stone is raised by means of a wheel and axle and pulleys, a part of the resistance is due to friction in the various parts of the machinery. Extra work must be done because of friction, and this results in heat, which consists in an increased motion of the molecules. This portion of the energy is thus dissipated throughout the mass of matter and into space. The energy is not now available by man for the performance of other work. It is not lost in the sense that it has been destroyed, but only in the sense that it was not applied in increasing the potential energy of the stone. The energy which is transferred to the stone is available, for it may, by proper connection with a machine, be made to do work while it descends.

Only a small part of the total energy of the world is available for work, for, no matter how great the quantity may be, it is only while energy is being transferred that work can be done. If all water were at the same level, there would be no rivers or water-falls; but when there is a difference of level, the fall is accompanied by a change from potential to kinetic energy, resulting finally in heat at the bottom of the precipice. This energy in a sense may be said to be lost because it is not available for the performance of work. If, however, a wheel of proper construction is placed in the path of the *falling* water, much of the energy may be made to do useful work. It is only while the water is falling that its energy is available.

In a similar manner, if all bodies were of the same temperature, none of the great store of molecular energy would be available. But when the temperature of one body is higher than that of another, then, in the process of transference of heat, energy becomes available and work may be done. The steam engine, as will be shown later, is a device for utilizing the molecular energy of steam while heat is being transferred from a hot to a cold body.

Numerous examples of this kind may be given, showing that the constant tendency is to diminish the quantity of potential energy and produce a condition of uniformity under which no energy would be available for the performance of work.

The sun is the chief source of energy on the earth. To it alone we are indebted for that store of potential energy which makes life possible.

Problems.

1. A ladder 30 feet long stands at an inclination of 30° to a vertical wall. How much work will be done in carrying a mass of 50 pounds to the top of the ladder?

2. A force of 5000 dynes is applied at the end of a rope to drag a mass along a horizontal surface. The rope is inclined 35° to the surface. How much work is done in moving the mass a distance of two metres?

3. If one joule of energy is expended in lifting 1 kg., to what height will the mass be raised? (g = 980).

4. What is the horse-power of an engine capable of lifting 2 tons of brick to the top of a 50-foot building in 5 minutes?

5. A mass of 200 g. is thrown vertically downward with a velocity of 50 $^{cm}/_{sec}$ at a point where the acceleration due to gravity is 980.5 $^{cm}/_{sec^2}$. If its fall is not obstructed for 10 sec. what is its energy at the end of that time?

6. A cylinder whose mass is 2 kg. and radius 3 cm. rotates on its own axis ten times per second. What energy does it possess?

1. 1299 ft.-lbs.
2. 819,150 ergs.
3. 10.2 cm.
4. 1.21 h.-p.
5. $9.712(10)^9$ ergs.
6. $1.776(10)^7$ ergs.

53. The Simple Pendulum.—A simple pendulum consists of a particle suspended from a point and capable of oscillation under the influence of gravity. The rod or chord connecting the particle to the point from which it is suspended is supposed to be without weight. Gravity exerts no influence upon any part of the pendulum except the suspended particle.

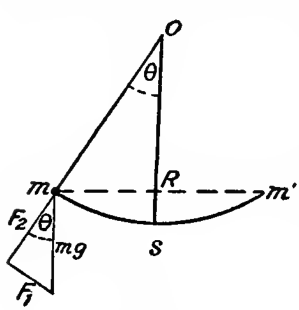


FIG. 83.

Such a pendulum cannot be realized in actual construction, but a near approach to it can be made by suspending a body, whose mass is supposed to be concentrated at a point, by a fine thread or wire of negligible mass.

Let a mass m, Fig. 83, be suspended from o, and let its position at any instant during oscillation make an angle θ with its position of rest at os. The force of gravity acting on m is mg, directed vertically downward. This may be resolved into two components, F_1 and F_2. The latter causes only a tension of the thread and has no effect in moving m along the arc ms. The other component, F_1, acts in a direction tangent to the arc at the point m, and thus is the part of the force of gravity that causes the pendulum to swing. This force is called the *force of restitution*, because by its action the pendulum is given an acceleration, either positive or negative, which restores it to the position of rest.

When the pendulum swings to m or m_1, the mass m is raised a distance sR, which we will call h in this discussion, and its potential energy is then mgh. When it falls to the lowest position, s, the mass m is moving in a horizontal direction with a velocity v. The energy is then all kinetic and is expressed by $\frac{1}{2}mv^2$. At the extreme limit of the oscillation the energy is all potential. At the lowest point it is all kinetic. At intermediate points it is partly potential and partly kinetic. The total quantity of energy is the same at all points of the swing. Hence

$$mgh = \tfrac{1}{2}mv^2$$

or

$$v^2 = 2gh \tag{80}$$

Equation (80) shows that the velocity of the horizontal motion at s is the same as the vertical velocity of a mass falling from R to s.

The force of restitution is expressed in terms of force of gravity and displacement by

$$F_1 = mg \sin\theta \tag{81}$$

Now, in S. H. M. the force of restitution as well as the resulting acceleration must be proportional to the displacement. Here the force varies as $\sin\theta$ and not as θ. Hence the motion of the pendulum is not a S. H. M. The displacement is the distance from s to m measured along the arc,—*i.e.*, θ measured in radians,—and, according to equation (81), F_1 varies as the sine of this angle.

If, however, the amplitude is small, so that θ is not greater than 2° or 3°, θ and $\sin\theta$ will differ so little that one may be used for the other without appreciable error.

The period of a simple pendulum may be determined in terms of its length and the value of g in the following manner: The force of restitution, F_1, Fig. 84, is

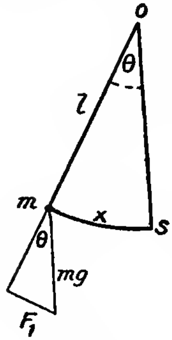


FIG. 84.

$$F_1 = mg \sin \theta$$

It has been shown in equation (40) that the acceleration in S. H. M. is

$$\frac{4\pi^2 x}{P^2}$$

where x is the displacement of the particle. Hence the force of restitution is

$$\frac{4\pi^2 mx}{P^2}.$$

In Fig. 84, x is the displacement as measured along the arc sm. Since θ is measured in radians,

$$x = l\theta$$

and we may write for the force of restitution

$$F_1 = \frac{4\pi^2 ml\theta}{P^2}$$

Substituting this value in (81),

$$\frac{4\pi^2 ml\theta}{P^2} = mg \sin \theta \qquad (82)$$

Since the arc is assumed to be so small that θ and $\sin \theta$ will not sensibly differ,

$$\frac{4\pi^2 l}{P^2} = g$$

whence
$$P = 2\pi\sqrt{\frac{l}{g}} \qquad (83)$$

54. The Physical Pendulum. — A *physical* or *compound* pendulum is one in which the mass is distributed over the entire body of the pendulum, as a bar of wood or metal suspended from o, Fig. 85. It is evident that, as this bar oscillates, the particles near o would, as simple pendulums, move faster and those near s more slowly than they do when all are rigidly connected. There must then be some point between o and s where a particle oscillates naturally,—*i.e.*, as it would if it were the particle of a simple pendulum. The length of the compound pendulum is the distance from this point to the point of suspension, or, in other words, it is the length of a simple pendulum which has the same period of oscillation.

Fig. 85.

The compound pendulum is the only kind that can be realized in construction, and hence the only kind that is actually used. Any body suspended so that its centre of gravity is below the point of support is a compound pendulum.

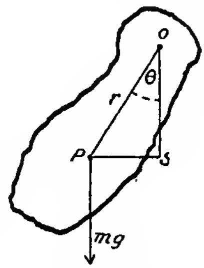


Fig. 86.

The problem, then, is to find the length of a simple pendulum that will vibrate in the same time. Let a body be suspended from o, Fig. 86, and let its centre of gravity be at P. The force of gravity will be mg. The moment of force causing rotation about o is

$$mg \cdot \overline{Ps}$$

Let r be the distance from the point of suspension to the centre of gravity, then

$$\overline{Ps} = r \sin \theta$$

and $$mg \ \overline{Ps} = mgr \sin \theta$$

this is the restoring moment when the angle is θ.

In the discussion of moment of inertia it was shown that

$$Fr = IA \qquad \text{(see equation 45)}$$

—*i.e.*, the moment of force (Fr) tending to produce rotation is equal to the product of moment of inertia I and angular acceleration A. Hence

$$mgr \sin \theta = IA \tag{84}$$

or $$\frac{I}{mr} = \frac{g \sin \theta}{A} \tag{85}$$

Suppose the whole mass to be concentrated at the point which vibrates naturally, and let the distance of this point from o be represented by l. This is the length of a simple pendulum of the same period. The moment of inertia of this mass at the distance l from the axis is ml^2, and r has, under this assumption, become l. Hence

$$\frac{ml^2}{ml}=l=\frac{g\sin\theta}{A} \tag{86}$$

hence $$l=\frac{I}{mr} \tag{87}$$

Substituting this value of l in equation (83),

$$P=2\pi\sqrt{\frac{I}{mgr}} \tag{88}$$

or, since $\frac{I}{m}$ is the square of the radius of gyration, we may write

$$P=2\pi\sqrt{\frac{k^2}{gr}} \tag{89}$$

From this it appears that, if the square of the radius of gyration is divided by the distance from the point of suspension to the centre of gravity, the quotient will be the length of a simple pendulum of the same period.

The equation for the pendulum shows that the period of vibration is independent of the mass and, within certain limits, of the amplitude, but varies directly as $\sqrt{l}$ and inversely as $\sqrt{g}$.

The pendulum furnishes a very accurate means of determining the value of g at any locality. By a change in the form of (83)

$$g=\frac{4\pi^2 l}{P^2} \tag{90}$$

hence, if the values of l and P are accurately determined, g can readily be calculated.

The point where the whole mass of the pendulum may be supposed to be concentrated without change in the period—*i.e.*, the point which vibrates naturally—is called the centre of *oscillation*, as c in Fig. 85. This point is also called the centre of *percussion*, because, if an impulse is applied here in a direction at right angles to the line from o to c, the axis of suspension will not be strained as it would be if struck above or below that point.

55. Reversibility of Compound Pendulum.—If a compound pendulum is reversed and suspended from its centre of oscillation, the period of vibration will not be changed. To prove this it will be shown that the length of the equivalent simple pendulum is not changed by the reversal. Let *os* be a rod suspended from *o*, its centre of gravity being at *G* and centre of oscillation at *c*. Let *r* be the distance from *o* to *G* and let l_1 be the length of an equivalent simple pendulum. It has been shown that the length of an equivalent simple pendulum is $\frac{k^2}{r}$ (see equation 89). It has also been shown that the square of the radius of gyration about any axis is equal to the square of the radius of gyration about a parallel axis through the centre of gravity increased by the square of the distance between the axes (see equation 48). Hence

$$l_1 = \frac{k^2 + r^2}{r} \tag{91}$$

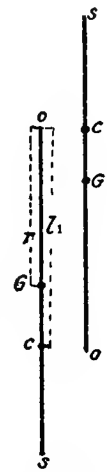


FIG. 87.

or
$$k^2 = r(l_1 - r) \tag{92}$$

where k^2 is the square of the radius of gyration about *G*. Now let the pendulum be reversed, and suspended from *c*. The distance from *c* to *G* is $l_1 - r$. Let l_2 be the length of the equivalent pendulum after reversal, then

$$l_2 = \frac{k^2 + (l_1 - r)^2}{l_1 - r} \tag{93}$$

just as in (91) above. Substituting the value of k^2 from (92) in (93),

$$l_2 = \frac{r(l_1 - r) + (l_1 - r)^2}{l_1 - r} = r + l_1 - r = l_1 \tag{94}$$

Hence the length, and consequently the period, is the same in either position.

This principle is utilized in finding the length of a simple pendulum whose period is the same as that of the compound one. In Fig. 88 is shown a physical pendulum, usually known as Kater's pendulum, which consists of a brass rod having an adjustable knife-edge near each end, and cylindrical masses which may be shifted on the bar. It is possible by proper adjustment to find positions of the knife-edges at different distances from the centre of gravity such that the period will be the same when the pendulum is supported by either knife-edge. The

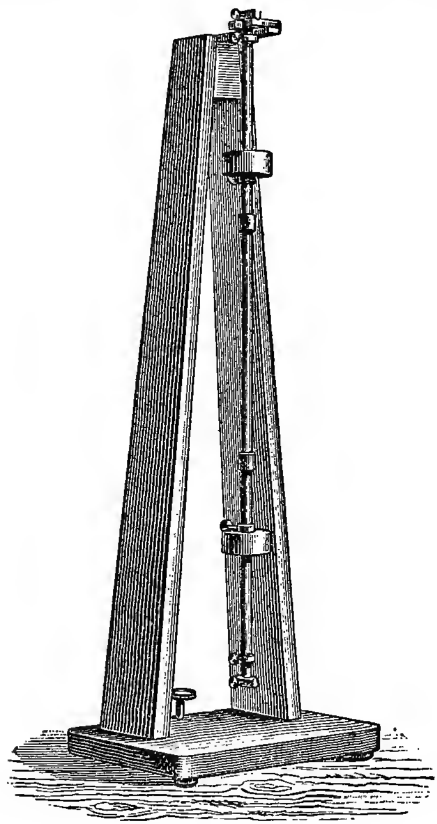


FIG. 88.

distance between the knife-edges can then be measured. This distance is the length of a simple pendulum of the same period.

56. Use of Pendulum for Measurement of Time. — It has been shown that the vibrations of a pendulum are practically isochronous,—*i.e.*, the period is independent of the amplitude. By proper mechanism a pendulum may be made to record its own vibrations and so is an excellent means of keeping time. As the pendulum swings from side to side, it allows a tooth of the escapement wheel to pass the pallet at each swing. The escapement wheel is connected through a train of wheels to the weights or springs which are the source of energy, and also to the hands of the clock. The motion of the hands is thus controlled and regulated, but not operated, by the pendulum.

Were it not for friction and resistance of the air, a pendulum once started would never come to rest. To keep the pendulum going, a wire extends from the pallet down a short distance along the pendulum rod, by which a slight impulse is given while the bob is at the lowest point of each swing. It is essential that the impulse be given while the pendulum is at the lowest point of its swing, L, Fig. 89. Let oa be a force applied while the pendulum has a position A. There will then be a component ab which will act in conjunction with the force of gravity, and consequently, during the time of the impulse, the force of restitution would be increased. The force F, however, acts in a horizontal direction and so has no component with or against the force of gravity.

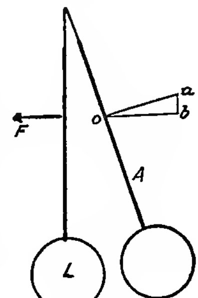


FIG. 89.

Problems.

1. The bob of a pendulum while passing its lowest point has a velocity of 100 cm/sec. To what height will it rise where $g = 980$ cm/sec²?

2. The displacement of a simple pendulum is 30° and its mass is 10 g. What is the force of restitution?

3. If in a certain locality a pendulum 99.3 cm. long beats seconds, what is the value of g?

4. If a pendulum loses 20 sec. per day at a place where g is 980.3 cm/sec², what is its length?

5. The length of a uniform cylindrical brass rod is 216.7 cm., its radius is 7.2 mm., and its mass is 2977 g. Its moment of inertia about an axis perpendicular to it through its centre of gravity is $\frac{1}{3}ma^2 + \frac{1}{4}mb^2$, where m = mass, a = half the length, and b = the radius. If this rod is suspended at one end and made to oscillate as a physical pendulum, what will the period be? ($g = 980$).

1. 5.1 cm.
2. 4900 dynes.
3. 980.05 cm/sec².
4. 99.37 cm.
5. 2.4 sec.

57. Machines. — A machine is any contrivance through which energy may be advantageously expended in doing work. A machine is simply a medium for the transmission of energy. It receives energy by virtue of work done upon it or energy transmitted to it, and then may expend this energy in doing work. The kind of energy transmitted may be very different from that received, but the quantity, including the so-called lost energy, is exactly the same. A machine of itself can neither do work nor assist in doing work. It is only a convenient medium through which work may be done. This principle of conservation of energy in machines makes the "perpetual motion" machine impossible.

Work is defined as the product of a force or resistance by a distance. If the force applied *to* a machine is F, and the force applied *by* the machine is R, also if the respective distances through which they move are S_f and S_r, then

$$FS_f = RS_r \qquad (95)$$

This equation expresses the machine principle and states the equality between energy received and energy expended. Also, since each member of the equation contains two factors, either factor may be changed in value provided the other is at the same time changed in an inverse ratio. In this fact consists the chief advantage in the use of machines. Thus, let FS_f represent a certain quantity of work. It is possible by use of a machine to make F one-fifth as great, for example, and at the same time make S_f five times as great, the amount of work remaining the same.

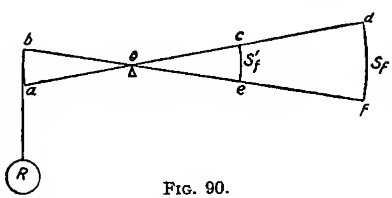


FIG. 90.

This principle may be further illustrated by reference to Fig. 90. Suppose it is desired to do the work of lifting a weight through the distance ab. This may be done by use of a simple form of machine,—the lever. If the force F is applied at c, c and a being equidistant from o, then F and R must move through equal distances and so F equals R as before. But if F is applied at d, od being, say, twice oa, then the arc df, or S_f, is twice the arc *ab*, or S_r. Hence $F = \frac{R}{2}$. This illustrates how a machine may be an *advantage*, and also that, whatever changes may be made in F, the value of S_f changes in such a manner that the product of the two is unchanged.

58. Mechanical Advantage.—Mechanical advantage is a ratio expressing the number of times the force is multiplied by the use of a machine. It is the ratio of R to F. The ratio of S_f to S_r gives the same result. Also, the ratio of any parts upon which the values of R and F depend, such as the arms of levers and the radii of wheel and axle, may be used to determine the mechanical advantage. If, for example, the ratio in any case is found to be 24, this means that the resistance against which the machine is capable of working is 24 times as great as the force applied to the machine.

It is not to be understood from the term *advantage* that a force applied to a machine is always increased by that machine. The number expressing the advantage may be a fraction. It is often desirable that F be greater than R, but in that case S_f is proportionately less than S_r.

59. Kinds of Machines.—Machines are usually classified as *simple* and *compound*, the compound being a combination of simple machines.

Simple machines are of two kinds,—namely, the *lever* and the *inclined plane*. All simple machines may be classed with one or the other of these two. Thus, pulleys and wheel and axle are levers, while the wedge and screw are inclined planes. The ordinary hand pump is a simple machine of the lever form, by which the energy expended in the operation of the pump is transmitted to the water which is raised from the well. The dynamo is a simple machine of the lever form, which causes a flow of electricity. The various kinds of compound machines are combinations of levers and inclined planes.

60. Levers.—A lever is a mechanical device such that a force applied at one point produces or tends to produce rotation about an axis called the fulcrum, against a resistance which tends to prevent such rotation.

The commonest form of lever is a straight or bent bar, the axis or fulcrum having various positions relative to the points of application of the force and the resistance.

Levers are usually divided into three classes, distinguished by the relative positions of the force F, the resistance R, and the fulcrum o. When o is between F and R, the lever is one of the first class; when R is between F and o, second class; when F is between R and o, third class.

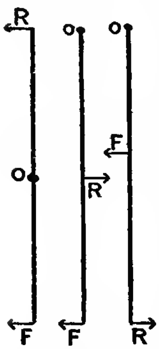


FIG. 91.

The mechanical advantage in any case is determined by the ratio of the distance from F to o to that of R from o.

If the direction of the force is not perpendicular to the lever, then, as already explained under the subject of moments, only that component which is perpendicular is to be considered as effective in producing rotation. Thus, in a bent lever aob, Fig. 92, let a force F be applied at a, its line of direction making an angle φ with the arm oa. The only part of F that causes rotation about o is the component ac, equal to $F \sin \varphi$; hence

$$F \sin \varphi \,.\, \overline{oa} = R \,.\, \overline{ob}$$

or, since oe, which is perpendicular to the line of direction of F, is equal to $\overline{oa} \sin \varphi$, or oa equals $\dfrac{oe}{\sin \varphi}$,

$$F \,.\, \overline{oe} = R \,.\, \overline{ob} \qquad (96)$$

61. Pulleys.—A pulley is usually a grooved wheel or disk, called a sheaf, supported in a frame called a block. All pulleys are levers of either the first or the second class. In Fig. 93, A, the pulley is supported at T. It is then said to be **fixed,** and a force F applied at one end of a rope passing over the sheaf will have the same effect as if applied at b,—one end of the lever ab. The fulcrum of the lever is at o, the centre of the pulley, hence F and R are equal. A fixed pulley may thus be considered a lever of the first class in which the mechanical advantage is unity. As the wheel is turned, the infinite number of such levers of which the wheel is composed come successively into use in the position ab.

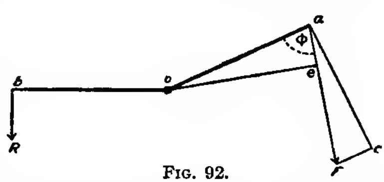


FIG. 92.

Although neither F nor S_f is changed in magnitude in passing to R and S_r, the direction of the motion is different. A weight, for example, may be raised by a force directed downward.

A pulley is said to be **movable** when it is supported in a bight of a cable or rope one end of which is fastened to a support. This, as shown in Fig. 93, B, is the use of a lever of the second

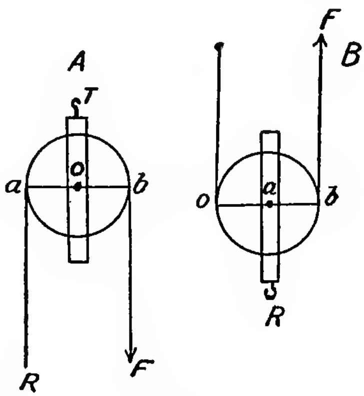


FIG. 93.

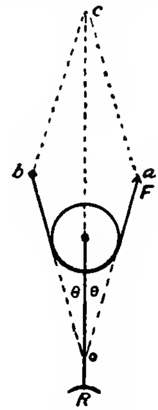


FIG. 94.

class, for the resistance is applied at the middle of the wheel and the force-arm ob is twice as long as the resistance-arm oa. Hence in a movable pulley

$$R = 2F$$

In case the strands of the cable are at an angle to each other as shown in Fig. 94, let θ be the angle which oa makes with the vertical line oc, R being the force of gravity. The tension of the rope at any point is F, and the angle which ob makes with the vertical is also θ. Hence the vertical component directed upward on each side of the pulley is $F \cos \theta$. The resultant is their sum, which is

$$2F \cos \theta$$

Hence, since the equilibrant is equal to the resultant,

$$R = 2F \cos \theta$$

or
$$F = \frac{R}{2 \cos \theta} \tag{97}$$

The same result may be obtained by the addition of vectors in the usual manner. Thus, since *oa* and *ob* may be considered two equal vectors whose divergence is 2θ, each having the value F, the resultant and also the equilibrant R may be found by

$$\begin{aligned} R &= \sqrt{F^2+F^2+2F^2 \cos 2\theta} \\ &= \sqrt{2F^2+2F^2(2\cos^2\theta-1)} \\ &= \sqrt{4F^2\cos^2\theta} \\ &= 2F\cos\theta \\ \therefore F &= \frac{R}{2\cos\theta} \end{aligned}$$

Fig. 95.

It is shown by equation (97) that if θ becomes zero its cosine is unity and

$$F=\frac{R}{2}$$

which is the case when the strands are parallel. But if θ becomes $\frac{\pi}{2}$ (90°), then the cosine is zero and

$$F=\frac{R}{o}=\infty$$

—*i.e.*, it is impossible to apply sufficient force at the end of a rope to hold it in a straight horizontal position while a weight is suspended from it.

Pulleys may be combined in a variety of ways. Several sheaves may be combined in one block, as shown in Fig. 95. The cable is continuous, one end being fastened to either the fixed or the movable block and the other threaded through the grooves of the sheaves in both blocks. The force is then applied at the free end. The resistance R, which may be a weight or any other resistance, is supported by the number (n) of strands between the two blocks. Since F is the tension of any strand of the rope,

$$F=\frac{R}{n} \tag{98}$$

The mechanical advantage in this arrangement is n, and the distance through which F moves is n times as great as that through which R moves.

62. The Wheel and Axle.—The wheel and axle is a form of lever of the first, second, or third class. If W, Fig. 96, is the wheel and A is the axle, then a force F applied at the periphery of the wheel will balance R applied at the periphery of the axle, when

$$F\ .\ \overline{ob} = R\ .\ \overline{oa}$$

This is a lever of the first class.

If R is applied at c and directed upward, the arrangement will be a lever of the second class, and the mechanical advantage is the same as before.

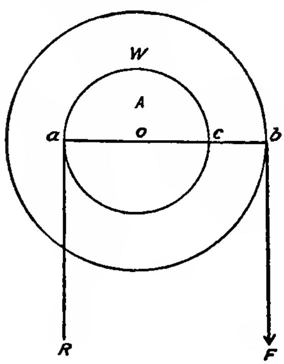


FIG. 96.

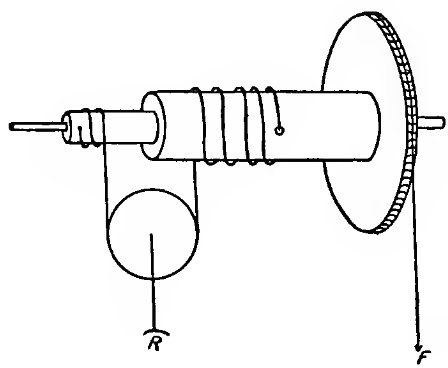


FIG. 97.

If F is applied at c and directed downward while R is applied at b and directed upward, the arrangement is a lever of the third class, and the mechanical advantage is $\frac{oc}{ob}$, while in the first two cases it is $\frac{ob}{oc}$.

A modification of the simple form of wheel and axle consists of two axles or drums of different radii attached to a wheel, and all rotating on a common axis, as shown in Fig. 97. As the wheel is turned, one end of the rope is wound on the larger drum and the other end is unwound from the smaller one, a movable pulley being suspended from the middle part of the rope. This arrangement is called the **differential wheel and axle.** To calculate the value of F for any given value of R or *vice* *versa*, let r be the radius of the wheel, r' that of the larger drum, and r'' that of the smaller drum. Then

$$Fr = \frac{Rr' - Rr''}{2}$$

$$\text{or} \qquad R = \frac{2Fr}{r' - r''} \qquad (99)$$

A machine constructed on this principle is used in shops where heavy masses are to be lifted. It is known as the differential pulley or **chain hoist.** As shown in Fig. 98, a continuous chain is thrown over the larger wheel, then looped up over the smaller one. This forms two free loops, L and P. Into one of these a movable pulley is placed, while the other may be grasped when force is to be applied at F. The mechanical advantage is calculated just as for the differential wheel and axle above, the larger wheel here serving for both wheel and larger drum of Fig. 97. Hence

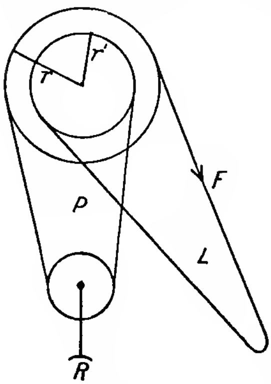


FIG. 98.

$$Fr = \frac{Rr}{2} - \frac{Rr'}{2}$$

$$\text{or} \qquad R = \frac{2Fr}{r - r'}$$

This equation shows that the mechanical advantage may be enormously great if r and r' are made nearly equal, for as the quantity $r - r'$ approaches zero, the value of R approaches infinity. Thus a small force exerted through a great distance may be employed to raise a great weight through a small distance.

63. The Inclined Plane. — An inclined plane is a plane so inclined to the direction in which a body is to be moved that a mechanical advantage is obtained. Examples of this kind of machine are the screw, the wedge, a cam, a plank with one end elevated, a roadway up a hill, and so on.

The principle in this machine is the same as in all others,— namely, the quantity of work necessary to cause a given displacement against a given resistance is not changed by a change in the method by which that work is accomplished. This is on the assumption that no energy is lost in friction or otherwise.

Let a mass m, Fig. 99, be moved from A to B against a force of gravity which is directed vertically downward. When the mass reaches B it will have been raised a distance CB against an opposing force mg. The amount of work, then, is the same whether the mass is moved along AB or CB, but in the former case a smaller force is exerted through a longer distance. An agent possessing small power may thus accomplish work which would not be possible without the use of a machine.

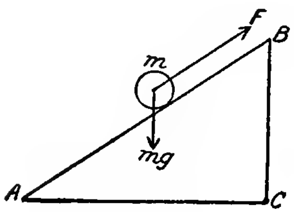


FIG. 99.

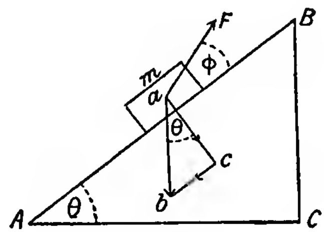


FIG. 100.

The mechanical advantage in the use of the inclined plane depends on the angle of inclination and the line of direction of the force and resistance. Let m, Fig. 100, be a mass which may be moved without friction along the incline AB. Let F be a force acting on m, its line of direction making angle φ with the incline. Let the resistance here considered be the force of gravity. The vector ab may be taken to represent the total weight of m, and this may be resolved into ac and cb. The component ac is normal to the inclined plane and has no effect on motion along the incline, but cb is parallel to the incline and directed toward A. The resistance along the incline is therefore

$$R \sin \theta$$

while the component of F in the opposite direction is

$$F \cos \varphi$$

Hence, for equilibrium,

$$F \cos \varphi = R \sin \theta$$

or

$$F = R \frac{\sin \theta}{\cos \varphi} \qquad (100)$$

The same result may be obtained by using the general equation for work. Thus,

$$F \cos\varphi \,.\, \overline{AB} = R \,.\, \overline{CB}$$

or

$$F = R\frac{CB}{AB\cos\varphi}$$

Since $\dfrac{CB}{AB} = \sin\theta$,

$$F = R\frac{\sin\theta}{\cos\varphi}$$

If the direction of F is parallel to the plane, φ becomes zero and its cosine is unity, hence

$$F = R\sin\theta$$

or, since $\sin\theta$ is the ratio of the height h to the length l of the inclined plane,

$$F = R\frac{h}{l} \qquad (101)$$

If the force is applied in a direction parallel to the base AC, the angle φ becomes equal to the angle θ, hence

$$F = R\frac{\sin\theta}{\cos\theta} = R\tan\theta = R\frac{h}{b} \qquad (102)$$

where b is the length of the base.

64. The Wedge.—A wedge is an inclined plane and its use involves the principles just described. A common form of wedge consists of two inclined planes placed base to base as shown in Fig. 101. Let a force F be applied in a direction co so as to separate two bodies A and B in directions at right angles to the direction of the force. Let θ be the inclination of each plane of the wedge and the resistance R perpendicular to the faces. Then, according to the principle of work,

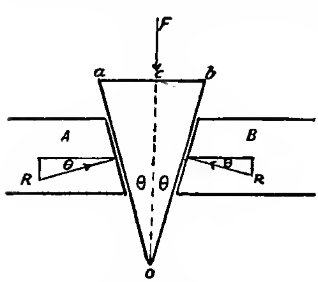


FIG. 101.

$$F \,.\, \overline{co} = R\cos\theta \,.\, \overline{ab}$$

But, since $ab=2co \tan\theta$

$$F=2R\cos\theta\tan\theta$$

or $$F=2R\sin\theta \tag{103}$$

65. The Screw.—The screw is an inclined plane by which a great mechanical advantage may be readily obtained.

If the right-angled triangle ABC, Fig. 102, made of thin flexible material, be wrapped on the cylinder L so that BC is parallel to the axis of L, the edge AB will mark the position of the threads of a screw. The distance between two consecutive threads, measured parallel to the axis of the cylinder, is the *pitch* of the screw. If the line Ab will go once around the cylinder, ab is the pitch. Let r' be the radius of the cylinder, then

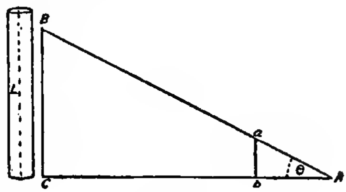


Fig. 102.

$$ab=2\pi r'\tan\theta \tag{104}$$

If a screw is turned by a force F, Fig. 103, applied at a distance r from the axis of the screw, then, applying the principle of work, and knowing that for each turn of the screw an advance equal to the pitch is made against the resistance R,

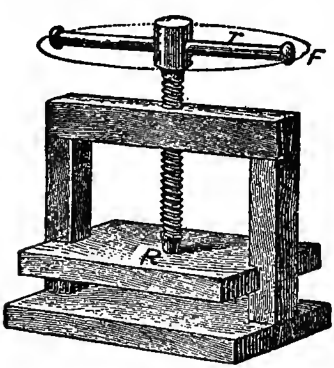


Fig. 103

$$F\,.\,2\pi r=R\,.\,\text{pitch} \tag{105}$$

By comparison with equation (104),

$$F\,.\,2\pi r=R\,.\,2\pi r'\tan\theta$$

or $$F=R\frac{r'}{r}\tan\theta \tag{106}$$

By use of either (105) or (106) the mechanical advantage may be found.

66. Friction.—When there is a relative motion between two bodies that are in contact, the *resistance* to this motion resulting from the contact is called *friction.*

Friction is observed on every hand. In all movements of liquids and gases friction enters into the calculation. This will be more definitely considered in a later discussion.

The most important cases of friction are those of solids on solids. Friction is encountered in the operation of all machines. As a result of friction much energy is lost, as has already been explained. In the operation of most machines friction is, as far as possible, avoided by lubrication, by ball bearings, by pivotal bearings, and in many other ways.

Friction is not, properly speaking, a force, for it does not exist until there is relative motion of bodies in contact, and when the direction of the motion changes, the direction of the resistance also changes. Since, however, the effect is the same as if an active force were exerted, the term force of friction is often employed. Friction between solids may be classified as *sliding* and *rolling*, the former being subdivided into *static* and *kinetic*.

67. Sliding Friction.—To illustrate a simple case of sliding friction, let a block, of mass m, rest on a horizontal plane AB, Fig. 104. The pressure P of the block, normal to the plane, is mg. Instead of the force of gravity it may be any other pressure normal to the plane. Let a force F act parallel to the plane,

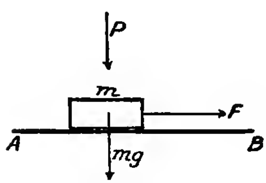


Fig. 104.

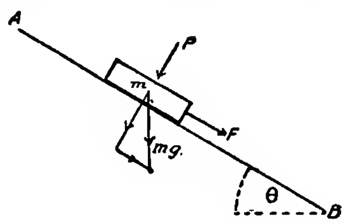


Fig. 105.

then, as found by experiment, the relation between F and P for any given surfaces is such that when F is just sufficient to start the motion of the block, the ratio of F to P is a constant. This is a case of *static friction*, for it is the friction while standing and just on the point of starting. This constant ratio is called the coefficient of friction, and for static friction may be designated by μ. Then

$$\mu=\frac{F}{P} \tag{107}$$

—*i.e.*, friction is independent of the area of the surfaces in contact, and varies only with the pressure normal to the surface in any given case under consideration. When the value of μ is once determined, the value of F for any given value of P can readily be found.

One method of finding μ for any given material is to elevate one end of a plane, AB, Fig. 105, until the block m just begins to slide. The inclination of the plane is called the *angle of repose*. Let θ be this angle. The pressure normal to AB is

$$mg \cos \theta$$

and the force which causes the block to slide is

$$mg \sin \theta$$

Hence $$\mu = \frac{mg \sin \theta}{mg \cos \theta} = \tan \theta \tag{108}$$

Thus the coefficient of static friction is equal to the tangent of the angle of repose.

68. Kinetic Friction.—The friction between two solids when actually in motion relative to each other is called *kinetic friction*. This is usually less than static friction. The coefficient of kinetic friction can be found in a manner just described for static friction. The inclination of the plane, Fig. 105, is varied so that the block, once started, continues to slide with a uniform motion. Since the movement is not accelerated, the friction must just equal the component of the force of gravity which causes the sliding. Hence, if κ is the kinetic friction,

$$\kappa = \frac{mg \sin \theta}{mg \cos \theta} = \tan \theta \tag{109}$$

For illustration, suppose a mass of 1000 g. slides uniformly down a plane when θ is 32° Then

$$\kappa = \tan 32° = .6249$$

If now the same plane be placed horizontally, the force required to drag the mass along may be found from (109). Thus

$$\kappa = \frac{F}{P}$$

or $$F = \kappa P = .6249 \times 1000 = 624.9 \text{ g.}$$

—*i.e.*, a force of 624.9 g. will be required to drag the mass along with a uniform motion. If a force of 724.9 g. be applied, then 100 g. of the force will be expended in giving the mass an accelerated motion, and since

$$F = ma$$

$$100 \times 980 = 1000a$$

or $$a = 98\ {}^{cm}/_{sec^2}.$$

69. Rolling Friction. — When a wheel, cylinder, or any circular body rolls on a horizontal plane, there is a resistance which causes the body to come to rest. The cause of the resistance is a depression in the plane at the point where the wheel is in contact with it, and also a slight flattening of the wheel at this point. The resistance is not friction, then, in the sense just explained.

Let a wheel of mass m be rolled along a plane P, Fig. 106. Just in front is a slight bulge which is virtually an inclined plane up which the wheel must ascend. The resistance is $mg \sin \theta$, which is equal to a force that would cause the wheel to be in equilibrium on the incline. The smaller θ becomes, the more nearly this resistance vanishes.

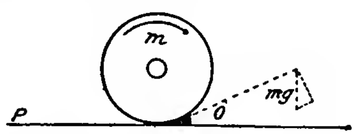


FIG. 106.

Resistance due to "rolling friction" is small compared to that of sliding friction, if the plane and wheel do not readily yield to pressure. An extreme example of resistance of rolling friction is the movement of a carriage wheel through a bed of sand or gravel.

70. Uses of Friction. — Friction is commonly considered as something to be avoided or reduced to a minimum. It, however, like all other conditions in nature, is an advantage in many ways. It makes possible the use of belts in transmitting energy from pulleys. Brakes applied to wheels would be useless without friction. Walking on a pavement or driving on a street or highway would be hazardous if there were no friction. Numerous examples of this kind may be given.

A special use of friction for determining the power of a machine is here described. The power of a steam engine, for example, may be found experimentally by use of a **friction dynamometer.** Let A, Fig. 107, be a broad-faced wheel attached to the shaft of an engine. A strap thrown over the wheel is fastened at one end to the spring scales s, and weights are hung on the other end. Sufficient weight is used to cause the engine to work at its full capacity against the friction of the strap on the wheel. When the wheel turns in the direction indicated by arrows, the friction tends to lift the weight w and relieve the strain of the spring in s. Hence the reading of s subtracted from the known weight w is the force of friction R. Hence

$$R=w-s$$

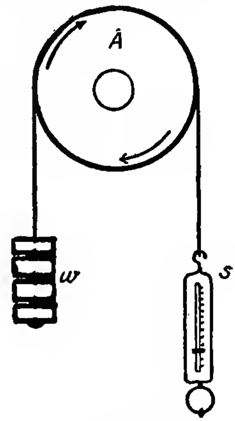


FIG. 107.

Let the number of rotations per minute be n, then the distance through which the force of friction is exerted is

$$2\pi rn$$

where r is the radius of the wheel. The work done is therefore

$$2\pi rnR$$

or $$2\pi rn(w-s)$$

Since one horse power (h. p.) is 33,000 foot-pounds per minute, then if r, w, and s are measured in feet and pounds, and n is the number of revolutions per minute,

$$\text{h.p.}=\frac{2\pi rn(w-s)}{33000} \qquad (110)$$

71. Efficiency.—Efficiency of a machine is the ratio of the quantity of energy which a machine transmits to that which it receives. Since some friction is always present, the efficiency can never be unity. If, for example, 500 foot-pounds of work are done on a machine and it in turn can do but 300 foot-pounds, its efficiency is 60 per cent. Any reduction of friction increases the efficiency of a machine.

Problems.

1. A weight of 50 kg. is suspended from the block of a single movable pulley. The strands of the cable are at an angle of 70° to each other. What force applied at one end of the cable will support the weight?

2. What is the mechanical advantage in a chain hoist where the radii of the wheels are 10 and 11 inches and the weight is suspended from a single movable pulley?

3. The elevation of an inclined plane is 37° and the direction of a force applied to hold a body on the incline makes an angle of 40° with the plane. What is the mechanical advantage?

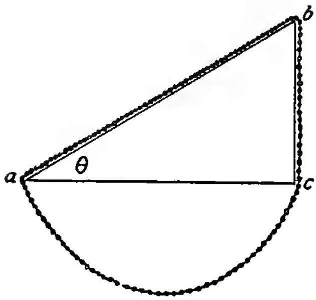


FIG. 108.

4. A chain is hung over an inclined plane in the manner shown in Fig. 108. Show that the force of gravity tending to cause the portion *ba* to slide down the incline is equal to that on *bc*, and that there will be no movement of the chain even when friction is completely eliminated.

5. How much energy has been lost in friction if a mass of 100 g. sliding down a plane 80 cm. high acquires a velocity of 300 cm/sec?

6. The slope of an inclined plane is 16°. What force will be required to drag a mass of 20 kg. up the incline when the coefficient of kinetic friction is .385?

7. If the radius of the body of a cylindrical screw is 1.5 inches, and the slope of the thread is 24°, what force applied to the best advantage 3 feet from the head of the screw will cause a pressure of 2 tons?

1. 30.52.
2. $\frac{1}{22}$.
3. .786 : 1.
4. ——.
5. 3,340,800 ergs.
6. 12.9128 kg.
7. 74.2 lbs.

CHAPTER III

SOLIDS

72. Constitution of Matter.—The scientific conception of a body of matter is that it is composed of a great number of very small particles called molecules. These particles are never in permanent contact; consequently all substances are porous in the sense that there are spaces between the molecules. The particles are in a continual state of agitation or rapid motion, colliding with and rebounding from their neighbors or other bodies with which they come in contact. Many of the properties and phenomena of matter are the result of molecular arrangement and relations. Crystallization, tenacity, temper, rigidity, heat, expansion, elasticity, and many other subjects might be classed under the general head of **molecular physics.**

The molecule is considered to be the smallest particle into which a substance can be divided without destroying the identity of that substance. Thus, a molecule of limestone, $CaCO_3$, is limestone—as much so as a ton of it. If, however, this molecule is separated into its constituents by any chemical process, it becomes calcium, carbon, and oxygen. These parts are called **atoms,** and were originally supposed to be the limit of divisibility of matter, as the word "atom" indicates. There is now, however, very strong evidence that the atom is composed of many small parts called corpuscles or **electrons.** These may be the ultimate particles of which all matter is composed.

This conception of matter furnishes a convenient model with which the mind can grasp and explain many of the observed phenomena of nature. The fact that by its aid satisfactory explanations can be made and new truths discovered, is sufficient justification for its existence.

73. States of Matter.—Matter is found in various states, depending on the relation and condition of its molecules. A **solid** is a body which will, under ordinary conditions, retain its shape and size by virtue of its molecular structure. The molecules of a solid can not, apparently, move beyond a limited space in which they vibrate. To furnish a mental picture, the molecules of a solid may be considered as so related that they form a rigid framework. Within certain limits which are different for different substances, the framework is of itself able to withstand the stress to which it is subjected. It may yield, but will return to the original position when the stress is removed. This property is called elasticity and is discussed in succeeding paragraphs.

A **liquid** is a substance in such a state that its molecules appear to move among their neighbors without any permanent restraint. A liquid when not confined by a solid will change its shape under the influence of a force however small. Liquids when exposed to air or other gases, or in a vacuum, have a definite free surface and are capable of being formed into drops. In these two respects liquids are clearly distinguished from gases.

A **gas** is a substance in such a state that the molecules *appear* to repel each other and to move with freedom from point to point throughout the body of the substance. As a consequence, the shape of a gas is that of the total interior of a containing vessel.

Many substances may easily be made to assume any one of the three states — solid, liquid, or gas — by application of the proper quantity of heat.

Some bodies in their natural state partake of the nature of both solids and liquids. Such substances as ice and asphaltum, for example, have the rigidity of solids when subjected to a momentary stress, but if the stress is continued for a time they exhibit properties of liquids and will slowly flow. If asphaltum is placed in a funnel, it will, under the force of gravity, slowly yield and adapt itself to the shape of the funnel, flowing on through like a liquid. Such substances are said to be **viscous.**

Ether is a substance which is assumed to fill all space, including even the interspaces of the molecules of a body. Ether is not matter in the ordinary sense of the term,—*i.e.*, our senses furnish no evidence of the objective existence of ether; but there is strong evidence that ether exists and that it possesses some of the most important properties of matter. For example, light is known to be, not a substance, but a periodic disturbance of some kind in the ether. If light, then, is transmitted as a wave motion on ether, the ether must possess elasticity, for otherwise there would be no restoring force after the deformation produced by the passage of a wave. Also, it is known that light has a finite speed of about $3(10)^{10}$ cm. per second; hence the medium on which it travels must possess inertia, for otherwise time would not be required. There are no direct methods by which ether may be studied, and it is not known whether it is continuous or not.

The substance which remains in a tube from which the air has been almost completely exhausted exhibits properties not observed in other states of matter. Sir William Crookes, who made an extensive study of this subject, referred to the condition, when a current of electricity was passed through it, as a "fourth state of matter." This so-called fourth state appears to be one in which the constituent parts of atoms have been separated from their ordinary group arrangement and made to flow in a stream through the medium within the tube.

74. Elasticity of Solids. — Most solids are to some extent elastic, though some, such as lead and gold, are so slightly elastic that they are called **plastic,**—*i.e.*, even a slight stress causes a permanent deformation. Others, such as ivory and steel, are very elastic,—*i.e.*, when the force which causes their deformation is removed, they will regain their original shape and volume. A substance like rubber is fairly elastic, but is remarkable for its **limit of elasticity,**—*i.e.*, it may be greatly deformed and yet will recover its shape when the deforming force is removed.

Three kinds of elasticity may here be considered. 1. *Elasticity of volume*, where all parts of a body are subjected to an equal and normal stress. A body subjected to hydrostatic pressure is an example of this kind of stress. 2. *Shearing elasticity*, where stress causes a change in shape without change in volume, as, for example, the torsion of a rod. 3. *Longitudinal elasticity*, where the stress is in only one direction while at right angles to this the body is free to expand or contract, as, for example, a wire subjected to a longitudinal stress.

The coefficient of elasticity is the ratio of a stress to the resulting strain. Thus, if a stress is denoted by F and the strain by S, the coefficient k may be expressed by

$$k=\frac{F}{S} \qquad (111)$$

75. Volume Elasticity.—Let a body be subjected to a uniform normal stress over its entire surface. Call this stress P. Let V be the original volume and v the change of volume resulting from the stress. Then the change per unit volume is

$$\frac{v}{V}$$

and this is a measure of the strain. Then the volume coefficient of elasticity is

$$k=\frac{\text{stress}}{\text{strain}}=\frac{P}{\frac{v}{V}}=\frac{VP}{v} \tag{112}$$

or

$$v=\frac{VP}{k} \tag{113}$$

When the value of k, a constant quantity, is once determined for any given substance, the decrease in volume for any given pressure may easily be found.

The pressure is usually given in dynes per square centimetre and the volume in cubic centimetres. For example, a pressure of one atmosphere is 76 cm. of mercury, or 1033.6 g. per square centimetre, or $1.013(10)^6$ dynes. A volume of water subjected to this pressure is known to decrease in volume by $5(10)^{-5}$ of the original volume. Using 1 c.c. of water and substituting in (112)

$$k=\frac{1.013(10)^6\times 1}{5(10)^{-5}}=2.02(10)^{10}.$$

This quantity, $2.02(10)^{10}$, is the volume elasticity of water. It is sometimes called the *bulk modulus*. The formula shows that it is the ratio of the pressure in dynes per square centimetre to the resulting change in volume per cubic centimetre. Volume elasticity is the only kind that liquids and gases can have, while solids have all three kinds named above.

The volume elasticity of a gas is found in a manner described in §§ 94 and 177. That for a solid may be calculated by equation (124). The volume elasticity of liquids may be determined by means of a piezometer, which, in the form shown, Fig. 109, consists of a strong glass tube filled with water and provided with a screw and plunger for increasing the pressure. The liquid under examination is in the bulb B, which is provided with a fine capillary stem open at the top. The capacity of B and the capillary tube are known, and any change of volume of the liquid due to pressure may be indicated by the movement of a short thread of mercury in the stem. The pressure is calculated from the rise of liquid in the manometer M, which is a glass tube closed at the top but open at the bottom and filled with air. When the volume of air is reduced one-half, for example, the pressure is twice as great. The apparent change of volume of the liquid must be corrected for the decrease in the capacity of the bulb due to compression of the bulb, for although the hydrostatic pressure is equal on the inside and outside, yet each small cube of which the walls may be supposed to consist is pressed into smaller volume and hence the total volume is decreased just as if the bulb were solid glass. Hence the correction is added.

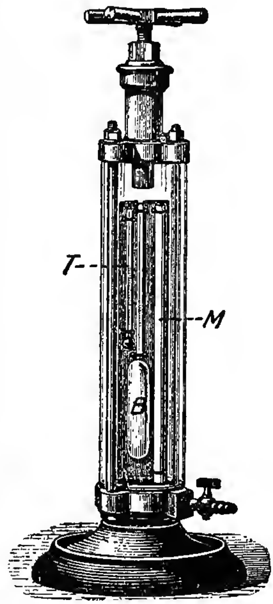


FIG. 109.

The piezometer may also be used to find the bulk modulus of solids.

76. Shearing Elasticity.—If several metal disks were piled one on top of another, a force applied in a horizontal direction to any one of them would cause a sliding of one relative to the others. If the disks were welded together and a similar force then applied, the effect would be to cause one portion of the solid to slide on an adjacent portion, but because of the rigidity of many solids the extent of the sliding is very limited. The change in the relative position of the molecules is called a *shear*. The stress is called a shearing stress, and the effect of the stress is a shearing strain. There is no change in the volume of the body as a result of the strained condition, for the molecules in each imaginary layer of the solid have not changed their distances from one another and the layers have come no closer together. If the strain does not exceed a certain limit, the molecules will return to their original positions when the force is removed. This ability to recover from a shearing strain is called *shearing elasticity.*

77. Coefficient of Shearing Elasticity. — The coefficient of shearing elasticity, also called the coefficient of rigidity, is the ratio of the force per unit area to the strain produced by that force,—*i.e.*, it is the ratio of the stress to the strain.

To find an expression for this coefficient in terms of measurable quantities, let a thin cylindrical shell C, Fig. 110, be fastened at one end to a support A and let a force be applied at the other end to twist it through an angle θ. Each layer of the cylinder will thus be made to slide on an adjacent layer, the amount of sliding or shear between any two successive layers being the same. The angle φ in radians is taken as a measure of the strain.

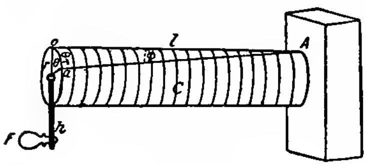


FIG. 110.

