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NICOMACHUS OF GERASA

INTRODUCTION
TO
ARITHMETIC

TRANSLATED INTO ENGLISH
BY
MARTIN LUTHER D'OOGE

WITH STUDIES IN GREEK ARITHMETIC
BY
FRANK EGLESTON ROBBINS
AND
LOUIS CHARLES KARPINSKI

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IN

COMMEMORATION

OF THE

LONG AND INTIMATE FRIENDSHIP

BETWEEN TWO KINDRED SPIRITS

EDWARD WALDO PENDLETON

MARTIN LUTHER D'OOGE

PREFACE

PROFESSOR Martin Luther D'Ooge died suddenly on September 12, 1915, leaving unfinished a work on the *Introduction to Arithmetic* by Nicomachus. His translation of the Greek text was complete, but the supporting studies had not been commenced.

As soon as possible after his death, colleagues of Mr. D'Ooge in the University of Michigan took up the unfinished task, and their work combined with his appears in this volume. Mr. Karpinski contributed Chapters I, III, IV and the greater part of Chapter X of Part I, together with the first section of Part III, *Extensions of a Theorem of Nicomachus;* Mr. Robbins made the final revision of Mr. D'Ooge's translation and prepared the rest of the volume. At first it was proposed to present a revised Greek text, but this proved to be impracticable without too great delay.

Sincere thanks are due to Mrs. Edward Waldo Pendleton, whose generous help made the publication of the volume possible. We are under much obligation also to our colleagues, who have rendered assistance on many points. A special mention of indebtedness should be made to the University Editor, Dr. Eugene S. McCartney, for his scholarly assistance in the preparation of the manuscript for the press.

FRANK EGLESTON ROBBINS
LOUIS CHARLES KARPINSKI

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PART I

STUDIES IN GREEK MATHEMATICS

CHAPTER I

THE SOURCES OF GREEK MATHEMATICS

ARITHMETIC is fundamentally associated by modern readers, particularly by scientists and mathematicians, with the art of computation. For the ancient Greeks after Pythagoras, however, arithmetic was primarily a philosophical study, having no necessary connection with practical affairs. Indeed the Greeks gave a separate name to the arithmetic of business, λογιστική; of this division of the science no Greek treatise has been transmitted to us. In general the philosophers and mathematicians of Greece undoubtedly considered it beneath their dignity to treat of this branch, which probably formed a part of the elementary instruction of children. The evidence for the existence of treatises on the fundamental operations is very insecure and vague, resting upon a passage of Diogenes Laertius [1] and a citation by Eutocius.[2]

So far as the content of the logistic is concerned, our main source of information is the scholium [3] on Plato's *Charmides*, 163 E. This scholium is undoubtedly based on the lost work of Geminus, although it may be through the medium of Anatolius.[4] A passage in Proclus [5] which explicitly mentions Geminus touches analogous points.

The scholium is as follows: "Logistic is the theory which deals with numerable objects and not with numbers; [6] it does not, indeed, consider number in the proper sense of the term, but assumes 1 to be unity, and anything which can be numbered to be number (thus in place of the triad, it employs 3; in place of the decad, 10), and discusses with these the theorems of arithmetic.

[1] *Vitae Philosophorum*, VIII. 12, where a certain Apollodorus is designated as ὁ λογιστικός, which may mean, as Cantor thinks, that he was a *Rechenmeister*.

[2] In the *Commentary on the Measurement of the Circle by Archimedes* (in Heiberg, *Archimedis Opera Omnia cum Commentariis Eutocii*, Leipzig, 1881, vol. III, p. 302, line 4), he mentions the λογιστικά of a certain Magnus or Magnes.

[3] Here quoted on the basis of the text given in C. F. Hermann's edition of Plato (Leipzig, 1907), vol. VI, p. 290. See also Hultsch, *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, Berlin, 1864, pp. 247–248.

[4] Tannery, *La Géométrie Grecque*, Paris, 1887, pp. 48–49.

[5] Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, pp. 38, 1—42, 8 (ed. Friedlein).

[6] Compare the similar distinction made by Aristotle.

"It treats, then, on the one hand, that which Archimedes called 'The Cattle Problem,' and on the other hand, 'melite' and 'phialite' numbers, the one discussing vials (measures, containers) and the other flocks; and when dealing with other kinds of problems it has regard for the number of sensible bodies and makes its pronouncements as though it were for absolute objects.

"It has for material all numerable objects, and as subdivisions the so-called Greek and Egyptian methods for multiplication and division, as well as the summation and decomposition of fractions, whereby it investigates the secrets lurking in the subject-matter of the problems by means of the procedure that employs triangles and polygons.

"It has for its aim that which is useful in the relations of life and in business, although it seems to pronounce upon sensible objects as if they were absolute."

The philosophical arithmetic of the Greeks, ἀριθμητική, of which the arithmetic of Nicomachus is a specimen, corresponds in a measure to our number theory; the subject was designed for mature students as a preparation for the study of philosophy, and was not at all intended for children. *Arithmetica* is, as the name indicates, the study of that which is implied in number. This branch of arithmetical science developed along two quite distinct lines. On the one hand we have the rigid, mathematical discussion of the properties of numbers, involving the forms of proof and the rigor of the demonstrational geometry, which is the great contribution of Greece to science; on the other hand we have a mystical development, ascribing even magical powers and life-properties to numbers. This pseudo-science which employs the results, but not the demonstrations of the rigid science, is commonly termed arithmology.[1] Greek arithmetic must be considered, then, from the point of view of the philosopher and theoretical mathematician, rather than from that of our elementary schools.

Arithmetic was intimately connected by the early Greeks with both geometry and music. The treatise on arithmetic by Euclid, as found in the seventh, eighth, and ninth books of the *Elements*,[2] is wholly from the geometrical standpoint. This point of view is reflected in many ways in later treatises, that of Nicomachus, for instance, which considered arithmetic as an independent science. The intimate con-

[1] See Chapter VII, pp. 90 ff., for a discussion of arithmology and of the share of Nicomachus in it.

[2] Our references to the *Elements* of Euclid will be to the English edition by Sir T. L. Heath, *The Thirteen Books of Euclid's Elements*, three volumes, Cambridge, 1908.

nection between arithmetic and music accounts, in some measure, for the complete and even tedious discussion of ratios in the Greek treatises on arithmetic. In consequence, our consideration of the origins of Greek arithmetic will necessarily touch incidentally not only the processes of computation of the Greeks, but also geometry, music, and even other sciences, as related to the sciences of the older civilizations.[1]

For the sources of the early Greek arithmetical sciences we must look to Egypt and to Babylon, possibly even beyond to India and China. Evidence of the exchange of ideas between Greece and Egypt, and between Greece and Babylon, has accumulated so much in recent years as to show a degree of intimacy long unsuspected.[2] In the early centuries of the Christian era, knowledge of Greek astronomy was carried to India; traces of reciprocal influence in ancient times are not wanting, although any detailed statement must await more accurate information of the historical development of Hindu learning. The sciences, biological, physical, and mathematical, as well as the fine arts and technical arts, are involved in the interchange of ideas between Orient and Occident, but our interest is centered upon the mathematical sciences. In this field the Oriental science served primarily as a directive force, determining the topics which for centuries occupied the attention of Greek mathematicians.

In mathematics and astronomy the early traces of Oriental influence cover a wide range of ideas, touching at the lower point the simplest operations of computation and at the upper point the development of complicated astronomical theories. At the outset we may say that one extraordinary achievement in mathematics remains undisputedly Greek in its origin, namely, the development of logical, demonstrative geometry. Writers[3] who confound with the whole of science the systematization of the sciences achieved by the Greeks, together with this process of logical demonstration, entirely mistake the nature of science and the processes of its progress. Science is concerned with the problems involved in comprehending the universe in which we live. Science involves inevitably the knowledge of numbers and form, or

[1] For more complete discussion of arithmetic and logistic, see Heath, *A History of Greek Mathematics* (Oxford, 1921), vol. I, pp. 13–16.

[2] F. Cumont, *The Oriental Religions in Roman Paganism* (Chicago, 1911); and *Astrology and Religion among the Greeks and Romans* (New York, 1912); Milhaud, *Nouvelles Études sur l'Histoire de la Pensée Scientifique* (Paris, 1911), pp. 41–133.

[3] Like John Burnet, *Greek Philosophy, Part I, Thales to Plato* (London, 1920), pp. 4 ff.

mathematics, as well as the sciences of material things and life-processes. This science begins with primitive man, and develops as man develops.

The processes of computation in Greece were closely allied to those of Egypt. The abacus with its counters for reckoning, which was in wide use among the ancient Greeks,[1] had its counterpart, according to Herodotus,[2] in Egypt. While no trace of any Egyptian abacus has been found, Plato's statement that in Egypt "systems of calculation have been actually invented for the use of children" suggests that the Egyptians may have invented the abacus for the purpose for which it is now used in our primary schools.

The 'Egyptian methods' of multiplication and division, mentioned in the scholium on Plato's *Charmides* quoted above, are now known to us through the preservation and publication of the Ahmes manual,[3] an Egyptian arithmetic which dates from about 1700 B.C. Multiplication is effected by repeated doubling. Division is the inverse of multiplication, effected by doubling and re-doubling the divisor until the dividend can be obtained by summation of the appropriate doubles. Thus the product of 27 times 57 is obtained as follows:

57	1′
114	2′
228	4
456	8′
912	16′
1539	

The multiplication of 27 times 57 is treated as $16 + 8 + 2 + 1$ times 57. The accent marks to indicate which numbers are to be summed appear in the papyrus. Were 1539 to be divided by 27, the same series of doubles would be written, and the required summands would be obtained by subtraction from the dividend or by inspection. A multiplier or quotient involving fractions would be treated in the same way; thus, to multiply 57 by 27, $\frac{1}{2}$, $\frac{1}{4}$, the numbers $28\frac{1}{2}$ and $14\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively of 57, would appear among the summands to be added. Multiplication by 10 was sometimes included, without any doubling.

The most distinctive feature of the Egyptian arithmetic is the

[1] Herodotus, II. 36; Aristotle, *Constitution of Athens*, 68, 3 ff.; Plutarch, *Vita Catonis Minoris*, 70; Sextus Empiricus, *Adversus Mathematicos*, IX. 194.

[2] Herodotus, *ibid.*

[3] Eisenlohr, *Ein mathematisches Handbuch der alten Aegypter* (*Papyrus Rhind des British Museum*), Leipzig, 1891; T. Eric Peet, *The Rhind Mathematical Papyrus*, London, 1923.

restriction of the operations with fractions to unit fractions, i.e. those having one as the numerator; $\frac{2}{3}$ is the single exception. Thus $\frac{3}{4}$ was written as $\frac{1}{2}\ \frac{1}{4}$, and $\frac{2}{7}$ as $\frac{1}{4}\ \frac{1}{28}$; the juxtaposition indicates that the fractions are to be summed. Now the Greek symbolism for fractions includes special devices for writing such unit fractions, together with a separate symbol for the fraction $\frac{2}{3}$.[1] The little that we know of ancient computation, supported by definite indications of later documents, shows the intimate connection between the Greek and Egyptian methods of treating fractions. Thus, Euclid has a special term for a unit fraction,[2] while in the works of Hero of Alexandria and Diophantus series of unit fractions in true Egyptian form are common.[3] Furthermore, in the *arithmetica* the superparticular is definitely connected with the notion of a unit fraction.

Mention has been made of the 'summation and decomposition of fractions.' In the absence of any treatise on logistic from the classical period, the meaning of the scholiast's phrase is revealed to us only by later documents. A Greek papyrus of the eighth century A.D., found at Akhmim in Egypt,[4] includes unit fractions entirely after the manner of the Ahmes manual. The products of $\frac{2}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, and $\frac{1}{10}$ by the integers from 1 to 10, and by the tens to 90 are written in terms of unit fractions. A fragment of the same nature appears in the ancient Egyptian manual, giving the product of $\frac{2}{3}$ by $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{2}$, $\frac{1}{9}$, $\frac{1}{7}$, $\frac{1}{11}$, and $\frac{1}{5}$, as well as the products separately, of $\frac{1}{3}$ by $\frac{2}{3}$, and $\frac{1}{2}$ by $\frac{1}{3}$, by $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{7}$, and $\frac{1}{11}$, and $\frac{1}{5}$ by $\frac{1}{4}$. The distribution problems by Ahmes of 1, 2, 3, 6, 7, 8, and 9 loaves of bread among ten people are arithmetically analogous.

Undoubtedly we have here the 'decomposition' process into unit fractions; this also appears in the introductory material of the Egyptian manual wherein the fractions having 2 as a numerator and odd numbers to 99 as denominators are resolved into unit fractions.[5] Some of the same numerical operations are found also in two letters

[1] Heath, *Diophantus of Alexandria, A Study in the History of Greek Algebra* (second edition, Cambridge, 1910), pp. 44–47.

[2] *Elements*, Book VI, definition 3: "A number is a *part* of a number, the less of the greater, when it measures the greater; but *parts* when it does not measure it." Heath adds (vol. II, p. 280) that "by the expression *parts* (μέρη, the plural of μέρος) Euclid denotes what we should call a *proper fraction*."

[3] Heath, *Diophantus*, p. 46.

[4] J. Baillet, *Le Papyrus Mathématique d'Akhmim, Mémoires Publiées par les Membres de la Mission Archéologique Française au Caire* (Paris, 1892), T. IX, pp. 1–89.

[5] Similar fractions in Michigan Papyrus, No. 621; described by L. C. Karpinski, *Isis*, vol. V (1922), pp. 20–25, with facsimile. The text was published by F. E. Robbins, *Classical Philology*, vol. XVIII (1923), pp. 328–333.

of the fourteenth century written by the Greek monk Nicolas Rhabdas Atarvasda of Smyrna, who gives further the reverse process of 'summation' of unit fractions into ordinary ones.[1] Doubtless it was early recognized that for multiplication and division by a series of unit fractions the combination of the set into a single common fraction was desirable. Nicolas explains the process of combination.

Europe continued to employ the unit fractions for many centuries. Leonard of Pisa in the thirteenth century includes in his famous *Liber Abbaci* a table for decomposition into unit fractions,[2] and employs them frequently. The Arabs [3] and the Hindus, too, used Egyptian methods, although not exclusively, in their discussions of fractions, and traces of the Egyptian process of multiplication are preserved to this day among the Russian peasants.

Plato makes a statement about Egyptian mathematics which shows not only his own respect for Egyptian methods of instruction, but also brings to light certain Egyptian problems which may have had to do with the problems on containers, 'phialite numbers,' mentioned in the scholium on the *Charmides* already cited. Plato says (*Laws*, 819):

"All freemen, I conceive, should learn as much of these various disciplines as every child in Egypt is taught when he learns his alphabet. In that country, systems of calculation have actually been invented for the use of children, which they learn as a pleasure and amusement. They have to distribute apples and garlands, apportioning the same number either to a larger or smaller number of persons. . . . Another mode of amusing them is by taking vessels of gold, and brass, and silver, and the like, and mingling them or distributing them without mingling; as I was saying, they adapt their amusement to the numbers in common use, and in this way make more intelligible to their pupils the arrangements and movements of armies and expeditions, and in the management of a household they make people more useful to themselves, and more wide awake; and again in measurements of things which have length, and breadth, and depth, they free us from that ludicrous and disgraceful ignorance of all things which is natural to man."

[1] Paul Tannery, *Notices sur les Deux Lettres Arithmétiques de Nicolas Rhabdas, Notices et Extraits de Manuscrits de la Bibliothèque Nationale* (Paris, 1886), vol. XXXII, pp. 121–252.

[2] *Scritti de Leonardo Pisano,* published by B. Boncompagni, *Il liber abbaci* (Rome, 1857), vol. I, p. 79.

[3] See Karpinski, *The Algebra of Abu Kamil Shoja ben Aslam,* in *Bibliotheca Mathematica* (third series), vol. XII, pp. 52–54.

The problems of the Ahmes papyrus on the distribution of loaves of bread among ten people,[1] and the problem to which we shall recur on the distribution of 100 loaves of bread according to the terms of an arithmetical series, are certainly suggestive of the type of problems of distribution to which Plato had reference. The following problem[2] in the Egyptian papyrus is doubtless one of the type dealing with containers ('phialite'): "I pour (from my container) three times; I add $\frac{1}{3}$ and $\frac{1}{3}$; I fill it up. What part of the measure have I?" In the Greek anthology are found a series of problems on the distribution of apples and nuts, and problems on the weights of bowls, which involve linear equations in one and two unknown quantities.[3]

The intimate connection between Greek logistic and Egyptian arithmetic can hardly be seriously questioned. So far as Greek *arithmetica* is concerned here again we find that the Greeks were inspired by their Oriental predecessors. The available Babylonian and Egyptian documents in the exact sciences are as yet extremely limited; our present information is more or less accidental, and by no means comprehensive. So far as early Egyptian mathematical science is concerned, we are largely dependent upon the Ahmes papyrus. But these few surviving documents give indications of development along many different lines of mathematical thought. Their content is, as we have already partially indicated, quite in harmony with the Greek traditions concerning Egyptian and Babylonian science. In view of this correspondence and of further definite indications of real progress in mathematical thinking among the Egyptians, we are warranted in giving some credence to the Greek traditions concerning Oriental science which are not yet confirmed by indigenous evidence.

Arithmology is closely related to the occult sciences, astrology, alchemy, and magic. While alchemy is undoubtedly a comparatively late development, the Oriental source of its theories is unquestioned.[4] Between the industrial arts of Egypt and Babylon and the development of theories of alchemy, there is an intimate connection, as Berthelot has shown. Furthermore this authority even asserts that Thales may have taken from Babylonian myths his theory that water is the material cause of all things.[5]

[1] Eisenlohr, *op. cit.*, pp. 71–74; Peet, *op. cit.*, pp. 78–79.

[2] Eisenlohr, *op. cit.*, pp. 63–65; Peet, *op. cit.*, pp. 70–72.

[3] See Heath, *A History of Greek Mathematics*, vol. I, pp. 15–16; vol. II, pp. 441–443; Tannery, *Diophanti Alexandrini, Opera omnia*, vol. II (Leipzig, 1895), pp. 43–72.

[4] Berthelot, *Les Origines de l'Alchimie* (Paris, 1885), Chapter III.

[5] *Op. cit.*, p. 251.

So far as the origin of the signs of the zodiac and the star-symbols is concerned, Oriental and Occidental contributions are not separable, and the same is true of other scientific ideas.[1] Astrology was born and bred in the temples of the Babylonians. The desire to forecast the future and equally the desire to establish a connection between the marvels of the beautiful heavens of the East and the events on the mundane sphere resulted in the cultivation of astrology. The devotion to the art constituted the first scientific study of the stars.[2] "The observations which the priests of the ancient Orient gathered with indefatigable patience inspired the first physical and astronomical discoveries, and just as in the period of scholasticism, the occult sciences [astrology and magic] led to the exact sciences. But these, by making evident later the vanity of the marvellous illusions by which they were nourished, destroyed the foundations of astrology and magic to which they owed their birth." [3]

The observations of the Egyptian and, more particularly, of the Babylonian astronomers, furnished a mass of material which was used by the Greeks.[4] Ptolemy and Hipparchus utilized the observations and the computations of the Chaldeans, mentioning specifically certain eclipses observed; [5] Theon of Smyrna discusses the different types of treatment of astronomical problems by Egyptians as compared with the Babylonians; [6] Diodorus Siculus notes that both the Egyptian priests and the Chaldeans were skilled in the prediction of eclipses.[7] As scientific observers of celestial phenomena the Babylonians compare favorably with the greatest of the Greek astronomers. Further than this, the evidence of their ability to use the data intelligently is indisputable. The determination of the period and mean motion of the moon, the determination of the lengths of the seasons and of the year, the determination of the period of eclipses and the periods (*ephemerides*) of the planets, and a host of minor deductions were derived by the scientists of the Orient from their data. The most

[1] Franz Boll, *Sphaera, Neue griechische Texte und Untersuchungen zur Geschichte der Sternbilder* (Leipzig, 1903), p. 461.

[2] Cumont, *Les Religions Orientales dans le Paganisme Romain* (Paris, 1909). See also his *Astrology and Religion among the Greeks and Romans* (New York, 1912).

[3] Cumont, *Les Religions Orientales dans le Paganisme Romain*, p. 235.

[4] Heath, *Aristarchus of Samos, The Ancient Copernicus, A History of Greek Astronomy to Aristarchus* (Oxford, 1913), Chapter III.

[5] Ptolemy, *Composition Mathématique de Claude Ptolémée* (Almagest), ed. Halma (Paris, 1813), vol. I, Book IV, pp. 216, 244–247, 267.

[6] Theon of Smyrna, p. 177, 9 ff., Hiller.

[7] Diodorus Siculus, *Bibliotheca Historica*, I. 50; II. 30.

notable advance in astronomy in Babylon was undoubtedly made during the period in which the science was making real progress in Greece; indebtedness was mutual, but independent scientific progress on both sides is incontestably established.

To deny to Babylon, to Egypt, and to India their part in the development of science and scientific thinking is to defy the testimony of the ancients,[1] supported by the discoveries of modern authorities.[2] The efforts which have been made to ascribe to Greek influence the science of Egypt, of later Babylon, of India, and later of the Arabs, do not add to the glory that was Greece. How could the Babylonians of the golden age of Greece have taken over the developments of Greek astronomy? This would have been possible only if they had arrived at a stage of development in astronomy which would have enabled them properly to estimate and appreciate the work which was to be absorbed. There has never been any question concerning the nature and origin of such feeble beginnings of science as are found among the American Indians. As regards the Babylonians, the Hindus, and the civilization of Europe in the time of Alexander the Great and up to 600 A.D., the problem is entirely different. These are peoples who had reached approximately the same stage of development. The admission that Greek astronomy immediately affected the astronomical theories of Babylon and India carries with it the implication that this science had attained somewhat the same level in these countries as in Greece. Without serious questioning we may assume that a significant part of the science of Babylon and Egypt that was developed during the times which we think of as Greek was indigenous. Nor do we thereby detract from the real greatness of Greece. The Hellenic civilization remains as an integral and vital part of all civilization, and not as something apart.

Turning to the *arithmetica* proper, we may first inquire as to the Egyptian attempts at systematization of the science. The Ahmes manual in itself is evidence of a noteworthy step in this direction, since it establishes the fact that the body of ideas which we now group under the name 'mathematics' was recognized as a separate field by

[1] Hipparchus and Ptolemy, Theon of Smyrna, and Diodorus, as cited above; Herodotus, II. 109; Berosus, fragg. 17 ff. in C. Müller, *Fragmenta Historicorum Graecorum*, vol. II, pp. 509 f.; Clemens Alexandrinus, *Stromata*, II. 4; Pliny, *Naturalis Historia*, VI. 121; VII. 193.

[2] Heath, Berthelot, Boll, Cumont, as cited above; Kugler, *Die babylonische Mondrechnung* (Freiburg, 1900), pp. 50–51; 203–211; Epping, *Astronomisches aus Babylon* (Freiburg, 1889), pp. 183–190.

the Egyptians. While no definitions of number, as such, have as yet been found, Iamblichus informs us that Thales gave the classical definition of it as a collection of units, and the definition of the unit, arithmetically, as one of a group, "following the custom of the Egyptians with whom he studied." [1] Furthermore, the distinction between even and odd, fundamental in the *arithmetica*, is implicit in the Egyptian manual. For example, the first part of the work is devoted to a table for the conversion into unit fractions of fractions with odd denominators from 5 to 99 and with 2 as numerator. This table in and of itself marks real progress in systematization.

The decad, which is prominent in the Pythagorean arithmetic, also receives, in a way, particular attention in the Ahmes papyrus, for 10 appears over and over again in the problems of the Egyptian manual.[2]

Attention to arithmetical and geometrical series was given both in early Babylon and in early Egypt. The single reference which we have, as yet, to the arithmetical and geometrical series in Babylon is found in a moon tablet [3] deciphered by Hincks. This gives the geometric series 5, 10, 20, 40, 80 followed by the arithmetical series, 80, 96, 112, 128, . . . 240.

In the Egyptian manual we have much more than the simple appearance of arithmetical and geometrical series. The discussion of arithmetical and geometrical progressions reveals an unexpected familiarity with rules which we now express by algebraical formulas, a familiarity which has not received adequate appreciation. The essential points of the two formulas which we have for the nth term and the sum of the arithmetical series, $a, a + d, a + 2d, a + 3d, \ldots$, appear from the problems to have been familiar to the Egyptians. Comparatively intricate problems are handled with the ease and intimacy born of long acquaintance.

The problem numbered 40 by Eisenlohr reads: "To distribute 100 loaves of bread among 5 people so that $\frac{1}{7}$ of the (total of the) first three equals that of the last two. What is the difference?" The solution shows that it is understood that the loaves are to be distributed in arithmetical progression.

"Following instructions, the difference $5\frac{1}{2}$," is the next somewhat cryptical suggestion of the manual. I hold that this reference implies

[1] *In Nicomachi Arithmeticam Introductionem Liber*, p. 10, 8 (Pistelli).
[2] Eisenlohr, *op. cit.*, 208, 211, 216, 217, 218, 219, *et passim*.
[3] *The Literary Gazette*, Aug. 5, 1854, with reference to Tablet K 90 of the British Museum.

definite rules of procedure in such problems, leading to the difference $5\frac{1}{2}$, if unity be taken as the first term, under the conditions proposed. Our common procedure, in analytical solution of this problem, leads to the result, $d = 5\frac{1}{2}\,a$ or $d = 5\frac{1}{2}$ if a is 1. Even if the method of arriving at this value for d be that of 'false position,' the procedure which, being adaptable to similar problems, arrives definitely and surely at the complete solution of the proposed problem must be regarded as scientific.

From this point the solution follows the lines of previous problems. With 1 as the first term and $5\frac{1}{2}$ as the difference, the terms are 1, $6\frac{1}{2}$, 12, $17\frac{1}{2}$, and 23, having 60 as a sum. To complete this to the required 100 loaves there must be added 40, or $\frac{2}{3}$ of 60. After it has been noted that this is the case, there is added to each of the numbers in the discovered series $\frac{2}{3}$ of itself, a process that gives $1\frac{2}{3}$, $10\frac{2}{3}\ \frac{1}{6}$, 20, $29\frac{1}{6}$, and $38\frac{1}{3}$ as the series fulfilling the required conditions.

A second problem involving an arithmetical series is entitled "Instructions for the difference in distribution." The solution opens with the phrase, "If you are told," which was later adopted by Arabic mathematicians, and is not uncommon even today. "If you are told, [distribute] 10 measures of grain to 10 people so that the difference in [the amount received by] each person as compared with the next one is $\frac{1}{8}$ of a measure of grain. I take the mean, one measure. I subtract 1 from 10, leaving 9. I take $\frac{1}{2}$ of the difference, $\frac{1}{16}$, and take it nine times. This gives $\frac{1}{2}\ \frac{1}{16}$, which I add to the mean. From this take away $\frac{1}{8}$ measure for each person in order to arrive at the goal. Following instructions: $1\frac{1}{2}\ \frac{1}{16}$, $1\frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, $1\frac{1}{4}\ \frac{1}{16}$, $1\frac{1}{8}\ \frac{1}{16}$, $1\frac{1}{16}$, $\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, $\frac{1}{2}\ \frac{1}{4}\ \frac{1}{16}$, $\frac{1}{2}\ \frac{1}{8}\ \frac{1}{16}$, $\frac{1}{2}\ \frac{1}{16}$, $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, together 10." The solution of this problem as given by the Egyptian manual should be compared step by step with the solution by the ordinary procedure with the formulas of our elementary algebra; the close correspondence is too striking to be regarded as wholly accidental.

No one could ask that the ancient Egyptians should have modern formulas with a literal symbolism, for this advance was not made in Europe until the end of the sixteenth century of the Christian era. The similarity in method is, however, highly significant, revealing a development in analytical thinking that is not equalled for many centuries. In effect, we have in these problems the first term of an arithmetical series regarded as a function of the common difference, under given conditions, and the last term as a function of the mean and the difference. This is true functional thinking whose like is hardly met again until Archimedes.

The single illustration of a geometrical series confirms the implications of the solutions found in the problems involving arithmetical series. The text is extremely concise, and possibly mutilated:

"A ladder

	1	2,801	scribe	7
	2	5,602	cat	49
	4	11,204	mouse	343
together		19,607	sheaf	2,401
			grain	16,807
			together	19,607."

At the right we have the summation of the series 7, 49, 343, 2,401, and 16,807 by actual addition; at the left we have the summation of the same series $7 \times 2{,}801$, with the multiplication effected in the usual manner. Now our formula for the summation of this series gives

$$7\,\frac{7^5 - 1}{7 - 1}, \text{ or } 7 \text{ times } 2{,}801.$$

Some three thousand years after Ahmes an Italian mathematician of prominence, Leonard of Pisa, includes in his arithmetic the same series with one further term. He effects the solution in precisely the two ways selected by his Egyptian predecessor. In India, too, powers of 7 received special attention. The words, or illustrations, which accompany the numbers suggest the nursery rhyme concerning the old woman going to St. Ives.

So far as geometrical formulas are concerned the Egyptians had definite methods for finding the area of triangles, rectangles, trapezoids, and circles, as well as for finding the volume of cylinders.[1] Recently an Egyptian document has been published which gives the expression for the volume of a truncated pyramid.[2] Further there have been found in Egyptian papyri problems concerned with determining the sides of a rectangle when the area and the ratio of the sides are given.[3] All of this material connects directly with the geometry of Greece.

Iamblichus asserts that Pythagoras brought the harmonical progres-

[1] Eisenlohr, *op. cit.*, pp. 75–115.

[2] B. Touraeff, *The Volume of the Truncated Pyramid in Egyptian Mathematics*, *Ancient Egypt*, 1917, pp. 100–102; Karpinski, *An Egyptian Mathematical Papyrus in Moscow*, *Science*, vol. 57, 1923, pp. 528–529.

[3] H. Schack-Schackenburg, *Der Berliner Papyrus 6619*, *Zeitschrift für ägyptische Sprache*, vol. XXXVIII, pp. 135–140, and vol. XL, pp. 65–66.

sion to Greece from Babylon, but no Babylonian record confirming the statement has yet been found.

The squares and cubes of numbers also received particular attention in Babylon, and tables of squares and cubes have been found. In Egypt the relation $3^2 + 4^2 = 5^2$ appears to have been used in the laying out of right angles by means of a stretched rope. Democritus, an able Greek mathematician of the fifth century B.C., boasts: "So far as the laying out of lines is concerned, no one has surpassed me, not even the rope-stretchers of ancient Egypt."[1] Michael Psellus of the eleventh century mentions the Greek equivalents of the Egyptian names of the higher powers, first power to twelfth power; it is supposed that the statement is based upon the lost commentary upon the *Arithmetic* of Diophantus by Hypatia. Again the reference, although not confirmed by available Egyptian material, adds to the probability of mathematical developments in addition to those with which we happen, almost accidentally, to be familiar.

The mystical element in Greek arithmetic is undoubtedly also of Oriental origin. "It must be remembered that at Babylon a *number* was a very different thing from a *figure*. Just as in ancient times and, above all, in Egypt, the *name* had a magic power, and ceremonial words formed an irresistible incantation, so here the number possesses an active force, the number is a symbol, and its properties are sacred attributes."[2] This attitude we see occasionally in Nicomachus, and to a more pronounced degree in later mediaeval times.

The purpose of this introduction is to show the Oriental inspiration and origin of many of the Greek developments in mathematics. The assertion, which has been seriously made by Burnet,[3] that all science is Greek in its origin, is shown to be not at all in accordance with the facts. The well-established tradition[4] of Babylonian and Egyptian influence upon the science of early Greece is confirmed by a mass of self-supporting evidence, naturally not confined to one branch of science, which has been illustrated above with particular reference to arithmetic. Greece retains the right to enjoy the profound admiration of the world of science, but the Orient, also, must be credited with contributions worthy of note.

[1] Clement of Alexandria (ed. Potter), p. 357.
[2] Cumont, *Astrology and Religion among the Greeks and Romans* (New York, 1912), p. 30.
[3] John Burnet, *Greek Philosophy, Part I, Thales to Plato* (London, 1914), pp. 4 ff.
[4] Bretschneider, *Die Geometrie und die Geometer vor Euklides* (Leipzig, 1870), pp. 3-35.

CHAPTER II

THE DEVELOPMENT OF THE GREEK ARITHMETIC BEFORE NICOMACHUS

ONLY a slight acquaintance with Greek mathematics is necessary to convince one that the *Introduction to Arithmetic* of Nicomachus is but a restatement of facts which were common property not only in Nicomachus's own generation but even long before him, and that, except for the few unimportant propositions the discovery of which our author with pardonable pride claims for himself, the book is largely unoriginal.[1] This naturally leads to the inference that the *Introduction* must be closely connected with other mathematical treatises, which served as the fountains whence Nicomachus drew his supply. Because so little remains of this literature, it is difficult to demonstrate the hypothesis in detail; few, however, will question its general truth.

A few words concerning the purpose of the *Introduction*, and the type of books of which it is a representative, will make clearer its necessarily dependent position among mathematical books, and explain why it became famous in spite of little originality. Iamblichus, when he refers to the *Introduction* as the ἀριθμητικὴ τέχνη, or *Art of Arithmetic*,[2] exactly describes it, and properly locates it in literature. The *Introduction* belongs, then, among the *artes* or τέχναι, concise, practical descriptions and systematic expositions of the principles of various arts and sciences, a type of treatise exceedingly common in ancient times,[3] and one which, save in a few well-known exceptions, made scant claim to originality.

[1] Cf. the estimate of Gow, *History of Greek Mathematics* (Cambridge, 1884), p. 94.

[2] P. 4, 12 ff. (Pistelli). See p. 80.

[3] The name was most often applied to texts of rhetoric, to mark the superiority of this over all other arts. E. M. Cope, *Introduction to Aristotle's Rhetoric* (London, 1867), pp. 1, 17, and notes, discusses this and gives examples of the use of τέχνη, πραγμάτεια, μέθοδος, and ἐπιστήμη, all of which were used in about the same sense, as "a system or body of rules and principles" of any art. The τέχνη of Korax was "the earliest theoretical Greek book, not merely on Rhetoric, but in any branch of art" (R. C. Jebb, *Attic Orators*, vol. I, p. cxxi), and Aristotle's lost συναγωγὴ τεχνῶν was a collection of such material. Isocrates refers to rhetorical treatises under this name: λοιποὶ δ' ἡμῖν εἰσιν οἱ πρὸ ἡμῶν γενόμενοι καὶ τὰς καλουμένας τέχνας γράψαι τολμήσαντες (*Oratio* XIII. 19). His own (fragmentary) τέχνη is collected in the Benseler-Blass edition (Leipzig, 1904), vol.

Designed for the use of students, they aimed to present in small compass and with accuracy, clearness and completeness, the elements of a subject, so that it might easily be comprehended and put into practice, in which regard they may best be compared to the modern school and college text-book. The scholars of ancient times, like their modern brethren, did not publish the results of special research in books of this character; and just as our school-books differ from such monumental works as *The Origin of Species*, so we must consider that the *Introduction to Arithmetic* differs from the great original treatises of Diophantus and Heron.

Even without presenting new material, Nicomachus found it possible, therefore, to win fame by writing an 'art of arithmetic.' Because in clearness, conciseness, compendiousness, orderly arrangement and adaptability for scholastic use, it satisfied the demands of seekers after education or general information, it remained the standard work of its class for many centuries. Independence was not, and did not need to be, one of its virtues. To understand it we must survey the historical processes out of which developed the science of which it is an epitome.

