THE EINSTEIN EQUATIONS OF THE GRAVITATIONAL FIELD FOR AN ARBITRARY DISTRIBUTION OF MATTER

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1. Einstein assumes that the trajectories of the motion of a mass particle in a gravitational field are given as solutions of the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds} = 0 \qquad i, j, k = 1, 2, 3, 4. \tag{1.1}$$

where (x^1, x^2, x^3) denote the position of the particle at the time x^4 and s is a parameter. The Γ's are the Christoffel symbols with respect to a symmetric tensor g_{ij}, i.e.,

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha}\left(\frac{\partial g_{\alpha j}}{\partial x^k} + \frac{\partial g_{\alpha k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha}\right)$$

g^{ij} is the cofactor of g_{ij} in the determinant

$$g = \begin{vmatrix} g_{11}\cdot & \cdot & \cdot\ g_{14} \\ & & \\ g_{41}\ \cdot & \cdot & \cdot\, g_{44} \end{vmatrix}$$

divided by g. The quantities g_{ij} are called the components of the gravitational field and are given as solutions of the field equations

$$B_{ij} = 0 \tag{1.2}$$

where $B_{ij} = B^\alpha_{ij\alpha}$, and

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{\alpha k}\Gamma^\alpha_{jl} - \Gamma^i_{\alpha l}\Gamma^\alpha_{jk}.$$

(1.2) holds at a point of the field at which no matter is present. As is customary I shall consider the tensor g_{ij} to determine a Riemann space and shall consequently speak of the metric structure of this space.

2. The Schwarzschild solution of the field equations gives as the metric structure of the space-time manifold for the case of a stationary gravitating mass, an expression of the form

$$ds^2 = \left(1 - \frac{A}{r}\right)dl^2 - \left[\frac{dr^2}{1 - A/r} + r^2(\sin^2\theta d\varphi^2 + d\theta^2)\right]. \tag{2.1}$$

This metric is referred to a spherical coördinate system in which the mass M is located at the origin. If we denote by K the Newtonian gravitational constant and by c the velocity of light, then

$$A = 2KM/c^2, \qquad l = ct, \qquad t = \text{time}.$$

Since the Newtonian potential V due to the mass M is

$$V = -KM/r, \tag{2.2}$$

the expression (2.1) may be written

$$ds^2 = \left(1 + \frac{2V}{c^2}\right) dl^2 - \left[\frac{dr^2}{1 + 2V/c^2} + r^2 (\sin^2 \theta d\varphi^2 + d\theta^2)\right]. \tag{2.3}$$

3. Let us now consider the gravitational field produced by any *stationary* distribution of matter. The Newtonian potential of the field at a point P at which no matter is present is given by

$$V = -K \int \rho/r_p . dv$$

where ρ is the density at a point which is a distance r_p from P and the integration is over the entire field. The equipotential surfaces are the surfaces

$$V = \text{constant} \tag{3.1}$$

If we denote by n the direction normal to an equipotential surface in which the resultant force acts, then this force is $-\partial V/\partial n$. The potential and the gravitational force at any point P of the gravitational field are the same as though due to a mass C_p/K located a distance R from P at a point O on the normal to the equipotential surface at P provided that C_p and R have the values

$$C_p = \frac{V^2}{\partial V/\partial n}, \quad R = -\frac{V}{\partial V/\partial n}$$

where V is the potential at the point P. Now the element of distance ds of the space-time manifold depends on the potential V of the gravitational field. We should therefore expect that the element of distance *ds at the point P* would be that due to the mass C_p/K located at O, which is given by the Schwarzschild solution. This leads to the form

$$ds^2 = \left(1 + \frac{2V}{c^2}\right) dl^2 - (dx^2 + dy^2 + dz^2) - \left(\frac{1}{1 + 2V/c^2} - 1\right)(\alpha dx + \beta dy + \gamma dz)^2 \tag{3.2}$$

with reference to rectangular coördinates. (α, β, γ) are the direction cosines of the normal to the equipotential surface at the point (x, y, z). But the form (3.2) will not satisfy the field equations although it is exactly the form that we must have if a solution of the general case is to exist which is of the same *nature* as the Schwarzschild solution for a single particle. (3.2) appears to be more justifiable than the approximation given by Eddington (Cf. Eddington, *The Mathematical Theory of Relativity*, 1923, §46, equ. 46.2) since this form is not of the nature of an approximation. According to the usual custom we would consider that the trajectories of the motion of a mass particle in the gravitational field are geodesics in the Riemann space defined by (3.2) i.e., that the Γ's in (1.1) are Christoffel symbols based on the form (3.2). In the following paragraph, however, we are giving a different method for the calculation of the Γ's.