Let f be the total force applied directly to the surface of the cylinder at a distance r from the centre, and n the rigidity coefficient. Then, according to the definition,

$$n = \frac{\frac{f}{\text{area}}}{\varphi} \tag{114}$$

Let t be the thickness of the shell and r its radius, then the area of one end of the shell is $2\pi rt$. The area here is that of a cross section of the material of the shell. Hence

$$n = \frac{f}{2\pi rt\varphi} \tag{115}$$

The linear distance oa is θr or φl, the length l being great compared with oa. Hence

$$\varphi l = \theta r$$

or

$$\varphi = \frac{\theta r}{l}$$

Substituting this value of φ in (115),

$$n = \frac{fl}{2\pi r^2 t\theta} \tag{116}$$

This total force f is applied to the shell at a distance r from the centre and so its moment is fr. Let a force F be applied at a distance h from the centre, by means of a crank rigidly fastened to the end of the cylinder as shown in the figure. Then its moment is Fh. These two moments are equal, for they produce the same strain of the cylinder, hence

$$Fh = fr$$

or

$$f = \frac{Fh}{r}$$

Substituting this value of f in (116),

$$n = \frac{Fhl}{2\pi r^3 t\theta} \tag{117}$$

Thus n is expressed in terms that are capable of experimental determination.

It is more common, however, to find n by use of a solid rod. Such a rod may be considered as composed of many cylindrical shells whose radii differ by only the very small quantity t,—the thickness of each shell,—as shown in Fig. 111. The moment Fh is different for each shell and the total moment required to produce a strain φ in the solid rod is the sum of all the moments required to produce the same strain in the various shells. From (117),

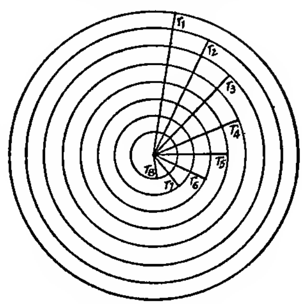


FIG. 111.

$$Fh = \frac{2\pi n\theta r^3 t}{l}$$

The only two variable quantities here are Fh and r. Hence, for a solid rod,

$$\overline{Fh}_1 + \overline{Fh}_2 + \overline{Fh}_3 + \ldots = \frac{2\pi n\theta t}{l}(r_1^3 + r_2^3 + r_3^3 \ldots)$$

or

$$\overline{Fh} = \frac{2\pi n\theta r^4}{4l} = \frac{\pi n\theta r^4}{2l} \tag{118}$$

whence

$$n = \frac{2\overline{Fh}l}{\pi r^4 \theta} \tag{119}$$

The process of deriving this final result is given in appendix 5.

78. Longitudinal Elasticity. Young's Modulus. — When a rod or wire is stretched, as by a weight suspended from one end of it, any minute portion of the material is changed in both shape and volume. The wire is increased in length, decreased in diameter, and increased in volume. Consequently there is both a shearing and a volume elasticity.

The elasticity of wires, rods, and pillars, when subjected to a longitudinal stress, is of such importance in practical work that another coefficient called *Young's modulus* has been defined to suit these conditions.

Young's modulus is the ratio of the stress (force) per unit area of cross section, to the strain (elongation) per unit length. Let F be the force, L the length, A the area of cross section, and l the change in length. Then, if Y represents Young's modulus,

$$Y = \frac{\frac{F}{A}}{\frac{l}{L}} = \frac{FL}{Al} \tag{120}$$

When the value of Y is once determined for any given kind of material, it can thereafter be used to find the elongation l whenever any given rod of that material is subjected to a known stress. This is evidently a valuable coefficient in most structural iron work.

The elongation here considered is only within the elastic limit. The stress must not produce permanent deformation. Young's modulus applies not only to stretching but also to compression, as when a pillar is made to support a structure.

One method of finding the value of Y for any given material is to select a wire of that material and measure its diameter. From this the area of its cross section A can be calculated. A convenient value of L for experimental purposes is about 1 m. or more. The elongation l may be found by some such apparatus as that shown in Fig. 112. Two wires are fixed to a beam and from their lower ends are hung two metal rings. The rings support a delicate spirit level which is pivoted at one end to one of the rings but rests on the point of a micrometer screw at the other end. Any convenient weights are placed in the pans c and d to straighten the wires preliminary to the experiment proper. The wire b is the one upon which the experiment is to be made. The level is now adjusted and known weights are placed in pan d. The level is again adjusted by turning the micrometer screw, and the elevation required to restore the spirit level to its former position is the elongation for that stress (weight).

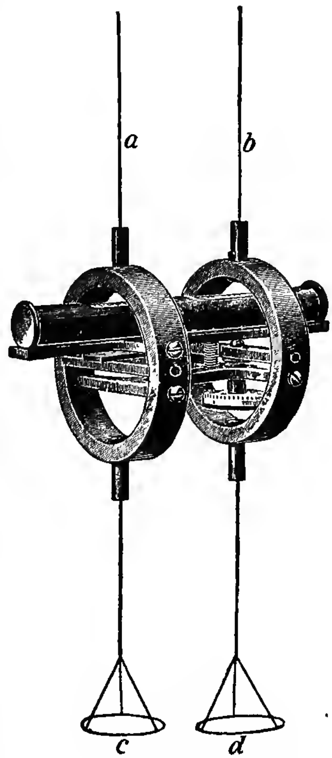


FIG. 112.

The advantage of having two wires fastened to the same beam is that if the beam itself yields to the weight, both rings will be lowered an equal distance and so there will be no relative motion on this account. The wire b, of course, must not be stretched beyond its elastic limit. When the weights are removed from d the wire should recover its original length.

Young's modulus may also be found by observing the distance through which a bar will bend when under stress. Let a bar of length l, breadth b, and depth d be supported at the ends and a weight w hung from the middle point, then if B is the bending, experiment shows that

$$B \propto \frac{wl^3}{bd^3}$$

or
$$B = c\frac{wl^3}{bd^3}$$

where c is a constant whose value may be shown to be $\frac{1}{4Y}$, Y being Young's modulus.

Hence
$$B=\frac{wl^3}{4Ybd^3}$$

or
$$Y=\frac{wl^3}{4Bbd^3} \tag{121}$$

If the bar is clamped at one end and loaded at the other,

$$Y=\frac{4wl^3}{Bbd^3} \tag{122}$$

79. Value of k in Terms of Y and n.—It is difficult to determine by experiment the coefficient of volume elasticity, k, of solids; but if Y., which is easily found, and the coefficient of rigidity, n, are known, the value of k may be found from

$$Y=\frac{9nk}{3k+n} \tag{123}$$

or
$$k=\frac{nY}{9n-3Y} \tag{124}$$

80. The Torsion Pendulum.—A torsion pendulum is not a pendulum at all in the sense the term has already been used, for its operation is not dependent on gravity but on the elasticity of a twisted wire. If a heavy cylinder, for example, is suspended from a wire and is turned on its axis through a few degrees and then released, it will execute torsional vibrations whose period depends on the dimensions and rigidity of the wire and the moment of inertia of the cylinder. The vibrations are simple harmonic, for the force of restitution is proportional to the displacement, in accordance with Hooke's law for elastic bodies. Consequently the acceleration at any point of the vibration is, from equation (40),

$$A=\frac{4\pi^2\theta}{P^2} \tag{125}$$

where A is the acceleration, P is the period, and θ is the displacement.

By equation (45)

$$A=\frac{Fh}{I} \tag{126}$$

where h is put in place of r to avoid confusion in later formulæ.

Combining (125) and (126),

$$\frac{4\pi^2\theta}{P^2}=\frac{Fh}{I}$$

or
$$\frac{Fh}{\theta}=\frac{4\pi^2 I}{P^2} \tag{127}$$

whence
$$P=2\pi\sqrt{\frac{I}{\frac{Fh}{\theta}}} \tag{128}$$

The expression $\frac{Fh}{\theta}$ is called the *moment of torsion*, and is a constant for a wire of any given length and radius. The constancy results from the fact that in an elastic body the strain (θ) is proportional to the stress (Fh).

81. Use of a Torsion Pendulum in finding I.—If a cylinder of known moment of inertia is suspended from a wire, Fig. 113, and the period of torsional vibration counted, the value of $\frac{Fh}{\theta}$ may be determined once for all for that particular wire by use of equation (127). If any other body is then suspended from this same wire in place of the cylinder, its moment of inertia can be found by observing P and substituting in (127).

Other methods of finding I by use of the torsional pendulum may be found in manuals for laboratory work.

82. Use of Torsional Pendulum for finding n.—From equation (119),

$$n=\frac{2\overline{Fh}l}{\pi r^4\theta}$$

or
$$\frac{Fh}{\theta}=\frac{\pi n r^4}{2l}$$

From (127),

$$\frac{\overline{Fh}}{\theta}=\frac{4\pi^2 I}{P^2}$$

hence
$$\frac{\pi n r^4}{2l}=\frac{4\pi^2 I}{P^2}$$

Fig. 113.

or
$$n=\frac{8\pi l I}{r^4 P^2} \tag{129}$$

83. The Torsion Balance.—A torsion balance is a mechanism by which it is possible to measure a moment of force tending to twist a wire. A light rod, *ab*, Fig. 114, is suspended from a fine wire and the whole enclosed in a glass case. Any horizontal rotation of *ab* is resisted by the rigidity of the wire, and the *angle* through which the wire is twisted is proportional to the *force* which causes the twist, as required by Hooke's law for elastic bodies. Hence two forces may be compared by measuring the angle through which they will separately cause *ab* to rotate. By means of a small mirror attached to *ab* at *o*, minute deflections may be read by use of a telescope and scale, or the upper end of the wire may be provided with a torsion head by which it is possible to read the angle through which the wire must be twisted at the upper end to balance the moment or torque at the lower end.

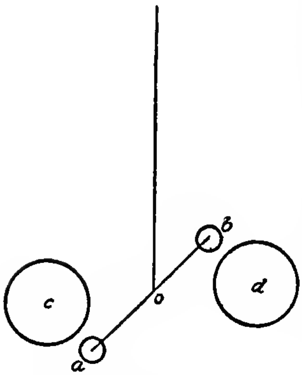


FIG. 114.

It has been shown in equation (118) that the twisting moment is

$$Fh = \frac{\pi n \theta r^4}{2l} \tag{130}$$

This shows that a force applied at *a* or *b*, tending to produce rotation in a horizontal plane, varies directly as the fourth power of the radius. If the radius is reduced one-half, for example, the force F need be only one-sixteenth as great to produce the same angular displacement. Thus by using very fine wire for suspension, very small forces may be detected.

It was by use of an instrument of this kind that Cavendish first determined the value of the gravitation constant G. Referring to equation (55), let the two small masses at *a* and *b* (Fig. 114) be denoted respectively as m and m'. Let two large masses, M and M', be placed at *c* and *d*. The force of gravitation between *a* and *c* at one end of *ab* and *d* and *b* at the other end will produce a couple which will cause a twisting of the wire.

Since the moment of a couple is the product of either force by the distance between the two forces, the moment in this case is

$$G\frac{mM}{r^2}\ .\ \overline{ab}$$

where r is the distance between the centres of a and c or of b and d.

Let the angle through which the wire is twisted be denoted by φ. The angle through which a unit moment of force (1 dyne at a distance of 1 cm. from the axis) will twist the wire is determined by a preliminary experiment. This may be done by causing ab, with masses m and m' attached, to vibrate as a torsion pendulum, the large masses M and M' meanwhile being removed. The moment of inertia of the system is known or can readily be found, and the period can be counted. Consequently the value of θ produced by unit moment can be deduced by use of equation (127), for all the terms of the second member of this equation are known, and, assuming that Fh is unity, the value of θ is found.

If unit moment of force will cause a deflection of θ radians, then a moment which will cause a deflection φ is $\frac{\varphi}{\theta}$, for the force varies directly as the angle through which the wire is twisted. Hence

$$G\frac{mM}{r^2}\overline{ab}=\frac{\varphi}{\theta} \tag{131}$$

or

$$G=\frac{r^2\varphi}{mM\theta\overline{ab}} \tag{132}$$

The value given for G (§ 37) was found by Boys, who used, instead of a fine metal wire, an exceedingly fine fibre of quartz. When quartz is heated to a white heat in an oxyhydrogen flame, it may be spun out into an exceedingly fine thread. Quartz also possesses the excellent property that it is not so subject to *fatigue* as metals are,—*i.e.*, it will promptly and completely recover from a strain.

84. Impact of Elastic Bodies. — When two elastic bodies collide, they will suffer a deformation as a result of the impact. Since they are elastic, they will regain their original shape and in doing so will react on each other.

Let this action and reaction be in a straight line joining the centres of the bodies, and let v_1 and v_2 be the respective velocities before impact, and u_1 and u_2 after impact. During the time of impact the force exerted by either mass is met by an equal and opposite force from the other mass, no matter what the velocity or mass of the bodies may be. Each body will therefore receive the same impulse, since equal forces are exerted during equal times. Consequently the change of momentum in each body will be the same in magnitude, but the direction will be opposite. Hence

$$m_1(v_1-u_1)=-m_2(v_2-u_2) \tag{133}$$

The negative sign may be placed before either one of the members of this equation, as it simply indicates that the quantities are oppositely directed. By rearrangement of terms of (133),

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2 \tag{134}$$

which shows that the sum of the momenta before impact is the same as that after impact, the addition being made algebraically.

In case the bodies are perfectly elastic, the sum of their kinetic energies will not be changed by the impact; hence

$$\tfrac{1}{2}(m_1v_1^2+m_2v_2^2)=\tfrac{1}{2}(m_1u_1^2+m_2u_2^2) \tag{135}$$

or

$$m_1(v_1^2-u_1^2)=m_2(u_2^2-v_2^2) \tag{136}$$

Dividing (136) by (133) and substituting the value of u_2 or u_1 thus found in (134), the velocity after impact, in case of perfectly elastic bodies, is found to be

$$u_1=\frac{2m_2v_2+v_1(m_1-m_2)}{m_1+m_2} \tag{137}$$

and

$$u_2=\frac{2m_1v_1+v_2(m_2-m_1)}{m_1+m_2} \tag{138}$$

There are no solids, however, that are *perfectly* elastic. Newton showed that the relative velocities of two spheres before and after impact bear to each other a constant ratio. The relative velocity before impact is $v_1 - v_2$; and after, $u_1 - u_2$. The constant ratio is

$$\frac{u_1 - u_2}{v_1 - v_2}$$

This quantity is a constant fraction called the *coefficient of restitution.* It is independent of the mass and also of the velocity, as long as the force of the impact does not produce permanent deformation.

The coefficient of restitution is usually denoted by e, and since the relative velocity before impact is opposite to that after, we may write

$$-\frac{u_1 - u_2}{v_1 - v_2} = e \tag{139}$$

or

$$\frac{u_2 - u_1}{v_1 - v_2} = e \tag{140}$$

Comparing (140) with (134) and solving for velocity after impact,

$$u_1 = \frac{m_1 v_1 + m_2 v_2 + m_2 e(v_2 - v_1)}{m_1 + m_2} \tag{141}$$

and

$$u_2 = \frac{m_1 v_1 + m_2 v_2 + m_1 e(v_1 - v_2)}{m_1 + m_2} \tag{142}$$

When the bodies are perfectly elastic, $e = 1$, and these values of u_1 and u_2 become the same as in (137) and (138).

In case of impact between a sphere and a fixed surface, as when a steel ball is dropped on an anvil, m_2 may be considered infinite and v_2 zero. Substituting these values in (141) and considering that $m_1 + \infty = \infty$, and also that a finite quantity divided by ∞ is zero, the equation becomes

$$u_1 = -ev_1 \tag{143}$$

This equation states that the velocity after impact is a certain fractional part of that before, and that the direction is opposite.

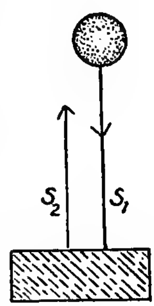


FIG. 115.

By the use of (143) the coefficient of restitution may be determined experimentally by noting the distance s_1, Fig. 115, through which a sphere of a given material is allowed to fall upon a fixed plane, and the distance s_2 through which it rebounds. Since $v=\sqrt{2gs}$, the velocity varies as the square root of the space, hence

$$e=\frac{u_1}{v_1}=\frac{\sqrt{s_2}}{\sqrt{s_1}} \tag{144}$$

85. Impact of Inelastic Bodies. — In case the bodies are inelastic, they will adhere to each other and will not rebound after impact. The bodies will then have the same velocity, which we will here denote by v_r, retaining v_1 and v_2 with the same meaning as in the paragraph above.

Since momenta are the same before and after impact whether the bodies are elastic or non-elastic,

$$m_1v_1+m_2v_2=(m_1+m_2)\,v_r \tag{145}$$

or
$$v_r=\frac{m_1v_1+m_2v_2}{m_1+m_2} \tag{146}$$

If m_2 is at rest at the time of the impact, $m_2v_2=0$, and (146) becomes

$$v_r=\frac{m_1v_1}{m_1+m_2} \tag{147}$$

or
$$v_1=\frac{m_1+m_2}{m_1}v_r \tag{148}$$

Equation (148) suggests a use for the **ballistic pendulum.** In Fig. 116, B is a block of wood suspended so that it may freely swing. This constitutes the pendulum. Suppose it is at rest and its mass is m_2. If a bullet is shot into the block, the bullet and block will move with velocity v_r. Hence, if the mass (m_1) of the bullet is known, its velocity may be calculated from (148) when v_r has been determined by experiment. If the apparatus is so constructed that the block will, when it swings, push a light rider i along the arc, the vertical height s through which it has been raised may be measured, and the velocity with which it started can be calculated by

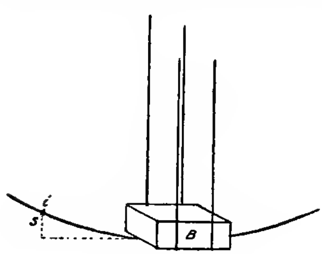


FIG. 116.

$$v_r=\sqrt{2gs}$$

Hence (148) may for this purpose be written

$$v_1 = \frac{m_1 + m_2}{m_1}\sqrt{2gs} \tag{149}$$

If in equation (145) the two masses are equal and m_2 is at rest at the time of the impact,

$$v_r = \frac{m_1 v_1}{2m_1} = \tfrac{1}{2}v_1 \tag{150}$$

This is as would be expected, for, since the total momentum is not changed by impact, then, if the mass is doubled, the velocity will be one-half as great.

Whether the bodies are elastic or inelastic, the total kinetic energy due to the motion of the masses is less after impact, for part of the energy is converted into molecular motion (heat).

Problems.

1. If the coefficient of rigidity of steel is $8.2(10)^{11}$ and Young's modulus is $22(10)^{11}$, what is the volume elasticity?

2. A right cylinder weighing 10 kg., radius 3 cm., is suspended from a wire. Its period of torsional vibration about its own axis is 5 sec. Find the moment of torsion of the wire.

3. A lead cylinder weighing 8 kg., 16 cm. in length and 7.5 cm. in diam., is suspended by a steel wire 126 cm. long and .72 mm. in diam. It is found that 10 periods of torsional vibration are executed in 1 min. 51.2 sec. Find the rigidity of the wire. (Lengths must be in centimetres, masses in grams, and time in seconds.)

4. A brass rod is 101 cm. long and its diameter is 6.39 mm. It is fastened at the ends, one end being attached to the axis of a wheel 14.8 cm. in diam. A weight of 1000 g. hung from the periphery of the wheel twists the end of the rod through 7°. Find the rigidity of the brass rod. (Angular displacement must be in radians.)

5. Find Young's modulus of a brass wire .1 cm. in diam. and 60 cm. long when a load of .2 kg. will stretch it .0149 cm.

6. Two spherical bodies are equal in mass and are assumed to be perfectly elastic. One is at rest and the other moves toward it in a line connecting the centres of the two spheres. Show by use of equations (137) and (138) that after impact the spheres will have exchanged velocities.

7. A marble, dropped from a height of 64 cm. upon a solid mass, rebounds to a height of 49 cm. What is the coefficient of restitution?

8. Two masses of lead move directly toward each other. One has a velocity of 500 cm/sec and a mass of 20 g., the other a velocity of 200 cm/sec and a mass of 15 g. What will be their velocity after impact?

9. If the suspended mass of a ballistic pendulum weighs 3.7 kg. and is raised 2 cm. by the impact of a bullet which weighs 120 g., what is the velocity of the bullet?

1. $23.1(10)^{11}$.
2. 71061 dyne-cm/radian.
3. $8.58(10)^{11}$.
4. $3.66(10)^{11}$.
5. $10(10)^{11}$.
6. ———.
7. .875.
8. 200 cm/sec.
9. 1993.08 cm/sec.

CHAPTER IV

GASES

86. Fluids. — The term *fluids* includes both liquids and gases. A distinguishing property of fluids is that they cannot sustain a shearing stress. They yield to the least force which tends to produce a change of shape. Both liquids and gases, however, resist any force tending to compress them into smaller volume.

As a result of the fact that a fluid cannot sustain a tangential or shearing stress, it follows that the forces exerted at any point in a fluid at rest are equal in all directions, for otherwise there would be motion. It also follows that the pressure of a fluid must be normal (perpendicular) to the walls of a containing vessel, for otherwise there would be an unbalanced component of force, as *ab* or *dc*, Fig. 117, which would cause motion of the liquid. For a similar reason, the free surface of a still liquid is horizontal,—*i.e.*, the surface is perpendicular to the direction of the force of gravity. Pascal's law and the principle of Archimedes are results of this general property of fluids.

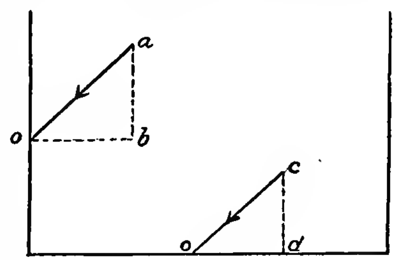


FIG. 117.

87. Character of a Gas.—The molecules of a gas appear to repel each other, and a gas will occupy the total interior of a containing vessel. If the vessel be enlarged, the same quantity of gas will continue to fill the vessel. When air, for example, is pumped from a receiver, the pump does not "draw" the air out, but only increases the space into which the air may expand at each upward stroke of the piston.

The theory that a repellant force exists between the molecules of a gas was entertained by many scientists even as late as the middle of the nineteenth century A.D. This is now replaced by the *kinetic theory*, by which the various phenomena of gases can be more satisfactorily explained.

88. The Kinetic Theory of Gases.—In accordance with the kinetic theory, the molecules of which a body of gas is composed are in rapid motion. There is no force of repulsion between them, but when they collide with one another or strike the walls of a containing vessel, they rebound according to the laws of impact of perfectly elastic bodies. The molecules, except during impact, move on in a straight line. Thus a gas expands as a result of the free motion of its molecules.

The pressure which a gas exerts upon the walls of a vessel is not due to the application of a steady force, as was once supposed, but to the impacts of countless numbers of molecules. When air, for example, is compressed, its pressure is increased because there are more molecules per unit volume to strike the walls of the containing vessel. The impulse which moves the piston of a steam engine is the sum of all the impulses received from the rapidly moving molecules of steam.

89. Pressure of a Gas.—To calculate the magnitude of the pressure of a gas on the sides of a vessel according to the kinetic theory, let a unit cube, Fig. 118, 1 c.c. in capacity, be filled with a gas. The molecules are regarded as having a varying velocity resulting from impacts with one another and against the sides of the vessel, but the mean velocity of a great number of them may be considered to be constant as long as the temperature and pressure are constant. In 1 c.c. of any gas at 15° C. and at standard pressure, there are about $3.6(10)^{19}$ molecules. Since the total energy of this great number of molecules is assumed to be constant, the average velocity will also be constant. This may be illustrated by the method of finding the average time of a human life. The average duration of life of a few hundred men, chosen at random, would not be reliable data for all. But if the lives of several million men are considered, the average for all may be determined with a fair degree of accuracy, such that, if each lived only during the average period, the death rate in the world would not be changed. In a similar manner we may determine the average velocity of the

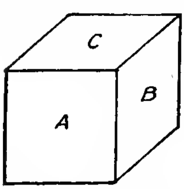


FIG. 118.

molecules of a body of gas such that if possessed by each molecule the effect would be the same as that which is observed. Let $\bar{v}$ represent this velocity, and suppose a single molecule of mass m and velocity $\bar{v}$ rebounds from side to side in a line perpendicular to the side A. The molecule is assumed to be perfectly elastic, and therefore it will rebound from the walls with the same velocity it had before impact. During the interval between impacts on the side A the molecule must move twice across the cube, hence the number of impacts per second on this side is $\frac{\bar{v}}{2}$. During each impact not only is the motion of the molecule completely stopped, but also an equal motion in the opposite direction is imparted. Consequently the total change of momentum resulting from each impact is $2m\bar{v}$, and the change per second is

$$2m\bar{v}\frac{\bar{v}}{2}=m\bar{v}^2$$

Let the whole number of molecules be n. One-third of them may be considered as moving in a direction perpendicular to A, another third perpendicular to B, and the remainder perpendicular to C. The motions are, in fact, in every direction, but may all be resolved into these three. Collisions may occur, but the effect on the sides of the vessel will be the same as if the molecules had passed on without affecting each other, for they are assumed to be perfectly elastic. Considering, then, that $\frac{n}{3}$ molecules strike $\frac{\bar{v}}{2}$ times per second against the side A, and the total change of momentum at each impact of a molecule is $2\ m\bar{v}$, the total change of momentum—*i.e.*, the total impulse on the side A—is

$$\frac{n}{3}\cdot\frac{\bar{v}}{2}\cdot 2m\bar{v}=\tfrac{1}{3}nm\bar{v}^2$$

This is the expression for the time rate of change of momentum.

The relation between impulse and momentum has been expressed by equation (42) as

$$Ft=mv$$

whence $$F=\frac{mv}{t}$$

which shows that force is measured by the time rate of change of momentum. Hence the force or pressure p exerted by the molecules on the side A is

$$p = \tfrac{1}{3} nm\overline{v}^2 \qquad (151)$$

But the product of the number (n) of molecules by the mass (m) of each is the total mass M of the gas. Hence (151) may be written

$$p = \tfrac{1}{3} M\overline{v}^2 \qquad (152)$$

Since M is the mass per unit volume, it is the density (ρ) of the gas, and (152) may be written

$$p = \tfrac{1}{3} \rho\overline{v}^2 \qquad (153)$$

or

$$\overline{v} = \sqrt{\frac{3p}{\rho}} \qquad (154)$$

By use of (154) it is possible to calculate the average velocity of the molecules of a gas when the pressure and density are known. Thus, for example, 1 c.c. of hydrogen, at 0° C. and under standard atmospheric pressure of $1.013(10)^6$ dynes per square centimetre, has a mass of $8.96(10)^{-5}$ g. Hence

$$\overline{v}^2 = \frac{3 \times 1.013(10)^6}{8.96(10)^{-5}}$$

or

$$\overline{v} = 184{,}100 \ {}^{\text{cm}}/_{\text{sec}}$$

90. Avogadro's Law. — According to this important law, announced early in the nineteenth century by an Italian named Avogadro, *equal volumes of gases under the same conditions of temperature and pressure contain the same number of molecules.* The experimental facts of chemistry lead up to an establishment of this law, for the ratio of the densities of two gases under the same conditions of temperature and pressure is the same as the ratio of their combining equivalents or molecular weights. It follows that the number of molecules in two equal volumes is the same, and their difference in density results from a difference in the weights of the individual molecules.

Proof of Avogadro's law, based on the kinetic theory of gases, may be given as follows:

Consider a cubic centimetre of each of two gases under the same conditions of temperature and pressure. Let them be designated as 1 and 2. Since their pressures are equal, then, from (151),

$$\tfrac{1}{3}n_1m_1\overline{v}_1^2=\tfrac{1}{3}n_2m_2\overline{v}_2^2 \tag{155}$$

Since the gases are at the same temperature, the average kinetic energies of the individual molecules are equal; hence

$$\tfrac{1}{2}m_1\overline{v}_1^2=\tfrac{1}{2}m_2\overline{v}_2^2 \tag{156}$$

A comparison of (155) and (156) shows that

$$n_1=n_2$$

This fact, in turn, serves as a basis for the determination of **molecular weights** in chemistry. Molecular weight, so called, is not an absolute weight, but a ratio. Thus, if the molecular weight of hydrogen is assumed to be 2, that of oxygen is 32. If, then, the density of any gas is determined, its molecular weight may be found by comparison with the density of another gas of known molecular weight, assuming that the number of molecules in the two gases is the same.

91. Dalton's Law.—Many phenomena of gases indicate that the distance between their molecules is much greater than the diameter of the molecules themselves,—*i.e.*, the space actually occupied by the molecules is small as compared to the volume of the gas. On this assumption it would be expected that different gases would readily mingle with each other, apparently occupying the same space at the same time. If some ether or other liquid that will readily evaporate be introduced into a closed vessel already filled with air, just as much of the liquid will evaporate as when no air is present. If a quantity of other liquid, alcohol say, be now introduced, it will evaporate in the presence of both air and ether just as it would do in a vacuum, except that the rate of evaporation is slower when other gases or vapors are present. A gas or vapor acts as a vacuum to another gas. Such facts show that there is between the molecules of a gas ample room for molecules of another gas.

It has been shown above that the pressure of a gas is due to the activity of the molecules. It is a natural inference, then, that if two or more gases are enclosed in the same vessel, the molecules of each would continue their motion, and their impacts against the sides of the vessel, either direct or indirect, would not be changed. In case of collision with one another there would be the exchange of velocity which results from the impact of perfectly elastic bodies, so that the result will be the same as if the impact had been made directly upon the sides of the vessel. Consequently the pressure would be the sum of the pressures which each would exert if it occupied the space alone.

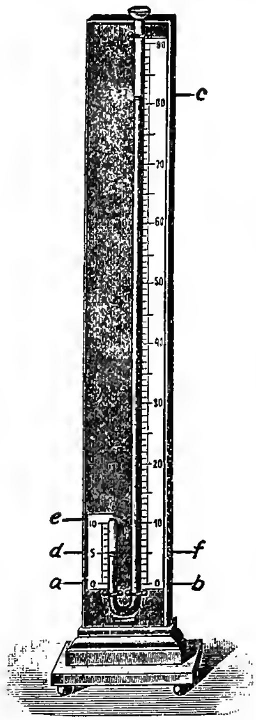


Fig. 119.

Dalton was the first to investigate this subject, and the result of his experiments may be stated as follows: *The quantity of a liquid which will evaporate into a given space is the same, for the same temperature, whether the space is a vacuum or is already filled by a gas, and the pressure exerted by a mixture of two or more gases or vapors is the sum of the pressures which each would exert if it occupied the space alone.*

92. Boyle's Law. — The relations of pressure, volume, and density of a gas were first systematically investigated by Robert Boyle about the middle of the seventeenth century A. D. By use of a glass tube of the form shown in Fig. 119, a quantity of gas enclosed in *ae* may be subjected to various pressures by pouring mercury into the long arm *bc*. Let a small amount of mercury be first introduced and adjusted so that it stands in each arm at the level *ab*. A certain volume of gas is thus entrapped in *ae* and its pressure on the mercury at *a* must be equal to that in the other arm at *b*. If, now, mercury is poured into the long arm until it stands at *c*, it will also rise in the short arm to some point, *d*,—*i.e.*, the gas will be compressed to the volume *de*. The original pressure on the gas was the pressure of the atmosphere, but now there is an additional pressure of the column of mercury *fc*. By varying the pressure in this manner and noting the corresponding volume in each case, Boyle was able to announce that *the product of the pressure and volume of a gas is a constant quantity if the temperature is constant.* This law was announced several years later by Mariotte, and it is known on the continent of Europe as Mariotte's law.

If P is pressure and V is volume, then

$$PV = k \tag{157}$$

where k is a constant.

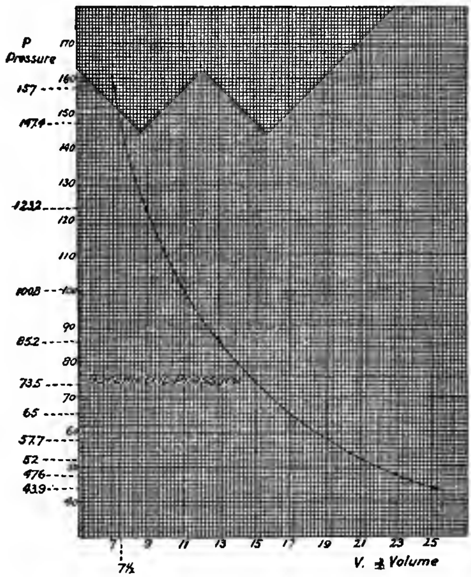


FIG. 120.

The same result may also be deduced from the kinetic theory of gases. Thus, it has been shown that

$$P = \tfrac{1}{3}\rho\overline{v^2}$$

or, since density is equal to mass divided by volume,

$$PV = \tfrac{1}{3}M\overline{v^2} \tag{158}$$

The second member of this equation is a constant quantity if $\overline{v^2}$ is constant. This is the case when the temperature does not change. Thus Boyle's law may be derived from a theoretical consideration alone.

The data obtained from an experiment such as that described above may be conveniently recorded on co-ordinate paper as shown in Fig. 120, the pressures being the ordinates and the volumes the abscissæ. This may be called the *PV* diagram. At all points of the curve thus plotted, *PV* is the same or varies only slightly due to errors of experiment. This constant product is a distinguishing property of a curve called the equilateral hyperbola. The data in the figure were obtained from an experiment when the barometric pressure was 73.5 cm. of mercury and the volume of air at that pressure was 15 c.c.

More elaborate experiments than those of Boyle were made later by other investigators, principally Regnault and Amagat, who subjected gases to much greater pressure. They showed that hydrogen was less compressible than Boyle's law would require, for the product *PV* became larger and larger as the pressure increased. Amagat experimented with other gases also, and found that, although at moderate pressures the gases were more compressible than Boyle's law would require, at high pressure they, like hydrogen, became less and less compressible as the pressure increased,—*i.e.*, a greater pressure was necessary to produce the same diminution of volume.

This is what would be expected from the kinetic theory of gases. When the gas is subjected to a moderate pressure, the molecules occupy but little of the space in which the gas is enclosed. Their mean free path is limited, practically, only by the sides of the vessel. If, however, the pressure is greatly increased, so that the space occupied by the molecules becomes an appreciable part of the volume of the gas, the mean free path is diminished. Since the molecules are assumed to have sensible dimensions, then when there is an impact of *a* and *b*, for example, against the wall of the vessel, Fig. 121, the centre of the molecules would not become coincident with the wall by nearly the radius of the molecules. Consequently as the pressure is increased the number of impacts on the sides increases at a greater rate than the increase in the number of molecules per unit volume, for the molecules are assumed to remain of the

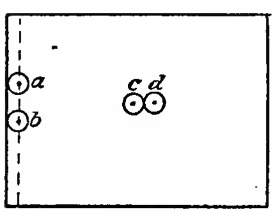


Fig. 121.

same size and their radii bear a greater ratio to a short distance than to a longer one. Also, when the molecules are crowded by pressure they will come into more frequent collision with one another, and their mean free path will be less for this reason also, for the centres of c and d, for example, Fig. 121, do not become coincident on collision, but rebound from each other so that the space cd is not traversed by either one. This decrease in distance causes a more frequent reversal of momentum and consequently a greater pressure.

The fact that at moderate pressures the product of pressure by volume is less than Boyle's law requires, is probably due to the attractive force between the molecules,—a force which is not negligible in comparison with the impacts of the molecules at that stage of compression. It may be shown that this force would vary inversely as the square of the volume. Its effect is to decrease the pressure due to the impacts of the molecules.

93. Equation of Van der Waals.—An equation which expresses more accurately the relation of volume to pressure of gases is

$$\left(p+\frac{a}{v^2}\right)(v-b)=\text{constant (temp. const.)} \qquad (159)$$

This is known as Van der Waal's equation. The value of b, a constant, depends on the size of the molecule, while the value of a, another constant, depends on the attractive force between the molecules. For carbon dioxide, for example, Van der Waals gives $a=.00874$ and $b=.0023$ for a specified mass and temperature.

It has been shown above that the virtual decrease in the volume due to the fact that molecules have sensible dimensions increases the pressure, hence b is subtracted from v to obtain a value which will hold true for v in Boyle's equation. Likewise, since the intermolecular attraction depends on the mutual force between the attracting and attracted molecules, the force varies as the square of the number of molecules,—*i.e.*, as the square of the density or inversely as the square of the volume. Expressed in formula, $F \propto \frac{1}{v^2}$ or $F=\frac{a}{v^2}$ where a is a constant.

Since this force decreases the pressure, $\frac{a}{v^2}$ must be added to p.

If in equation (159) v becomes very large,—*i.e.*, if the gas becomes very rare,—$\frac{a}{v^2}$ becomes negligible in comparison with p, and b has a similar relation to v. When a gas is in this condition Boyle's law may be assumed to express the correct relation of p and v.

94. Elasticity of Gases.—As already explained, a gas can have only *volume* elasticity, expressed as a ratio of the *stress* to the strain per unit volume,—that is,

$$\frac{p}{\frac{v}{V}} = \text{constant}, k$$

where p is the *change* of pressure which caused the volume V to decrease by a volume v. Let the change of pressure and the consequent change of volume be very small. P will then become $P+p$ and V will be $V-v$. According to Boyle's law,

$$PV = (P+p)(V-v)$$

or

$$PV = PV - Pv + pV - pv$$

Since p and v are very small, their product may be neglected, and therefore

$$P = \frac{pV}{v} = \frac{p}{\frac{v}{V}} = k \tag{160}$$

It appears therefore that the coefficient of volume elasticity of a gas is equal to the pressure to which the gas is at any time subjected. This may be roughly verified by substituting in (160) values given in Fig. 120, keeping in mind that p is *change* of pressure. This coefficient is constant for the same pressure, but is different at different pressures. The greater the pressure, the greater the elasticity. If the added pressure p is very small, the original pressure is practically the total pressure,—*i.e.*, the pressure which just equals the counter-pressure of the gas. This may be called *isothermal elasticity*, since the temperature is assumed to be constant during the change of volume.

If the gas is suddenly compressed so that heat is developed, the product of pressure by volume is increased and so likewise the elasticity. An opposite effect will be obtained if the gas is suddenly expanded. The discussion of elasticity under these conditions is deferred to the chapter on heat (§ 177).

95. Pressure of the Atmosphere.—After the invention of the air-pump by Von Guericke, it was demonstrated by experiment that the atmosphere exerts great pressure on bodies at the surface of the earth. Air is matter, and so it is subject to gravitational forces which bind it as a great gaseous envelope on the earth.

The weight of a small mass of air causes very little pressure on the bottom of a vessel in which the air is confined, but a great number of such masses piled one on top of another to a height of several miles would press at the bottom with a force equal to the weight of all.

One cubic centimetre of air at 0° C. and at sea-level atmospheric pressure in latitude 45° weighs .001293 g., or one litre weighs 1.293 g. If the air were of the same density at all altitudes, the pressure at the surface of the earth would be the weight of 1 c.c. times the height of the atmosphere in centimetres. But since air is very compressible, it is most dense in the lower layers and becomes more and more rare in the upper regions.

The pressure of the atmosphere at any point may be found by balancing it against a column of mercury. If a glass tube A, Fig. 122, about 80 cm. long and closed at one end, is filled with mercury and then inverted with its open end in a pool of the same liquid, the mercury will stand at a certain height h above the surface in the vessel. The pressure at the bottom of this column is

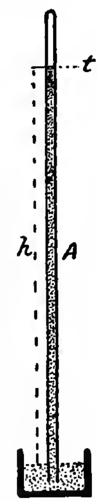


Fig. 122.

$$p = \rho g h$$

where ρ is the density and g is the force of gravity in dynes per gram. This pressure is in equilibrium with atmospheric pressure, and so is a measure of that pressure.

The proper **unit of pressure** is 1 dyne per cm.^{2}, but it is often more convenient to employ as a unit the weight of 1 c.c. of mercury at 0° C. in latitude 45°, where the value of g is 980.6. The mass of this cube is 13.596 g., hence

$$p = 13.596 \times 980.6 \times 1 = 13332.24 \text{ dynes}$$

Pressure is frequently indicated by simply naming the height of a column of mercury. Thus, a pressure of 73 cm. of mercury means that the pressure per square centimetre is the weight of 73 c.c. of mercury. This is true, no matter what the actual cross section of the column may be.

Another unit often employed is **one atmosphere,** which is the pressure at the bottom of a column of mercury 76 cm. high. Expressed in dynes, this unit is

$$76 \times 13{,}332.24 = 1{,}013{,}250 \text{ dynes}$$

For most purposes it is a sufficient approximation to say

$$\text{one atmosphere} = 1.013(10)^6 \text{ dynes}$$

A pressure of 75 cm. of mercury is almost exactly $(10)^6$ dynes, or one megadyne. This is sometimes used as a unit of pressure and is called the *barie.*

96. The Barometer.—A barometer is an instrument used to indicate atmospheric pressure. The principle on which it operates has been explained in the preceding section. One of its standard forms is illustrated in Fig. 123. A glass tube, t, is filled with mercury and inverted with its open end in a cistern partly filled with the same liquid. The upper part of the cistern is a glass cylinder, but the bottom is a leather bag N. The graduations at the top of the tube show the correct height of the column of mercury only when the mercury in the cistern is at the zero level indicated by the "ivory point" h. When air pressure increases, mercury will be forced into the tube and consequently the surface in the cistern will be below the zero level, but when pressure decreases mercury will run back into the cistern and the surface will rise above the zero level. For this reason it is necessary, before reading the barometric height, to raise or lower the level in the cistern to the zero point. For this purpose a screw, o, is provided, by means of which the bottom of the leather bag may be raised or lowered until the point h just touches the surface of the mercury.

The glass tube is enclosed in a tube of brass upon which are placed the graduations in millimetres or fractions of an inch, or both. For convenience and accuracy of reading, a sliding vernier is provided, as shown in Fig. 124. The lower edge of the vernier is moved down to the top of the meniscus of mercury, and, to avoid error of parallax, a screen at the back of the tube is fixed to and moves with the vernier. The proper position for reading is reached when the top of the meniscus and the lower edge of both screen and vernier are in the same plane. The scale is then read up to the zero on the vernier, in a common form of which, as shown in the figure, 25 divisions are equal in length to

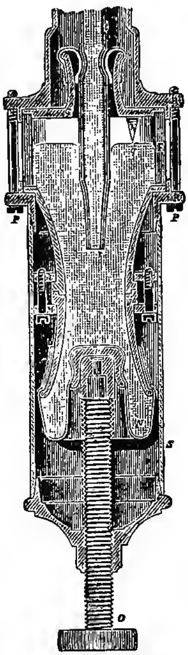


FIG. 123.

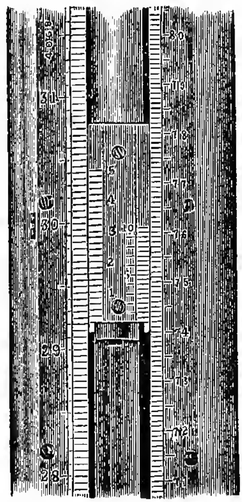


FIG. 124.

24 on the scale in English units, while 20 vernier divisions cover 19 mm. in metric measure. One inch on the scale is divided into 20 equal parts, hence each division on the vernier is $\frac{1}{25}$ of $\frac{1}{20}$ or .002 inch shorter than one of the scale divisions,—*i.e.*, the least count is .002 inch. The figures 1, 2, 3, 4, and 5 on the vernier, therefore, show the number of hundredths of an inch to be added to the scale reading. On the metric side the least count is $\frac{1}{20}$ or .05 mm. The height of the mercurial column as shown in the figure is 74.18 cm. or 29.212 in.

97. Corrections of Barometric Readings.—To reduce the reading of a barometer to standard conditions, several corrections must be made, as follows:

1. In most instruments the scale is made on the brass tube. If this scale is correct at 0° C., its reading at $t°$ will be too low, for each millimetre space has expanded and so the number of such spaces equal to the height of the mercury column is not so great as at 0° C. If, for sake of illustration, each millimetre should expand to twice its true length, there would need be only half the number to cover the same space, and the true height would be twice the reading. Such an extreme expansion, of course, cannot occur, but the coefficient of linear expansion of brass is .0000178,—*i.e.*, brass will expand .0000178 of its length when its temperature is raised 1° C. The length of each millimetre at $t°$ will then be $1+.0000178\,t°$, and the number of these false units decreases, for any given length, in the same proportion as the length of each increases. Hence the ratio of the observed to the true height is equal to the ratio of the true to the false scale division. This may be expressed by

$$\frac{h_t}{h_0}=\frac{1}{1+.0000178\,t°}$$

or

$$h_0=h_t(1+.0000178\,t°) \qquad (161)$$

where h_t is the observed height and h_0 is the height at 0° C. Thus the correct reading as far as the brass scale is concerned may be obtained.

2. When the temperature rises, the density of the mercury becomes less. Consequently the column of mercury must rise to a greater height to balance the same atmospheric pressure. The height of the column at 0° C. is less than at any higher temperature for the same pressure. The coefficient of cubical expansion of mercury is .0001818,—*i.e.*, the volume of 1 c.c. of mercury at $t°$ is $1+.0001818\,t°$ cubic centimetres. Since the height of the column varies inversely as the density, and the density varies inversely as the volume,

$$\frac{h_0}{H_0}=\frac{\rho_H}{\rho_h}=\frac{1+.0001818\,t°}{1}$$

where h_0 is the height as measured by the corrected brass scale, H_0 is the correct height when the temperature of the mercury is 0° C., ρ_H is the density at 0° C., and ρ_h the density at temperature $t°$ From this relation,

$$H_0 = \frac{h_0}{1 + .0001818\,t°} \tag{162}$$

But the value of $\frac{1}{1 + .0001818\,t°}$ is $1 - .0001818t° + (.0001818t°)^2 - (.0001818t°)^3$, etc. Since the coefficient is very small, all powers above the first may, without sensible error, be neglected, and we may write

$$H_0 = h_0(1 - .0001818\,t°) \tag{163}$$

By substituting the value of h_0 as found above,

$$H_0 = h_t(1 + .0000178\,t°)(1 - .0001818\,t°) \tag{164}$$

Thus by use of equation (164) correction is made for temperature effects in both the brass scale and the mercury. It is not necessary to consider the expansion of the glass tube, for pressure depends only on the height of the mercury, being independent of the area of cross section.

3. The barometric height may also be reduced to a corresponding height for the same pressure at sea level in latitude 45°, where g is 980.6. At any point of observation where the value of g is greater, the height of the column will be less; if g is less, the height will be greater. Hence to reduce any observed reading to that at sea level, latitude 45°, it is only necessary to take the fractional part expressed by $\frac{g}{g_{45}}$, the numerator being the value of g at the point of observation, and the denominator the value at sea level in latitude 45°. Equation (164) then becomes

$$H_0 = \frac{h_t g}{g_{45}}(1 + .0000178\,t°)(1 - .0001818\,t°) \tag{165}$$

The value of g for various points on the surface of the earth is given in the tables, or a fairly accurate value can be found from

$$g = 978(1 + .00531\,\sin^2 L) \tag{166}$$

where L is the latitude of the place.

4. Correction should also be made for depression resulting from capillarity. When the diameter of the column is about 2.5 cm., the depression is negligible, but for smaller tubes a correction should be made. This can best be done by comparison with some standard instrument or by reference to tables of correction provided for this purpose.

98. Glycerin Barometer.—Any liquid may be used in the construction of a barometer, but, all things considered, mercury is the best. A water barometer would not be satisfactory, because it would need to be more than 34 feet high (13.6×30 in.), and water vapor would fill the vacuum at the top of the tube, thus causing a depression of the column which would vary with every change of temperature. An excellent barometer, however, can be made by use of glycerin. The density of this liquid is 1.28 $^{g}/_{cc}$; hence, when the mercury column is 30 inches high, the glycerin would be raised by the same pressure to a height of 318.5 inches. For this instrument, then, the tube must be nearly 27 feet long. Its construction is possible where the lower part of the tube may extend into a room or basement below, only the upper end being exposed in the room where the readings are to be made. The great advantage in the use of this instrument is that, since its height is about 10.5 times as great as that of mercury, its variation in height for change of pressure is also 10.5 times as great. A change of 1 cm. in the height of mercury corresponds to a change of 10.5 cm. in the height of glycerin. Glycerin does not evaporate, and so the Torricellian vacuum at the top of the tube is maintained.

99. The Aneroid Barometer.—The *aneroid* barometer is so called because no liquid is used in its construction. It consists of a cylindrical metal box of German silver, Fig. 125, with a flexible, corrugated top. The air is nearly all pumped from the box, and the collapse of its sides is prevented by a stiff spring, R, attached to the central post on the top. Any increase in the pressure of the air will cause a further depression of the top, while a decrease will permit the top to spring out. In either case the movement will be such as to restore equilibrium between the air pressure on one hand and the elastic force of the top and spring on the other. This movement is multiplied by means of a system of levers, and is communicated to a hand which is made to move over a graduated dial. The chief advantages of the aneroid are portability and sensitiveness. If carefully calibrated by comparison with a mercurial barometer, it will record variations in atmospheric pressure with a fair degree of accuracy.

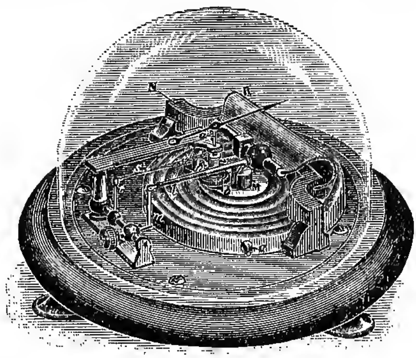


FIG. 125.

100. Mechanical Air-pumps.—A mechanical air-pump in its simplest form does not differ in principle from the common lifting pump used in raising water. A receiver, R, Fig. 126, fits air-tight on a plate, P. A tube leads from the interior of the receiver to the base of the pump. There are two valves, a and b, both of which open upward. By raising the piston the air pressure is removed from the top of the valve a and the impacts of the molecules beneath cause it to open. A portion of the air from the receiver thus passes into the space o. When the piston is pushed down, a is closed by the excess of molecular impacts from above, and b is opened by an excess of impacts from below. This operation is repeated in each successive stroke of the piston, a certain constant fractional part of the air remaining in the receiver being removed by each cycle of operations. Let the volume of the receiver, including the connecting tube, be V, and the volume of the cylinder between the limits of motion of the piston be v. When the piston is raised in the first stroke, the air in the receiver expands to a volume $V+v$. Now, when the piston is pushed

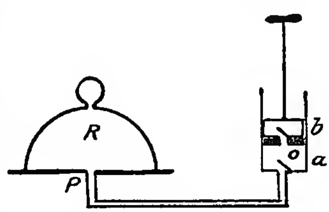


FIG. 126.

down, a certain volume v of the air which had expanded to $V+v$ escapes. Consequently the fractional part of the total mass, m, removed by the first stroke is

$$\frac{v}{V+v}m$$

The mass remaining after the first complete stroke is, therefore,

$$m-\frac{v}{V+v}m=\frac{V}{V+v}m$$

The sum of these is seen to be the original mass. By a second cycle of operations the same fractional part of the remaining air will be removed,—*i.e.*,

$$\frac{v}{V+v}\cdot\frac{V}{V+v}m=\frac{Vv}{(V+v)^2}m$$

and the mass remaining after the second cycle is

$$\frac{V}{V+v}m-\frac{Vv}{(V+v)^2}m=\frac{V^2}{(V+v)^2}m$$

In a similar manner it may be shown that the mass remaining after the third cycle is

$$\frac{V^3}{(V+v)^3}m$$

and so on for any number of strokes, the exponent of $\frac{V}{V+v}$ always being that number. When the number of strokes is n, the mass of air remaining in the cylinder is

$$\left(\frac{V}{V+v}\right)^n m$$

Since this coefficient of m expresses the fractional part of the original mass occupying the volume V, it also expresses the fractional part of the original density and pressure. If the operation of this pump were exactly as here assumed, any degree of vacuum could be obtained by increasing the number of strokes. But, for mechanical reasons, a limit is reached at the end of a few strokes. It is difficult to prevent leaks between the piston and the walls of the cylinder, and after a certain degree of exhaustion has been reached, the pressure of air in the receiver is not sufficient to raise the valve a.