Greek interest in the topics dealt with by arithmetic can be traced back to the very dawn of all their science among the Ionians of the sixth century before Christ. Tradition credits Thales with the introduction of geometry into Greece from Egypt, and even designates the theorems which were his discoveries.[1] For the present purpose, however, it is more important to note that the astronomical problems with which Thales is said to have dealt — the length of the year,[2] the prediction of an eclipse,[3] the determination of the apparent size of the sun as $\frac{1}{720}$ of the complete circle of the heavens,[4] and the determination of the equinoxes,[5] — are all fundamentally arithmetical, a fact which far

II, p. 275. The title of one of Protagoras's works was τέχνη ἐριστικῶν (Diogenes Laertius, VIII. 55). The pseudo-Hippocratean *On the Art* was, however, a defense of medicine. 'Introductions,' εἰσαγωγαί, also, were of much the same nature as the τέχνη; examples are, besides those written by Nicomachus, that of Alcinous to the Platonic philosophy, and that of Porphyry to the Aristotelian.

[1] Cf. Burnet, *op. cit.*, p. 20; Eudemus in Proclus, *In Euclid. I*, pp. 157, 10; 250, 20; 299, 1; 352, 14 (Friedlein). Cf. Heath, *A History of Greek Mathematics*, vol. I, pp. 128–137.

[2] Diogenes Laertius, I. 24, 27; T. L. Heath, *Aristarchus of Samos*, p. 21; *A History of Greek Mathematics*, vol. I, pp. 137–139.

[3] Diogenes Laertius, I. 23; Theon of Smyrna, p. 198, 16 (Hiller); Heath, *Aristarchus of Samos*, pp. 13–18; Diels, *Die Fragmente der Vorsokratiker*, vol. I³, p. 7, no. 5.

[4] Diogenes Laertius, I. 24; Heath, *op. cit.*, p. 22, where the point is disputed.

[5] Diogenes Laertius, I. 23; Heath, *op. cit.*, p. 20.

outweighs the statement of the late writer Iamblichus that Thales gave a definition of number based upon Egyptian tradition.[1]

Not Ionia, however, but Magna Graecia, was destined to be the real birthplace of Greek arithmetic; for although the Ionians undoubtedly set in motion the wonderful series of scientific researches that culminated in Plato and Aristotle, the immediate successors of Thales do not seem to have been much concerned with mathematics, and Pythagoras and his school must receive the credit for laying its real foundations. Just what was the measure of their service in the development of arithmetic it is hard to say, because, as is generally admitted, no Pythagorean treatises were published until the time of Philolaus, that is, for nearly a century after the lifetime of Pythagoras himself.[2] Still there is enough to show that many of the subjects treated by Nicomachus were known in the earliest days of the sect.

Pythagoras himself is said to have determined the numerical ratios of the fundamental musical concords,[3] a statement which implies a knowledge of the ratios in general; the discovery of the *tetraktys*, too, was said to be his, and was commemorated by the customary oath of the Pythagorean brethren,[4] in view of which it is clear that figurate numbers, certainly triangles, squares and heteromecic numbers, were known in the earliest days.[5] The classification of numbers into odd and even, and perhaps some of the others which we observe in Nicomachus, may also safely be ascribed to Pythagoras and his group. Although the traditions that credit Pythagoras with a knowledge of the three common proportions,[6] if not their introduction into Greece, are late, it is not improbable, since he was acquainted with ratios, that he knew them.

No inconsiderable portion of Nicomachus's material, then, was al-

[1] See p. 127.

[2] Cf. Diogenes Laertius, VIII. 85. Iamblichus, *Vita Pythagorica*, 199, distinctly says that no one had met with any Pythagorean writings until Philolaus (*floruit ca.* 440 B.C.) published his books. A written text-book might have circulated within the community itself, of course, but there is no evidence that they used books of any sort.

[3] Cf. Nicomachus's account of his discovery, *Manuale Harmonicum*, c. 6, and Theon of Smyrna, p. 56, 9 (Hiller); Burnet, *op. cit.*, pp. 45 ff. In general see Heath's chapter on "Pythagorean Arithmetic," *History*, vol. I, pp. 65 ff.; Leon Brunschvig, *Les Étapes de la Philosophie Mathématique* (Paris, 1922), pp. 33 ff.; Aldo Mieli, *Le Scuole Ionica Pythagorica ed Eleata* (Florence, 1916), pp. 236 ff.

[4] Quoted by Theon of Smyrna, p. 94, 6–7 (see Hiller, *ad loc.*, for other citations).

[5] Cf. Burnet, *op. cit.*, pp. 52–54.

[6] Nicomachus, *Introduction*, II. 22. 1; Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 118, 23 (Pistelli).

ready the property of the first Pythagoreans. It is another question, however, whether they formulated their doctrines and put them in writing, and for the answer we must rely wholly upon our ideas of likelihood, for there is no tradition of the existence, in any form, of an *ars arithmetica* among the fraternity. Undoubtedly it was necessary to instruct their novices in the elements of arithmetic, and so it is highly probable that in time their arithmetic came to assume a fixed and definite form; but in view of the aversion to written records which seems to have characterized the early Pythagorean school, it is also likely that this formulation took at first the aspect of an oral tradition and can hardly have been written down, in any case, before the time of Philolaus. In any event, even though we cannot assume that Nicomachus possessed documents that directly emanated from this group, their pioneer service to the science of arithmetic can hardly be overrated.

The later generations of the early Pythagorean school abandoned, at least partially, the policy of secrecy and began the publication of written records; thus it is easier to determine the extent of their arithmetical knowledge. In general, if we cannot quote the actual words of the members of the sect, we can be confident that the traditions handed down by trustworthy authorities have an authentic documentary basis. The two authors most worthy of attention are Philolaus and Archytas of Tarentum.

From the fragments of Philolaus's book *On Nature* (περὶ φύσεως) some idea may be gained of the extent to which his *ars arithmetica* had been developed. In one [1] he states that there are two 'proper' classes of number, odd and even, and the even-odd (ἀρτιοπέριττος) combining the characteristics of these; another is a passage of considerable length dealing with the numerical ratios of harmony,[2] which shows that his nomenclature of the ratios and probably his doctrine of them were substantially those of later times. Traditions about him allow the scope of his arithmetic, as we are to conceive it, to be greatly increased. Speusippus, we are told,[3] used Philolaus's works as the chief source of his book *On Pythagorean Numbers* (περὶ Πυθαγορικῶν ἀριθμῶν) and treated therein both plane and solid figurate numbers and the five forms assigned to the elements of the universe. A passage quoted from the latter half of the treatise mentions odd and even

[1] Diels, *Die Fragmente der Vorsokratiker*, vol. I[3], p. 310, frag. 5.

[2] Diels, *op. cit.*, p. 310, frag. 6.

[3] *Theologumena Arithmeticae*, p. 61 (Ast).

numbers, primes and composites, multiples and submultiples, ratios, and pyramidal numbers, besides many other matters more akin to arithmology or to geometry.

The tradition of Nicomachus [1] that Philolaus used the expression 'harmonic proportion,' and referred to the cube as 'geometric harmony,' and that of Iamblichus,[2] ascribing to him the use of the so-called 'musical proportion' 6, 8, 9, 12, show that the proportions also were included in his arithmetical knowledge, although only three kinds are reputed to have been known to him.[3] By the time of Philolaus, therefore, we may assume that the *ars arithmetica* was practically complete in all its essentials.

On the other hand, perhaps no written codification of it had as yet been made. We know that Philolaus wrote much, but we hear of no arithmetic among his books; there are also references to the 'Pythagorean tradition' concerning various matters of arithmetic and the allied sciences,[4] but none coupled with the name of an 'art of arithmetic.' Such books may have been compiled, but probably for the most part the Pythagorean doctrines of arithmetic occurred in philosophical or musical contexts, if we may judge by the fragments of Philolaus, for much as the school valued mathematics, it was nevertheless to them a means to the end of philosophizing about the nature of the universe. The mathematics of the Pythagoreans, also, was even at this early day deeply tinged with arithmological speculations, which detract from its value as pure science; and the tendency persisted among their descendants for many centuries, as we may observe in the writings of Nicomachus, Theon, and Iamblichus.

Archytas, who is far more important than Philolaus in the history of arithmetic, lived fully a generation later and was a somewhat older contemporary of Plato.[5] Of course we may assume that he was familiar with all the science of arithmetic as it was known to Philolaus, and it

[1] *Introduction*, II. 26. 2.

[2] *In Nicomachi Arithmeticam Introductionem*, p. 118, 20–22 (Pistelli).

[3] Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 100, 19 (Pistelli), says the others were discovered later.

[4] E.g., Theon of Smyrna, p. 47, 8 ff. (Hiller), admits that the arithmetic he has thus far presented is taken from 'Pythagorean tradition'; see also what he says about the proportions, p. 116, 3 ff. Some of the makers of this tradition may have belonged, of course, to the Alexandrian period, like Myonides and Euphranor (see on *Introduction*, II. 22. 1; 28. 6). Note also that Nicomachus vaguely mentions 'other writers' on arithmetic (see p. 29), some of whom may have been early Pythagoreans.

[5] His *floruit* is given as *ca.* 400–365 B.C. For his life, see Diogenes Laertius, VIII. 79 ff.

is probable that it underwent refinement at his hands. What is more important, he quite certainly wrote an *ars arithmetica*, but we cannot be certain whether it was an independent book, or, as is more likely, a part of a more voluminous work on musical theory. At any rate it is the first example of this type of literature for the existence of which there is more than mere conjectural testimony.

The chief evidence that Archytas compiled an *ars arithmetica* is found in a detailed proof in Euclidean form of the proposition that no number, that is, no rational number, can be a mean between n and $n + 1$; this is quoted by Boethius and ascribed to Archytas; it occurs also in the Euclidean *Sectio Canonis*.[1] The theorem finds a direct application in musical theory in the proof that the 'tone,' the numerical ratio of which is $9:8$, cannot be halved,[2] and so it is very probable that it originally occurred, just as it is now found in Euclid, in a treatise on harmony. But in point of mathematical refinement it is far above the level of Nicomachus, and whatever its subject, the book of which it was a part is of high importance in the history of arithmetic.

Another fragment of Archytas, dealing with the proportions, is likewise notable for its scientific accuracy of expression. This is quoted by Porphyry [3] as follows: "Archytas, in explaining the means, wrote the following: 'There are in music three means; the first is the arithmetical mean, the second is the geometrical, and the third is the subcontrary mean which is called harmonical. The mean is arithmetical when the three terms are in proportion according to the following excess: the quantity by which the first exceeds the second is the same precisely as that by which the second exceeds the third. In this proportion it is found that the ratio of the greater terms is smaller, and the ratio of the smaller terms is greater. There is a geometrical mean when the first term is to the second as the second is to the third; here the ratio of the greater terms is identical with that of the lesser. The subcontrary mean, which we call harmonical, exists when the first term exceeds by a fraction of itself the second, while the second exceeds the third by the same fraction of the third. In this proportion the ratio of the greater terms is greater, and of the lesser is less.'"

We may note that in addition to defining the proportions Archytas

[1] Boethius, *De Institutione Musica*, III. 11; cf. P. Tannery, *Un Traité Grec d'Arithmétique Antérieur à Euclide*, *Bibliotheca Mathematica*, 3 Folge, vol. VI, pp. 225 ff. See also Euclid, *Sectio Canonis*, in Von Jan, *Musici Scriptores Graeci* (Leipzig, 1895), p. 152; Heath, *History*, vol. I, p. 90.

[2] Euclid, *Sectio Canonis*, c. 16, p. 161 (Von Jan).

[3] *In Ptolemaei Harmonica*, p. 267; Diels, *Die Fragmente der Vorsokratiker*, vol. I^3, pp. 334–335.

states what is the peculiar characteristic of each; this is precisely the procedure of Nicomachus in the *Introduction*. Archytas perhaps herein furnished a model for future writers of *artes*.[1]

Either in the same work or in another, Archytas seems to have written upon certain of the fundamental conceptions of arithmetic; for it is reported that he and Philolaus used the terms 'monad' and 'one' indiscriminately, and that he believed the monad to partake of the nature of both odd and even,[2] and we may note also the passage which Nicomachus quotes in the *Introduction*, I. 3. 4. Archytas was besides a pioneer in mechanics and an able geometrician.

The first two fragments cited above, together with several others which need not be mentioned here, seem, to judge from their similarity of subject, to have come from the same work of Archytas, and Diels, who groups them under the title *Discourse on Harmony* (ἁρμονικός, *sc.* λόγος), has probably made the best conjecture as to both name and nature of the work, for an underlying interest in music pervades them all. This book, however, was also cited by the ancients as *On Mathematics* or *On Music*.[3] Blass's conjecture as to the character of the book,[4] that it was a comprehensive one on mathematics, with a general introduction and sections dealing with harmony and the other mathematical sciences, and that the ancients cited it by the subjects of the various parts, is on the whole less likely than the theory which conceives it to have been a treatise dealing primarily with music, but containing a subsidiary discussion of arithmetic as a necessary introduction to the theory of harmony.

The section on arithmetic must have been a systematic *ars arithmetica;* whether or not it was a complete one must remain in doubt. It is at least true, as Tannery declares, that the proposition preserved by Boethius would naturally be one of a series like that in the seventh book of Euclid's *Elements*, and we may be assured that ratio and proportion also were systematically dealt with. Tannery adds that most probably this could not have been original with Archytas, but is evidence for an *ars arithmetica* already existent in his time. However

[1] Cf. Nicomachus, *Introduction*, II. c. 23 ff. Note that similar definitions of the means are given by Plato in *Timaeus*, 31 C – 32 A, and 36 A-B.

[2] Theon of Smyrna, pp. 20, 19; 22, 5 ff. (Hiller).

[3] περὶ μαθηματικῆς, Porphyry, *In Ptolemaei Harmonica*, p. 236, introducing the fragment quoted by Nicomachus, *Introduction*, I. 3. 4, as from the ἁρμονικός. Diels's fragment 3 is cited as from περὶ μαθημάτων by Iamblichus. The fragment on the proportions, cited by Porphyry and quoted above, purports to come from the περὶ τῆς μουσικῆς.

[4] *Mélanges Graux*, pp. 583–584.

that may be, the name of Archytas is the first with which is associated something definite in the history of the development of a formal arithmetic, and it is in consonance with his undoubted eminence as a mathematician to assume that it was by his own original efforts that he accomplished a great share of the contributions already mentioned.[1]

Plato and the Academic philosophers who succeeded him were hardly less concerned with mathematics than Archytas and the Pythagoreans. Although but one proposition, that which specifies that between two plane numbers as extremes one mean can be found, but that there must be two between two solid numbers,[2] is definitely linked with the name of Plato in the later tradition of the *ars arithmetica*, he nevertheless exercised in another way his influence upon its form, for we see that Nicomachus planned his *Introduction* so as to explain the mathematical principles involved in the difficult Platonic passages concerning the world-soul in the *Timaeus* and the marriage-number in the *Republic*.[3]

The esteem in which Plato held mathematical studies is sufficiently seen from the importance he attaches to them in his account of education in both the *Republic* and the *Laws*,[4] and by his constant references to things mathematical. Undoubtedly they were the subject of instruction and discussion in the Academy, and we might well conjecture that for such purposes some formal outline of the subject was prepared, if nothing more than a set of lecture notes, but there is no positive information that a book of the sort existed. Some of the matters which it would contain can be inferred, however, from the mathematical references of the dialogues.

Like Nicomachus, Plato gives a list of allied mathematical sciences, the most famous of which is that in the *Republic*, which contains arithmetic, plane geometry, solid geometry, and astronomy.[5] Music

[1] The opinion of T. L. Heath of the importance of Archytas's contribution is to be seen from the following quotation from *The Thirteen Books of Euclid's Elements*, vol. II, p. 295: "We have then here a clear indication of the existence at least as early as the date of Archytas (about 430–365 B.C.) of an *Elements of Arithmetic* in the form which we call Euclidean." Cf. Heath, *History*, vol. I, pp. 212–216.

[2] Cf. *Introduction*, II. 24. 6 and Plato, *Timaeus*, 32 A.

[3] Cf. *Introduction*, II. 24. 6.

[4] *Republic*, 522 C ff.; *Laws*, 817 E. See Burnet, *op. cit.*, pp. 224 ff.

[5] Plato's arithmetic includes also logistic. Other sciences mentioned are those of weighing and measuring, στατική and μετρική, besides those spoken of by Nicomachus. The passages referring to these sciences are *Republic*, 522 C ff.; *Laws*, 817 E; *Philebus*, 55 E ff.; *Alcibiades*, 126 C; *Euthydemus*, 290 B; *Protagoras*, 356 D ff.; *Gorgias*, 453 E; *Hippias Minor*, 367 ff.; *Politicus*, 284 E; *Theaetetus*, 198 A ff.

is omitted, but astronomy is distinguished, as by Nicomachus, as a science that treats of bodies in motion. Arithmetic is twice defined as the science that deals with the odd and even,[1] and Plato distinguishes the mathematics of the philosopher, which deals with abstract numbers and quantities, from the arithmetic of ordinary life which manipulates concrete units that are not always the same.[2] Among the fundamental ideas of mathematics which Plato discusses is the nature of arithmetical number,[3] of 'one,'[4] and of the odd and even,[5] together with mention of addition,[6] greatness and smallness,[7] multitude,[8] and the counting process.[9] In the *Parmenides*,[10] moreover, it is demonstrated that number must exist, and in the *Theaetetus*[11] there occurs a set of three axioms which underlie all arithmetical computation.

The Platonic dialogues contain also a surprising number of references to the classifications of number and the topics which fall under the head of relative number in the typical *ars arithmetica;* the terminology, too, is the same as that of Nicomachus. Plato usually divides number into odd and even,[12] but a more exhaustive classification, including even-times even, odd-times odd, even-times odd, and odd-times even, occurs in one place. The even series is a *στίχος*, just as in Nicomachus;[13] there are mentioned 'parts' (*μέρη*),[14] aliquot parts (*μόριον*),[15] 'measures' (*μέτρον*),[16] rationals and surds (*ῥητά, ἄρρητα*),[17] powers (*δυνάμεις*)[18] and roots (*δυνάμεναι*),[19] solid and plane numbers (*στερεοί, ἐπίπεδοι*) with their varieties, squares, cubes, and oblongs (*ἴσον ἰσάκις, κύβος, προμηκής*),[20] and their sides (*πλευρά*) and dimensions (*ἀπόστασις*),[21] addition and division (*σχίσις*,

[1] *Theaetetus*, 198 A; *Gorgias*, 453 E.
[2] *Philebus*, 55 E ff.
[3] *Philebus*, 56 D ff.
[4] *Parmenides*, 137 D.
[5] *Phaedo*, 103 E ff.
[6] *Phaedo*, 96 E ff.
[7] *Phaedo*, 100 E.
[8] *Parmenides*, 144 A.
[9] *Theaetetus*, 198 C; to count (ἀριθμεῖν) is "to see how great a number it chances to be" (σκοπεῖσθαι πόσος τις ἀριθμὸς τυγχάνει ὤν).
[10] 143 ff.
[11] 155 A.
[12] *Gorgias*, 453 B; *Hippias Maior*, 303 B; *Politicus*, 262 E; *Theaetetus*, 198 A; *Republic*, 510 C. The fourfold classification occurs in *Parmenides*, 143 E; it should be compared to the Euclidean subdivision (*Elements*, Book VII, definitions) into the same classes, and contrasted with the threefold classification (of the even) of Nicomachus, *Introduction*, I. 8. 3.
[13] *Phaedo*, 104 B; cf. *Introduction*, II. 17. 3–4, etc.
[14] *Theaetetus*, 204 E ff.; *Parmenides*, 151 B.
[15] *Timaeus*, 36 B.
[16] *Parmenides*, 151 B; *Philebus*, 25 A.
[17] *Hippias Maior*, 303 B; *Republic*, 546 C.
[18] *Theaetetus*, 147 D ff. δυναστευόμεναι also occurs, *Republic*, 546 B.
[19] *Republic*, *loc. cit.*
[20] *Republic*, 546 C; *Theaetetus*, 147 E ff.; *Timaeus*, A 32. Note also the terminology πλέων ἐλαττονάκις, ἐλάττων πλεονάκις, *Theaetetus*, *loc. cit.*, which is much like that of Nicomachus in *Introduction*, II. 17. 6. Plato also uses ἀπό as the name for a square; *Meno*, 83 D; Nicomachus, *Introduction*, I. 9. 6, etc.
[21] *Republic*, 546 B.

πρόσθεσις),[1] and the 'pythmen' or smallest representative of any form of ratio (*πυθμήν*).[2]

With regard to relative number, the fundamental idea of equality is often mentioned and is defined as "that which neither exceeds nor is exceeded" (*τὸ μήτε ὑπερέχον μήτε ὑπερεχόμενον*),[3] and the major classes of relations between numbers, the greater and the less, occur no less frequently.[4] From the multitude of examples that may be cited, it is clear that Plato's nomenclature of the individual ratios was the ordinary one.[5] He uses the notions 'greater' and 'less' to illustrate relativity,[6] but at the same time assigns ratios a place in the division of the finite.[7] Finally, if we consider that all mention of them is purely incidental, the treatment of proportion (*ἀναλογία*) is especially complete. Definitions of the arithmetic and harmonic types occur in the form found also in Archytas and Nicomachus,[8] and the geometric is seen in the simile of the divided line [9] and elsewhere.

We have already noted that the theorem concerning the number of means necessary between plane and solid numbers as extremes may well be a contribution of Plato himself; and furthermore the use of the so-called 'musical' proportion by Plato in *Timaeus* 36 A, though it had doubtless been employed by the Pythagoreans before him, led the authors of *artes*, like Nicomachus in *Introduction*, II. 29, and of commentaries on the *Timaeus* to devote much space to its discussion.

From this wealth of arithmetical material in the works of Plato later authors might surely have borrowed. Whether or not they did so, its presence in Plato is another link in the chain of evidence proving the gradual development of a standard form of statement for arithmetical matters, and shows perhaps as well that, although the arithmetic of Plato is substantially that of Archytas, further refinement of definition and classification was constantly going on, in which the Platonic school bore a share.

To prove this more definitely, we could wish to have more extensive documents from the Academy than the scanty remains we actually possess; for although little but the titles of books is left, yet these

[1] *Phaedo*, 97 A.

[2] *Republic*, 546 C; cf. on *Introduction*, I. 19. 6.

[3] *Phaedo*, 97 A.

[4] *Republic*, 438 B; *Charmides*, 168 B; *Parmenides*, 140 C, 150 D, 151 B.

[5] *Charmides*, 168 B; *Phaedo*, 105 A; *Meno*, 83 B, 84 E; *Timaeus*, 36 A; *Theaetetus*, 154 C; *Republic*, 546 C.

[6] *Republic*, 438 B.

[7] *Philebus*, 25 A.

[8] *Timaeus*, 36 A.

[9] *Republic*, 509 D ff.; *Timaeus*, 31 C – 32 A (a proportion of equality).

indicate that the interest in mathematics continued high after the master's death. Speusippus's book *On Pythagorean Numbers* has already been mentioned; and we may note that Xenocrates's works included *On Mathematics*, *On Geometry*, *On Numbers*, and *Theory of Numbers*.[1] Philip of Opus, the reputed author of the Platonic *Epinomis*, treated the subject of polygonal numbers in a book which is now lost, but which may well have influenced subsequent discussions of the polygonals; he is said to have written upon arithmetic as well, but this work also is not now extant.[2]

The celebrated astronomer and mathematician Eudoxus was also intimately connected with the Platonic group and is known to have made important contributions to the theory of proportion. The treatment of this subject found in Euclid, in fact, is now regarded as due to him, and as Heath remarks, it is "equally applicable to geometry, arithmetic, music, and all mathematical science." [3] Since the works of Eudoxus have unfortunately all perished, we cannot be certain that his theory of proportion occurred in a book on arithmetic, but the reference made by Theon, as noted above, to the school of Eudoxus, and Iamblichus's statement that he defined number as 'limited multitude' [4] make it somewhat probable. His success as an astronomer has tended to obscure whatever he may have accomplished in other fields of mathematics. We may also recall that the philosopher Democritus, an older contemporary of Plato, was much interested in mathematics, and that a book entitled *Numbers* (ἀριθμοί) is among those which he is said to have written.[5]

Aristotle's share in the making of the τέχνη was, to judge from his influence traceable in Nicomachus, no small one; it was concerned chiefly with the definition of the fundamental concepts of mathematics. His antiquarian interests also led him to write an essay upon the Pythagoreans, now lost,[6] and to bring into his extant works frequent discussions of them and their theories.[7] He is still our best informant

[1] See p. 89, n. 1.

[2] Cf. Βιόγραφοι, *Vitarum Scriptores Graeci Minores* (ed. Westermann, 1845), p. 446.

[3] *Op. cit.*, vol. II, p. 112. The date of Eudoxus was about 408–355 B.C. See Diogenes Laertius, VIII. 86, on his life. See also Heath, *History*, vol. I, pp. 321–334.

[4] See p. 127.

[5] See Burnet, *op. cit.*, pp. 193 ff., and Diels, *Die Fragmente der Vorsokratiker*, vol. I[3], p. 390, for a collection of the titles of the mathematical works of Democritus.

[6] Πυθαγορικός or περὶ τῶν Πυθαγορείων; see Zeller, *Die Philosophie der Griechen* (ed. 5), vol. I, part 1, p. 280, note 1.

[7] The first book of the *Metaphysics* gives an extended account of them. References abound, however, throughout the *Metaphysics*, *Physics* and other works.

about Pythagoreanism, and probably he, like the doxographers and biographers of the philosophers,[1] preserved much that was of interest to Nicomachus. Aristotle's pupil, Theophrastus, wrote books dealing with geometry, music, and astronomy, besides one book each of *Arithmetical Inquiries* and *On Numbers*,[2] but we are wholly ignorant of the character of his work.

So great was the respect of the Greeks for the genius of Plato that his dialogues very soon became a subject of study in their higher education, and scholars began to write commentaries upon them. The Platonic commentaries undoubtedly must be taken into account in any study of the evolution of such books as the *Introduction*, for in order to elucidate the mathematical passages of Plato it was often necessary to set forth in detail the principles upon which the explanation rested, and so, although they may not have embraced a complete *ars arithmetica*, parts of them dealing with special subjects — for example, with ratios — were really incomplete *artes* and were of great use to compilators like Nicomachus. This is particularly true of commentaries on the *Timaeus*, the most mathematical of all the Platonic dialogues.

We shall see that one of these commentaries, the *Platonicus* of Eratosthenes,[3] was very probably a source, direct or indirect, of Theon of Smyrna, if not of Nicomachus himself, and that another, by Adrastus,[4] was a primary source of Theon. To what extent still others may have exercised influence it would be rash to try to say, for few of them survive. We can form an idea of the lost commentaries only through the extant ones, those of Plutarch,[5] Chalcidius, and Proclus, and by means of the citations found in these and other authors. It is known, however, that Crantor, Xenocrates, Eudorus, Clearchus, Theodorus, Panaetius and Posidonius commented on the *Timaeus*, besides Plutarch, Eratosthenes, Adrastus, Chalcidius and Proclus;[6] and we must grant the possibility that any one of them may have contributed something to the *ars arithmetica*.

[1] For the doxographers, cf. Diels's collection of texts, *Doxographi Graeci*. Arius Didymus was a writer of this type, and Sotion compiled *Lives of the Philosophers*.

[2] Cf. Diogenes Laertius, V. 2. 50.

[3] Eratosthenes, *c.* 275–194 B.C., was librarian at Alexandria. On the book, cf. E. Hiller, *Philologus*, vol. XXX, pp. 60 ff., who shows that it was a commentary on the *Timaeus*.

[4] On this commentary, cf. E. Hiller, *Rheinisches Museum*, vol. XXVI, pp. 582 ff.; Zeller, *op. cit.* (ed. 4), vol. III, part I, p. 806, note; Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. I (ed. 3), p. 433.

[5] *De Animae Procreatione in Timaeo.*

[6] Th. Martin, in his edition of the *Timaeus*, has collected the names of the commentaries and the information about them that we have.

Before we leave the commentaries, it must be remarked that two of the surviving *artes*, those of Theon and of Nicomachus, have a very close connection with this class in that both authors relate their discussion of mathematics to the study of Plato, Theon openly, through the title and the introduction of his book, Nicomachus by frequently reminding the student that certain of his chapters will be of use toward understanding the mathematics of Plato.[1] That is, the two treatises both belong to the class of the *artes* and are at the same time related to the Platonic commentaries and handbooks. It is essential that this relationship be constantly borne in mind in order to form a correct idea of the literary ancestry of the *Introduction*. Not only is it a handbook of arithmetic, but it presupposes, like Theon's work, that its user is to hear lectures on the *Timaeus* and the *Republic*, and is designed to assist him. Theon's introductions to arithmetic, geometry, astronomy and music all have this as their avowed purpose. Remembering that Nicomachus wrote other 'introductions' also,[2] we may perhaps say that he did in four treatises what Theon did in one.

Euclid [3] marks an undoubted epoch in the history of the *ars arithmetica*. The *Elements* is the first extant written work which completely covers the ground of elementary arithmetic, definitions and propositions alike; and needless to say the *Elements* was likewise in many respects a model for later compilers to follow. We cannot here pause to show how fully the seventh, eighth and ninth books of the *Elements* fulfill the requirements of the *ars arithmetica*, and indeed it may be assumed that the work is too well known to require such a demonstration. The methods of Euclid and Nicomachus, it may be remarked, are very different, in that Euclid always, Nicomachus never, offers proofs for his propositions; and in this respect the *Elements* probably differed from most of the *artes*, for as a class they seem to have been descriptive rather than based on demonstration.

Another mathematician who probably contributed to this tradition

[1] Cf. *Introduction*, II. 2. 3; 21. 1; 24. 6; 28. 1. In these passages it is stated that the material given will be useful in the study of Platonic theorems, or the passages read and dealt with (ἀναγνώσματα) in the schools; a similar purpose will be suspected also when Nicomachus halts to point out the bearing of arithmetical propositions upon metaphysical theory; e.g., *Introduction*, I. 23. 4; II. 17. 2; 18. 4.

[2] See p. 81.

[3] The dates of Euclid's life are not exactly known. He was later than Plato's disciples, earlier than Eratosthenes and Archimedes, according to Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, p. 68 (Friedlein), and is usually placed in the reigns of the first Ptolemies. See Heath, *History*, vol. I, pp. 354–357.

was Hypsicles.[1] He, like Philip of Opus, is known to have worked upon the subject of polygonal numbers, and one of his propositions bears close resemblance to a Nicomachean passage.[2] The name of the book in which this occurred, however, is unknown, and likewise we have no information upon a treatise on arithmetic which he is reported to have written.

Euclid, Eudoxus, Hypsicles and Eratosthenes, who has been mentioned in another connection, are practically the only mathematicians whom we can specify as having very probably been concerned in the development of the *ars arithmetica* from the fourth century before Christ to the time of Nicomachus. Yet in this age the Greek science of mathematics was in its most flourishing state, centering especially about Alexandria, and it is hardly to be imagined that the *Elements* of Euclid was the only book of the type of the *artes* which was written in this period. Nesselmann has already noted this serious lacuna in our knowledge of arithmetical history.[3]

That the production of books upon elementary arithmetic, however, really went on during this time is attested by many proofs, to be drawn partly from Nicomachus and Theon of Smyrna, partly from others, especially Philo, which by their cumulative evidence lead one to believe that most of what Nicomachus has written was commonly found in the books of his time, and furthermore that this elementary mathematical knowledge existed in a form fairly well fixed and generally accepted. It was not, apparently, accompanied by Euclidean proofs; at least the references in Nicomachus and Philo make no mention of them, and, if anything, suggest the opposite. To sum up the whole matter, it is extremely probable that the period between Euclid and Nicomachus witnessed the final development of the *ars arithmetica*, by the work of many hands, into a form greatly resembling the *Introduction to Arithmetic* itself.

Much of the evidence for this conjecture comes, as has been remarked, from Nicomachus, who, without giving names, several times refers to arithmeticians engaged, as it would seem, in work much like his own. He says, for example, in *Introduction* II. 14. 5, that he must introduce the subject of truncated and bitruncated pyramids and the

[1] Hypsicles lived about 180 B.C. His work included arithmetical progressions as well as polygonal numbers. See Gow, *op. cit.*, p. 87, and Cantor, *op. cit.*, vol. I, pp. 358 ff., especially p. 361; Heath, *History*, vol. II, pp. 213–218.

[2] *Introduction*, II. 11. 4 (cf. the notes).

[3] *Geschichte der Algebra*, vol. I, p. 219.

like, which may be met with 'in the theoretical treatises' (ἐν συγγράμμασι μάλιστα τοῖς θεωρηματικοῖς); and again, touching upon scalene numbers, he records that 'certain ones' or 'others' used various names for them, 'wedges,' 'wasps,' 'altars.'[1]

Somewhat more definite information may be derived from what is stated in II. 22. 1 and 28. 6 about the history of the varieties of proportion. In the former passage he says that, whereas the three chief forms and their subcontraries were known to Pythagoras, Plato and Aristotle, the 'moderns' (οἱ νεώτεροι) devised four more to make up the sacred number, 10; and, furthermore, that in the latter the three subcontraries came into use among the 'writers of commentaries and members of schools' (ὑπομνηματογράφοι, αἱρετισταί) after Plato and Aristotle, while 'certain ones' discovered the remaining four. It could easily be inferred even on this basis alone that the immediate successors of Plato and Aristotle, the Academics and Peripatetics, and the Platonic and Aristotelian commentators, were the ones who dealt especially with the three subcontraries, and because the sacred number 10 was taken into account that the 'moderns' mentioned were Pythagoreans.

Iamblichus[2] fortunately confirms this suspicion by informing us that Eudoxus and his followers invented the three subcontraries, which with the three original forms were in use up to the time of Eratosthenes, and that thereafter the Pythagoreans Myonides and Euphranor introduced the rest. Nothing further is known of these two men other than that they flourished between the times of Eratosthenes and Nicomachus. It cannot be assumed that they were purely scientific writers — the presumption is rather that they were not — nor that they dealt with the whole subject of mathematics. The circumstance, however, proves for Nicomachus a disposition to take account of existing work and casts light, though feebly, on the problem of his sources.

Certain utterances of Theon of Smyrna are similar in purport to those of Nicomachus just mentioned. We have already noted that Theon speaks vaguely of a 'Pythagorean tradition' to which he was deeply indebted.[3] In other passages he says that 'some' regarded the

[1] II. 16. 2. Similarly with reference to the arithmetical proportion he writes that one of its peculiar properties has escaped 'the majority,' whereas another is recorded by 'all previous authors' (II. 23. 6). Cf. also II. 13. 1.

[2] See on II. 22. 1; 28. 6. Note also that Moderatus probably dealt with all the varieties of proportion.

[3] See p. 20, n. 4.

monad as first of the series of odd numbers, and with regard to the elements of number he mentions the opinions of 'later' writers as opposed to 'the followers of Pythagoras,' and those of Archytas and Philolaus in contrast to 'the majority.'[1]

The testimony of Philo Judaeus is even more important, for it shows that, although he cannot have seen the works of Nicomachus or of Theon, he nevertheless knew a surprising number of the topics which they employ, and we can hardly account for this fact save on the ground that he was acquainted with an *ars arithmetica* which presented these matters in practically the same form as they did. Nothing will more clearly demonstrate this than a brief recapitulation of the arithmetical material in his works. The examination of the Philonic corpus shows that he distinguished between the monad and one;[2] that he had an idea comparable to that of Nicomachus concerning 'sameness' and 'otherness' in number, embodied in the monad and dyad or in 'odd' and 'even';[3] and that he distinguished even, odd, and even-times odd numbers,[4] primes[5] and perfect numbers.[6]

Of the subjects treated by Nicomachus under the general head 'relative number' Philo mentions equality, inequality, excess and deficiency,[7] and ratios, including multiples, superparticulars and superpartients, with their specific forms.[8] Concerning figurate numbers, he touches upon the distinction between the dimensional and the indimensional, upon the four 'bounds' of things — point, line, surface and solid, each of which he defines[9] — and the numbers corre-

[1] Pp. 20, 5 ff.; 20, 19 ff.; 21, 24 (Hiller).