4. Let us assume that all the quantities of the gravitational field, i.e., g_{ij}, Γ^i_{jk}, B^i_{jkl} B_{ij}, etc., at any point P are the same as though due to the mass C_p/K at the point O. The quantities of the gravitational field at the point P can then be calculated from the form

$$ds^2 = \left(1 - \frac{2C_p}{c^2r}\right)dl^2 - (dx^2 + dy^2 + dz^2) - \frac{1}{r^2}\left(\frac{1}{1-2C_p/c^2r} - 1\right) [(x-l)dx + (y-m)dy + (z-n)dz]^2 \quad 4.1$$

where $r^2 = (x-l)^2 + (y-m)^2 + (z-n)^2$; (l,m,n) are the coördinates of the point O. In the differentiations which are involved in calculating the quantities of the gravitational field C_p and (l,m,n) are to be treated as constants. If after all differentiations we replace C_p and (l,m,n) by their values as functions of (x,y,z) we shall obtain the quantities of the gravitational field as functions of the coördinates. *These quantities will satisfy the field equations (1.2).* The use of these values of the Γ's to describe the motion of a mass particle in the gravitational field according to the equations (1.1) would seem to express the principle that the force on the particle at the point P is due to the mass C_p/K at the point O associated with P. We cannot say, however, that the Γ's and B's are derivable from a fundamental tensor g_{ij}.

5. We have considered that the matter producing the gravitational field is stationary. If more than one body is present, however, the matter will move in the gravitational which it creates. The only example of such a system in nature is the universe itself. In this case we consider the motion of the matter as known and determine the Newtonian potential V which will now involve the time. The equipotential surfaces (3.1) will be equipotential surfaces at a particular time l. The quantities of the gravitational field will depend on the time but in the calculation of these quantities from the form (4.1) the time will be held constant during differentiation.

6. The assumption on which the calculation of the Γ's that we have just proposed depends, is essentially the following: *In an infinitely small space-time region of the gravitational field a coördinate system can be selected with reference to which space possesses a Schwarzschild structure.* The particular coördinate system associated with the neighborhood of any point P will be the spherical coördinate system (or its equivalent) with center on the normal to the equipotential surface through the point P

at the point O, which is a distance R from P. In case of a varying potential field our assumption requires the time to be held constant throughout the neighborhood of P, so that the quantities of the gravitational field are calculated as explained in the preceding paragraph.

In this connection we recall a very similar assumption which is made in the general theory of relativity, namely, that in infinitely small regions, and for coördinate systems with reference to which the bodies considered are unaccelerated, the special theory of relativity holds. This contains the assumption that space is euclidean with respect to this coördinate system. But we cannot use these coördinate systems here since their determination would require a knowledge of the motion of a particle in the gravitational field which we wish to calculate.

7. The equations (1.1) define a geometry of paths and give a definition of covariant differentiation based on the functions Γ^i_{jk}. (Cf. Eisenhart and Veblen, these PROCEEDINGS, **8,** pp. 19–23 (1922)). This geometry will become a Riemann geometry, i.e., the paths will become geodesics with respect to a tensor g_{ij} if

$$g_{ij,k} = 0$$

where $g_{ij,k}$ is the covariant derivative of g_{ij}, i.e.,

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{\alpha j}\Gamma^\alpha_{ik} - g_{i\alpha}\Gamma^\alpha_{jk}.$$

The necessary and sufficient condition for the existence of the tensor g_{ij} (cf. Veblen and Thomas, "The Geometry of Paths," *Trans. Amer. Math. Soc.*, to appear in 1924) is the algebraic consistency of the infinite set of equations

$$\begin{aligned} g_{i\alpha}B^\alpha_{jkl} + g_{\alpha j}B^\alpha_{ikl} &= 0 \\ g_{i\alpha}B^\alpha_{jklm_1} + g_{\alpha j}B^\alpha_{iklm_1} &= 0 \\ \cdots\cdots\cdots\cdots\cdots & \\ g_{i\alpha}B^\alpha_{jklm_1\cdots m_n} + g_{\alpha j}B^\alpha_{iklm_1\cdots m_n} &= 0 \\ \cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots & \end{aligned} \qquad 7.1$$

where $B^\alpha_{jklm_1\cdots m_n}$ is the nth covariant derivative of B^α_{jkl}. Hence, *the space-time trajectories of the motion of a mass particle in a gravitational field which depend on* Γ*'s calculated from* (4.1) *will be geodesics with respect to some tensor* g_{ij} *if and only if the sequence* (7.1) *be algebraically consistent.*

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