An improved mechanical pump, known as the "geryck" or Fleuss pump, is free from many of the defects of the ordinary pump. Its construction is illustrated in Fig. 127. A pipe, *A*, leads to the vessel which is to be exhausted. The air passes into the annular space *B* and thence, without any obstruction, through the port *p* into the space above the piston. A leather bucket, *C*,*C*, forms part of the piston and is held against the sides of the cylinder by the pressure of the air and oil above. The piston valve *E* operates only during the first stages of exhaustion, and when the vacuum is less than about 1.3 cm. it becomes inactive. A pipe, *F*, leads from the annular space *B* to the bottom of the cylinder, the purpose of which is to prevent a great difference of pressure between the upper and lower sides of the piston at the beginning of the ascent during the first few strokes. Otherwise there would be a vacuum below the piston and full atmospheric pressure above. The air can pass freely from *B* into the cylinder, there being no valves to be pushed open. When the piston is raised, the port *p* is closed, and all air thus entrapped in the cylinder is carried up through the valve *G* into the upper chamber of the cylinder. The valve *G* is held on its seat by a spring *K*. When the piston reaches the upper end of its stroke, it raises the valve *G* and holds it open while all the air below passes through, the two bodies of oil meanwhile becoming one. Thus, it is not possible for any

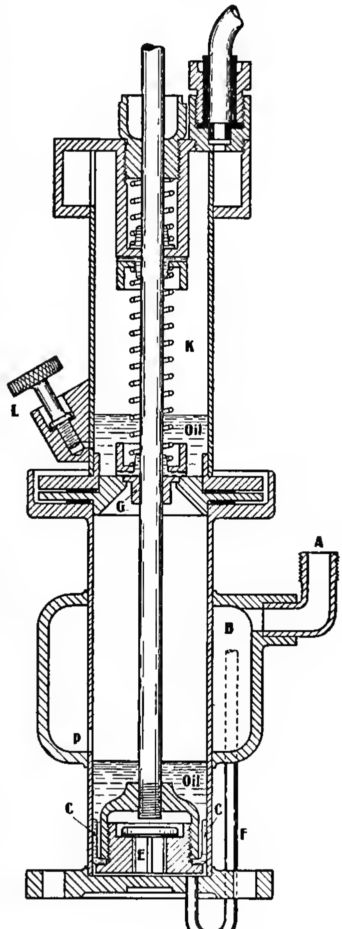


FIG. 127.

air to pass the piston. If oil leaks through at the valve or between the piston and cylinder, it is picked up during the next stroke of the piston.

With this style of pump a very good vacuum may be obtained. When two are joined in series, as shown in Fig. 128, and the air is made to pass through a drying tube filled with phosphoric pentoxide before it enters the pump, a vacuum of about $\frac{1}{5000}$ mm.

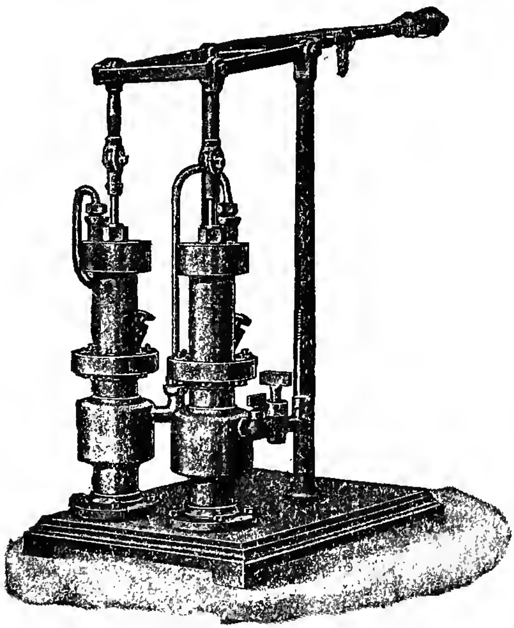


FIG. 128.

may be obtained. The rapidity with which a vacuum may be produced gives this pump a great advantage over mercury pumps for many purposes.

101. Mercury Air-pump.—If the space at the top of a barometer tube is made to include the vessel which is to be exhausted of air, a very good vacuum can be obtained. A common form of pump constructed on this principle, and known as the Sprengel pump, is illustrated in Fig. 129. Mercury from the reservoir *F* is made to fall drop by drop into the top of the tube *T*. The vessel to be exhausted is attached at *A*. Each drop of mercury carries before it a quantity of air which escapes at o. As the exhaustion progresses the quantity of air between the drops becomes less and less until there will be a continuous column of mercury in T, equal to the height of the barometric column. The vacuum in C and also in the vessel attached to A will then be a Torricellian vacuum, so called, at the top of a barometer. The loop B is longer than the height of a barometric column, so that no air can enter C through the tube leading from the reservoir even when all the mercury has run out of F.

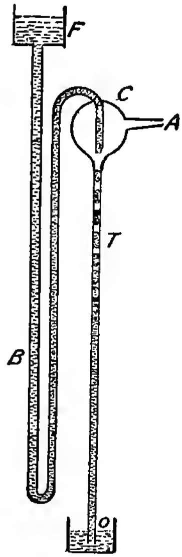


FIG. 129.

102. Diffusion of Gases.—If two or more gases are enclosed in the same vessel, each gas will in a short time be distributed to all parts of the vessel, just as if the other gases were not present,—*i.e.*, there will be a uniform mixture of all the gases. This process is called *diffusion*. The rate of diffusion may be deduced from a consideration of the velocity with which a gas will issue from a small orifice in the side of a vessel. Let a body of gas, G, Fig. 130, do work by expanding and thus exerting pressure on the piston o. Let the pressure per unit area be p and the area of the piston A. Let the piston be moved by the expanding gas through the distance x. The work done is expressed by

$$W = pAx$$

W being the work and pA the total pressure or force. The product Ax is the change of volume, hence the work is equal to the product of pressure by change of volume. If, now, instead of moving the piston, the same increase of volume is produced by allowing the gas to escape from a small orifice, work will be done in giving kinetic energy to a stream of gas. The mass of the issuing stream is $V\rho$, where V is the volume, equal to Ax above, and ρ is the density. Hence the kinetic energy is

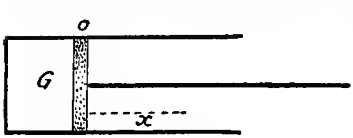


FIG. 130.

$$E_k = \tfrac{1}{2} V\rho v^2$$

where v is the velocity of the stream. Hence

$$\tfrac{1}{2}V\rho v^2 = pAx$$

or
$$v = \frac{\sqrt{2p}}{\sqrt{\rho}} \qquad (167)$$

From this it is seen that the velocity of the gas as it issues from the orifice varies inversely as the square root of the density of the gas.

This deduction may be shown to be consistent with the kinetic theory of gases, for, as already shown, there will be $\frac{\bar{v}}{2}$ impacts of each molecule upon one of the walls of a unit cube in which the gas is confined. If n is the total number of molecules and one-third of them move in each of the three possible directions,

$$\frac{n}{3}\,\frac{\bar{v}}{2} = \frac{n\bar{v}}{6}$$

is the total number of impacts upon one side of the vessel, $\bar{v}$ being the average velocity of the molecules. Now, if a small hole of area s is made in this side of the vessel, the number of molecules that formerly formed impacts against that portion of the side will now issue from the hole. The rate at which the gas will escape is, then,

$$\frac{n\bar{v}s}{6}$$

The only variable quantity in this expression is $\bar{v}$, n being, according to Avogadro's law, the same for all gases under the same conditions. Consequently the rate of escape of the gas is proportional to the average velocity of the molecules. But it has been shown in equation (154) that

$$\bar{v} = \frac{\sqrt{3p}}{\sqrt{\rho}}$$

—*i.e.*, the rate of escape of the gas is inversely proportional to the square root of the density.

To illustrate this principle, let a vessel, Fig. 131, be divided into two compartments by a porous partition, C, which may be made of unglazed porcelain or plaster of Paris. A manometer,

T, partly filled with liquid passes through a cork into the chamber V. If a stream of hydrogen is made to flow through the bottom of the bottle into the space O, it will diffuse through the partition into V more rapidly than the gas there will pass in the opposite direction, and the increased pressure in V will be shown by the manometer. If V is filled with oxygen and O with hydrogen, the rate of diffusion toward V will be four times as great as from V to O. If V is filled with hydrogen, the manometer will show decrease of pressure. In a short time, however, the liquid will stand at the same height in each arm of the manometer, showing that diffusion is complete.

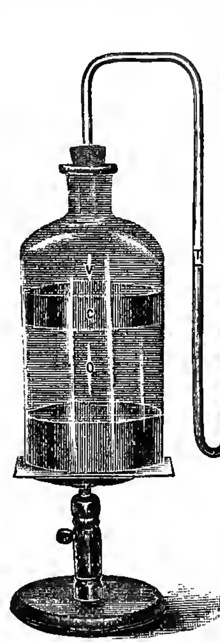


FIG. 131.

The atmosphere is a mechanical mixture of several gases of different density, but each gas is uniformly distributed by diffusion.

103. Buoyancy of Air.—According to the well-known principle of Archimedes, a body is buoyed up by a force equal to the weight of the fluid which a body displaces. When the body is dense, as iron or copper, the effect of buoyancy of air is not always apparent; but a light body of large volume, as a balloon inflated with hydrogen, will be buoyed up by a force greater than the weight of the body. The force of buoyancy is just as great in case of a solid mass of iron as large as the balloon, but it is not so apparent.

For accurate weighing allowance must be made for buoyancy of air. If a body is placed in one of the pans of a balance and is counterpoised by a known weight on the other pan, both are buoyed up by a force equal to the weight of the air which they displace. The true weight (weight in vacuum), less the weight of air displaced, is the apparent weight. Let x be the true weight of the body and ρ its density. Also let w represent the standard weight in the other pan, and ρ_1 its density. Let the density of air be a. If the body had the same density as air, its apparent weight in air would be zero. If twice as dense as air, its apparent weight would be $\frac{a}{2a}x$. Whatever the density ρ may be, the loss of weight is $\frac{a}{\rho}x$. Likewise the standard weight will lose $\frac{a}{\rho_1}w$. Since the apparent weights in the pans are balanced,

$$x - \frac{a}{\rho}x = w - \frac{a}{\rho_1}w$$

or

$$x\left(1 - \frac{a}{\rho}\right) = w\left(1 - \frac{a}{\rho_1}\right)$$

If both sides of this equation be divided by $1 - \frac{a}{\rho}$,

$$x = w\left(1 + \frac{a}{\rho} - \frac{a}{\rho_1} + \frac{a^2}{\rho^2} \text{ etc.}\right)$$

or

$$x = w + wa\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \text{nearly} \qquad (168)$$

for the square or higher powers of $\frac{a}{\rho}$ and $\frac{a}{\rho_1}$ may be considered negligible quantities.

To illustrate the use of this equation, suppose a mass x of aluminum is balanced in air by a brass weight marked 500 g. Then

$$x = 500 + 500 \times .001293\left(\frac{1}{2.6} - \frac{1}{8.5}\right)$$

or

$$x = 500.174 \text{ g.}$$

—*i.e.*, the weight in vacuum is .174 g. more than the weight in air.

If the density of the object is the same as that of the standard weight,

$$wa\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = 0$$

hence

$$x = w$$

and no correction is necessary.

If the density of the object is greater than that of the standard weight, the correction must be subtracted from w, as equation (168) shows.

Problems.

1. If the average velocity of the hydrogen molecules at a pressure of $1.013(10)^6$ dynes and at 0° C. is 184,100 $^{cm}/_{sec}$, what is the average velocity of oxygen molecules under the same conditions?

2. If the atmosphere were all as dense as it is at sea-level, latitude 45°, what would be its height where the pressure is 76 cm. of mercury?

3. Calculate the value of g in the latitude where you live. (Use (166).)

4. A mercurial barometer is inclined 10° to the vertical. The reading is 73.2 cm. What would be the reading in a vertical position?

5. What part of the mass of air remains in a receiver after five cycles of the piston of an air-pump, the ratio of the capacities of the pump and receiver being $\frac{1}{4}$?

6. A sphere 10 cm. in diameter weighs 523.6 g. in air of density .0012 $^{g}/_{cc}$. What would the sphere weigh in vacuum, the standard weights being brass?

7. A glass tube used in sounding is 24 inches long and is covered on the inner walls with a brown pigment which becomes white when in contact with sea water. The tube, open end down, is lowered in water in which the pressure at a depth of 33 feet is equal to one atmosphere. On raising the sounder it is found that the pigment is white for a distance of 18 inches from the open end. What is the depth of the sea water?

1. 46025 $^{cm}/_{sec}$.
2. 5 miles, approx.
3. ———.
4. 72.08.
5. .328, approx.
6. 524.155.
7. 99 ft. of water.

CHAPTER V

LIQUIDS

104. Liquid Pressure.—One cubic centimetre of pure water at its greatest density, 4° C., weighs almost exactly one gram. Water, like other liquids, is almost incompressible, hence the pressure at any given depth, h, measured in centimetres, is hg dynes per square centimetre. For any other liquid of density ρ the pressure at a depth h is ρgh. It has been shown that at any point in a fluid at rest the pressure is equal in all directions, and in liquids the pressure at any point may be taken as proportional to the depth. The pressure here considered is only that due to the weight of the liquid.

105. Transmission of Pressure.—*The pressure per unit area exerted on a fluid enclosed in a vessel is transmitted to every equal unit area on the interior of the vessel.* This principle is known as Pascal's law. It is a direct consequence of the fact that fluids do not resist a shearing stress. Let C be a cylinder filled with fluid and let a and A be the respective areas of the pistons as shown in Fig. 132. If the small piston is thrust against the fluid with a pressure p per unit area, every equal unit area within the cylinder will be subjected to the same increase of pressure. Any area s within the body of the fluid is subjected to an increased pressure ps. The total pressure of the small piston is pa, and that on the large piston pA. Thus, whatever pressure is exerted at o is multiplied $\frac{A}{a}$ times at R, for

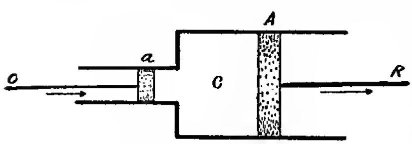


FIG. 132.

$$pa \cdot \frac{A}{a} = pA$$

The mechanical advantage in this case is the ratio of the areas of the pistons. This is the principle of the **hydrostatic press.** By making A large, the total pressure may be enormously increased, but the distance moved is less in proportion as the area of A is greater. Hence the hydrostatic press conforms to the general law for machines.

106. Pressure of a Liquid on the Walls of a Vessel.—Since the pressure resulting from force of gravity is proportional to the depth of a liquid, and also since the pressure per unit area at any depth is transmitted to every equal unit area below that depth, it follows that the pressure on the horizontal bottom of a vessel is

$$P = h\rho g A_b \tag{169}$$

where h is the vertical height and A_b is the area of the bottom. It also follows that this pressure is independent of the shape of the vessel, for, according to Pascal's law, if the box B, Fig. 133, has a tube T inserted at one side, or at any point, and the whole be filled with liquid to a height l, the pressure on the bottom of the vessel is the same as if, instead of the tube, the sides of B had been extended up to the level l and the whole filled with the same liquid. Whatever the shape of the vessel may be, the pressure on the bottom, or any part of it, may be found by equation (169).

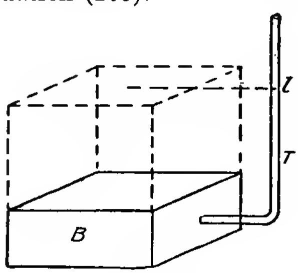


Fig. 133.

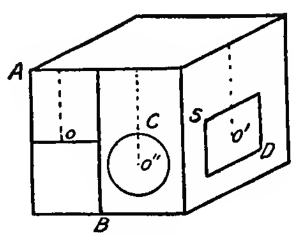


Fig. 134.

Pressure on the side of a vessel is zero at the surface of a liquid and maximum at the bottom. Reference is here made only to pressure due to weight. Since this pressure increases uniformly with the depth, one-half the sum of the zero and maximum pressures is the average pressure per unit area,—*i.e.*, the average is at a point of which the depth is one-half the depth of the liquid. If then the side of the vessel is of such a shape that a horizontal line through its centre divides the area into equal upper and lower parts (*e.g.*, square, rectangle, circle, etc.), the total pressure is the average pressure times the area. In Fig. 134 a rectangular box is supposed to be filled with a liquid of density ρ. The total pressure on AB, a portion of one side, is the area of AB times the pressure at o. The total pressure on SD is its area times the pressure at o' So also the total pressure on the circle C is its area times the pressure at o''. When the area of the side considered is not symmetrically divided by a horizontal line through its centre, the total pressure cannot be found in this manner. If, for example, the side is in form of a triangle, as ABC, Fig. 135, the total pressure on the triangle when the vessel is full of liquid is the area of ABC times the pressure at o, the distance from A to o being two-thirds of the altitude of the triangle. (Appendix 1.)

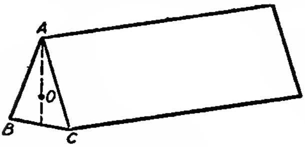


FIG. 135.

107. Buoyancy of Liquids. — *A body immersed in a liquid is buoyed up by a force equal to the weight of the liquid displaced.* This is known as the principle of Archimedes and is applicable to all fluids. Buoyancy is a result of the fact that pressure increases with depth and at any given depth is equal in all directions. Let abc be a rectangular block the upper surface of which is at a depth d below the surface of the liquid. The pressure in dynes on the top of the block is $bcd\rho g$, while the upward pressure on the bottom is $bc(d+a)\rho g$. The excess of upward pressure is, therefore,

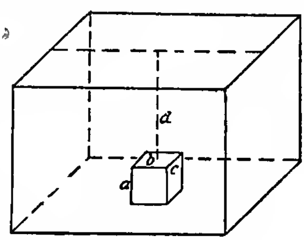


FIG. 136.

$$bc(d+a)\rho g - bcd\rho g = abc\rho g$$

and this is the weight of liquid displaced by the block. In case the block floats partly submerged, the depth d becomes negative and there is no pressure on the top of the block, hence the excess of upward pressure on the bottom is

$$bc(a-d)\rho g$$

or

$$abc\rho g - bcd\rho g$$

which is again the weight of the liquid displaced.

108. Density and Specific Gravity. — The density of a substance is the quantity of matter in the unit volume. Specific gravity is the ratio of the mass of a substance to the mass of an equal volume of water. In the metric system of units density and specific gravity are numerically the same, since the mass of 1 c.c. of water at 4° C. is assumed to be 1 g. Thus, we say, for example, that the density of brass is 8.5 $^{g}/_{cc}$, or the specific gravity is 8.5, the latter being a pure number.

109. Density of Solids.—A common method of finding the density of a solid is to take the ratio of the weight of the solid in air to its loss of weight when immersed in water, for, according to the principle of Archimedes, the loss of weight is the weight of an equal volume of water. This ratio is the specific gravity, which, multiplied by the density of water, 1 $^{g}/_{cc}$, gives the density.

The specific gravity in reference to any **liquid other than water** may be found in the manner just described for water, and the density is then

$$\rho=\frac{W_a}{W_l}\rho_l \qquad (170)$$

where ρ is the density of the solid, W_a its weight in air, W_l the weight of an equal volume of the liquid, and ρ_l the density of the liquid.

In case a **solid floats** on water it may be submerged by attaching to it a sinker, and its density is then found by the equation

$$\rho=\frac{W_a}{W_s-W_{ss}} \qquad (171)$$

where W_a is the weight of the solid in air, W_s the weight of solid and sinker, the latter being submerged, and W_{ss} the combined weight when both are submerged. The difference between W_s and W_{ss} is evidently the buoyancy of the water on the solid, and hence is the weight of a volume of water equal to that of the solid.

When only small **fragments** of a solid are obtainable, the density may be found by placing the fragments in a specific-gravity bottle. Let W_f be the weight of fragments and bottle less the weight of the bottle, W_w the weight of bottle when filled with water less the weight of the bottle, and W_{fw} the weight of the bottle when filled with both fragments and water less the weight of the bottle. Then

$$\rho = \frac{W_f}{W_f + W_w - W_{fw}} \tag{172}$$

for the fragments displace their own volume of water.

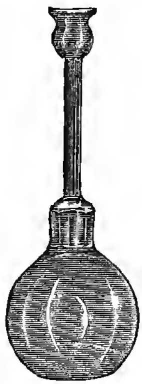


FIG. 137.

FIG. 138.

The **Nicholson's hydrometer** may often be used to find the density of solids. This instrument consists of a hollow metal cylinder, Fig. 138, supporting a pan above and one below. It floats in water in an upright position. The solid under consideration is placed on the upper pan, and known weights are added until the hydrometer sinks to a mark on the slender stem below the upper pan. The solid is then removed, and known weights are put in its place to restore the mark again to the surface of the water. The weight last added is the weight of the solid in air. Now place the solid on the lower pan beneath the water. The weight which must be added to that already in the upper pan to restore the mark to a position at the surface of the water is the weight of water displaced by the solid. From these data the density can at once be determined.

110. Density of Liquids. — In liquids as in solids there are numerous ways of finding density. In all, however, the standard for comparison is water,—*i.e.*, the density is found to be a certain number of times that of water, which is 1 $^{g}/_{cc}$. One method is to find the weight of a solid when it is immersed in water and then when immersed in a liquid the density of which is sought. The ratio of the loss of weight in the liquid to the loss of weight in water is the ratio of the weights of equal volumes of two liquids, one of which is water, and so is the density of the liquid. A convenient instrument made expressly for finding the density of liquids in the manner just described is illustrated in Fig. 139, and is known as **Mohr's balance.** A hanger containing a thermometer is suspended by a fine platinum wire from one end of a beam and just balances a counterpoise at the other. When the hanger is immersed in a liquid, certain riders which accompany the instrument are placed on the graduated beam to restore the balance. The density of the liquid is then read directly from the position of the riders.

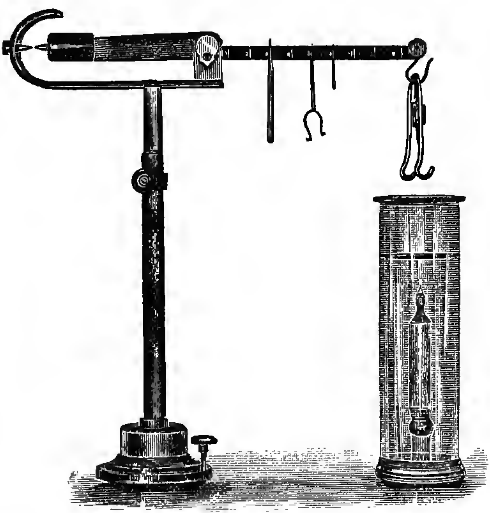


Fig. 139.

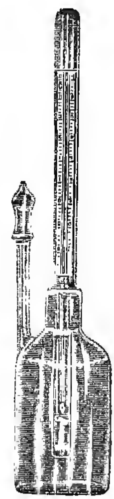


Fig. 140.

A convenient method of finding the weights of equal volumes of two liquids, one of which is water, is by use of **specific-gravity bottles,** also called pyknometers. In one form, Fig. 140, the bulb of a thermometer projects into the liquid. A ground glass stopper forms a part of the stem of the thermometer. The bottle is filled with liquid to the top of a capillary tube, which is covered with a glass cap to prevent evaporation. If W_l is the weight of the liquid and W_w the weight of an equal volume of water at temperature $t°$, the density of the liquid is

$$\rho_l = \frac{W_l}{W_w}\rho_w \qquad (173)$$

where ρ_w is the density of water at the temperature $t°$. At 4° C. the density of water is nearest 1 g/cc. For any other temperature its density may be found in appendix 20. The temperature should be considered when exact results are desired.

For rapid though generally not very accurate determinations of the density of liquids, **hydrometers** are used. These are of two kinds. 1. Hydrometers of constant volume, such as the Nicholson's hydrometer already described, where the same volume of liquid is always displaced by sinking the mark on the stem to the same point. The use of this instrument in finding the density of liquids is apparent. 2. Hydrometers of variable volume, Fig. 141, where the volume of liquid displaced is inversely as the density. These consist usually of glass tubes loaded with mercury or shot at the slower end to cause them to float upright. The weight of the hydrometer is constant, and it will sink only as far as is necessary to displace its own weight of liquid. These instruments are made in a variety of styles suited to special purposes. The graduations on the stem of the hydrometer may be so spaced that the density can be read directly from the mark at the surface of the liquid. By making the stem slender the divisions may be widely separated for even small differences of density, for the more slender the stem the greater the distance it must rise or sink to produce any given change of displacement. The range, however, is decreased in the same proportion as the scale divisions are increased.

FIG. 141.

Since the volume of liquid displaced by any given hydrometer varies inversely as the density of the liquid, the spaces on direct reading scales become shorter and shorter as the density increases.

Scales with equal divisions may be employed, but the density, if desired, must be calculated from the reading in each determination.

The divisions on the scale of **Baumé's hydrometer** are equal in length. In forming such a scale it is necessary to know the depth to which a hydrometer will sink in two liquids of known density. Baumé chose water for one liquid and brine (15 parts of salt in 85 of water) for the other. Marks were made on the stem at the surface of pure water and at the surface of the salt solution. The space between these two marks was divided into 15 equal parts, and these graduations were continued throughout the length of the stem (Fig. 141). Let ρ_w be the density of the water and ρ_s that of the salt solution. Also let V_0 be the volume of liquid displaced when the hydrometer sinks to the zero mark,—*i.e.*, in this case, when it is floating in pure water. Then, if one of the scale divisions is taken as a unit of volume and n is the number of divisions from zero down to the surface of a liquid,

$$V_0\rho_w = (V_0 - n)\rho_s$$

for each member of this equation is the weight of the hydrometer. Hence

$$V_0 = \frac{n\rho_s}{\rho_s - \rho_w}$$

In the solutions here considered, $\rho_w = 1$, $\rho_s = 1.116$, and $n = 15$. Hence

$$V_0 = \frac{15 \times 1.116}{.116} = 144.3$$

For any other liquid denser than water,

$$(144.3 - N)\rho = 144.3\rho_w$$

for again each member of the equation is the mass of the hydrometer. Hence

$$\rho = \frac{144.3}{144.3 - N}\rho_w \qquad (174)$$

where N is the reading of the scale and ρ is the density sought.

For liquids lighter than water the zero mark is near the lower end of the stem and the hydrometer sinks deeper as the density decreases. To determine the divisions on this scale Baumé made a solution of 10 parts salt in 90 parts water, and marked the stem zero at the surface of the salt solution, and 10 at the surface of water. Then, using the same letters as above,

$$V_0\rho_s = (V_0 + 10)\rho_w$$

or

$$V_0 = \frac{10\rho_w}{\rho_s - \rho_w}$$

If $\rho_w = 1$ and $\rho_s = 1.085$, then

$$V_0 = \frac{10}{.085} = 117.6$$

Hence for any other liquid lighter than water

$$(117.6 + N)\rho = (117.6 + 10)\rho_w$$

or

$$\rho = \frac{127.6}{117.6 + N}\rho_w \tag{175}$$

111. Twaddell's Hydrometer. — Baumé's hydrometer is in common use, but is both unscientific and inconvenient where specific gravity is desired. There is no connection between the

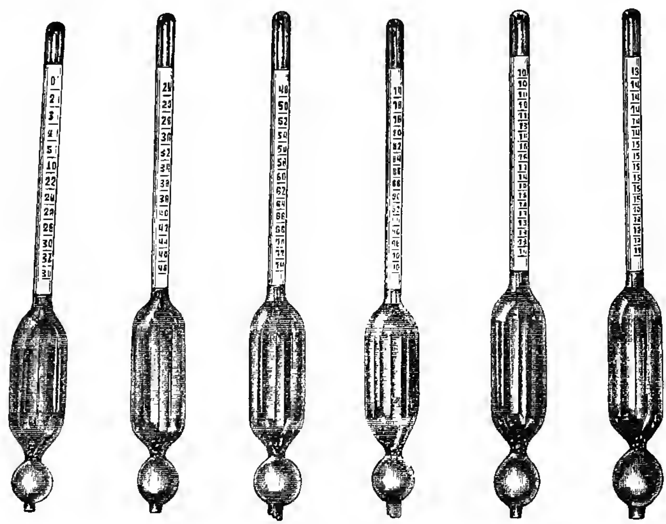


FIG. 142.

readings on a Baumé hydrometer and specific gravity, though the latter may be deduced from the former in the manner shown in § 110. In the proper use of this instrument any given number on the scale, as 30 or 70 Baumé, should indicate a desired strength of solution without reference to specific gravity. The use of whole numbers and equal scale divisions is an advantage in the arts.

The divisions on *Twaddell's hydrometers* have a direct relation to the specific gravity of the liquid, each division indicating a difference of .005 in the specific gravity. To avoid the inconvenience of a long stem, there are usually six spindles, as shown in Fig. 142. The total number of divisions is from 0 to 174, each spindle continuing the reading of the one before it. In water at 15.5° C. the first spindle sinks to 0. At this mark, then, the specific gravity is unity (1). Hence, when the Twaddell reading is known, an easy mental calculation will give the specific gravity. Thus, 20 Twaddell is

$$1+(20\times.005)=1.10 \text{ sp. gr.}$$

and 174 Twaddell is

$$1+(174\times.005)=1.87 \text{ sp. gr.}$$

The divisions are shorter as the density increases, but, since they are distributed over six spindles, even the shortest may be conveniently read.

112. Equilibrium in Case of Buoyancy.—The weight of liquid displaced by a floating body is equal to the weight of the body. The centre of gravity of the displaced body of liquid is called the centre of buoyancy. The centre of gravity of a body and the centre of buoyancy of the liquid displaced may be so related in position that any one of the three states of equilibrium—stable, unstable, or neutral—may obtain. Let a uniform rod of wood be floated on water as shown in Fig. 143, *A*. The centre of gravity of the rod is at g and of buoyancy at b. The force of gravity is directed downward, and that of buoyancy upward. While b is in the line of direction of gravity,—*i.e.*, directly beneath g,—the rod will be in unstable equilibrium, for the least disturbance of this relation of g and b will result in a couple, $F_g \cdot \overline{db}$ or $F_b \cdot \overline{db}$, which will cause the rod to topple over.

In case of the floating sphere B, the equilibrium is neutral, for b will remain directly beneath g in any position of the sphere.

To secure stable equilibrium of a floating body it is necessary that there be a righting couple,—*i.e.*, a couple which will restore b to a position in the line of direction through g. In the figure C represents a ship rocked to one side so that b is at b', and there is a couple, $F_g \cdot \overline{gs}$ or $F_b \cdot \overline{sg}$, tending to right the ship. A vertical line through b' intersects ah at a point, m, called the *metacentre*. Whenever the metacentre is above g, the equilibrium is stable. When m and g coincide, the equilibrium is neutral, but when m is below g, unstable. The bodies A and B may be made stable by loading them with some dense substance at points below g. The ballast of ships serves this purpose.

It is evident that the shape of the keel is an important matter in the construction of ships, that the centre of buoyancy may, when the ship lists, be shifted in such a manner as to result in a restoring couple.

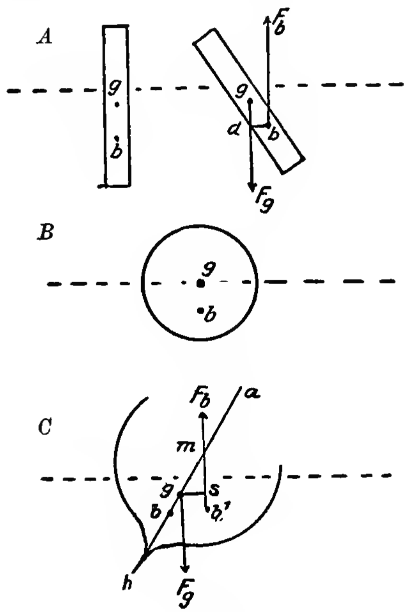


FIG. 143.

113. Valve Pumps for Liquids.—If a bucket is lowered into water, a valve in the bottom will open while the bucket is filling, but will close when the bucket is lifted. Let this bucket be fitted into a tube, T, which extends down to water or other liquid. If the tube is closed except through the valve v, a short up-and-down movement of the bucket, here called the "sucker," will lift water to the top of the tube. If the sucker is above the level of liquid in the reservoir, pressure of the atmosphere performs a necessary part of the operation of the pump, causing liquid to pass through v and follow the sucker even to the height of a water barometer, or the barometric height of whatever liquid is being pumped. If both valves are beneath the liquid, air pressure is not essential to the operation of the pump. This device is called a **lifting pump.** (Fig. 144.)

In Fig. 145 are shown the essential parts of a **force pump.** Two valves are necessary in this as in all pumps. A solid plunger, *P*, takes the place of the sucker of a lifting pump, and when it is raised a quantity of liquid is entrapped above the valve *v* and

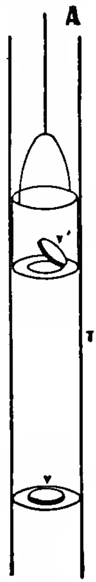


FIG. 144.

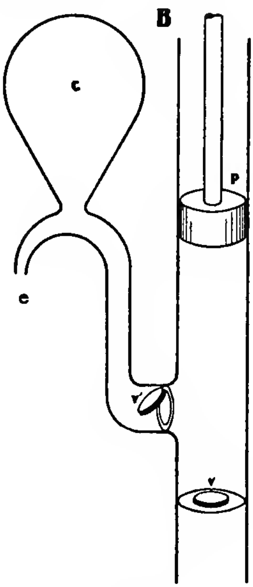


FIG. 145.

then *forced* out through *v'* by a downward thrust of the plunger. The vessel *C*, an expansion of the outlet tube, is full of air, which serves as a cushion to the liquid and causes a steady flow from *e*.

114. The Siphon.—A siphon is a contrivance by which force of gravity and pressure of the atmosphere are employed in the transference of fluids. Let two barometer tubes be filled with mercury and inverted as shown in Fig. 146. Let *A* be at a higher level than *B*. The atmospheric pressure at *A* is balanced by the pressure of the mercurial column *Ac*, and at *B* by the column *Bd*. If now the tubes are connected at the top by a cross tube, *T*, thus forming an inverted U-tube, the pressure of the air at A will cause mercury to fill the space T and join the top of the column at d. The column of mercury on one side of the U-tube is then Ac, and on the other Be. Let Ac be represented by h and Bd by h^1, and let de be x. Also let the atmospheric pressure at A be P_a, and at B, P_b. Then

$$P_a - hg\rho = P_b - h'g\rho$$

for the height of the columns are proportional to the pressures at A and B. There will then be an unbalanced force, $x\rho g$, directed toward B, which will cause the liquid to flow in that direction. The tube T may cross from Ac to Bd at any point between c and A, the extreme height being the barometric height of the liquid which fills the siphon. The value of h^1 is always such that

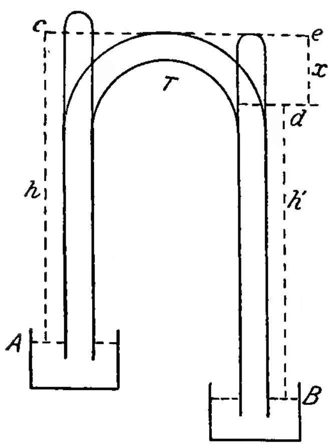


Fig. 146.

$$\frac{h'}{h} = \frac{P_b}{P_a}$$

or

$$h' = \frac{P_b}{P_a} h$$

—*i.e.*, h' is the height to which the pressure at B is capable of raising the liquid, no matter what the actual height of the siphon may be. In a short siphon—*i.e.*, one of little altitude—P_b and P_a are exerted for the most part against each other. The difference between h' and the greatest height at which the siphon will operate for any given fluid is x, where $x\rho g$ is the force causing the flow. Since B is in a lower stratum of air than A, P_b is a little greater than P_a, but for fluids that are denser than air

$$P_b - P_a < x\rho g$$

and the flow will be toward B, *i.e.*, in the direction of the force $x\rho g$. If, however, the fluid is less dense than air, hydrogen for example

$$P_b - P_a > x\rho g$$

and the flow will be toward A. To transfer hydrogen from A to B the whole apparatus shown in the figure should be inverted.

115. Efflux of Liquids.—If an opening be made in the side or bottom of a vessel filled with liquid, the velocity of efflux will depend, for the most part, on the depth of the liquid. Let a vessel, A, Fig. 147, be filled to a height, h, above e, with a liquid of density ρ. If, now, 1 c.c. of the same kind of liquid is forced into A through e, the amount of work done is equal to that required to lift a column of h cubic centimetres of the liquid 1 cm. high. This is the same as lifting 1 c.c. to a height of h centimetres. If m is the mass of 1 c.c., the work done or the increase of potential energy of the whole mass is mgh ergs. The actual rise of liquid in the vessel is inversely proportional to the area of the surface, but in any case the increase of energy is equal to the work done in introducing liquid at e. If now the liquid is allowed to flow from e, the potential energy of a mass m may be assumed to be all converted to kinetic energy, $\frac{1}{2}mv^2$. Hence

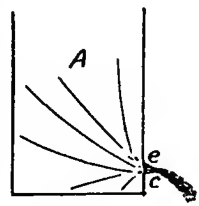


FIG. 147.

$$mgh = \tfrac{1}{2}mv^2$$

$$\text{or} \qquad v^2 = 2gh \tag{176}$$

Thus it appears that the velocity of efflux is the same as that of a body falling freely from the surface of the liquid to the orifice.

It would seem that if the area of the orifice is a and the velocity of efflux is v, the volume of liquid which would flow out in time t would be avt. But because portions of the liquid in the vessel move from all directions toward the orifice, the issuing stream is contracted at a point, c, called the *vena contracta*. Let the area of cross section at this point be A_c, then the volume of efflux in any given time is A_cvt, the value of which is about .62 of avt. If a short tube (about two or three times as long as the diameter) is fitted to the orifice, the rate of flow is increased to about .82 avt. In this discussion the viscosity (internal friction) of the liquid has not been considered.

116. Velocity of Efflux in Terms of p and ρ.—The pressure p which causes the flow of liquid from an orifice depends on the depth h and the density ρ of the liquid, and may be expressed by

$$p=\rho gh \tag{177}$$

The pressure of the atmosphere downward on the surface of the liquid and inward at the orifice may for most cases be considered as balanced. From equation (177)

$$h=\frac{p}{\rho g} \tag{178}$$

Substituting (178) in (176),

$$v^2=\frac{2gp}{\rho g}$$

or $$v=\sqrt{\frac{2p}{\rho}} \tag{179}$$

which is the same expression as that for the velocity of a gas escaping from a small orifice.

117. Lateral Pressure of a Moving Stream.—The pressure per unit area at a depth h is mgh, and this, as shown above, is the potential energy of a unit mass (m) in vessel A, Fig. 148. Hence the potential energy per unit mass is equal to the pressure. If, then, the potential energy becomes wholly kinetic, the pressure p becomes zero. Let B be a tube, small as compared with A, and at the end of B let the liquid issue from a small orifice, e. If the velocity of efflux at e is the same as that which would be acquired by a body falling freely through a distance h,—*i.e.*, from the surface of the liquid to a point on a level with the orifice,—all the potential energy of that portion of the liquid becomes kinetic, and so there is no lateral pressure at that point. For, let P be the pressure at any point in a flowing stream, then

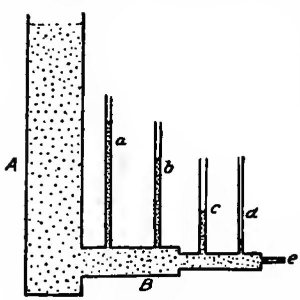


Fig. 148.

$$P=p-\tfrac{1}{2}mv^2$$

or $$P=p-\tfrac{1}{2}\rho v^2 \tag{180}$$

since m is the mass of unit volume. The pressure while the liquid is not flowing is p. If the value of v in (179) be substituted in (180)

$$P=p-p=0 \tag{181}$$

The velocity of the flow of liquid through B is less than at e, for the same volume passes a larger cross sectional area. The energy of the liquid in B is therefore partly potential and partly kinetic, the pressure being found by equation (180).

118. Viscosity. — Viscosity is that property of fluids by virtue of which they partake in some measure of the properties of solids. A perfect liquid would offer no resistance to a shearing stress, but all liquids and gases do to some extent resist such a stress. Work must be done in sliding one layer of liquid on an adjacent layer. The degree of resistance offered to a shearing stress is a measure of the viscosity of a substance. There is a coefficient of rigidity for fluids as well as for solids.

No distinct line of division can be made between liquids and solids. Some liquids, such as alcohol and ether, are very mobile, —*i.e.*, comparatively free from viscosity. Others, such as molasses, heavy oils, pitch, and molten glass, are distinctly viscous. Other substances, such as ice and asphaltum, are classed as solids, but will permanently change their shape when subjected to a continued stress. A great mass of ice will flow down a channel just as a river of water does, though much more slowly. A lead bullet placed on a block of asphaltum will in time sink to the bottom, while a cork placed beneath the block will rise to the top. Thus a substance which is brittle when subjected to a sudden stress, behaves like a liquid under a continued stress.

When a liquid, such as water, flows through a tube or pipe, it adheres to the walls of the pipe, forming a layer over which the liquid flows. When a pipe is full of flowing water, the part having the greatest velocity is at the centre of the pipe, while that adhering to the walls does not flow at all. Consequently the rate of flow is independent of the nature of the tube. It is plain from this that viscosity will retard the rate of flow, and the greater the viscosity the greater the retardation.

The coefficient of viscosity may be defined as the tangential force per unit area required to move one plane with unit velocity parallel to another plane which is fixed, the space between them being filled with a viscous fluid. As shown in Fig. 149, let a and b be planes 1 cm. apart, the space between them being filled with a liquid which adheres to the planes. Then the force in dynes required to move the plane a with a velocity of 1 $^{cm}/_{sec}$, divided by the area of a, is the coefficient of viscosity of the liquid.

It may be shown that the volume of liquid under pressure that will flow through a tube in a given time is

$$V=\frac{\pi p r^4 t}{8l\varphi} \qquad (182)$$

where V=volume, r=radius, t=time, l=length of tube, p= difference of pressure at the ends of the tube, and φ=coefficient of viscosity. Hence the value of φ for any liquid may be determined experimentally by measuring all the other terms of (182).

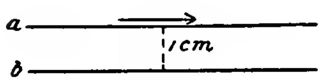


FIG. 149.

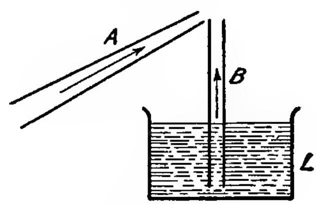


FIG. 150.

Viscosity is also apparent in case of a relative motion of bodies of gas. A jet of gas will drag with it other gas that is adjacent to it. The cause of viscosity of gases may be explained in accordance with the kinetic theory, for when one layer moves adjacent to another, molecules from the moving mass would cause the adjacent mass to move in the same direction, while those from the mass of gas at rest would retard the moving mass. This may be illustrated by considering the case of two trains of cars on parallel tracks, one train being at rest or in slow motion relative to the other, and both being loaded with small bags of sand. If while the trains are side by side numerous bags are thrown from one to the other, the motion of the slow train will be increased while that of the other will be retarded. The train having the more rapid motion would thus appear to drag the other with it.

The effect of viscosity of gases is apparent in many phenomena. If, for example, a strong blast of air is forced from A, Fig. 150, across the top of the tube B, there will be a decrease of pressure at the sides of the blast, and surrounding air will be set in motion. A liquid in L will then be raised in B by the unbalanced atmospheric pressure and dashed into spray by the blast of air. Atomizers are operated on this principle.

A light ball may be supported on a blast of air as shown in Fig. 151. The jet of air sets in motion the adjacent layers, thus creating a partial vacuum. Air then flows in from all sides and is carried up by the swiftly moving current. Thus, the pressure at the sides of the current is less than atmospheric pressure, and if the ball is disturbed by a slight lateral pressure, it will be promptly pushed back in position.

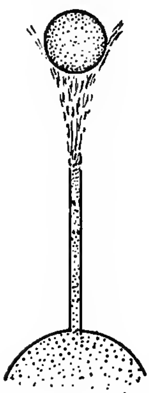


Fig. 151.

The curving of a ball when thrown in the proper manner is the result of viscosity of air. If a light ball is thrown from o, Fig. 152, and at the same time is made to spin in the direction indicated by the arrows, the ball will curve in the direction indicated by the broken line. If three slender poles, a, b, and c, are placed upright in the same plane, the ball may easily be thrown so that it will pass to the left of a, the right of b, and the left of c. The cause of this deviation from a straight line is found in the fact that one side of the ball, l, is moving with a velocity equal to that of the ball plus that due to its rotary motion, while the forward velocity at r is the difference of these two quantities. Air adhering to the ball drags adjacent air with it and consequently is more crowded at l than at r,—*i.e.*, the ball meets more resistance on the side l and hence is deflected toward r. A simple experimental illustration of this fact may be made by suspending a tennis ball or any light ball by a string. Then if, by means of a turning table or otherwise, the ball is made to rotate rapidly while at the same

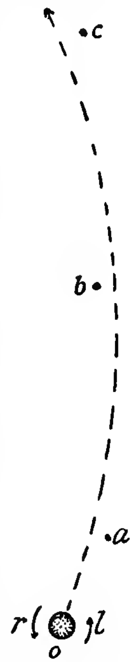


Fig. 152.

time it swings as a pendulum, its deviation from a fixed plane of vibration is at once apparent.

119. Surface Tension.—Each molecule of a liquid is assumed to exert an attractive force upon other molecules which are near. This attraction is called cohesion. The distance through which the cohesive force acts is exceedingly small, being, according to the estimate of Quincke, $5(10)^{-6}$ cm. Each molecule may within this limit be considered the centre of a sphere through which its influence is exerted. A molecule within the body of a liquid, as *a*, Fig. 153, is in equilibrium in reference to forces exerted upon it by surrounding molecules.

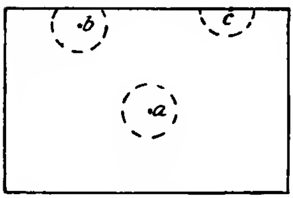


Fig. 153.

The distance of *b* from the surface is less than the radius of its sphere of influence, consequently there is a resultant force acting on *b* and directed toward the body of the liquid. At *c* the resultant force is maximum. This unbalanced force acting on each molecule at or near the surface of a liquid produces a condition known as *surface tension*. The surface behaves in some respects as if it were a stretched membrane or skin. A body of liquid, such as a drop of water or a pellet of mercury, will, except for the distorting influence of gravity, become spherical in shape, just as it would do if covered with a stretched elastic membrane. The reason for this is the fact that a sphere has a smaller surface than any other geometrical solid of the same volume.

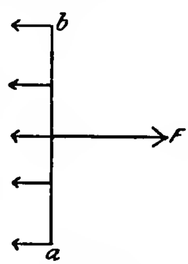


Fig. 154.

120. Unit Surface Tension.—A convenient unit for measurement of surface tension is the force per unit length which will balance the tension. In c.g.s. units it is the number of dynes per centimetre. If *ab* is a length *l* of surface under consideration and *F* is a force equal to the tension, then the surface tension *T* is

$$T = \frac{F}{l} \qquad (183)$$

121. Surface Tension Compared to an Elastic Membrane.—If a wire, *abcd*, bent in the form shown in Fig. 155, is dipped into water or other liquid and then lifted a short distance by a force, F, a film of liquid will fill the space enclosed by the wire and the surface of the liquid. Although this film is thin, yet in comparison with the range of molecular attraction it has considerable thickness, and the force due to surface tension is operative on both sides of the film. Hence the force F in this case is

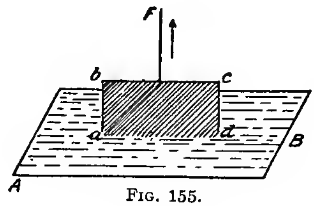


FIG. 155.

$$F = 2T \cdot \overline{bc}$$

By measuring F and $\overline{bc}$ a rough determination of T may be made.

This film differs from a stretched elastic membrane, such as a sheet of rubber, in several particulars. In the first place, the film of liquid will, when released, contract indefinitely. Secondly, the force exerted by the film is independent of the thickness, for there are only two surfaces whatever the thickness may be. This is true at least until the film becomes so thin that the molecules on the two surfaces come within the limits of attraction of each other. Thirdly, considering only the tension, the force F, Fig. 155, is constant for the various heights through which the film may be raised from the liquid if the width of the film is constant. In case of a rubber band the force must be increased with the distance through which it is stretched, while a liquid film increases in length and area by the addition of more molecules from the body of the liquid.

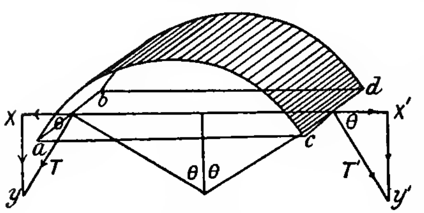


FIG. 156.

122. Pressure due to Surface Tension.—When the surface of a liquid is plane, the surface tension does not cause any lateral pressure, just as a stretched sheet of rubber would exert no pressure upon a plane which it covers. If, however, the surface is curved, there will be a component of the tension which will cause pressure in the liquid on the concave side of the surface.

Consider a cylindrical surface as shown in Fig. 156. Let T be the tension of the film, and let the body of liquid under consideration be bounded above by the curved surface and below by the plane *abcd*. Let the width of the surface, *ab* or *cd*, be 1 cm. The vectors T and T', representing the surface tension, may each be resolved into two forces, xy and $x'y'$ being the components that produce pressure within the liquid on the plane *abcd*. Since these components are equal, parallel, and in the same direction, the total pressure is their sum. But

$$xy = T \sin \theta$$

and

$$x'y' = T' \sin \theta$$

Since T and T' are equal, the total pressure is

$$2T \sin \theta$$

The total area of the plane *abcd* is *ac* times *cd*. But *cd* is, for sake of simplicity, taken as 1 cm., and *ac* is equal to $2r \sin \theta$, r being the radius of the curved surface. If p is the pressure per unit area, then

$$2pr \sin \theta$$

is the total pressure on *abcd* as a result of the tension of the film. Hence

$$2T \sin \theta = 2pr \sin \theta$$

or

$$p = \frac{T}{r} \tag{184}$$

Hence the increase of pressure per unit area in passing from the convex to the concave side of a liquid film varies directly as the tension and inversely as the radius.