[2] *Quaestiones et Solutiones in Genesim*, IV. 110; cf. Lydus, II. 5. This is not in Nicomachus, but is treated at length by Theon.

[3] Suggestions of this occur in *Quaestiones et Solutiones in Genesim*, II. 12; IV. 110.

[4] *Ibid.*, IV. 199; *De Mundi Opificio*, 3; *De Septenario*, 6; *De Decalogo*, 6; *Quaestiones et Solutiones in Genesim*, III. 38. The threefold classification is found in older Pythagorean works and in *Theologumena Arithmeticae*, p. 3 (Ast).

[5] *De Decalogo*, 7.

[6] *De Mundi Opificio*, 34: τέλειον καὶ τοῖς αὑτοῦ μέρεσιν ἰσούμενον (cf. Nicomachus's definition); *ibid.*, 3; *Legis Allegoriae*, I. 2; *De Decalogo*, 7; *Quaestiones et Solutiones in Genesim*, III. 38; *Quaestiones et Solutiones in Exodum*, II. 87; *De Vita Mosis*, III. 5. Philo mentions only 6 and 28 as perfect numbers.

[7] *De Justitia*, 14; *Quis Rerum Divinarum Heres Sit*, 28.

[8] Various specific ratios are named, as *De Mundi Opificio*, 15, 30, 37; *Quaestiones et Solutiones in Genesim*, II. 5; III. 49, 56; IV. 71; *Quod Deterius Potiori Insidiari Soleat*, 19. Musical concords, *De Mundi Opificio*, 15, 31; *De Vita Mosis*, III. 11; *Quaestiones et Solutiones in Genesim*, IV, 27.

[9] *De Decalogo*, 7; *De Mundi Opificio*, 10, 16, 34; *De Congressu Quaerendae Eruditionis Gratia*, 26; *De Somniis*, I. 5; *Quaestiones et Solutiones in Exodum*, II. 93.

sponding to them,[1] triangular, square, pentagonal, hexagonal, heptagonal numbers,[2] and heteromecic.

He knew that the addition of successive odd numbers produces the squares, and that of even numbers the heteromecic numbers,[3] and he states more fully than Nicomachus himself the theorem concerning the occurrence of squares, cubes and cube-squares in analogous series.[4] Among solid numbers he mentions pyramids[5] and cubes[6] and in the matter of proportions he is informed as to the three chief forms[7] and their union in the series 6, 8, 9, 12.[8] This brief synopsis shows that far more of the arithmetic of Nicomachus was in the hands of Philo than either Plato, Archytas or Philolaus possessed. This could hardly have been so unless we imagine these doctrines to have been written over and over again in books which were in ordinary circulation. By Philo's time not only the topics, but even the Nicomachean form of statement, must have been commonly known.

We may rest assured that activity in the production of *artes arithmeticae* did not cease during the two or three centuries immediately preceding the lifetime of Nicomachus, although we are not directly informed about it. The part played by the Neo-Pythagorean sect, of which Nicomachus himself was a member, remains to be discussed. So far as the later Pythagoreans occupied themselves with mystical and theological speculations in the realm of arithmology, they can be conceded very little credit for the development of the arithmetical τέχνη upon its scientific side; yet it will be seen that even the arithmological doctrines made their way, to some extent, into the *artes*. On the other hand, we have already found reason to believe that many of the unknown writers of *artes* were Pythagorean.

From the titles of their books and such fragments as remain, it is clear that the interests of the Neo-Pythagoreans were directed more toward philosophy and mysticism than to mathematics proper. Evidently their books were more closely related to the *Theologumena*

[1] *De Mundi Opificio*, 16, 32; *De Decalogo*, 7; *Quaestiones et Solutiones in Exodum*, II. 93, 94.

[2] All mentioned in *Quaestiones et Solutiones in Genesim*, I. 83; cf. *ibid.*, I. 91; II. 5; III. 56.

[3] *Quaestiones et Solutiones in Genesim*, II. 5, 12, 14.

[4] *De Mundi Opificio*, 30, 36.

[5] *Ibid.*, 16.

[6] *De Decalogo*, 7; *Quaestiones et Solutiones in Genesim*, II. 5; III. 49, 56.

[7] Philo defines them: *De Decalogo* 6; *De Mundi Opificio*, 37.

[8] Cf. *Introduction*, II. 29. Philo calls this the πλίνθιον (*laterculus*): *De Mundi Opificio*, 37; *Quaestiones et Solutiones in Genesim*, I. 91; III. 38; IV. 27. He also makes use of the similar series 6, 9, 12, 18 (mentioned in *Theologumena Arithmeticae*, but not in the *Introduction*).

Arithmeticae than to the *Introduction*, although even the latter contains references to some of them which prove that in the midst of extraneous matter they may have contained topics relating to arithmetic, or to the traditions of the early Pythagoreans which would be of value to writers of *artes*. The following are of some interest, either from their titles or because they are cited by Nicomachus:[1]

Androcydes, *On the Pythagorean Symbols* (Περὶ Πυθαγορικῶν συμβόλων),
Aristaeon, *On Harmony* (Περὶ ἁρμονίας),
Butherus, *On Numbers* (Περὶ ἀριθμῶν),
Eubulides, author of a work (title unknown) dealing with Pythagoras,
Hippasus,[2]
Kleinias,
Megillus, *On Numbers* (Περὶ ἀριθμῶν),
Prorus, *On the Hebdomad* (Περὶ τῆς ἑβδομάδος).

Aside from the fact that Nicomachus himself was a Neo-Pythagorean and that the works of Philo seem to presuppose an *ars arithmetica* of decidedly Neo-Pythagorean cast, the little that is known of the career of Moderatus of Gades,[3] one of the most eminent of the school, confirms the conjecture that this group was actively interested in the *ars arithmetica*. The only known work of Moderatus was called *Pythagorean Lectures*,[4] and its fragments show it to have been less extravagant than most of the writings of the school, and to have approached the study of certain fundamental questions of the theory of numbers in a decidedly scientific spirit. Moderatus's treatment

[1] Androcydes: *Introduction*, I. 3. 3; *Theologumena Arithmeticae*, p. 40 (Ast); Aristaeon is usually identified with the 'Aristaeus' of *Theologumena Arithmeticae*, p. 41; Butherus: see Stobaeus, *Eclogae* I, *Prooemium*, 5; Eubulides: *Theologumena Arithmeticae*, p. 40; Hippasus is frequently cited, but the writings attributed to him are generally thought spurious; Kleinias, Megillus, and Prorus: *Theologumena Arithmeticae*, pp. 17, 27, and 43, respectively. Zeller, *op. cit.* (ed. 4), vol. III, part 2, pp. 115 ff., and Chaignet, *Pythagore et la Philosophie Pythagoricienne* (Paris, 1873), vol. I, pp. 165 ff., may be consulted for longer lists of such works and for theories of their dates and compositions.

[2] The real Hippasus belonged, of course, to the early school. The writings attributed to him, however, should be dealt with as products of the Neo-Pythagoreans.

[3] Usually assigned to the time of Nero on the ground that he was a teacher of Lucius, a contemporary of Plutarch. Cf. also Porphyry, *Vita Plotini*, 20; Stephanus Byzantinus, s.v. Γάδειρα.

[4] Πυθαγορικαὶ σχολαί; cf. Porphyry, *Vita Pythagorica*, 48; Simplicius, *Physica*, p. 50, B; Stobaeus, *Eclogae* I, *Prooemium*, 8–9 (vol. I, p. 21, ed. Wachsmuth-Hense).

of the monad and one influenced a chapter of Theon of Smyrna,[1] and, like Nicomachus, he probably discussed ten forms of proportions.[2] Although little more is known of the scope and nature of the *Pythagorean Lectures*, it is at any rate clear that Moderatus occupied himself with the themes of the *ars arithmetica* and may well have produced an *ars* of his own.

The survey of the history of the *ars arithmetica* which we have just made is in itself enough to show how hopeless is the attempt to assign to each topic of the *Introduction to Arithmetic* its precise source. The *Introduction* brings together the results of a gradual growth; there is hardly one of its subjects but had been discussed and written about dozens, if not hundreds, of times. One may naturally assume, of course, that the more eminent mathematicians, who had made especially famous contributions to the art, were the real sources of Nicomachus when he deals with the subjects of their special study, and such probabilities can be specified with reasonable accuracy; his general sources, however, can only be regarded as the indeterminate mass of previous arithmetical writing.

It cannot even be positively stated that Nicomachus used Euclid's *Elements*. There can be no question of course that he knew it. But the two works are of entirely different character, Euclid defining and demonstrating, Nicomachus defining and laying down general principles with abundant illustration and explanation. If in any respect Euclid could have served as a basis for Nicomachus, we should at once think of his definitions, but even here it will be observed that there are many divergences between the two. In Euclid's *Elements*, however, there were at least a pattern of arrangement and an example showing what subjects needed exposition. It is perhaps in this general way, if at all, that we are to look for a relation between the two. It may be confidently stated, nevertheless, that Euclid did not serve as the only model, nor even as the principal model, for Nicomachus.

With regard to his actual sources Nicomachus himself offers small help; the only authors he mentions by name are Androcydes, Archytas,

[1] Cf. Moderatus in Stobaeus, *loc. cit.*, and Theon of Smyrna, pp. 18, 3–9 and 19, 7–8 and 12–13 (Hiller).

[2] Cf. Proclus, *In Timaeum* (Diehl), vol. II, p. 18, 29 ff., who says, *inter alia*, ἵνα γὰρ παρῶμεν μὲν τὰς ἄλλας μεσότητας ἃς οἱ νεώτεροι προστιθέασι, τοὺς Νικομάχους λέγω, τοὺς Μοδεράτους καὶ εἴ τινες ἄλλοι τοιοῦτοι, περὶ δὲ τῶν τριῶν τὰ νῦν μεσοτήτων εἴπωμεν ἀφ' ὧν καὶ ὁ Πλάτων ὑφίστησι τὴν ψυχήν, κτλ. This leads to the inferences, first, that Moderatus was actively interested in arithmetical subjects in much the same way as Nicomachus, and second, since their names are so coupled, that he discussed, like Nicomachus, ten forms of proportion.

Aristotle, Eratosthenes, Plato and Philolaus, besides Pythagoras, to whom he refers in a general way.

Three of these, Androcydes, Archytas, and Philolaus, are Pythagoreans, and Plato is treated as practically a member of the school. The two former are quoted in I. 3. 4–4 to substantiate the well-known Pythagorean principle of the necessity of mathematics, and immediately thereafter Plato is cited to the same effect. But the Pythagorean mathematicians have left their imprint upon the *Introduction* in far more essential matters; to say nothing of the cosmological topics found in the introductory chapters, to them must be ascribed the fundamental conception of number as a balanced and harmonized fabric of 'odd' and 'even,' 'same' and 'other,' which makes itself especially felt in certain chapters of Book II, and in the treatment of the dyad, as well as the monad, as an element of number.[1]

Less important matters are the remarks about the nomenclature of the harmonic proportion (II. 26. 2) attributed to Philolaus, the 'Pythagorean' definition of 'odd' and 'even' (I. 7. 3), and the references to the 'sacred' decad (II. 22. 1). The fact is that the whole *Introduction* has a distinctly Pythagorean coloring through the underlying assumption that the numbers influence their derivatives. Just what were the books most powerful in influencing Nicomachus cannot be said; certainly Philolaus's work, Περὶ φύσεως, was one, but the long list of Neo-Pythagorean documents must not be forgotten.

That Aristotle was a source for Nicomachus cannot be doubted, but it was not so much mathematical material as logical method that was derived from the Stagirite. To be sure, Aristotle seems to have been consulted upon the definition and classification of the ultimate subjects of mathematics, number and quantity,[2] but he is also present on nearly every page in the distinctions of genus and species and in the formation of arguments.[3]

Eratosthenes, of course, makes his appearance in connection with his famous 'sieve' (I. 13), but there are two other places where with the aid of Theon we can detect a suggestive similarity. This is seen most clearly in the theorem of the 'three rules' for deriving the various ratios from equality;[4] and there was perhaps another kindred matter in which Eratosthenes led the way for Nicomachus. Theon reports

[1] See pp. 115 ff. [2] See pp. 112 ff.

[3] See on *Introduction*, I. 2. 3–4; 3. 1; 4. 2, 3; 14. 2; 23. 5; II. 20. 2.

[4] See on *Introduction*, I. 23. 4

much of what Eratosthenes had to say in demonstrating that equality is the element of ratio.[1] Now one of Nicomachus's own favorite themes is that equality is the element of relative number.[2] There is doubtless no way of demonstrating a direct relation in these matters, and in fact it is not certain that even for the 'sieve' proposition Nicomachus went to Eratosthenes himself.

Plato is quoted many times by Nicomachus, and was most influential in the establishment of his philosophic doctrines, as we have had occasion to observe.[3] On the mathematical side, the most important proposition probably taken from this source is the one which states that there must be two geometrical means between cubes and one between squares; Plato had also been one of the first to make use of the proportion dealt with in II. 29.

There is little further that can be definitely said about the source of Nicomachus, but no one can doubt that our author drew from many others who must for us remain nameless, the authors of the *artes arithmeticae* for the existence of which in his day we have seen that there is sufficient evidence. Nicomachus can hardly fail to leave the impression with the reader that he is working over familiar ground, and the assumption that there was an extensive literature, now lost, of the type described in the preceding pages, is the only satisfactory explanation of this fact. His task was the collection and organization of material, not its creation; and his fame in antiquity must have been based on the skill with which he performed this work and on the usefulness of the book which he produced.

Before passing, however, from this topic to the influence of Nicomachus upon later writers, we must examine his relation to Theon of Smyrna. Concerning the personality and history of this author far less is known than even the meager facts of the life of Nicomachus, but his date may be confidently placed in the first quarter of the second century of our era.[4] Theon was the author of a book dealing with the mathematical matters needful for the reading of Plato,[5] in the course of which he draws freely upon Thrasyllus, Eratosthenes, and especially the Peripatetic Adrastus, and in addition seems to have

[1] Theon, p. 82, 22 ff. (Hiller).
[2] See *Introduction*, II. 1, and on I. 23. 4.
[3] See pp. 92, etc.
[4] See p. 72. Cf. Heath, *History*, vol. II, pp. 238–244.
[5] *Expositio Rerum Mathematicarum ad Legendum Platonem Utilium*, ed. E. Hiller, Leipzig, 1878 (whose paging is cited). See also J. Dupuis, *Théon de Smyrne*, Paris, 1892.

made some reference to Moderatus of Gades.[1] The book is interesting as an illustration of the methods then in vogue of helping Platonic students in their tasks, and inasmuch as there is evidence that Nicomachus took account of such writings, a comparison of the two authors, even though not wholly conclusive, cannot but be enlightening.

In scope and in the selection of the topics treated, Theon and Nicomachus are in close agreement, as the following parallel arrangement shows:

THEON (ed. Hiller)	NICOMACHUS (ed. Hoche)
On the necessity of mathematics, p. 1, 1 ff. Note the occurrence of the same Platonic passages, e.g.,	
p. 2, 16–19: cf. *Epinomis*, 992 B;	I. 3. 5.
p. 3, 8–15: cf. *Republic*, 527 D;	I. 3. 7.
p. 5, 11–13: cf. *Republic*, 526 D	I. 3. 7 (p. 8, 13–14).
and p. 12, 10 ff.: Philolaus.	II. 19. 1.
On the natural order of mathematical studies, p. 16, 24 ff.[2]	I. 5. 1–2.
Cf. especially p. 18, 1.	I. 4. 1 (p. 9, 7).
On one and the monad, p. 18, 3 ff.	
Definitions of number,[3] p. 18, 3 f.	I. 7. 1.
The monad, p. 18, 5 ff.	Not in Nicomachus.
Cf., however, this passage: *καλεῖται δὲ μονὰς ἤτοι ἀπὸ τοῦ μένειν ἄτρεπτος καὶ μὴ ἐξίστασθαι τῆς ἑαυτῆς φύσεως· ὁσάκις γὰρ ἂν ἐφ' ἑαυτὴν πολλαπλασιάσωμεν τὴν μονάδα, μένει μονάς* (p. 19, 7 ff.).	Cf. . . . *καὶ μονὰς ἐκ παντὸς μόνη τοῦ ἀριθμοῦ ἑαυτὴν πολλαπλασιάσασα οὐδὲν πλέον ἑαυτῆς γεννᾷ*, II. 6. 3 (p. 84, 22 ff.), and cf. II. 17. 4 (p. 110, 8 ff.).
Further discussion of one and the monad[4] and of the Pythagorean doctrine of the elements of number, pp. 19, 21 ff.	Not in Nicomachus.
Cf. *ἁπλῶς δὲ ἀρχὰς ἀριθμῶν οἱ μὲν ὕστερόν φασι τήν τε μονάδα καὶ τὴν δυάδα*, p. 20, 5 ff.	Cf. *τοῦ γὰρ ἁπλῶς καὶ καθ' αὑτὸ ποσοῦ μονὰς ἦν καὶ δυὰς τὰ ἀρχικώτατα στοιχεῖα, κτλ.*, II. 1. 1 (p. 74, 5 f.).
'Odd' and 'even,' p. 21, 20–24.	I. 7. 1–2.

[1] Cf. p. 33; also G. Borghorst, *De Anatolii Fontibus* (Berlin, 1905), pp. 12 ff.; G. Altmann, *De Posidonio Timaei Platonis Commentatore* (Berlin, 1906), pp. 19 ff.; and Schmekel, *Die Philosophie der mittleren Stoa* (Berlin, 1892), p. 409, n. 3, especially upon the source of the passage in Theon, pp. 93, 16–106, 11 (Hiller).

[2] For a detailed comparison with Nicomachus, see p. 113, n. 4.

[3] Here (p. 18, 3–9; p. 19, 7–8; 12–13) occur parallels (almost word for word) with Moderatus in Stobaeus (see p. 34); the other lines can be regarded as comment and explanation.

[4] In this passage (p. 19, 21 – 20, 9) also there is agreement with Moderatus (in Stobaeus).

Theon	Nicomachus
Is the monad odd or even?[1] p. 21, 24 – 22, 16.	Not in Nicomachus.
The natural series increases with the constant difference 1 but decreasing ratios, p. 22, 17 – 23, 5.	Not in Nicomachus.[2]
Cf. the notion that the 'odd' is 'sameness'; the 'even,' 'difference,' p. 22, 10 ff.	General agreement in II. 17. 2; 18. 4.
Prime and Composite, p. 23, 6 ff.	Nicomachus disagrees.
Classification:[3] Prime Absolutely Relatively Composite Absolutely Relatively	(1) His classification is as follows: Prime Relatively prime Composite (2) It refers to odd numbers only; Theon includes all numbers in these classes.
Absolute primes defined, p. 23, 9. They are also called γραμμικοί, εὐθυμετρικοί, περισσάκις περισσοί, *ibid.*, 12.	I. 11. 2 (different phraseology). These terms do not occur in the *Introduction*.[4]
Odd numbers only are prime, *ibid.*, 23.	Agrees.
Status of 2, p. 24, 5.	Not in Nicomachus.
Relatively prime numbers, *ibid.*, 8.	I. 13. 1.
Absolutely and relatively composite numbers, *ibid.*, 16. (Here are discussed the status of 1 and 2; plane and solid numbers classed here.)	I. 12. 1 (absolutely composite numbers: but the class does not include even numbers as in Theon). No discussion of the other topics.
Varieties of the 'even'; the even-times even, p. 25, 5.	I. 8. 3–4.
The even-times odd, *ibid.*, 19.	I. 9. 1.
The odd-times even, p. 26, 5.	I. 10. 2.
ἰσάκις ἴσοι and ἀνισάκις ἴσοι as kinds of composite numbers, *ibid.*, 14.	Not included in the formal classification; the former term is used in I. 19. 19 and for the latter, cf. note on that section.

[1] Citing Aristotle's Πυθαγορικός (now lost), and Archytas.

[2] Nicomachus notes, however, that in any arithmetical series the ratios are greater in the smaller terms (*Introduction*, II. 23. 6). This involves of course the general idea stated by Theon.

[3] This classification agrees with Euclid (*Elements*, VII, *Def.*) against Nicomachus.

[4] But in *Theologumena Arithmeticae*, p. 44 (Ast): μονωτάτη γὰρ οὔτε πλάτος ἐπιδέχεται εὐθυμετρικὴ οὖσα καὶ μόνου τοῦ ὁμωνύμου ἐπιδεκτικὴ μέρους, κτλ. (*sc.* the number 7). There is some doubt about the authorship of this portion of the *Theologumena Arithmeticae*. Euclid (*Elements* VII,

THEON	NICOMACHUS
Heteromecic numbers defined, p. 26, 21.	II. 17. 1; 18. 2.
They are always even, p. 27, 1.	Cf. II. 19. 1; 17. 2; 20. 3.
Formation by adding evens, *ibid.*, 8.	
Formation by multiplication, *ibid.*, 15.	Cf. II. 18. 2.
Reason for the name, *ibid.*, 20.	Cf. II. 18. 3 (p. 114, 1); 17. 3 (p. 109, 24); 18. 1 (p. 112, 22).
Parallelograms (i.e., προμήκεις), p. 27, 23.	Cf. II. 18. 2.
Squares, p. 28, 3.	II. 9.
Formation by adding odd numbers, *ibid.*	II. 9. 3, etc.
Formation by multiplication, *ibid.*, 13.	II. 18. 3 (not a direct statement).
Make a geometrical proportion with the heteromecic numbers as mean terms. The converse is not true. *Ibid.*, 16.	II. 19. 4. The converse is not stated.
Oblong (promecic) numbers defined, p. 30, 8.	II. 18. 2.
For their classification, see on *Introduction*, II, 18, 2.	Nicomachus does not classify in Theon's manner.
Plane numbers are the product of two factors, p. 31, 9.	Plane numbers are mentioned, but not this definition. Cf. note on II. 7. 3.
Triangular numbers, p. 31, 13. (Here follows a digression on the squares and 'heteromeces,' merely repeating previous material.)	II. 8.
Repetition of methods of forming squares,[1] with several new theorems, p. 34, 1.	
Squares are alternately even and odd, *ibid.*, 3.	Not in Nicomachus.
The gnomons of the other polygonals are numbers of the natural series occurring at intervals of $n - 2$ terms, *ibid.*, 6.	II. 11. 4.
Occurrence of the squares and cubes in the series of multiples, *ibid.*, 16.	II. 20. 5.

Def. 11) uses περισσάκις περισσός to denote the product of an odd number by an odd number, not in the sense in which Theon employs it.

[1] Several topics, of which this is an instance, are duplicated in Theon. The Nicomachean parallels are given only with the first occurrence.

Theon	Nicomachus
Special properties of the squares, p. 35, 17 ff.	None of them given by Nicomachus.
They are either divisible by 3 or can be if 1 is subtracted; likewise by 4; even squares that are divisible by 3 when 1 has been subtracted are divisible by 4; those that are divisible by 4 when 1 has been subtracted are divisible by 3 (Theon neglects to add "or become divisible by 3 when 1 is subtracted"); some squares are divisible by both 3 and 4; finally, the square which is not divisible by 3 or 4 admits both divisors when 1 is subtracted.[1]	
Miscellaneous matter not found in Nicomachus: Squares and heteromecic and promecic numbers are plane. Solids have three factors (a repetition). They are called from their resemblance to the space they measure, p. 36, 3.	
Similar numbers, *ibid.*, 12.	Not in Nicomachus.
Polygonal numbers, p. 37, 7.	
Definition of gnomons, *ibid.*, 11.	Not in Nicomachus.
The side of the triangle equals the last gnomon added, *ibid.*, 13.	II. 8. 3: The side of the triangle equals the number of gnomons taken.
The monad potentially triangular, *ibid.*, 15.	II. 8. 2.
Cyclic, spherical, recurrent numbers, p. 38, 16.	II. 17. 7.
Derivation of squares repeated, p. 39, 10.	
Pentagons defined, p. 39, 14.	II. 10.
Hexagons defined, p. 40, 1.	II. 11.
Derivation of other polygonals and rule for their gnomons, *ibid.*, 14 (as in p. 34, 6).	
Two triangles make a square, p. 41, 3.	II. 12. 1–2.
Solid numbers classified as to the equality of 3, 2 or no dimensions, *ibid.*, 8.	Not in Nicomachus.

[1] Iamblichus (see p. 129) includes this.

THEON	NICOMACHUS
Cubes = $a \times a \times a$;	II. 15. 2; 16. 1.
'Altars,' = $a \times b \times c$;	II. 16. 2; 17. 6.
'Plinths' = $a \times a \times b$, $(b < a)$;	
'Beams' = $a \times a \times b$, $(b > a)$.	
Pyramidal numbers; truncated pyramids, p. 42, 3.	II. 13 and 14, much fuller.
(Theon reports that some called the pyramid trapezium.)	Not in Nicomachus.
Pleuric and diametric numbers, *ibid.*, 10.	Not in Nicomachus.
Perfect, superabundant, deficient numbers, p. 45, 9.	I. 14–16.
The perfection of 3, p. 46, 14. (Other material intervenes.)	Not in *Introduction*, but cf. *Theologumena Arithmeticae*, p. 15 (Ast).
Ratio defined, p. 73, 16.	II. 21. 3.
Ratio exist between homogeneous things only.[1]	I. 17. 4–5 states that equality exists only between homogeneous things, but no such statement occurs regarding ratio.
'Terms' defined, p. 74, 8.	ὅρος is used, but not defined.
Proportion defined, *ibid.*, 12.	II. 21. 2 (differs; *vide ad loc.*).
Equal, greater, and less ratios,[2] *ibid.*, 15.	Not formulated; but cf. I. 17. 2, 6, etc.
The ratios of music.	Cf. I. 5. 1; II. 26; 29. 4.
The ratios of arithmetic, p. 76, 1.	I. 17. 4. (See the note on I. 17. 7.)
Multiples, *ibid.*, 8.	I. 18. 1–3 (much fuller).
Superparticulars, *ibid.*, 21.	I. 19. 1–5 (much fuller).
Superpartients, p. 78, 6.	I. 20 (much fuller).
Multiple superparticular and superpartient, *ibid.*, 23.	I. 22–23 (much fuller).
'Ratio of number to number,'[3] p. 80, 7.	Not in Nicomachus.
Root ratios (πυθμένες λόγοι), *ibid.*, 15.	No separate treatment, but cf. I. 19. 6; 20. 1; 21. 1; II. 19. 3, etc.
Intervals and ratios contrasted,[4] p. 81, 6.	No separate treatment. Cf. on II. 6. 3.
Definitions of proportions, and of continuous and disjunct proportions, p. 82, 6.	II. 21, 2, 5, 6.
Eratosthenes on equality as the element of ratio, *ibid.*, 23.	Cf. II. 1. (Equality is the element of relative number.)
Epinomis, 991 E cited, p. 84, 7.	Cited I. 3. 5.

[1] Based on Adrastus. See on I. 17. 4.
[2] See on I. 17. 7.
[3] With reference to Plato, *Timaeus*, 36 B.
[4] A passage based on the Πλατωνικός of Eratosthenes; cf. p. 237, n. 3, and Hiller, *Philologus*, vol. XXX, pp. 60 ff.

Theon	Nicomachus
ἀναλογία and μεσότης differentiated, p. 84, 15.	Not treated.
Proportions: the three chief kinds,[1] with definitions, p. 85, 8.	II. 21 (much fuller).
Here intervenes a section on the ratios of harmony (p. 85, 22; cf. p. 75, 7). Thrasyllus's division of the canon, and the Pythagorean aspects of the tetraktys and of the numbers of the first decade.[2]	
Proportions: geometric, arithmetic, harmonic, subcontrary, fifth, sixth, and 6 others subcontrary to them, p. 106, 12.	II. 22. 1 (but the fourth is called simply 'fourth' and the remaining kinds number 4, not 6. In II. 28. 3, the name 'subcontrary' is given for 'fourth')
ἀναλογία, properly the geometric alone.[3]	II. 24. 1.
Repetition of the short definitions.	
Equality is the first and elementary ratio (from Adrastus; [cf. p. 82, 23]), p. 107, 10.	
Citation of Eratosthenes for the prooemium of the proposition, of the 'three rules' and of Adrastus for the actual rules, p. 107, 15.	I. 23 (cf. the note on I. 23. 4).
Converse of the 'three rules,' p. 110, 19.	II. 2.
Figures: point, line, plane, solid, and varieties, p. 111, 14.	Not formally included.
Varieties of solids (geometrical treatment); parallelopipeda, rectangular parallelopipeda, p. 112, 26: Cubes (dimensions a, a, a); Plinths (dimensions a, a, b where $b < a$); Beams (dimensions a, a, b where $b > a$); Scalene (dimensions a, b, c).	No geometrical treatment of these.
Proportions; general definition of μεσότης, p. 113, 9.	Does not occur.

[1] Citing Adrastus, who is never mentioned by Nicomachus.

[2] This material can be paralleled largely in the *Theologumena Arithmeticae*, but not in the *Introduction*.

[3] Adrastus is again cited.

THEON	NICOMACHUS
Arithmetic proportion, *ibid.*, 18.	II. 23. (See the note on II. 28. 1 for a general comparison of the proportions used by Nicomachus and by Theon.)
Geometric proportion, p. 114, 1 ff.	II. 24.
Harmonic proportion, p. 114, 14.	II. 25.
'Subcontrary' proportion, p. 115, 5.	II. 28. 3.
Fifth variety of proportion, p. 115, 12.	II. 28. 4.
Sixth variety of proportion, *ibid.*, 20.	II. 28. 5.
How to find the various means between two given terms, p. 116, 8.	II. 27. 7.
Arithmetic; has both methods given by Nicomachus, and this: "Add the halves of the extremes."	
Geometric; Parallels Nicomachus, p. 140, 4–5, but not p. 140, 6, and adds a geometrical method.	
Harmonic; gives methods for finding the mean, first, when the extremes are in double ratio; second, in triple; third, in any ratio (but this is identical with the first method).	
His first method agrees strikingly with Nicomachus (who gives this alone, p. 118, 5): ἐὰν μὲν οὖν ἐν διπλασίῳ λόγῳ πρὸς ἀλλήλους δοθῶσιν οἱ ἄκροι, οἷον ὁ ιβ′ καὶ ὁ ς′ τὴν ὑπεροχὴν τοῦ μείζονος παρὰ τὸν ἐλάττονα οἷον τὰ ς′ ποιήσαντες ἐπὶ τὸν ς′ καὶ τὸν γενόμενον λς′ παραβαλόντες παρὰ τὸν σύνθετον ἐκ τῶν ἄκρων οἷον παρὰ τὰ ιη′ καὶ τὸ πλάτος τῶν λς′ οἷον τὰ β′ προσθέντες τῷ ἐλάττονι, τουτέστι τῷ τῶν ς′, ἕξομεν τὸ ζητούμενον.	p. 140. 8 ff.: ἁρμονικὴν δέ, τῶν ἄκρων τὴν διαφορὰν ποιητέον ἐπὶ τὸν ἐλάττονα καὶ τὸν γενόμενον παραβλητέον ἐπὶ τὸν σύνθετον ἐκ τῶν ἄκρων, εἶτα τὸ πλάτος τῆς παραβολῆς προσθετέον τῷ ἐλάττονι, καὶ ἔσται ὁ γινόμενος ἁρμονικὴ μεσότης.

The differences between Theon and Nicomachus are easily discerned in this summary. Theon omits the elaborate Pythagorean introduction and the incidental dissertations of a similar nature; unlike Nicomachus (cf. *Introduction*, I. 19. 21; II. 19. 20), he does not use tables and point out numerical properties from them; he does not give such elaborate illustrations and explanations, but there are really few important propositions that he leaves out. We may note among them the definitions of I. 7. 3 ff., the theorem that $a = \frac{(a-1)+(a+1)}{2}$ except when $a = 1$ (I. 8. 1–2); the 'sieve' of Eratosthenes (I. 13); the discussion of the derivation of superparticulars from multiples and of the combinations of ratios in II. 3–5; the proposition concerning the addition of a triangular number to any sort of polygonal (II. 12. 3 ff.); and certain observations upon squares and heteromecic numbers (II. 19. 2–3; 20. 1). On the other hand one may easily note the large number of topics in Theon which Nicomachus omits. (See Theon, pp. 18, 5 ff.; 19, 21 ff.; 21, 24 – 22, 16; 34, 3; 35, 17 ff.; 37, 11; 42, 10 ff.; 73, 16 ff.; 80, 7 and 15; 81, 6 ff.; 82, 23 ff.) It is particularly surprising to see that Nicomachus does not attempt to discuss the monad as Theon does.

But even if one takes all these differences into consideration, there is great similarity between the two works; the most notable point is that, notwithstanding their agreement in subjects treated, and the fact that they say substantially the same thing about most subjects, there is after all slight verbal likeness between them. The only passage in which verbal likeness is especially marked is that to which attention is called at the very end of the table above (Theon, p. 118, 5; Nicomachus, p. 140, 8). It is justifiable to infer, therefore, that neither writer served as the direct source of the other, and this must remain for the present the only certain fact regarding their relationship to each other. We may also conjecture that behind them, to some extent, lay the same sources, a supposition supported by the verbal parallels noted. But it does not immediately follow that Nicomachus, like Theon, employed Adrastus; had he done so, we might have looked for greater similarity in the texts. From the fact that the two quote the same passages of Plato not much can be inferred; they must be simply the usual quotations made by schoolmen to support their statements by Platonic authority, and it chances that Theon and Nicomachus have the same position to sustain.[1]

[1] The Platonic passages which Theon and Nicomachus quote in common are enumerated on pp. 37 and 41. In none of the cases does either writer agree completely with our Platonic text, but although they sometimes agree with each other in their variations, their differences are far more striking, and the only conclusion warranted is that each independently quotes loosely from memory the gist of Plato's words. They certainly cannot have used a common source, such as, for instance, a collection of Platonic passages or a commentary which employed similar inaccurate quotations. The most noteworthy verbal agreement is in the citation of *Republic*, 527 D, where

Whether Theon is to be treated as a successor or a predecessor of Nicomachus is, as has been pointed out before, a difficult question. It is most probable that their lives were contemporary in part, and farther than that, it is useless to conjecture. In view of the fame of Nicomachus, it seems hardly possible that Theon could have written after the publication of the *Introduction* without quoting it, but we have seen that there is no suggestion of a direct connection between the two men. They must have been nearly contemporary.

Nicomachus (I. 3. 7) has *ὡς ἡδὺς εἶ, ὅτι ἔοικας δεδιέναι μὴ ἄρα ἄχρηστα ταῦτα τὰ μαθήματα προστάττοιμι.* Theon (p. 3, 8) reads *ἡδὺς εἶ, ὅτι ἔοικας δεδιέναι μὴ ἄχρηστα τὰ μαθήματα προστάττοιμι,* and Plato *ἡδὺς εἶ . . . ὅτι ἔοικας δεδιότι τοὺς πολλούς, μὴ δοκῇς ἄχρηστα μαθήματα προστάττειν.*

Nicomachus and Theon agree in using peculiar syntax which does not occur in the original; but in the remainder of the passage their texts are widely divergent. The initial phrase, perhaps, was commonly quoted in their day in the form which they both used.

CHAPTER III

THE MATHEMATICAL CONTENT OF THE GREEK ARITHMETICA

If we restrict the meaning of the term *arithmetica* so as to include only the *ars arithmetica*, which, as is explained more fully elsewhere, refers, when technically used, to the fundamentals of the science of arithmetic systematically stated, we may obtain an idea of its mathematical content by a comparison of three texts, Nicomachus's *Introduction*, Euclid's *Elements* (Books VI, VIII, and IX), and Theon's treatise *On Mathematical Matters Useful for Reading Plato.* Without such restriction the term, applying to the Greek science of arithmetic in general, would include all the developments of higher mathematics such as those made by Diophantus. Since, however, Nicomachus's book is an 'arithmetic' in the narrower sense, we shall not be forced to go so far afield in this summary.

In the first place, the differences existing between Euclid on the one hand and Nicomachus and Theon on the other are to be observed. The latter two are generally in agreement in their selection of topics and in their manner of presentation; Euclid differs in both these respects. His material is stated in the form of propositions, each subjected to logical proof, a thing wholly lacking in Nicomachus and Theon, who confine themselves to setting forth principles and illustrating them. Euclid's material itself, moreover, is not confined to the propositions treated by the other two, although there is much overlapping between the two groups, but Euclid makes a far more systematic study of the matters he introduces. In the third place, Euclid employs the geometrical form of illustration in the proof of his arithmetical propositions, and makes use of lines in much the same way that modern mathematics uses its literal algebraic symbolism. This process complicates his solutions and often involves him in long demonstrations of matters which are more or less self-evident when stated in algebraic terms. Nicomachus and Theon, since both avoid any attempt at proof, do not share this habit of Euclid, but deal with the numbers directly; only when they come to the discussion of plane and solid numbers do they make their approach through geometry. Furthermore, Euclid does not share the philosophical proclivities, or, more accurately, the Pythagorizing tendencies, of Nicomachus and Theon, and consequently holds himself continually to a more strictly scientific level. It is for this reason that he is more successful in general than Nicomachus in making a scientific classification of numbers. All three mathematicians, however, agree in confining themselves, in their arithmetic, to the use of rational numbers, although many of the propositions of Euclid's books (notably V and X) are stated with reference to irrational quantities. Some of these, in Book VII, are applied to the discussion of rational numbers.

The following outlines roughly sketch the plans according to which the work of Nicomachus, Euclid, and Theon, respectively, are arranged. In the preceding chapter will be found an analysis of Theon's treatise, with the parallel passages of Nicomachus indicated.[1]

Nicomachus	Theon	Euclid
		Book VII
Philosophical introduction	Philosophical introduction	Definitions
1. Number per se	Definitions	Greatest common divisor
2. Relative number	Number per se, including plane and solid numbers	Propositions dealing with submultiples and fractions, with reference to the propositions on proportion in Book V
3. Plane and solid numbers	Ratios	Prime and relatively prime numbers
4. Proportions	Proportions	Least common multiple
		Book VIII
		Properties of numbers in progression and proportion
		Cubes, squares, and their relations
		Book IX
		Cubes and squares, continued
		Geometrical series
		Prime numbers
		Odd and even numbers
		The perfect number

[1] See pp. 37 ff.

Nicomachus and Theon, it will be seen, follow plans closely similar, and their topics fall into a scheme ordered from the philosophic point of view; Euclid does not proceed in the same way. The primary subdivision of material in the first two is based on the distinction between absolute and relative number, which they carefully separate; Euclid mingles the two inextricably, but keeps uppermost the relations between numbers. In setting forth the subject in detail we shall follow the order adopted by Nicomachus, with the exception of the philosophical introduction, which is lacking in mathematical interest.

I. Number per Se

The definitions of 'number' given by the Greek mathematicians are discussed elsewhere. Nicomachus, in I. 7, limits himself to defining number and its two subdivisions, odd and even; in Euclid (VII, def. 1) we have also a definition of the monad (unity), and in Theon a lengthy discussion of this subject (p. 18, 5 – 21, 19, Hiller). His reference (p. 20, 7 ff.) to the Pythagorean doctrine of the 'principles' or 'origins' (ἀρχαί) of number shows a surprising similarity to the most modern conception of number; the Pythagoreans, he says, considered all the terms of the natural series 'principles,' so that "for example 'three' (the triad) is the principle of three's among sensible objects, and 'four' (the tetrad) of all four's," and so on.

Nicomachus, in I. 8, states that any integer is one half the sum of the two integers on each side of it:

$$m = \frac{(m - 1) + (m + 1)}{2}.$$

Since zero is not included in the numerical system,[1] unity, in Nicomachus's eyes, occupies a unique position, and is therefore regarded by him as the 'natural starting point of all numbers.' Theon introduces here the problem of determining whether unity is odd or even; his treatment of the problem takes the form of a logical demonstration, which is all the more remarkable because both he and Nicomachus usually avoid this method: "Some have said that unity is odd. For the odd is opposite to the even; unity is therefore either odd or even. It would not be even; for it cannot be divided into equal halves, nor even divided at all. Therefore unity is odd. And if you add an even

[1] When he speaks philosophically, Nicomachus denies that even 1 and 2 are real numbers, and begins the series with 3; see pp. 116 ff.

number to an even number, the sum is even;[1] but if unity be added to an even number, the sum is odd; unity is therefore not even, but odd."[2] Euclid does not thus give special treatment of unity, but in IX. 21–31, he studies the relations of the two classes of numbers, odd and even, proving that the sum of any number of even numbers is even (IX. 21), and that the sum of odd numbers is even if the number of terms is even, and odd if the number of terms is odd, and then dealing in similar fashion with differences and products of odd and even numbers. This material, curiously, is not introduced by Nicomachus until, in II. 24. 10, he is led to speak of the products of squares, cubes, and heteromecic numbers, and introduces incidentally, by way of illustration, certain propositions regarding the products of even and odd numbers:

(1) $2n \times 2m$ is even;
(2) $(2n \pm 1)(2m \pm 1)$ is odd;
(3) $2n(2m \pm 1)$ or $(2n \pm 1)2m$ is even.

The classes of even numbers, according to Nicomachus (I. 8. 3 ff.), are three:

(1) Even times even, that is, numbers of the form 2^n;
(2) Odd times even, of the form $2^n(2k + 1)$, $n > 1$;
(3) Even times odd, of the form $2(2k + 1)$.

Theon has the same classification, but Euclid omits the odd times even and adds another class, the odd times odd, namely, those "measured by an odd number an odd number of times," as $(2m + 1)(2n + 1)$. The name 'odd times odd' occurs in Theon as an alternative designation of the primes (p. 23, 14 ff.). Furthermore, Euclid is not precise in this classification, a fault for which Iamblichus, in his *Commentary on Nicomachus*, criticizes him. Since he defines the even times even as "a number measured by an even number an even number of times" (VII, def. 8), and the even times odd as one "measured by an even number an odd number of times" (def. 9), it is obvious that the former class can include numbers of two forms 2^n, or, $2^n(2k + 1)$, $n > 1$, and the latter might likewise include the two types, $2(2k + 1)$, or, $2^n(2k + 1)$, where k is a positive integer. It is evident from the algebraic forms that the two classes overlap, and the proposition stated in IX. 34, which deals with certain numbers that are both

[1] Cf. Euclid, IX. 21.

[2] P. 21, 24 ff.

even-times even and even-times odd, makes it clear that Euclid recognized the possibility of a number falling in both classes.[1]

The arithmetic of Nicomachus includes a lengthy discussion of all these types of the even numbers. In I. 8. 12 he notes that

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots + 2^{n-1} = 2^n - 1,$$

and that this theorem will be useful in the discussion of perfect numbers,[2] but many of his observations are more or less self-evident. For example, in the series

$$1, 2, 4, 8, 16, 32, 64 \cdots,$$

it is evident that in any sequence of three of these terms the square of the mean will equal the product of the extremes, or,

$$(2^n)^2 = 2^{n-1} \cdot 2^{n+1},$$

and similarly, if each member of the series be multiplied by the same integer, the property persists. Further, in the series,

$$1, 3, 5, 7, 9, 11, \cdots 2n - 1,$$

evidently the middle term of any sequence of three terms equals half the sum of the extremes, and in any sequence of four the two means equal the two extremes.

The subdivision of the odd (I. 11 ff.) is a matter of more difficulty to Nicomachus. As in the former case, he sets up three classes, and designates them as prime, composite and relatively prime; the latter are pairs of composite numbers that are prime to each other. To disregard 2, the primes are indeed odd, but there is no reason why composite numbers should be treated as a class of the odd. The classification belongs, of course, to all numbers rather than to the odd alone, and it is so regarded by Theon and Euclid, who add to the list another class, 'relatively composite' numbers (i.e., composite numbers that have a common measure). Strictly speaking, none of the authors need have specified more than the first two subdivisions, for all the 'relatively prime' and 'relatively composite' numbers are composite. As to 2, Nicomachus is silent:[3] but Theon somewhat doubtfully recognizes it as prime,[4] and it satisfies Euclid's definition of the prime.

[1] For a fuller discussion of the difficulties centering about this matter, see Heath's *The Thirteen Books of Euclid's Elements*, vol. II, pp. 281 ff.

[2] Theon omits this theorem, while Euclid proves it.

[3] But he names 3 as the first prime number (I. 11. 2).

[4] He does not name 2 in his list of primes, which begins with 3 (p. 23, 11); he states that only the odd are prime (*ibid.*, 23); yet he admits that 2 shares the characteristics of the primes (p. 24, 5) and finally says that it is "called odd-like because it shares the qualities of the odd" (*ibid.*). His stand in the matter is indecisive. See Heath, *op. cit.*, vol. II, p. 285.

To find the prime numbers the celebrated method of Eratosthenes, known as his 'sieve,' is employed (I. 13. 2 ff.). This involves no mathematical principle of any difficulty, but is, notwithstanding, the only systematic method of locating them. The odd numbers are conceived as written down; then every third number beyond 3 is stricken out, every fifth beyond 5, every seventh beyond 7, and so on, leaving only numbers which are prime; after the multiples of n have been rejected, only prime numbers remain up to n^2. Nicomachus makes it clear that the process consists in discarding multiples.

In the same chapter (sections 10 ff.) he takes up the Euclidean problem of the greatest common divisor, but with the announced purpose of determining whether any two given numbers (in practice limiting himself to odd numbers) are prime to each other or not. The process consists in dividing the greater by the smaller,[1] and then the smaller by the remainder after the division, until, if the two numbers are prime, the final remainder is unity, or until the remainder is found to be an exact divisor of the preceding divisor. In this case, the remainder is the greatest common factor of the two original numbers. Euclid devotes VII. 1–3 to this subject, and Theon omits it.

In Euclid, the subject of prime and relatively prime numbers receives a more detailed treatment in VII. 20–32 and again in Book IX. The matters proved in VII. 20–32, stated algebraically, are as follows, if we always assume a and b relatively prime to each other:

VII. 20: *If* $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$, *then* $c = ka$, *and* $d = kb$.

VII. 21: *If* $\frac{a}{b} = \frac{c}{d}$, *then* $a < c$, *and* $b < d$.

VII. 22: *Given* $\frac{a}{b} = \frac{c}{d}$, *and given that, for any values of* c *and* d, $a < c$ *and* $b < d$, *then* a *and* b *are relatively prime to each other.*

VII. 23: *If* a *and* kb *are prime to each other, then* a *and* b *are prime to each other.*

VII. 24: *Given* a *and* c *prime to* b, *then* ac *is prime to* b.

VII. 25, 27: *Given* a *prime to* b, *then* a^2 *is prime to* b, *also* a^2 *is prime to* b^2, *and* a^3 *prime to* b^3.

VII. 26: *Given* a *and* c *both prime to* b *and* d, *then* ac *is prime to* bd.

[1] Nicomachus's 'division' is really subtraction continued until the remainder is smaller than the subtrahend.

VII. 28: *Given a prime to b, then $a + b$ is prime to a and to b, and also if $a + b$ is prime to a and b, a is prime to b.*

VII. 29, 30: *If a is a prime number, it is prime to any number which does not contain it as a factor, and conversely.*

In two propositions (31 and 32) Euclid shows that every number is prime, or has a prime factor. In Book IX, propositions 11–17 deal with primes and relative primes in continued proportion, and IX. 20 proves that the number of primes is infinite.

The ancient *arithmetica* commonly divided numbers in yet a third way, into perfect, superabundant and deficient numbers. Again we find that Nicomachus restricts to one subdivision, the even, a classification which properly belongs to number in general; and although Theon speaks of 'numbers,' not specifically even numbers, as being subject to this classification, it may be noted that all the examples he cites are even numbers. Nicomachus and Theon define all three classes, but Euclid touches upon only the perfect number, defining it (VII, def. 22) and subjecting to proof the method of its discovery (IX. 36). As usual, Nicomachus and Theon state only the method. Nicomachus mentions four perfect numbers, 6, 28, 496, and 8,128. If, in I. 17. 3 and 7, he means to imply that a perfect number is to be found in each order of the powers of 10, he is mistaken; and he certainly is incorrect in asserting that all perfect numbers end alternately in 6 and 8 (*ibid.*). It is proved by Euclid, IX, 36, that every number of the form $2^{n-1}(2^n - 1)$, if $2^n - 1$ is prime, is perfect; the first ten, viz.: for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61$, and 89, have actually been calculated.

II. Relative Number

The fundamental relations of number, as the Greek *arithmetica* states, are equality and inequality, and the latter is further divided into two classes, the greater and the less. Furthermore, each of these latter has five subdivisions, those of the greater being

multiples, as, $mn : n$
superparticulars, as $n + 1 : n$
superpartients, as $n + k : n$, $k > 1$
multiple superparticulars, as $mn + 1 : n$, $m > 1$
multiple superpartients, as $mn + k : n$, both m and k being greater than 1.

The subdivisions of the less are the reciprocal ratios of these, and are designated by the same names with sub- (Greek, ὑπο-) prefixed; for example, submultiple. Each of these is described by Nicomachus at considerable length (I. 17–23) and more briefly by Theon; Euclid has no parallel discussion, but it may be recalled that he gives definitions (VII, defs. 3, 4) of the terms 'part' and 'parts' (μέρος, μέρη), the former having the same meaning as submultiple $\left(\text{e.g.}\ \frac{1}{3}, \frac{1}{n}\right)$, and the latter being equivalent to our term 'proper fraction' $\left(\text{with numerator greater than unity, as}\ \frac{2}{3}, \frac{m}{n}\right)$. For the former term Nicomachus prefers to substitute μόριον, and the use of this word in composition, in the term ἐπιμόριος, 'superparticular,' a number which contains another smaller number once, plus one μόριον of it (I. 19. 1), shows clearly the proper meaning of the word.

In I. 19. 9, Nicomachus presents our ordinary multiplication table, but not for use as such; it is rather to serve as a table of multiples. Such tables are of common occurrence in the *Introduction*, but neither Euclid nor Theon makes use of them. In his discussion of this table, Nicomachus notes two propositions which may thus be expressed (I. 19. 9):

$$(1)\ n^2 + (n + 1)^2 + 2n(n + 1) = (2n + 1)^2;$$
$$(2)\ (n - 1)n + (n + 1)n + 2n^2 = (2n)^2.$$

As a fitting termination to Book I, Nicomachus states a general principle whereby all forms of inequality of ratio may be generated from a series of three given equal terms; it was designed to show that equality is the 'root and mother' of all forms of inequality.[1] The first number of the given series is taken as the first term of the new series; the sum of the given first and second terms gives the new second term; and the new third term is derived by adding the given first and third terms plus twice the given second term. Thus, given a, a, a, we first obtain a, $2a$, $4a$. Given a, na, n^2a, we obtain a, $(n + 1)a$, $(n + 1)^2a$, a geometric progression with the ratio $(n + 1)$. Given n^2a, na, a, we obtain n^2a, $n(n + 1)a$, $(n + 1)^2a$, in which each term has the ratio $n + 1 : n$ to the preceding. The reverse process is elaborated in II. 2, showing that equality may finally be obtained from

[1] On the history of this proposition see on I. 23. 4.

any given series illustrating any form of inequality. Theon also gives these rules and their converse (p. 107, 10 ff.)

In II. 3–4, Nicomachus sets forth a method for deriving from the successive series of multiples the series of superparticulars; from the doubles come the series with ratio $3:2$; from the triples, those with ratio $4:3$; and in general, if the multiple series is $1, n, n^2, n^3, \cdots$, the ratio is $n+1:n$. Algebraically expressed, the series and derivative ratios are:

$$
\begin{array}{lllll}
1 & n & n^2 & n^3 & n^4 \cdots \\
 & n+1 & n^2+n & n^3+n^2 & n^4+n^3 \\
 & & n^2+2n+1 & n^3+2n^2+n & n^4+2n^3+n^2 \\
 & & & n^3+3n^2+3n+1 & n^4+3n^3+3n^2+n \\
 & & & & n^4+4n^3+6n^2+4n+1
\end{array}
$$

In his discussion Nicomachus remarks upon two facts as especially strange and significant, first, that the number of integral superparticulars arising from a given term in the series is limited, and second, that the number of superparticulars that may be derived from a given term can be determined, as it will agree with the exponent of the power of n in the multiple series. With the algebraic notation it is easy to see that the derivation of integral superparticulars must come to a halt when an expression whose numerator is not divisible by n is reached, for example, $\frac{(n+1)}{n}(n+1)$, and that, given n^k, from it there will arise a series of k numbers increasing in the ratio $n+1:n$, thus:

$$n^k,\ (n+1)n^{k-1},\ (n+1)^2n^{k-2},\ (n+1)^3n^{k-3}, \cdots (n+1)^k.$$

The matter dealt with in II. 5, the combination of ratios, calls forth a lengthy discussion, but in our notation the facts that

$$\tfrac{3}{2}\times\tfrac{4}{3}=\tfrac{2}{1},\ \tfrac{3}{2}\times\tfrac{2}{1}=\tfrac{3}{1},\ \tfrac{3}{1}\times\tfrac{4}{3}=\tfrac{4}{1}$$

are self-evident.

III. Plane and Solid Numbers

The remainder of Nicomachus's second book is devoted to the plane and solid numbers, including a rather detailed treatment of the squares, cubes and heteromecic numbers, followed by the general subject of proportions.

The doctrine of plane, or polygonal, numbers set forth by both Nicomachus and Theon is based upon the possibility of arranging the component units of any number in various regular forms in a plane or planes. Thus, first, the monad, unity, may be regarded as a geometrical point; any greater integer, beginning with 2, if its units are arranged along a line, may be regarded as linear; beginning with 3, the numbers can be arranged in a plane so as to form the various plane figures; and from 4 on, the numbers can be arranged in three dimensions, after the fashion of solids. There is reason to believe that this was a typically Pythagorean view of the matter, and older than Aristotle's time.[1]

Euclid's conception of the plane and solid numbers is very different, and is based upon the possibility of representing numbers by lines of lengths corresponding to the units in the number, the lines in turn being used as the sides of rectangular plane and solid figures. The plane number, according to his definition (VII, def. 16), is the product of the two (linear) numbers which are used as sides; similarly, solid numbers are the product of three numbers (def. 17). Euclid's view, therefore, excludes triangles, pentagonals, and the like among plane numbers, and pyramids among solids, and coincides with the theory held by Nicomachus and Theon only when rectangular figures and solids are in question. Euclid's plane and solid numbers, that is, really represent areas and volumes; those of Nicomachus and Theon are simply arrangements of points in space.[2]

Nicomachus shows in II. 8 that the triangular numbers, thus conceived, result from the summation of the terms of the natural series $1, 2, 3, 4, 5, 6, \cdots n$, giving

$$\frac{n(n+1)}{2}$$

as the nth triangular number; squares may be derived in two ways, by squaring the successive terms of the natural series, or by the summation of the terms of the natural series with difference 2, i.e., $1, 3, 5, 7, 9, \cdots (2n-1)$, giving n^2 as the nth square number; the pentagons arise from the summation of the arithmetical series with difference 3, i.e., $1, 4, 7, 10, 13, \cdots (3n-2)$, so that the nth pentagon is $\frac{n(3n-1)}{2}$. To generalize, the polygonal number of the order k (k-gonal number) is derived from the series

$$1, 1+(k-2), 1+2(k-2), 1+3(k-2), \cdots 1+(n-1)(k-2),$$

and the nth k-gonal number will be

$$\frac{n}{2}(kn-2n-k+4).$$

[1] See Aristotle, *Metaphysica*, 1092 b 10.

[2] Heath, *op. cit.*, vol. I, pp. 287 ff., gives a detailed comparison of the two theories.

Nicomachus carries the discussion up to the octagonal and indicates that the process of derivation may be indefinitely extended. It is noteworthy that he himself develops a generalization in this connection. The terms of the arithmetical series from the summation of which the polygonals are derived were technically called gnomons and Nicomachus (probably following Hypsicles) states that the constant difference between the gnomons will be $k - 2$, k being the number of sides (or angles) of the polygon in question (II. 11. 4).

In II. 12 there follow some theorems dealing with the polygonal numbers. Both Nicomachus (II. 12. 2) and Theon (p. 41, 3) state that the sum of two consecutive triangular numbers is a square, and conversely every square is the sum of two triangular numbers. That is, taking the $(n - 1)$st and the nth triangular numbers,

$$\frac{(n-1)n}{2} + \frac{n(n+1)}{2} = n^2.$$

Nicomachus next states that any triangular number added to the square number next after it in order in the parallel series of squares gives a pentagonal; that is, for the $(n - 1)$st triangle and the nth square,

$$\frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2},$$

which is the nth pentagonal. He proceeds to put this principle in general form, to the effect that any triangle added to the following k-gonal number produces the $(k + 1)$-gonal number, or in algebraic terms, for the $(n - 1)$st triangle and the nth k-gonal number,

$$\frac{n(n-1)}{2} + \frac{n}{2}[2 + (n-1)(k-2)] = \frac{n}{2}[2 + (k-1)(n-1)].$$

Nicomachus sets down the polygonal numbers in tabular form, and notes what is obvious in the table, viz., that the polygonal numbers occupying the same position in their respective series, the nth triangle, the nth square, the nth pentagon, and so on, form an arithmetical series with the $(n - 1)$st triangular number as their difference, but as usual he does not offer any proof. In algebraic form we have

$\frac{n}{2}[2 + (n-1)(k-2)]$, the nth k-gonal number,

$\frac{n}{2}[2 + (n-1)(k-1)]$, the nth $(k + 1)$-gonal number,

with the difference $\frac{n(n-1)}{2}$, the $(n - 1)$st triangular number.

So far as the solid numbers are concerned, the pyramids and the truncated pyramids, the cubes, wedges, the parallelopipeds, the heteromecic and the promecic numbers, we have here no real mathematical principles involved, but simply definitions. The recurrent or spherical numbers are worthy of note; numbers which end in 1, 5 and 6 when multiplied by themselves produce other numbers ending in the same digit, and such are called spherical.

From an arrangement of the squares and the heteromecic numbers (which have the form $k(k+1)$) in parallel array, Nicomachus discovers, in II. 19, certain general relationships whose proof offers no difficulty with modern algebraic symbolism.

$$\begin{array}{lllllll} 1 & 4 & 9 & 16 & 25 & 36 \cdots n^2 & (n+1)^2 \\ 2 & 6 & 12 & 20 & 30 & 42 \cdots n(n+1) & (n+1)(n+2) \end{array}$$

The ratio of the nth heteromecic number to the nth square is obviously $n+1:n$ (II. 19. 3), and the difference is $n(n+1) - n^2 = n$. Similarly the $(n+1)$st square less the nth heteromecic number is $(n+1)^2 - n(n+1) = n+1$. Furthermore, the nth heteromecic number is a mean proportional between the nth and the $(n+1)$st squares; and if two successive squares be added with twice the intermediate heteromecic number, a square number results:

$$n^2 + (n+1)^2 + 2n(n+1) = (2n+1)^2.$$

Nicomachus is quite amazed by the fact (II. 19. 4) that the sums $1+2$, $2+4$, $4+6$, $6+9$, $9+12$, $12+16$, $\cdots$ give the triangular numbers in regular sequence:

$$n^2 + n(n+1) = (2n+1)n, \text{ or } \frac{2n(2n+1)}{2},$$

a triangular number; and similarly

$$n^2 + (n-1)n = (2n-1)n = \frac{(2n-1)2n}{2} = \frac{(2n-1)(2n-1+1)}{2},$$

the triangular number which precedes the one just given.

After noting (II. 20. 1) that n^2+n and n^2-n give in each case a heteromecic number, Nicomachus introduces a proposition the discovery of which is commonly credited to him. He states (II. 20. 5) that if the odd numbers be written down in order,

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \cdots$$

the first number is the 'potential cube' (cube of unity); the second two, $3 + 5$, make the cube of 2; the following three, $7 + 9 + 11$, make the cube of 3, and so on.

1,	3, 5,	7, 9, 11,	13, 15, 17, 19,	21, 23, 25, 27, 29, ...
1^3	2^3	3^3	4^3	5^3

A theorem of the first importance from the standpoint of the modern theory of numbers, relating to the divisibility of the square numbers, is found in Theon, but not in Nicomachus or Euclid. According to this proposition, every square is either divisible by 3, or becomes so when diminished by unity, and is similarly divisible by 4. Theon gives no proof of this, of course, in accordance with his usual practice. Any number when divided by 3 gives either 1, -1 (or 2), or 0, as remainder; and hence it may be written either as $3n$ or $3n \pm 1$, or $3n \pm 2$; the square is of the form

$$9n^2 \text{ or } 9n^2 \pm 6n + 1 \text{ or } 9n^2 \pm 12n + 4.$$

In the first case the square is divisible by 3, and in the second and third cases it becomes divisible by 3 when 1 is subtracted. Similarly any even number, when squared, contains the factor 4, and any odd number, being of the form $2n + 1$, when squared, may be written

$$4n^2 + 4n + 1,$$

which is divisible by 4 when unity is subtracted. Apparently Theon desired to divide all square numbers into four classes, viz., those divisible by 3 and not by 4; by 4 and not by 3; by 3 and by 4; and by neither 3 nor 4. In modern mathematical phraseology all square numbers are termed congruent to 0 or 1, modulus 3, and congruent to 0 or 1, modulus 4. This is written:

$$n^2 \equiv 1 \pmod{3},$$
$$n^2 \equiv 0 \pmod{3},$$
$$n^2 \equiv 0 \text{ or } 1 \pmod{4}.$$

This is the first appearance of any work on congruence which is fundamental in the modern theory of numbers.

Another important subject, omitted by Nicomachus, but studied by Theon, is that of 'side and diagonal' numbers, which may be introduced at this point, inasmuch as it bears some relation to the theories about plane numbers. Theon says (p. 42, 10 ff.):

"Even as the numbers are potentially invested with the essential principles of triangles or quadrilaterals, of pentagons or other figures, so also we find that the essential principles of sides and diagonals appear in numbers in accordance with the ultimate principles of their being;[1] for it is these which harmonize the configurations. Then just as unity is the starting point of all (geometrical) figures, according to the supreme generative principle, so also the ratio of the diagonal to the side is found in unity."

A pair of side and diagonal numbers consists of two numbers, y and x, so related that

$$x^2 - 2y^2 = +1,$$

or,

$$x^2 - 2y^2 = -1.$$

In other words, a series of increasing side and diagonal numbers are rational approximations to the sides and hypotenuses of increasing isosceles right triangles, and in each case the sum of the squares of the sides differs numerically from the square of the hypotenuse by unity. Theon gives the general rule which establishes such a series, starting from any given solution. Taking y, x, b as any solution, he then states that another solution is given by $x + y$, $x + 2y$, in which the first of the pair represents the side and the second always the diagonal. Thus 1, 1, a potential algebraic solution, gives 2, 3 as a second solution; 2, 3 generates by the rules 5, 7, and from this, in turn, 12, 17 is derived; to this point Theon carries the numerical illustration. The successive values of $\frac{x}{y}$ are increasingly refined approximations to the square root of 2; thus,

$$\tfrac{1}{1}, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}$$

differ in square from 2 by

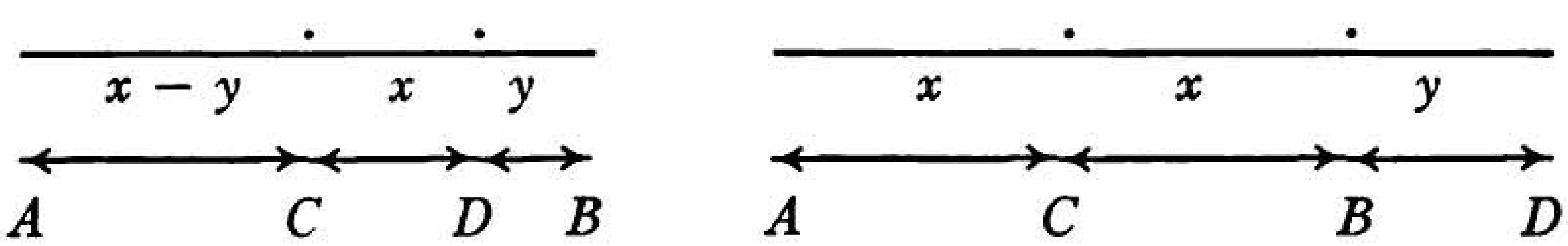
$$-1, +\tfrac{1}{4}, -\tfrac{1}{25}, +\tfrac{1}{144},$$

and continuing the series, we would have

$$\tfrac{41}{29}, \tfrac{99}{70}, \tfrac{239}{169}, \cdots$$

as further and closer approximations of the square root of 2; the value $\frac{239}{169}$ is correct to the fourth decimal place.

Zeuthen[2] was the first to note that a generalized geometrical statement of this theory is implicit in Euclid II. 9 and 10. In effect Euclid states that



[1] κατὰ τοὺς σπερματικοὺς λόγους. This is the technical terminology of the Stoics.

[2] Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, 1886, pp. 27–28.

if C is the mid-point of AB, and D any other point in the line (internal in II. 9, and external in II. 10), then

$$\overline{AD}^2 + \overline{DB}^2 = 2\,\overline{AC}^2 + 2\,\overline{CD}^2,$$

or,

$$\overline{AD}^2 - 2\,\overline{AC}^2 = 2\,\overline{CD}^2 - \overline{DB}^2;$$

taking the values x and y, respectively, as indicated in the diagrams, we have

$$(2x + y)^2 - 2(x + y)^2 = 2x^2 - y^2.$$

The theorem is ascribed by Zeuthen to the Pythagoreans, and confirmation of its Pythagorean origin, as well as of the correspondence between the geometrical propositions and the arithmetical theory, is found in Proclus's *Commentary on Plato's Republic*,[1] in which Proclus refers to this method of forming 'side and diagonal numbers' as Pythagorean, and says that the same is proved graphically in the second book of the *Elements*. Further, Proclus alludes to the Pythagorean distinction between 'rational' and 'irrational' diameters, to which Plato makes reference in the *Republic*[2] when he contrasts the 'rational diameter of 5' with the 'irrational,' having in mind 7 as opposed to the irrational number $\sqrt{50}$.[3] The discussion above touches only one of many possible references to essentially arithmetical or algebraic theorems which are concealed in geometrical form in the first six or last four books of Euclid's *Elements*. It likewise gives evidence of the fact that, although Theon's treatise was not pedagogically so successful as the *Introduction*, yet it is much superior to that work from the mathematician's point of view.

IV. Proportions

Nicomachus distinguishes the proportions involving four distinct terms, called disjunct, and those involving only three terms, or continuous; there are two means in the first type and only one in the second. Given a, b, c, d as the terms of an arithmetical proportion, the propositions he sets forth correspond to the following algebraic formulations:

[1] Procli Diadochi, *In Platonis Rem Publicam Commentarii* (Leipzig, 1901), vol. II, cc. 23, 27; see Heath, *op. cit.*, vol. II, pp. 395–401.

[2] 546 C. This became known during the Middle Ages as the "Rule of Nicomachus"; thus $98^2 = (98 - 2)(98 + 2) + 2^2$. However, Nicomachus does not give such illustrations.

[3] The discussion here closely follows Heath, *loc. cit.*

(1) $a + c = 2b$,
(2) $a + c = b + d$,
(3) $\frac{a}{a} = \frac{b - a}{c - b} = 1$,
(4) $ac = b^2 - (c - b)^2$, or $(a - x)(a + x) + x^2 = a^2$,
(5) $\frac{b}{a} > \frac{c}{b} > \frac{d}{c}$, when $d > c > b > a$.

The fourth of these propositions, which states that the product of the extremes differs from the square of the mean by the square of the common difference, came to be known in the Middle Ages as the *Regula Nicomachi*.[1] It is more readily grasped if the terms are written as $a - k$, a, $a + k$, when evidently

$$(a - k)(a + k) = a^2 - k^2.$$ [2]

The theorems stated by Nicomachus concerning four numbers, a, b, c, d, which form a geometric proportion, are as follows:

$$d : b = c : a;$$

$$\frac{d}{c} = \frac{b}{a} = \frac{d - c}{c - b};$$

When $\frac{d}{c} = \frac{c}{b} = \frac{b}{a} = \frac{2}{1}$, *then* $d - c = c$, *and* $c - b = b$;

When $\frac{d}{c} = \frac{c}{b} = \frac{b}{a} = \frac{3}{1}$, *then* $d - c = 2c$, *and* $c - b = 2b$;

When $\frac{d}{c} = \frac{c}{b} = \frac{b}{a} = \frac{4}{1}$, *then* $d - c = 3c, \cdots$;

When $\frac{d}{c} = \frac{c}{b} = \frac{b}{a} = \frac{5}{1}$, *then* $d - c = 4c, \cdots$;

$$\frac{a}{b} = \frac{b}{c},\ b^2 = ac;$$

$$\frac{a}{b} = \frac{c}{d},\ bc = ad.$$

Between n^2 and $(n + 1)^2$, only one geometric mean, viz., $n(n + 1)$, is possible; between n^3 and $(n + 1)^3$, there are two, viz., $n^2(n + 1)$ and $n(n + 1)^2$. These propositions, which Nicomachus ascribes to Plato, are also proved by Euclid (VIII. 11–12). Nicomachus goes on to

[1] O'Creat, *Twelfth Century*. See p. 60, n. 2.

[2] This formula can be used in squaring numbers near to 100, and other numbers also; thus, $94^2 = (94 - 6)(94 + 6) + 6^2$, or $88 \times 100 + 36$.

state that the product of two squares is a square, and that of two cubes a cube, but the product of a square by a heteromecic number, or that of a cube by a heteromecic number, can never be, respectively, either a square or a cube.

Nicomachus says that three numbers, as a, b, c, are in harmonic proportion when $\frac{c}{a} = \frac{c-b}{b-a}$; from this definition it follows that if $c - b = kc$, then $b - a = ka$, and this is the definition given by Plato for the harmonic progression. Nicomachus uses this result, numerically, stating also the general theorem. Further, given a, b, c in harmonic proportion, with $c > b$, it follows that $\frac{c}{b} > \frac{b}{a}$ and that $b(c + a) = 2\,ac$. The particular proportion given by 6, 8, 12, the numbers of the faces, vertices, and edges of a cube, is taken as the type of harmonic proportion par excellence, since the three fundamental concords of music, the diapason (with the numerical ratio 2 : 1 or 12 : 6), the diatessaron (4 : 3 or 8 : 6), and the diapente (3 : 2 or 12 : 8), are contained in these numbers.