If the film is curved in two directions at right angles to each other, the total pressure is the sum of the pressures resulting from the surface tension in two directions. If r is the radius of curvature in one direction and r' that in another, then

$$p = T\left(\frac{1}{r} + \frac{1}{r'}\right) \tag{185}$$

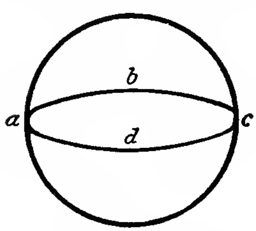


Fig. 157.

If a body of liquid is in form of a sphere, as a falling drop of water, or, better, a drop of olive oil suspended in a mixture of water and alcohol of the same density, the pressure p may be found in a manner similar to that given above. In Fig. 157 let *abcd* be a plane through the centre of the sphere. The total force exerted by the surface tension upon *abcd* is the tension T per unit length times the circumference of the circle. This is expressed by

$$2\pi r T$$

where r is the radius of curvature. The area upon which this pressure is exerted is the area of the circle *abcd*,—*i.e.*, πr^2. Hence

$$p\pi r^2 = 2\pi r T$$

or

$$p = \frac{2T}{r} \qquad (186)$$

which is the same as equation (185), since r and r' are in this case equal.

If the sphere were a soap bubble instead of a continuous mass of liquid, there would be a tension on both sides of the film, and so in this case

$$p = \frac{4T}{r} \qquad (187)$$

the radii of the inner and outer surfaces being considered equal.

123. Angle of Contact.—When a liquid adheres to a solid with more force than its particles cohere, it is said to wet the solid. When a wet solid is placed in a liquid, as shown in Fig.

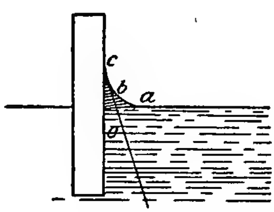


FIG. 158.

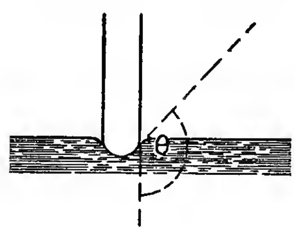


FIG. 159.

158, the surface of the liquid and the film on the solid become continuous. As a consequence, the liquid near the solid will be raised, and its surface will be curved along *abc*. The pressure on the concave side of this curve is greater than that on the convex side, the difference being equal to the weight of liquid raised above the hydrostatic level. The angle at which the liquid meets the solid is called the *angle of contact*. marked θ in the figure. The size of the angle depends on the character of the liquid, the surface of the solid, and the nature of a third fluid, usually air, which rests on the liquid under consideration. When a liquid thoroughly wets a solid, the angle of contact is zero. When the liquid does not wet the solid, as when a plate of glass is dipped in mercury, the surface of the liquid is depressed. The effect is as if the glass were pressed down on a stretched membrane on the surface of a liquid. The surface tension in this case depresses the liquid below the hydrostatic level, and the angle of contact is greater than 90°. For mercury and glass in air, θ is about 137° (Fig. 159.)

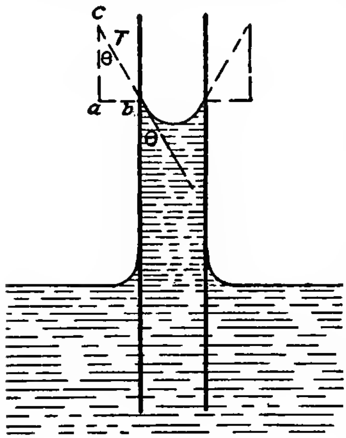


FIG. 160.

124. Capillary Action due to Surface Tension.—The phenomena of surface tension were first investigated in reference to the rise of liquids in capillary tubes. Let a tube having a fine bore be wet with a liquid, say a glass tube wet with water, and let it be placed vertically with its lower end in the liquid. The liquid will rise to a height such that its weight is equal to the surface tension at the top of the column. In case of water and glass the angle of contact is zero, but to make the consideration general the angle will be assumed to be θ. The component of T which is effective in lifting the liquid is ac, which is equal to $T \cos \theta$, Fig. 160. This is the tension per centimetre, and the number of centimetres to be here considered is the circumference of the bore of the tube. Hence the upward force due to the surface tension is

$$2\pi r T \cos \theta$$

This force is balanced by the weight of the column of liquid, which is

$$\pi r^2 h \rho g$$

Hence

$$2\pi r T \cos \theta = \pi r^2 h \rho g$$

or

$$h = \frac{2T \cos \theta}{r \rho g} \qquad (188)$$

where h is the height of the column, T the tension, r the radius of the tube, ρ the density of the liquid, and g the acceleration due to gravity.

In case of water in a glass tube,

$$\theta = 0$$

and $$\cos\theta = 1$$

hence $$h = \frac{2T}{r\rho g} \tag{189}$$

This equation shows that the elevation of a liquid varies inversely as the radius of the tube. The same is true for depression when the liquid does not wet the tube.

The value of h is the height of the liquid up to the bottom of the meniscus plus $\frac{1}{3}r$, for the quantity of liquid above the lowest point of the meniscus is equal to a cylinder the base of which is the area of cross section of the column and the height $\frac{1}{3}r$.

By accurate measurements of h and r the value of T for various liquids may be found by equation (188).

125. Some Surface Tension Phenomena. — The area of the surface of a liquid will always be as small as possible as a result of the force which constantly tends to drag molecules from the surface into the body of the liquid, and so the shape and position of the liquid will be such that as few as possible of the molecules will be on the surface.

A soap bubble will contract into a smaller and smaller sphere and finally become a flat membrane across the bowl of a pipe or other instrument by which the bubble was blown.

A soap film across the wide end of a funnel will, if the funnel is wet, move to a position at the small end where the area of the surface of the film is least.

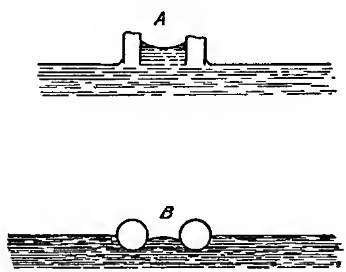


FIG. 161.

If two light bodies, such as short pieces of a match, are placed on the surface of water, about one centimetre apart, they will come together. As shown in Fig. 161, A, the water wets the pieces and is raised between them by surface tension.

Thus, the pressure on the convex side of the surface is less than in the surrounding water, and the bodies are pushed together. The same movement is noticed when a light floating body is near the side of a vessel.

If the bodies are not wet by the liquid, as when two pieces of metal are floated on mercury, the liquid between them is depressed, as shown in Fig. 161, *B*, and the bodies are pushed together by the liquid outside which is at a higher level.

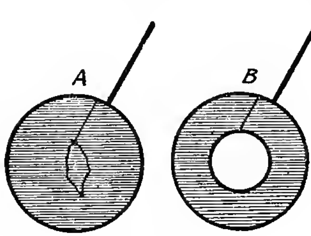


FIG. 162.

If a soap film is supported on a ring or loop of wire, and a loop of fine thread is floated on the film, as in Fig. 162, *A*, then if that portion within the thread is broken, as may easily be done with a hot wire, the thread will be pulled into the form of a perfect circle, as shown at *B*.

126. Diffusion of Liquids. — When two liquids which are miscible are placed in contact with each other, they will slowly mingle or diffuse until the whole body of liquid is homogeneous. The process is similar to diffusion of gases. In gases, however, molecules move with little interference from neighboring molecules and so the diffusion is rapid, while in liquids the molecules are never beyond the sphere of influence of other molecules, hence their rate of diffusion is slow.

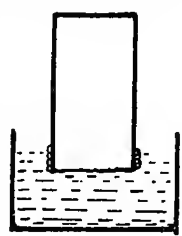


FIG. 163.

The subject of diffusion was first investigated by Graham in 1850 A.D., and he found that such substances as acids, bases, and salts, which may in most cases be reduced to a crystalline form, will diffuse rapidly, while such substances as gums, albumen, and starch, which are amorphous, diffuse slowly. He therefore classified substances as *crystalloids* and *colloids*.

A colloid will prevent the passage through it of another colloid but will permit the passage of a crystalloid. If, then, a mixture of crystalloids and colloids be placed in a vessel the bottom of which is a colloidal membrane, such as parchment paper, and the whole is placed so that the membrane is beneath the surface of pure water, Fig. 163, the crystalloids will pass into the water, but the colloids will be retained in the vessel. This operation is called **dialysis.**

127. Osmotic Pressure. — When liquids which diffuse are separated by a partition which prevents the passage of one and permits that of the other, a difference of pressure will be maintained on the two sides of the partition. This is known as *osmotic pressure*, and may be illustrated by tying a piece of bladder, coat of intestine, or other animal membrane over the mouth of a thistle tube, then filling the tube with copper sulphate and placing it so that the membrane is beneath the surface of water. Water will pass through the membrane into the tube more rapidly than the copper sulphate passes out, and the rise of liquid in the tube is a measure of the increased pressure on that side of the membrane.

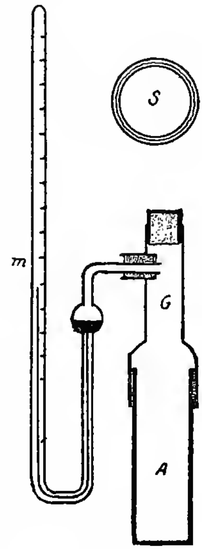


FIG. 164.

Accurate determinations of osmotic pressure have been made by Pfeffer and others by use of **semipermeable membranes** which permit a free passage of water but completely prevent the passage of certain substances dissolved in water. Such membranes are made in accordance with a principle discovered by Traube, that a mixture of certain substances causes a precipitate which is pervious to water but impervious to certain other substances including the two from which the precipitate was formed. Pfeffer prepared such a membrane by filling an unglazed porcelain cup, Fig. 164, *A*, with a solution of potassium ferrocyanide and then suspending the cup in a solution of copper sulphate. The two liquids would meet within the walls of the cup and precipitate ferrocyanide of copper, which would appear as a brown ring in a cross section of the walls as shown in the figure, *s*. This membrane may also be made to form an inner lining of the cup. To the top of the cup is cemented a glass tube, *G*, to which is attached a manometer, *m*. If the cup and tube be now filled with a solution, of sugar say, and suspended in a bath of water, the water can pass in but the sugar cannot come out. The increase of pressure within the tube is called the osmotic pressure of the solution.

Determinations of osmotic pressures of various substances, using solutions of various strengths and at various temperatures, show the following results:

1. Boyle's law for gases is true also for substances in solution, for osmotic pressure is proportional to concentration,—*i.e.*, the pressure is inversely proportional to the volume of liquid in which the substance is dissolved.

2. Avogadro's law for gases also holds for solutions, for when a number of grams equal to the molecular weight of a substance is dissolved in a given quantity of water or other liquid, at constant temperature, the osmotic pressure is the same whatever the substance may be,—*i.e.*, the pressure of different substances in solution is the same when the volume, temperature, and number of molecules are the same.

3. Charles's law for gases, that pressure is proportional to absolute temperature, is found to be as truly applicable to substances in solution.

It thus appears that substances in solution exert the same pressure as they would if converted to gases under the same conditions.

In certain solutions which are conductors of electricity—*i.e.*, electrolytes—the osmotic pressure is abnormal, being, in very dilute solutions, about twice as great as would be expected from the number of molecules in solution. This is the result of dissociation of molecules into ions, and will be further discussed under the subject of electrolytes.

Problems.

1. A body weighing 100 g., density 3 g/cc, is immersed in a liquid the density of which is 1.84 g/cc. What will the body then weigh?

2. A mass of 50 g., density 2.6 g/cc, weighs 23.076 g. less when immersed in a liquid. What is the specific gravity of the liquid?

3. A piece of wood weighs 25 g. A sinker weighing 226 g., density 11.3 g/cc, is attached to it and both in water weigh 181 g. What is the density of the wood?

4. A bottle containing 60 g. of sand weighs, when filled with water, 220 g. When filled with water alone it weighs 180 g. Find the density of the sand.

5. A tube contains 2 g. of mercury in 10 cm. of its length. Find the radius of the tube.

6. A cubical box contains 1000 c.c. of water, density 1 g/cc. A tube, also filled with water, extends 10 m. from the top of the box at angle of 30° to the horizontal. Find the total pressure on the bottom of the box.

7. With what velocity will water issue from an orifice when the pressure is 100 g. per square centimetre, viscosity not being taken into account?

8. What volume of glycerin at 20° C., the coefficient of viscosity then being 8.3, will flow through a tube 1 m. long and 1 cm. radius in 1 min., the difference of pressure at the ends of the tube being 1000 dynes?

9. If a soap solution of density ρ rises to a height h in a capillary tube of radius R_t, what, in these terms, will be the pressure on the interior of a bubble of radius R_b formed of the same solution?

1. 38.667 g.
2. 1.2.
3. .5 g/cc.
4. 3 g/cc.
5. .0684 cm.
6. 51000 g.
7. 443 cm. approx.
8. 28.4 c.c.
9. $p=2\rho gh\frac{R_t}{R_b}$

CHAPTER VI

HEAT

128. Heat and Temperature.—Heat is the state or condition of a body in reference to the kinetic energy of the particles of which a body is composed. As already explained, the molecules of all bodies are in motion, and it is the energy possessed by virtue of this motion that we call heat. This form of energy includes not only that due to the molecule as a whole, but also that due to the motions of the atoms and electrons within the molecule.

Temperature is the degree of activity of the particles of which a body is composed.

For a long time there were two theories in regard to the nature of heat. The commonly accepted theory up to the beginning of the nineteenth century was that heat is a fluid the total quantity of which is constant. This supposed fluid was called **caloric.** It was assumed to be self repellent and that it would consequently spread throughout a body of matter. Bodies were hot or cold according as caloric was present or absent. The various phenomena of heat were explained in accordance with this hypothesis. But those who believed that heat is not a substance but a condition of a substance finally brought forth experimental evidence, as will be shown later under thermodynamics, which completely overthrew the caloric theory. It is now well established that heat is a certain quantity of energy which a body may possess by virtue of the invisible motion of its particles, and temperature is the degree to which such energy exists in any given mass of matter.

The terms hot and cold have no absolute significance in physical science, but are used to describe sensations. A body is said to be hot or cold when its temperature is higher or lower than that of the human body. These terms may be used in a relative sense. Thus, ice may be said to be hot in relation to liquid air or liquid hydrogen, and a red-hot piece of iron may be considered cool in comparison with the electric arc.

Heat affects nearly all the properties of matter. Inertia and mass are not changed by a change of temperature, but volume, elasticity, tenacity, conductivity, state as to solid, liquid, or gas, and so on, are all changed when heat is added or withdrawn from a body.

Some of the phenomena and laws of gases, such as the kinetic theory and Boyle's law explained in a previous chapter, are, in a sense, heat phenomena, but there the purpose was to explain the nature of a gas, while in this chapter attention is directed to the effects when heat is increased or diminished, and the relation of heat to dynamics.

129. Expansion.—One of the most noticeable effects of heat is change in the dimensions of a body when a quantity of heat is added or withdrawn. The expansion which results from an increase of heat may be considered as *linear*, *superficial*, or *cubical.* Expansion is always cubical, but each dimension may be considered separately.

The **coefficient of expansion** is the increase per unit length, area, or volume, when the temperature rises from 0° to 1° C.

Consider first the **linear expansion** of solids. Let λ be the coefficient, L_0 the length at 0° C., and L_t the length when the temperature is raised to $t°$. Then, assuming for the present that the rate of expansion is the same for each degree,

$$\lambda = \frac{L_t - L_0}{L_0 t} \tag{190}$$

Thus, the coefficient of linear expansion is the fractional part of the total length at zero which a body increases in length when the temperature is raised from 0° to 1° C.

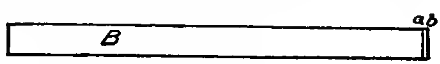


FIG. 165.

From (190) the length at any temperature is expressed by

$$L_t = L_0 + L_0 \lambda t$$

or

$$L_t = L_0(1 + \lambda t) \tag{191}$$

In case of solids the value of the coefficient is very small, and hence L_0 need not be at 0° C., but may be the length at any given temperature. If a bar, B, Fig. 165, is increased in temperature from 0° to 1° C., it will increase in length ab, say .00001 of its length. Then if the temperature is further increased from 1° to 2°, the increase in length will be as before and .00001 of ab in addition. This increase in the length of ab is, for most purposes, negligible. Hence, if a bar of iron, for example, is 100 cm. long at 20° C., its length at 100° C. is found with sufficient exactness by

$$L_{100} = 100(1 + .000012 \times 80)$$

The coefficient of linear expansion of solids may therefore be defined as the increase in length per unit length per degree.

For gases and liquids, particularly for gases, the coefficient is comparatively large, and consequently, as will be shown later, the increase per degree must be taken as a constant part of the volume at zero.

The coefficient of **superficial expansion** of solids is the increase in area per unit area per degree. Let this be represented by a, then

$$A_t = A_0(1 + at) \qquad (192)$$

where A is area.

Since unit area is the square of unit length, equation (191) may be written

$$L^2_t = L^2_0(1 + 2\lambda t + \lambda^2 t^2)$$

or

$$A_t = A_0(1 + 2\lambda t + \lambda^2 t^2)$$

But, since λ is a very small fraction, being of the order .00001, the quantity $\lambda^2 t^2$ is negligible, and so

$$A_t = A_0(1 + 2\lambda t) \qquad (193)$$

—*i.e.*, the coefficient of superficial expansion of solids may be considered as equal to twice that of linear expansion.

The coefficient of **cubical expansion** of solids is the increase in volume per unit volume per degree. Let this be represented by v, then

$$V_t = V_0(1 + vt) \qquad (194)$$

where V is volume.

Since volume is the cube of the unit of length, equation (191) may be written

$$L^3_t = L^3_0(1 + 3\lambda t + 3\lambda^2 t^2 + \lambda^3 t^3)$$

or

$$V_t = V_0(1 + 3\lambda t)$$

since $3\lambda^2t^2$ and λ^3t^3 are negligible, for reasons given above. Hence the coefficient of cubical expansion of solids may be considered as equal to three times that for linear expansion.

130. Determination of λ. — Numerous methods have been devised for the experimental determination of the coefficient of linear expansion. In all of them the effort is made to measure exactly the increase in length of a given rod for a certain change of temperature. One method is illustrated in Fig. 166. Micrometer microscopes are first focussed on fine lines near the ends of a rod, *ab*, while ice-water or water at a known temperature

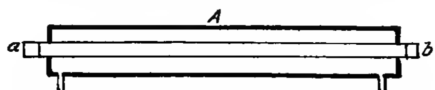


FIG. 166.

is made to flow through the jacket *A*. Then steam at temperature t° is in turn passed through the jacket and the micrometer screws are turned till the spider lines again coincide with the lines on the bar. The sum of the changes in the readings of the micrometers is the increase in length, and this, divided by the length at 0° C. and the number of degrees through which the temperature is raised, is the average coefficient of linear expansion for the range of temperature used, as shown by equation (190).

131. Expansion of Liquids.—In case of liquids cubical expansion is the only kind that is ordinarily considered. Since liquids are contained in vessels which also expand or contract with change of temperature, the apparent change in the volume of the liquid is the difference between its change and that of the vessel. This is called the *apparent expansion.* The actual expansion of the liquid is called its *absolute expansion.* The rise of mercury in a thermometer, for example, is a case of apparent expansion, the absolute expansion of mercury being considerably greater than that of glass.

If a large bulb with capillary tube attached, Fig. 167, is filled with a liquid, the effects resulting from temperature changes may be observed by changing the bulb from one bath to another of different temperature. Thus, in a certain experiment the bulb was filled with water and placed in a bath at 10.8° C. The top of the water in the stem came to rest at *b*. The bulb was then transferred to another bath at a temperature of 24° C., and the water in the stem at once dropped from b to a, a distance of 8 mm. From a the liquid then gradually moved up to c, a distance of 162 mm. above a. The drop from b to a was caused by an increase in the capacity of the bulb before heat was conducted to the liquid within. A moment later the water began to receive heat and, since its coefficient of expansion is greater than that of glass, the water was forced up the tube. The thread ac is thus the total increase in the volume of the water. If the bulb could be first warmed without any heat being communicated to the liquid and the liquid then raised to the same temperature, the volume ac would be the absolute expansion. This divided by the total volume of liquid and the number of degrees rise in temperature would be the coefficient of absolute expansion. Approximate results may be obtained in this manner, but it is better to observe the apparent expansion bc and to this add the increase in the capacity of the bulb, for it is clear that if the volume of the bulb had remained constant, the liquid would have risen higher by a distance ab. Hence, if V is the original volume of the liquid, and also the capacity of the bulb, v_g the coefficient of expansion of the glass, v_a the coefficient of apparent expansion, v the coefficient of absolute expansion, and t the rise in temperature, then

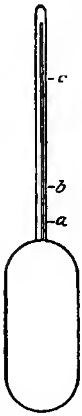


FIG. 167.

$$Vvt = Vv_at + Vv_gt$$

$$\text{or} \qquad v = v_a + v_g \tag{195}$$

—*i.e.*, the coefficient of absolute expansion of a liquid, as measured in this manner, is the sum of $v_a + v_g$.

The value of v_g may be found by filling the bulb with a liquid of known coefficient of cubical expansion, mercury for example, and thus finding the value of v_a for mercury. Then, knowing v and v_a, the value of v_g is at once calculated by (195). The volume per unit length of the stem may be found by weighing a measured length in it of a liquid of known density, then the volume for this length would equal the mass divided by the density.

An improved method, not affected by the expansion of the containing vessel, was devised by Dulong and Petit and later improved by Regnault for finding the absolute coefficient of expansion of liquids. Two tubes, *ab* and *cd*, Fig. 168, are connected at the top by a horizontal tube, *ac*. Side tubes at the bottom, *bg* and *de*, are connected to each other as shown. The vertical tubes are filled with liquid to the top of the arm *ac*, a small hole *n* insuring that the height in the tubes will not be greater than at *n*. The tube *t* communicates with an air-compressor, so that the pressure on the surface of the liquid at *k* and *f* may be varied at will. The tubes *ab* and *cd* are surrounded by vessels which may be filled with ice, or water at any desired temperature. Suppose *cd* is packed in ice at 0° C., and *ab* in water at *t*°. Then, since the heights of columns of liquid that balance each other are inversely as the densities of those liquids,

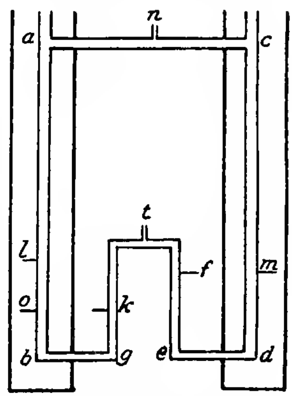


FIG. 168.

$$\frac{h_0}{h}=\frac{\rho}{\rho_0} \tag{196}$$

where h_0 and ρ_0 are the height and density of the column at 0° C., h and ρ being the height and density of the warm column. The upper surfaces of both columns are in the horizontal line *ac*, and the pressure at *k* is the same as that at *f*. Hence the difference of level at *f* and *k* must result from the greater density of the cold liquid. If the portion *ef* has the same temperature as the cold liquid, it will just balance the portion *md* in the tube, consequently the pressure at *f* is due to the column *cm*. Likewise, if *gk* has the same temperature as the warm column, the pressure at *k* is that due to the column *ao*. Hence, if *cm* is represented by h_0 and *ao* by h,

$$h_0 g\rho_0 = hg\rho \tag{197}$$

or

$$\frac{h_0}{h}=\frac{\rho}{\rho_0}$$

By measuring the difference in level between f and c, which is h_0, and between k and a, which is h, the ratio of the densities may easily be found.

From (194), the equation for cubical expansion,

$$V = V_0 (1 + vt)$$

Since volume in any case is equal to mass divided by density,

$$V = \frac{m}{\rho}$$

and so the equation for cubical expansion may be written

$$\frac{m}{\rho} = \frac{m}{\rho_0} (1 + vt)$$

or $$\frac{\rho_0}{\rho} = \frac{h}{h_0} = 1 + vt$$

hence $$v = \frac{h - h_0}{h_0 t} \qquad (198)$$

where v is the absolute coefficient of cubical expansion.

If the liquids in both ef and gk are at the temperature of the cold body, dm is balanced by ef as before, but gk now requires a longer column than bo, say bl, to balance it. Hence, taking depth to represent pressure on each side,

$$ab - bl = cd - ef,$$

and, by adding $bl = gk$,

$$ab = cd - ef + gk = cd - kf$$

From this it is seen that if the total length ab is measured and called h, the same distance less the vertical distance between k and f is h_0. These values can then be used in (198).

The coefficients of cubical expansion of liquids are considerably larger than those for solids (see table 24 in appendix), hence for accurate work it may be necessary to consider the coefficient as a fractional part of the volume at 0° C. only.

132. Maximum Density of Water.—Water differs from other liquids in that it is most dense when its temperature is about 4° C. When the temperature of a body of water is raised or lowered from this point, expansion occurs. The density of water at 0° C. and that at 8° C. are very nearly the same. Since the cooling of a body consists in a decrease of molecular activity, a body must always contract in volume when it is cooled. If it expands when cooled, some other cause must be assigned for the effect. In case of water it appears that a rearrangement of molecules begins at about 4° C., and, as the cooling continues, more and more molecules take positions in the crystalline form peculiar to ice, an arrangement which requires more room. This is sometimes expressed by saying that "ice molecules" and "water molecules" exist together when the temperature is between 0° C. and 4° C. The density of water at various temperatures is given in table 20 of the appendix.

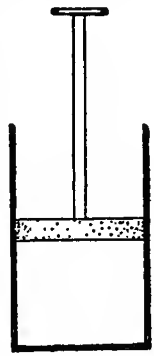


FIG. 169.

133. Expansion of Gas.—When the thermal expansion of a gas is considered, it is necessary to designate the pressure to which the gas is subjected, for a gas differs from a solid or liquid in being very compressible. Let a body of gas, Fig. 169, be enclosed in a cylinder under the pressure of the atmosphere and the weight of the piston. If the gas is now heated, the piston will be raised,—*i.e.*, the volume will be increased under constant pressure. The volume at any temperature may then be expressed by

$$V_t = V_0(1+vt) \tag{199}$$

where v is the coefficient of cubical **expansion at constant pressure.**

If the piston had been fixed in position, the applciation of heat would have caused an increase of pressure but no change of volume. In this case the pressure at any temperature would be expressed by

$$P_t = P_0(1+\beta t) \tag{200}$$

where β is the coefficient of pressure. Thus we may find the coefficient at constant pressure or at **constant volume.** Careful experiments show that for a perfect gas these two coefficients are numerically the same. The value of v may be found experimentally from (199), where

$$v = \frac{V_t - V_0}{V_0 t}$$

and the value of β from (200), where

$$\beta = \frac{P_t - P_0}{P_0 t}$$

The value of β, the coefficient of pressure, can be more easily and accurately found, and hence it, instead of v, is nearly always determined. The method consists in enclosing a mass of gas—air, for example—in a bulb and subjecting it to a pressure such that, whatever the temperature of the gas may be, the volume is kept constant. Thus let B, Fig. 170, be a glass bulb filled

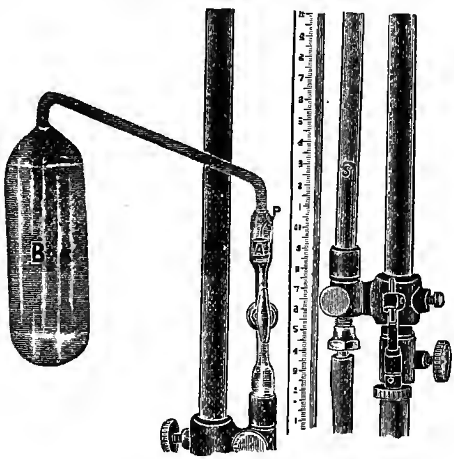


FIG. 170.

with dry air and connected to A by a tube of small bore. The tubes A and S with the connecting rubber tube are filled with mercury as shown. A pointer is fused into the glass at P to indicate the exact height to which the mercury in A must be kept that the volume of air may be constant. If B is now surrounded by snow or shaved ice, the temperature of the air within will become 0° C. The tube S is then raised or lowered until the pointer P just touches the surface of mercury in A. The difference of level in A and S is then read by means of a cathetometer. This difference added algebraically to the height of the barometer gives, in centimetres of mercury, the pressure of the air in the bulb. Call this pressure h_0. Then let the bulb be surrounded by steam at, say, 100° C. The pressure of the gas will increase and it will be necessary to raise S a considerable distance to bring the mercury again to the pointer. This difference of level added to the barometric height gives the pressure at 100° C. Call this pressure h_{100}. Then

$$\beta = \frac{P_t - P_0}{P_0 t} = \frac{h_{100} - h_0}{h_0 t} \qquad (201)$$

for the pressure is proportional to the height of the column of mercury. Careful experiments of this kind show that

$$\frac{h_{100} - h_0}{h_0} = .366 \qquad (202)$$

Now, if this range of temperature from the freezing of water to the boiling of water at a pressure of 76 cm. of mercury be divided into 100 equal steps or degrees, then for each degree

$$\beta = \frac{h_{100} - h_0}{100 h_0} = .00366 = \tfrac{1}{273} \qquad (203)$$

—*i.e.*, for a rise or fall of one such degree as we have assumed the pressure or volume of a gas is increased or diminished $\frac{1}{273}$ of the volume at the temperature of melting ice. Such a scale of temperature is known as the *Centigrade scale.*

134. Law of Charles.—A law known as the law of Charles, also as the law of Gay-Lussac, states that *the thermal coefficient of all true gases is* .00366 *or* $\frac{1}{273}$,—*i.e.*, an increase in temperature of $\frac{1}{100}$ of the range from the freezing to the boiling of water under standard pressure increases the volume or pressure $\frac{1}{273}$. Since this coefficient is large as compared to that for solids, it is always to be taken of the volume at the freezing temperature of water,—*i.e.*, 0° C. Thus, suppose it is desired to know how much 100 c.c. of a gas at 20° C. will increase in volume if its temperature is raised to 50° C., the pressure being constant. From equation (194)

$$V_t = V_0(1 + \tfrac{1}{273}t)$$

$$\therefore\ 100 = V_0(1 + \tfrac{20}{273})$$

or $\qquad V_0 = 93.174$ c.c. = volume at 0° C.

Then $\qquad V_{50} = 93.174(1 + \frac{50}{273}) = 110.24$ c.c. at 50° C.

A better method of making such a calculation is given in the next section.

135. Absolute Temperature.—It is evident from equation (200) that since

$$P_t = P_0(1 + \tfrac{1}{273}t)$$

then, if t becomes -273,—*i.e.*, 273° C. below the temperature of melting ice,—the pressure becomes

$$P_{-273} = P_0(1 + (-\tfrac{273}{273}))$$

$$\therefore P_{-273} = zero$$

This means that all molecular motion at that point would cease, and, since temperature, which is the degree of molecular activity, cannot be further reduced, this point in the centigrade scale is called the *absolute zero.* The number of degrees of temperature reckoned from the absolute zero is the *absolute temperature.* Thus, for any temperature t as reckoned from 0° C. the absolute temperature is $273 + t$, which will here be denoted by τ. Now, from equation (200)

$$P_t = P_0\left(1 + \frac{t}{273}\right) = P_0\left(\frac{273 + t}{273}\right) \tag{204}$$

$$\therefore P_t = \frac{P_0\tau}{273} \tag{205}$$

—*i.e.*, the pressure of a gas at constant volume is proportional to the absolute temperature, for P_t and τ are the only variable quantities in (205).

It has been shown that if the pressure is constant the volume will vary as does the pressure when the volume is constant, hence we may also write

$$V_t = \frac{V_0\tau}{273} \tag{206}$$

—*i.e.*, the volume of a gas varies directly as the absolute temperature. Hence the illustrative problem of the preceding section may be more simply solved by making the volumes proportional to the absolute temperature, thus

$$\frac{x}{100} = \frac{273 + 50}{273 + 20}$$

$$\therefore x = 110.24 \text{ c.c. at } 50° \text{ C.}$$

136. Laws of Boyle and Charles Combined.—If a body of gas at 0° C. and of volume V_0 is under a pressure P_0, then if the pressure is changed to P the resulting volume will be, say, V_1. By Boyle's law

$$P_0V_0=PV_1$$

or $$V_1=\frac{P_0V_0}{P}$$

the temperature being constant during this change. If then this volume V_1 is kept at constant pressure while the temperature is changed, the volume will be changed to, say, V. Then, by equation (206), substituting the value of V_1 for V_0,

$$V=\frac{P_0V_0\tau}{273P} \tag{207}$$

whence the volume at any pressure and temperature may be found if the volume under standard conditions is known. A temperature of 0° C. and pressure of 76 cm. of mercury are standard conditions for a gas.

It is often desirable to reduce a gas to standard conditions when the volume at any given pressure and temperature is known. This may readily be done by use of (207). Thus,

$$V_0=\frac{273PV}{P_0\tau} \tag{208}$$

where P_0 is 76 cm.

Equation (207) may be written in the form

$$PV=\frac{P_0V_0\tau}{273} \tag{209}$$

in which $\frac{P_0V_0}{273}$ is a constant for any given gas and is usually designated by R. Hence (209) may be written

$$PV=R\tau \tag{210}$$

for unit mass of gas. For a mass m,

$$PV=mR\tau \tag{211}$$

or $$P=\frac{m}{V}R\tau=\rho R\tau \tag{212}$$

where ρ is the density of the gas.

When ρ, τ, and P of any gas are known, the value of R can be found once for all for that gas. For air the value of R is $2.872(10)^6$; for hydrogen, $4.14(10)^7$; for oxygen, $2.59(10)^6$; for nitrogen, $2.96(10)^6$.

Problems.

1. What must be the length of an iron wire or rod at 20° C. that it may just fit into a space of 100 cm. when its temperature is 100° C.? (Coef. = .0000117.)

2. If the steel bars of a gridiron pendulum are 85 cm. long, how long should the brass bars be that the length of the pendulum may not change with change of temperature? (Coef. of steel = .000012, coef. of brass = .000018.)

3. A copper wire 100 m. long is found to be 3.44 cm. shorter when its temperature falls 20° C. What is its coefficient of expansion?

4. If a metre bar of nickel steel is correct in length at 0° C., its coefficient of expansion being .0000012, what is the correct length of a metal rod which when compared with the standard bar at 20° C. measures 800 mm.?

5. A sheet of lead is 3×20 feet in area and its coefficient of linear expansion is .000028. What will be its change in area if the temperature is raised 40° C.?

6. The mass of a certain body of oxygen is 6 g. The temperature is 22° C. and the pressure is 74.3 cm. of mercury. What is the volume of the gas?

7. The capacity of a glass bulb is 50 c.c. It is filled with a liquid. The capillary stem attached to the bulb is uniform in bore and 10 cm. of its length will hold .1 c.c. of liquid. When the bulb is heated 10° C. the liquid rises 23.725 cm. in the stem. What is the absolute coefficient of expansion of the liquid if the coefficient of linear expansion of the glass is .0000085?

8. A certain volume of gas at 0° C. and under a pressure of 74 cm. of mercury is heated to 100° C. and its pressure is then found to be 101.084 cm. What is its coefficient of cubical expansion?

9. If 2500 c.c. of a gas is under a pressure of 82 cm. of mercury and at a temperature of 20° C., what will be its volume under standard conditions?

10. What is the value of the constant R for a gas which under standard conditions has a density of 1.5 g. per litre?

11. What is the density of oxygen gas at —73° C. and under a pressure of 10 atmospheres?

1. 99.906 cm.
2. 56.666 cm.
3. .0000172.
4. 800.0192 mm.
5. 19.35 sq. in.
6. 4629.1 c.c.
7. .0005.
8. .00366.
9. 2513.2 c.c.
10. 2.474.
11. .019 g. per c.c.

137. Thermometry. — A thermometer is an instrument by which the degree of molecular activity of one body may be compared with that of another,—*i.e.*, it is an instrument for the comparison of the temperatures of bodies. The temperature sense may be employed for this purpose with a fair degree of accuracy in the comparison of two bodies of the same kind when their temperatures do not differ widely from that of the human body. If, however, the two bodies have the same temperature but are of different materials, as wood and metal, the metal will feel much cooler than the wood if both are cooler than the hand, and warmer if both are warmer than the hand. This results from the fact that metal conducts heat more rapidly. The unreliability of the temperature sense may be illustrated by placing one hand in hot and the other in cold water for a short time, then transferring both to tepid water. The sensation in one hand will be markedly different from that in the other, though the temperature is the same.

That temperature may be accurately and reliably measured, it is necessary to select some property of matter that is, as nearly as possible, always affected in the same manner by the same changes of temperature. The properties commonly selected are (1) *cubical expansion of gases or mercury*, (2) *change in electrical conductivity of platinum when the temperature is changed*, (3) *the change in electromotive force when there is a change of temperature at the junction of dissimilar metals which form part of a conducting circuit*, (4) *the change in the energy radiated from a hot body when its temperature changes*, *and* (5) *the displacement of maximum radiation to shorter wave lengths when temperature rises* (§ 166).

138. Hydrogen Thermometer.—There are two thermometers in very common use, one being dependent on the cubical expansion of a gas, usually hydrogen, nitrogen, or air, and the other dependent on the cubical expansion of a liquid, usually mercury. The former consists of a bulb filled with a gas and operated as shown above in Fig. 170. When the bulb is filled with hydrogen, the apparatus is known as a hydrogen thermometer. The International Committee of Weights and Measures, in 1887, adopted the hydrogen thermometer as the standard.

By equation (200)

$$t = \frac{P_t - P_0}{\beta P_0} \qquad (213)$$

hence, if the pressure of the hydrogen at the temperature of melting ice is P_0 and at the temperature of boiling water under standard condition of pressure is P_t, and if this range of temperature be divided into 100 equal parts, the temperature of melting ice being called zero, then the temperature at boiling point is

$$t = \frac{P_{100} - P_0}{\beta P_0} \qquad (214)$$

or, if pressure is expressed in height h of a column of mercury,

$$t = \frac{h_{100} - h_0}{\beta h_0} \qquad (215)$$

The value of β has been very carefully determined and found to be .0036625, which is equal to $\frac{1}{273}$ to within .0000005. Hence the temperature at the boiling point of water at 76 cm. pressure is

$$t = \frac{273(h_{100} - h_0)}{h_0} \qquad (216)$$

Hydrogen is used as the standard because it most nearly complies with Boyle's law, and its coefficient of thermal expansion is the same at different pressures, at least through a considerable range in pressure.

If $P_t - P_0 = \frac{1}{273}P_0$, then, from equation (213),

$$t = \frac{273P_0}{273P_0} = 1° \text{ C.}$$

—*i.e.*, an increase in the pressure of the gas at constant volume by $\frac{1}{273}$ of the pressure at zero indicates a change of 1° C. in temperature.

If $P_t - P_0 = \frac{100}{273}$ of the pressure at zero, then

$$t = \frac{100 \times 273P_0}{273P_0} = 100° \text{ C.}$$

Thus any other temperature on this scale may be found if the pressure at zero and the pressure at that temperature are known. Thus, suppose the pressure P_0 or h_0 is 78 cm., including, of course, the atmospheric pressure, and P_t or h_t, the pressure at the given temperature, is 90 cm., then

$$t=\frac{(90-78)273}{78}=42^\circ \text{ C.}$$

In determining the temperature by this method a correction must be made for the expansion of the bulb and also for the portion of gas in the stem which is not subjected to the change of temperature.

139. Mercury-in-glass Thermometers. — Gas thermometers are used chiefly as standards and are useful in testing the accuracy of other thermometers. But, for obvious reasons, they are not employed in the ordinary determinations of temperature. The mercury thermometer is in most common use, and the principle upon which it is based is that the apparent change in the volume of a mass of mercury in glass may be taken as a measure of the change of temperature which caused the change of volume. Mercury is selected as the best liquid for this purpose because (1) its rate of expansion is nearly constant at any temperature between its freezing and boiling points, (2) it is a good conductor of heat, (3) it does not wet the glass, (4) it remains a liquid through a wide range of temperature (about —39° C. to 357° C.), and (5) it is opaque and can readily be seen.

In form this thermometer consists of a glass tube with a fine bore and having a bulb blown on one end. It may be filled by inserting the open end in mercury and heating the bulb to drive out some of the air. When the bulb cools, a small quantity of mercury is forced by air pressure into the stem and bulb. This is then boiled, and thus the whole bulb and tube are filled with vapor of mercury. When this vapor condenses, the air pressure from without will fill the bulb with mercury. The thermometer is then heated to the highest temperature which it is intended to measure and the open end is sealed with a blowpipe. When the mercury contracts, there will be a vacuum above the thread in the stem.

Since the bore of the stem is very fine,—scarcely visible except as it is magnified by the convex surface of the glass,—a small change in the volume of the mercury in the bulb causes a very perceptible rise or fall of the thread in the stem. But the observed change of volume is only apparent, for the glass bulb also changes in volume, and it is only because the coefficient of cubical expansion of mercury is about .00018 while that for glass is only about .000025 that mercury rises in the stem when the bulb is heated. The hollow bulb will increase in volume just as if it were solid glass.

140. Graduation of Thermometers. — The temperature of melting ice is chosen as a point of reference; hence, if the bulb of a thermometer is packed in pure crushed ice or snow, the thread of mercury in the stem will sink to a certain point and become stationary while the ice is melting. This point is marked zero on the **centigrade scale.** The bulb and stem are then immersed in a bath of steam above boiling water under a pressure of 76 cm. of mercury. The mercury rises and becomes stationary at a point which is marked 100° C. on the centigrade scale. The intervening space is then divided into 100 equal parts or degrees, these being in some cases divided into fifths or tenths of a degree. This thermometer is almost exclusively employed for scientific purposes, and to some extent also for industrial and domestic purposes.

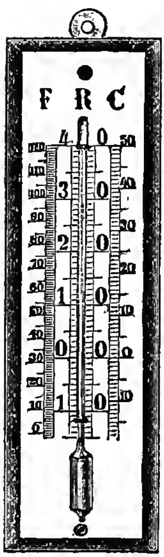


Fig. 171.

Another form of graduation, called the **Fahrenheit scale,** is commonly used in England and America for commercial and meteorological purposes. In it there are 180 divisions between the temperature of steam and melting ice, the zero being 32 of these divisions below the temperature of ice. The boiling point of water on this scale is therefore 212° F. when the pressure is 76 cm.

Another scale, named after **Réaumur,** is used in Germany. It consists of 80 divisions between the melting point of ice, which is zero, and the boiling point of water, which is 80° R. These three styles of scales are shown together in Fig. 171.

In transferring from one scale to another, it is only necessary to consider that 100° C., 180° F., and 80° R. all indicate the same range of temperature. Hence for the same range the *number* of degrees C. is $\frac{5}{9}$ of the number F. and $\frac{5}{4}$ of the number R. The number F. is $\frac{9}{5}$ of the number C. and $\frac{9}{4}$ of the number R. Likewise any given range R. equals $\frac{4}{5}$ C. or $\frac{4}{9}$ F. To transfer the reading from any one of the three scales to another, it is only necessary to multiply by the proper ratio. The result will be the reading in reference to the ice line. If, however, the transfer is from the F. scale to either of the others, 32 must first be subtracted to obtain a reading F. in reference to the ice line, while, if the transfer is to F. from either C. or R., the reading of C. or R. is first multiplied by the proper ratio, which gives the reading F. in reference to the ice line, and 32° is then added to obtain a reading in reference to 0° F. This is the procedure whether the readings are positive or negative.

141. Calibration of Thermometers. — If a mercury-in-glass thermometer is to be used for exact measurement of temperature, it should be calibrated,—*i.e.*, its error at any given temperature must be noted and recorded so that proper corrections may be made whenever the thermometer is used. One method of calibration is to compare the readings to the temperature as indicated by a hydrogen thermometer when both are subjected to the same temperature, as when the mercury thermometer and the bulb of the hydrogen thermometer are placed in the same bath. By varying the temperature of the bath, the errors on the mercury scale as indicated by the hydrogen thermometer may be noted. This operation is tedious, and a much easier method is to compare one thermometer with another which has been carefully calibrated and can therefore be used as a standard for many purposes.

Another method, fully described in laboratory manuals, consists in finding the error of the boiling and zero points by immersing the thermometer first in a bath of steam and then packing the bulb in pure melting ice. The bore of the tube is then tested for variation in cross section by separating a short thread of mercury and measuring its length as it is moved from point to point along the tube. With these data it is possible to make on coördinate paper a chart which shows at a glance the correction which should be made at any temperature.

One source of error in mercury thermometers is a slow change in the volume of the bulb which continues for a long time after the glass has been intensely heated in the process of manufacture.

The bulb becomes smaller and smaller with comparative rapidity during the first few weeks, but may slowly continue this so-called secular change for several years.

Even after the glass has thoroughly recovered from the effects of the first heating, if it is transferred from steam to ice there will be a depression of the zero point, showing that the glass does not at once recover its volume. Different kinds of glass behave differently in this respect, but good thermometers made of the same kind of glass agree very closely with one another.

The sensitiveness of a mercury-in-glass thermometer may be indefinitely increased by increasing the capacity of the bulb and decreasing the bore of the tube. With the **Beckmann thermometer,** shown in Fig. 172, a change of $\frac{1}{100}$° C. may be read on a graduated scale. The range, however, is greatly reduced, being here only about 5° C. The instrument is so constructed that this range may be selected at any place from several degrees below 0° C. to a number of degrees above 100° C., according to the particular temperature at which the change is to be noted. By aid of an auxiliary scale, Fig. 172, *B*, the thermometer may be set for any desired temperature, by warming the bulb till mercury rises to that temperature as marked on the auxiliary scale and then, by inverting the thermometer, the excess of mercury may be passed around the bend in the tube. The fine thread of mercury will then show the changes at that temperature. To find the exact temperature and the value of 1° on the Beckmann scale, comparison must be made with a standard thermometer.

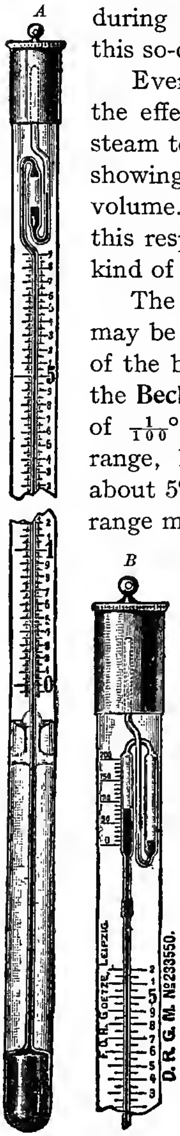


Fig. 172.

142. Thermometers for Special Purposes.— It is often desirable to know the highest or lowest temperature reached during a given period of time. An instrument so constructed that it will automatically make such a record is called a **maximum and minimum** thermometer. One form of such instrument is shown in Fig. 173. It consists of two thermometers placed in a horizontal position. The maximum thermometer—the lower one in the figure—is filled with mercury. As the temperature rises, a thread of mercury is pushed along in the stem, but when the temperature falls, the thread will sepa-

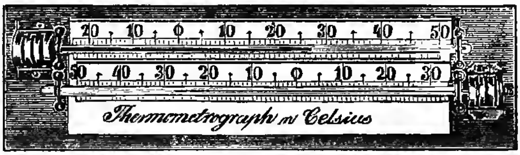


FIG. 173.

rate just above the bulb as a result of a constriction of the bore at that point. Thus, the top of the thread indicates the highest temperature reached. The minimum thermometer—the upper one—is filled with alcohol, and a small glass rod with rounded ends is placed in the bore to serve as an index. When the thread of alcohol rises it passes by the index without moving it, but when the temperature falls, the index is drawn toward the bulb by the surface tension at the end of the thread of alcohol. Thus the top of the index shows the lowest temperature reached. Both thermometers are set by holding the instrument on end with the mercury bulb down.

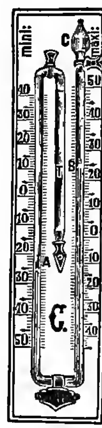


FIG. 174.

Another form of instrument for recording the highest and lowest temperature is known as **Six's thermometer.** This consists of a single tube bent in the form shown in Fig. 174. The lower bend of the tube contains mercury, while the remainder of the tube, the bulb *T*, and part of *C* are filled with alcohol. At the top of each arm of mercury is an index which is a small glass tube with flattened ends. Within the tube is an iron wire. When the temperature rises, the alcohol in *T* expands, thus pushing the mercury and the index *B* toward *C*. The lower end of *B* will then indicate the maximum temperature. When the temperature falls, the alcohol in *T* will contract and the mercury will raise *A* but leave *B* in place. The lower end of *A* will thus show the minimum temperature. The space above the liquid in *C* is filled only with the vapor of alcohol, which gives room for the expansion of the liquid and also supplies the pressure needed to support a difference in level of the mercury column when the temperature falls. By use of a magnet the glass tubes are brought back to the mercury.

Another thermometer, of special construction for convenience of physicians in taking the temperature of the human body, is called the **clinical thermometer,** Fig. 175. In this instrument

FIG. 175.

the scale need extend over only a few degrees, say from 90° to 110° F. Consequently the stem is short and the mercury does not reach the scale until the temperature is about 94° F. At a point just above the bulb there is a constriction in the tube, so that the thread of mercury which rises in the stem will remain there. The thermometer can then be taken from the mouth and read at leisure, for when the mercury in the bulb contracts it will break at the constriction. By shaking or swinging, the mercury in the stem may be returned to the bulb.

The expansion of any substance may be used for thermometric purposes, though, as has been shown, some substances are much better for this purpose than others. The expansion of metals is sometimes used to indicate temperature, according to principles illustrated in Fig. 176. Two bars of metal of different

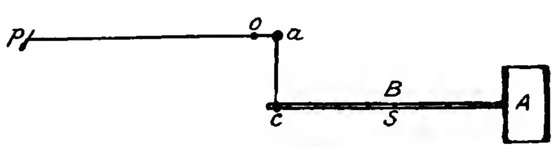


FIG. 176.

coefficients of expansion are riveted or brazed together, as *B* and *S*, brass and steel, one end being fastened to a firm support, *A*. Since *B* will expand more than *S* (see appendix 24), then, when the bar is heated, the end *c* will move downward; when cooled, upward. This motion may be multiplied by a lever, *ap*, pivoted at *o*. A pencil at *p* may be made to trace a line on a revolving drum. The drum is slowly rotated by clock-work, and the line is traced on paper specially prepared for the instrument, horizontal lines indicating temperature and vertical lines time. This instrument is called a thermograph, and is useful in making a record of temperature changes during any chosen interval of time.

143. Pyrometry.—Mercury under a pressure of one atmosphere will boil at 357° C. and in vacuum will boil at lower temperature. For this reason an ordinary mercury-in-glass thermometer cannot be used to measure temperatures which are much above 300° C. Air thermometers may be used at higher temperatures, but are not suitable for ordinary determination because of the difficulties in their manipulation.