The definition of these three types of proportion as given by Theon presents an interesting variation of the Nicomachean definition. They are:

$$\frac{c-b}{b-a} = \frac{c}{c}, \textit{ arithmetical},$$

$$\frac{c-b}{b-a} = \frac{c}{b}, \textit{ geometrical},$$

$$\frac{c-b}{b-a} = \frac{c}{a}, \textit{ harmonic}.$$

There may also be noticed at this point the series of propositions introduced by Euclid in Book VII, in which he applies to arithmetic the principles of proportion already stated in Book V. They are as follows:

$$a = \frac{a}{b} \cdot b; \qquad \textit{VII. 4}$$

$$\textit{If } a = \frac{1}{n}\,b, \textit{ and } c = \frac{1}{n}\,d, \textit{ then } a \pm c = \frac{1}{n}\,(b \pm d); \qquad \textit{VII. 5, 7}$$

$$\textit{If } a = \frac{m}{n}\,b, \textit{ and } c = \frac{m}{n}\,d, \textit{ then } a \pm c = \frac{m}{n}\,(b \pm d); \qquad \textit{VII. 6, 8}$$

$$\textit{If } a = \frac{1}{n}\,b, \textit{ and } c = \frac{1}{n}\,d, \frac{a}{c} = \frac{b}{d}; \qquad \textit{VII. 9}$$

If $a = \frac{m}{n}b$, *and* $c = \frac{m}{n}d$, $\frac{a}{c} = \frac{b}{d}$; VII. 10

If $a > c$, *and* $b > d$, *and* $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a-c}{b-d} = \frac{a}{b}$; VII. 11

If $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'}, \cdots$, *then* $\frac{a+b+c+d+\cdots}{a'+b'+c'+d'+\cdots} = \frac{a}{a'}$; VII. 12

If $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a}{c} = \frac{b}{d}$; VII. 13

If $\frac{a}{b} = \frac{a'}{b'}$, *and* $\frac{b}{c} = \frac{b'}{c'}$, *and* $\frac{c}{d} = \frac{c'}{d'}, \cdots$, *then* $\frac{a}{d} = \frac{a'}{d'}$; VII. 14

If $\frac{1}{a} = \frac{m}{ma}$, *then* $\frac{a}{m} = \frac{a}{ma}$; VII. 15

$a \cdot b = b \cdot a$. VII. 16

Given a, b, c,

$\frac{b}{c} = \frac{ab}{ac}$, *and* $\frac{b}{c} = \frac{b \cdot a}{c \cdot a}$; VII. 17, 18

If $\frac{a}{b} = \frac{c}{d}$, *then* $ad = bc$, *and conversely.* VII. 19

Another very important theorem of Euclid dealing with proportions is IX. 35, in which he stops just short of determining the formula for the summation of a geometric series. If the series be stated as

$$a, ar, ar^2, ar^3, ar^4, \cdots ar^{n-1}, ar^n,$$

giving $(n + 1)$ terms, and if we take S_n as the sum of n terms, this proposition proves that

$$\frac{ar - a}{a} = \frac{ar^n - a}{S_n},$$

or, in effect, that

$$S_n = \frac{ar^n - a}{r - 1}.$$

Archimedes gives the 'sum to infinity' of a geometric series with ratio 1 : 4, but Euclid nowhere takes up this type of arithmetical discussion. Euclid utilizes IX. 35 for proving his proposition in IX. 36 concerning the perfect number, showing that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.

Both Nicomachus and Theon give rules for inserting the different kinds of means between two numbers; their methods differ but slightly. Those of Nicomachus are as follows:

For the arithmetical mean, x must be chosen so that

$$x - a = b - x.$$

For the harmonical mean, x must be chosen so that

$$b - x = k \cdot b,$$

and
$$x - a = k \cdot a.$$

For the geometrical mean, x must be chosen so that

$$\frac{b}{x} = \frac{x}{a}.$$

To choose x scientifically, Nicomachus adds further that for the arithmetical mean

$$x = \frac{a + b}{2},$$

or,
$$x = \frac{b - a}{2}, \text{ wherein } b > a;$$

for the geometric mean,
$$x = \sqrt{ab},$$

or (incorrectly),
$$x = a \cdot \frac{b}{2a};$$

for the harmonic mean,
$$\frac{(b - a)a}{b + a} + a, \text{ where } b > a.$$

Three other proportions are defined as follows (given $c > b > a$):

$$\frac{c}{a} = \frac{b - a}{c - b}, \text{ the fourth type;}$$

$$\frac{b}{a} = \frac{b - a}{c - b}, \text{ the fifth type;}$$

and
$$\frac{c}{b} = \frac{b - a}{c - b}, \text{ the sixth type.}$$

In connection with the fifth type, Nicomachus gives as numerical illustrations 2, 4, 5, and then makes the incorrect assertion that "to this form of proportion it is likewise peculiar that the product of the greater by the middle term is double that of the greater by the lesser term," an accident due to this particular selection of numbers.

For the remaining four proportions, Nicomachus gives only the definitions and therefore avoids further blunder; they involve no mathematical points. In the closing chapter, however, the 'most perfect of all proportions' is presented, that is, such a series of four numbers a, b, c, d that

$$a:b=c:d,$$

while at the same time a, c, and d form an arithmetical progression and a, b, and d form a harmonical progression. Nicomachus refers to the possibility of the end terms being cubes, but if this be the case, it would lead to irrational means; and again, he does not note, and apparently is not aware of the fact, that two of the given conditions are sufficient to determine the third. For if we are given

$$\frac{a}{b}=\frac{c}{d}$$

and $$d-c=c-a,$$

it follows that $$d-\frac{da}{b}=\frac{da}{b}-a \text{ (substituting for } c \text{ its value)},$$

or that $$\frac{d}{a}=\frac{d-b}{b-a},$$

whence, by definition, d, b, and a form a harmonical progression.

This concludes our review of the content of the Greek *arithmetica*, from the mathematical point of view. There can be no doubt that if Nicomachus is to be judged, as a mathematician, simply with reference to his *Introduction to Arithmetic*, he must yield place to both Euclid and Theon. Euclid himself is far less successful in the portions of the *Elements* which deal with arithmetic than in the strictly geometrical parts, and in fact we should look rather to books like the second, fifth, and twelfth for propositions — geometrically stated, it is true, but related to the field known as 'geometrical algebra' — which are of fundamental importance in the development of analysis. Theon's work is disjointed in its arrangement and far briefer than the *Introduction*, yet in general it shows a deeper appreciation of the mathematical points involved in the subject-matter. For Nicomachus we can only say that, in spite of his mistakes and his philosophical prejudices, he gives the most complete discussion of the matters customarily included in the elements of the Greek arithmetical science.

CHAPTER IV

GREEK ARITHMETICAL NOTATION

THE language of any nation undoubtedly reflects, both in its vocabulary and in its structural peculiarities, the mental development of the people. The relationship is, of course, reciprocal; the mental development influences the language, and the language influences the unfolding of thought. Particularly in philosophy and in literature this type of influence is apparent. The French philosophy and literature are in accord with the genius of the French language, reflecting the clarity and gracefulness of the speech; the German philosophy and literature are in striking contrast, reflecting obscurity, and ponderosity of the German language possibly inevitable in profound philosophical speculation.

In mathematical development the choice of a numerical notation might, at first thought, be supposed to be a matter of indifference so far as progress in scientific thinking is concerned. However, even as language influences the development of philosophy and literature, even more does notation directly affect mathematical progress. The Greeks were unfortunate in their choice of mathematical notation; apparently they realized the deficiency of their early system, for about 500 B.C., a new one was adopted, radically different in principle from the old, but even more awkward from the mathematical point of view.[1] The comparative lack of progress in analysis in Greece may be attributed in a large measure to the clumsy systems of notation which were in use.

The numerals employed in Greece are of three separate and distinct types, viz., the geometrical numerals, initial letter numerals, and alphabetic numerals. The oldest of these are found in the recently discovered Minoan writings, which far antedate the classical Greek

[1] Sir T. L. Heath, *History of Greek Mathematics*, vol. I, pp. 37–39, has recently attempted to show that the Greek notation did not adversely affect their arithmetic. Heath's assertion that we reckon "with words" is not correct; computation in arithmetic and algebra is by symbols. In looking at $\frac{3}{2}$ we do not say "three times two"; we think "6" immediately.

period, and may be assigned to the second millennium B.C. The system employed is more closely related to the pictorial system of the Egyptians than to the systems of later Greece. The forms [1] are as follows:

) or | = 1,	))))) or ||| over || = 5,
• = 10,	
\ or / = 100,	⁘ = 40,
◊ = 1,000,	V = $\frac{1}{4}$ (probably).

The symbols for one and for ten are strikingly like the curvilinear numerals employed in ancient Babylon; the semicircular mark was used there also for 1, and the complete circle for 10.

The second system of numerals adopted in Greece was based upon the initial letters of the corresponding numeral words; but exception was made for the unit, which was represented, as commonly everywhere, by a simple vertical stroke.[2]

Π (or Г)	from πέντε	for	5;	compare *pentagon*.
Δ	from δέκα	for	10;	compare *decagon*.
H	from ἑκατόν	for	100;	compare *hektaliter*.
X	from χίλιοι	for	1,000;	compare *kilogram*.
M (or Mᵘ)	from μύριοι	for	10,000;	compare *myriad*.

Г soon replaced Π for five; combinations of Г with the symbols for 10, 100, 1,000, and 10,000, were used to represent 50, 500, 5,000 and 50,000, thus: ГΔ, ГH, ГX, and ГM.

These numerals occur frequently in Attic inscriptions and have therefore received the name Attic numerals. A description of them was given by a Byzantine grammarian of the second century after Christ, Herodian, and in consequence the designation 'Herodianic numerals' has frequently been employed.

The Attic system was replaced about 500 B.C.–300 B.C. (date uncertain) by another type of alphabetic numerals in which nine letters are used to represent the nine units, nine other letters to represent the nine tens, and nine more letters to represent the nine hundreds, as follows:

[1] Evans, *Scripta Minoa* (Oxford, 1909), p. 258.

[2] Heath, *op. cit.*, vol. I, pp. 29–64; the best and most recent discussion of Greek arithmetic.

α	β	γ	δ	ε	ϝ	ζ	η	θ
1	2	3	4	5	6	7	8	9
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
10	20	30	40	50	60	70	80	90
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
100	200	300	400	500	600	700	800	900
͵α	͵β	͵γ	...					
1,000	2,000	3,000	...					

The ordinary Greek alphabet included only 24 letters, so three characters borrowed from older Greek alphabets were employed to designate the remaining numbers. These characters, called **ἐπίσημα**, are the ones which are used for 6 (**στίγμα** or digamma), for 90 (**κόππα**), and for 900 (**σαμπῖ**). To distinguish from a word a group of letters used with numerical significance, a horizontal line was commonly placed over the numerals; an accent after the final numeral letter of any number was also a distinguishing mark.

To extend the system to numbers from 1,000 to 999,999 a multiplication by 1,000 of any of the numbers above was represented by the corresponding character with an accent mark placed before it and a little below the line; the myriad symbol was retained and with any of the foregoing numerals written above it represented the given number of myriads.

A multiplication example in these numerals is recorded by Eutocius, a writer of the sixth century A.D., in a commentary on the works of Archimedes;[1] this is one of the few even comparatively ancient illustrations of the fundamental operations as performed with Greek numerals:

ψπ'	780
ψπ'	780
$\overset{\mu\theta}{M}\overset{\epsilon}{M}$͵ϛ'	490,000 + 50,000 + 6,000
$\overset{\epsilon}{M}$͵ϛ͵ϛυ'	50,000 + 6,000 + 6,000 + 400
$\overset{\xi}{M}$ηυ'	600,000 + 8,000 + 400

[1] Eutocius, *Commentarius in Dimensionem Circuli*, in *Archimedis Opera Omnia cum Commentariis Eutocii* (ed. Heiberg, 1881), vol. III, p. 290.

The use of letters with numerical significance was common among the Semitic peoples, with whom, indeed, the system may have originated. The Arabs continued to use their letters in this way long after our present Hindu-Arabic system became known to them. According to this scheme of representation of numbers, a numerical value is attached to any word. Sometimes for purposes of secrecy or mysticism a play upon the numerical value of a word or name was indulged in; the Hebrews particularly had a fancy for this kind of juggling, giving it the name gematria. The passage in Revelation, xiii, 18, "Here is wisdom. Let him that hath understanding count the number of the beast: for it is the number of a man: and his number is six hundred three score and six," refers to the fact that some name in Hebrew, or Greek letters or even Roman numerals, has the numerical value 666 (or 616).[1]

Apparently the alphabetic type is simpler than the Attic type of numerals, for such a number as 725 is written more compactly $\overline{\psi\kappa\epsilon}$, than in Attic symbols, 𐅅HHΔΔΓ. However the gain in writing these numbers is entirely overbalanced by the loss of simplicity in the addition and multiplication tables. The connection between 7, 70, and 700 is preserved in the form of writing these numbers in the Attic style: Γ II, 𐅄ΔΔ, 𐅅HH; whereas $\zeta = 7$, $o = 70$, and $\psi = 700$ have no connecting link so far as the representation is concerned. The Greek multiplication table with the alphabetic numerals consisted of 378 combinations; in this system 2×3, 2×30, 2×300, 20×3, 20×30, 20×300, 200×3, 200×30, 200×300 are entirely distinct combinations, as is evident when these products are written as they had to be learned, $\beta \times \gamma$, $\beta \times \lambda$, $\beta \times \tau$, $\kappa \times \gamma$, $\kappa \times \lambda$, $\kappa \times \tau$, $\sigma \times \gamma$, $\sigma \times \lambda$, and $\sigma \times \tau$. The complexity in addition is not quite as great, including only 135 combinations as opposed to 45 with 9 digits. In these facts we probably have one reason for the lack of progress along arithmetical lines as compared with the attainments of the Greeks along other mathematical lines.

The representation of fractions in the alphabetic system was effected in several different ways. Probably the most common was to write the numerator first with one accent and then the denominator repeated, each time with two accent marks; thus $\iota\zeta'\ \kappa\epsilon''\ \kappa\epsilon''$, for $\frac{17}{25}$.

[1] This passage in *Revelation* is now known to be early commentary (second century or even first), not original text. See Sanders, *The Number of the Beast in Revelation, Journal of Biblical Literature*, vol. XXXVII (1918), pp. 95–99.

Another device was to write the denominator as a kind of exponent attached to the numerator, $\iota\zeta^{\kappa\varsigma}$. Unit fractions were represented by the simple denominator written but once and distinguished by an accent, usually doubled. This distinction attaching to the unit fractions accords with the Egyptian practice, and the same may be said of the representation of both $\frac{2}{3}$ and $\frac{1}{2}$ by special symbols, outside the regular system. For $\frac{2}{3}$ the symbol is ω'', and for $\frac{1}{2}$, $C^{\vee\prime}$ or C.[1]

Archimedes (250 B.C.) and Diophantus (250 A.D.) may be mentioned among the Greek mathematicians who did not pay particular attention to unit fractions while Heron of Alexandria, the great mechanician, was among those who did. A Greek papyrus of the eighth century A.D., found at Akhmim, employs the unit fractions quite in the ancient Egyptian manner and includes the separate symbols for $\frac{1}{2}$ and $\frac{2}{3}$.

A further peculiar notation with alphabetic numerals was noted by Maximus Planudes[2] in the thirteenth century, but without any indication of the date and place of origin. This system consists in writing the 27 letters each with two dots superimposed to represent the corresponding number of myriads, thus $\ddot{\beta}$ for 20,000, $\ddot{\gamma}$ for 30,000. The system can be indefinitely extended, and with three tiers of dots, for example, represents myriads of myriads of myriads; thus $\overset{\vdots\vdots}{\iota}$ for 100,000,000,000,000. Nicolas Rhabdas of Smyrna (end of thirteenth century) explains the system and states that it can be extended even to infinity.[3] The suggestion of place value is evident, and a somewhat similar device with superposed and also with subscript dots was used with the Hindu-Arabic numerals by the Arabs;[4] thus, $\ddot{5}$ for 500, $\dot{5}$ for 50.

A great mathematician like Archimedes or Diophantus could largely overcome the difficulties created and inherent in the various Greek mathematical notations. However for the rank and file of those who occupied themselves with mathematical studies, the notations proved an insurmountable barrier to progress in the development of arithmetical and algebraical ideas.

[1] Hultsch, *Metrologicorum Scriptorum Reliquiae* (Leipzig, 1864), vol. I, pp. 173–174.

[2] Gerhardt, *Études Historiques sur l'Arithmétique de Position* (Programm, Berlin, 1856), p. 19.

[3] P. Tannery, *Notices sur les deux Lettres Arithmétiques de Nicolas Rhabdas*, in *Notices et Extraits des Mss. de la Bibliothèque Nationale*, vol. XXXII, p. 147.

[4] Smith-Karpinski, *The Hindu-Arabic Numerals* (Boston, 1911), pp. 66–67.

CHAPTER V

THE LIFE OF NICOMACHUS

HISTORY has been most unkind to Nicomachus of Gerasa, author of the *Introduction to Arithmetic*. During his lifetime he enjoyed, apparently, the highest reputation as a mathematician, and after his death he continued to be studied, directly or indirectly, by generation after generation of schoolboys, yet scarcely a word has come down to us to tell what sort of man he was, or where he lived, and under what circumstances, or even what were the years of his birth and death.

The period of Nicomachus can be fixed, within certain limits, by indirect evidence.[1] Nicomachus does not say much about his contemporaries. He does mention Thrasyllus,[2] a celebrated writer on music who lived under Tiberius, but fails to make reference either to Theon of Smyrna or to Claudius Ptolemy; and we are told that Apuleius of Madaura honored him by translating into Latin his book on arithmetic.[3] Thrasyllus furnishes us with one limit, the reign of Tiberius, and Apuleius, who lived in the time of the Antonines, the other. It seems improbable that Nicomachus would have failed to mention Ptolemy in his book on harmony, if the latter had already attained to fame by the time that he wrote. As to the period of Ptolemy we have more satisfactory data in the form of astronomical observations made and reported by himself. The earliest and latest of these fall in 125 and 151 A.D., respectively.[4] About Theon of Smyrna not enough is

[1] For discussions of Nicomachus's date, cf. Hoche's edition, p. iv, n., where the pertinent citations are collected; Von Jan, *Musici Scriptores Graeci* (Leipzig, 1895), pp. 211 ff.; Nesselmann, *Geschichte der Algebra* (Berlin, 1842), pp. 188 ff.; Zeller, *Die Philosophie der Griechen* (4th ed.), vol. III, part 2, p. 124, n. 3; Heath, *History*, vol. I, pp. 97–112.

[2] *Manuale Harmonicum*, I, p. 24.

[3] Cassiodorus, *De Artibus ac Disciplinis Liberalium Litterarum*, c. IV, *De Arithmetica* (Migne, *Patrologia Latina*, vol. LXX, pp. 1204 ff.), p. 1208 B: . . . *arithmetica disciplina quam apud Graecos Nicomachus diligenter exposuit. Hunc primum Madaurensis Apuleius, deinde magnificus vir Boetius Latino sermone translatum Romanis contulit lectitandum.*

[4] Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. I (3d edition, Leipzig, 1907), p. 415.

known to warrant any conclusions in relation to the date of Nicomachus. Theon probably flourished in the first part of the second century,[1] and his book contains much matter that is parallel to Nicomachus's own work, yet it is unsafe to argue for the dependence of either of these authors upon the other.[2] The inconclusiveness of the *argumentum ex silentio* is proverbial. We are warranted, therefore, merely in saying that the period of Nicomachus's life fell somewhere between the middle of the first century and the middle of the second century after Christ.

In the manuscripts of his works and in the scholia of Johannes Philoponus upon the *Introduction*, Nicomachus is referred to as a Gerasene.[3] The most prominent city having the name Gerasa was located in Palestine, in the Decapolis, some thirty miles southeast of the Lake of Tiberias; it was, therefore, close to the region where Christianity had its birth. This was probably the Gerasa from which Nicomachus came, and upon that supposition a few surmises can be ventured about the environment of his youth. There is a tradition to the effect that Alexander the Great, in the course of his campaigning, left behind at this place a group of his veterans (γέροντες), and that from this circumstance the place got its name. Whatever may be the truth of this story, it suggests the inference that Greeks predominated in this neighborhood.

Not far from Gerasa is placed the episode of the cure of the demoniac who was possessed by the 'Legion,' and the name Gerasa is found in this connection in some of the Biblical manuscripts.[4] At any rate, the swine which are reported to have rushed into the lake were being tended not more than a score of miles from Gerasene territory; the swine were not likely to be a product of a region where Jewish traditions held sway.

[1] Cf. the dates of the astronomical observations probably made by Theon; Cantor, *op. cit.*, vol. I, p. 433.

[2] Von Jan, *op. cit.*, p. 211, holds that Nicomachus refers to Theon in the *Manuale Harmonicum* without calling him by name; he sets the *Manuale* before 170 A.D. and Nicomachus in the middle of the second century.

[3] *Scholia* (ed. Hoche): Γερασηνὸς δὲ λέγεται ἀπὸ τῆς πόλεως ἧς ὄνομα Γέρασα· ἔστι δὲ περὶ Βόστραν καὶ Ἀραβίαν. Γέρασα δὲ λέγεται ἀπὸ τοῦ τοὺς συστρατεύσαντας τῷ Ἀλεξάνδρῳ γέροντας καὶ μὴ δυναμένους πολεμεῖν ἐκεῖ τὴν οἴκησιν ποιήσασθαι.

[4] In Mark, v. 1–20, the episode is referred to Gerasene territory, but modern critics identify the Gerasa in question with a place called Gersa, or Khersa, on the shore of the lake. Cf. Matthew, viii. 28–34; Luke, viii. 26–39. Westcott and Hort read Γερασηνῶν in Mark and Luke, Γαδαρηνῶν in Matthew; but the manuscripts of Matthew and Luke vary, giving both these readings and Γεργεσηνῶν besides.

The remains of the city, moreover, monumental and written, attest a Greek civilization strongly colored by the Roman influence that came with the tightening of Rome's grasp upon the Near East. Most of the inscriptions from the place are in Greek; and there are, besides remains of extensive walls, an imposing Roman arch, a circus, a theater and a *naumachia* of a provincial city of considerable importance, which without doubt enjoyed a varied and lively existence, including all that the Greco-Roman civilization had to offer.[1] Furthermore, the name Nicomachus (Νικόμαχος) is pure Greek.

We may imagine that Nicomachus spent his early years in Gerasa and attended the school of the grammar-master, where he would learn to write, read and sing; to figure a little, and to enjoy to some extent the works of the classical authors. What was the social position of his family it is impossible to say, but there is no suggestion in his writings to warrant the conclusion that his surroundings in life were other than comfortable. It is interesting to note that among the Gerasene inscriptions the name Nicomachus (spelled Νεικόμαχος) occurs thrice, and in each case it would appear that the bearer was a person of affluence.

The Nicomachus who comes nearest in time to the author of the *Introduction* is the father of a certain Demetrius, who, either in the year 149 A.D., or 255 A.D., according to the era upon which the date is to be based, set up an altar bearing an inscription in honor of two Augusti — Antoninus Pius and Marcus Aurelius, or Valerian and Gallienus, as the case may be.[2] The other Nicomachus, with the surname Claudius, was a commissioner (ἐπιμελητής) of the city in 231 A.D.[3] In view of the Greek custom of naming a child from its grandfather, there is of course a possibility that these men may have been related to Nicomachus the mathematician, but the name was common and the chances are slight. Whether his family was engaged in trade, and sent goods to Rome or to Alexandria or into the East, or was connected with officialdom, local or imperial, or belonged to the class of landed proprietors, they seem to have been able to give to the young Nicomachus the best sort of training obtainable for the career he chose. As

[1] For a description of the present remains and citation of passages dealing with Gerasa, see Pauly-Wissowa, *Real-Encyclopädie*, *s.v.* Gerasa.

[2] *Inscriptiones Graecae ad Res Romanas Pertinentes*, vol. III, No. 1343: ἔτους βισʹ Δαισίου / αʹ. ὑπὲρ τῆς τῶν / Σεβαστῶν σωτηρίας / θεῷ Αραβικῷ ἐπηκόῳ / Δημήτριος Ἀλκίου / τοῦ καὶ Νεικομάχου / τὸν βωμὸν ἀνέθη / κεν.

[3] *Ibid.*, Nos. 1360–61, referring to the same person.

Iamblichus says, "The man is great in mathematics and as instructors therein had those that were most skilled in the subject."[1]

It is hardly to be assumed that Nicomachus could find in Gerasa itself all the advantages which he seems to have enjoyed. His home city was of course responsive to those influences which in those times went out from the centers of culture, — Athens, Rhodes, Tarsus, and Alexandria, among the nearest. Rome had made all parts of the empire easily accessible; knowledge of what was being done in these university cities must have been current in the Hellenistic towns of the East, and it is easy to see how a desire may have been aroused in Nicomachus to devote himself to the study of philosophy.

In regard to the course of his studies, we know nothing of a certainty, but the balance of probability points to Alexandria as the place to which Nicomachus would naturally go to acquire the training which he sought. The choice of that center of learning would also explain the type of his thinking, for in the first century after Christ Alexandria was the most famous seat of Pythagoreanism in the world. There the old doctrines were being revived, and new treatises were being put in circulation under old names; in Alexandria, in short, the Neo-Pythagorean movement received, if not its initial impulse, at least its chief encouragement.

The claim of Alexandria to be the real birthplace and center of this philosophical movement can be disputed only by Rome; but a brief survey of the early history of Neo-Pythagoreanism is enough to prove that the Egyptian capital has by far the better right to the distinction. There are, to be sure, scattered literary references to Pythagoreans in other parts of the world during the centuries following the disaster in Magna Graecia which brought the independent existence of the school to an end; the New Comedy took as a butt those who followed 'the Pythagorean way of life.'[2] There were too traces of a Pythagorean tradition in southern Italy,[3] but the former certainly counted for nothing in a philosophical way, and the latter neither deserve to rank with the greater sects contemporary with them nor form a necessary link in the chain connecting the renascent Pythagoreanism with

[1] *In Nicomachi Arithmeticam Introductionem*, p. 4, 14 (ed. Pistelli, Leipzig, 1894): ὅ τε γὰρ ἀνὴρ μέγας ἐστὶν ἐν τοῖς μαθήμασι καὶ καθηγεμόνας ἔσχε περὶ αὐτῶν τοὺς ἐμπειροτάτους ἐν τοῖς μαθήμασι.

[2] See Zeller, *op. cit.*, vol. III, part 2, pp. 93 ff.

[3] *Ibid.*, pp. 97 ff. The interest of the Platonic schools and of Aristotle himself was of far more weight in preserving Pythagorean doctrine than the influence of the obscure men who professed to be Pythagoreans; see p. 88.

the old. The first prominent person in the later time to be called definitely a Pythagorean is the Roman, Nigidius Figulus.[1] There is also a dim tradition about the Sextii at Rome, who were Pythagoreans,[2] but the meager reports about them indicate that they made little of the essential features of the doctrine, much more of its external observances.

Apparently even before the interest at Rome had been awakened, the renaissance at Alexandria [3] had started. Inasmuch as the beginnings of the movement cannot be connected with the names of noble Romans like Figulus, we do not know much about it, but many facts indicate how important this city was as a seat of Neo-Pythagoreanism. With Alexandria is probably to be connected the unknown Pythagorean of the first century before Christ quoted by Alexander Polyhistor, who is one of our best sources of information. Alexandria was the home of Arius and Eudorus, who are connected with the early history of the movement, and of Sotion; in its neighborhood were established the Therapeutae, who built up a system highly colored by Pythagoreanism; and perhaps the best evidence of all is that the Alexandrian philosophy of Philo Judaeus, who flourished early in the first century of the Christian era, is deeply influenced by it.

Very probably many of the pseudonymous writings, of which Zeller has collected a long list,[4] and the period of whose composition he would make begin with the last half of the first century before Christ, originated in Alexandria. At no place in the ancient world could a compiler of such books have found a better place in which to work; for here was the famous library, part of which was burned in 47 B.C., but which had been restored before Nicomachus's time through the liberality of Antony and the transfer hither of the royal Pergamene collection. All the written material on the early history of Pythagoreanism must have been available to the scholars working here, and Nicomachus in his time would find a complete apparatus at his disposal.

Furthermore, it is absolutely certain that Alexandria had been for a long time, and still was, the center of Greek mathematical interests.[5] Nearly all of the famous mathematicians who lived after the date of the founding of the city are associated in one way or another with

[1] Zeller, *ibid.*, p. 109.

[2] Chaignet, *Pythagore et la Philosophie Pythagoricienne* (Paris, 1873), vol. II, p. 301, n. 1.

[3] On the importance of Alexandria in the history of Neo-Pythagoreanism, see Zeller, *op. cit.*, vol. III, part 2, p. 114; Chaignet, *op. cit.*, p. 301.

[4] *Ibid.*, p. 115, n. 3.

[5] See Cantor, *op. cit.*, vol. I, p. 427.

Alexandria — Euclid, Eratosthenes, Apollonius of Pergae, and Heron; and their successors were continually offering instruction in the science.

Thus, the center at once of later Pythagoreanism and of mathematical study, Alexandria clearly furnished the most fitting environment for the training and subsequent career of Nicomachus, but if we picture him to ourselves as having lived and written there, it must not be forgotten that we have no positive testimony to that effect. Perhaps it is significant, however, that he is once referred to as an authority upon Egyptian festivals,[1] but, as we shall see, we cannot be sure that our Nicomachus is meant.

One bit of evidence regarding the life of Nicomachus he himself gives. His *Manual of Harmony* is dedicated to an unknown lady, apparently of high degree, having been written at her request. Seeing that it is the only autobiographical touch in all his extant works, it will be worth while to quote the passage. In speaking of the treatise that he is presenting, he says, "But I must spur on all my zeal, most noble and august lady, since it is you that bid me."[2] Again, modestly depreciating the *Manual* in favor of the more scholarly work which he plans later to substitute for it, he writes, "And, if the gods are willing, just as soon as I shall have leisure and a rest from my journeyings, I will compile for you a better and more detailed *Introduction* dealing with this very subject; . . . and, so that you may the more easily follow the argument, I will take my beginning, say, from the same point as that at which I began your instruction when I was expounding the subject to you."[3]

How much light would be thrown on the life of Nicomachus if we but knew the name of this lady! We cannot tell whether she was Greek or Roman. Was Nicomachus but a 'Greekling,' a household philosopher to noble dames, a holder of pet poodles, like the butts of Lucian's satire?[4] Such a supposition makes but a sorry figure of him; it condemns him to a career of humiliation of which there is no suggestion in the tone of his writings. The constant mention therein of matters that arise in the student's reading of the philosophers — that is, in the ordinary course of higher studies — seems to indicate that Nicomachus was engaged in educational work, and ought to

[1] See p. 80.

[2] P. 237, 15 (Von Jan's edition).

[3] *Ibid.*, p. 238, 6 ff. Other references to the lady and the promise to her, *ibid.*, pp. 242, 11 ff.; 260, 4 ff.; 261, 17; 264, 1.

[4] *De Mercede Conductis.*

outweigh any disparaging inferences that might be made from these references to the unknown lady.

At the same time the references show that he associated with members of the upper classes of society and could adopt the attitude of a man of the world. Nicomachus was called upon to give instruction to the noble lady to whom he writes, and valued her good will so highly as to compose a manual at her request. It would not be impossible for him to do this and at the same time to maintain his more dignified position as an independent teacher of the higher subjects. A Pythagorean would be more interesting, perhaps, to the nobility than other philosophers; but we must not assume that the only motive of Nicomachus's noble friend was mere curiosity. She seems to have been serious in her inquiries; in order to use the proffered *Manual* she would have to possess no mean knowledge of the theories of mathematics and music.

Nicomachus speaks also of journeys which necessitate a postponement of work on another and larger *Harmonic Introduction*. His language implies a journey which he was obliged to make, the object of which was known to his correspondent. He was, we infer, a busy man; others besides those in his own neighborhood, perhaps, wished to hear him lecture, although we know nothing of the errand that put him to the inconvenience of travel. At any rate we conclude that he was a man of affairs, and of some eminence, befriended by the mighty; he was a man, too, who knew how to play his part successfully in such a character, and was not content merely to cultivate learning in a corner. Lucian's remark, "You reckon like Nicomachus!" shows that he did in fact achieve such fame that his name was synonymous with mathematical skill.[1]

We have a bit of interesting testimony to the reputation of Nicomachus after his death which may reflect also upon the fame that he won in his lifetime. Proclus, who died 485 A.D., is said to have been convinced that he himself was one of the 'golden chain,' or succession of true philosophers, who, as it were, connected men with heaven after the fashion of the golden chain which Homer mentions. It was revealed to him in a dream, we are told, that the soul of Nicomachus was incarnate in him.[2] This is a clear implication that Nicomachus too

[1] *Philopatris*, 12: καὶ γὰρ ἀριθμέεις ὡς Νικόμαχος ὁ Γερασηνός.

[2] Marinus, *Vita Procli*, 28: ὅτι τῆς Ἑρμαϊκῆς εἴη σειρᾶς σαφῶς ἐθεάσατο (*sc.* Proclus) καὶ ὅτι τὴν Νικομάχου τοῦ Πυθαγορείου ψυχὴν ἔχοι ὄναρ ποτὲ ἐπίστευσεν.

was regarded as a link in the 'golden chain.' That he was reckoned among the 'illustrious men' of the Pythagorean sect we know on the authority of Porphyry;[1] if any further evidence be needed, we have but to point to the reputation borne by his works and to the number of commentaries that scholars wrote upon them.

[1] Quoted by Eusebius, *Historia Ecclesiae*, VI. 9. 8: *συνῆν τε* (*sc. Origen*) *γὰρ ἀεὶ τῷ Πλάτωνι τοῖς τε Νουμηνίου καὶ Κρονίου, Ἀπολλοφάνους τε καὶ Λογγίνου καὶ Μοδεράτου Νικομάχου τε καὶ τῶν ἐν τοῖς Πυθαγορείοις ἐλλογίμων ἀνδρῶν ὡμίλει συγγράμμασιν.*

The praise of Isidore of Seville (*Etymologiae*, III. 2. 1), by whom Nicomachus is coupled with Pythagoras himself as a mathematician, shows the view of a later time. See also Cassiodorus as cited, p. 71.

CHAPTER VI

THE WORKS OF NICOMACHUS

THE fame of Nicomachus rests chiefly upon his writings. Only two works by him are preserved to us in their entirety, the *Manuale Harmonicum*[1] and the *Introduction to Arithmetic*.[2] Of a third, the *Theologumena Arithmeticae*,[3] we have a large part, which gives us a far more accurate judgment of Nicomachus's philosophy than we should otherwise have had; this work, as it stands, is one of the best sources of information about Neo-Pythagoreanism.

Besides these three works, Nicomachus was certainly the author of several other books; modern scholars have credited him with the authorship of additional works which he probably did not write. A full list of them, including both those properly ascribed to him and those that are either doubtfully or wrongly assigned to him, is as follows:

1. An *Introduction to Geometry* (Γεωμετρικὴ εἰσαγωγή). — Nicomachus certainly wrote a book with this title, for he refers to it in his *Introduction to Arithmetic*.[4]

2. A *Life of Pythagoras*. — Nicomachus is quoted by both Porphyry and Iamblichus, in their biographies of Pythagoras,[5] and, as a prominent member of the sect, he is likely to have compiled a life of the master.

3. Another and larger work on music. This would be the book which, as we have seen, Nicomachus promised to write; because of certain citations of Nicomachus by musical writers upon matters that are not to be found in the present *Manuale Harmonicum*, Von Jan, its latest editor, thinks it probable that this work was actually written.[6]

[1] Here cited in Von Jan's edition, *Musici Scriptores Graeci*, Leipzig, 1895.

[2] Here cited in Hoche's edition, Leipzig, 1866.

[3] A fuller discussion of this work follows; see pp. 82 ff.

[4] See II. 6. 1.

[5] Porphyry, *Vita Pythagorae*, 20 (p. 27, 3); 59 (p. 50, 12). In the second instance (the story of Damon and Phintias), he says that Nicomachus followed Aristoxenus. Iamblichus, *Vita Pythagorica*, XXXV, 251.

[6] Von Jan, *op. cit.*, pp. 223 ff., collects the evidence on the matter.

4. A work on the interpretation of Plato (Πλατωνικὴ συνανάγνωσις). This and the following are more doubtful than those that have already been mentioned. The evidence in the present case consists simply of Nicomachus's words in the *Introduction*, II. 24. 11: ταῦτα δὲ τῆς οἰκείας σαφηνείας ἐπιλήψεται ἐν τῇ Πλατωνικῇ συναναγνώσει κατὰ τὸν τοῦ λεγομένου γάμου τόπον ἐν τῇ Πολιτείᾳ ἀπὸ προσώπου τῶν Μουσῶν παρεισαγομένου. This might be taken to refer to the title of a book; but it more likely means simply the school lectures on Plato, and in the lack of further evidence is best so taken.

5. *On Egyptian Festivals* (Περὶ ἑορτῶν Αἰγυπτίων). — The only evidence for this is a citation in Athenaeus, beginning: "And Nicomachus says in the first book *On Egyptian Festivals*."[1] The objection is, of course, that another Nicomachus may have written the book.[2] There is no other reference to it, save a doubtful one in Johannes Laurentius Lydus, *De Mensibus*.[3]

6. A larger work on arithmetic,[4] besides the *Introduction* and the *Theologumena Arithmeticae*. Only by the misconception of the meaning of Iamblichus has such a book been attributed to Nicomachus. At the beginning of his commentary Iamblichus states that he will not write a new book on arithmetic because he finds that this has already been done so well by Nicomachus in his *Art of Arithmetic* (ἀριθμητικὴ τέχνη).[5] As he continues, he draws freely upon the *Introduction* and the *Theologumena*, but not, as far as can be judged, from any other Nicomachean source. It is perfectly clear that in these words he has referred to the *Introduction* by a somewhat unusual title, but one which could fairly be given to it. Those who have used this passage as testimony for the existence of a separate work with this title have been misled.

[1] *Deipnosophistae*, XI. 55, 478 A: Νικόμαχος δ' ἐν πρώτῳ περὶ ἑορτῶν Αἰγυπτίων φησί· Τὸ δὲ κόνδυ ἐστὶ μὲν Περσικόν, τὴν δὲ ἀρχὴν ἣν Ἕρμιππος ἀστρολογικὸς ὡς ὁ κόσμος ἐξ οὗ τῶν θεῶν τὰ θαύματα καὶ τὰ καρπώσιμα γίνεσθαι ἐκ γῆς· διὸ ἐκ τούτου σπένδεσθαι. The passage as it stands seems very confused; the editor, Kaibel, says of it, "non intellego."

[2] Von Jan, *op. cit.*, p. 233, finds it uncertain.

[3] IV, 46. Roether (in his edition, Leipzig, 1827), *ad loc.*, suggests that the citation in Lydus is to this work; it may equally well be, however, that Lydus was quoting the *Theologumena Arithmeticae*, of which he certainly made use in *De Mensibus*, III. 51.

[4] Ast (*Theologumena Arithmeticae*, pp. 160, 205), Fabricius, *Bibliotheca Graeca*, vol. III, p. 632, and others named by Nesselmann, *op. cit.*, p. 217, n. 53, believe in the existence of such a work, but Nesselmann clearly exposes the error. Cf. also Von Jan, *op. cit.*, p. 232.

[5] P. 4, 12 ff. (ed. Pistelli): εὑρίσκομεν δὴ πάντα κατὰ γνώμην τῷ Πυθαγόρᾳ τὸν Νικόμαχον περὶ αὐτῆς ἀποδεδωκότα ἐν τῇ ἀριθμητικῇ τέχνῃ. Note also the scholium of Philoponus on the title of the *Introduction*, εἰσαγωγὴ ἐπιγέγραπται ὡς πρὸς τὰ γεγραμμένα αὐτῷ θεολογικὰ ἤτοι μεγάλα ἀριθμητικά. This refers, of course, to the *Theologumena Arithmeticae*.

7. A *Life of Apollonius of Tyana*.[1] — This also may be dismissed for lack of evidence; the only testimony regarding it has been misinterpreted. Sidonius Apollinaris says, "Since you urged, I have sent you the life of the Pythagorean Apollonius, not in the form in which the elder Nicomachus wrote it down from the account of Philostratus, but as Tascius Victorianus set it forth from the outline of Nicomachus."[2] Now, as Philostratus certainly lived long after Nicomachus, the 'elder Nicomachus' who is mentioned cannot be the one in whom we are at present interested; and it is hardly possible that the name Nicomachus, twice used in the passage quoted, does not refer to the same person each time. Even if the Nicomachus of the second reference is not the same as the first, there is no reason to identify him with the Gerasene.

8. A work dealing with astronomy.[3] The evidence for such a book is inconclusive, consisting merely of the following remark of Simplicius: ". . . unless the hypothesis of eccentric circles was devised by the Pythagoreans, as not only certain others recount, but also Nicomachus, and Iamblichus following Nicomachus."[4] A statement such as this could easily have found a place in the life of Pythagoras which Nicomachus seems to have written; but in case he did write an astronomical work, it fills up for him a series of four introductions to the four mathematical sciences, arithmetic, geometry, astronomy, and music, corresponding to the four divisions of the book of Theon of Smyrna, and doubtless designed, as was the latter, for the use of students about to begin their higher studies. Nicomachus certainly called his books on arithmetic, geometry, and music 'introductions,'[5] and so the title of this work, if it existed, was probably *Introduction to Astronomy*. Perhaps, as Zeller[6] supposes, the books written by Nicomachus were all parts of what was called "Collection [or Collections] of Pythagorean Opinions."

In the further consideration of Nicomachus we shall not find it necessary to make frequent reference to the *Manuale Harmonicum*,

[1] Von Jan, *op. cit.*, p. 234, strangely accepts this as Nicomachean.

[2] VIII. 3: *Apollonii Pythagorici vitam non ut Nicomachus senior e Philostrati, sed ut Tascius Victorianus e Nicomachi schedio exscripsit, quia iusseras, misi.*

[3] Von Jan, *op. cit.*, p. 223, accepts this as Nicomachean on the evidence stated.

[4] *In Aristotelis de Caelo*, p. 227, Karsten (= *Scholia Bekkeri*, p. 503 b): εἰ μὴ ἄρα ἡ τῶν ἐκκέντρων κύκλων ὑπόθεσις ὑπὸ τῶν Πυθαγορείων ἐπενοήθη, ὡς ἄλλοι τέ τινες ἱστοροῦσι καὶ Νικόμαχος καὶ Νικομάχῳ κατακολουθῶν Ἰάμβλιχος.

[5] He applied the term εἰσαγωγή to the larger work on music which he says he intended to write. See *Manuale Harmonicum*, p. 238, 6 ff.

[6] *Op. cit.*, vol. III, part 2, p. 124, note 2.

but it is very important to take into account the theories advanced in the *Theologumena Arithmeticae*, where there is found a treatment of number very different from that of the *Introduction*. We must, therefore, determine how far the *Theologumena Arithmeticae*, in its present form, may be used as valid evidence for Nicomachus's views. A full discussion of this question has never been presented, nor can such a discussion be attempted here. It would involve a careful comparison of Iamblichus's citations of the book, in his commentary on the *Introduction*, and a study of the sources, in addition to a comparison of Photius's epitome of the *Theologumena* with the longer form of the book. For the purposes of this study, however, probably the latter alone will yield sufficiently conclusive results.

The title *Theologumena Arithmeticae*,[1] as is well known, is today applied to two different ancient works, that of Nicomachus, and an anonymous treatise published in 1817 by Ast, and believed by many to have been compiled by Iamblichus, an assumption which is on the whole a likely one.[2] The work of Nicomachus is not known to us at first hand, but we have an epitome of its contents by Photius,[3] and Ast's *Theologumena* was in part derived from it, as may be deduced from stylistic evidence, from the actual citation of Nicomachus in it, and from comparison with Photius. Photius's account is summary in the extreme; for the most part he simply lists the epithets which the Pythagoreans applied to the numbers; and he seldom adds the reasons why they were bestowed. Ast's text contains much more material of the latter sort, and if it can safely be used for evidence, it is plainly desirable to do so.

Ast himself compared the epitome given by Photius with the text which he was editing, and stated the conclusion (p. 157) that his *Theologumena* was a different work from that of Nicomachus. "Besides," he says, "Photius took much from Nicomachus's *Theologumena* which you will seek here in vain; and there is the additional fact that in this *Theologumena* Nicomachus's *Introduction to Arithmetic* and his own *Theologumena* are often cited (as c. I, p. 4, 23; c. X, p. 42, 8). So without doubt a philosopher of the later time compiled our *Theologumena*, taking from the mathematical works of Nicomachus, Anatolius (cf. c. II, p. 7, 7; III, p. 14, 22; VI, p. 33; VII, p. 41, 7; X, p. 63, 23),

[1] I have adopted this form of the Latin title, rather than *Theologumena Arithmetica*, as more correctly representing the Greek, Θεολογούμενα τῆς ἀριθμητικῆς.

[2] Zeller holds this view.

[3] *Bibliotheca*, *Codex* 187 (edition of I. Bekker, Berlin, 1824).

and others, whatever seemed useful to him." He then names a number of other scholars who held a somewhat similar opinion, some of them inclining to assign the compilation to Iamblichus. From a statement of Iamblichus himself we know that he had in mind the composition of such a work.[1]

Ast's opinion seems, in substance, to have been adopted by most scholars. Yet all that he says may easily be true and at the same time the text which he published may be for the most part of Nicomachean origin. If Iamblichus, as is very probable, was the compiler of the *Theologumena*, we need only to inspect his *Commentary on the Introduction of Nicomachus* to be assured that in all likelihood he would not do violence to his source. Although in the *Commentary* he has added a considerable amount to the Nicomachean original, he contradicts it in nothing and adds nothing essential of his own. Some of the additions, in fact, are apparently from the *Theologumena Arithmeticae*, and everything combines to show the author's respect for Nicomachus. He makes it clear, in fact, that he does not intend to depart from the model he has selected, because he does not think it right to deprive Nicomachus of the honors he has won as an arithmetician and because he himself could not in any case do better independently.[2] The same motives would determine his course in making a compilation of the *Theologumena*.

However much truth may be at the basis of Ast's remarks about the *Theologumena Arithmeticae*, we may still see in the treatise a compilation based almost entirely upon Nicomachus, aside from the obvious citations of Anatolius. For if some of the epithets of Photius are not found in Ast's text, this is sufficiently accounted for by the fact that the latter has been both edited and abbreviated; conversely, if Ast's text preserves something not mentioned in Photius, it is to be remembered that Photius is even more summary. The writer holds the view that the *Theologumena* of Ast is in fact mainly Nicomachean in origin, and that it may be used, with due discretion, to illustrate his opinions. It may be well to set forth briefly the grounds for such a belief. A laborious study of the question would doubtless correct certain details, but the following statement seems to be supported by the evidence.

[1] *Commentary on the Introduction of Nicomachus*, p. 118, 15 (Pistelli): ὅταν καὶ τῶν ἄλλων ἀπὸ μονάδος μέχρις αὐτῆς ἀριθμῶν ἑκάστου ἐπανθήματα εὐθὺς ἑξῆς μετὰ τήνδε τὴν εἰσαγωγὴν δεικνύωμεν.

[2] *Ibid.*, p. 5, 13 ff. (Pistelli).

Ast's *Theologumena Arithmeticae* falls into ten chapters, each dealing with one of the numbers in the first decade; although Nicomachus's treatise consisted of two books, according to Photius, it may have followed much the same arrangement. The only two sources of the compiler actually known are Nicomachus and Anatolius, if the numerous slight citations are not taken into consideration; for the latter were probably found already quoted in the original sources. Since we possess the full text of Anatolius's *On the Decad and the Numbers within It*,[1] it is a simple matter to extract from the whole the portions that were taken from him. Although the compiler omitted much of the Anatolian material at his disposal, probably because he preferred to incorporate the same topics from other sources, and though he has varied its order and phrasing somewhat and made a few additions, there is on the whole no reason to think that he used a text substantially different from the one published by Heiberg. The Anatolian sections of the *Theologumena Arithmeticae* are thus distributed:[2]

I C and E, p. 6: ὅτι Ἀνατόλιος . . . μηδενὸς ἀριθμοῦ; the intervening sentence is not in Heiberg's text; then ὅτι τὴν μονάδα . . . τοὺς ἐν αὐτῷ ἀριθμούς.
II A, p. 7: beginning of chapter to λόγον τὸν ἀναλογίᾳ.
III B, p. 14: as indicated in Ast's text.
IV C, p. 23: καλεῖται δὲ αὐτή to end.
V A, p. 24: beginning of chapter to ὡς δηλοῖ τὸ διάγραμμα.
VI A, p. 33: beginning of chapter to διαστάσεις σωμάτων ἕξ.
VII A, p. 41–42: as indicated in Ast's text.
VIII B, pp. 55–56: as indicated in Ast, to Ἐρατοσθένης φησίν.
IX B, p. 58: ἐννεὰς ἀπὸ περισσοῦ to end.
X C, pp. 63–64: as indicated in Ast's text.

Of the rest, a great part is undoubtedly Nicomachean. In certain sections, either because there is correspondence with the epithets cited by Photius from Nicomachus, or because Nicomachus is actually named, there can be no question. The passages most clearly Nicomachean are the following:

I B: p. 4, καὶ ὅτι τὸν θεόν to the first Anatolian citation.[3]

[1] See p. 90, n. 8.

[2] In the following tabulation, for convenience, the sections of Ast's text are in each case referred to by a Roman numeral showing the chapter (monad, dyad, etc.) and a letter indicating the order of the sections.

[3] Beginning at this point, the first 11 epithets given by Photius occur in order; the other 18 are not mentioned. Nicomachus is cited at the beginning and there is no sign of internal breaks.

II D: p. 8, ὅτι δυὰς λέγεται to end.[1]
III A: p. 12, beginning of chapter to the Anatolian section, p. 14.
III C: p. 15, top to p. 16, end of chapter.[2]
IV B: p. 22, ὅτι Αἰόλου φύσιν to the Anatolian matter.[3]
V B: p. 24, end of Anatolian section to the end of the chapter.[4]
VI B: p. 33, after the Anatolian section, to p. 38, ὅτι ἑπτὰ τῶν σφαιρῶν κτλ.[5]
VII B: p. 42, after the Anatolian section, to p. 48, ὅτι καὶ τὰ σπέρματα κτλ.[6]
VIII A: p. 54, beginning, to the Anatolian section, p. 55.[7]
IX A: p. 56, beginning, to the Anatolian section, p. 58.[8]
X A: p. 58, beginning, to the reference to Speusippus, p. 61.[9]

There can be little question of referring the foregoing portions to Nicomachus. This leaves in question the following:

I A: p. 3, beginning, to the first Nicomachean section, p. 4.
I D: a few lines between the Anatolian sections on p. 6.
I F: p. 7, end of the Anatolian section to end of the chapter.[10]
II B: δύο γὰρ ἀριθμῶν σχέσις πανσχήμων ἐστίν, p. 8, at the end of the Anatolian section.
II C: ἀπολείπεται δή κτλ., p. 8, to the Nicomachean section, p. 8.
IV A: p. 16, beginning, to p. 22, the Nicomachean section.
VI C: p. 38, after the Nicomachean section, to end of chapter.
VII C: p. 48, after the Nicomachean section, to end of chapter.
VIII C: p. 56, ἀρχὴ τῶν μουσικῶν λόγων, κτλ. to end of chapter.
X B: p. 61, ὅτι καὶ Σπεύσιππος . . . to the Anatolian section.

[1] Parallels Photius; the 27th of his 51 epithets for the dyad is the last cited, and, of the first 27, 9 are omitted. This shows that there was much abbreviation.

[2] Both these sections parallel Photius throughout (citing 12 of 50 epithets), but the order is varied. About half of Photius's material relates to divinities and all such has been neglected. Nicomachus's name heads Section C.

[3] A short section, but with many parallels to Photius.

[4] Of the epithets in Photius the first 15, and the 19th, 21st, 23rd, 24th and 25th are given. With the mention of the 5 elements (p. 25; also in Photius) begins a list of pentadic groups in nature with another parallel with Photius (p. 26 bottom, ἀνεικίαν) at the end. The following passage on 5 as Justice is confirmed for Nicomachus both by Photius and by its use in Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 16, 11 ff. Thence the parallelism with Photius continues, and the last paragraph is headed by Nicomachus's name. This is probably quoted verbatim, the rest summarized.

[5] Of the epithets in Photius, Nos. 1, 3–6, 12, 14, 15, 18–20, 24–27 are given.

[6] Caption contains Nicomachus's name; no sign of breaks. The first two epithets of Photius occur (p. 43, bottom) and on p. 44 the next two, τύχη, καιρός; after which the demonstration that 7 is effective in the working of the world and on human life (justifying καιρός) does not afford quotable epithets to Photius.

[7] Out of Photius's 17 epithets this contains Nos. 1–3, 9–12, 16–17.

[8] Out of the 22 epithets of Photius, the 1st–4th, 7th–11th, 13th, and 20th–22nd occur. There is also a reference at the beginning of the section to the topic of 5 as Justice already met.

[9] Photius gives but 16 epithets; the 1st and 3rd–16th occur in the passage.

[10] Chapter II probably should begin here.

Of the passages cited above, I D and F and II B are very short, and VIII C, from its character, is unimportant for the present purpose. The same may be said of X B, the quotation from Speusippus; it makes very little difference whether Nicomachus had already cited the passage or the compiler found it elsewhere. The other more important sections, I A, II C, IV A, VI C, VII C, are set down as doubtful chiefly because in them it is difficult to find consistent correspondence with Photius. This is not necessarily proof that they are not Nicomachean, either in whole or in part. At the beginning of each chapter Nicomachus seems to have devoted a paragraph or two to introductory remarks of a general nature concerning the number in question, and these naturally enough might not include specific epithets. The latter of course are what Photius was interested in citing, as a glance at his report shows; consequently, he discovered little in the first few paragraphs to record. This may be the reason why I A, II C and IV A contain no parallels in Photius; each, if Nicomachean, must have stood at the beginning of the chapter.

In the section I A, there is much that recalls the language of both the *Introduction* and the Nicomachean sections of the *Theologumena Arithmeticae*. One is inclined to think it Nicomachean in origin, but subjected to the editing of Iamblichus. The reference to the 'Introduction' could mean Iamblichus's *Commentary*, as well as our *Introduction*.[1] When the writer says that in the *Introduction* the monad was seen to be both 'pleuric and diametric,' he refers to something mentioned not by Nicomachus, but by Iamblichus. Again, the reference to 'the lambdoid figure at the beginning of the *Arithmetic*' cannot mean the *Introduction*, but is explained by Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 11, 13 ff.

Something similar may be said of II C, save that there are fewer signs of editing here than in the former case. Although with the exception of IV A, they do not come at the head of chapters, the suspected sections IV A, VI C, and VII C may be said to have failed of parallels in Photius for a like reason; they are made up rather of accounts of the potencies of numbers in nature than of epithets. Evidences of Nicomachean origin are shown particularly in IV A. It contains on p. 22 references to Heracles and Hermes, both mentioned in Photius,[2] and

[1] Cf. Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 125, 14 ff. (Pistelli).

[2] In Photius, however, *after* αἰόλα (*sic*); cf. *Theologumena Arithmeticae*, *ibid.*, Αἰόλου φύσις, after which Heracles is again mentioned; and the latter is the real parallel.

several references to other portions of the book which seem clearly enough Nicomachean.[1] In VII c, after a long series of instances to show that the heptad is influential in natural phenomena, — in human birth, life, and illnesses, — the epithets Ἀθηνᾶ, καιρός, τύχη are given, which occur together in Photius.

The examination of the text, then, has resulted in showing sections I B, II D, III A and C, IV B, V B, VI B, VII B, VIII A, IX A, and X A to be quite certainly Nicomachean, and IV A probably so; whereas of I A, II C, VI C and VII C, it can at least be said that there is no reason why they should not be ascribed to him. Only I D, I F, II B and VIII C, with the quotations of Speusippus, are really left uncertain. The conclusion to be drawn seems to be that, aside from the Anatolian passages, Ast's *Theologumena Arithmeticae* is based almost entirely on Nicomachus; at least the most important parts seem to be, and the portions of which less can be said are chiefly enumerations of groups of certain numbers in natural phenomena which need concern us very little because after all they are non-essential in the discussions which are to be entered upon.

While this may be granted, the work of the editor can be seen in many places. The tell-tale ὅτι often shows that he has omitted or abbreviated, and there is perhaps reason to believe that he has added somewhat, if section I A be taken as a criterion. But in general the treatise published by Ast gives the impression of being an integral work, however mutilated, in which the same underlying notions and the same vocabulary are uniformly used. One of the peculiarities which pervades the entire work is its predilection for etymologies — perhaps 'puns' would be the better word — and, as has been seen, there are frequent cross references from one part of the work to another. Certainly all things point to the assumption that the *Theologumena Arithmeticae* of Nicomachus underlies it all.

[1] E.g., cf. p. 19: περισσοειδὴς γὰρ πολλάκις ἡμῖν ὤφθη ἡ δυάς, κτλ.; with p. 11, οὐκ ἀριθμὸς δὲ ἡ δυάς, κτλ. (source doubtful); also p. 20, the reference to αἰων, χρόνος, καιρός, ὥρα, which is repeated, p. 23 (top), probably Nicomachean. At p. 21, the topic that 4 is the last of the initial series of perfect numbers; compare with this p. 13 at the beginning of the chapter on the triad.

CHAPTER VII

THE PHILOSOPHY OF NICOMACHUS

ANY discussion of the philosophy of Nicomachus is necessarily incomplete. The *Introduction to Arithmetic* gives, to be sure, more information about the religious and philosophical doctrines of the author than its title would lead one to expect, but this, after all, is meager. Furthermore, in dealing with the *Theologumena Arithmeticae* there is always the danger that the compiler has cited some one other than Nicomachus, however thoroughly we may be convinced that Nicomachus is the chief source and that the sentiments there found, whatever their source, would be acceptable to him. But the chief difficulty lies in the fact that Nicomachus was a Neo-Pythagorean.[1]

Modern readers find this sect hard to understand for several reasons: they were the inheritors of a tradition already confused and complicated by the most varied associations; they were mystics by temperament, satisfied to see deep meanings in the time-worn formulas that had come down to them, and not always careful to explain all that they felt and believed; and our information regarding them is itself fragmentary.

We are prone to emphasize the gap that lies between the old Pythagorean school and the new; in reality, in one way or another, a continuous tradition maintained itself down to the time of Nicomachus, and beyond. The Pythagorean school died, to be sure, in the sense of losing its independent existence, but Pythagoreanism did not die; and those to whom the survival of its doctrines was committed during this interval were often sympathetic enough. This can certainly be said of Plato and of his immediate successors, Speusippus[2] and Xeno-

[1] On the Neo-Pythagoreans in general see Zeller, *Die Philosophie der Griechen* (4th ed.), vol. III, part 2, pp. 92 ff.; Chaignet, *Pythagore*, vol. II, pp. 215 ff.

[2] Speusippus was the author of a book called περὶ Πυθαγορικῶν ἀριθμῶν, quoted (with this title) in *Theologumena Arithmetica*, pp. 61 ff. (Ast). Half of this book dealt with the varieties of the plane and solid numbers and the forms of the elementary corpuscles; the rest was devoted to the decad and its virtues. The *Theologumena Arithmetica* (*ibid.*) speaks of Speusippus's interest in Pythagorean lore, especially in Philolaus. Diogenes Laertius, IV. 1. 5, cites among his works a Μαθηματικός. Cf. Chaignet, *op. cit.*, vol. II, p. 294.

crates,[1] all of whom show distinctly Pythagorizing tendencies; in a less degree it is true of Aristotle, to whose antiquarian interest and collections we owe a large part of our information about Pythagoreanism.[2] In this way the doctrines of Pythagoras become imbedded in the literature which was the basis of education and the subject of learned commentary in the Hellenistic period. The exegesis of the *Timaeus* in particular led to much use of number symbolism.[3]

Furthermore, the doctrines of Pythagoras did not remain all this time in a static condition; the Academics, in particular, added their own speculations to those of the older school. So it happens that when men again began to claim the name of Pythagorean and to disseminate anew the doctrines of the sect, they found the latter necessarily modified by the development to which they had been subjected. Platonism, Aristotelianism, and Stoicism, all of which had been brought into association with the Pythagorean doctrines, left their mark upon the philosophy of Nicomachus and his fellows; after the advances in thought made by these schools, the Neo-Pythagoreans must needs state their position in the terminology that had now become universal.

An even greater modification, perhaps, was in the directing of the activities of the new Pythagorean school into more bookish ways. With the exception of Apollonius of Tyana, who seems to have been a man of quite different temperament from the more easy-going Nicomachus, they were not so much concerned with purifications and ascetic rules as with intricate arguments about the virtues of numbers.[4] One finds no mention of these things in the *Theologumena* or the *Introduction*. Nicomachus would be known as a Pythagorean only by his absorbing interest in numbers, his reverence for Pythagoras, Philolaus, Archytas and the rest, and his repetition of the things which they had

[1] Diogenes Laertius, IV. 2. 13, lists among his writings Πυθαγόρεια, Τῶν περὶ τὰ μαθήματα βιβλία, Περὶ γεωμετρῶν, Περὶ ἀριθμῶν, Ἀριθμῶν θεωρία, Τῶν περὶ ἀστρολογίαν, Περὶ γεωμετρίας.

His famous definition of the soul, 'number moving itself,' indicates his Pythagorean leanings. Cf. Chaignet, *op. cit.*, vol. II, p. 293.

[2] Cf. Chaignet, *op. cit.*, vol. I, p. 21.

[3] Plutarch's essay, *De Animae Procreatione in Timaeo* (especially 12 ff.), well illustrates this fact. There is a widespread belief that Posidonius's exegesis of the *Timaeus* dealt largely with such matters, and was the source of a series of treatises on the numbers, in which a Stoic flavor is to be distinguished; cf. Schmekel, *Die Philosophie der mittleren Stoa*, pp. 405 ff.; G. Borghorst, *De Anatolii Fontibus* (Berlin, 1905), pp. 55 ff.; G. Altmann, *De Posidonio Timaei Platonis Commentatore* (Berlin, 1906); but R. M. Jones, *The Platonism of Plutarch*, pp. 76–77 (especially n. 21), points out that the matter is not proved.

[4] Cf. Zeller (4th ed.), vol. III, part 2, p. 161: "Doch werden dieselben (i.e., asceticism, purifications) in unseren Fragmenten weder so haüfig berührt, noch gehen diese in ihren Anforderungen so weit, als man erwarten möchte."

said. His philosophy is, in fact, eclectic, with a Pythagorean background.

In Nicomachus, we clearly find that infusion of mysticism which manifests itself in the discovery of divinity in numbers and leads to rhapsodies over their virtues. As the author of the *Theologumena*, Nicomachus takes a place in the long series of writers who compiled treatises upon what may be called the 'theology of numbers,' to use the name which he himself employed, or 'arithmology,' a term revived by M. Armand Delatte, a recent writer on Pythagorean topics.[1]

The beginnings of arithmology are to be found as parts of general works in the earliest literature of the sect. In this the mathematical properties of the first ten numbers were already likened to and identified with physical properties and sometimes with the gods.[2] Thus Philolaus identified seven with 'the leader of the universe'[3] and had established a series of identifications for the numbers above 4.[4] Another example of the most ancient form of arithmology is the identification of the odd with the male and the even with the female. After the Περὶ φύσεως of Philolaus come the Περὶ τῆς δεκάδος of Archytas[5] and a work by Speusippus[6] in which apparently arithmology was more frankly the main theme; in the following years it is probable that it made its way into the Platonic commentaries,[7] although there is great obscurity in its history at this point.

The greatest development in arithmology took place, however, in the period extending from the second century before Christ down to the time of Neo-Platonism. The beginnings of this revivified interest, which eventually gave rise to numerous treatises,[8] more or less com-

[1] *Études sur la Littérature Pythagoricienne* (Fasc. 217, *Bibliothèque de l'École des Hautes Études*, Paris, 1915). On page 139 M. Delatte defines arithmology as "ce genre de remarques sur la formation, la valeur, et l'importance des dix premiers nombres, où se mêlent la saine recherche scientifique et les fantaisies de la religion et de la philosophie." See *ibid.* on the history of the name.

[2] For notices of old Pythagorean arithmology, see Aristotle, *Metaphysics*, I. 5, and XIII. 4, 1078 b 21.

[3] Philo, *De Mundi Opificio*, 33; Lydus, *De Mensibus*, II. 11.

[4] *Theologumena Arithmeticae*, p. 55 (Ast).

[5] Cited by Theon of Smyrna, p. 106, 10 (Hiller). [6] See p. 19. [7] See p. 27.

[8] This list contains three books devoted to arithmology alone: Nicomachus, *Theologumena Arithmeticae* (in Photius, *Codex* 187); the *Theologumena Arithmeticae* (Iamblichus?) edited by Ast, and Anatolius, Περὶ δεκάδος καὶ τῶν ἐντὸς αὐτῆς ἀριθμῶν (ed. Heiberg, *Annales Internationales d'Histoire*, Congrès de Paris, 1900, 5e. sect., *Histoire des Sciences*, pp. 27 ff.). Works containing arithmological material incidentally: Varro, *Hebdomades* (or, *De Originibus*; in Gellius, III. 10; the heptad only); Philo Judaeus, Περὶ ἀριθμῶν (lost, but much material of the kind is found in *De Mundi Opificio*, *Legis Allegoriae*, *Quaestiones et Solutiones in Genesim*, etc.);

plete, on the first decade of numbers, are not surely known, but there seems to have been an earlier compilation upon which both Varro and Philo drew and which was also the ancestor of the treatises of Theon, Anatolius, and some others. The interrelations of these authors have not yet been fully determined, but it is clear that one work lay back of them all.[1] The name of Posidonius has been suggested as their source,[2] and very probably in his commentary on the *Timaeus* Posidonius used material of the kind; still, inasmuch as this unknown original arose in the very period when promiscuous Neo-Pythagorean falsifications were being issued under the names of ancient authors, it is entirely possible that Posidonius figures in this tradition in some other capacity. However this may be, the series of writings mentioned continues the tradition of arithmology in about the same style as before, making much of the mathematical virtues of numbers and their physical analogies.

When Nicomachus, as the author of the *Theologumena Arithmeticae*, enters this literary succession, it is somewhat to one side of the usual current and as the representative of a different tendency which had already made its appearance and begun a development parallel to that of the older tradition. The first known representatives of this later style of arithmology are the two documents studied by M. Delatte, the Ἱερὸς λόγος in Doric prose and the *Hymn to Number* in Ionic verse, attributed to Pythagoras and Orpheus respectively, and supposed by M. Delatte to have arisen among the Pythagorean group in Italy.[3] M. Delatte does not specify their date, and indeed this is hardly possi-

Theon of Smyrna, p. 99, 24 ff. (complete, Hiller); Clemens Alexandrinus, *Stromata* VI. 16 (heptad; based on Hermippus of Berytus, Περὶ ἑβδομάδος); Chalcidius, *Commentarius in Timaeo;* Macrobius, *Commentarius in Somnium Scipionis;* Martianus Capella, *De Nuptiis Philologiae et Mercurii*, VII (complete); Lydus, *De Mensibus* (various chapters make an almost complete account); Favonius, *Commentarius in Somnium Scipionis;* Hierocles, *Commentarius in Carmen Aureum.* Furthermore, there are brief notices in the scholia upon Aristotle, in Plutarch, Sextus Empiricus, Stobaeus, etc., and the short *Anecdota* published by M. Delatte, *op. cit.*, pp. 167 ff. We know of other lost works probably to be included; see p. 33.

[1] The studies of Borghorst and Altmann (see p. 89) touched upon this question. The writer also has studied phases of the question in two papers, *Posidonius and the Sources of Pythagorean Arithmology, Classical Philology*, vol. XV (1920), pp. 309–322, and *The Tradition of Greek Arithmology, ibid.*, vol. XVI (1921), pp. 97–123. However the matter is decided, it is clear that the same compilation ultimately lies back of Varro, Philo, Anatolius, Theon, Clement (hence Hermippus), Chalcidius and Lydus; perhaps it also influenced some of the others, though it was not their main source. Of course, in the vicissitudes of the tradition various changes came about and the accounts thus grouped are by no means identical.

[2] See p. 89, n. 3.

[3] *Op. cit.*, pp. 191 ff.; on their origin, see pp. 206 f., 211.

ble, but in all probability they arose at about the same time as the source previously mentioned.

It is characteristic of this branch of the tradition to introduce many further identifications of the numbers with the gods, and that too under their cult names and stock epithets; arithmology thus became truly a theology of numbers, and great ingenuity was expended in the development of its complicated fabric. It is here that Nicomachus, as an arithmologist, belongs, and M. Delatte has demonstrated his use of sources of this kind. He is not forgetful, to be sure, of the older identifications, and was undoubtedly influenced by that side of the tradition as well. Thus he succeeded in the end in gathering and combining dozens of identifications for each number, the heirlooms of the previous centuries, each with its mystic meaning; and it is no wonder that the result is a perplexing mass in which it is hard to disentangle the many threads. But obscured as they are by mystic forms of utterance and confused by the presence of several types of thought, Nicomachus's general philosophical principles can still be determined with some degree of coherency, albeit only upon the physical or cosmological side, to the exclusion of ethics.[1]

First, Nicomachus is a dualist. He states his position in a way that recalls Plato's distinction between "that which ever exists, having no becoming" and "that which is ever becoming, never existent," [2] rather than the Stoic active and passive principles, *δραστικόν* and *παθητικόν*. On the one hand there are the "real things . . . which exist forever changeless and in the same way in the cosmos, never departing from their existence even for a brief moment," and on the other "the original eternal matter and substance" which was entirely "subject to deviation and change." [3] It is not, however, original matter to which Nicomachus devotes his greatest attention, but rather material

[1] The most interesting, and almost the only, Nicomachean fragment with an ethical bearing is found in the *Theologumena Arithmeticae*, p. 32 (Ast), and definitely ascribed to him: "When men are injured, they are willing that there should be gods; but when they do an injury, they are not willing. They are injured, therefore, so that they may wish gods to exist; for unless they wish gods to exist they will not be constant; if then the reason why men are constant is the desire that gods exist, and they so desire whenever they are injured, the injury is to be sure a bad thing, but it is an expediency of nature, and an expediency of nature is a good thing, and nature is good, the same thing as providence. So harm comes to men in accordance with providence."

[2] *Timaeus*, 27 D.

[3] *Introduction*, I. 1. 2: *ὄντα δὲ τὰ κατὰ τὰ αὐτὰ καὶ ὡσαύτως ἀεὶ διατελοῦντα ἐν τῷ κόσμῳ καὶ οὐδέποτε τοῦ εἶναι ἐξιστάμενα οὐδὲ ἐπὶ βραχύ;* I. 1. 3: . . . *τῆς ἐξ ἀρχῆς ἀιδίου ὕλης καὶ ὑποστάσεως . . . ὅλη γὰρ δι' ὅλης ἦν τρεπτὴ καὶ ἀλλοιωτή.*

things as found in the world, that is, matter impressed with form; and it is correspondingly harder to determine his views concerning matter per se. The investigator here finds difficulties similar to those encountered in the exegesis of the *Timaeus*, and fewer data to work upon.

Nicomachus held to a theory of the elements similar to that of Plato, including the doctrine that the corpuscles of the elements have the forms of the regular solids and may be reduced to triangles,[1] but whether the 'original matter' of I. 1. 3 did or did not consist of the elements in mixture he does not explain. From the brief description quoted above, however, one may suspect that it did not as yet have even elementary form, and thus it would resemble the so-called 'secondary matter' of the *Timaeus*. The recurrence of the same fundamental problems in connection with both Nicomachus and Plato, as well as the obvious quotations and reminiscences of Platonic phrases in the *Introduction*, suggests strongly the marked influence of Plato upon Nicomachus in these matters.[2]

But if original matter is not thoroughly discussed by Nicomachus, at least in his extant works, there is more to be learned about material things, upon which Nicomachus centers his attention in the *Introduction*, and which — rather than matter per se — he commonly contrasts with the eternal entities. Material things are a combination of matter (evidently matter reducible to elements) with form, and the manner of combination is referred to simply as a 'presence with'

[1] Four elements are mentioned in *Introduction*, II. 1. 1, but five, including 'ether,' in *Theologumena Arithmeticae*, p. 25 (Ast), with their corpuscular forms. Nicomachus declares in II. 7. 4 that the triangle is the element of plane figures (cf. *Theologumena Arithmeticae*, p. 18) and devotes II. 12 to proving it the element of polygons. Cf. also *Theologumena Arithmeticae*, p. 8: διόπερ ἡ πρώτη σύνοδος αὐτῶν πρῶτον ὡρισμένον πλῆθος ἀπετέλεσε, στοιχεῖον τῶν ὄντων, ὃ ἂν εἴη τρίγωνον μεγεθῶν τε καὶ ἀριθμῶν σωματικῶν τε καὶ ἀσωμάτων.