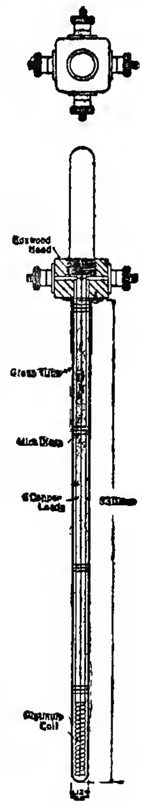


Fig. 177.

A department of physical science devoted to laws and methods for the measurement of high temperatures is known as **pyrometry.** Not only for scientific but also for many industrial purposes, it is often a great advantage to know with some degree of accuracy the temperature when it is high. For example, in annealing furnaces, porcelain kilns, glass furnaces, and steel castings, the best results are obtained only within a comparatively narrow range of temperature.

One form of thermometer, capable of measuring temperatures from about —200° C. up to about 1200° or 1400° C., is the **platinum resistance thermometer.** This, shown in Fig. 177, consists of a coil of platinum wire, from which leads extend to the top, where they are connected to two of the binding posts. Lying close to these leads is a loop of wire having exactly the same resistance as the leads and connected to the other two of the four posts at the top. The coil and leads are enclosed in a porcelain or glass tube. For high temperatures porcelain is used. The end of the tube containing the coil of platinum is placed in a furnace, molten metal, or whatever it may be of which the temperature is desired. The method by which the temperature is obtained is shown in the diagram, Fig. 178. The platinum coil T is connected, by means of cables, P,P, to one arm of the bridge, while the compensating loop is connected by cables C,C, of the same resistance as P,P, to binding posts at a gap in the adjacent arm. The purpose of the compensating loop and cables is, that, whatever change in resistance may result from a change of temperature in the leads and cables in one arm, the same change will be made in an adjacent arm, and so the balance of the bridge will not be disturbed on this account. Only the change of resistance in the coil T will

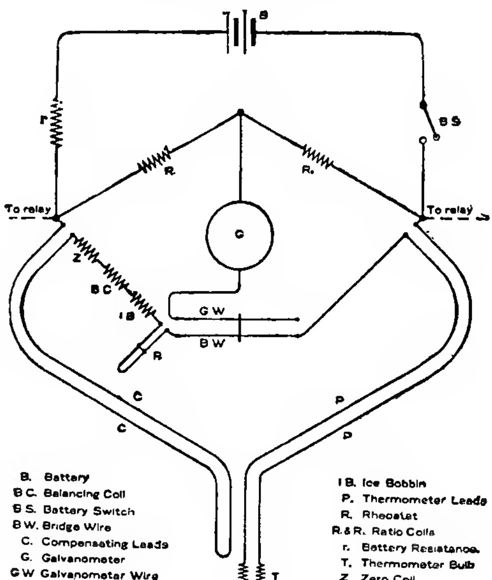


FIG. 178.

affect the bridge. The bridge may be balanced by adjusting the resistance in BC and R. Then any change in the temperature of the coil will be indicated by the position on the bridge wire, BW, where contact must be made with the galvanometer to maintain the balance of the bridge.

If now the resistance of the platinum coil is measured while it is at the temperature of melting ice, R_0, and then when at the temperature of boiling water under a pressure of 76 cm., R_{100}, the change in resistance per degree centigrade is

$$\frac{R_{100}-R_0}{100}$$

and so the temperature for any other resistance, R_t, is

$$T_{pt}=(R_t-R_0)\div\frac{R_{100}-R_0}{100}=\frac{100\,(R_t-R_0)}{R_{100}-R_0} \qquad (217)$$

where T_{pt} represents the temperature as measured by the platinum thermometer. If the change of resistance from 0° C. to 100° C. is one ohm, then the temperature is found by

$$T_{pt}=100(R_t-R_0) \qquad (218)$$

Other substances of known temperature should also be used in calibration,—for example, sulphur, which boils at 445° C.

The cables may be of any length, so that the measuring device may be at any convenient distance from the heated body whose temperature is being measured.

Another valuable instrument in pyrometry is the **thermo-electric thermometer.** The principle in this thermometer is, that, if two wires or bars of dissimilar metals are joined at one end and made part of a conducting circuit, then, if the temperature is changed at the juncture, a current of electricity will be caused to flow in the circuit. The electromotive force caused by heating the joint is proportional to the temperature. A delicate voltmeter will indicate the electromotive force, and so by proper calibration with known temperatures the scale of the voltmeter may be marked in degrees and the temperature is then read directly from the voltmeter. As shown in diagram, Fig. 179, two wires forming the thermo-electric couple are fused together at one end and enclosed in a tube.

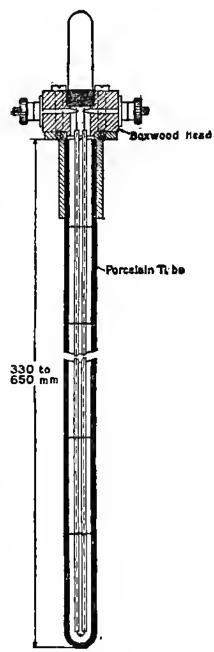


FIG. 179.

Since this end of the couple must be raised to the high temperature which is to be measured, it must be made of the most infusible metals. A common form of couple consists of platinum for one element and an alloy of platinum and iridium or platinum and rhodium for the other. These may be used up to temperatures of 1400°

or 1600° C. For lower temperatures less infusible metals may be used, such as copper and constantan. This style of pyrometer is in common use. It is suitable for any temperature within the range 500° C. to 1600° C. It can be used to determine the temperature of small quantities of a substance and is less expensive than the platinum resistance thermometer.

Both the thermo-electric and the platinum thermometers may be made to record temperature on a rotating drum through any desired interval of time.

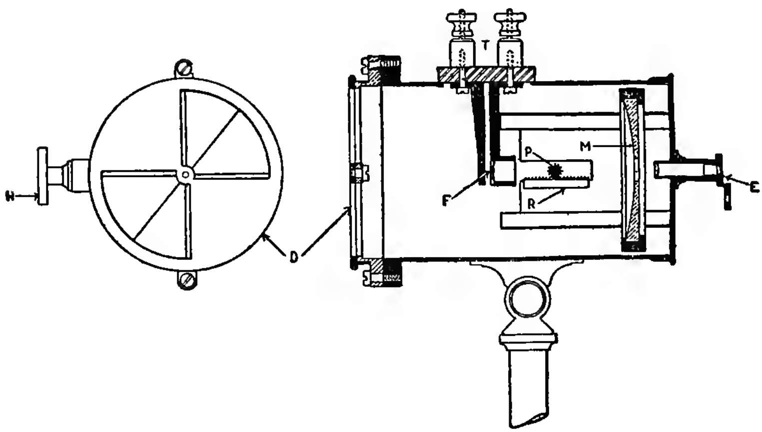


Fig. 180.

A third style of instrument is known as the **radiation pyrometer.** The principle underlying the use of this instrument is a law known as the law of Stefan and Boltzmann,—viz., the total energy radiated from a black body is proportional to the fourth power of the absolute temperature of that body. If T is absolute temperature and E is the radiant energy,

$$E \propto T^4$$

This pyrometer, as shown in Fig. 180, consists of a tube in which is mounted a concave mirror, M, and a thermo-couple, F. Radiations from a hot body fall upon the mirror and are focussed on F. This heats the thermo-couple and causes a current of electricity to flow through a galvanometer, as explained above in case of the thermo-electric thermometer. The observer sights through E and makes sure that the image of the hot body overlaps F on all sides. The deflection of the galvanometer needle is proportional to the energy focussed upon the thermo-couple, and the energy, as just stated, is proportional to the fourth power of the absolute temperature. Hence, if R_1 and R_2 are the deflections corresponding to T_1 and T_2, the divisions on the galvanometer scale being uniform, then

$$\frac{T_1}{T_2}=\frac{\sqrt[4]{R_1}}{\sqrt[4]{R_2}} \tag{219}$$

Thus, if a body having a known temperature, T_1, causes a deflection, R_1, then the temperature, T_2, of another body which causes a deflection R_2, can be calculated. In this manner the scale of the galvanometer may be graduated in degrees centigrade and the temperature read directly.

The advantages of this thermometer are that it may be used to determine any temperature from 500° C. up to the highest temperatures known, while the thermo-couple itself need not be raised to more than about 100° C. in any case. Also, the heated body may be of any size and the thermometer at any distance, provided only the image is large enough to cover entirely the thermo-couple. When the thermometer is set closer to the heated body, more radiant energy falls upon the mirror, but at the same time the image is larger, so that the same amount of energy falls upon the constant area of the couple.

The law of Stefan and Boltzmann is true only in case the heated body is black when cold,—*i.e.*, the body must be capable of emitting waves of all lengths. For this reason the radiation pyrometer gives what is called *black body temperature*. This is the true temperature if the body is black, but the temperature of a bright body, such as platinum, when determined in this manner, is less than the true temperature. The temperature of iron or of a furnace as observed through an opening in one side is very close to the true temperature. The black body temperature is just as useful as the true one if it is known to indicate a state which is desired.

Other pyrometers (§ 168) adapted to special uses are also employed in pyrometry and are described in special works on that subject.

Problems.

1. If a certain mass of gas at constant volume is under standard conditions of temperature and pressure, what increase of temperature will increase the pressure to 100 cm. of mercury?

2. What increase of temperature will be required to cause a confined mass of gas to exert twice as great a pressure?

3. If the temperature changes 25° C., what is the change F.?

4. If the reading on the Fahrenheit scale is 12°, what is the reading C.?

5. If the sum of the readings of all three thermometers (C., F., and R.) for the same temperature is 150, what is the temperature C.?

6. Calculate the absolute zero as expressed on the Fahrenheit scale.

7. A glass flask of 200 c.c. capacity is filled with dry hydrogen at 0° C. and 76 cm. pressure. The pressure is constant while the temperature is raised 100° C. How much gas, measured under standard conditions, will flow out, the coefficient of cubical expansion of glass being .000025?

8. What correction in degrees C. should be made for the expansion of the glass bulb of a hydrogen thermometer, the pressure of the gas at constant volume being increased by heat from 76 cm. to 86 cm.?

1. 86.2° C.
2. 273° C.
3. 45° F.
4. $-11\frac{1}{9}$° C.
5. 32.78° C.
6. -459.4° F.
7. 53.25 c.c.
8. .279° C.

144. Calorimetry.—A quantity of heat is a definite physical magnitude and so is capable of measurement. For this purpose some convenient and practical unit must be selected. The quantity of heat required to melt one gram of ice would be a good unit, but would be inconvenient and impractical in many ordinary determinations. The joule (10^7 ergs) is a unit of energy and might be adopted as the unit quantity of heat, for heat is a form of energy. Again, the unit might be the heat resulting from the passage of a known current of electricity through a conductor of known resistance. While any of these might be adopted as a standard, just as a hydrogen thermometer is the standard in temperature, yet for practical use the unit that is nearly always employed is *the amount of heat that will raise the temperature of* 1 *g. of water* 1° *C.* This unit is called the *gram calorie*, often simply the *calorie.* The amount of heat required to raise 1 kg. of water 1° C. is called the *large calorie.*

It is found that the quantity of heat needed to change the temperature of 1 g. through 1° C. is not exactly the same at different temperatures, being least at about 30° C. and increasing from that point toward 0° C. or 100° C. The difference is slight, and for most purposes the quantity needed to change 1 g. 1° C. may be taken as $\frac{1}{100}$ of that required to change the temperature of 1 g. 100° C. For very precise work it may be necessary to specify that the calorie used was the quantity of heat needed to change the temperature of 1 g. of water from 0° to 1°, from 4° to 5°, or whatever temperature is selected. There is an advantage in taking as the calorie the amount of heat needed to change the temperature of 1 g. from 10° to 11° C., for then the mechanical equivalent of heat (§ 171) is almost exactly 42,000,000 ergs or 4.2 joules. The British thermal unit (B.T.U.) is the quantity of heat required to raise the temperature of 1 lb. of water 1° F.

145. Thermal Capacity.—It is a matter of common experience that the temperatures of equal masses of different substances will often be very different although each may receive the same quantity of heat. That substance in which the temperature is least changed is said to have the greatest capacity, just as in case of several vessels the one which is least filled by the same amount of water has the greatest capacity. The quantity of heat that will change the temperature of a body 1° C. is the thermal capacity of that body. Thus, if 475 cal. of heat will raise the temperature of 1 kg. of copper 5° C., the thermal capacity is

$$475 \div 5 = 95 \text{ cal.}$$

Thus 1000 g. of copper has the same thermal capacity as 95 g. of water.

146. Specific Heat.—The specific heat of a substance is its thermal capacity per unit mass. Thus, if H is the quantity of heat applied to a substance, t the change in temperature, and m the mass, then the specific heat, s, is

$$s = \frac{H}{mt} \qquad (220)$$

Since the value of s for water is unity, the specific heat of any substance is the ratio of the quantity of heat required to change its temperature 1° C. to the quantity required to change the same mass of water 1° C. at a chosen standard temperature. The average specific heat of glass, for example, is about .2, which means that two calories of heat would change the temperature of a given mass of glass as much as ten calories would change the same mass of water.

147. Molecular Heat.—The number of calories required to raise one gram-molecule of a substance 1° C. is called the molecular heat. A gram-molecule of a substance is a number of grams equal to the molecular weight of that substance,—*e.g.*, 2 g. of hydrogen, 44 g. of CO_2, 400 g. of mercury, and so on. The difference in specific heat of various substances is due in part to the difference in the number of molecules in unit mass of those substances. But by taking a gram-molecule of each the number of molecules is the same in all, and the molecular heat for certain groups of substances is then found to be very nearly the same. Thus, the specific heat of oxygen at constant volume is .2175 cal. per gram per degree. The molecular weight of oxygen is 32 g., hence the molecular heat is

$$.2175 \times 32 = 6.95$$

Likewise the specific heat of hydrogen at constant pressure is 3.409, and the molecular weight is 2, hence the molecular heat is

$$3.409 \times 2 = 6.82$$

Similarly for CO, the molecular heat is

$$.245\ (12 + 16) = 6.86$$

For a number of simple substances in a gaseous condition the molecular heat is about the same as in these illustrations, but for solids, liquids, and the more complex gases the molecular heat is much greater. This is due to the fact that the heat applied to a body is expended not only in increasing the motion of translation of the molecules, but also in doing *internal work*, such as the separation of the molecules against a force of cohesion, a change in the rate of motion of the atoms and electrons within the molecules, change in the structure of the molecule, or any work other than that which changes the energy of the molecule in translatory motion. In case of simple gases the molecules are assumed to be very nearly independent of one another, and hence very little internal work need be done. Their molecular heat is consequently expressed by a small number. In other substances the heat must not only increase the translatory motion but also do internal work. The specific heat of copper, for example, is .095, and its molecular weight is 126.8, hence the molecular heat is

$$.095 \times 126.8 = 12.05$$

For steam the molecular heat is

$$.4805\ (2+16) = 8.64$$

A number of substances similar in structure to copper will have about the same molecular heat, while another group of substances will be classed with steam in this respect,—*i.e.*, the difference in molecular heats is due only to the quantity of internal work which must be done.

The relation between specific heat and atomic weight was pointed out by Dulong and Petit early in the nineteenth century, and they announced a law which bears their name, that *the product of specific heat by atomic weight is a constant for all elementary substances.* This, as shown above, is approximate only for certain groups of substances, being about 6.4 for elementary solids and 3.4 for elementary gases. These numbers are one-half of the molecular heats.

148. Change of Specific Heat with Change of Temperature.—The specific heat of most substances increases with increase of temperature. For gases, water, and most solids this change is small, and for most purposes the average specific heat between 0° and 100° C. is a sufficient approximation. In case of most liquids, however, the change is not negligible, and the specific heat of carbon in form of diamond has been shown to be about four times as great at 300° C. as at 0° C.

149. Water Equivalent.—It is often an advantage to know the number of grams of water that will have the same thermal capacity as a given mass of any other substance. This is called the *water equivalent* of the substance, and may be found by multiplying *mass* by *specific heat.* Thus, the water equivalent of 100 g. of copper is 9.5 g. In this way the equivalent of a containing vessel may be added to the water contained, the whole then being treated as water.

150. Latent Heat.—When solids which are crystalline are heated, the temperature will rise till a point called the melting point is reached. The temperature then becomes stationary until all the solid becomes liquid. The quantity of heat required to change 1 g. of such a solid at melting point to a liquid at the same temperature is called the **latent heat of fusion.** If the liquid be further heated, the temperature will again rise until the boiling point is reached, where the temperature again becomes stationary and remains so until all the liquid is converted to vapor. The amount of heat needed to convert 1 g. of a liquid at boiling point to vapor at the same temperature is called the **latent heat of vaporization.** Each crystalline substance has a definite latent heat for both fusion and vaporization, and these values for ice and water are greater than for any other substance. (Table 24.)

The calorists explained these phenomena by saying that a certain quantity of heat became *latent* when a substance changed its state. The term is retained, but the explanation is that the heat which is called latent is expended, not in increasing the motion of translation of the molecules, but in the separation of the molecules against forces which bind them together. Heat energy applied in this manner becomes potential energy, and does not then affect the thermometer which indicates only the energy of molecular motion. When the operation is reversed, as when steam is changed to water or water to ice, all this potential energy will be restored in form of heat energy, just as the potential energy of a mass of iron or stone lifted to a certain height may all be recovered in form of heat or other energy by allowing the mass to fall to its original position.

The latent heat of fusion of ice has been determined many times by various experimenters, and a fair average of the results is that about 80 cal. of heat are required to melt 1 g. of ice,—*i.e.*, the latent heat of ice is 80 cal.

If 100 g. of ice at 0° C. is mixed with 200 g. of water at 100° C., the water equivalent of the containing vessel being a part of the 200 g., the resulting temperature, correction being made for radiation, will be about 40° C. The heat lost by the hot water is expended in melting the ice and raising the temperature of the water resulting from the melting of the ice from 0° to 40° C.

Since the loss of heat on one side is equal to the gain on the other, then, if L is the latent heat of ice, we may write

$$200\,(100-40)=100L+100\times 40$$

$$\therefore\ L=80 \text{ cal. per gram.}$$

In a similar manner it may be shown, by condensing steam in a given mass of water and noting the change in temperature, that the latent heat of vaporization of water at 100° C. is very nearly 536 cal. per gram. Some careful experimenters obtain as high as 536.6 cal.

A convenient instrument for determination of latent heat of vaporization is that known as Berthelot's apparatus, shown in Fig. 181. A small quantity of liquid, about 50 c.c., is put into the vessel *a* and heated by a ring burner beneath. The vapor passes down a tube to the ground joint *J*, where it enters a coil and is condensed by the surrounding water in the calorimeter. The condensed liquid is collected in the bulb *b*, and the amount is determined by weighing the coil and bulb before and after the experiment. The water in the calorimeter is stirred by the loop of wire, and its rise of temperature is noted. From these data the latent heat of vaporization is readily found by use of an equation similar to that given above for latent heat of fusion of ice.

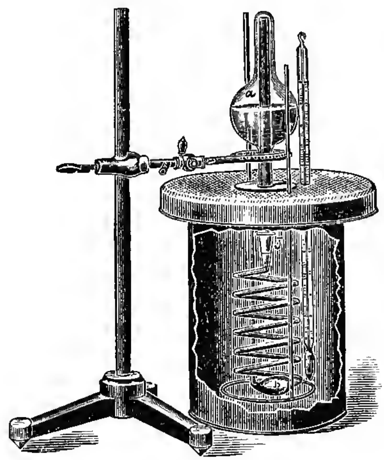


FIG. 181.

151. Specific Heat by Method of Mixture.—A common method of finding the specific heat of solids and liquids is by immersing in water a certain mass of a substance at a high temperature and noting the resulting change of temperature in both the substance and the water. If equal quantities of water at different temperatures are mixed, the resulting temperature will be the average of the two; but if some other liquid or a solid is mixed with colder water, the water will be raised only a few degrees while the other substance will fall many degrees. Let W_w be the weight of the water, W_c the weight of the calorimeter, t_w the change in the temperature of the water, W_b the weight of the solid body or liquid, t_b its change of temperature, s_c the specific heat of the calorimeter, and s the specific heat of the body; then, if all the heat lost by the body is gained by the water and calorimeter,

$$W_w t_w + W_c t_w s_c = W_b t_b s$$

$$\therefore\ s = \frac{W_w t_w + W_c t_w}{W_b t_b} \tag{221}$$

In this equation no allowance is made for radiation. Errors due to radiation may for the most part be prevented by surrounding the calorimeter with non-conducting material or by taking the temperature of the calorimeter as much below that of the room as it will be above at the end of the experiment. If correction for radiation is to be made, it may be done as explained in § 169.

152. Specific Heat by the Method of Melting Ice.—A method employed by Black was to heat a body to the temperature of boiling water, or some other known temperature, and then place it in a cavity in a block of ice, covering all with a slab of ice. A certain quantity of the ice will be melted, and water, ice, and the body will after a time have a temperature 0° C. If now the water be collected by use of a sponge or filter paper, and weighed, it is evident that

$$W_w L = W_b s t_b$$

$$s = \frac{W_w L}{W_b t_b} \tag{222}$$

where L is the latent heat of fusion of ice and the other letters have the same values as in the preceding paragraph. It is evident that exact results cannot be expected by this method.

153. Bunsen's Ice Calorimeter.—An excellent instrument for finding the specific heat of solids and liquids was devised by Bunsen. The principle involved is that when ice melts its volume is considerably reduced—about one-twelfth. A glass tube, t, is fused to a bulb, B, as shown in Fig. 182. From the bottom of B a side tube rises to f. The bulb B is filled with water which has been carefully boiled to exclude all air bubbles. The lower end of B and all of the side tube are filled with mercury. A glass tube of fine bore, about one metre long and graduated in millimetres, is connected by a ground-glass joint to f through a stopcock, C. By turning the stopcock the mercury in the funnel g may be allowed to run out to any desired point in M, and by turning the cock to another position connection with g is closed and the mercury in M is joined to that in f. The whole is enclosed in a double-walled vessel and packed in snow or crushed ice, only the graduated tube and the tops of f and t extending above. By running a stream of ice-water into t, the temperature of the water in B is soon reduced to about 0° C.

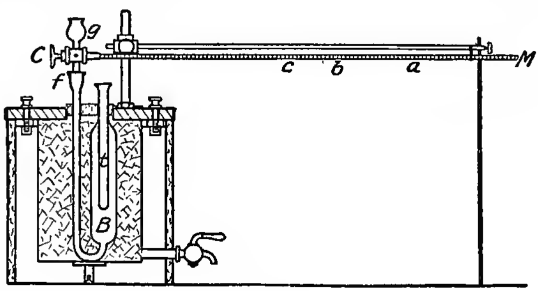


FIG. 182.

Then, by passing through t a stream of alcohol which is cooler than 0° C., a part of the water in B will be frozen. The alcohol may be made cooler than 0° C. by use of a mixture of ice and crystallized calcium chloride or ice and sodium chloride. The bulb B then contains ice and water both at 0° C. and kept so by the surrounding snow. If now any warm body is dropped into t, some of the ice will be melted and so its volume becomes less. Consequently the thread of mercury in M will be withdrawn through a certain distance. To determine the value of the divisions on M, let a mass of water w at temperature $t°$ be dropped into the tube t, and suppose that as a result the mercury withdraws through 200 mm. The amount of heat given out by the hot water is wt cal., for its temperature fell from $t°$ to 0° C. Hence the value of each division on the scale would be

$$\frac{wt}{200}$$

or, to make the conditions general, suppose the end of the thread of mercury moved from a to b. Then the number of calories indicated by each scale division would be

$$\frac{wt}{a-b}$$

Now, if a quantity, w_1, of some other substance at a temperature $t_1°$ and of specific heat s is dropped into the tube, additional ice will be melted and the mercury will further withdraw to some point c,—*i.e.*, through $b—c$ divisions. The number of calories of heat given to the ice by the substance is w_1t_1s cal.; hence again the value of each scale division is

$$\frac{w_1t_1s}{b-c}$$

Hence $$\frac{wt}{a-b}=\frac{w_1t_1s}{b-c}$$

or $$s=\frac{b-c}{a-b}\cdot\frac{wt}{w_1t_1} \qquad (223)$$

154. Specific Heat by the Method of Cooling.—The specific heat of a liquid may be found by noting its time of cooling as compared with the time for water when the rate is the same in both cases. A vessel in form of a bottle having thin metallic walls is suspended in air within another vessel which is surrounded with water or crushed ice, Fig. 183, the temperature of which may be assumed to remain constant. The bottle is first filled with a mass of water, m, at a temperature which may be observed by the thermometer which passes through the stopper and which also serves as a handle. The time, T, required for the contents of the bottle to cool through $t°$ is noted. The bottle is then filled with a liquid of mass m_1, the specific heat of which is sought, and the time T_1 is noted during which the temperature falls through the same range, $t°$. Since the conditions and range of temperature are the same in both cases, the amount of heat that escapes from

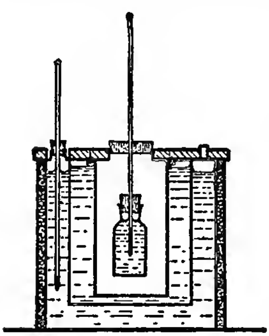


FIG. 183.

the bottle in each case is proportional to the time. Hence

$$\frac{mt}{m_1ts}=\frac{T}{T_1}$$

where s is the specific heat sought, and, since t is the same in both cases,

$$s=\frac{mT_1}{m_1T} \qquad (224)$$

The mass of the liquids must include the water equivalent of bottle and thermometer.

155. Specific Heat by Electric Heating.—An excellent method of finding the specific heat of some liquids is by use of two electric calorimeters similar to that shown in Fig. 184. Two heavy copper wires, covered with a coat of shellac to protect them from the action of the liquids, extend from binding posts down to a coil of resistance wire near the bottom of the calorimeter. If the resistance of the coil is about 2 or 3 ohms, a current of 5 amperes will cause it to become hot and thus heat the liquid in which it is placed. The two calorimeters are exactly alike and the current is passed through them in series. The liquid must be thoroughly stirred and the temperature is read on a delicate thermometer which passes through the cover. One calorimeter is partly filled with water of mass m, and the other with liquid of unknown specific heat the mass of which is m_1. The two masses are made such that the rise in temperature is very nearly the same in each calorimeter. This may be done by a preliminary trial or may be calculated when the specific heats are approximately known. By doing this the water equivalent of the calorimeters and the correction for radiation may be neglected. Let $t°$ be the rise of temperature of the water and t_1 that of the other liquid. Since each calorimeter will receive the same quantity of heat in the same time,

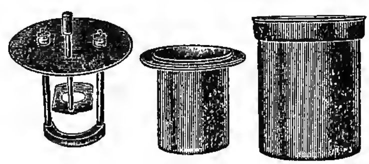


Fig. 184.

$$mt=m_1st_1$$

where s is the specific heat sought. Hence

$$s=\frac{mt}{m_1t_1} \tag{225}$$

156. Specific Heat of Gases.—Gases have two specific heats, (1) specific heat at constant pressure, which may be denoted by C_p, and (2) that at constant volume, C_v.

If a mass of gas is enclosed in a horizontal cylinder one end of which is closed by a movable piston, the pressure on the enclosed gas is that of the atmosphere, which during the time of the experiment may be considered constant. If now the gas is heated, it will expand and move the piston against this pressure. Hence an amount of work will be done equal to the product of the pressure by the change of volume of the gas (§ 102). Thus, not only is the gas heated but a certain amount of work is done beside, both being at the expense of the heat applied. Consequently, when a gas under pressure is made to expand, more heat must be applied to change its temperature through any given number of degrees than when it is confined to a constant volume. When the process is reversed,—*i.e.*, when heat is taken from the gas,—the quantity will be not only that which caused a rise in temperature but also that which was expended in work.

If the gas is confined in a vessel which does not permit a change of volume, less heat is required to cause a given rise of temperature, for no external work is done. Hence the value of C_p is always greater than C_v.

Specific heat of gases cannot be accurately determined by simple devices such as are used for solids and liquids, for the thermal capacity of the containing vessel is large compared with that of the gas itself, and so it is difficult to obtain reliable data for the gas which contains only a small part of the total heat.

The operation for finding C_p usually consists in passing a large quantity of gas continuously through a long spiral coil immersed in a bath of hot water or other liquid. The gas, heated as it passes through this coil, is then passed on through a second coil immersed in the water of a calorimeter. The product of the mass of water by its rise of temperature gives the number of calories of heat received from the gas. The mass of gas is calculated from its volume and pressure. The product of its mass by its fall in temperature while passing through the calorimeter and by its specific heat is the number of calories of heat given to the water. By equating the calories given up by the gas and those received by the water, the specific heat of the gas is readily found. By this method a large quantity of gas may be used, and by passing it slowly through the coils the pressure may be kept practically constant.

Specific heat of a gas at constant volume is difficult to determine by experiment, for reasons mentioned above; but by use of the **steam calorimeter,** invented by Jolly, of Dublin, fairly reliable results have been obtained. This consists of a vessel, *B*, Fig. 185, into which steam is admitted through a tube, *s*. The mass of steam condensed on any object within *B* is a measure of the quantity of heat needed to raise it to the temperature of the steam. For gases two hollow globes of the same mass and material are suspended within the calorimeter, one from each end of the beam of a delicate balance. Both globes are exhausted. If steam is now admitted and the balance remains undisturbed, the heat capacity of the two globes is the same, for the same amount of steam is condensed by each. If the balance is disturbed, weights are added to compensate for the difference. One of the globes is now filled with a gas under a pressure of 30 or 40 atmospheres, that the mass may be as great as possible. The other globe remains exhausted. Steam is again admitted, and the condensation on the globe filled with gas will be increased, for the gas must receive sufficient heat to raise its temperature to that of the steam. The weight needed to restore the balance is the mass of water condensed by the gas. Knowing the mass of gas m_g, its rise in temperature $t°$, and the mass of steam condensed m_c,

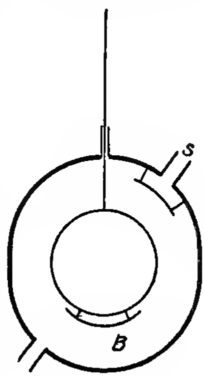


FIG. 185.

$$m_g ts = Lm_c$$

$$\therefore s = \frac{Lm_c}{m_g t} \qquad (226)$$

where s is the specific heat of the gas and L is the latent heat of steam. No allowance for the thermal capacity of the globe need be made, for the globes are alike in this respect, nor does the change of buoyancy due to immersion in steam affect the balance when two globes are used.

The steam calorimeter may be used in finding the specific heat of solids and liquids as well as of gases. The solid may be suspended by a fine wire from one end of the beam and counter-balanced by weights at the other end.

Specific heat of a gas at constant volume may be deduced from the fact that the ratio of C_p to C_v is a constant quantity for any given gas. Thus

$$\frac{C_p}{C_v}=\gamma$$

where γ is a constant quantity, being 1.41 for air. The value of γ may be found from the velocity of sound in any gas; the value of C_p is found in the manner described above, and C_v may then be calculated from

$$C_v=\frac{C_p}{\gamma} \tag{227}$$

This ratio is more fully discussed in § 175.

Problems.

1. If 50 g. of copper at 100° C. immersed in 50 g. of water at 20° C. raise the temperature of the water to 26.94° C., what is the specific heat of the copper?

2. If a copper calorimeter weighs 50 g. and contains 80 g. of water at 18° C., what will be the temperature of the water after 10 g. of melting ice have been strired into it?

3. How much steam at 100° C. must be condensed in 1 kg. of water at 20° C. to raise the temperature of the water to 75° C.?

4. How much heat would be required to melt a block of ice 20×30×50 cm.?

5. A room measures 3×4×6 m. The air within the room is at a temperature of 15° C. and under a pressure of 72 cm. How many degrees will the temperature of the air be raised by condensing 1 kg. of steam at 100° C. to water at 100° C. in the steam radiators?

1. .095.
2. 7.65° C.
3. 98.04 g.
4. 2367.18 large calories.
5. 26.6° C.

157. Fusion and Solidification. — The state of a substance, solid, liquid, or gas, is mainly dependent on temperature. Iron, for example, is known as a solid because its temperature is commonly such that it is found in that state. Mercury for the same reason is usually known as a liquid, and hydrogen as a gas. But any of these substances may be changed to any one of the three states by proper changes in temperature and pressure.

In case of most crystalline substances there is a definite temperature known as the melting point. When this point is reached, a solid will begin to change to the liquid state. While the solid is melting, the temperature will be constant, for the heat energy applied is expended in causing a change of state. During the process of melting, the solid and liquid exist side by side in equilibrium. More heat simply changes some of the solid to liquid, and if some heat is abstracted, a portion of the liquid will change back to the solid state. The quantity of heat per unit mass required to produce this change of state is called the latent heat of fusion. (§ 150.) (Appendix 24.)

Substances which are amorphous,—not crystalline,—such as glass, rosin, solder, paraffin, etc., gradually soften and finally become liquid as the temperature is raised, but there is no exact point at which they may be said to fuse.

The melting point of alloys is lower than that of the metals which are fused together to form the alloy. Thus, by melting together tin and lead in different proportions, solder of different degrees of hardness may be made. Rose's fusible metal, composed of 4 parts bismuth, 1 part tin, and 1 part lead, melts at 94° C. Wood's fusible metal, 4 parts bismuth, 2 parts lead, 1 part tin, and 1 part cadmium, by weight, melts at 60.5° C. There appears to be a change in the grouping of molecules of an alloy so that not so much heat is needed to change the state.

Substances which have a definite melting point will in some instances also change abruptly in volume at the moment of change of state. Cast iron has practically the same volume in the solid or liquid state, hence it will take the exact form of the pattern in moulding. Bismuth and antimony increase slightly in volume when they solidify. Most substances decrease in volume when they change from liquid to solid. Water more than any other substance increases in volume when it solidifies.

One cubic centimetre of water at 0° C. will in form of ice have a volume 1.0907 c.c., an increase of more than 9 per cent. This is a fact of great economic value in nature.

It may readily be inferred from what has just been said that pressure affects the melting point of those substances that change in volume on change of state, for pressure would either assist or hinder that change of volume which accompanies the change of state. Phosphorus, for example, increases in volume when it is melted. It will when all pressure is removed melt at about 44° C., but under a pressure of 2000 kg. per sq. cm. it melts at about 97° C. Ice, on the other hand, melts at a lower temperature under pressure. Professor James Thomson showed, from theoretical considerations (see equation 259), that a pressure of one atmosphere lowers the melting point of ice .0075° C. Lord Kelvin later verified this by experiment. Under

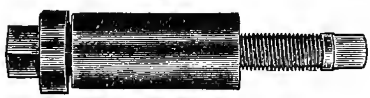


FIG. 186.

a pressure of 1000 atmospheres water will not freeze above —7.5° C. Hence, if a strong vessel is filled with water and closed, the water will either remain a liquid or the vessel will be burst when the temperature is reduced below 0° C. Many illustrations of the effect of pressure on the melting point of ice might be given. If a strong iron cylinder, Fig. 186, be filled with fragments of ice, both cylinder and ice being below 0° C., and pressure be applied by screwing in the plug which exactly fits the bore of the cylinder, the ice will be melted, but when the pressure is then removed the water will by regelation become one solid block of ice. In accord with this same principle, snowballs are formed by the pressure of the hands, but if the snow is very cold the pressure may not be sufficient to cause any melting. An experiment due to Bottomley consists in suspending a weight from each end of a wire thrown over a block of ice. The ice beneath the wire is melted by pressure and flows to the upper side where it is frozen. Thus the wire will in time pass through the ice and leave the block as solid as before. The formation of "ground ice" at the bottom of streams or at points where there are eddies in the current results from the fact that water at such points may be near the freezing temperature, and pieces of ice carried by the water are driven against the bottom. The ice is first melted at the point of concussion and then at once is frozen and so adheres to the bottom. Other pieces in a similar manner are frozen to this and so the mass accumulates.

158. Freezing Point of Solutions.—It is a matter of common observation that a liquid when pure will freeze at a higher temperature than when it contains foreign substances in solution. The principles underlying this subject were investigated in the last quarter of the nineteenth century by the French chemist François Marie Raoult. He found experimentally the lowering of the freezing point which resulted from a solution of acids, bases, and salts in water, acetic acid, benzene, and other solvents. He used very dilute solutions, less than one gram-molecule in 2 kg. of the solvent. The advantage in this is that a considerable quantity of ice can be formed without greatly changing the concentration, the range of lowering can be made 1° C. or less and so a delicate thermometer can be used, and dissociation if such occurs is most nearly complete in very dilute solutions. From this it is possible to calculate the lowering which would occur in a 1 per cent. solution, assuming that the rate of lowering would continue. Thus different solutions would all be reduced to a standard for comparison. The lowering for 1 per cent. solution—*i.e.*, 1 g. of a substance in 100 g. of the solvent—is called the coefficient of lowering. Raoult's expression for finding this is

$$A = K\frac{P}{P' \times 100} \qquad (228)$$

where A is the coefficient of lowering, K is the lowering observed in the experiment, P is the mass of the solvent, and P' is the mass of the substance dissolved. The product of this value of A by the molecular weight M of the dissolved substance is the molecular lowering T,—*i.e.*, T is the lowering which would be obtained if one gram-molecule of the substance had been dissolved instead of 1 g. This is expressed by Raoult in the equation

$$MA = T \qquad (229)$$

As a concrete illustration of this procedure, suppose 40 g. of H_2SO_4 is dissolved in 1000 g. of water and that the observed lowering is 1.56° C. Then, by equation (228),

$$A = 1.56\frac{1000}{40 \times 100} = .39$$

The molecular weight of H_2SO_4 is $2+32+64=98$. Hence, from equation (229),

$$T = .39 \times 98 = 38.22$$

The values of T throughout a great variety of solutions cluster about two numbers, one of which is twice as great as the other. For solutions in water the numbers are 18.5 and 37. For the same number of physical molecules in a given solvent the lowering is the same whatever the character of the substance may be. Variation in experimental results can usually be explained as resulting from special causes. The purpose of multiplying A by M in equation (229) is that the lowering can thus be obtained for the same number of molecules in all cases, for in equal masses of two substances the numbers of molecules vary inversely as the molecular weights. Distinction is made between a chemical molecule and what is often called a physical molecule, the latter of which may consist of a grouping of, or may be a part of, a chemical molecule. The greater the number of physical molecules the greater is the lowering of freezing point. When water is used as the solvent the values of T are either 37 or 18.5. For all strong acids and bases and all salts of alkalies the number is 37. These are also the substances which cause abnormally great osmotic pressure (§ 127) and an abnormal elevation of the boiling point (§ 161). These are also the solutions known as electrolytes,—*i.e.*, conductors of electricity. In explanation of this difference in the behavior of solutions, Svante Arrhenius in 1887 announced the dissociation theory now generally accepted. In accordance with this theory, the molecules of the dissolved substance in an electrolyte separate into ions, which are atoms or groups of atoms charged with positive or negative electricity. Thus H_2SO_4 in water will separate into $\overset{+}{H}$ and $\overset{-}{SO}_4$. Each ion then acts as a physical molecule, and the molecular lowering is therefore twice as great as when such dissociation does not occur. In the experimental work of finding the freezing point the solution is placed in a test tube around the bulb of a delicate thermometer, the whole being surrounded by a freezing mixture. When the temperature is sufficiently reduced small flakes or granules of ice will appear. The difference between this temperature and that at which the pure solvent freezes is the lowering due to the substance in solution.

Since it is the solvent that freezes, the remaining liquid is a more concentrated solution than before. If the temperature is further reduced, more ice will be formed, and so on till the solution is saturated. Any further withdrawal of heat does not reduce the temperature, but causes more of the solvent to freeze and a precipitation of some of the substance in solution. This may be continued till the whole becomes a solid,—a mechanical mixture called cryohydrate.

It was pointed out by Raoult that the laws governing molecular lowering could be used to determine molecular weight. For illustration, the value of T for all salts of alkalies is 37. Then, if the coefficient of lowering is found for any salt of this kind, by equation (228), the molecular weight is

$$M=\frac{T}{A} \tag{230}$$

159. Evaporation.—The process by which many substances slowly and quietly change to a vapor is known as evaporation. The process is chiefly observed in the change of volatile liquids to aeriform fluids, as in case of water, alcohol, ether, etc. Some solids, such as snow, ice, iodine, etc., may change to an aeriform state by sublimation,—*i.e.*, they appear to evaporate directly without change to a liquid.

Evaporation is a result of molecular motion within a substance, and hence may be considered as a heat phenomenon. It has already been explained that when a substance is in a gaseous state the molecules are widely separated from one another as compared with the diameter of the molecules themselves, the interspace being something like 100 times greater than the diameter. Hence there is freedom of motion and but little constraint from neighboring molecules. In solids and liquids, however, the molecules are within the range of attraction of one another. In liquids a molecule has freedom of motion through the mass of a substance, but whatever position it may have, it is subjected to the attractive influence of its neighbors (§ 119).

The molecules of a liquid are in motion in all directions as long as the substance contains any heat energy. Only at the theoretical, absolute zero are all supposed to be at rest. If, then, molecules move up to the surface of a liquid, as countless numbers are doing, their escape into the space above will in most cases be prevented by the attraction of their neighbors below. Consequently a liquid has a definite surface and a surface tension. It may readily happen, however, that some molecules moving with greater speed than others will leap from the surface and will free themselves from the attractive force of their neighbors. In this way a mass of liquid may under proper conditions of temperature and pressure be completely changed to a vapor. Not only do molecules of the liquid leap into the space above, but a number of those of the vapor re-enter the liquid. As long as the former is in excess, evaporation will continue. When the number leaving the liquid is equal to the number that re-enter it, the vapor is said to be saturated. Then, although evaporation still continues, the quantity of liquid is not diminished.

A liquid is cooled by evaporation. This is as would be expected, for those molecules that are moving with greatest velocity are the ones most likely to leap from the surface, hence there is a decrease in the average kinetic energy of the molecules that remain in the liquid. Hence when a liquid evaporates rapidly there will be a rapid fall in temperature. This fact is utilized in the manufacture of ice. Ammonia gas is liquefied by cooling and compression and is then allowed to evaporate rapidly into a coil which is submerged in strong brine. The temperature of the brine is thus reduced below 0° C. and is then made to flow about the metal moulds which contain the water to be frozen. The same gas is returned to the pump, where it is again liquefied and the operation is repeated.

In accordance with the theory of the cause of evaporation, it is plain that the rate at which a liquid will evaporate depends on (1) the area of the surface exposed, (2) the temperature of the liquid, (3) the removal of the vapor as soon as it appears at the free surface, as by fanning or any movement of air, and (4) decrease of pressure on the surface of the liquid.

160. Vapor Pressure.—The molecules of a vapor are in rapid motion, and so will, like a gas, exert a pressure on the walls of a containing vessel. The vapor of each liquid will when saturated exert a pressure known as its vapor pressure at that temperature. While a saturated vapor is in presence of its liquid, any change in the volume of the vapor will not change the pressure, for any decrease of volume only changes some of the vapor to liquid and any increase of volume only allows some liquid to vaporize. This fact may be shown experimentally by use of the apparatus shown in Fig. 187. A U-shaped glass tube, each arm of which is about 76 cm. long, is closed at one end by a stopcock. By inclining the tube to one side with the end *a* beneath the surface of mercury, an aspirator or pump may be used to fill one arm of the tube. If the stopcock is now closed and the tube placed upright, the difference in height of the mercury columns in the arms will show the atmospheric pressure in centimetres of mercury. If now the tube *a* above the stopcock is filled with ether or some other volatile liquid, and the cock is turned so that only a drop or two of the liquid is admitted to the vacuum above *c*, the heights of the mercury in the arms will change, as shown in *B*. Sufficient ether is admitted so that a small quantity in the liquid form may be seen at *c′*. The vapor is then saturated, and the difference in the levels *c′* and *b′* subtracted from the difference in level of *c* and *b* is the vapor pressure expressed in centimetres of mercury. If this pressure were equal to the pressure of the atmosphere, *c′* and *b′* would be at the same level.

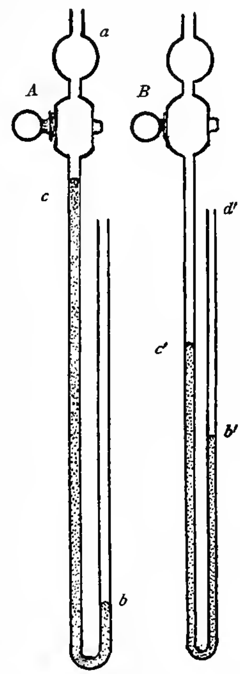


FIG. 187.

Now let some mercury be poured in at *d′*. Both *c′* and *b′* will rise, but their difference of level will be unchanged. As *c′* rises, more vapor will change to liquid, but the pressure will not change. (Appendices 28 and 29.)

161. Boiling Point.—When a mass of liquid is heated, not only is evaporation increased but at a certain temperature bubbles of vapor formed within the liquid rise to the surface. The liquid is then said to boil, or to be in a state of ebullition. When boiling begins, temperature becomes constant, and all heat applied to maintain the process of boiling becomes latent heat of vaporization (§ 150). The temperature at which boiling begins is called the boiling point under the conditions present. Boiling point is greatly modified by pressure. This is as would be expected, for bubbles of vapor cannot form until there is equilibrium between vapor pressure and external pressure. In fact boiling point may be defined as such an equilibrium. It will be noted in appendix 28 that water boils at 100° C. when the pressure of the atmosphere is 76 cm., because the vapor pressure at that temperature is also 76 cm. The same table shows that when the external pressure is 45 cm. water will boil at 83° C. When a liquid is confined in a vessel,—*e.g.*, water in a boiler,—it may be heated far above the boiling point, for the pressure of steam above the water prevents vaporization. Any decrease of steam pressure, as when a valve to the engine is opened, permits some water to vaporize. When a confined mass of water contains all the heat necessary for its vaporization, it will, unless the containing vessel is strong enough to prevent it, explode with terrific violence, for 1 c.c. of water will when vaporized in air occupy nearly 1500 c.c. of space. It is safe to heat water to the boiling point, 100° C., only because an additional 536 calories must be added to convert each gram of it to a vapor, and this under ordinary conditions is a slow process.

Many other conditions also modify the boiling point. The nature of the material of a vessel and the roughness or smoothness of the interior walls may cause a difference of several degrees. In a glass vessel the temperature of boiling water may be as much as 3° C. higher than in a metal one. A few tacks or a small quantity of sand thrown into a vessel of hot water will lower the boiling point 1° C. or more. Water must contain nuclei of some kind, such as air, which is nearly always found in water, before bubbles of vapor can be formed. Pure water from which all air has been removed may be heated far above the boiling point. It is then in an unstable condition and liable to vaporize all at once,—*i.e.*, it will explode.

Bubbles of vapor formed at the bottom of a vessel are under the pressure of water above them and also the pressure due to surface tension of the bubble. For these reasons, the temperature is higher than need be for equilibrium with atmospheric pressure, but when the bubble breaks at the surface, the vapor at once expands and is thus cooled to the true boiling point. For these reasons, the bulb of a thermometer is suspended in the vapor above a liquid, and not in the liquid, when the true boiling point is sought.

The boiling point of solutions was investigated by Raoult, and his results were published in 1887–8 A.D. His researches showed that when one gram-molecule of a non-volatile substance is dissolved in 100 g. of a solvent, the molecular lowering of vapor pressure is independent of the nature of the substance,—*i.e.*, is dependent only on the number of physical molecules present; that in dilute solutions the lowering is proportional to the concentration; and that for a given solvent there is nearly a constant ratio between the molecular lowering of the freezing point and that of vapor pressure. Since vapor pressure is lowered by the presence of substances in solution, the boiling point is raised, for a higher temperature is then necessary to cause equilibrium between the vapor and the external pressure.

Some substances in solution cause an abnormal lowering of vapor pressure. These are the same substances as those that cause an abnormal osmotic pressure or an abnormal depression of freezing point. These phenomena are due to dissociation (§ 158).

From a knowledge of the lowering of vapor pressure or, what is more easily determined, the elevation of the boiling point, it is possible to calculate molecular weights of soluble substances that are non-volatile. The vapor above a solution is that of the pure solvent, and hence the bulb of a thermometer must be placed in the solution to obtain its boiling temperature.

162. Isothermals of a Vapor.—According to Boyle's law, the product of pressure by volume of a gas is constant as long as the temperature does not change. A curve plotted on the pressure-volume diagram for a gas which is very nearly true to Boyle's law is shown in Fig. 120. This is the curve for a given mass of air at 22° C. If air had been at a higher or lower temperature, other similar curves would have been formed either farther from or closer to the axes of reference. Such curves are called isothermals, for the temperature remains constant during the changes in volume and pressure. In case of a vapor, however, the curve is different, because at the point of saturation a decrease of volume is not accompanied by an increase of pressure (§ 160). Thus, in Fig. 188 let the point *a* represent the pressure and volume of an unsaturated vapor. As the pressure increases the volume will decrease, forming the curve *ab*, which is thus far much like the curve of a gas.

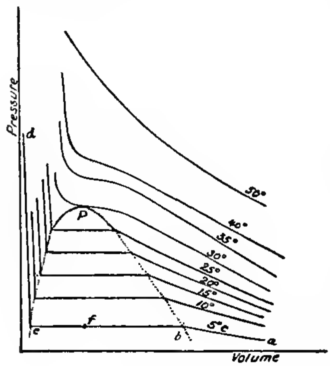


FIG. 188.

Further decrease of volume does not increase the pressure (§ 160), but will convert the saturated vapor at *b* to a liquid at *c*. This change is therefore represented by the horizontal line *bc*, in which the vapor and liquid exist together with a distinct plane of separation. If the state is *f*, the ratio of the quantity of vapor to the quantity of liquid will be as *cf* to *fb*. When all is in the state *c* a further increase of pressure results in only a slight diminution of volume, as indicated by *cd*, for liquids are only slightly compressible. The curve *abcd* is an isothermal of a vapor. If successively higher temperatures are taken, a similar series of changes in volume and pressure will produce similar isothermals, but the horizontal parts of the curves will become shorter and shorter, for there must be a greater decrease of volume before condensation begins and the liquid has greater volume at higher temperature. The point at which the horizontal line becomes infinitely short is called the critical point. If a line is passed through the points of saturation and the points of complete conversion to liquid, as shown by the dotted line in Fig. 188, the highest point of the curve thus formed is the critcial point *P*. The line *Pb* is sometimes called the steam line and *Pc* the water line. The temperature of that isothermal which passes through the critical point is called the critical temperature. The pressure corresponding to the point *P* is called the critical pressure, and the volume of unit mass of a substance in this state is the critical volume. As the temperature is increased more and more above the critical one, the isothermals show that the substance complies more and more closely with Boyle's law.

The distinction between a gas and a vapor is as follows: A substance in a gaseous form and at a temperature above the critical one is called a gas, but below the critical temperature the same substance is called a vapor.

That a gas may be converted to a liquid it is necessary to cool it below the critical temperature and then subject it to a pressure sufficient to saturate the vapor at that temperature. No amount of pressure will liquefy a gas above the critical temperature. Oxygen, for example, must be cooled to −118° C. and put under a pressure of about 50 atmospheres before it will become liquid. All well-known gases have been liquefied.