[2] On 'secondary matter' in Plato, cf. Zeller, *op. cit.* (6th ed.), vol. II, part 1, p. 729; Bäumker, *Das Problem der Materie* (especially pp. 142 ff.). In the course of his opening chapters of the *Introduction* Nicomachus cites Plato thrice, *Timaeus*, 27 D (I. 2. 1), the pseudo-Platonic *Epinomis*, 991 D (I. 3. 5), *Republic*, 526 D ff. (I. 3. 7). He refers most often to the psychogony of the *Timaeus* (*Introduction*, II. 2. 3; 18. 4; 24. 6) but also to the marriage number, in the *Republic* (II. 24. 11). Plato is mentioned also in II. 22. 1; 28. 6. For similarities of terminology, cf. the description of ὄντα, *Introduction*, I. 1. 2: τὰ κατὰ τὰ αὐτὰ καὶ ὡσαύτως ἀεὶ διατελοῦντα . . . καὶ οὐδέποτε τοῦ εἶναι ἐξιστάμενα, and *Phaedo*, 78 D: αὕτη ἡ οὐσία . . . πότερον ὡσαύτως ἀεὶ ἔχει κατὰ ταὐτὰ ἢ ἄλλοτ' ἄλλως; also *Sophist*, 248 A, etc.; *Republic*, 380 D: εἴπερ τι ἐξίσταιτο τῆς αὑτοῦ ἰδέας; *Cratylus*, 439 E: μηδὲν ἐξιστάμενον τῆς αὑτοῦ ἰδέας. With Nicomachus's term, ὁμωνύμως ὄντα, used of material objects (cf. p. 2, 14; 3, 15, Hoche), cf. *Sophist*, 234 B, ὁμώνυμα τῶν ὄντων; *Parmenides*, 133 D, τὰ δὲ παρ' ἡμῖν ταῦτα ὁμώνυμα ὄντα ἐκείνοις. The word μετουσία, however, used in this connection by Nicomachus (pp. 2, 14; 3, 14, Hoche) is not Platonic, but μετέχει (p. 3, 16, Hoche) can be freely paralleled; e.g., *Phaedo*, 101 C.

matter or a 'sharing in' the forms.[1] Their material nature subjects these objects to constant change — becoming, decay, growth, diminution, change in general — in which they imitate matter;[2] their only stability, a merely relative one, comes from the formal element in them.[3]

The impression of form upon a given quantum of matter halts, as it were, for a brief moment the ceaseless flow of change in which matter is involved; of it then we can say 'It is' and call it 'a thing.'[4] But it follows at the same time that individual existence is possessed by material things, not naturally nor in their own right, but due entirely to the form which they share; they exist on sufferance, as it were. Nicomachus describes it by calling them ὁμωνύμως ὄντα, 'existent under the same names,' because both their being and the name by which they are known belong not to them but to their forms.[5] The latter, in contrast, are 'really existent' or 'properly existent' things (κυρίως ὄντα, ὄντως ὄντα).

It remains to inquire more closely into the nature of the forms themselves. Eternal, immaterial, without beginning, these never change; their nature is fixed.[6] To describe them more definitely, they are "qualities, quantities, configurations, largeness, smallness, equality, relations, actualities, dispositions, places, times, all those things, in a word, whereby the qualities in each body are comprehended."[7] In listing these abstracts, Nicomachus is apparently giving random examples, with no serious attempt to cite only the higher and more gen-

[1] *Introduction*, I. 1. 2 (p. 2, 13, Hoche): *ταῦτα ἂν εἴη τὰ ἄυλα καὶ ὧν κατὰ μετουσίαν ἕκαστον λοιπὸν τῶν ὁμωνύμως ὄντων καὶ καλουμένων τόδε τι λέγεται καὶ ἔστι*, and I. 2. 1: *καθ' ὅσον αὐτῶν μετέχει.*

[2] I. 2. 1: *τὰ δὲ ἐν γενέσει τε καὶ φθορᾷ καὶ αὐξήσει καὶ μειώσει καὶ μεταβολῇ παντοίᾳ καὶ μετουσίᾳ φαίνεται διηνεκῶς τρεπόμενα καὶ λέγεται μὲν ὁμωνύμως ἐκείνοις ὄντα, καθ' ὅσον αὐτῶν μετέχει, ἔστι δὲ τῇ ἑαυτῶν φύσει οὐκ ὄντως ὄντα· οὐδὲ γὰρ τὸ βραχύτατον ἐπὶ ταὐτοῦ διαμένει, ἀλλ' ἀεὶ μεταβαίνει παντοίως ἀλλασσόμενα, κτλ.*

[3] This 'stability' is practically the 'sameness' of which Nicomachus speaks: see p. 99.

[4] *τόδε τι* in I. 1. 2 (quoted in n. 1, above); cf. note *ad loc.* This is an Aristotelian formula, but it is to be noted that Nicomachus finds reality, not in these individual objects, but in the ideal entities, wherein he is more a Platonist than an Aristotelian.

[5] Cf. I. 2. 1 (quoted in n. 2, above) and I. 1. 2 (quoted in n. 1, above).

[6] I. 2. 1: *ἀλλ' ἐκεῖνα μὲν ἄυλα καὶ ἀΐδια καὶ ἀτελεύτητα καὶ διὰ παντὸς ὅμοια καὶ ἀπαράλλακτα πέφυκε διατελεῖν, ὡσαύτως τῇ αὐτῶν οὐσίᾳ ἐπιδιαμένοντα, καὶ ἕκαστον αὐτῶν κυρίως ὂν λέγεται* (the rest of the section is quoted above, n. 1).

[7] *Introduction*, I. 1. 3: *τὰ δὲ περὶ αὐτὴν ἢ καὶ σὺν αὐτῇ θεωρούμενα ἀσώματα, οἷον ποιότητες, ποσότητες, σχηματισμοί, μεγέθη, μικρότητες, ἰσότητες, σχέσεις, ἐνέργειαι, διαθέσεις, τόποι, χρόνοι, πάντα ἁπλῶς οἷς περιέχεται τὰ ἐν ἑκάστῳ σώματι, ὑπάρχει καθ' ἑαυτὰ ἀκίνητα καὶ ἀμετάπτωτα, συμβεβηκότως δὲ μετέχει καὶ παραπολαύει τῶν περὶ τὸ ὑποκείμενον σῶμα παθῶν.*

eral ideas;[1] in general they seem to represent only predicables, doubtless suggested by the Aristotelian categories but neither identical with them nor employed in the same way; Nicomachus speaks of them in a manner that reminds one rather of the independently and eternally existing Platonic idea.[2] In this detail of his system he has probably united Platonic and Aristotelian theory, but in general, hitherto, Platonic terminology and doctrine have predominated. We shall, however, encounter plain evidence of Stoic influence.

The Stoic influence is manifest in Nicomachus's doctrines about God. In the dim picture of divinity which Nicomachus gives us,[3] we note resemblance to the mysterious deity of the *Timaeus;* it is primarily a world-creating God and one kindly disposed to the world, characterized by providence,[4] but clearly Stoic touches have been laid on the portrait, and the result is not far different from what is observed in Philo Judaeus, upon whom similar influences worked. God contains in himself all the ideal forms, which, as we have seen, are the essence of things and secure them and the world in general whatever stability they have.

In a remarkable passage of the *Theologumena Arithmeticae,*[5] which

[1] If this is a genuine attempt to list categories, it is unsuccessful, for according to Nicomachus himself some of them should fall under the head of others in the list; e.g., ποσότης (equivalent to τὸ ποσόν) belongs under μέγεθος (I. 2. 5); ἰσότης is a σχέσις (II. 6. 3, p. 84, 17 and 21).

[2] The Aristotelian list is thus given in *Categories*, 4, 1 b 25: τῶν κατὰ μηδεμίαν συμπλοκὴν λεγομένων ἕκαστον ἤτοι οὐσίαν σημαίνει ἢ ποσὸν ἢ ποιὸν ἢ πρός τι ἢ ποὺ ἢ ποτὲ ἢ κεῖσθαι ἢ ἔχειν ἢ ποιεῖν ἢ πάσχειν.

Cf. *Topica*, I. 9, 1036 b 21. Nicomachus differs from Aristotle, it may be observed, in making these abstracts and conferring upon them eternal ideal existence independent of the material things with which they are connected. It is best to see in this a fusion of Platonism with Aristotle; but it may be remembered that Archytas, in Nicomachus's day, passed as the author of a work on the categories. It is surprising, too, that οὐσία is omitted, if this is an Aristotelian list.

[3] God is mentioned only twice in the *Introduction*, I. 4. 2; 6. 1.

[4] Plato was not the first to conceive of teleology, but his influence in bringing this notion into subsequent cosmological speculation was enormous. The present writer has discussed this matter in another place (*The Hexaemeral Literature* [Chicago, 1912], p. 3, n. 1, etc.). The assertion of the *Timaeus* (29 A) that God took an 'eternal,' and hence good, pattern for the world underlies the whole dialogue and makes teleology, or providence, a leading motif. In Nicomachus we have the direct statement that "the providence of the world-creating God wrought all things" (*Theologumena Arithmeticae*, p. 43 (Ast): ἡ τοῦ κοσμοποιοῦ θεοῦ πρόνοια τὰ ὄντα πάντα ἀπειργάσατο, γενέσεως μὲν ἀρχὴν καὶ ῥίζαν ἀπὸ τοῦ πρωτογόνου ἑνὸς ποιησαμένη τοῦ πάντα εἰς ἀπόμαξιν καὶ ἀφομοίωσιν ἰόντος ἀνωτάτου καλοῦ, συμπληρώσεως δὲ τελείωσιν καὶ κατάκλεισιν ἐν αὐτῇ τῇ δεκάδι), and we must certainly infer that Nicomachus conceived it to be out of good-will for the world that God followed a predetermined plan in creation. (See p. 107.)

[5] P. 4 (Ast): καὶ ὅτι τὸν θεὸν φησιν ὁ Νικόμαχος τῇ μονάδι ἐφαρμόζειν, σπερματικῶς ὑπάρχοντα πάντα τὰ ἐν τῇ φύσει ὄντα, ὡς αὐτῇ ἐν ἀριθμῷ ἐμπεριέχεται δυνάμει τὰ δοκοῦντα ἐναντιώτατα κατ' ἐνέργειαν εἶναι πᾶσιν ἁπλῶς ἐναντιότητος τρόποις, καθὼς αὐτὴ ἀρρήτῳ τινὶ φύσει πανειδὴς οὖσα ὤφθη παρ' ὅλην τὴν ἀριθμητικὴν εἰσαγωγὴν . . . ὡς δὲ οὐκ ἄνευ αὐτῆς σύστασις ἁπλῶς τινός, οὕτως

may be cited confidently because it is definitely referred to Nicomachus, the monad is compared with God, who, it is stated, is "in a seminal sense all things in nature," just as in the *Introduction* the monad is observed to embody potentially all numerical forms;[1] again, "it (*sc.* the monad) generates itself and is generated from itself, is self-ending, without beginning, without end, and appears to be the cause of enduring, as God in the realm of physical actualities is in such manner conceived of as a preserving and guarding agent of nature."

It is noteworthy, too, that σπερματίτης λόγος occurs among the epithets of the monad in the epitome of Photius. The likeness of the Nicomachean God to the Stoic Divine Fire which contained "all the σπερματικοὶ λόγοι, by which all things in accord with Fate come into being"[2] must be apparent, although God, to Nicomachus, is not fire. God sums up in himself the λόγοι, the principles of all things, and we shall probably not err in ascribing to Nicomachus, likewise, the doctrine that the ideal forms are the thoughts of God, eternally present in his mind.

The monad was likened, in the passage cited above, to God's mind, which is said to be "that thing in God which is the leader both in creation and in every art and in every reasoning," because, of course, it contains the forms, which are its thoughts; and again, in the same passage, God is called 'artistic word' or 'reason' (τεχνικὸς λόγος). This is the point of resemblance between Nicomachus and Philo, in whose *De Mundi Opificio* the νοητὸς κόσμος, the ideal world which was the pattern of this, is definitely declared to be God's thought,[3] and it

δὲ χωρὶς αὐτῆς γνώρισις οὐτινοσοῦν, ὡς φωτὸς καθαροῦ κυριωτάτης πάντων ἁπλῶς οὔσης, καὶ ἡλιοειδοῦς καὶ ἡγεμονικοῦ, ἵν' ἐοίκῃ καθ' ἕκαστον τούτων τῷ θεῷ, καὶ μάλιστα, καθὸ φιλοτικὴ καὶ συστατικὴ καὶ τῶν πολυμιγῶν καὶ πάνυ διαφορωτάτων, ὡς ἐκεῖνος ἐξ οὕτως ἀντικειμένων ἁρμόσας καὶ ἑνώσας. τὸ δὲ πᾶν ἑαυτήν γε μὴν γεννᾷ καὶ ἀφ' ἑαυτῆς γεννᾶται, καὶ αὐτοτελὴς καὶ ἄναρχος καὶ ἀτελεύτητος καὶ διαμονῆς αἰτία φαίνεται, καθὼς ὁ θεὸς ἐν τοῖς φυσικοῖς ἐνεργήμασι τοιοῦτος ἐπινοεῖται διασωστικὸς καὶ τῶν φύσεων τηρητικός. λέγουσιν οὖν ταύτην οὐ μόνον θεόν, ἀλλὰ καὶ νοῦν καὶ ἀρσενόθηλυν· νοῦν μέν, ὅτι τὸ ἐν θεῷ ἡγεμονικώτατον καὶ ἐν κοσμοποιίᾳ καὶ ἐν πάσῃ ἁπλῶς τέχνῃ τε καὶ λόγῳ, εἰ καὶ μὴ ἐπιφαίνοιτο καθ ἕκαστον ὅλης δι' ἐνεργείας, νοῦς ἐστί, ταυτότης τις ὢν καὶ ἀμετάτρεπτος δι' ἐπιστήμης, ὡς αὐτὴ πάντα περιειληφυῖα ἐν ἑαυτῇ κατ' ἐπίνοιαν, εἰ καὶ κατ' ἔκστασιν ἐν τοῖς τῶν ὄντων εἴδεσιν, ὡς λόγος τις τεχνικὸς ἐοικὼς τῷ θεῷ, καὶ οὐ καθισταμένη τοῦ καθ' ἑαυτὴν λόγου, οὐδὲ μεθίστασθαι ἄλλον τινὰ ἐῶσα, ἀλλὰ ἄτρεπτος ὡς ἀληθῶς καὶ μοῖρα Ἄτροπος.

[1] For example, cf. *Introduction*, I. 16. 8; II. 8. 3; 9. 2; 10. 2; 11. 1–2; 13. 8; 14. 1; 15. 3.

[2] *Placita Philosophorum*, I. 7. 33 (= Diels, *Doxographi Graeci*, 305): οἱ Στωϊκοὶ νοερὸν θεὸν ἀποφαίνονται, πῦρ τεχνικὸν ὁδῷ βαδίζον ἐπὶ γένεσιν κόσμου, ἐμπεριειληφὸς πάντας τοὺς σπερματικοὺς λόγους, καθ' οὓς ἕκαστα καθ' εἱμαρμένην γίνεται. The σπερματικοὶ λόγοι or 'seed principles' are the active, formal, divine element which, acting upon material things, produce their proper natures in the same way that seeds produce plants.

[3] *De Mundi Opificio*, 5: καθάπερ οὖν ἡ ἐν τῷ ἀρχιτεκτονικῷ προδιατυπωθεῖσα πόλις τὴν χώραν ἐκτὸς οὐκ εἶχεν, ἀλλ' ἐνεσφράγιστο τῇ τοῦ τεχνίτου ψυχῇ, τὸν αὐτὸν τρόπον οὐδὲ ὁ ἐκ τῶν ἰδεῶν

is likewise the point of departure from the cosmology of the *Timaeus*, where the eternal pattern (παράδειγμα) is independent of God's mind.

The same conclusion may be inferred from the statement already quoted to the effect that God is "the preserving and guarding agent of natures in the realm of physical actualities," which is substantiated by another passage that designates him as "cause of 'sameness' and unchanging persistence" and matter as the cause of change in things.[1] It is of course as the source of form, that which alone gives stability to material existence, that God is so denominated.[2] We shall find further evidence when we discuss the cosmogony of Nicomachus.

Thus far, nothing has been encountered which could not have been uttered with equal propriety by any eclectic philosopher; but we must turn to the consideration of numbers, wherein Nicomachus's Pythagoreanism begins to make itself felt. In addition to the former statement that the eternal things, the forms, are quantities, qualities, arrangements and the like, we find Nicomachus also asserting that the world is ordered on a numerical basis.[3]

The proper way to reconcile these two views seems to be by declaring numbers to be a superior kind of forms, out of which the other forms are made and under which they are classified. This appears to be the meaning of the statement that number was 'the real eternal essence,' [4] and certain passages of the *Theologumena Arithmeticae* lead to the same conclusion, that numbers are the highest forms and that the properties seen primarily in them are also the essential properties of things in the world, conferred upon them by number.

In one place, speaking of the hexad, Nicomachus says that the Pythagoreans "revered it with distinguished praises, saying that the world is endowed with soul in accordance with it, and harmonized, and that

κόσμος ἄλλον ἂν ἔχοι τόπον, ἢ τὸν θεῖον λόγον τὸν ταῦτα διακοσμήσαντα, κτλ.; 6: εἰ δέ τις ἐθελήσειε γυμνοτέροις χρήσασθαι τοῖς ὀνόμασιν, οὐδὲν ἂν ἕτερον εἴποι τὸν νοητὸν εἶναι κόσμον ἢ θεοῦ λόγον ἤδη κοσμοποιοῦντος.

[1] *Theologumena Arithmeticae*, p. 8 (Ast): ἀντιδιαστελλομένη (sc. ἡ δυάς) παραπλησίως τῇ τοῦ θεοῦ φύσει κατὰ τὸ αὐτὴν μὲν τῆς μεταπτώσεως καὶ μεταβολῆς ἐμποιητικὴν τοῖς οὖσι νομίζεσθαι, τὸν δὲ θεὸν ταυτότητος καὶ ἀμεταπτώτου διαμονῆς.

[2] See p. 99 for further discussion of the connection between 'sameness' and the ideal forms.

[3] *Introduction*, I. 4. 2; 6. 1.

[4] The whole passage (I. 6. 1) is so important that it may well be cited: πάντα τὰ κατὰ τεχνικὴν διέξοδον ὑπὸ φύσεως ἐν τῷ κόσμῳ διατεταγμένα κατὰ μέρος τε καὶ ὅλα φαίνεται κατὰ ἀριθμὸν ὑπὸ τῆς προνοίας καὶ τοῦ τὰ ὅλα δημιουργήσαντος νοῦ διακεκρίσθαι τε καὶ κεκοσμῆσθαι βεβαιουμένου τοῦ παραδείγματος οἷον λόγον προχαράγματος ἐκ τοῦ ἐπέχειν τὸν ἀριθμὸν προϋποστάντα ἐν τῇ τοῦ κοσμοποιοῦ θεοῦ διανοίᾳ, νοητὸν αὐτὸν μόνον καὶ παντάπασιν ἄυλον, οὐσίαν μέντοι τὴν ὄντως τὴν ἀΐδιον, ἵνα πρὸς αὐτὸν ὡς λόγον τεχνικὸν ἀποτελεσθῇ τὰ σύμπαντα ταῦτα, χρόνος, κίνησις, οὐρανός, ἄστρα, ἑξελιγμοὶ παντοῖοι.

animals and plants get completeness and persistence and careful health by its joining them and its share in their birth and its beauty and virtue, and the like, and they set about it with this line of reasoning:

"The original universe, lacking order and shapeless and totally devoid of the things that give distinction according to the categories of quality and quantity and the rest, was organized and arranged most clearly by number as the most authoritative and artistic form, and gained a share in a harmonious exchange and flawless consistency in accordance with its desire for and its receiving the impression of the peculiar properties of number." [1] From this one judges that all categorical distinctions are based ultimately upon number.

Other passages assert that the universe is patterned after number; for example, in the *Theologumena Arithmeticae*, p. 58 (Ast), we read: "We have often anticipated ourselves in saying that the devising Mind wrought with reference to the resemblances and likenesses of number, as to a pattern that was perfect, the fabric and composition of the world and of all that is in it." [2] This is confirmed as Nicomachean by the fact that the same things are said in *Introduction*, I. 4. 2 and I. 6. 1, whence we learn that arithmetic was preëxistent as a cosmic pattern in the mind of the creator, and that the material world was formed with reference to this model.[3]

A distinction, which is left all too vague by our sources, is made, however, at this point. The number that preëxisted in God's mind and was the basis of creation, a wholly conceptual and immaterial number, is not the same as that of which science treats, the number that is constantly found in connection with material things and which measures them, their arrangements and their movements. This number is called 'scientific' (ἐπιστημονικός), and Nicomachus

[1] P. 33 (Ast): *μετὰ δὲ τὴν πεντάδα τὸν ϛ′ εὐθὺς ἀριθμὸν ἐναργεστέροις ἐσέμνυνον ἐγκωμίοις, ἐπιλογιζόμενοι δείγμασιν οὐκ ἀμφιβόλοις, κατ' αὐτὴν ἐμψυχῶσθαι καὶ καθηρμόσθαι τὸν κόσμον, τυχεῖν τε ὁλότητος καὶ διαμονῆς ἐπιμελοῦς τε ὑγιείας καὶ τὰ ζῷα καὶ τὰ φυτὰ συνόδῳ τε καὶ ἐπιγονῇ καὶ καλλονῇ καὶ ἀρετῇ, καὶ τῶν τοιούτων· ἐπεχείρουν δὲ οὕτως ἐπάγοντες· ἡ τῆς ἐξ ἀρχῆς ἀιδίου ὕλη ἀκοσμία καὶ ὅσον ἐπ' αὐτῇ ἀμορφία στέρησίς τε πάντων ἁπλῶς τῶν τρανωτικῶν, κατά τε ποιὸν καὶ ποσὸν καὶ τὰς λοιπὰς κατηγορίας, ἀπ' ἀριθμοῦ ὡς κυριωτάτου καὶ τεχνικοῦ εἴδους ἐκρίθη, καὶ διεκοσμήθη τρανώτατά τε καὶ ἐμμελοῦς ἐξαλλαγῆς καὶ ἀκολουθίας ἀκηράτου ἔτυχε μετασχοῦσα κατ' ἔφεσιν καὶ ἀπόμαξιν τῶν ἀριθμοῦ ἰδιωμάτων.*

[2] *πολλάκις ἔφθημεν εἰπόντες τὸν τεχνικὸν νοῦν πρὸς τὰς ἀριθμοῦ ἐμφερείας καὶ ἀφομοιώσεις ὡς πρὸς παράδειγμά τι παντελὲς ἀπεργάσασθαι τὴν τοῦ κόσμου καὶ τῶν ἐν κόσμῳ πάντων κατασκευήν τε καὶ σύστασιν, κτλ.*

[3] Nicomachus does not say that the plan *consisted of* numbers, nor that the ideal elements of it *were* numbers; it was founded on number. Cf. p. 108.

sharply distinguishes it from the divine number in the discussion of I. 6. He deals only with scientific number in the *Introduction;* the divine number, which forms the pattern for the inferior variety as well as for all mundane things, we find dealt with in the *Theologumena Arithmeticae*, where are set forth the specific functions of the numbers in the universe and the divinities with which they are identified because of fancied likeness. This divine number, we find, is decadic; the first ten terms show all the properties found in its entire fabric and the terms after 10 are simply a repetition.[1]

The divine numbers influence things through their properties, conveying like qualities to the objects which they affect.[2] One might perhaps assert that these properties are, therefore, more elementary than the numbers themselves, and presuppose a possible analysis of the divine number. This is true, but only to a certain extent. There are elements even of the divine number, and fundamental properties on which the characters of the various terms of the decade are founded; but these elements are themselves in the decade, and the fundamental properties are made identical with the elements of number. The elementary things are the monad and dyad, 'sameness' and 'otherness,' 'odd' and 'even.'

The origins (ἀρχαί) of number, and indeed of all things, are the 'same' and the 'other,' or 'sameness' and 'otherness.'[3] They are the formal principles, which, when they enter into the composition of things, cause them either to persist in the same fashion, preserving their identity, or, in the case of 'otherness,' to change from their original forms and assume others.[4] The former is characteristic of ideas and principles (εἴδη, λόγοι) because these keep uniform those

[1] *Theologumena Arithmeticae*, p. 25 (Ast): ὅτι δὲ ἄρχεται μὲν ἀπὸ μονάδος, τελειοῦται δὲ ὁ ἀριθμὸς εἰς ι', λεχθήσεται προϊοῦσιν; p. 59 (they call the decad) πᾶν δὲ ὅτι ἀριθμὸς φυσικῶς πλείων οὐδείς ἐστιν, ἀλλ' εἴ γέ τις ἐπινοεῖται, κατὰ παλινοδίαν ἐπ' αὐτὸν πως ἀνακυκλεῖται· ἑκατοντὰς γὰρ δέκα δεκάδες καὶ χιλιὰς δέκα ἑκατοντάδες καὶ μυριὰς δέκα χιλιάδες, καὶ ἄλλων ἕκαστος οὕτως ἢ εἰς ἢ εἴς τινα τῶν ἐντὸς αὐτῆς ἀναποδισθήσεται παλινοδούμενος.

[2] This notion of the influence of numbers on things is at the heart of Pythagoreanism. Iamblichus, *In Nicomachi Arithmeticam Introductionem Liber*, p. 78, 20 ff. (Pistelli), speaking of the monad and dyad, well illustrates the point: ὡς γὰρ τὸ θερμὸν θερμαίνειν πέφυκε τὰ πλησιάζοντα καὶ τὸ ψυχρὸν ψύχειν καὶ τὸ ὑγρὸν ὑγραίνειν, οὕτως καὶ αἱ τῶν ὄντων ἀρχαὶ ἄμικτοι τῶν ἄλλων δυνάμεων οὖσαι πάντα τὰ μεταλαμβάνοντα αὐτῶν κατὰ τὰς οἰκείας δυνάμεις ῥυθμίζουσι.

[3] These principles are treated most fully in *Introduction*, II. 17. 1; 18. 1 and 4; 19. 1 and 20. 2.

[4] This seems to be the significance of the terms ταυτότης, ἑτερότης, and it might be inferred from *Introduction*, I. 23. 4, especially καὶ ὥσπερ ὑπὸ σφραγιστῆρός τινος ἢ μέτρου πάντα τὰ ἐμπίπτοντα μεταλαμβάνει τῆς ὁμοιότητος καὶ ὁμωνυμίας. He is here praising 'equality' especially, but this is aligned with 'sameness' and 'inequality' with 'otherness.'

things whereon they are impressed; whereas 'otherness' inheres particularly in matter.[1]

Whether 'otherness' be matter itself or not, it is hard to say,[2] for Nicomachus often writes as though it were; at any rate, it is represented as a tendency as old as matter and inseparable from it, bound into it by the force of Necessity, a cosmic power that is always arrayed against the efforts of 'sameness' and the order which 'sameness' represents and strives to bring about. These are by no means original doctrines with Nicomachus, but were taken by him — as he substantially admits in II. 18. 4 — from the 'same' and the 'other' ('indivisible' and 'divisible') of Plato, and the 'limiting' and 'unlimited' of Philolaus.

The *Timaeus* is, in fact, viewed in one light, the history of the contest between the two forces of the Deity and Necessity, 'sameness' and 'otherness.' So Nicomachus would have us view the constitution of the world. It is made up of opposite and warring elements, the forces of 'sameness' and 'otherness,' which enter it through its dependence on numbers; but these have been reduced to harmony by the operation of the mathematical principles on which the world is constructed and governed; for the system of numbers is itself a harmony.[3] Both elements are needful to make the universe, for according to the old saying of Philolaus, harmony is "the unification of the diverse and the concord of the discordant," and there must, therefore, be opposite elements in any cosmos.

'Sameness' is further held by Nicomachus to be identical with the monad and 'otherness' with the dyad.[4] An explicit statement of this will be found in *Introduction*, II. 18. 1; indications less clear occur elsewhere. This was not Nicomachus's own theory, and consequently he does not demonstrate it formally, but it is easy to see what line of reasoning he would adopt. The fact that 1 multiplying itself or

[1] It is to be noted that in several passages Nicomachus uses the terms περαίνοντα, ἄπειρα, old Pythagorean names, for ideas and matter (the 'limiting' and the 'unlimited'). In II. 18. 4 they are quoted from Philolaus and then used independently by Nicomachus. Cf. also περαίνεται, I. 23. 4.

[2] One may perhaps be misled by the identifications given by Photius in his brief summary of the *Theologumena Arithmeticae*. This is so brief that it cannot be certain that Nicomachus stated matters in the way reported. It is much safer to observe carefully the expressions of the *Introduction* and of Ast's text where it is reasonably certain that Nicomachus is cited.

[3] Cf. *Introduction*, II. 19. 1; I. 6. 2 ff.

[4] *Introduction*, II. 18. 1; also 17. 2; 18. 4. *Theologumena Arithmeticae*, p. 4, Ast (quoted p. 95, n. 5) says that they call the monad Mind, and that is ταυτότης τις.

another number causes no change in numerical value,[1] whereas 2 or any other multiplier makes the product different from the multiplicand, had long been regarded as significant, and is cited by Nicomachus himself as evidence for the elementary character of the monad. But in 'theological' language it leads to the identification of the Monad with Prometheus: "Wherefore they also call it Prometheus, the maker of life, from its never running forward in any way, and alone never departing from its own principle, nor allowing other things to do so, giving them a share in its own peculiar traits; for by however many intervals it be increased or by however many it increases (*sc.* others), it prevents them from running forward."[2] There is here a characteristic Pythagorean play on the name Prometheus (πρόσω μὴ θεῖν, 'not to run forward').

We see, then, that the reason for the identification of the monad with 'sameness' is that the monad causes 'sameness.' Another way of stating the principle is that 'sameness' is generically (γενικῶς) present in the monad; there is no difference between 'sameness' and 'oneness,' and the 'same' and the 'one' are identical.[3] Nicomachus's meaning is admirably illustrated by a passage from Aristotle: "And these are called 'same' in the preceding way, but other things are called 'same' absolutely, as 'one' is. For 'same' is predicated of those things whose matter is one, either in kind or in number, as well as of those whose essence is one; and so it is evident that 'sameness' is a kind of 'oneness' of being, either of a plurality of objects or of one thing when one employs it as a plurality, as for example when one says that a thing is equal to itself, for then he employs it as two things."[4] In other words, 'one' is, like 'same,' the essential quality of the things which fall under the class 'sameness,' and can, therefore, be identified

[1] I.e., $1^n = 1$; $m \times 1^n = m$.

[2] *Theologumena Arithmeticae*, p. 5 (Ast): διὸ καὶ Προμηθέα μυθεύουσιν αὐτήν, δημιουργὸν ζωότητος, ἀπὸ τοῦ πρόσω μηδενὶ τρόπῳ θεῖν μηδὲ ἐκφοιτᾶν τοῦ ἰδίου λόγου μονωτάτην, μηδὲ ἄλλα ἐᾶν, μεταδιδοῦσαν τῶν ἰδιωμάτων ἑαυτῆς· ὁπόσαις γὰρ ἂν αὐξηθῇ ἀποστάσεσιν ἢ ὁπόσας ἂν αὐξήσῃ, θεῖν πρόσω κωλύει καὶ μεταπίπτειν εἰς τὸν ἐξ ἀρχῆς ἑαυτῆς τε κἀκείνων λόγον. The text in the last few words is doubtless corrupt.

[3] *Introduction*, II. 20. 2: ἱκανὸν καὶ τοῦτο τεκμήριον τοῦ ταυτότητος καὶ ἑτερότητος μετέχειν τὰ εἴδη ἀμφότερα, ἑτερότητος μὲν ἀορίστως, ταυτότητος δὲ ὡρισμένως, γενικῶς μὲν μονάδα καὶ δυάδα, ὑποβεβηκότως δὲ περισσὸν μὲν ταυτότητος διὰ τὸ μονάδι ὁμογενὲς εἶναι, ἄρτιον δὲ ἑτερότητος διὰ τὸ δυάδι.

[4] *Metaphysics*, IV. 8, 1018 a 4 ff.: καὶ τὰ μὲν οὕτως λέγεται ταὐτά, τὰ δὲ καθ' αὑτά, ὥσπερ καὶ τὸ ἕν· καὶ γὰρ ὧν ἡ ὕλη μία ἢ εἴδει ἢ ἀριθμῷ ταὐτὰ λέγεται, καὶ ὧν ἡ οὐσία μία. ὥστε φανερὸν ὅτι ἡ ταυτότης ἑνότης τίς ἐστιν ἢ πλειόνων τοῦ εἶναι ἢ ὅταν χρῆται ὡς πλείοσιν, οἷον ὅταν λέγῃ αὐτὸ αὑτῷ ταὐτόν· ὡς δυσὶ γὰρ χρῆται αὐτῷ. Cf. the collocation in Aristotle, *Categories*, 4 a 10, ταὐτὸν καὶ ἓν ἀριθμῷ.

with the class in a fundamental way. The case of the dyad and 'otherness' is similar; 'twoness' and 'otherness' are in the last analysis identical.[1]

This identification of the monad and dyad with 'sameness' and 'otherness' is employed further in Nicomachus's system. The monad and the dyad confer respectively 'sameness' and 'otherness' upon the numbers in the generation of which they play a leading part, that is, the odd numbers and squares on the one hand, and the even and heteromecic numbers on the other.[2] But whereas the monad and dyad partake of 'sameness' and 'otherness' directly or generically, γενικῶς, the latter do so in a secondary way, ὑποβεβηκότως, or as species, — εἰδικῶς, belonging to the genera founded on the monad and dyad.[3] They are characterized by additional properties, whereas the others are pure 'sameness' and 'otherness.' And if 'sameness' and 'otherness' thus enter the numbers through their elements, the monad and dyad, it can be readily seen that they enter the constitution of all things in general, because the world and all in it are, according to Nicomachus, fashioned upon a numerical basis.[4] That is, they are ἀρχαὶ τῶν ὅλων, 'elements of the universe.'[5]

Upon these comparatively sane beginnings is built the complicated structure of further identifications among the numbers, dependent both upon these principles and others discovered among the properties of numbers, or even upon the fancied etymologies of their names;[6] all the numbers of the decade are thus dealt with in turn. One object of identification suggests another similar to it; the mention of a god suggests the epithets and functions of that god; before the end is reached Nicomachus has propounded a series of astounding length in which hopeless confusion reigns. The same epithets are shared by several numbers,[7] the same number receives contradictory

[1] Aristotle does not carry out this point in the passage quoted.

[2] The monad is the fundamental form of the odd numbers (II. 17. 2) and through them of the squares (II. 17. 3; 19. 1); thus both partake of 'sameness' (II. 18. 1; 20. 2, 5). In the same way the dyad forms the even numbers directly, and through them the heteromecic numbers.

[3] Cf. II. 20. 2, already quoted. [4] See p. 108.

[5] *Introduction*, II. 17. 2.

[6] The following from the *Theologumena Arithmeticae* will serve as illustrations (cited by Ast's pages): μονάς-μένειν (3), δυάς-διϊέναι (8), δύη-ἐδυάσθη, δύσις (12), δίκη-δίχη, Ἶσις-ἴσος, Διομήτωρ-Διὸς μήτηρ, Ῥέα-ῥύσις (12), τριάς-ἀτειρής (15), τρεῖν (14), τετράς-τετλάς (23), ἑξάς-ἕξις (48), Ἥρα-ἀήρ (58), δεκάς-δεχάς (59). As this was a favorite device of the arithmologists, it would be easy to cite similar examples from other Pythagorean sources.