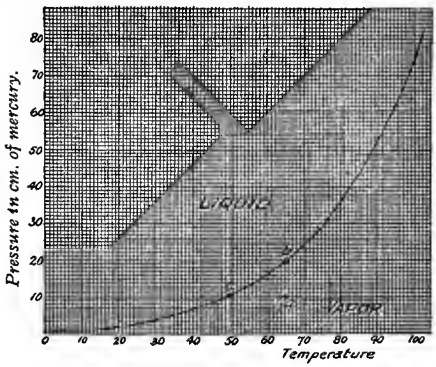


FIG. 189.

This whole subject of the behavior of vapors as compared wth gases was investigated by Andrews, of the University of Glasgow, previous to the year 1869. His diagram for carbon dioxide is similar to that shown in Fig. 188. At a temperature of 13.1° C. and a pressure of 49 atmospheres the CO_2 was changed to liquid. At 21.5° C. a pressure of 60 atmospheres was required to effect the same result. The critical temperature was found to be 30.9° C. At 31.1° C. the isothermal still showed considerable deflection, but the substance remained a homogeneous mass,—*i.e.*, did not become part vapor and part liquid, as it would have done during part of the process if below the critical temperature. At higher temperatures the deflections became less and less until at 48.1° C. they entirely disappeared.

The state of a saturated vapor is often conveniently represented on a **pressure-temperature diagram.** The curve in Fig. 189 is plotted from the table in appendix 29. The point *a* represents a state of unsaturated vapor at 65° C. and a pressure of 10 cm. It is plain from this diagram that a vapor in this state may be saturated either by increasing the pressure to *b*,—*i.e.*, to 20 cm.,—or by reducing the temperature to *c*,—*i.e.*, to 49° C. Whenever the state may be represented by a point on the saturation curve, the vapor is saturated.

In Fig. 190 any point on the curve *Pc* represents a state in which a vapor and its liquid are in equilibrium,—*i.e.*, they can exist together without any of either passing over into the state of the other. This is the saturation curve of Fig. 189. Any point in *Pb* represents the pressure and temperature at which a state of equilibrium exists between the solid and the liquid. In this diagram *Pb* is drawn for ice and water, and, since ice melts at a slightly higher temperature when pressure is decreased, the line will slope a very little downward toward the right. Some solids, such as ice, will evaporate and pass into vapor by a process called sublimation. This process will continue until there is equilibrium between the pressure resulting from the tendency of the solid to pass into vapor and the counter pressure of the vapor against the solid. Any point on the line *Pa* represents a state of equilibrium between a solid and its vapor at that pressure and temperature. A point common to these three lines represents a state called the **triple point.** Here the vapor is in equilibrium with its liquid, the liquid with its solid, and the solid with its vapor. This condition may be realized by a simple experiment first performed by Leslie. A shallow metal dish containing 3 or 4 c.c. of water is supported over another dish containing strong sulphuric acid. These are placed on the plate of an air-pump and covered with a shallow receiver. By exhausting the air the water will rapidly evaporate and its vapor is in large measure absorbed by the acid. Satura-

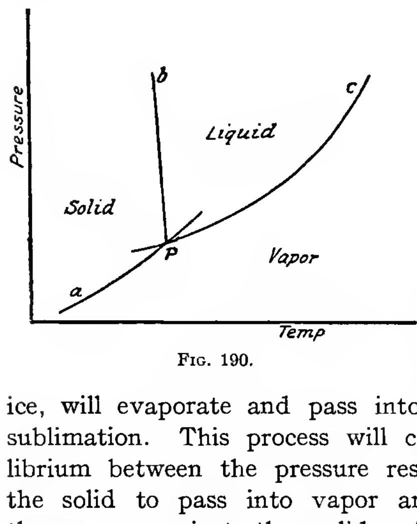


FIG. 190.

tion of vapor in the receiver is thus prevented and the water will be rapidly cooled to the freezing point where its vapor pressure is only .46 cm. The triple point has then been reached, and the water is observed to boil and freeze at the same time.

163. Humidity. — Humidity is the state of the atmosphere in reference to the amount of water vapor it contains. Relative humidity is the ratio between the mass of vapor actually contained in a given quantity of air and the mass which it would contain if it were saturated. This is evidently the same as the ratio of the vapor pressures or vapor densities on the assumption that Boyle's law holds true for vapors. Absolute humidity is the mass of vapor in the unit volume of air. Relative humidity is of greater importance and is usually designated simply as humidity. Air in its natural state always contains more or less water vapor, which may be brought to a state of saturation by a reduction of temperature, thus causing clouds, fog, and dew. The dryness of air, however, depends not so much on the quantity of vapor present as on the nearness to saturation. An increase of temperature will cause air to appear dry though the quantity of vapor remains the same.

That which relates to the determination of humidity is called **hygrometry,** and an instrument used for this purpose is a hygrometer. The three classes of instruments of most importance are (1) chemical hygrometers, (2) dew-point hygrometers, and (3) wet and dry bulb hygrometers.

In use of a **chemical hygrometer** a quantity of air of known volume, temperature, and pressure is passed through a drying tube which contains calcium chloride, pumice stone soaked in sulphuric acid, or phosphorus pentoxide. The increased weight of the tube is the actual amount of vapor in the air used. The ratio of this quantity to that which the air would hold if saturated is the humidity. (Table 30.) This method is accurate, but is somewhat difficult and tedious and is seldom used except for special scientific purposes.

The **dew-point hygrometer** is in common use. There are several different styles of instruments of this kind. A good form is one shown in Fig. 191, known as the Alluard hygrometer. The face of this instrument is polished nickel. The surface D is the front part of a metal tube which is filled with ether. One thermometer shows the temperature of the ether and the other that of the air. By attaching a long rubber tube to A, it is possible by use of a bulb to force a stream of air bubbles up through the ether. Thus the temperature is rapidly lowered and a mist appears on the surface D. The temperature of the ether when the mist first appears is the dew-point. The thermometers are read through a telescope at a distance, so that the humidity may not be affected by the breath or heat of the body. Knowing the dew-point and the temperature of the air, it is seen from the diagram, Fig. 189, that the pressure of a saturated vapor at these temperatures would be represented by ordinates erected at the proper points of temperature on the abscissa and limited above by the saturation curve. The ratio of the ordinate at dew-point to the one at the temperature of the air is the humidity, for the longer ordinate is the pressure which the vapor would exert at that temperature if saturated. Instead of using the curve the pressures may be taken from table 29 in the appendix.

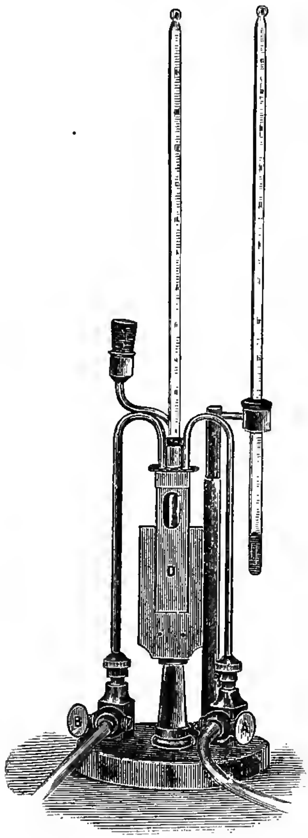


FIG. 191.

The **wet and dry bulb hygrometer** is also in common use. One form of it is shown in Fig. 192. Two thermometers are mounted as shown, and the bulb of one of them is covered by a hollow wick which extends into a vessel of water. If the air were saturated with moisture no water would evaporate and the two thermometers would show the same temperature. But in proportion as the air is dryer the rate of evaporation will be greater and consequently the reading of the wet bulb thermometer will be lower. Evaporation at the maximum rate will occur when the air is moving about 10 feet per second. After the temperatures have become stationary the thermometers are read. Then, by reference to psychrometrical tables prepared by long observation and by comparison of this instrument with dew-point hygrometers, the humidity may be directly found.

164. Transference of Heat.—There are three ways by which heat is distributed or moved from one point to another,—viz., conduction, convection, and radiation. There is but one way, in fact, by which heat as such distributes itself through a body, and that is by conduction, where heat energy is passed from molecule to molecule. Convection is an efficient method in the distribution of heat, as when portions of heated gases or liquids are displaced by buoyancy, thus causing a circulation which brings the colder portions of the fluid in contact with the source of heat. Convection involves the use of some agent outside of the heated body to effect the transference. The heated body is carried from place to place. A hot mass of iron carried into a cold room would be a method of distribution according to the principles of convection. The uses of convection are numerous, as in the heating of water by the application of heat to the bottom of a vessel; the heating of buildings by the circulation of hot air or hot water; the circulation of the atmosphere and the movements of ocean currents. In case of radiation a heated body sets up ethereal vibrations which are not heat, though they possess energy at the expense of the body whence they came, and when they are arrested by another body heat energy will appear again. Thus heat energy may be transferred through the agency of ether waves. Conduction and radiation are more fully discussed in the sections which follow.

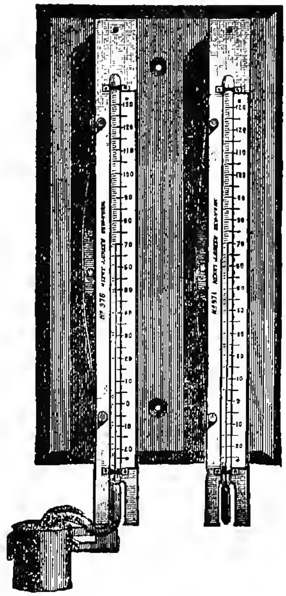


FIG. 192.

165. Conduction. — While we say that heat is conducted from molecule to molecule throughout a mass of matter, yet it is not known by what mechanism this is accomplished. There appears to be an intimate relation between heat and electricity, as is evidenced by the fact that a good conductor of heat is also a good conductor of electricity. It has been suggested that the "roaming electrons," which when set in motion along a conductor cause what is known as a current of electricity, are also the agents by which heat is conducted. Silver and copper are the best conductors of electricity and also of heat. A list of conductors and non-conductors of electricity stand in nearly the same order as for heat.

Heat conductivity is usually measured by the number of calories of heat that will pass through a cubic centimetre of a substance in one second when the difference of temperature on two opposite faces is 1° C. It is plain that more heat, Q, will flow in proportion as the area of the faces a, the time T, and the difference of temperature, t_1-t_2, are greater. Also Q will be less in proportion as the distance, l, between the two faces is greater. Hence

$$Q \propto \frac{a(t_1-t_2)T}{l}$$

If k is the constant for any given substance,—*i.e.*, the conductivity as just defined,—then

$$Q=k\frac{a(t_1-t_2)T}{l} \qquad (231)$$

where $Q=k$ when all the other terms in the equation become unity. From (231)

$$k=\frac{Ql}{a(t_1-t_2)T} \qquad (232)$$

By use of an apparatus like that shown in Fig. 193 the value of k may be found experimentally. A rod of copper, c, is enclosed at one end in a jacket through which steam from the tube a is made to flow. The other end is surrounded by a coil through which a steady stream of water flows. The thermometers t_1 and t_2 are fitted into holes in the rod at a distance, l, from each other. The whole is packed in abestos or other non-conducting material, that radiation may be prevented. Practically all the heat communicated to the rod by the steam will be conducted to the water at the other end. After the steam and water have been flowing for some time the readings of the thermometers will become stationary. Then, by collecting a mass, M_w, of

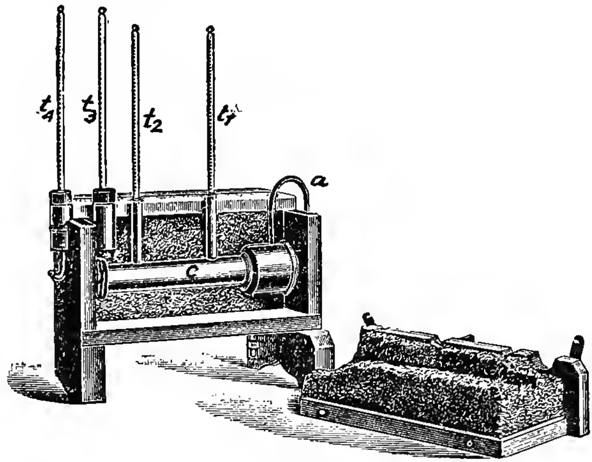


FIG. 193.

water which flows in time T around the end of the rod, the quantity of heat which flows through the rod in that time can be measured. The temperature of the water has been raised $t_4 - t_3$ degrees; hence

$$Q = M_w(t_4 - t_3) \tag{233}$$

Equation (232) may then be written

$$k = \frac{M_w (t_4 - t_3)\, l}{a\,(t_1 - t_2)\, T} \tag{234}$$

in which all the terms in the right-hand member may be determined by the experiment.

The rate at which the temperature of a bar of metal or other substance will rise when heat is applied at one end of it depends not only on thermal conductivity but also on another property called **diffusivity,**—*i.e.*, the rate at which heat spreads, causing a rise of temperature in the cooler parts of the body. This depends on the specific heat of the body as well as on the conductivity. A body of large coefficient of thermal conductivity and also large specific heat may diffuse heat more slowly than one of low conductivity and very small specific heat. Let s be the specific heat and ρ the density of the substance, then $s\rho$ is the quantity of heat required to raise the temperature of unit volume 1° C. But if the quantity of heat measured by k is applied to this unit volume, the temperature will be raised $t°$. Then

$$k = s\rho t$$

$$\text{or} \qquad t = \frac{k}{s\rho} \tag{235}$$

Thus it is seen that the rise of temperature is directly proportional to conductivity and inversely proportional to specific heat. The ratio expressed by the right-hand member of equation (235) is called the diffusivity.

166. Radiation.—Radiation is a process by which a heated body sets up waves in the surrounding ether. The body is thus cooled by a loss of heat energy which then appears as energy of wave motion. Ether serves as a medium for the transference of heat, but the medium itself is not heated. If several bodies at different temperatures are placed apart from each other in an enclosure from which the air is exhausted, all will in time have the same temperature. The transference could not have been by convection for there was no air in the vessel, nor could it have been by conduction for the bodies were not in contact. One may feel the warmth of a fire at a distance from it, though the intervening air may be at freezing temperature. Heat energy of the sun is transformed into that of ether waves which move with the velocity of light, $3(10)^{10}$ cm, through about $1.5(10)^8$ kilometres of space to the earth, where the energy is converted back to heat.

According to **Prevost's theory** of exchanges, all bodies are constantly giving out ether waves whether there are other bodies to receive the waves or not. When several bodies of different temperatures are considered in their relation to one another, the hotter bodies radiate more heat than they receive from the cold ones, so in time there will be thermal equilibrium. Radiation still continues, but each body receives as much heat as it loses and hence there is no change of temperature.

There is abundant evidence that heat and light waves are identical in character, the only difference being the length of the waves. Both are transmitted on the ether and travel with the same speed. Both may be reflected, refracted, or dispersed. They are distinguished only by the difference in effects produced by waves of different length. When the wave length is .0004 mm. a sensation of violet is produced in the eye. As the waves grow longer and longer all different shades of color will result until red is reached, where the wave length is .00076 mm. These are only the limiting values of wave lengths that affect the eye. Much longer waves, formerly called heat waves, may be detected below the red of the spectrum.

167. Source of Ether Waves.—Waves set up in any medium have their origin in a vibrating body. A vibrating bell or tuning-fork sets up waves in the air, but waves of heat and light are transmitted on ether. These travel with a speed of $3(10)^{10}$ cm. per second; hence, if the wave length is .00004 cm., as in case of violet light, the number of waves per second must be $3(10)^{10} \div 4(10)^{-5} = 7.5(10)^{14}$ This then must be the number of vibrations of the particles which cause waves of violet. It is thought that the vibrating particles which cause heat and light radiations are not the molecules but the minute corpuscles of which the molecules are composed. Waves of ether may vary in length all the way from .00001 cm. to several kilometres in length.

168. Measure of Radiant Energy.—Instruments used in the measurement of etherial radiations are called radiometers. Various forms of radiometers have been devised, the most common of which are here described.

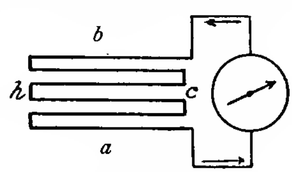


Fig. 194.

The **thermopile,** shown in Fig. 195, consists of a number of thermo-couples made preferably of bars of antimony and bismuth joined so that the electromotive force at each joint will be in the same direction. In Fig. 194 three couples are thus connected, the circuit from the first to the last bar being closed by a conducting wire and galvanometer. If the joints at the ends h are heated, a current of electricity will flow from bismuth to antimony, while at the ends c the current is from antimony to bismuth. When the ends c are

0° C. and h is 100° C., the electromotive force for these metals is only about .01 volt for each couple, but by joining a large number of these couples a very small difference of temperature may be detected by use of a sensitive galvanometer. The thermopile shown in Fig. 195 consists of 49 couples. Their ends are blackened, that they may more completely absorb the radiant energy which falls upon them. In the instrument shown, where the galvanometer is not particularly sensitive, the radiations from a candle flame at a distance of 10 feet will cause a deflection of several divisions on the galvanometer scale.

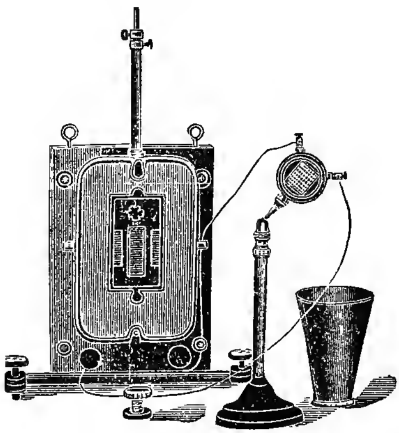


Fig. 195.

The **bolometer** is an instrument invented by Langley for the detection and measurement of small changes in temperature. It consists essentially of a Wheatstone bridge, Fig. 196, in one arm of which is inserted a strip of platinum foil covered with lampblack. The bridge is first balanced and the galvanometer shows no deflection, then when radiations from a heated body fall upon the blackened strip P, the resistance in that arm is increased, thus throwing the bridge out of balance. The resultant deflection in the galvanometer is a measure of the change of temperature of the platinum strip. The intensity of radiation from various sources may thus be compared. A change of .0001° C. may be detected.

The **radiomicrometer** is an instrument devised by D'Arsonval and later improved by Boys. It is a delicate D'Arsonval galvanometer in which the movable coil consists of a single turn of pure copper wire suspended between strong magnets, Fig. 197. One end of the coil is soldered to a light block of antimony, a, and the other to a similar block of bismuth, b. These are joined to a thin strip of copper, d, which extends a short distance below and is covered with lampblack. Radiations falling on d will heat the junction of the thermo-couple, thus causing a current of electricity to flow through the coil *C*. This causes the coil to turn in the magnetic field, and the distance through which

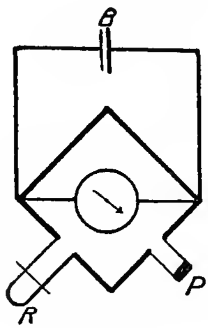


Fig. 196.

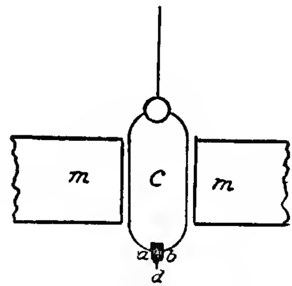


Fig. 197.

it turns is a measure of the intensity of radiation. This instrument may be made exceedingly sensitive.

Crookes's radiometer, shown in Fig. 198, consists of a light frame, at the ends of the arms of which are mounted light disks of mica blackened on one side. This is supported so that it will freely rotate in a glass bulb from which the air is exhausted to a vacuum of about one-thousandth of a centimetre of mercury. When radiant energy from some heated body passes through the glass into the bulb, the blackened faces are heated more than the polished ones, and the whole frame will rotate with the blackened faces moving away from the source of heat. The cause of the rotation is that when the molecules of air next to the black surfaces are heated they leap away with increased energy and by their reaction cause the disk to move in an opposite direction. Since the air is rare, the mean free path of the molecule is comparatively long. Hence the pressure due to this increased molecular motion is not communicated through the mass of air to the opposite side of the disk, as would be the case at ordinary density.

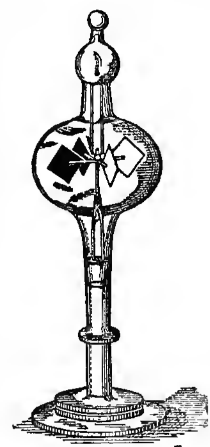


Fig. 198.

Professor E. F. Nichols has modified this instrument by suspending a horizontal arm from a fine quartz fibre, a mica disk blackened on one side being supported at each end of the arm. In the wall of the glass vessel is a fluorite window which freely admits radiations of all wave lengths. One of the blackened disks is opposite the window, and when it is repelled as explained above the quartz fibre is twisted through an angle which may be measured by aid of a light mirror attached to the arm. This is the most sensitive of all radiometers.

169. Laws of Radiation.—When a certain quantity, Q, of radiant energy falls upon a body, a portion of it, q, will be absorbed and converted to heat. The ratio $\frac{q}{Q}$ is called the **coefficient of absorption,** or simply the absorption. If in any case this ratio is unity, the body will absorb all the incident waves and convert them into heat. Such bodies are said to be perfectly black, or simply "black bodies." A body covered with lampblack is nearly a "black body," though in no case is the ratio exactly unity.

The ratio of the quantity of heat emitted by a body to the quantity which it would emit if it were a "black body" is called the **emission.**

For any given body the absorption and emission are numerically the same. This law was first deduced by Balfour Stewart and Kirchhoff, and is sometimes known as the **Stewart-Kirchhoff law.** It may be directly deduced from Prevost's law of exchanges, for when a body is in thermal equilibrium with its surroundings its absorption and emission must be the same or its temperature would change. The relation between emission and absorption may be illustrated by a simple apparatus like that shown in Fig. 199. A glass tube, bent in the form shown, communicates with the interior of the metal cylinders p and b and is partly filled with a colored liquid. The cylinders are full of air. When the cylinders are at the same temperature, the liquid in the tube will be stationary. The face of b is covered with lampblack while that of p is polished. The face B of the central drum C is black and P is polished. Now if C is filled with boiling water or other hot liquid, no change will be noted in the tube; consequently the air in the two cylinders is equally heated.

The blackened face b absorbs all the energy radiated by the polished face P, while p absorbs only a portion of that from B. But the faces p and P are alike, and P emits as much heat energy as p absorbs. Hence the ratio of heat emitted by P to that which it would emit if it were black is the same as the ratio of the amount absorbed by p to that which fell upon it,—*i.e.*, p absorbs as much energy as it would emit if its temperature were the same as P.

The power of absorption possessed by matter may be explained as a kind of **resonance.** When the motion of the minute particles of which matter is composed is synchronous with the ether waves which fall upon a body, the effect is increased motion or heat. Consequently we would expect that the waves which are thus absorbed would be the ones which would be emitted when the body is heated. Thus a body which transmits red light must absorb the waves which would give the higher colors of the spectrum. If then this same body is sufficiently heated, it will give out a bluish-green light. Heated carbon will emit waves of all lengths; hence when carbon is cold it will absorb waves of all lengths and will as a consequence be black. Lampblack and platinum black absorb nearly all the radiations that fall upon them. Polished silver reflects nearly all radiations incident on it, absorbing only about .02 of them. Rock salt transmits about .92 of the incident radiations, absorbing practically none.

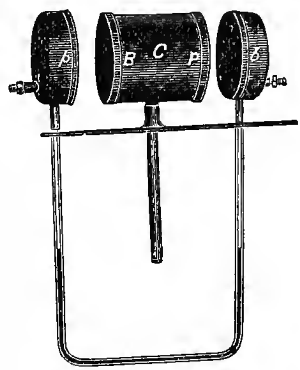


FIG. 199.

The **Stefan-Boltzman law,** already referred to under pyrometry (§ 143), states that the total energy emitted by a black body is proportional to the fourth power of the absolute temperature. By total energy is meant that due to waves of all lengths. Radiation from such a body is independent of the nature of the substance and depends on temperature only. If E represents the total energy, τ the absolute temperature, and c a constant for any conditions that are selected, then

$$E = c\tau^4 \quad (236)$$

By use of a radiation pyrometer operated according to these principles, it is possible to determine the temperature of distant bodies. Thus, the temperature of the sun is found to be about 6000° C. This, however, is the "black body" temperature. Estimates from other data give 7000° C. or more. The temperature of the electric arc is 3500° C., and, since the body is black, this is about the true temperature. The Nernst lamp is about 1950° C. and the incandescent lamp about 1500° C.

The **displacement law,** first formulated by Wien and often known by his name, states that if the radiations from a "black body" which is heated to a high temperature are dispersed so as to form a spectrum, there will be one color,—*i.e.*, a train of waves of definite wave length,—where the energy of radiation is maximum,—*i.e.*, greater than for any other train of waves. As the temperature of the body rises, this region of maximum radiation shifts to waves of shorter wave length. Wien has shown that if λ is the wave length of that train of waves where radiation is maximum and τ is the absolute temperature, then

$$\lambda\tau = c, \text{ a constant} \qquad (237)$$

After c is once determined and the value of λ found by observation, τ is readily calculated. This principle is applied in one form of optical pyrometer.

A body at a high temperature will lose heat by radiation more rapidly than when it is cooler. **Newton's law** of cooling is that the rate of cooling is proportional to the difference in temperature between the body and its surroundings. This law, however, is not exact, as may be seen from the following data of an experiment.

Difference of temperature.	Rate of cooling in degrees per min.	Ratio.
68.3° C.	1.95	35
42.6° C.	1.10	39
28.5° C.	.65	44
19.8° C.	.40	50

If Newton's law were rigidly correct, the ratios in the third column would be constant, but the rate of cooling decreases more rapidly than does the difference in temperature. If, however, the difference in temperature is small, only a few degrees, as is the case in many laboratory experiments, the law may be applied to determine the loss of heat due to radiation. This may be done, with a fair degree of approximation, as follows: Suppose correction is to be made for the loss of heat of a calorimeter where the temperature of a mass of water or other liquid has been raised by the introduction of a hot body. The temperature is noted at short intervals during the rise, and the average of these gives the mean temperature t_w of the calorimeter. The temperature of the surrounding air t_a may be considered constant or its mean may be determined. Then the mean difference of temperature during the time of the experiment is $t_w - t_a$. If the time required for the experiment is n minutes, the number of calories, q, lost during that time is

$$q = c\,(t_w - t_a)\,n \tag{238}$$

The constant of radiation, c, may be found after the temperature of the calorimeter begins to fall by noting the number of calories lost per minute when the difference of temperature between the calorimeter and its surroundings is 1° C. For example, suppose that in 3 minutes the temperature falls .6° C. and that the mean difference of temperature between the calorimeter and air during that time is 5° C., then if the mass of water including the water equivalent of the vessel is 300 g.,

$$c = \frac{300 \times .6}{3 \times 5} = 12 \text{ cal.}$$

This is the loss in one minute when the difference in temperature is 1° C. Hence equation (238) gives the loss in n minutes when the difference in temperature is $t_w - t_a$.

170. Thermodynamics.—Thermodynamics, as the name implies, treats of the relation between heat energy and mechanical or other forms of energy. The motion of the small particles of which a mass is composed represents a certain quantity of energy, just as truly as does the motion of the mass as a whole. Any kind of energy may be converted into molecular motion (heat), and this heat energy is exactly equal to the energy expended in producing it. Heat may in turn be often converted into mechanical motion. In all such changes there is an exact equivalence between energy expended and energy received for the same amount of heat.

The fact, now well established, that heat phenomena are included in the general law of **conservation of energy** is one of the greatest and most important generalizations of modern science. The principle of conservation of energy is, that, while energy may appear in a great variety of forms,—heat, electrical, mechanical, wave motion, etc.,—yet the sum total of all the energy in the universe is a constant quantity, and the various forms are so correlated that whenever energy of one form disappears an exact equivalent in some other form appears. This is a fundamental principle in physical science, and includes heat energy as well as other kinds.

Up to the beginning of the nineteenth century a generally accepted theory stated that heat was an imponderable, self-repellent fluid called **caloric.** A body was hot when it contained a large quantity of caloric. Water had a great capacity for heat because it could hold a large quantity of caloric. Friction caused a rise of temperature because the abraded particles had less capacity for heat than when in the solid body. A piece of metal was heated by concussion because the impact increased the density of the metal and so the caloric was squeezed out. Thus, heat was assumed to be a material substance. An analogous theory, also generally accepted at that time, explained combustion as the escape of a material substance called **phlogiston.** Such theories prevailed up to 1800 A.D. and were not abandoned till about the middle of the nineteenth century. Even in an edition of the "Encyclopædia Britannica" published in 1856 heat is defined as a "material agent of peculiar nature."

The first to combat publicly the caloric theory was **Benjamin Thompson** (Count Rumford), a native of Massachusetts, U.S.A., but at that time superintending the construction of cannon at Munich, Germany. He in 1798 performed a simple experiment which led to very important results. He noticed that the cannon were heated by boring, and was able, by using a blunt drill, to produce a large quantity of heat while only a small quantity of the metal in form of powder or shavings was bored away. As long as his engine continued to turn the drill, heat was produced. He rightly concluded that the heat was communicated to the cannon by the motion of the drill and at the expense of work done by his engine. According to the calorists the amount of heat should be in proportion to the quantity of shavings or abraded particles, but Rumford showed that the capacity of the shavings was the same as that of the solid metal. The calorists in reply insisted that while the temperature might rise as before, yet the quantity of heat was less. The claims of Count Rumford and others who believed as he did were ridiculed by the calorists for about forty years. But a line of experimental evidence has firmly established the modern theory of heat and thermodynamics.

In 1799 **Sir Humphry Davy** performed a simple but convincing experiment which the calorists could not satisfactorily explain. By rubbing together two blocks of ice he showed that the ice will be melted. The calorists admitted that water has a greater capacity for heat than ice has. Whence, then, does the heat come if not from the motion of rubbing?—for the experiment can be successfully performed in a vacuum or in air below the temperature of melting ice. In this case the friction did not diminish the capacity for heat.

Later a series of most careful and painstaking experiments were performed by James Prescott Joule, of Manchester, to test the claim that heat is a form of energy due to motion of particles within a body and to determine the number of units of energy in a thermal unit. Later still, in 1878–9, Rowland, of Johns Hopkins University, made a similar but more exact determination as described in the next section. Then Griffiths (1893), Schuster and Ganon (1894), and Callendar and Barnes (1899) made similar tests, using the heat effects of electric currents.

171. Mechanical Equivalent of Heat.—The mechanical equivalent of heat is the number of units of energy which when converted into heat will produce one thermal unit. Thus, it may be the number of foot-pounds of energy required to raise the temperature of 1 lb. of water 1° F. at any chosen temperature of the water, or it may be the number of ergs required to raise the temperature of 1 g. of water 1° C. at any selected temperature.

Joule employed a variety of experiments in his endeavor to find this mechanical equivalent. His methods were direct,—*i.e.*, he converted a measurable quantity of mechanical energy into heat and then measured the quantity of heat produced. Thus the heat effects due to work done in compressing air, in expansion of air, in friction of mercury, friction of iron plates, friction of water, etc., were measured by the rise of temperature in a given mass of water or other liquid. The chief value of Joule's work was not so much his numerical results as the fact that the constancy of his results showed that heat is only another form of energy, and the quantity of heat measured in energy units is equal to the quantity of work expended in producing it.

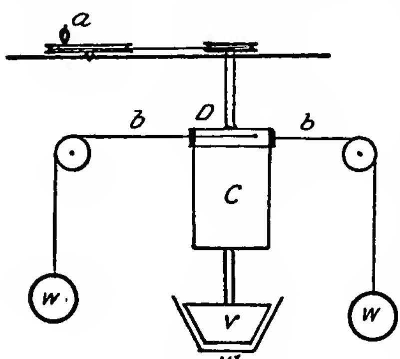


FIG. 200.

In Joule's later and more accurate work he used an apparatus illustrated by diagram in Fig. 200. Paddles are attached to the vertical axis of the calorimeter C and are made to rotate in a measured quantity of water. Projecting strips fastened to the interior walls of C prevent the movement of the water as a body along with the paddles. The calorimeter is supported mainly from beneath by the vessel v, which floats on water. Thus friction of the bearings is greatly reduced. The paddles were made to rotate by turning the wheel a. Cords, b, b, passed tangentially from opposite sides of a circular rim at the top of C, over pulleys, and to their ends were attached weights, w, w. When the paddles are turned with sufficient speed, the weights are just sufficient to prevent any rotation of C. By means of a recorder attached to the vertical axis, the rotations in a given time are automatically counted. To calculate the work done, let r be the radius of the rim D, then the circumference is $2\pi r$. Although the weights do not rise or fall, yet the work done in turning the paddles through one revolution is just the same as if the weights had been raised through a distance $2\pi r$,—*i.e.*, the work is the same as if the paddles had been stationary and the calorimeter had been turned by a force $2w$ exerted through a distance $2\pi r$. Hence

$$\text{work} = 2\pi r \cdot 2w = 4\pi r w \qquad (239)$$

and in n revolutions

$$\text{work} = 4\pi rnw \tag{240}$$

If the quantity of water including the water equivalent of the calorimeter is M and the rise of temperature is t°, the mechanical equivalent, J, is

$$J = \frac{4\pi rnw}{Mt} \tag{241}$$

This is the energy per unit mass per degree,—*i.e.*, the mechanical energy per caloric of heat.

Joule gave as his result

$$J = 772.65 \frac{\text{ft.-lbs.}}{\text{lb. } 1^\circ \text{ F.}} \text{ for water at } 61.69^\circ \text{ F.}$$

This expressed in c. g. s. units and centigrade degrees is

$$J = 4.167(10)^7 \frac{\text{ergs}}{\text{g. } 1^\circ \text{ C.}} \text{ at } 16.5^\circ \text{ C.}$$

Dr. Joule used a mercury-in-glass thermometer which by comparison with recent standards is found to be inaccurate. A recalculation with correction for errors in temperature gives as the results of Joule's experiment

$$J = 4.173(10)^7$$

Henry Augustus Rowland (1848–1901) devised an apparatus similar in principle to that of Joule but much better in construction. The paddles were rapidly turned by means of an engine and thus the time of each experiment was shortened. He repeated his experiment thirty times, varying the temperature of the water and using different thermometers. His original values were

Temperature.	J (in ergs).
5° C.	$4.212(10)^7$
10° C.	$4.200(10)^7$
15° C.	$4.189(10)^7$
20° C.	$4.179(10)^7$
25° C.	$4.173(10)^7$
30° C.	$4.171(10)^7$
35° C.	$4.173(10)^7$

It was by this series of experiments that Rowland established the fact that the capacity of water for heat is different for different temperatures. As seen in the table above, the value of J diminishes with rise of temperature to 30° C. and then begins to increase. The change in capacity is shown by the change in the number of ergs required to raise the temperature 1° C. Rowland's results are probably the most reliable of all the various determinations of J, and we may accordingly give as a very close approximation

$$J = 4.187(10)^7 \frac{\text{ergs}}{1\text{ g. } 1^\circ\text{ C.}} \text{ between } 15^\circ \text{ and } 16^\circ\text{ C.}$$

Thus, our thermal unit may be defined in terms of dynamic units as 4.187 joules of energy, or, if a temperature of 10° C. is chosen, 4.2 joules.

Other experimenters, principally those mentioned at the end of the preceding section, sought by indirect methods to determine the value of J,—*i.e.*, they measured the heat effects of a current of electricity when passed through a wire submerged in water. If the strength of current, i, is measured in amperes, and the resistance, R, in ohms, then the energy expended in heating the wire in T seconds is

$$\text{energy} = i^2RT \text{ joules} \tag{242}$$

The number of calories of heat, Q, received by the water is determined by the mass of water and its rise in temperature. Hence

$$JQ = i^2RT(10)^7 \tag{243}$$

from which the value of J can be calculated.

Values obtained by this and similar methods are

Griffiths, $4.187(10)^7$ at 25° C.
Schuster and Gannon, $4.190(10)^7$ at 19.1° C.
Callendar and Barnes, $4.190\ (10)^7$ at 15° C.

172. Laws of Thermodynamics. — There are two general principles underlying the phenomena of heat energy, known as the laws of thermodynamics.

1. *Whenever mechanical energy is transformed into heat, the heat energy thus produced is exactly equal to the mechanical energy which disappeared.* This may be expressed by

$$E = JQ$$

where E is the mechanical energy expended or work done and Q is the number of calories of heat which appears as a consequence. This law is only a statement that heat is included in the general law of conservation of energy.

2. *Heat cannot of itself pass from a cold to a hot body.* This law states the direction of the flow of heat,—*i.e.*, always from a body of higher to one of lower temperature. Since it is only during the transference of heat that its energy becomes available, it is impossible to devise any machine which will derive any mechanical effect from a body which is colder than all surrounding bodies. The cold body may contain a considerable quantity of energy, but no way is known by which that energy may be made available except by the transference of heat from it to a body which is still colder.

173. Difference of Specific Heats of Gases.—It has been shown (§ 136) that

$$PV = Rm\tau$$

If now the pressure is constant while the temperature is raised one degree, the volume will become V_1. Hence

$$PV_1 = Rm(\tau + 1) \tag{244}$$

The difference between these equations is

$$P(V_1 - V) = Rm \tag{245}$$

But $m = 1$ g., since specific heat is the quantity of heat needed to raise 1 g. 1° C. Hence in this consideration

$$P(V_1 - V) = R \tag{246}$$

This is the work done, for $P(V_1 - V)$ is the product of pressure by change of volume.

The cause of the difference in specific heat at constant pressure, C_p, and that at constant volume, C_v, is the external work done when the pressure is constant (§ 156). Hence, expressing this difference in mechanical units,

$$J(C_p-C_v)=P(V_1-V)=R \qquad (247)$$

The value of R may be found when the pressure, volume, and temperature of a gas are known (§ 136); C_p can be reliably determined by experiment; hence C_v can be calculated.

174. Effect of Intermolecular Forces.—When a gas expands without doing any external work, its temperature as a whole does not change, for it has lost none of its energy. This is on the assumption that the energy of the gas consists only in the motion of its molecules. But if there is cohesion between the molecules, then when the gas expands some of the kinetic energy of the molecules would become potential, for work would have to be done in effecting a separation against cohesion. The effect would be a fall of temperature,—*i.e.*, a lowering of kinetic energy. If, on the other hand, there is repulsion between molecules, this would add to kinetic energy in case of expansion and the temperature would rise. Dr. Joule attempted to test this matter experimentally by compressing gas in one vessel and allowing it to escape through a small tube into another vessel from which the air was exhausted. Both vessels were immersed in a water bath. By this arrangement no external work would be done during expansion, but Joule was not able to detect any change in the temperature of the water and concluded that if there was any change it was very slight. In a later experiment, with an improved apparatus and more delicate thermometers, Dr. Joule and Lord Kelvin performed what is known as the **porous plug** experiment. They passed air and other gases through a copper coil immersed in a water bath at constant temperature and allowed the gas to escape through a plug of cotton wool, the difference of pressure on the two sides of the plug being one atmosphere. The temperature of the gas on each side of the plug was measured by delicate thermometers. An air-pump operated by an engine was made to do the external work which the gas would have had to do if it had expanded without assistance. Hence any change of temperature would be due to work done because of intermolecular forces. By this arrangement it was possible to maintain continuous expansion of a stream of gas. The porous plug prevents a rapid motion of the gas, which if permitted would represent a certain amount of kinetic energy obtained from the heat energy on the other side of the plug. The results for a few gases are

Gas.	Temperature before passing through the plug.	Change of temperature.
Air	17.1	—0.255° C.
Air	91.6	—0.203° C.
Oxygen	8.7	—0.317° C.
Oxygen	93.0	—0.165° C.
CO_2	12.8	—1.207° C.
CO_2	19.1	—1.144° C.
CO_2	91.5	—0.69° C.
Hydrogen	6.8	+0.89° C.
Hydrogen	90.2	+0.46° C.

All gases except hydrogen showed a fall of temperature, indicating a cohesion of the molecules. Hydrogen increases slightly, indicating a repellent force between its molecules. An ideal gas is one in which there are no intermolecular forces. This is a condition which is not found to exist, but with a knowledge of the deviation in any case, an actual thermometer may be compared with ideal conditions.

The greater the difference of pressure on the two sides of the plug, or the lower the temperature of the gas, the greater is the change of temperature on passing through the plug. Thus, as seen in the table above, oxygen at 93° C. fell .165° C., while at 8.7° C. it fell .317° C. This principle is utilized in the liquefaction of air, oxygen, hydrogen, and other gases. Although hydrogen increases in temperature when it expands without doing external work, yet at about −80° C. the effect is reversed and it is cooled. The gas to be liquefied is compressed by powerful pumps and allowed to escape by a small opening corresponding to the porous plug. It is then made to return over the tube through which it advanced, thus cooling the gas, which by escape from the orifice is still further cooled. This constitutes what is called a **regenerative process,** which is continuous as long as the pumps are operated. When the critical temperature (table 26) is reached, the gas begins to change to the liquid state. By this process liquid air is produced in large quantities, and other gases long considered permanent have been liquefied. In this manner

Professor Dewar in 1898 liquefied hydrogen and by allowing it to evaporate under reduced pressure changed it to the solid state. The hydrogen must first be cooled by liquid air or other cold liquid, and then it can be further cooled by the regenerative process. The boiling point of liquid air under one atmosphere is at first − 192° C., but later, after the nitrogen has escaped, − 182° C. Hydrogen boils at − 253° under a pressure of one atmosphere. In the year 1908 Professor Onnes, of the University of Leyden, succeeded in changing helium to the liquid state. He was able to obtain a considerable quantity of this gas from monazite sand. By cooling with liquid hydrogen and by use of the regenerative process he obtained about 60 c.c. of liquid helium, the density of which he found to be .15 and the temperature of the boiling point 4.5° C. on the absolute scale. This is the lowest temperature known.

Temperature as low as − 200° C. may be measured with a platinum thermometer. For still lower temperature the hydrogen thermometer is reliable provided the pressure is well above the critical pressure.

175. Ratio of Specific Heats of Gases.—When a gas is heated, the energy thus expended (1) will increase the translatory motion of the molecules,—*i.e.*, give them greater kinetic energy due to their increased velocity; (2) may do external work, as when a gas expands against pressure; (3) may increase the rotary or vibratory motion within the molecule, a change which is probably proportional to the change of translatory motion; or (4) may do work in separating molecules against intermolecular forces. The fourth will ordinarily consume but a small quantity of heat, as is shown by the porous plug experiment; hence we may consider theoretically what the ratio $\frac{C_p}{C_v}$ should be if all heat were expended as indicated in (1) and (2), and compare this result with experimental results. The comparative influence of (3) may thus be shown.

The kinetic energy of any mass m moving with velocity $\overline{V}$ is

$$e = \tfrac{1}{2} m \overline{V^2} \qquad (248)$$

If m is the mass of one molecule of a gas and $\overline{V}$ is the mean velocity, the energy e of n molecules is

$$e = \tfrac{1}{2} mn \overline{V^2} \qquad (249)$$

Let n be the number in 1 g. of the gas, then mn is 1 g.; hence

$$e=\tfrac{1}{2}\overline{V^2} \text{ or } \overline{V^2}=2e \tag{250}$$

One gram is taken because we are here discussing specific heat.

By § 89,

$$PV=\tfrac{1}{3}\overline{V^2}$$

$$\therefore\ PV=\tfrac{2}{3}e \tag{251}$$

Now if this gas is heated 1° C. under constant pressure, its volume will increase by a certain amount, say v, and its energy will increase to e'. Hence

$$P(V+v)=\tfrac{2}{3}e' \tag{252}$$

Subtracting the preceding equation,

$$Pv=\tfrac{2}{3}(e'-e) \tag{253}$$

But Pv is the product of pressure by change of volume, and therefore it is the external work done by expansion under constant pressure. The total change of energy is $e'-e$, hence two-thirds of this is expended in doing external work. Now, C_p includes the external work, hence

$$\frac{C_p}{C_v}=\frac{e'-e+\frac{2}{3}(e'-e)}{e'-e}=1+\tfrac{2}{3}=1.666=\gamma \tag{254}$$

The value of γ for various gases and vapors may be found from the velocity of compressional waves passing through them, as when sound passes through air. The velocity of such a wave varies directly as the square root of the elasticity E of the gas and inversely as the square root of the density ρ. Hence

$$\text{velocity of wave}=\frac{\sqrt{E}}{\sqrt{\rho}}$$

But since the compressions and rarefactions are so rapid that heat has not time to enter or leave the gas during the passage of the wave, the elasticity is adiabatic, and so is not equal to the pressure (§§ 94, 176), but to γP. Hence

$$\text{velocity of wave}=\frac{\sqrt{\gamma P}}{\sqrt{\rho}}$$

From this the value of γ can be determined. For such gases as argon, helium, and vapor of mercury, experiment shows a value very nearly equal to 1.66. In these gases the molecules consist of a single particle of matter,—*i.e.*, they are monatomic,—hence the heat would all be expended in increasing translatory motion and doing external work. In a gas such as oxygen, O_2, the value of γ is 1.41. In air, 1.41. In ether, with its complex molecule, $C_4H_{10}O$, the value of γ is only 1.03. Thus, as might be expected, the greater the complexity of the molecule the greater the amount of heat absorbed in producing motion within the molecule.

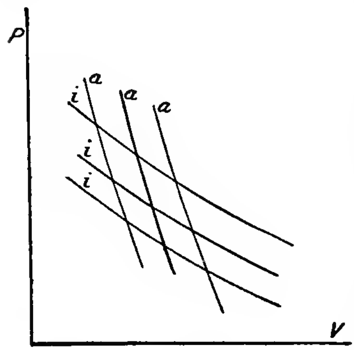


FIG. 201.

176. Adiabatic Expansion.—When a gas expands without receiving or losing any heat, the expansion is said to be adiabatic (*α-διά-βαίνειν*, not to pass through). It is evident that if a gas is compressed and the heat resulting from the external work done on the gas is not allowed to escape, the pressure will increase more rapidly with decrease of volume than when the process is isothermal. Hence if curves are drawn on the pressure-volume diagram, Fig. 201, for

$$PV = \text{constant, isothermal,}$$

and $$PV^\gamma = \text{constant, adiabatic,}$$

the adiabatic curves *a*, *a*, *a* will show a more rapid rise in pressure than the isothermals *i*, *i*, *i*. If the gas expands in doing external work, the pressure rapidly falls, for no heat is received from the outside.

177. Elasticity of Gases. — Since a gas may expand either isothermally or adiabatically, there will as a result be two coefficients of elasticity. It has already been shown that the isothermal elasticity (§ 94) is equal numerically to the original pressure P, for if

$$\frac{p}{\frac{v}{V}} = E_t$$

then, as the increase of pressure p and the resulting change of volume v approach the limit zero, the actual pressure on the gas becomes P; hence

$$P = E_t$$

The adiabatic elasticity is greater than P because all the heat energy is retained in the gas. This coefficient, as shown in appendix 13, is equal to the product of pressure by the ratio $\frac{C_p}{C_v}$, for since

$$PV^{\gamma} = \text{a constant},$$

$$E_{\eta} = \gamma P \qquad (255)$$

178. Carnot's Cycle. — The modern theory of thermodynamics may be said to have been established by such scientists as Lord Kelvin, Helmholtz, Clausius, and Rankine about the year 1850, yet the foundations upon which these men built

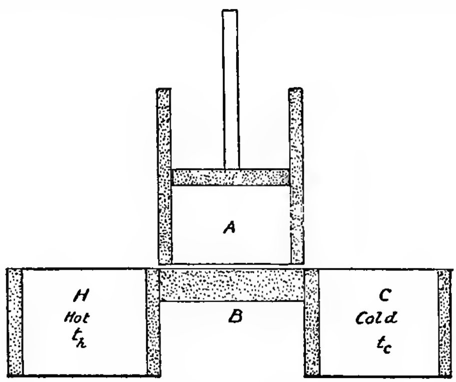


Fig. 202.

were laid by Sadi Carnot (1796–1832), a French physicist who sought to increase the efficiency of the steam engine. The so-called Carnot's engine is an imaginative one, devised for the study of ideal conditions which can never be fully realized but to which the practical engine is an approach. The true nature of heat was not known to Carnot, but a study of the principles which he announced has led to valuable results in the way of a knowledge of the limits of a heat engine and of thermodynamic principles in general.

To understand Carnot's cycle suppose a quantity of air, steam, or other gas or vapor, called the working substance, is enclosed in a cylinder, A, Fig. 202, the walls and piston of which are adiabatic but the bottom a perfect conductor. Assume also that the tops of the hot body H and the cold body C are perfect conductors, the top of B being adiabatic. Let the gas in A have a pressure and volume represented by the point a on the pressure-volume diagram Fig. 203. **First,** having the gas at the temperature of the cold body, place the cylinder on B and press the piston down till the temperature rises to that of the hot body. Here work is done on the gas and, since no heat escapes, the change is represented by the adiabatic line ab. The gas is now in the state b. **Second,** move the cylinder over on H and allow the gas to expand. In this change external work will be done by the gas and heat will flow from H into A and keep the temperature constant. The volume will increase and pressure will fall. This change is represented by the isothermal bc. **Third,** move the cylinder back to B and allow the expansion to continue till the temperature falls to that of the cold body. There is an increase of volume and a rapid fall of pressure. This is represented by the adiabatic line cd. **Fourth,** place the cylinder on the cold body C and push the piston down to the original starting point. Here work is done on the gas, but without increase of temperature, for heat flows freely into C. One cycle is now complete and the gas is in the same state as at the start. The change was from a to b, b to c, c to d, and thence back to a. The capacity of the bodies H and C are assumed to be so great that their temperature is practically unchanged by the loss or gain of heat.

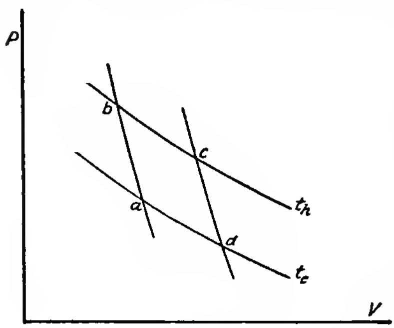


FIG. 203.

Some heat was transferred from H to C during the cycle of operations, but not all of that taken from H was delivered to C. An inspection of Fig. 203 shows that the lines da and ab represent changes produced by work done *on* the gas, while bc and cd show changes while work was done *by* the gas. If the mean pressure along each of these four lines is multiplied by the change of volume, the products will be the work done on or by the gas.

The difference is the area *abcd*, which is the excess of work done by the gas.

Thus it is seen that of the heat taken from H part was expended in doing external work and the balance was delivered to the cold body. It is also apparent from the diagram that the greater the difference between the hot and cold bodies, the greater will be the area of *abcd*,—*i.e.*, the greater the external work. This is true no matter what the working substance may be, for the work is accomplished only by the expenditure of the heat energy.

The **efficiency** of an engine is defined as the ratio of the work done to the energy received, or, in other words, the ratio of the work done by the engine to what it would do if all the energy which it received were expended in work. Let Q_1 be the number of calories of heat received from the hot body, and Q_2 that delivered to the cold body, then the work W, expressed in ergs, is

$$W = J(Q_1 - Q_2) \tag{256}$$

and the efficiency E is

$$E = \frac{J(Q_1 - Q_2)}{JQ_1} = \frac{Q_1 - Q_2}{Q_1} \tag{257}$$

If all the heat were expended in work, Q_2 would be zero and the efficiency would be $\frac{Q_1}{Q_1} = 1$. Efficiency could not be greater than unity.

179. Reversible Cycle. — By use of an external agent it is possible to reverse Carnot's cycle and restore to H, Fig. 202, the heat which was transferred to C. Starting at a, Fig. 203, let the gas expand isothermally while the cylinder is on C. Heat will flow from C into the gas and the change will be *ad*. Then place the cylinder on B and push the piston down till the temperature rises to t_h. This change is represented by *dc*. Then place the cylinder on H and push the piston still lower. This gives the isothermal *cb*. Now place the cylinder on B and let the gas expand till the temperature is t_c. The cycle is thus completed in a reverse direction, a quantity of heat, Q_2, has been transferred from C to H, and the excess of work done by the external agent over that done by the gas is represented by the area *abcd*.

Carnot's cycle is exactly reversible because of the ideal conditions which are assumed for his engine. In practical operations, however, it is not possible to avoid conduction and radiation of heat and friction of the moving parts of an engine. These cannot be reversed by reversing the engine, and so the working substance cannot be brought to its original state.

180. Carnot's Theorem.—Carnot showed that all reversible engines working between the same temperatures have the same efficiency, and no other engine can have a greater efficiency under the same conditions of temperature. He proved this by showing that the operation of such an engine would be contrary to the second law of thermodynamics. Suppose an engine, M, assumed to be more efficient, is coupled to a Carnot's engine, C, so that C is made to run backward as explained in the preceding article. Let C take Q_2 units of heat from the cold body and deliver Q_1 units to the hot body. Let M take Q_1 units from the hot body and, since M is more efficient, deliver less than Q_2, say Q_3, units to the cold body. By this operation the quantity of heat in the hot body will remain the same, but that in the cold one will grow less and less. The engine M would thus be able to operate C and do external work besides. Work would be done by using up the heat of the cold body. But this is contrary to experience,—*i.e.*, to the second law. Hence no engine can be more efficient than a reversible one. For the same reason no form of reversible engine can be more efficient than another. This theorem, in other words, states that an engine is most efficient when all the heat received from the hot body and from external work done on the working substance is used in changing the state of the working substance,—*i.e.*, there is no loss by conduction, radiation, etc.

181. Thermodynamic Scale of Temperature. — In all thermometers which have thus far been described temperature is measured by changes depending on some property of a substance, as expansion of a gas or mercury, change in electric conductivity, etc. Exact determinations of temperature by these methods is very difficult, because of the many different conditions which modify the results. If, however, there were a perfectly reliable standard with which ordinary thermometers could be compared, the latter could be used as effectively as the standard. A perfect gas—*i.e.*, one in which there are no intermolecular forces, and consequently one which is true to Boyle's law—would make such a standard, but such a gas does not exist.

It has been shown above that the efficiency of a Carnot's engine is independent of the character of the working substance, and depends only on the difference of temperature of the hot and cold bodies between which the engine works. The amount of work done by a working substance in one cycle depends only on the difference of temperature between the isothermals of that cycle.

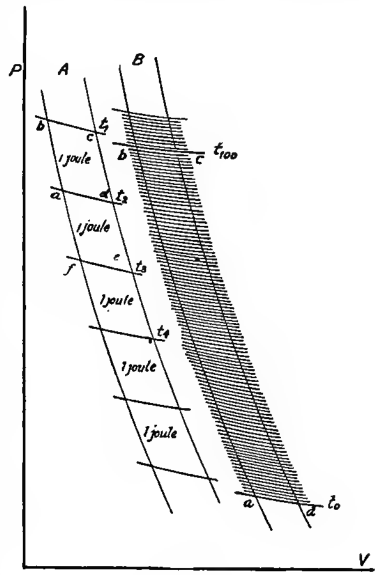


FIG. 204.

Based on this fact, Lord Kelvin proposed a thermodynamic scale of temperature which is independent of the substance used. Thus, in Fig. 204, *A*, let a Carnot's engine be of such dimensions that, when working between temperatures t_1 and t_2, 1 joule of work will be done in completing the cycle *abcda*. Then $t_1 - t_2$ might be chosen as a degree on some new scale. If the same engine does 2 joules of work in the cycle *fbcef*, $t_1 - t_3$ would be two degrees on this scale, and 3 joules would be three degrees, and so on. To make this scale correspond with the centigrade scale, let the two isothermals of a cycle be t_{100} and t_0,—*i.e.*, the temperature of boiling water and that of melting ice. Then the area *abcd*, Fig. 204, *B*, will be the work done in one cycle between these limits of temperature. Now, if isothermals are drawn dividing this area into 100 equal areas, each isothermal will differ from the next one by 1° C.

If the temperature of the hot body remains constant and that of the cold body is lowered until all the heat, JQ_1 ergs, taken from the hot body is expended in work during the cycle, then the

$$\text{efficiency} = \frac{Q_1 - Q_2}{Q_1} = 1$$

for Q_2 has become zero. This could occur only in case the cold body contains no heat. It could not then be any colder, for the efficiency cannot be greater than unity. This low temperature is the **absolute zero** on the thermodynamic scale.

Lord Kelvin showed by experiment that this scale of temperature coincides very closely with the centigrade scale of a gas thermometer and would agree exactly if the gas were perfect. Hence the temperature of melting ice on this scale is almost exactly 273 and that of boiling water 373.

The hydrogen thermometer most nearly coincides with the thermodynamic scale, the difference being due to intermolecular forces in the gas. This very slight difference was determined by the porous plug experiment (§ 174), and corrections may be made accordingly in exact measurements.

By inspection of the diagram, Fig. 204, *B*, it is seen that if each small area included by two adiabatics and two adjacent isothermals be considered as a unit of heat measured in ergs, the number of such units at any temperature, counting from absolute zero, is to the number at any other temperature as the corresponding absolute temperatures are to each other, for each change of one degree corresponds to a change of one unit of heat. Hence, if τ_1 and τ_2 are the absolute temperatures,

$$\frac{Q_1}{Q_2}=\frac{\tau_1}{\tau_2} \tag{258}$$

$$\frac{Q_1-Q_2}{Q_1}=\frac{\tau_1-\tau_2}{\tau_1} \tag{259}$$

Thus efficiency is expressed in terms of temperature.

182. Entropy.—In considering a reversible cycle such as is represented in Fig. 203, where *ab* and *cd* are adiabatics intersected by isothermals, let the point *a* represent the state of a body in reference to pressure, volume, and temperature. If the pressure is increased, no heat being allowed to escape, the state may be changed to *b*. Let Q be the total quantity of heat in a substance in state *a*,—*i.e.*, the total number of thermodynamic units as measured from absolute zero; also let τ be the absolute temperature at *a*. Then, as Q is increased, τ will increase in the same ratio; therefore $\frac{Q}{\tau}$ will be constant between

a and b. For illustration, let the total quantity of heat at a be 22,800 ergs and the absolute temperature 285° C. Then for each increase of 1° C. there is an increase of 80 ergs of heat energy. Hence

$$\frac{22800}{285}=\frac{22880}{286}=\frac{22960}{287}\text{ etc.}=80$$

Whatever the value of Q may be, the ratio

$$\frac{Q}{\tau}=\text{constant} \tag{260}$$

Entropy may be defined as that quantity which remains constant during an adiabatic change, just as temperature is that which remains constant during an isothermal change. For this reason an adiabatic line is often called isentropic.

Although entropy cannot be measured by any instrument, as temperature can be measured by a thermometer, yet it is a distinct physical quantity which can be calculated when the pressure, volume, and temperature are known.

In passing from b to c, Fig. 203, τ remains constant but Q increases, hence the ratio $\frac{Q}{\tau}$ increases. From c to d the ratio is constant, for Q and τ decrease at the same rate. From d to a the temperature is again constant while Q decreases, so the ratio of Q to τ decreases. If Q_1 is the increase of heat in passing from b to c, and Q_2 is the loss of heat in passing from d to a, then, according to equation (258),

$$\frac{Q_1}{\tau_1}+\frac{Q_2}{\tau_2}=0 \tag{261}$$

—*i.e.*, in a complete cycle of this character the sum of the ratios $\frac{Q}{\tau}$ is zero.

A series of changes which form a complete reversible cycle may be represented by a closed curve, as in Fig. 205. The area enclosed represents work done, and may be considered as composed of an infinite number of Carnot's cycles such as *abcda*. Since equation (261) is true for each cycle, it is true for all the cycles of which this area is composed. Hence

$$\Sigma\frac{q}{\tau}=0 \tag{262}$$

where q is an infinitely small change in the quantity of heat gained or lost in the small cycle and τ is the absolute temperature at which the change was made. By making the Carnot's cycles infinitely small the change of state becomes continuous, as shown by the smooth curve. Equation (262) shows that the increase of entropy during one part of the cycle is equal to the decrease during another part, so that on the completion of the cycle the entropy is the same as at the beginning.

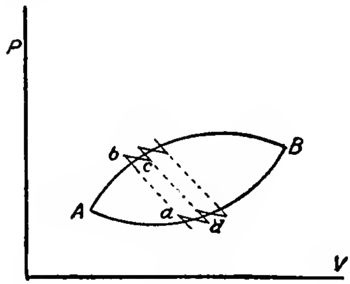


FIG. 205.

The natural **zero of entropy** is the state of a substance devoid of all heat, but it is more convenient to select a certain state arbitrarily as a standard and calculate the change of entropy from that standard, just as the temperature of melting ice is assumed to be 0° C. while in fact the temperature is 273° C. Thus in Fig. 203 let a be the point of reference for entropy, then the entropy at c will be the sum of an infinite number of very small additions of heat each divided by the temperature at which the addition was made. If entropy is represented by η,

$$\eta_c = \Sigma \frac{q}{\tau} \tag{263}$$

This is the case no matter whether we pass from a to c by the path ab, bc or by ad, dc, and in the continuous change shown in Fig. 205 the entropy at B in reference to A as a standard is also expressed by equation (263),—*i.e.*, entropy depends only on pressure, volume, and temperature, and is independent of how or by what means the change of state was effected, just as a weight raised from the ground to a certain height will possess a definite amount of potential energy in reference to the ground no matter how or by what path it was raised.

In the particular case shown in Fig. 203 it is seen that in changing from a to b there is no change of entropy, but from b to c τ remains constant and change of entropy is proportional to the increase in the quantity of heat. Here $\Sigma q = Q$, where

Q is the total heat added between b and c; hence

$$\eta_c = \frac{Q}{\tau}$$

but in changes along a path where both q and τ are constantly changing, change of entropy will be found by adding an infinite number of these ratios as expressed by equation (263).

Mechanical energy is constantly being changed into heat, and heat is constantly passing by conduction, convection, and radiation from bodies at higher to those at lower temperature. If q_1 is heat transferred from a body at temperature τ_1 to another body at temperature τ_2, the entropy of the former will be diminished $\frac{q_1}{\tau_1}$ and the latter will be increased $\frac{q_1}{\tau_2}$. But since τ_2 is less than τ_1, the entropy of the second body will be increased more than the first is decreased. Clausius expressed this by saying that the **entropy of the universe tends to a maximum.** If the maximum is ever reached, there will no longer be any available energy, for there will be no high or low and hence no transference from one to the other.

The sun is the great disturber of equilibrium, and as a result we have at present, available for work, fuel, food, running water, wind, water waves, and direct solar radiation. The amount of radiant energy intercepted by the earth is only about $\frac{1}{2200000000}$ part of the total energy sent out continually from the sun, yet, according to experiments made at points where the sun's rays are vertical, the energy value of solar radiations on each square meter of earth's surface is about one horse-power. The radiations falling on one-half the whole surface of the earth are the same as would fall vertically on the area of a great circle of the earth. This area is about $128(10)^{12}$ square metres. Hence the total energy received from the sun in any given time is equivalent to the work which could be done by $128(10)^{12}$ horse-power in the same time.

183. The Steam Engine. — In a steam engine the working substance is vapor of water at a high temperature and pressure. A cycle of operations is performed similar to that of a Carnot's cycle, but the walls of the steam engine are not adiabatic, and so there is considerable conduction and radiation of heat. Such heat is lost as far as efficiency is concerned. The boiler may be considered the hot body and the condenser the cold body. The amount of work done will be, allowing for loss of heat, the difference between the thermal energy received from the boiler and that given to the condenser. The maximum efficiency is expressed by equation (259).

In the **reciprocating engine** steam is admitted first to one side and then to the other of a piston, thus causing a forward and backward motion. As shown in Fig. 206, steam is admitted

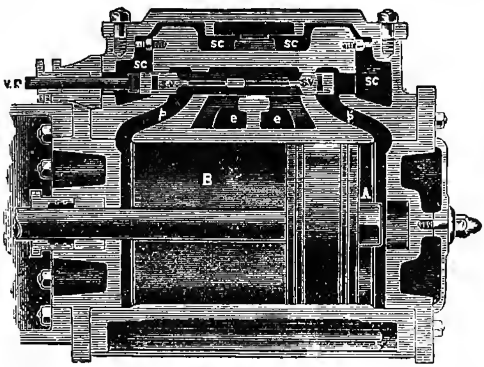


Fig. 206.

to the steam chest *sc* above the slide valve *sv*. When *sv* is in the position shown, steam at boiler pressure is admitted through port p', thus driving the piston toward B. Before this stroke is completed, the slide valve moves back and covers the port p', and the steam in A continues to expand to the end of the stroke. During this time the port p has been closed, but the steam in B, used in the previous stroke, may escape beneath the slide valve and out through the exhaust pipe e. The exhaust port is closed shortly before the completion of the stroke, and the vapor thus entrapped in B serves as a cushion for the piston. Then the port p is opened to boiler pressure, and the same operation is repeated, but in opposite ends of the cylinder.

When the engine is working under full load, more heat energy is needed than when the load is light. Various devices are employed to regulate the admission of steam at a rate adapted to the work being done. One method of accomplish-ing this end is to regulate the flow of steam into the steam chest by use of a governor like that described in § 35. Another method is to regulate the motion of the slide valve so that for light work less steam will be admitted to the cylinder, and the work will be done by expansion rather than by direct boiler pressure. This may be accomplished by use of a **governor** like that shown in Fig. 207. A stiff spring fastened to the fly-wheel carries a heavy mass, *m*. When the wheel rotates, *m* is thrown out toward the rim. This moves the arm *a* and brings the eccentric

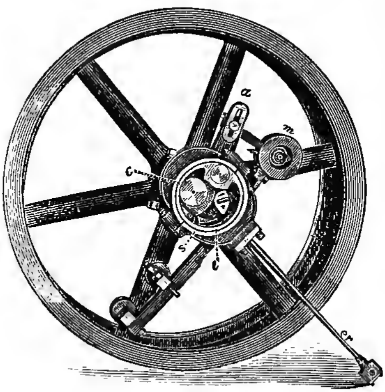


Fig. 207.

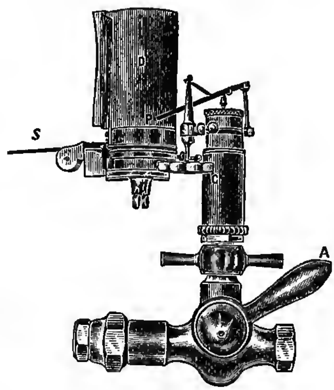


Fig. 208.

ring *e* to a position more nearly concentric with the axis *c*. Around the eccentric and sliding upon it is a metal strap to which the eccentric rod *er* is attached. If *e* were exactly concentric with *c*, the eccentric rod would have no motion and the slide valve would rest over both steam ports. If *e* is at maximum eccentricity, the slide valve would admit steam at boiler pressure through the whole stroke of the piston. Thus, a constant speed with a changing load is effected by regulating the distance through which the slide valve moves.

The work which is actually being done by the steam in the cylinder may be determined by use of a steam-engine **indicator,** shown in Fig. 208. Steam is admitted from either end of the engine cylinder to a small cylinder, *C*. The steam pressure raises a small piston against a stout spring, and thus the pencil *P* is made to move up or down according to pressure. At the

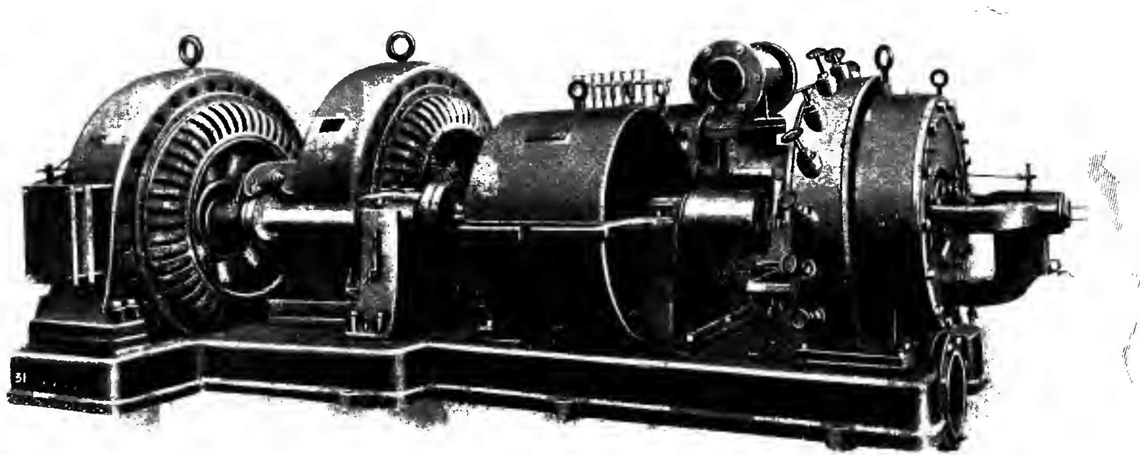


Fig. 211.

spring gives the average pressure of steam—usually in pounds per square inch. This times the area of the piston in square inches gives the total pressure. The total pressure times the distance in feet travelled by the piston per minute is the number of foot-pounds of work per minute. The indicated **horse-power** is then found by dividing by 33,000.

The exhaust steam still contains a large quantity of heat and so may do more work between its temperature and a lower one. If the steam is exhausted from the first or high pressure cylinder into another cylinder of the same construction but larger, additional work will be done. Such an arrangement is shown in Fig. 210. Both pistons are attached to the same axis and move in the same direction. The steam exhausted from one end of the high pressure cylinder passes into the opposite end of the lower pressure cylinder. The steam then passes into the condenser, or into a third, fourth, or even sixth cylinder, each larger than the preceding one, and then into the condenser.

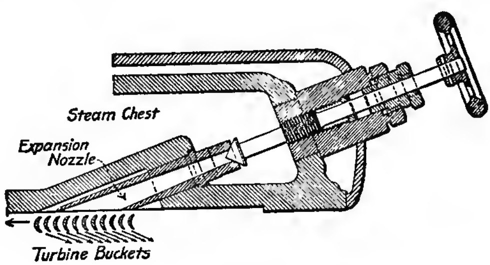


FIG. 212.

Instead of having the steam pressure produce a reciprocating motion of a piston, it may be directed with great velocity against a row of buckets or blades on the periphery of a wheel. Such constitutes a rotary engine or **steam turbine.** In one type of such engines, steam under high pressure is projected from a nozzle against a series of buckets attached to the circumference of a wheel, the wheel being keyed to the drive-shaft. These may be called impulse turbines. The De Laval turbine, illustrated in Fig. 211, is one of this type. The turbine proper is shown to the right of the figure. Steam may be admitted to the buckets at several points around the wheel as shown. A section of one of the nozzles is shown in Fig. 212. To the left of the turbine, Fig. 211, is shown the gear case. The turbine is run at high speed to secure proper efficiency. Here the angular velocity is reduced, and the motion is communicated to two shafts which operate the dynamos shown on the left of the figure.

Another type may be called impulse-reaction turbines, for they are driven not only by the impulse of the steam but also by the reaction resulting from a change in the direction of flow caused by numerous fixed and movable blades set in the path of the steam. This is illustrated in diagram, Fig. 213.

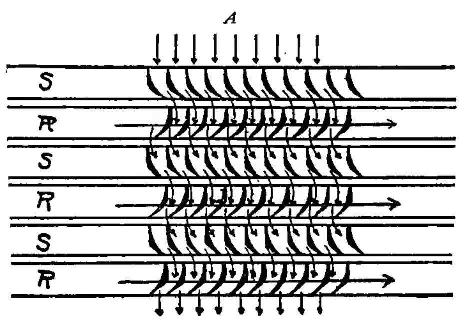


FIG. 213.

Steam under high pressure is admitted from the side A through a row of fixed blades, S, and directed against a row of movable ones, R, which are thus forced in the direction of the long arrows passing through them. The same operation is repeated, but with less intensity, through a number of successive rows, all fixed to the same spindle. In Fig. 214 is shown a spindle covered with rows of movable blades, and also one half of the casing. The fixed or guide blades are attached to the casing and fit in between the rows of movable ones. Steam is admitted at the smaller end of the casing and forced along the space between the casing and spindle,—*i.e.*, the space filled with alternate rows of fixed and movable blades. The Parsons turbines are of this type and are in common use, particularly for the propulsion of large steamships. The increase in size of the spindle as the velocity of steam decreases gives an advantage similar to that in case of compound cylinders. The turbine, when properly constructed

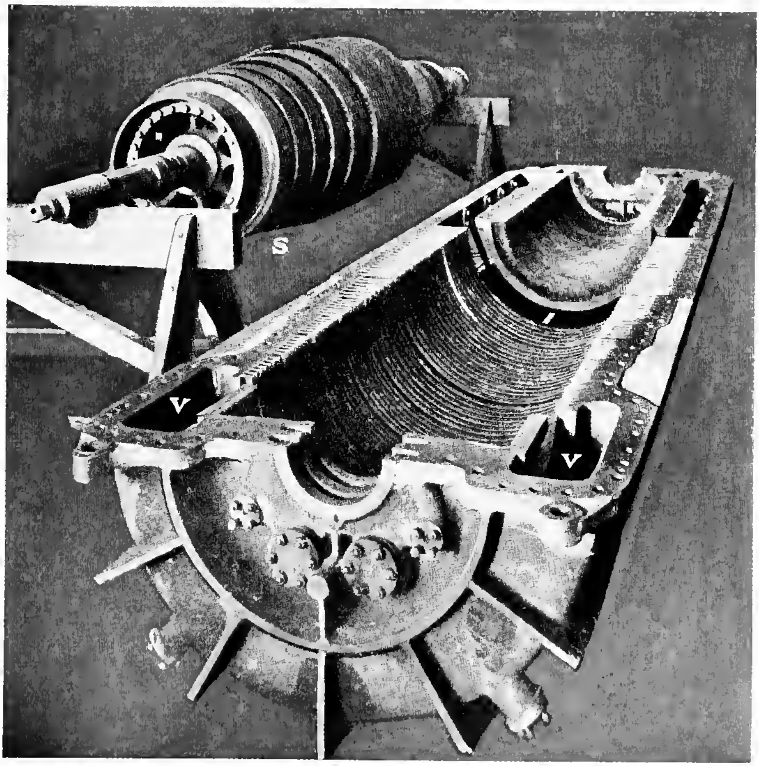


Fig. 214. (From Sothern's "The Marine Steam Turbine.")

and operated, is quite as efficient as other forms of engine and is without many of the objections to the reciprocating engines.

Instead of converting water to steam and then using the heat energy of steam to do work, the energy of fuel may be more directly used in internal combustion engines, usually called **gas engines.** Here the fuel in form of a gas is mixed with oxygen of the air and exploded in a cylinder. Thus the piston which closes one end of the cylinder is driven forward. Two important types of this engine are the four-cycle and the two-cycle. In the former there are four strokes of the piston for each explosion. (1) The outward movement of the piston draws into the cylinder a mixture of gas and air. (2) The backward motion compresses the mixture. (3) The explosion drives the piston out again. (4) The return motion exhausts the cylinder. There cannot in this type be more than one explosion for every two revolutions of the fly-wheel.

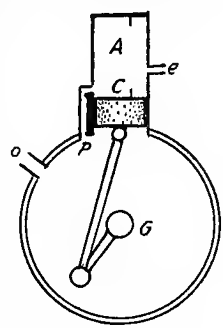


Fig. 215

In the two-cycle engine there may be an explosion in each revolution. In diagram, Fig. 215, *G* is a space enclosed by the crank case. When the piston *C* moves toward *A*, it will close the ports *p* and *e*, compress the gas in *A*, and draw gas through *o* into the crank case. Then the gas in *A* is exploded by an electric spark, and as *C* descends, the port *e* is first opened for the exhaust and then *p* is opened for the admission of fresh gas and air. This operation is then repeated. This type has the advantages of being without movable valves and of being lighter for the same power, while in the four-cycle type the combustion is more complete and the cylinder is not so highly heated.

The power of a gas engine may be computed in a manner very much like that for the steam engine. The average pressure is found from the indicator diagram and

$$\text{horse-power} = \frac{p \times l \times a \times e}{33000}$$

where p = average pressure per sq. in.
l = length of stroke in feet
a = area of piston in sq. in.
e = number of explosions per min.

Problems.

1. If when 1 c.c. of water at 100° C. is converted to steam under pressure of one atmosphere its volume increases to 1649 c.c., how much external work is done? How much internal work?

2. How much is the melting point of ice lowered by a pressure of one atmosphere?

$$\text{Use } \frac{Q_1-Q_2}{Q_1}=\frac{\tau_1-\tau_2}{\tau_1} \text{ (see equation 259)}$$

$$Q_1-Q_2=\text{work done}=P(v_1-v_2)$$

$$P=1.013(10)^6 \text{ dynes}$$

Change of volume on melting = .091 c.c. per c.c.

$Q_1=80\times 4.187(10)^7$ ergs = ergs of heat energy expended in melting 1 g.

$$\tau_1-\tau_2=\text{result sought}$$

$$\tau_1=273° \text{ C.}$$

3. How much ice at 0° C. would be melted by the energy in a mass of 20 kg. moving with a velocity of 3000 cm. per second?

4. If when the temperature of air is 23° C. the dew point is 8° C., what is the relative humidity?

5. Calculate the maximum efficiency of a steam engine working between temperatures 120° C. and 10° C. (Equation 259.)

6. Calculate the indicated horse power (I. H. P.) of a steam engine from following data:

Area of indicator diagram	2.7 sq. in.
Length of diagram	3.0 in.
Diameter of piston	20.0 in.
No. of spring	50
Revolutions per min. (R. P. M.) of fly-wheel	300
Length of stroke	2 ft.

7. How much work can be done by the energy obtained by the combustion of 50 litres of hydrogen at 20° C. and under a pressure of 10 atmospheres?

8. Calculate the value of J from

$$J(C_p-C_v)=R$$

and $$PV=mR\tau$$

Use air at 0° C., assuming it to be a perfect gas. $\frac{V}{m}=\frac{1}{\rho}$ where ρ is density of air under standard conditions. $P=1.013(10)^6$. $\tau=273$.

1. $1.67(10)^9$ ergs. $2.077(10)^{10}$ ergs.
2. .00751° C.
3. 26.866 g.
4. 38.3 per cent.
5. 28 per cent.
6. 514.08 h. p.
7. $5.96(10)^{13}$ ergs.
8. $4.19(10)^7$ ergs/1 g. 1° C.

INDEX OF APPENDIX

APPENDIX

PROOFS NOT FOUND IN THE BODY OF THE TEXT.

1. Pressure on the vertical side of a rectangular vessel filled with liquid.

It is required to find the total pressure on the side A, Fig. 216, of a liquid whose density is ρ and depth h. Since pressure is proportional to depth, if the liquid is considered as composed of elementary layers of width a and depth dx, then at any depth x the pressure is ρgx, and the pressure against the elementary strip is $\rho gaxdx$. The total pressure on all the strips is

$$\int_0^h \rho gaxdx$$
$$= \tfrac{1}{2}\rho gah^2$$
$$= \rho gah\frac{h}{2}$$
$$= \rho gA\frac{h}{2}$$

for ah is the total area A. Hence the total pressure on the side is the weight of a column of liquid whose dimensions are the area of the side considered and whose depth is half the depth of the liquid. If c.g.s. units are used, the result is found in dynes.

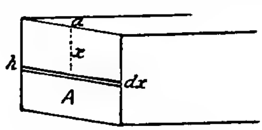


FIG. 216.

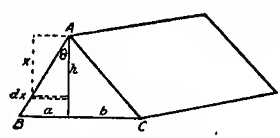


FIG. 217.

This is true of any area where a horizontal line through its centre divides the area into equal upper and lower parts.

In case the vertical side is a triangle, as shown in Fig. 217, the total pressure on that side is found by multiplying the area of the triangle by the pressure at two-thirds of the depth. Let ABC be the end of a prism filled with water. Let m be the length of the horizontal base and h the altitude. Also let a and b be the segments of m made by the vertical h. Then

$$\tan\theta = \frac{a}{h}$$

hence an elementary strip at depth x and of width dx has an area

$$\frac{a}{h} x dx$$

The pressure in water at depth x is x, hence the total pressure on this elementary area is

$$\frac{a}{h} x^2 dx$$

and the total pressure on this part of the triangle is

$$P_a = \int_0^h \frac{a}{h} x^2 dx$$

$$= \frac{ah^3}{3h} = \frac{ah^2}{3}$$

$$= \frac{1}{2} ah \cdot \frac{2h}{3}$$

In the same manner it may be shown that the pressure on the portion of the triangle having the base b is

$$P_b = \frac{1}{2} bh \cdot \frac{2}{3} h$$

Hence the pressure on the whole triangle is

$$P = \frac{1}{2}(a+b)h \cdot \frac{2}{3} h$$

or

$$P = \frac{1}{2} mh \cdot \frac{2}{3} h$$

The pressure is thus shown to be equal to the product of the area of the triangle ($\frac{1}{2}mh$) by the pressure at two-thirds of the depth. This result times the density would give the pressure for any other liquid.

2. To deduce a formula for the velocity of falling bodies when the acceleration varies inversely as the square of the distance.

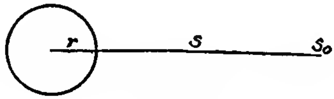


FIG. 218.

Let r be the radius of the earth and S any distance greater than r. Let g be the acceleration at the surface of the earth and a the acceleration at a distance S from the earth's centre. Then

$$\frac{a}{g}=\frac{r^2}{S^2}$$

since the force of gravitation, and consequently the acceleration, varies inversely as the square of the distance. Hence

$$a=\frac{gr^2}{S^2}$$

Hence, since $F=ma$, the force at a distance S is

$$F=\frac{mgr^2}{S^2}$$

The work done by this varying force is

$$W=\int F ds$$

$$\therefore W=\int \frac{mgr^2}{S^2} ds=mgr^2\int \frac{ds}{S^2}$$

Integrating between S and S_0,

$$W=mgr^2\int_S^{S_0}\frac{ds}{S^2}=mgr^2\Big|_S^{S_0}-\frac{1}{S}$$

$$\therefore W=mgr^2\left(\frac{1}{S}-\frac{1}{S_0}\right)$$

The energy of the falling body may also be expressed by

$$W=\tfrac{1}{2}mv^2$$

$$\therefore \tfrac{1}{2}mv^2=mgr^2\left(\frac{1}{S}-\frac{1}{S_0}\right)$$

$$\therefore v^2=2gr^2\left(\frac{1}{S}-\frac{1}{S_0}\right)$$

Thus the velocity acquired by a body in falling from a point S_0 to S may be found.

3. A mass swinging as a simple pendulum will at its lowest point have the same velocity as when it falls vertically from the same elevation.

In Fig. 219 let om be a pendulum whose length is l and point of suspension o. Let om be inclined $\theta°$ to the vertical. When m swings to P it will be moving with the same velocity as if it had fallen from Q to P. The general equation for velocity in terms of space is

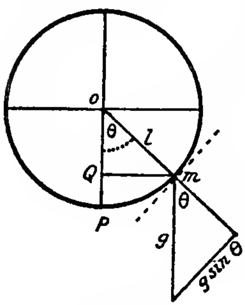


FIG. 219.

$$v^2=2gs$$

The value of s when the body falls vertically is QP. But $QP=l-oQ=l-l\cos\theta$ $=l(1-\cos\theta)$. Hence $v^2=2gl(1-\cos\theta)$ for velocity acquired in the vertical fall.

When m moves along the arc mP, the same general equation $(v^2=2gs)$ is applicable, but g becomes $g\sin\theta$ as shown in the figure, and this quantity varies with each small change in position, ds, along the arc mP. But $ds=ld\theta$, and the square of the velocity will then be the sum $2gl\sin\theta d\theta$ from θ to zero angle. That is

$$v^2=2gl\int_\theta^0 \sin\theta d\theta=2gl\Big|_\theta^0 \cos\theta=2gl-2gl\cos\theta$$
$$=2gl\,(1-\cos\theta)$$

This is the same result as that found above for the vertical fall from Q to P.

4. When force is applied to a turbine wheel in a direction perpendicular to the plane of rotation, the component causing rotation will be maximum when the blades are set at an angle of 45°.

Let F be the total force and x the effective component, then, as already shown in the text,

$$x = F \sin\theta \cos\theta$$

By differentiation,

$$\frac{dx}{d\theta} = F(\cos^2\theta - \sin^2\theta)$$

The turning effect of the force will be greatest when $\frac{dx}{d\theta}$ is zero, consequently the angle sought is one where

$$F(\cos^2\theta - \sin^2\theta) = 0$$

or

$$\cos\theta = \sin\theta$$

This is true only when the angle is 45°, for only then does $\cos\theta = \sin\theta$.

5. To find an expression for coefficient of rigidity in case of a solid cylindrical rod.

For the cylindrical shell it has been shown that

$$n = \frac{Fhl}{2\pi\theta r^3 t}$$

The t is here the thickness of the shell,—*i.e.*, the differential of the radius, dr, of the solid rod, and $\overline{Fh}$ is the $d\overline{Fh}$ when the solid rod is considered as composed of an infinitely large number of concentric shells. Hence

$$d\overline{Fh} = \frac{2\pi n\theta r^3 dr}{l}$$

$$\therefore Fh = \frac{2\pi n\theta}{l}\int_0^r r^3 dr$$

$$= \frac{2\pi n\theta r^4}{4l} = \frac{\pi n\theta r^4}{2l}$$

$$\therefore n = \frac{2Fhl}{\pi\theta r^4}$$

6. Show that the moment of inertia, I, about any axis is equal to I about a parallel axis through the centre of gravity plus the mass of the body multiplied by the square of the distance between the axes.

Let o, Fig. 220, be the centre of gravity of the mass. Draw rectangular coördinates through o. Let one axis, perpendicular to the paper, pass through o, and another, parallel to the former,

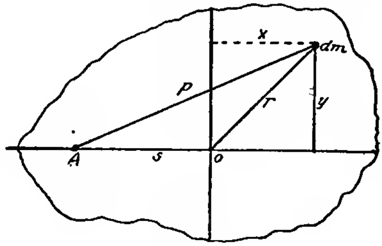


FIG. 220.

through A. Let I_o, be the moment of inertia in reference to the axis through o, and I_a in reference to that through A. Then it is to be shown that

$$I_a = I_o + ms^2$$

Let an elementary mass dm be chosen at a distance r from o, and p from A. Then

$$I_a = \int p^2 dm$$

but $$p^2 = y^2 + (x+s)^2 = y^2 + x^2 + 2xs + s^2$$

$$\therefore I_a = \int (y^2 + x^2 + 2xs + s^2) dm$$

$$= \int (y^2 + x^2) dm + 2s \int x dm + s^2 \int dm$$

$$y^2 + x^2 = r^2 \quad \therefore \int (y^2 + x^2) dm = \int r^2 dm = I_o$$

$$2s \int x dm = 0$$

for a body is in equilibrium about the centre of gravity, and hence the sum of the moments, xdm, is zero.

$$s^2 \int dm = s^2 m$$

$$\therefore I_a = I_o + ms^2$$

7. Find moment of inertia, I, of a circular disk with axis through its centre and perpendicular to its plane.

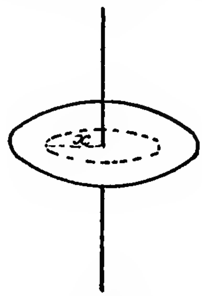


FIG. 221.

Let the mass of the disk be m and the radius r. Choose an elementary ring of radius x and width dx; then the total area of the surface of the disk is to the area of the elementary ring as the total mass is to the differential mass dm. That is,

$$\frac{\pi r^2}{2\pi x dx} = \frac{m}{dm}$$

$$\therefore dm = \frac{2mxdx}{r^2}$$

$$I = \int x^2 dm$$

Substituting the value of dm and integrating between $x = o$ and $x = r$,

$$I = \int_0^r \frac{2mx^3}{r^2} dx$$

$$= \Big|_0^r \tfrac{2}{4} m \frac{x^4}{r^2} = \tfrac{1}{2} mr^2$$

$$\therefore I = \tfrac{1}{2} mr^2$$

The moment of inertia about any other axis parallel to this one and distant s from it is

$$I' = m\left(\frac{r^2}{2} + s^2\right)$$

The radius of gyration, k, is such a distance from the axis that

$$mk^2 = I$$

$$\therefore\ k^2 = \frac{I}{m}$$

In case of this disk

$$k^2 = \frac{mr^2}{2m} = \frac{r^2}{2}$$

$$\therefore\ k = \frac{r}{\sqrt{2}}$$

The square of the radius of gyration in any case is the moment of inertia with the factor m omitted.

The expression $I = \frac{1}{2}mr^2$ is independent of the thickness of the disk, and hence is the expression for the moment of inertia of any cylinder rotating on its own axis.

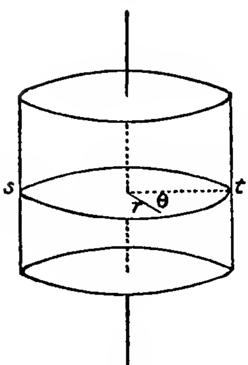


Fig. 222.

8. Another method of calculating the moment of inertia of a cylinder rotating on its axis is by use of polar coördinates.

Thus, in Fig. 222 let st be a section perpendicular to the axis, and let dA be a differential area in this plane. This area is located by (r, θ) and its value is

$$dA = rdrd\theta$$

for θ is expressed in radians, and therefore $rd\theta$ is the length of the arc in this elementary area, dA, and dr is the width. If the length of the cylinder is l, an elementary prism extending the whole length of the cylinder and having a cross section dA is

$$ldA = lrdrd\theta$$

and if density is uniform, the elementary mass is

$$dm = \rho lrdrd\theta \qquad (\rho = \text{density})$$

But $$I=\int r^2dm=\int r^3drd\theta\rho l$$

$$\therefore I=\rho l\int_0^R\int_0^{2\pi} r^3drd\theta$$

$$=\frac{\rho lR^4}{4}\int_0^{2\pi} d\theta$$

$$=\frac{\pi l\rho R^4}{2}=\pi R^2l\rho\frac{R^2}{2}=\tfrac{1}{2}mR^2$$

since $\pi R^2l\rho$ is the mass m, R being the radius of the cylinder.

9. To find moment of inertia of a sphere rotating on an axis through its centre. Let R be the radius of the sphere, and r the radius of an elementary section, st, perpendicular to the axis and at a distance x from the centre of the sphere. The thickness of this elementary section is then dx, and its mass is

$$\rho\pi r^2dx$$

where ρ is the density. The mass of this elementary section is the differential mass, dM, of the sphere. The moment of inertia of this elementary mass is therefore

$$\tfrac{1}{2}r^2dM$$

for it is a cylinder of length dx.

$$\tfrac{1}{2}r^2dM=\tfrac{1}{2}\rho\pi r^4dx$$

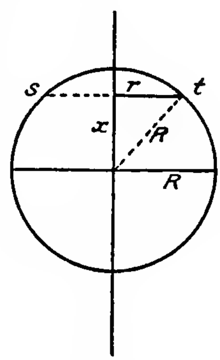


FIG. 223.

Hence the moment of inertia of the sphere is the sum of all the elementary sections between $+R$ and $-R$. That is

$$I=\tfrac{1}{2}\rho\pi\int_{-R}^{+R} r^4dx$$

$$=\tfrac{1}{2}\rho\pi\int_{-R}^{+R} (R^2-x^2)^2dx$$

Since $r^2 = R^2 - x^2$, as seen in the figure, $r^4 = (R^2 - x^2)^2$

$$\tfrac{1}{2}\rho\pi \int_{-R}^{+R} (R^4 - 2R^2x^2 + x^4)dx = \tfrac{1}{2}\rho\pi \;.\; \tfrac{16}{15}R^5 = \tfrac{8}{15}\pi\rho R^5 = \tfrac{4}{3}\pi R^3\rho \;.\; \tfrac{2}{5}R^2$$

But $\frac{4}{3}\pi R^3\rho$ is the mass, M, of the sphere.

$$\therefore I = \tfrac{2}{5}MR^2$$

10. Find the moment of inertia of a thin rectangular plate whose length is b and whose width is c, when the axis bisects b and is parallel to c.

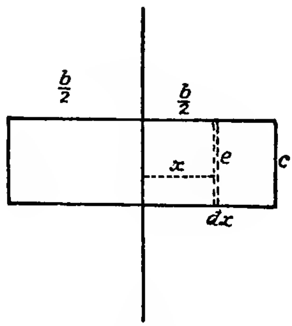


FIG. 224.

Let e be an elementary area at a distance x from the axis. Its area is cdx. The total area is bc. Let the total mass be m. The mass of the elementary part is dm. Hence

$$\frac{bc}{cdx} = \frac{m}{dm}$$

$$\therefore dm = \frac{m}{b}dx$$

$$I = \int x^2 dm$$

$$= \int \frac{m}{b} x^2 dx$$

Integrating between $-\dfrac{b}{2}$ and $+\dfrac{b}{2}$

$$I = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \frac{m}{b} x^2 dx$$

$$= \frac{m}{b} \Big|_{-\frac{b}{2}}^{+\frac{b}{2}} \tfrac{1}{3}x^3$$

$$= \frac{m}{b}\left(\frac{b^3}{24} + \frac{b^3}{24}\right) = \frac{mb^2}{12}$$

$$\therefore I = \frac{mb^2}{12}$$

11. Find I of a rectangular parallelopiped with axis through its centre and perpendicular to the side ab.

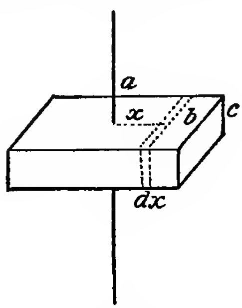


FIG. 225.

Let a be the length, b the breadth, and c the depth. Let m be the total mass. Let an elementary section at a distance x from the axis have a mass dm. Then

$$\frac{a}{dx} = \frac{m}{dm}$$

or $$dm = \frac{m}{a} dx$$

It has been shown above (10) that I for this elementary mass is $\frac{mb^2}{12}$ when the axis is through its centre and parallel to c. It has also been shown (6) that its moment of inertia in reference to another axis at a distance x and parallel to its own axis is $m\left(\frac{b^2}{12}+x^2\right)$. But the m of this elementary mass is the dm of the parallelopiped. Hence, substituting for m in $m\left(\frac{b^2}{12}+x^2\right)$ the value of $dm=\frac{m}{a}dx$, and integrating between $-\frac{a}{2}$ and $+\frac{a}{2}$,

$$I=\int_{-\frac{a}{2}}^{+\frac{a}{2}}\frac{m}{a}\left(\frac{b^2}{12}+x^2\right)dx$$

$$=\left|_{-\frac{a}{2}}^{+\frac{a}{2}}\frac{m}{a}\left(\frac{b^2x}{12}+\frac{x^3}{3}\right)\right.$$

$$=2\left(\frac{m}{a}\cdot\frac{b^2}{12}\cdot\frac{a}{2}\right)+2\left(\frac{m}{a}\cdot\frac{a^3}{24}\right)$$

$$\therefore I=m\frac{a^2+b^2}{12}$$

12. To prove that pv^γ is a constant quantity in case of adiabatic expansion of gases, γ being the ratio of specific heat at constant pressure to that at constant volume,—*i.e.*, $\frac{C_p}{C_v}$.

Let Q be the quantity of heat required to raise unit mass of a gas through dt degrees, the pressure p being constant. This heat will be expended both in raising the temperature and in increasing the volume of the gas. The quantity of heat required will be the same as if the gas had been kept at constant volume and the temperature raised dt degrees, and, in addition, the heat energy required to do the external work in increasing the volume by dv. This external work is pdv, expressed in ergs, or $\frac{pdv}{J}$ in calories (§ 102), J being the number of ergs per calorie. The increase of temperature per gram requires C_vdt calories where C_v is the specific heat at constant volume. Hence

$$Q=\frac{pdv}{J}+C_vdt$$

But in adiabatic expansion no heat is gained or lost by a gas; hence $Q=0$. Then

$$\frac{pdv}{J}+C_v dt=0 \quad (1)$$

By equation (210),

$$pv=R\tau$$

By differentiation,

$$pdv+vdp=Rd\tau$$

or

$$d\tau=\frac{pdv+vdp}{R}$$

But if p is kept constant, then

$$pdv=Rd\tau$$

or

$$\frac{pdv}{d\tau}=R$$

If $d\tau$ is 1° C.,

$$pvd=R=J\,(C_p-C_v)$$

since pdv is the excess of work done at constant pressure over that at constant volume when the mass and change of temperature are unity,—*i.e.*, it is the difference in specific heats times the number of ergs per calorie. Hence

$$d\tau=\frac{pdv+vdp}{J\,(C_p-C_v)}$$

Substituting this value of $d\tau$ in equation (1) above,

$$\frac{pdv}{J}+\frac{C_v pdv+C_v vdp}{J\,(C_p-C_v)}=0$$

or

$$C_p pdv-C_v pdv+C_v pdv+C_v vdp=0$$

or

$$\frac{C_p}{C_v}\,pdv+vdp=0$$

Let $\frac{C_p}{C_v}$ be represented by γ; then, dividing through by pv,

$$\gamma\frac{dv}{v}+\frac{dp}{p}=0$$

By integration,

$$\gamma\log v+\log p=0=\text{constant}$$

$$\therefore\ pv^{\gamma}=\text{constant}$$

13. Adiabatic coefficient of elasticity.

It has been shown that when no heat enters or leaves a body of gas,

$$PV^{\gamma}=\text{constant, } c \tag{1}$$

$$V^{\gamma}=\frac{c}{P} \tag{2}$$

Differentiating, $$\gamma V^{\gamma-1}dV=\frac{-cdP}{P^2} \tag{3}$$

or, substituting the value of c from (1) in (3)

$$\gamma V^{\gamma-1}dV=\frac{-PV^{\gamma}dP}{P^2} \tag{4}$$

$$\therefore \frac{dP}{dV}=\frac{P^2\gamma V^{\gamma-1}}{PV^{\gamma}}=\frac{P\gamma}{V} \tag{5}$$

$$\therefore \frac{\frac{dP}{dV}}{V}=E_{\eta}=P\gamma \tag{6}$$

Which shows that E_{η}, the elasticity at constant entropy, is equal to the pressure times *gamma*, γ, where γ = the ratio of specific heat at constant pressure to that at constant volume.

It has been shown that elasticity at constant temperature is

$$E_t=P \quad (\S\, 94) \tag{7}$$

Dividing (6) by (7)

$$\frac{E_{\eta}}{E_t}=\gamma=\frac{C_p}{C_v} \tag{8}$$

14. To calculate the change of entropy in case of a perfect gas.

Let Q be a function of pressure p and volume v, then by partial differentiation

$$dQ=\left(\frac{\partial Q}{\partial v}\right)_p dv+\left(\frac{\partial Q}{\partial p}\right)_v dp \tag{1}$$

Multiplying each quantity in the parentheses by $\frac{\partial \tau}{\partial \tau}$,

$$dQ=\left(\frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial v}\right)_p dv+\left(\frac{\partial Q}{\partial \tau}\frac{\partial \tau}{\partial p}\right)_v dp \tag{2}$$

But $\frac{\partial Q}{\partial \tau} = C_p =$ specific heat at constant pressure in the first parenthesis, but C_v in the second.

$$\therefore\ dQ = C_p\left(\frac{\partial \tau}{\partial v}\right)_p dv + C_v\left(\frac{\partial \tau}{\partial p}\right)_v dp \tag{3}$$

Since the gas is assumed to obey Boyle's law,

$$pv = R\tau \tag{4}$$

$$\therefore\ \left(\frac{\partial \tau}{\partial p}\right)_v = \frac{v}{R}$$

and $$\left(\frac{\partial \tau}{\partial v}\right)_p = \frac{p}{R}$$

Substituting these values in (3),

$$dQ = C_p \frac{p}{R} dv + C_v \frac{v}{R} dp \tag{5}$$

Dividing by $\tau = \frac{pv}{R}$,

$$\frac{dQ}{\tau} = C_p \frac{dv}{v} + C_v \frac{dp}{p} \tag{6}$$

Integrating and remembering that $\int \frac{dQ}{\tau} = \eta$, the change of entropy,

$$\eta = C_p \int_{v_2}^{v_1} \frac{dv}{v} + C_v \int_{p_1}^{p_2} \frac{dp}{p}$$

$$\therefore\ \eta = C_p \log \frac{v_1}{v_2} + C_v \log \frac{p_2}{p_1} \tag{7}$$

15. Dimensions of mechanical units.

Physical quantity.	Definition.	Dimension.	Derivation.
Length	l	L	
Mass	m	M	
Time	t	T	
Area	l^2	L^2	$L\times L$
Volume	l^3	L^3	$L\times L\times L$
Velocity	$\frac{l}{t}=v$	LT^{-1}	$\frac{L}{T}=LT^{-1}$
Linear acceleration	$\frac{l}{t}\div t$	LT^{-2}	$LT^{-1}\div T=LT^{-2}$
Angular velocity	$\frac{l}{t}\div r=\omega$	T^{-1}	$LT^{-1}\div L=T^{-1}$
Angular acceleration	$\frac{\omega}{t}$	T^{-2}	$LT^{-1}\div T=LT^{-2}$
Force	$F=ma$	LMT^{-2}	$M\times LT^{-2}$
Stress	$\frac{F}{l^2}$	$ML^{-1}T^{-2}$	$MLT^{-2}\div L^2$
Impulse	Ft	LMT^{-1}	$LMT^{-2}\times T$
Momentum	mv	LMT^{-1}	$M\times LT^{-1}$
Work	$Fs=w$	L^2MT^{-2}	$LMT^{-2}\times L$
Potential energy	Fs	L^2MT^{-2}	
Kinetic energy	$\frac{1}{2}mv^2$	L^2MT^{-2}	$M\times L^2T^{-2}$
Power	$\frac{W}{t}$	L^2MT^{-3}	$L^2MT^{-2}\div T$
Moment of force	Fl	L^2MT^{-2}	$LMT^{-2}\times L$
Moment of inertia	Σmr^2	L^2M	
Density	$\frac{m}{l^3}$	ML^{-3}	
Pressure	$\frac{F}{l^2}$	$L^{-1}MT^{-2}$	$LMT^{-2}\div L^2$

16. Dimensions of thermal units.

Θ = unit of temperature
Thermal unit = *e.g.* calorie
Dynamical unit = unit of energy *e.g.* erg.

Physical quantity.	Definition.	Dimensions.	Derivation.
Quantity of heat	mt° (thermal)	$M\Theta$	
Quantity of heat	$mt^\circ J$ (dynamic)	ML^2T^{-2}	$M\Theta \times L^2T^{-2}\Theta^{-1}$
Coefficient of thermal expansion ..	$\frac{l}{Lt^\circ}$	Θ^{-1}	$L \times L^{-1} \times \Theta^{-1}$
Thermal conductivity	$\kappa = \frac{QL}{ATt^\circ}$ (thermal)	$ML^{-1}T^{-1}$	$\frac{M\Theta \times L}{L^2T\Theta} = ML^{-1}T^{-1}$
Thermal conductivity	$\frac{QL}{ATt^\circ} J$ (dynamic)	$MLT^{-3}\Theta^{-1}$	$ML^{-1}T^{-1} \times L^2T^{-2}\Theta^{-1} = MLT^{-3}\Theta^{-1}$
Emissivity...	$\frac{Q}{ATt^\circ}$ (thermal)	$ML^{-2}T^{-1}$	$M\Theta \times T^{-1}L^{-2}\Theta^{-1} = ML^{-2}T^{-1}$
Emissivity...	$\frac{Q}{ATt^\circ} J$ (dynamic)	$MT^{-3}\Theta^{-1}$	$ML^{-2}T^{-1} \times L^2T^{-2}\Theta^{-1} = MT^{-3}\Theta^{-1}$
Capacity.....	$m \times$ sp. ht.	M	
Latent heat..	$\frac{Q}{m}$ (thermal)	Θ	$M^{-1} \times M\Theta = \Theta$
Latent heat..	$\frac{Q}{m} J$ (dynamic)	L^2T^{-2}	$\Theta \times L^2T^{-2}\Theta^{-1}$
Joule's equivalent..	$J = \frac{\text{energy}}{Q}$	$L^2T^{-2}\Theta^{-1}$	$ML^2T^{-2} \times M^{-1}\Theta^{-1} = L^2T^{-2}\Theta^{-1}$
Entropy.....	$\frac{Q}{\tau}$ (thermal)	M	$M\Theta \times \Theta^{-1} = M$
Entropy.....	$\frac{Q}{\tau} J$ (dynamic)	$ML^2T^{-2}\Theta^{-1}$	$M \times L^2T^{-2}\Theta^{-1}$

17. Moment of inertia, I.

Figure.	Position of axis of rotation.	I.	
Rod, slender, straight and uniform	⊥ to rod at center	$M\frac{l^2}{12}$	M = mass. l = length.
Solid cylinder	Coincident with axis of cylinder	$\frac{1}{2}Mr^2$	r = radius.
Solid cylinder	⊥ axis of cylinder through center of gravity	$M\left(\frac{r^2}{4}+\frac{l^2}{12}\right)$	l = length. r = radius.
Hollow cylinder	Coincident with axis	$M\left(\frac{r_1^2+r_2^2}{2}\right)$	r_1 = external radius. r_2 = internal radius.
Hollow cylinder	⊥ to axis and passing through center of gravity	$M\left(\frac{r_1^2+r_2^2}{4}+\frac{l^2}{12}\right)$	l = length.
Rectangular lamina $a\times b$	‖ to side b and through center	$M\frac{a^2}{12}$	
Rectangular lamina $a\times b$	⊥ to plane and through center	$M\left(\frac{a^2+b^2}{12}\right)$	
Sphere	Through center	$\frac{2}{5}Mr^2$	r = radius.
Spherical shell	Through center	$\frac{2}{3}Mr^2$	r = radius.
Rectangular bar	Through center and ⊥ to face $a\times b$	$M\left(\frac{a^2+b^2}{12}\right)$	

18. Density of solids in g/c.c.

Alum	1.7	Ivory	1.83—1.92
Aluminum, cast	2.56—2.58	Lead	11.3
Aluminum, wrought	2.65—2.8	Limestone	2.46—2.86
Asphaltum	1.1—1.2	Marble	2.5—2.8
Brass	8.1—8.7	Mica	2.6—3.2
Brick	2—2.2	Nickel	8.9
Copper	8.5—8.95	Oak	.8
Cork	.24	Osmium	21.4—22.4
Clay	1.8—2.6	Paraffin	.87—.91
Coke	1.1—1.7	Pine, white	.35—.5
Diamond	3.5—3.6	Platinum	21.5
Ebonite	1.15	Porcelain	2.3—2.5
Glass, crown	2.4—2.8	Quartz	2.65
Glass, flint	2.9—4.5	Silver	10.53
Gold	19.3	Sodium	.97—.99
Graphite	1.9—2.3	Tin	7.29
Ice at o°C.	.91	Zinc	7.15
Iridium	21.8—22.4	Mean density of the earth	5.576
Iron, cast	7.4	Surface density of the earth	2.56
Iron, wrought	7.8		
Iron, steel	7.8		

Density of liquids at 0° C. in g/c.c.

Alcohol	.81	Mercury	13.596
Bromine	3.187	Nitric acid	1.56
Carbon disulphide	1.293	Olive oil	.918
Ether	.736	Sea water	1.025
Glycerin	1.26	Sulphuric acid	1.85
Hydrochloric acid	1.27	Turpentine	.873
Linseed oil, boiled	.942		

Density of gases at 0° C. and 76 cm. pressure in g/c.c.

Air	.001293	Marsh gas	.00072
Ammonia	.00079	Nitrogen	.001257
Carbon dioxide	.001974	Oxygen	.00143
Chlorine	.003133	Steam 100° C.	.000581
Hydrogen	.0000895		

19. Density of air.

Density of dry air at 0° C. and under pressure of 76 cm. of mercury is .001293. Coefficient of expansion = .00367. Hence

$$\text{density} = \frac{.001293}{1 + .00367\,t} \cdot \frac{h}{76}$$

where t is the temperature and h is barometric pressure.

t°C.	72 cm.	73 cm.	74 cm.	75 cm.	76 cm.	77 cm.
10	.001181	.001198	.001215	.001231	.001247	.001263
11	1177	1194	1210	1226	1243	1259
12	1173	1189	1206	1222	1238	1255
13	1169	1185	1202	1218	1134	1250
14	1165	1181	1197	1214	1230	1246
15	.001161	.001177	.001193	.001209	.001225	.001242
16	1157	1173	1189	1205	1221	1237
17	1153	1169	1185	1201	1217	1233
18	1149	1165	1181	1197	1213	1229
19	1145	1161	1177	1193	1209	1224
20	.001141	.001157	.001173	.001189	.001204	.001221
21	1137	1153	1169	1185	1200	1216
22	1133	1149	1165	1181	1196	1212
23	1130	1145	1161	1177	1192	1208
24	1126	1141	1157	1173	1188	1204
25	.001122	.001138	.001153	.001169	.001184	.001200
26	1118	1134	1149	1165	1180	1196
27	1114	1130	1146	1161	1176	1192
28	1110	1126	1142	1157	1172	1188
29	1107	1122	1138	1153	1169	1184
30	1103	1119	1134	1149	1165	1180

When a correction is to be made for humidity the barometric reading h in equation above must be diminished by a quantity $.378e$, where e is the pressure of the vapor in the air. The value of e may be found by determining the dew-point and taking the corresponding pressure for saturated vapor.

20. Density of water and mercury in g/c.c.

Water free from air.

	Density			Density	
t° C.	Water	Mercury	t° C.	Water	Mercury
—10	.998145	13.6203	32	.99517	13.5169
— 9	8427	6178	33	485	5145
— 8	8685	6153	34	452	5120
— 7	8911	6129	35	.99418	13.5096
— 6	9118	6104	36	383	5071
— 5	.999218	13.6079	37	347	5047
— 4	9455	6055	38	310	5022
— 3	9590	6030	39	373	4998
— 2	9703	6005	40	.99235	13.4974
— 1	9797	5981	41	197	......
— 0	.999871	13.5956	42	158	......
1	9928	5931	43	118	......
2	9969	5907	44	078	......
3	9991	5882	45	.99037	......
4	1.000000	5857	46	.98996	......
5	.999990	13.5833	47	954	......
6	9970	5808	48	910	......
7	9933	5783	49	865	......
8	9886	5759	50	.98820	13.4731
9	9824	5734	55	582	......
10	.999747	13.5709	60	338	4488
11	9655	5685	65	074	......
12	9549	5660	70	.97794	4246
13	9430	5635	75	.97498	......
14	9299	5611	80	194	4005
15	.999160	13.5586	85	.96879	......
16	9002	5562	90	556	13.3764
17	8841	5537	95	219	......
18	8654	5513	100	.95865	3524
19	8460	5488	110	......	3284
20	.998259	13.5463	120	......	3045
21	8047	5439	130	......	2807
22	7826	5414	140	......	2569
23	7601	5390	150	......	2331
24	7367	5365	160	......	2094
25	.997120	13.5341	170	......	13.1858
26	6870	5316	180	......	1621
27	6600	5292	190	......	1385
28	6330	5267	240	......	13.0210
29	6050	5243	290	......	12.9041
30	.995770	13.5218	340	......	12.7873
31	5470	5194	360	......	12.7406

21. Reduction of barometric height to 0° C.

Ht=observed height of barometer. *a*=correction to be subtracted when rise of temperature is 1° C. The values of *a* are found by multiplying *Ht* by the difference between the linear coefficient of expansion of brass (.000019) and the coefficient of cubical expansion of mercury (.000181). The correction for any temperature is found by multiplying *a* by that temperature in degrees C.

Ht mm.	*a* mm.	*Ht* mm.	*a* mm.	*Ht* mm.	*a* mm.	*Ht* mm.	*a* mm.
400	.0651	500	.0813	600	.0975	700	.1137
410	.0668	510	.0830	610	.0992	710	.1154
420	.0684	520	.0846	620	.1008	720	.1170
430	.0700	530	.0862	630	.1024	730	.1186
440	.0716	540	.0878	640	.1040	740	.1202
450	.0732	550	.0894	650	.1056	750	.1218
460	.0749	560	.0911	660	.1073	760	.1235
470	.0765	570	.0927	670	.1089	770	.1251
480	.0781	580	.0943	680	.1105	780	.1267
490	.0797	590	.0959	690	.1121	790	.1283
						800	.1299

22. Coefficients of elasticity.

	Young's modulus	Simple rigidity	Volume elasticity
	Dynes per sq. cm.	Dynes per sq. cm.	Dynes per sq. cm.
Aluminum	$6.5(10)^{11}$	$2.5—3.3(10)^{11}$	$5.5(10)^{11}$
Benzene 16° C.	....	....	$1.12(10)^{10}$
Brass	$7.7—10.8(10)^{11}$	$3.1—4(10)^{11}$	$10.78(10)^{11}$
Copper	$8.5—12.3(10)^{11}$	$3.8—4.6(10)^{11}$	$15—16(10)^{11}$
German silver	$11.7—13.7(10)^{11}$	$4.2(10)^{11}$	....
Glass	$4—6(10)^{11}$	$2—2.4(10)^{11}$	$3.6—3.9(10)^{11}$
Iron, wrought	$19.3—20.9(10)^{11}$	$7.7—8(10)^{11}$	....
Mercury	....	....	$2.6(10)^{11}$
Platinum (drawn)	$15—16.6(10)^{11}$	$6.3(10)^{11}$	....
Silver	$7(10)^{11}$	$2.6(10)^{11}$	....
Steel	$20—21.6(10)^{11}$	$8—8.8(10)^{11}$	$18(10)^{11}$
Water 20° C.	....	....	$2.27(10)^{10}$

23. Surface tension of liquids in dynes per centimetre in contact with air.

Liquid.	Tension.	Liquid.	Tension.
Alcohol	23	Petroleum	29
Glycerin	63	Soap solution, saturated	25
Mercury	513	Turpentine	28.5
Olive oil	34.6	Water	73

24. Heat constants of solids and liquids.

Substance	Linear expansion	Cubical expansion	Specific heat	Melting point °C.	Boiling point °C.	Heat of fusion Cal.	Heat of vaporization Cal.
Air, liquid	......	......	.....	....	—192 to —182		
Alcohol	......	.00104	.58	...	78.3	....	210
Antimony	.000017	......	.049	632	1535		
Aluminum	.000023	......	.21	657			
Benzene	......	.00138	.38	...	80.3	...	94
Bismuth	.000016	......	.03	269	1413	12.64	
Brass	.000019	......	.093	900			
Cadmium	.000031	......	.055	320	780	13.6	
CO_2, liquid	......	......	.....	....	—78		
Copper	.000017	......	.095	1062 (in air)			
Glycerin	......	.00053	.576	....	290		
German silver	.000018	......	.095	1000			
Glass	.000008	......	.18	1100			
Gold	.000014	......	.0316	1064			
Hydrogen, liquid	......	......	.....	....	—253		
Ice	......	.000125	.46	0	....	80	
Iron	soft .000012; cast .000011	.....	.11	pure 1500; pig 1100	....	30	
Lead	.000029	......	.0306	327	1600	5.86	
Mercury	......	.00018	.033	—39	356.7	2.82	62
Nickel	.000013	......	.11	1435	....	4.6	
Phosphorus	......	......	.....	40	288	5.0	
Platinum	.0000089	......	.03	1760	...	27	
Silver	.000019	......	.055	955 (in air)	...	21	
Steel	.000011 to .000013	.....	.11	1375			
Sulphur	......	......	.163	115	445*	9.4	362
Tin	.000022	......	.054	232	1600	13	
Turpentine	......	.00105	.45	....	159	...	70
Water	......	......	1.00	....	100	...	536
Zinc	.000029	......	.0935	419	930	28	

* 76 cm. pressure.

25. Specific heat of gases and vapors.

C_p=sp. ht. at constant pressure.
C_v=sp. ht. at constant volume.

Gas or vapor.	C_p	C_v	$\frac{C_p}{C_v}$	Symbol.
Air, 0°–100° C.	.2374	.1690	1.405	O_2+N_2
Alcohol, ethyl, 108°–220° C.	.4534	.3991	1.136	C_2H_6O
Argon	....	....	1.63	Ar
Carbon dioxide, 15°–100° C.	.2025	.1545	1.311	CO_2
Ether, 69°–224° C.	.4797	.4657	1.03	$C_4H_{10}O$
Hydrogen, 12°–198° C.	3.409	2.417	1.41	H_2
Mercury	....	....	1.666	Hg
Nitrogen, 0°–200° C.	.2438	.1729	1.41	N_2
Water, 128°–217° C.	.4805	.3613	1.33	H_2O

26. Critical temperatures, pressures, volumes, and densities of gases.

Smithsonian Tables

Substance	Critical temperature	Critical pressure in atmospheres	Volume referred to air at 0° C. and 76 cm. pressure	Density in g/c.c.
Air	—140	39		
Alcohol (ethyl)	243.6	62.76	.00713	.288
Ammonia	130	115		
Argon	—121	50.6	......	1.5
Carbon dioxide	30.92	77	.0066	
Ether	19.7	35.77	.01584	.208
Hydrogen	—243	20		
Nitrogen	—146	35	......	.44
Oxygen	—118	50	......	.6044
Sulphur dioxide	155.4	78.9		
Water	370	195.5		

27. Thermal conductivity.

Expressed in calories of heat transmitted per sec. per sq. cm. through a plate 1 cm. thick when the difference of temp. on the two faces is 1° C.

Smithsonian Tables

Substance	Conductivity	Substance	Conductivity
Air 0° C.	.000568	Iron	.163
Aluminum	.353	Lead	.079
Brass	.229	Mercury	.021
Carbon	.000405	Nitrogen	.000524
Copper	.720	Oxygen	.000563
Cotton-wool	.000043	Silver	1.096
Cork	.000717	Sawdust	.00012
Felt	.000087	Water 4° C.	.00129
German silver	.079	Zinc	.303
Hydrogen 0° C.	.000327		

28. Pressure of saturated vapor in cm. mercury.

Temp.	Ethyl alcohol	Methyl alcohol	Carbon bisulphide	Ammonia	Mercury
0°C.	1.224	2.997	12.79	318.33	.00002
5	1.73	4.02	16.00	383.03	
10	2.378	5.38	19.85	457.40	.00005
15	3.244	7.14	24.42	543.34	
20	4.4	9.4	29.81	638.78	.0001
25	5.88	12.27	36.11	747.70	
30	7.81	15.89	43.46	870.10	.0003
35	10.26	20.39	51.96	1007.02	
40	13.37	25.94	61.75	1159.53	.0008
50	22.00	40.94	85.71	1515.83	.0015
60	35.03	62.43	116.45	1948.21	.0029
70	54.12	85.71	155.21	2467.55	.0052
100	169.2	240.51	332.51	4660.82	100° C., .027 350° C., 65.8

29. Pressure P of saturated water vapor at temperature t.

P = cm. of mercury.
$t°$ = centigrade.

$t°$	P	$t°$	P	$t°$	P	$t°$	P
—5	.32	27	2.65	59	14.20	91	54.58
—4	.34	28	2.81	60	14.88	92	56.68
—3	.37	29	2.978	61	15.58	93	58.84
—2	.39	30	3.155	62	16.32	94	61.07
—1	.42	31	3.34	63	17.08	95	63.38
0	.46	32	3.536	64	17.87	96	65.75
1	.49	33	3.74	65	18.69	97	68.19
2	.53	34	3.957	66	19.55	98	70.72
3	.57	35	4.183	67	20.44	98.2	71.23
4	.61	36	4.42	68	21.36	98.4	71.74
5	.65	37	4.67	69	22.32	98.6	72.26
6	.70	38	4.93	70	23.31	98.8	72.79
7	.75	39	5.20	71	24.34	99	73.32
8	.80	40	5.49	72	25.41	99.2	73.85
9	.857	41	5.79	73	26.51	99.4	74.38
10	.92	42	6.10	74	27.66	99.6	74.92
11	.979	43	6.43	75	28.85	99.8	75.47
12	1.046	44	6.78	76	30.08	100	76.00
13	1.116	45	7.14	77	31.36	100.2	76.55
14	1.19	46	7.52	78	32.68	100.4	77.10
15	1.27	47	7.91	79	34.05	100.6	77.65
16	1.354	48	8.32	80	35.46	100.8	78.21
17	1.44	49	8.75	81	36.93	101	78.77
18	1.536	50	9.20	82	38.44	110	107.54
19	1.635	51	9.67	83	40.01	120	149.13
20	1.739	52	10.15	84	41.63	130	203.03
21	1.85	53	10.66	85	43.30	140	271.76
22	1.966	54	11.19	86	45.03	150	358.12
23	2.089	55	11.75	87	46.82	160	465.16
24	2.218	56	12.32	88	48.67	170	596.17
25	2.355	57	12.93	89	50.58	180	754.64
26	2.499	58	13.55	90	52.54	190	944.27
						200	1168.90

30. Weight in grams of the aqueous vapor contained in a cubic meter of saturated air.

Smithsonian Tables

Temp. °C.	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
—20	1.078	0.992	0.913	0.839	0.770	0.706	0.647	0.593	0.542	0.496
—10	2.363	2.192	2.032	1.882	1.742	1.611	1.489	1.375	1.269	1.170
— 0	4.835	4.513	4.211	3.926	3.659	3.407	3.171	2.949	2.741	2.546
+ 0	4.835	5.176	5.538	5.922	6.330	6.761	7.219	7.703	8.215	8.757
10	9.330	9.935	10.574	11.249	11.961	12.712	13.505	14.339	15.218	16.144
20	17.118	18.143	19.222	20.355	21.546	22.796	24.109	25.487	26.933	28.450
30	30.039	31.704	33.449	35.275	37.187	39.187	41.279	43.465	45.751	48.138

31. Calories of heat produced by a combination with oxygen of 1 g. of the substance in first column.

Substance	Calories	Energy in ergs.
Alcohol	7000	$2.94(10)^{11}$
Carbon	8000	$3.36(10)^{11}$
Copper (forming CuO)	600	$2.52(10)^{10}$
Hydrogen	34000	$1.43(10)^{12}$
Iron (forming Fe_3O_4)	1576	$6.62(10)^{10}$
Magnesium	1721	$7.23(10)^{10}$
Phosphorus	5747	$2.41(10)^{11}$
Sulphur	2240	$9.41(10)^{10}$
Tin	1233	$5.18(10)^{10}$
Zinc	1314	$5.47(10)^{10}$

32. Freezing mixtures.

Smithsonian Tables

Substance	Proportionate part	Substance	Proportionate part	Temp. before mixing	Temp. after mixing
NH_4Cl	30	H_2O	100	13.3	— 5.1
$CaCl_2$ (cryst.)	250	H_2O	100	10.8	—12.4
NH_4NO_3	60	H_2O	100	13.6	—13.6
$CaCl_2$	30	Snow	100	—1	—10.9
$H_2SO_4+H_2O$ (66.1% H_2SO_4)	1	Snow	1.097	—1	—37
NH_4NO_3	1	Snow	1.31	0	—17.5
NaCl	33	Snow	100	—1	—21.3

33. Value of gravity in dynes.

Place	Latitude	Elevation in meters	g observed	g reduced to sea level
Berlin	52° 30′	49	981.26	981.27
Chicago	41° 49′	165	980.34	980.37
Cincinnati	39° 8′ 20″	245	979.99	980.03
Paris	48° 50′	67	980.96	980.97
Philadelphia	39° 57′ 06″	16	980.182	980.182
Singapore	1° 17′	14	978.07	978.07
Washington	38° 53′	10	980.10	980.10
Yakutat bay	59° 32′	4	981.82	981.82

34. Miscellaneous Data.

1 metre = 39.37 inches
1 kilometre = .62137 mile
1 inch = 2.5400 cm.
1 mile = 1.60935 kilometres
1 litre = 1.0567 U. S. quarts
1 U. S. gallon = 3.78543 litres
1 gram = 15.432 grains
1 kilogram = 2.2046 lb. avoir.
1 ounce avoir. = 28.3495 grams
1 lb. avoir. = 453.5924 grams
1 horse-power = 33,000 foot-pounds per minute
= 746 watts

	30°	45°	60°
Sine	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
Cosine	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
Tangent	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$

$\pi = 3.14159265$. Approx., 3.1416

$\pi^2 = 9.8696$

$\text{Log}_{10}\, \pi = .497149$

Base of natural or hyperbolic logarithms $= e = 2.71828$
Base of common logarithms = 10

$$\text{Log}_{10}\, a = \log_e a \times .4343$$
$$\text{Log}_e\, a = \log_{10} a \times 2.3026$$

Length, l, of seconds pendulum at sea level,

$$l = 39.012540 + .208268 \sin^2 \varphi \text{ inches}$$
$$= .990991 + .00529 \sin^2 \varphi \text{ metres}$$
$$\varphi = \text{latitude}$$

Acceleration, g, due to gravity per sec.^{2} mean solar time,

$$g = 32.086528 + .171293 \sin^2 \varphi \text{ feet}$$
$$= 977.9886 + 5.2210 \sin^2 \varphi \text{ centimetres}$$

Mean distance of earth to sun = 92,796,950 miles
Mean distance of earth to moon = 60.269 terrestrial radii
= 238,854.75 miles
= $3.844(10)^{10}$ cm.

Circumference of the earth = $4(10)^9$ cm. approx.
Radius of the earth = $6.4(10)^8$ cm. approx.

One radian = 57.2958°
= 57° 17′ 44.88″
One degree = .017453 radian

Length of mean solar day = 86,400 seconds
Length of sidereal day = 86,164.09965 mean solar seconds

35. Natural Sines and Cosines.

	SINE							
Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01454	0.01745	89
1	0.01745	0.02036	0.02327	0.02618	0.02908	0.03199	0.03490	88
2	0.03490	0.03781	0.04071	0.04362	0.04653	0.04943	0.05234	87
3	0.05234	0.05524	0.05814	0.06105	0.06395	0.06685	0.06976	86
4	0.06976	0.07266	0.07556	0.07846	0.08136	0.08426	0.08716	85
5	0.08716	0.09005	0.09295	0.09585	0.09874	0.10164	0.10453	84
6	0.10453	0.10742	0.11031	0.11320	0.11609	0.11898	0.12187	83
7	0.12187	0.12476	0.12764	0.13053	0.13341	0.13629	0.13917	82
8	0.13917	0.14205	0.14493	0.14781	0.15069	0.15356	0.15643	81
9	0.15643	0.15931	0.16218	0.16505	0.16792	0.17078	0.17365	80
10	0.17365	0.17651	0.17937	0.18224	0.18509	0.18795	0.19081	79
11	0.19081	0.19366	0.19652	0.19937	0.20222	0.20507	0.20791	78
12	0.20791	0.21076	0.21360	0.21644	0.21928	0.22212	0.22495	77
13	0.22495	0.22778	0.23062	0.23345	0.23627	0.23910	0.24192	76
14	0.24192	0.24474	0.24756	0.25038	0.25320	0.25601	0.25882	75
15	0.25882	0.26163	0.26443	0.26724	0.27004	0.27284	0.27564	74
16	0.27564	0.27843	0.28123	0.28402	0.28680	0.28959	0.29237	73
17	0.29237	0.29515	0.29793	0.30071	0.30348	0.30625	0.30902	72
18	0.30902	0.31178	0.31454	0.31730	0.32006	0.32282	0.32557	71
19	0.32557	0.32832	0.33106	0.33381	0.33655	0.33929	0.34202	70
20	0.34202	0.34475	0.34748	0.35021	0.35293	0.35565	0.35837	69
21	0.35837	0.36108	0.36379	0.36650	0.36921	0.37191	0.37461	68
22	0.37461	0.37730	0.37999	0.38268	0.38537	0.38805	0.39073	67
23	0.39073	0.39341	0.39608	0.39875	0.40142	0.40408	0.40674	66
24	0.40674	0.40939	0.41204	0.41469	0.41734	0.41998	0.42262	65
25	0.42262	0.42525	0.42788	0.43051	0.43313	0.43575	0.43837	64
26	0.43837	0.44098	0.44359	0.44620	0.44880	0.45140	0.45399	63
27	0.45399	0.45658	0.45917	0.46175	0.46433	0.46690	0.46947	62
28	0.46947	0.47204	0.47460	0.47716	0.47971	0.48226	0.48481	61
29	0.48481	0.48735	0.48989	0.49242	0.49495	0.49748	0.50000	60
30	0.50000	0.50252	0.50503	0.50754	0.51004	0.51254	0.51504	59
31	0.51504	0.51753	0.52002	0.52250	0.52498	0.52745	0.52992	58
32	0.52992	0.53238	0.53484	0.53730	0.53975	0.54220	0.54464	57
33	0.54464	0.54708	0.54951	0.55194	0.55436	0.55678	0.55919	56
34	0.55919	0.56160	0.56401	0.56641	0.56880	0.57119	0.57358	55
35	0.57358	0.57596	0.57833	0.58070	0.58307	0.58543	0.58779	54
36	0.58779	0.59014	0.59248	0.59482	0.59716	0.59949	0.60182	53
37	0.60182	0.60414	0.60645	0.60876	0.61107	0.61337	9.61566	52
38	0.61566	0.61795	0.62024	0.62251	0.62479	0.62706	0.62932	51
39	0.62932	0.63158	0.63383	0.63608	0.63832	0.64056	0.64279	50
40	0.64279	0.64501	0.64723	0.64945	0.65166	0.65386	0.65606	49
41	0.65606	0.65825	0.66044	0.66262	0.66480	0.66697	0.66913	48
42	0.66913	0.67129	0.67344	0.67559	0.67773	0.67987	0.68200	47
43	0.68200	0.68412	0.68624	0.68835	0.69046	0.69256	0.69466	46
44	0.69466	0.69675	0.69883	0.70091	0.70298	0.70505	0.70711	45
	60′	50′	40′	30′	20′	10′	0′	
	COSINE							

Natural Sines and Cosines.—(*Continued.*)

	COSINE							
Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
0	1.00000	1.00000	0.99998	0.99996	0.99993	0.99989	0.99985	89
1	0.99985	0.99979	0.99973	0.99966	0.99958	0.99949	0.99939	88
2	0.99939	0.99929	0.99917	0.99905	0.99892	0.99878	0.99863	87
3	0.99863	0.99847	0.99831	0.99813	0.99795	0.99776	0.99756	86
4	0.99756	0.99736	0.99714	0.99692	0.99668	0.99644	0.99619	85
5	0.99619	0.99594	0.99567	0.99540	0.99511	0.99482	0.99452	84
6	0.99452	0.99421	0.99390	0.99357	0.99324	0.99290	0.99255	83
7	0.99255	0.99219	0.99182	0.99144	0.99106	0.99067	0.99027	82
8	0.99027	0.98986	0.98944	0.98902	0.98858	0.98814	0.98769	81
9	0.98769	0.98723	0.98676	0.98629	0.98580	0.98531	0.98481	80
10	0.98481	0.98430	0.98378	0.98325	0.98272	0.98218	0.98163	79
11	0.98163	0.98107	0.98050	0.97992	0.97934	0.97875	0.97815	78
12	0.97815	0.97754	0.97692	0.97630	0.97566	0.97502	0.97437	77
13	0.97437	0.97371	0.97304	0.97237	0.97169	0.97100	0.97030	76
14	0.97030	0.96959	0.96887	0.96815	0.96742	0.96667	0.96593	75
15	0.96593	0.96517	0.96440	0.96363	0.96285	0.96206	0.96126	74
16	0.96126	0.96046	0.95964	0.95882	0.95799	0.95715	0.95630	73
17	0.95630	0.95545	0.95459	0.95372	0.95284	0.95195	0.95106	72
18	0.95106	0.95015	0.94924	0.94832	0.94740	0.94646	0.94552	71
19	0.94552	0.94457	0.94361	0.94264	0.94167	0.94068	0.93969	70
20	0.93969	0.93869	0.93769	0.93667	0.93565	0.93462	0.93358	69
21	0.93358	0.93253	0.93148	0.93042	0.92935	0.92827	0.92718	68
22	0.92718	0.92609	0.92499	0.92388	0.92276	0.92164	0.92050	67
23	0.92050	0.91936	0.91822	0.91706	0.91590	0.91472	0.91355	66
24	0.91355	0.91236	0.91116	0.90996	0.90875	0.90753	0.90631	65
25	0.90631	0.90507	0.90383	0.90259	0.90133	0.90007	0.89879	64
26	0.89879	0.89752	0.89623	0.89493	0.89363	0.89232	0.89101	63
27	0.89101	0.88968	0.88835	0.88701	0.88566	0.88431	0.88295	62
28	0.88295	0.88158	0.88020	0.87882	0.87743	0.87603	0.87462	61
29	0.87462	0.87321	0.87178	0.87036	0.86892	0.86748	0.86603	60
30	0.86603	0.86457	0.86310	0.86163	0.86015	0.85866	0.85717	59
31	0.85717	0.85567	0.85416	0.85264	0.85112	0.84959	0.84805	58
32	0.84805	0.84650	0.84495	0.84339	0.84182	0.84025	0.83867	57
33	0.83867	0.83708	0.83549	0.83389	0.83228	0.83066	0.82904	56
34	0.82904	0.82741	0.82577	0.82413	0.82248	0.82082	0.81915	55
35	0.81915	0.81748	0.81580	0.81412	0.81242	0.81072	0.80902	54
36	0.80902	0.80730	0.80558	0.80386	0.80212	0.80038	0.79864	53
37	0.79864	0.79688	0.79512	0.79335	0.79158	0.78980	0.78801	52
38	0.78801	0.78622	0.78442	0.78261	0.78079	0.77897	0.77715	51
39	0.77715	0.77531	0.77347	0.77162	0.76977	0.76791	0.76604	50
40	0.76604	0.76417	0.76229	0.76041	0.75851	0.75661	0.75471	49
41	0.75471	0.75280	0.75088	0.74896	0.74703	0.74509	0.74314	48
42	0.74314	0.74120	0.73924	0.73728	0.73531	0.73333	0.73135	47
43	0.73135	0.72937	0.72737	0.72537	0.72337	0.72136	0.71934	46
44	0.71934	0.71732	0.71529	0.71325	0.71121	0.70916	0.70711	45
	60′	50′	40′	30′	20′	10′	0′	
	SINE							

36. Natural Tangents and Cotangents.

	TANGENT							
Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01455	0.01746	89
1	0.01746	0.02036	0.02328	0.02619	0.02910	0.03201	0.03492	88
2	0.03492	0.03783	0.04075	0.04366	0.04658	0.04949	0.05241	87
3	0.05241	0.05533	0.05824	0.06116	0.06408	0.06700	0.06993	86
4	0.06993	0.07285	0.07578	0.07870	0.08163	0.08456	0.08749	85
5	0.08749	0.09042	0.09335	0.09629	0.09923	0.10216	0.10510	84
6	0.10510	0.10805	0.11099	0.11394	0.11688	0.11983	0.12278	83
7	0.12278	0.12574	0.12869	0.13165	0.13461	0.13758	0.14054	82
8	0.14054	0.14351	0.14648	0.14945	0.15243	0.15540	0.15838	81
9	0.15838	0.16137	0.16435	0.16734	0.17033	0.17333	0.17633	80
10	0.17633	0.17933	0.18233	0.18534	0.18835	0.19136	0.19438	79
11	0.19438	0.19740	0.20042	0.20345	0.20648	0.20952	0.21256	78
12	0.21256	0.21560	0.21864	0.22169	0.22475	0.22781	0.23087	77
13	0.23087	0.23393	0.23700	0.24008	0.24316	0.24624	0.24933	76
14	0.24933	0.25242	0.25552	0.25862	0.26172	0.26483	0.26795	75
15	0.26795	0.27107	0.27419	0.27732	0.28046	0.28360	0.28675	74
16	0.28675	0.28990	0.29305	0.29621	0.29938	0.30255	0.30573	73
17	0.30573	0.30891	0.31210	0.31530	0.31850	0.32171	0.32492	72
18	0.32492	0.32814	0.33136	0.33460	0.33783	0.34108	0.34433	71
19	0.34433	0.34758	0.35085	0.35412	0.35740	0.36068	0.36397	70
20	0.36397	0.36727	0.37057	0.37388	0.37720	0.38053	0.38386	69
21	0.38386	0.38721	0.39055	0.39391	0.39727	0.40065	0.40403	68
22	0.40403	0.40741	0.41081	0.41421	0.41763	0.42105	0.42447	67
23	0.42447	0.42791	0.43136	0.43481	0.43828	0.44175	0.44523	66
24	0.44523	0.44872	0.45222	0.45573	0.45924	0.46277	0.46631	65
25	0.46631	0.46985	0.47341	0.47698	0.48055	0.48414	0.48773	64
26	0.48773	0.49134	0.49495	0.49858	0.50222	0.50587	0.50953	63
27	0.50953	0.51320	0.51688	0.52057	0.52427	0.52798	0.53171	62
28	0.53171	0.53545	0.53920	0.54296	0.54673	0.55051	0.55431	61
29	0.55431	0.55812	0.56194	0.56577	0.56962	0.57348	0.57735	60
30	0.57735	0.58124	0.58513	0.58905	0.59297	0.59691	0.60086	59
31	0.60086	0.60483	0.60881	0.61280	0.61681	0.62083	0.62487	58
32	0.62487	0.62892	0.63299	0.63707	0.64117	0.64528	0.64941	57
33	0.64941	0.65355	0.65771	0.66189	0.66608	0.67028	0.67451	56
34	0.67451	0.67875	0.68301	0.68728	0.69157	0.69588	0.70021	55
35	0.70021	0.70455	0.70891	0.71329	0.71769	0.72211	0.72654	54
36	0.72654	0.73100	0.73547	0.73996	0.74447	0.74900	0.75355	53
37	0.75355	0.75812	0.76272	0.76733	0.77196	0.77661	0.78129	52
38	0.78129	0.78598	0.79070	0.79544	0.80020	0.80498	0.80978	51
39	0.80978	0.81461	0.81946	0.82434	0.82923	0.83415	0.83910	50
40	0.83910	0.84407	0.84906	0.85408	0.85912	0.86419	0.86929	49
41	0.86929	0.87441	0.87955	0.88473	0.88992	0.89515	0.90040	48
42	0.90040	0.90569	0.91099	0.91633	0.92170	0.92709	0.93252	47
43	0.93252	0.93797	0.94345	0.94896	0.95451	0.96008	0.96569	46
44	0.96569	0.97133	0.97700	0.98270	0.98843	0.99420	1.00000	45
	60′	50′	40′	30′	20′	10′	0′	
	COTANGENT							

Natural Tangents and Cotangents.—(*Continued.*)

	COTANGENT							
Deg.	0′	10′	20′	30′	40′	50′	60′	Deg.
0	Infini	343.77371	171.88540	114.58865	85.93979	68.75009	57.28996	89
1	57.28996	49.10388	42.96408	38.18846	34.36777	31.24158	28.63625	88
2	28.63625	26.43160	24.54176	22.90377	21.47040	20.20555	19.08114	87
3	19.08114	18.07498	17.16934	16.34986	15.60478	14.92442	14.30067	86
4	14.30067	13.72674	13.19688	12.70621	12.25051	11.82617	11.43005	85
5	11.43005	11.05943	10.71191	10.38540	10.07803	9.78817	9.51436	84
6	9.51436	9.25530	9.00983	8.77689	8.55555	8.34496	8.14435	83
7	8.14435	7.95302	7.77035	7.59575	7.42871	7,26873	7.11537	82
8	7.11537	6.96823	6.82694	6.69116	6.56055	6.43484	6.31375	81
9	6.31375	6.19703	6.08444	5.97576	5.87080	5.76937	5.67128	80
10	5.67128	5.57638	5.48451	5.39552	5.30928	5.22566	5.14455	79
11	5.14455	5.06584	4.98940	4.91516	4.84300	4.77286	4.70463	78
12	4.70463	4.63825	4.57363	4.51071	4.44942	4.38969	4.33148	77
13	4.33148	4.27471	4.21933	4.16530	4.11256	4.06107	4.01078	76
14	4.01078	3.96165	3.91364	3.86671	3.82083	3.77595	3.73205	75
15	3.73205	3.68909	3.64705	3.60588	3.56577	3.52609	3.48741	74
16	3.48741	3.44951	3.41236	3.37594	3.34023	3.30521	3.27085	73
17	3.27085	3.23714	3.20406	3.17159	3.13972	3.10842	3.07768	72
18	3.07768	3.04749	3.01783	2.98869	2.96004	2.93189	2.90421	71
19	2.90421	2.87700	2.85023	2.82391	2.79802	2.77254	2.74748	70
20	2.74748	2.72281	2.69853	2.67462	2.65109	2.62791	2.60509	69
21	2.60509	2.58261	2.56046	2.53865	2.51715	2.49597	2.47509	68
22	2.47509	2.45451	2.43422	2.41421	2.39449	2.37504	2.35585	67
23	2.35585	2.33693	2.31826	2.29984	2.28167	2.26374	2.24604	66
24	2.24604	2.22857	2.21132	2.19430	2.17749	2.16090	2.14451	65
25	2.14451	2.12832	2.11233	2.09654	2.08094	2.06553	2.05030	64
26	2.05030	2.03526	2.02039	2.00569	1.99116	1.97680	1.96261	63
27	1.96261	1.94858	1.93470	1.92098	1.90741	1.89400	1.88073	62
28	1.88073	1.86760	1.85462	1.84177	1.82906	1.81649	1.80405	61
29	1.80405	1.79174	1.77955	1.76749	1.75556	1.74375	1.73205	60
30	1.73205	1.72047	1.70901	1.69766	1.68643	1.67530	1.66428	59
31	1.66428	1.65337	1.64256	1.63185	1.62125	1.61074	1.60033	58
32	1.60033	1.59002	1.57981	1.56969	1.55966	1.54972	1.53987	57
33	1.53987	1.53010	1.52043	1.50184	1.50133	1.49190	1.48256	56
34	1.48256	1.47330	1.46411	1.45501	1.44598	1.43703	1.42815	55
35	1.42815	1.41934	1.41061	1.40195	1.39336	1.38484	1.37638	54
36	1.37638	1.36800	1.35968	1.35142	1.34323	1.33511	1.32704	53
37	1.32704	1.31904	1.31110	1.30323	1.29541	1.28764	1.27994	52
38	1.27994	1.27230	1.26471	1.25717	1.24969	1.24227	1.23490	51
39	1.23490	1.22758	1.22031	1.21310	1.20593	1.19882	1.19175	50
40	1.19175	1.18474	1.17777	1.17085	1.16398	1.15715	1.15037	49
41	1.15037	1.14363	1.13694	1.13029	1.12369	1.11713	1.11061	48
42	1.11061	1.10414	1.09770	1.09131	1.08496	1.07864	1.07237	47
43	1.07237	1.06613	1.05994	1.05378	1.04766	1.04158	1.03553	46
44	1.03553	1.02952	1.02355	1.01761	1.01170	1.00583	1.00000	45
	60′	50′	40′	30′	20′	10′	0′	
	TANGENT							

37. Common Logarithms.

N	0	1	2	3	4	5	6	7	8	9
1	00,000	04,139	07,918	11,394	14,613	17,609	20,412	23,045	25,527	27,875
2	30,103	32,222	34,242	36,173	38,021	39,794	41,497	43,136	44,716	46,240
3	47,712	49,136	50,515	51,851	53,148	54,407	55,630	56,820	57,978	59,106
4	60,206	61,278	62,325	63,347	64,345	65,321	66,276	67,210	68,124	69,020
5	69,897	70,757	71,600	72,428	73,239	74,036	74,819	75,587	76,343	77,085
6	77,815	78,533	79,239	79,934	80,618	81,291	81,954	82,607	83,251	83,885
7	84,510	85,126	85,733	86,332	86,923	87,506	88,081	88,649	89,209	89,763
8	90,309	90,849	91,381	91,908	92,428	92,942	93,450	93,952	94,448	94,939
9	95,424	95,904	96,379	96,848	97,313	97,772	98,227	98,677	99,123	99,564
10	00,000	00,432	00,860	01,284	01,703	02,119	02,531	02,938	03,342	03,743
11	04,139	04,532	04,922	05,308	05,690	06,070	06,446	06,819	07,188	07,555
12	07,918	08,279	08,636	08,991	09,342	09,691	10,037	10,380	10,721	11,059
13	11,394	11,727	12,057	12,385	12,710	13,033	13,354	13,672	13,988	14,301
14	14,613	14,921	15,229	15,534	15,836	16,137	16,435	16,732	17,026	17,319
15	17,609	17,897	18,184	18,469	18,752	19,033	19,312	19,590	19,866	20,140
16	20,412	20,683	20,952	21,219	21,484	21,748	22,011	22,272	22,531	22,789
17	23,045	23,300	23,553	23,805	24,055	24,304	24,551	24,797	25,042	25,285
18	25,527	25,768	26,007	26,245	26,482	26,717	26,951	27,184	27,416	27,646
19	27,875	28,103	28,330	28,556	28,780	29,003	29,226	29,447	29,667	29,885
20	30,103	30,320	30,535	30,750	30,963	31,175	31,387	31,597	31,806	32,015
21	32,222	32,428	32,634	32,838	33,041	33,244	33,445	33,646	33,846	34,044
22	34,242	34,439	34,635	34,830	35,025	35,218	35,411	35,603	35,793	35,984
23	36,173	36,361	36,549	36,736	36,922	37,107	37,291	37,475	37,658	37,840
24	38,021	38,202	38,382	38,561	38,739	38,917	39,094	39,270	39,445	39,620
25	39,794	39,967	40,140	40,312	40,483	40,654	40,824	40,993	41,162	41,330
26	41,497	41,664	41,830	41,996	42,160	42,325	42,488	42,651	42,813	42,975
27	43,136	43,297	43,457	43,616	43,775	43,933	44,091	44,248	44,404	44,560
28	44,716	44,871	45,025	45,179	45,332	45,484	45,637	45,788	45,939	46,090
29	46,240	46,389	46,538	46,687	46,835	46,982	47,129	47,276	47,422	47,567
30	47,712	47,857	48,001	48,144	48,287	48,430	48,572	48,714	48,855	48,996
31	49,136	49,276	49,415	49,554	49,693	49,831	49,969	50,106	50,243	50,379
32	50,515	50,651	50,786	50,920	51,055	51,188	51,322	51,455	51,587	51,720
33	51,851	51,983	52,114	52,244	52,375	52,504	52,634	52,763	52,892	53,020
34	53,148	53,275	53,403	53,529	53,656	53,782	53,908	54,033	54,158	54,283
35	54,407	54,531	54,654	54,777	54,900	55,023	55,145	55,267	55,388	55,509
36	55,630	55,751	55,871	55,991	56,110	56,229	56,348	56,467	56,585	56,703
37	56,820	56,937	57,054	57,171	57,287	57,403	57,519	57,634	57,749	57,864
38	57,978	58,092	58,206	58,320	58,433	58,546	58,659	58,771	58,883	58,995
39	59,106	59,218	59,329	59,439	59,550	59,660	59,770	59,879	59,988	60,097
40	60,206	60,314	60,423	60,531	60,638	60,746	60,853	60,959	61,066	61,172
41	61,278	61,384	61,490	61,595	61,700	61,805	61,909	62,014	62,118	62,221
42	62,325	62,428	62,531	62,634	62,737	62,839	62,941	63,043	63,144	63,246
N	0	1	2	3	4	5	6	7	8	9

Common Logarithms.—(*Continued.*)

N	0	1	2	3	4	5	6	7	8	9
43	63,347	63,448	63,548	63,649	63,749	63,849	63,949	64,048	64,147	64,246
44	64,345	64,444	64,542	64,640	64,738	64,836	64,933	65,031	65,128	65,225
45	65,321	65,418	65,514	65,610	65,706	65,801	65,896	65,992	66,087	66,181
46	66,276	66,370	66,464	66,558	66,652	66,745	66,839	66,932	67,025	67,117
47	67,210	67,302	67,394	67,486	67,578	67,669	67,761	67,852	67,943	68,034
48	68,124	68,215	68,305	68,395	68,485	68,574	68,664	68,753	68,842	68,931
49	69,020	69,108	69,197	69,285	69,373	69,461	69,548	69,636	69,723	69,810
50	69,897	69,984	70,070	70,157	70,243	70,329	70,415	70,501	70,586	70,672
51	70,757	70,842	70,927	71,012	71,096	71,181	71,265	71,349	71,433	71,517
52	71,600	71,684	71,767	71,850	71,933	72,016	72,099	72,181	72,263	72,346
53	72,428	72,509	72,591	72,673	72,754	72,835	72,916	72,997	73,078	73,159
54	73,239	73,320	73,400	73,480	73,560	73,640	73,719	73,799	73,878	73,957
55	74,036	74,115	74,194	74,273	74,351	74,429	74,507	74,586	74,663	74,741
56	74,819	74,896	74,974	75,051	75,128	75,205	75,282	75,358	75,435	75,511
57	75,587	75,664	75,740	75,815	75,891	75,967	76,042	76,118	76,193	76,268
58	76,343	76,418	76,492	76,567	76,641	76,716	76,790	76,864	76,938	77,012
59	77,085	77,159	77,232	77,305	77,379	77,452	77,525	77,597	77,670	77,743
60	77,815	77,887	77,960	78,032	78,104	78,176	78,247	78,319	78,390	78,462
61	78,533	78,604	78,675	78,746	78,817	78,888	78,958	79,029	79,099	79,169
62	79,339	79,309	79,379	79,449	79,518	79,588	79,657	79,727	79,796	79,865
63	79,934	80,003	80,072	80,140	80,209	80,277	80,346	80,414	80,482	80,550
64	80,618	80,686	80,754	80,821	80,889	80,956	81,023	81,090	81,158	81,224
65	81,291	81,358	81,425	81,491	81,558	81,624	81,690	81,757	81,823	81,889
66	81,954	82,020	82,086	82,151	82,217	82,282	82,347	82,413	82,478	82,543
67	82,607	82,672	82,737	82,802	82,866	82,930	82,995	83,059	83,123	83,187
68	83,251	83,315	83,378	83,442	83,506	83,569	83,632	83,696	83,759	83,822
69	83,885	83,948	84,011	84,073	84,136	84,198	84,261	84,323	84,386	84,448
70	84,510	84,572	84,634	84,696	84,757	84,819	84,880	84,942	85,003	85,065
71	85,126	85,187	85,248	85,309	85,370	85,431	85,491	85,552	85,612	85,673
72	85,733	85,794	85,854	85,914	85,974	86,034	86,094	86,153	86,213	86,273
73	86,332	86,392	86,451	86,510	86,570	86,629	86,688	86,747	86,806	86,864
74	86,923	86,982	87,040	87,099	87,157	87,216	87,274	87,332	87,390	87,448
75	87,506	87,564	87,622	87,679	87,737	87,795	87,852	87,910	87,967	88,024
76	88,081	88,138	88,195	88,252	88,309	88,366	88,423	88,480	88,536	88,593
77	88,649	88,705	88,762	88,818	88,874	88,930	88,986	89,042	89,098	89,154
78	89,209	89,265	89,321	89,376	89,432	89,487	89,542	89,597	88,653	89,708
79	89,763	89,818	89,873	89,927	89,982	90,037	90,091	90,146	90,200	90,255
80	90,309	90,363	90,417	90,472	90,526	90,580	90,634	90,687	90,741	90,795
81	90,849	90,902	90,956	91,009	91,062	91,116	91,169	91,222	91,275	91,328
82	91,381	91,434	91,487	91,540	91,593	91,645	91,698	91,751	91,803	91,855
83	91,908	91,960	92,012	92,065	92,117	92,169	92,221	92,273	92,324	92,376
84	92,428	92,480	92,531	92,583	92,634	92,686	92,737	92,788	92,840	92,891
N	0	1	2	3	4	5	6	7	8	9

Common Logarithms.—(*Concluded.*)

N	0	1	2	3	4	5	6	7	8	9
85	92,942	92,993	93,044	93,095	93,146	93,197	93,247	93,298	93,349	93,399
86	93,450	93,500	93,551	93,601	93,651	93,702	93,752	93,802	93,852	93,902
87	93,952	94,002	94,052	94,101	94,151	94,201	94,250	94,300	94,349	94,399
88	94,448	94,498	94,547	94,596	94,645	94,694	94,743	94,792	94 841	94,890
89	94,939	94,988	95,036	95,085	95,134	95,182	95,231	95,279	95,328	95,376
90	95,424	95,472	95,521	95,569	95,617	95,665	95,713	95,761	95,809	95,856
91	95,904	95,952	95,999	96,047	96,095	96,142	96,190	96,237	96,284	96,332
92	96,379	96,426	96,473	96,520	96,567	96,614	96,661	96,708	96,775	96,802
93	96,848	96,895	96,942	96,988	97,035	97,081	97,128	97,174	97,220	97,267
94	97,313	97,359	97,405	97,451	97,497	97,543	97,589	97,635	97,681	97,727
95	97,772	97,818	97,864	97,909	97,955	98,000	98,046	98,091	98,137	98,182
96	98,227	98,272	98,318	98,363	98,408	98,453	98,498	98,543	98,588	98,632
97	98,677	98,722	98,767	98,811	98,856	98,900	98,945	98,989	99,034	99,078
98	99,123	99,167	99,211	99,255	99,300	99,344	99,388	99,432	99,476	99,520
99	99,564	99,607	99,651	99,695	99,739	99,782	99,826	99,870	99,913	99,957
100	2.00000									
N	0	1	2	3	4	5	6	7	8	9

INDEX