[7] E.g., in Photius, *Codex* 187, the following epithets, among others, are assigned several numbers: νοῦς (1, 3), ὕλη (1, 2), Ἄτλας (1, 10), ἄξων (1, 5), ἁρμονία (2, 3, 4, 6), γάμος (3, 5, 6). The

titles.[1] How far the authors of such collections believed in what they wrote, and what good they felt they derived from their labor, perhaps only they could say. In some cases we can see that an identification is fairly reasonable and might have some meaning, but far more often it seems futile.

An interesting fragment of Moderatus of Gades says that the Pythagoreans used the numbers for the sake of clearness in teaching about first principles and forms, just as geometricians and schoolteachers use diagrams and characters: "And in this way they called the idea of 'sameness' and 'oneness' and 'equality,' and the cause of concord and sympathy in the universe and the cause of preservation of that the condition of which remains just the same, 'one'; for the 'one' of sensible things is such because it is unified in its parts and concordant by participation in the first cause. And the idea of 'otherness' and 'inequality' and of all that is divisible and in change, now in one state, now in another, they called the double ratio and the dyad, for such is the nature of 2 in sensible things also. And these principles do not form part of the doctrine of these men alone and not of the others; one may also observe that the other philosophers have handed down the tradition of certain forces which are unifying and dominant over the universe, and they too have certain principles of 'equality' and 'unlikeness' and 'otherness.' . . . And similarly with regard to the other numbers there is the same account; each one is arranged under certain functions. To take another case, something exists in nature that has a beginning, middle, and end. Of this form and nature they predicate the number 3. Wherefore everything that involves a mean they say is 'three-form.' And the successive numbers are embraced by a certain single idea and power; this they called decad, as it were 'receiver.'"[2]

The generalization of Moderatus was perhaps once true, and might remain true for the simpler identifications, but another explanation must be sought for the extreme cases. It is probably near the truth to say that the Pythagoreans developed on the basis of early arithmology a pseudo-science of peculiar attractions, to which it was easy

fact that various epithets of the same gods appear complicates matters still more; e.g., Ἀπόλλων, etc., of 1, Λοξίας of 3, ἑκάεργος of 9.

[1] For example, in Photius, *Codex* 187, the monad is both ἥλιος, ἀλαμπία and σκοτωδία; 2, ἴση and ἄνισος; 4, like the hero of the *Frogs*, both 'Heracles' and 'Dionysus.'

[2] δεκάς-δεχάς, an etymology claimed by the early Pythagoreans. The passage quoted is found in Porphyry, *Vita Pythagorae*, 49 (p. 33, 30 ff., Nauck).

to make additions by following the established rules. They were carried away by it; at the same time they were influenced by the superstitious regard for numbers that may still be observed in our own time, which must have been still stronger in the days when belief in demons, astrology, and compelling magic was not confined to the unlearned.[1] It is perhaps significant, in this connection, that the arithmological treatises frequently make reference to astrology and other superstitions.[2]

It would be a long and unprofitable task to set down here more than a brief outline of the arithmology of Nicomachus. Of the monad and the dyad something has already been said. In addition to its identification with God, Mind and the like,[3] the monad is also likened to the first-born Chaos of Hesiod, because it is the first of the numerical series.[4] Other epithets are 'Sun' and 'Apollo,'[5] 'Zeus's tower,' and 'matter in a sense,' because 2, the number consecrated especially to matter, is derived from the doubling of 1.

The dyad naturally derives most of its titles from its character as 'otherness,' the opposite of the monad.[6] Hence comes the title 'matter,' and thence in turn 'unequal,' 'excess,' 'deficiency';[7] but on the other hand 'equality' was seen in it because $2 \times 2 = 2 + 2$.[8]

[1] Tannery, *Pour l'Histoire de la Science Hellène*, Appendix II, pp. 379–80, suggests that the enumerations found under the captions of various numbers (e.g., 4 elements and seasons, 5 elements, zones, circles, senses, etc.) originated as a mnemonic device. Certainly if this is so, this origin was forgotten by the time of Nicomachus.

[2] E.g., in treating the number 7, it is pointed out that this number governs the critical periods (κλιμακτῆρες) of human life according to the 'Chaldeans' (i.e., astrologers): Varro in Gellius, III. 10; Clemens Alexandrinus, *Stromata*, VI. xvi. 143, 1; *Theologumena Arithmeticae*, p. 53 (Ast). Lydus, *De Mensibus*, II. 3 and 7, has references to 'the Chaldeans' or 'the Chaldean.' Philo does not fuse astrology with his arithmology, but we learn that he had a deep aversion on religious grounds to Chaldean astrology from the fact that he often condemns it (*De Migratione Abrahami*, 32, 33; *De Abrahamo*, 15; *De Nobilitate*, 5).

[3] Note that we nowhere get an absolute identification of the monad either with God or with the 'cosmic word.' The monad 'fits' or 'is like' these things, or 'is called by these names.'

[4] The epithets selected for mention in the following paragraphs are in the main taken from the lists preserved from Nicomachus by Photius; Ast's text often explains the reason for their bestowal. On the identification with 'chaos,' cf. Delatte, *op. cit.*, p. 142. Other titles perhaps to be connected with this are σύγχυσις, σύγκρασις, ἀλαμπία, σκοτωδία, χάσμα, τάρταρος, Στύξ, φρικωδία, ἀμιξία, βάραθρον ὑποχθόνιον, Λήθη.

[5] Probably explained by Lydus, *De Mensibus*, II. 3: ἀναφέρεται δὲ ἡ μονὰς εἰς Ἀπόλλωνα, τουτέστιν εἰς τὸν ἕνα ἥλιον, κτλ. Cf. also Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 13 (Pistelli); Stobaeus, *Eclogae*, *Prooem.*, 10 (vol. I, pp. 21–22, ed. Wachsmuth-Hense); Plutarch, *De Iside*, 10; Zeller, *op. cit.*, vol. III, 2, p. 139 (4th ed.); Chaignet, *op. cit.*, vol. II, 109. With this epithet probably go also πυράλιος, προφήτης, λόγιος.

[6] Cf. pp. 99 f.

[7] Cf. *Theologumena Arithmeticae*, pp. 10, 11; Lydus, *De Mensibus*, II. 6.

[8] *Theologumena Arithmeticae*, p. 10.

It is the root of all relativity, because 2 : 1 is the first ratio;[1] and a very old epithet, repeated by Nicomachus, is 'daring,' derived from the idea that 2 first 'dares' to separate itself from the original unity.[2] It is to be noted that neither the monad nor the dyad, the elements of number, were regarded as numbers in the proper sense of the term.[3] Among deities, Erato, Phanes, Zeus, Isis, Rhea-Demeter, Artemis, and Aphrodite were associated with the dyad.[4]

The triad derives its most distinctive epithets from the notion that after the two elements of number, 1 and 2, which are not really numbers, it is a true 'combination of monads,' the first actual number, because it has beginning, middle, and end. Hence its title 'mean,' and several others.[5] Leto, Hecate, Thetis, Athena and other divinities are assigned to it,[6] and this is the first number called 'marriage,' as a combination of odd and even numbers, 'male' and 'female' ($1 + 2$).[7]

The tetrad is a square,[8] is produced from $2 + 2$ and 2×2, and had long been reverenced as potentially the decad, because $1 + 2 + 3 + 4 = 10$.[9] Nicomachus took account of all these matters. Hermes, Hephaestus, Heracles, and Dionysus shared the tetrad: because the musical ratios are contained in it, it was called 'harmony.'[10]

[1] This epithet is given by Anatolius, Theon (p. 100, 9 ff.) and the *Theologumena Arithmeticae*, p. 8 *ad fin.*

[2] Photius; *Theologumena Arithmeticae, loc. cit.;* Lydus, *De Mensibus*, II. 6; Plutarch, *De Iside*, 75; Martianus Capella, p. 259, 3 (*motusque primi probamentum*). Lydus refers this to Pherecydes.

[3] See p. 117. This is why the dyad is called *μεταίχμιον πλήθους καὶ μονάδος* (cf. *Theologumena Arithmeticae*, p. 9, Ast), *ἀρχὴ τοῦ ἀρτίου μόνη, ῥίζα οὐ κατ' ἐνέργειάν πω, δύναμις, πόδες πολυπιδάκου* (*sic*) *Ἴδης* (?).

[4] For Erato, cf. *Theologumena Arithmeticae*, p. 11; Rhea, *ibid.*, p. 12; Isis, *ibid.* Etymology probably has much to do with the epithets Rhea and Isis (*ῥέω, ἴσος*). Matter is unstable and may be called 'fluid.'

[5] Probably this notion is at the bottom of the identifications *πρῶτος περισσός, μονάδων σύστημα, περατωτικὴ τῆς ἀπειρίας τῆς ἐν ἀριθμῷ, ὅμοιον, ταὐτόν, ὁμόλογον, ὡρισμένον.* *μεσότης*, however, is explained differently in *Theologumena Arithmeticae;* 3 stands between the greater and the less, and hence is 'mean,' thus: 2 is greater than the preceding number, 1, and 4 is less than the preceding numbers, $1 + 2 + 3$, while $3 = 1 + 2$.

[6] Hecate of course for her three forms; Leto possibly because 1 (Apollo) + 2 (Artemis) = 3 (their mother, Leto). Only the epithets *Τριτογένεια, Γοργονία* of Athena are given here.

[7] Cf. *Theologumena Arithmeticae*, p. 16.

[8] *Theologumena Arithmeticae*, p. 22, mentions this in connection with the epithet Hermes; it was also used by Anatolius, *Theon*, p. 101, 11; Philo, *De Mundi Opificio*, 16; Lydus, *De Mensibus*, IV. 44.

[9] That is, it was the tetraktys of the Pythagoreans, by which they swore; *Theologumena Arithmeticae*, pp. 18, 22. Perhaps to this fact are due the epithets in Photius, *τῶν φυσικῶν ἀποτελεσμάτων πηγή, κλειδοῦχος τῆς φύσεως, φύσις.*

[10] The topic harmony appears also in Anatolius, *Theon*, 101, 11 ff.; Philo, *De Mundi Opificio*, 15.

The pentad gains its glory from the fact that it is one half of 10,[1] the arithmetical mean between any two numbers whose sum is 10,[2] and the sum of an odd and an even number. The latter fact secures its place as 'marriage' and 'Aphrodite,' and it was of course pointed out that there are 5 elements and 5 zones on earth and circles in the heavens.[3] Nicomachus's most elaborate argument is devoted to establishing 5 as Justice.[4]

The hexad is the first 'perfect' number; it is the product of an odd by an even term, and 'cyclic' or 'spherical.' Its most important identification was with the 'soul,'[5] but it also was considered by Nicomachus as the marriage number par excellence, and 'Aphrodite.'[6]

Seven was treated with peculiar veneration, largely because of its connection with the moon and temporal periods, and hence, supposedly, with physical phenomena, such as those of birth.[7] This secured its titles 'chance' and 'due season' (**τύχη, καιρός**). Because of all the numbers of the decade it alone is both prime and does not as a factor produce any other number within the decade — neither generates nor is generated, as the Pythagoreans said — it is called 'Athena,' the virgin goddess.[8]

Eight, an unlucky number because children born in eight months were thought not to be viable,[9] is the first cube and hence was etymologically connected with Rhea-Cybele. Nine is a boundary, 'horizon' and 'ocean,' because after it there is repetition among the numbers;[10] it is identified with the Sun, Hephaestus, Hera, Prometheus, Apollo, Ares, and Artemis. The decad, since it embraces all the num-

[1] Hence called ἀμίθεος, διδυμαία, by Photius: cf. *Theologumena Arithmeticae*, p. 32; perhaps Δίκην οἷον δίχασιν, p. 31, for the same reason.

[2] Theon of Smyrna mentions this fact.

[3] Photius mentions the elements (τὰ στοιχεῖα τοῦ παντὸς κατὰ τὴν πεντάδα); cf. *Theologumena Arithmeticae*, p. 25; ζωνιαία, κυκλιοῦχος in Photius refer perhaps to the zones and circles; cf. *Theologumena Arithmeticae*, *ibid.*

[4] Cf. *Theologumena Arithmeticae*, pp. 27 ff.

[5] Cf. pp. 109 f.

[6] Cf. *Theologumena Arithmeticae*, p. 38. The point is that 6, as 2×3, is a more effective combination of 'male' and 'female' elements than 5, or $2 + 3$.

[7] The discussion of the *Theologumena Arithmeticae* takes the influence of 7 in gestation and in the ages of man as its chief theme here. The fact that the perfect number 28, which measures the lunar month, is $1 + 2 + 3 + 4 + 5 + 6 + 7$, was significant.

[8] Photius cites the name Athena and several epithets of the goddess here; cf. *Theologumena Arithmeticae*, p. 53. This was one of the commonest identifications, found in practically all the arithmologies. In fact the number 7 was the one upon which most efforts were expended by these writers.

[9] ἠλιτόμηνα in Photius; cf. *Theologumena Arithmeticae*, p. 55, and Lydus, *De Mensibus*, III. 51.

[10] Cf. *Theologumena Arithmeticae*, pp. 56–57.

bers and furthermore all numerical forms, was called perfect, and for these reasons Nicomachus gives it the epithets All, Cosmos, Universe, Faith, Necessity, Might, Fate, Eternity, Atlas, Unwearied God, Phanes, Sun, Urania, Memory, Mnemosyne.[1]

Our data regarding the cosmogonic theories of Nicomachus consist chiefly of two passages of the *Introduction* (I. 6 and I. 4. 2), and two in the *Theologumena Arithmeticae*. One of the latter, concerning the hexad, has already been quoted; the other, relating to the decad, follows:

"We have often anticipated ourselves in saying that the devising Mind wrought with reference to the resemblances and likenesses of number, as to a pattern that was perfect, the fabric and composition of the world and of all that is in it. But since the whole was an unlimited multitude, and the whole substance of number is not to be followed out, it was not reasonable . . . to use an incomprehensible pattern, but there was need of due measure, so that the artistically contriving God might be greater than the bounds and measures set before him in creation and hold sway over them, and neither compress in niggardly fashion nor unharmoniously exceed to a lesser or greater degree than the fitting; but a natural balance, mean, and wholeness existed above all in this (*sc.* the decad); for embracing in principle all things in itself, solids and surfaces, even and odd and even-times odd . . . it had of itself no peculiar or natural variance otherwise, save in the fact that all things ran toward it and circled into it. Reasonably therefore he used it as a measure for the whole, and as it were a gnomon and a straight edge for the setting forth (*sc.* of all), fitting (*sc.* them to it); wherefore things from heaven to earth are found both as wholes and in part to have their ratios of concord based upon it and to be ordered after it."[2] Then follow the identifications of 10, described above.

[1] Cf. *Theologumena Arithmeticae*, pp. 58–61, where most of them are explained.

[2] P. 58, Ast (the first sentence will be found on p. 98; the rest is as follows): ἐπεὶ δὲ ἀόριστον τὸ ὅλον πλῆθος ἦν καὶ ἀδιεξίτητος ἡ τοῦ ἀριθμοῦ πᾶσα ὑπόστασις, οὐκ ἦν εὔλογον οὐδ' ἄλλως ἐπιστημονικὸν ἀπεριλήπτῳ χρῆσθαι παραδείγματι, ἔδει δὲ συμμετρίας ἵνα τῶν προκειμένων αὐτῷ ὅρων καὶ μέτρων ὁ τεχνίτης θεὸς ἐν τῇ δημιουργίᾳ περιγένηται καὶ περικρατήσῃ, καὶ μήτε ἐπ' ἔλαττον μήτε ἐπὶ πλέον τοῦ προσήκοντος ἤτοι ἐνδεῶς συστείλῃ ἢ πλημμελῶς ὑπερεκπέσῃ· φυσική δέ τις συσταθμία καὶ μετριότης καὶ ὅλωσις ἐν τῇδε μάλιστα ὑπῆρχε· πάντα μὲν σπερματικῶς ἐντὸς αὑτῆς περιειληφυῖα, στερεὰ καὶ ἐπίπεδα, ἄρτιά τε καὶ περισσὰ καὶ ἀρτιοπέρισσα . . . μηδεμίαν δὲ ἰδιάζουσαν ἢ φυσικὴν ἄλλως παραλλαγὴν καθ' ἑαυτὴν ἔχουσα, ὅτι μὴ κατ' ἐπιδρομὴν καὶ ἀνακύκλησιν τὴν εἰς ἑαυτήν. εἰκότως μέτρῳ τῶν ὅλων αὐτῇ καὶ ὥσπερ γνώμονι καὶ εὐθυντηρίῳ ἐχρήσατο πρὸς [τὴν] πρόθεσιν ἁρμοζόμενος· διόπερ τοὺς κατ' αὐτὴν λόγους συμφώνους ἔχοντα τὰ ἀπ' οὐρανοῦ μέχρι γῆς ὁλοσχερέστερόν τε καὶ κατὰ μέρος εὑρίσκεται καὶ διακεκοσμημένα κατ' αὐτήν.

The general doctrine here set forth [1] appears to be essentially this: God, the supreme and original Being, has in his mind from all eternity number, all of the forms and properties of which are epitomized in the terms of the first decade. This is the divine or 'conceptual' number. In the process of world creation God devises an ideal plan for the universe, and this plan was most probably not an independently existing idea, but actually in God's mind. In it the numbers 'prevail,' that is, the 'sameness' of the forms of things is conferred by the monad, 'otherness' by the dyad, their actuality as distinguished from potentiality by the triad — for the peculiar property of the triad is to have 'beginning, middle and end,' and so to exist naturally and completely — and so on, down to completeness and order, which come from the decad.

From the language of the *Introduction* (I. 6. 1) we must certainly understand that this pattern is not number alone, nor are the ideas and forms in it identical with numbers; but simply that its components are determined, arranged and governed by number; and if the forms get 'sameness' from number, they thereby take from this source what is their essential nature. Following the outline of this plan, then, the material world is constructed; and because there was a numerical frame for its plan, the material world too is ordered upon the principles of arithmetic. The number which is seen in the world, and that which is seen in its structure, is that which must be studied by those who desire to gain a knowledge of the truth in things,[2] but it is not the divine number that existed from the first in God's mind. Although it is like that divine number which shapes the general plan, still it is on a lower plane, a quality of material things and associated with them; it is what we have found Nicomachus called 'scientific' number.

A problem of interest and importance is, whether in Nicomachus's cosmogony the act of creation was performed by God himself or by an intermediary divinity, a demiurge, according to the Platonic scheme. Nicomachus uses both the words *δημιουργός* and *θεός*, but there is very little evidence upon which to formulate a statement.[3] On the

[1] *Introduction*, I. 4. 3; 6. 1, are also taken into account in the following.

[2] *Introduction*, I. 2. 3–5.

[3] Cf. *Introduction*, I. 4. 2: *ἐν τῇ τοῦ τεχνίτου θεοῦ διανοίᾳ . . . ὁ τῶν ὅλων δημιουργός*; 6. 1: *τοῦ τὰ ὅλα δημιουργήσαντος νοῦ . . . ἐν τῇ τοῦ κοσμοποιοῦ θεοῦ διανοίᾳ.* On the use of *θεός*, see *Theologumena Arithmeticae*, p. 4 (Ast), *τὸν περὶ θεοῦ λόγον*; *τὸν θεόν* (22, cf. p. 5, 1, 7, 11, 18, p. 8, 13); *τῷ κοσμοποιῷ θεῷ*, p. 43, 34. For *δημιουργός* as an epithet of the monad, see p. 5, 21. Cf. also p. 59, 2, *ὁ τεχνίτης θεὸς ἐν τῇ δημιουργίᾳ*; p. 58, 28, *τὸν τεχνικὸν νοῦν*.

other hand, in the Nicomachean system there seems to be something to correspond more or less exactly with the world-soul of the *Timaeus*. Nicomachus devotes much attention to the elaborate mathematical account in the latter dialogue of the making of the world-soul and its distribution in the world,[1] but whether it is merely with a view to facilitating the interpretation of the passages or because he substantially agrees with the doctrine, it is hard to say. In the *Theologumena Arithmeticae* (p. 33 ff., Ast), in a context that is certainly Nicomachean, it is stated that the world is endowed with soul by virtue of the hexad. One portion of this passage has already been quoted, and we may add here the rest.

"In another way soul is capable of articulating and arranging body in the same general way that psychic form is capable of doing so to formless matter; and in general no number is more able to fit the soul than the hexad; no other would be called so much an articulation of the universe set up as maker of soul and discovered also capable of instilling the condition of life, whence it is called 'hexad.' That every soul is a harmonizing element, and that in harmony the most elementary concordant intervals are the sesquitertian and the sesquialter, by the combination of which the others are made up, is evident; for when it is present it makes peace, and orders well and fits best together the mingled opposites in the living thing, which yield and follow it, and it instills in this way health into the combination. . . . Indeed, so far as soul is present, it brings them together; but when it goes away, there comes about a breaking down of all the elements of the animal and a leaving of the ranks." [2]

He goes on to say that a half and a third part are necessary for the composition of the two elementary ratios of harmony mentioned, and that 6 has both; and in the succeeding remarks he states that the soul had to be solid and spherical. From 6 was derived one spherical

[1] Cf. p. 93, n. 2.

[2] P. 34: τρόπον δ' ἕτερον διαρθρωτικὴ καὶ συντακτικὴ σώματος ψυχή, καθάπερ ψυχικὸν εἶδος ἀμόρφου ὕλης, τῇ δὲ ψυχῇ τὸ παράπαν οὐδεὶς ἐφαρμόζειν δύναται μᾶλλον ἑξάδος ἀριθμός, οὐκ ἄλλος ἂν οὕτω διάρθρωσις τοῦ παντὸς λέγοιτο, ψυχοποιὸς ἱστάμενος, εὑρισκομένη καὶ τῆς ζωτικῆς ἕξεως ἐμποιητική, παρὸ ἑξάς. ὅτι μὲν ἁρμοστικὴ πᾶσα ψυχή, ἁρμονίας δὲ τὰ στοιχειωδέστατα σύμφωνα διαστήματα ἐπίτριτος καὶ ἡμιόλιος, ὧν κατὰ σύνθεσιν τὰ λοιπὰ συμπληροῦται, φανερόν· παρούσης μὲν γὰρ αὐτῆς εἰρηνεύει καὶ εὐτακτεῖ καὶ βέλτιστα ἐνήρμοσται τὰ ἐγκεκριμένα τῷ ζῴῳ ἐναντία, ὑπείκοντα καὶ ἀντακολουθοῦντα, καὶ διὰ τοῦτο ὑγείαν ἐμποιοῦσα τῷ συγκρίματι, θερμὸν ψυχρῷ, ὑγρὸν ξηρῷ, βαρὺ κούφῳ, πυκνὸν ἀραιῷ, καὶ τὰ ἐοικότα, ἃ χωρὶς ἁρμονίας τινὸς οὐκ ἂν συναναστρέφοιτο· συνυπάρχει γε μήν, ἐφ' ὅσον ψυχὴ πάρεστι, συναγωγὸς αὐτοῖς· ἐξελθούσης δ' αὐτῆς, διάλυσις τῶν ἐν τῷ ζῴῳ πάντων καὶ λειποταξία συμβαίνει. The etymology ἑξάς-ἕξις is evident.

number, from 5 another;[1] but the former was more suitable to soul than the latter, because 6 (2×3) is male-female or even-times odd, a combination of the male and female numbers, whereas 5 is male only. Of course the perfection of 6 was cited in this connection.[2]

The function of the world-soul is to establish harmonious working between the discordant elements composing the world; for, since they are constituted upon numerical principles, 'sameness' and 'otherness' enter the composition of things, the former on their formal, the latter upon their material, side, and confusion can be avoided only by the establishment of a harmony. The doctrine that harmony prevails in the universe is an old Pythagorean one.[3] There is not much material in what we have of Nicomachus to show exactly how he worked it out.

In the foregoing account of Nicomachus as a philosopher, the relation of his thought to Plato, Aristotle and the Stoics has been pointed out in a general way. It has been seen, too, that his philosophy draws very close in many respects to that of Philo Judaeus; in general he is typical of his age. It must be granted that his achievements as an arithmetician were more important than his philosophical triumphs, and the modern reader, failing perhaps to take sufficiently into account the fact that in Nicomachus's time the 'theology of numbers' was a well-recognized variety of speculation, bearing with it values which it is hard for us to comprehend, is likely to agree with the conclusion of Photius: "So it is necessary, it would seem, to spend and expend a whole life for the sake of this theological juggling with numbers, and to be a sober philosopher in mathematics in order to be able to talk consummate nonsense."

[1] Cf. *Introduction*, II. 17. 7.

[2] *Theologumena Arithmeticae*, p. 34, top (Ast). Cf. *Introduction*, I. 16. Perhaps it is not entirely fanciful to see in the aliquot parts of 6 (1, 2, 3) a similarity to the ingredients of the world-soul in the *Timaeus*. Nicomachus identified 1 with the 'same' and 2 with the 'other,' as we have seen, and 3 is the combination 1 and 2. Nicomachus of course does not say anything about this likeness.

[3] *Introduction*, II. 19. 1, where the note may be consulted.

CHAPTER VIII

NICOMACHUS'S PHILOSOPHY OF NUMBER

In the discussion of the philosophy of Nicomachus certain matters which belong with his theory of numbers have already been touched upon. In dealing further with that division of his doctrine, it will be found that though on basic matters he is in close agreement with non-Pythagorean writers, especially Aristotle, still the philosophic prejudices of his sect not infrequently intrude themselves and cause Nicomachus at times to bring non-mathematical elements into his arithmetical pronouncements. This, though fundamental, is after all not made the chief end in the composition of the *Introduction*, nor is it carried to an extent that would disturb the reader in ancient and medieval times, when all men to a surprising degree shared in the Pythagorean veneration of numbers; doubtless this is one characteristic of the book, together with its clearness, conciseness, and excellent organization, which caused it to survive when so many Pythagorean treatises have utterly perished.

Nicomachus's theory of number takes its start from the assumption of the fundamental qualities 'continuous' or 'discrete' for all things in the universe, of both the eternal and the transient classes, into which, as has already been seen, he made the first division of existent things.[1] Continuous things he called *ἡνωμένα*, 'unified,' or *ἀλληλουχούμενα*, 'holding to one another,' terms which testify to a notion that their parts are in direct contact, thereby unifying the object.[2] The discrete are *διηρημένα, ἐν παραθέσει, οἷον κατὰ σωρείαν*, 'separated into parts,' 'in the form of a setting forth side by side,' 'heap-like, as it were,'[3] terms which suggest the conception of objects ranged one beside the other without touching or merging, with the parts of the

[1] *Introduction*, I. 2. 4. *ποσότητες* and *μεγέθη* are among the eternals mentioned in I. 1. 3.

[2] In confirmation of this interpretation, cf. Aristotle, *Categoriae*, 4 b 25 ff.; where after classifying Quantum as discrete or continuous (see p. 113), he says: *τῶν μὲν γὰρ τοῦ ἀριθμοῦ μορίων οὐδείς ἐστι κοινὸς ὅρος πρὸς ὃν συνάπτει τὰ μορία αὐτοῦ . . ., ἡ δὲ γραμμὴ συνεχής ἐστιν· ἔστι γὰρ λαβεῖν κοινὸν ὅρον πρὸς ὃν τὰ μορία αὐτῆς συνάπτει, στιγμήν, καὶ τῆς ἐπιφανείας, γραμμήν, κτλ.*

[3] Cf. *Theologumena Arithmeticae*, p. 4, 30 (Ast): *τῶν ὅλων, ἐάν τε κατ' ἀλληλουχίαν ἐάν τε κατὰ παράθεσιν ἐπινοῶμεν αὐτὴν συνεστάναι. . . .* See also *ibid.*, p. 17, 5 ff.

collection keeping their identity; perhaps this helps to an understanding of what is meant by the word 'flow' (χύμα) in the definition of number.

Nicomachus next tells us tnat an object in the continuous class is called 'magnitude' (μέγεθος) and one in the discrete 'multitude' (πλῆθος); the two terms are used both as abstracts referring to the quality, and, as concretes, of objects of such natures.[1] Though these are mere statements and no discussion of them is found in our sources, we see from this that Nicomachus assumes continuity and discreteness to be qualities always associated with magnitude and multitude. He has not yet arrived at the definition of number, but only a single step further is needed. Magnitude and multitude per se are indefinite terms, 'simply great' or 'simply many,' as Nicomachus says.[2] Without further limitation they are infinite; the 'great' can be infinitely subdivided and the 'many' infinitely increased. With such things science cannot deal. We can know about them and scientifically deal with them only when it can be said 'how great' and 'how many' they are. This is not to assert any new quality of magnitudes and multitudes, but merely to impose a limitation upon them, to renounce dealings with the 'many' and the 'great' per se and to occupy ourselves with magnitudes and multitudes of limited and known extent.

It is in this sense, as opposed to μέγεθος and πλῆθος, that Nicomachus employs the terms τὸ πηλίκον (πηλικότης) and τὸ ποσόν (ποσότης), which may be translated 'quantity' and 'number.' Furthermore, as will be seen, the latter is practically synonymous with number (ἀριθμός), since the latter is defined as 'limited multitude.' In this account of the fundamentals of arithmetic, it is evident that Nicomachus is following Aristotle, although the terminology of the two differs.

Aristotle's words are sufficient evidence. "The term 'quantum' is used of that which may be divided into components, whereof either or each is naturally one thing or this thing (i.e., an individual thing). A quantum is multitude if it can be numbered, and a magnitude if it can be measured. And that is called a multitude which is potentially divisible into non-continuous things; a magnitude, into continuous. Of magnitude, that which is continuous in one direction is length;

[1] *Introduction*, I. 2. 4; here they are individuals, but in I. 2. 5, genera; cf. I. 16. 2; *Theologumena Arithmeticae*, p. 8, 21; p. 9, 28, 34.

[2] ἁπλῶς μέγας would be his expression; cf. ἁπλῶς μέγεθος, ἁπλῶς πλῆθος, p. 5, 7 (Hoche), and I. 14. 1. What follows is based on I. 2. 5.

in two, breadth; in three, depth. And of these, limited multitude is number; length, a line; breadth, a surface; depth, a body." [1]

Aristotle's term *τὸ ποσόν* is used in a different sense; it is quantum, a general term, and not limited multitude. But this is the only serious difference. Magnitude and multitude per se are *μέγεθος* and *πλῆθος* in both authors, and limited multitude *ἀριθμός*. Aristotle does not call limited magnitude, *τὸ πηλίκον*, as Nicomachus does, but uses rather the specific names body, surface, and line, and he prefers *συνεχής* to mean 'continuous,' not *ἀλληλουχούμενος*.[2] These points of variation are, however, very slight, and to offset them we can point to parallels in Aristotle to the doctrine that magnitude and multitude are infinite per se.[3] Altogether Nicomachus's dependence on him is obvious.

This division of objects into quantities and numbers, too, furnishes Nicomachus the basis for the determination of the subjects of the sciences that treat of them, and the result is the *quadrivium*, a term apparently first used by Boethius and famous throughout the Middle Ages.[4] Each half of the field claims two sciences; numbers absolute or per se belong to arithmetic; numbers in their mutual relations, to music; geometry treats of quantity at rest, and astronomy of quantity in motion. As a matter of fact, the first two at least of these

[1] *Metaphysica*, IV, 13, 1020 a 7 ff.: ποσὸν λέγεται τὸ διαιρετὸν εἰς ἐνυπάρχοντα, ὧν ἑκάτερον ἢ ἕκαστον ἕν τι καὶ τόδε τι πέφυκεν εἶναι. πλῆθος μὲν οὖν ποσόν τι ἐὰν ἀριθμητὸν ᾖ, μέγεθος δ' ἐὰν μετρητὸν ᾖ. λέγεται δὲ πλῆθος μὲν τὸ διαιρετὸν δυνάμει εἰς μὴ συνεχῆ, μέγεθος δὲ τὸ εἰς συνεχῆ· μεγέθους δὲ τὸ μὲν ἐφ' ἓν συνεχὲς μῆκος, τὸ δ' ἐπὶ δύο πλάτος, τὸ δ' ἐπὶ τρία βάθος. τούτων δὲ πλῆθος μὲν τὸ πεπερασμένον ἀριθμός, μῆκος δὲ γραμμή, πλάτος δὲ ἐπιφάνεια, βάθος δὲ σῶμα.

[2] Cf. the following Aristotelian passages: *Metaphysica*, 1054 a 22, τὸ μὲν γὰρ ἢ διῃρημένον ἢ διαιρετὸν πλῆθός τι λέγεται. Cf. *ibid.* 1015 b 36; *Physica*, 233 a 11, μέγεθος ἅπαν ἐστὶ συνεχές κτλ.; *De Caelo*, 268 a 4 ff.; *Categoriae*, 4 b 20, τοῦ δὲ ποσοῦ τὸ μέν ἐστι διωρισμένον μὲν οἷον ἀριθμὸς καὶ λόγος, συνεχὲς δὲ οἷον γραμμή, ἐπιφάνεια, σῶμα. . . . In Sextus Empiricus, *Adversus Mathematicos*, IV. 1, τὸ ποσόν is divided into μέγεθος in continuous things (συνεχῆ) and ἀριθμός in discrete (διεστῶτα).

[3] Cf. *Metaphysica*, 1066 b 7: ἔτι πῶς ἐνδέχεται καθ' αὑτὸ εἶναι ἄπειρον, εἰ μὴ καὶ ἀριθμὸς καὶ μέγεθος, ὧν πάθος τὸ ἄπειρον; 1085 b 24: ὑπῆρχε γὰρ . . . καὶ πεπερασμένον πλῆθος, ἐξ οὗ αἱ πεπερασμέναι μονάδες καὶ τοῦ ἑνός· ἔστι δὲ ἕτερον αὐτὸ πλῆθος καὶ πλῆθος ἄπειρον. Cf. also *De Caelo*, 271 b 33 ff. But no actually existent object in these genera can be infinite: *Metaphysica*, 1066 b 18: ἀλλ' ἀδύνατον τὸ ἐντελεχείᾳ ὂν ἄπειρον· ποσὸν γὰρ ἀνάγκη; similarly *Physica*, 204 a 28.

[4] Cf. Gow, *History of Greek Mathematics* (Cambridge, 1884), p. 72, n. 1. The division of the field in Theon of Smyrna, p. 15, 13 ff. (Hiller), is different: (*a*) arithmetic; (*b*) geometry, i.e., plane geometry; (*c*) stereometry, or solid geometry; (*d*) astronomy (ἀστρονομία), concerned with moving solids. Besides these four, music deals with "the movement, order, and harmony of the moving stars"; elementary harmony — the mathematical theory of the concords, etc., — is attached to arithmetic. Note that σφαιρική means astronomy, not stereometry, in Nicomachus; cf. the further references in I. 3. 7; 5. 2 and Philoponus on the latter passage.

definitions are arbitrary and artificial, and in practice Nicomachus disregards them, discussing in a long section of the *Introduction* relative number,[1] which should fall under music, and furthermore treating numbers in another part of the book after the fashion of geometrical planes and solids.[2] Nicomachus held, and demonstrated at length, that arithmetic is the fundamental and indispensable science, the basis of all the others.[3]

Nicomachus now comes to the formal, threefold definition of number.[4] In the first place, he states, number is 'limited multitude,' *πλῆθος ὡρισμένον*; secondly, it is 'a combination of monads,' *μονάδων σύστημα*; and thirdly, 'flow of number, composed of monads,' *ποσότητος χύμα ἐκ μονάδων συγκείμενον*. The first of these is identical with that which has already been defined as the subject of scientific study in the realm of multitude; it is equivalent to *ποσόν*, *ποσότης*, and has already been translated 'number.' Aristotle had stated his definition of number in approximately these terms,[5] and Eudoxus also is reported to have adopted it.[6] According to this definition number is simply a species of the genus multitude, with the differentia limitation. Of the nature of this limitation Nicomachus has nothing to say, although some explanation is really necessary.

In his second definition Nicomachus agrees with Theon of Smyrna [7] and if we are to regard the testimony of Iamblichus on the point, the definition was as old as Thales and was derived by him from Egyptian sources.[8] This, too, comes closest of the three to Euclid's definition, *τὸ ἐκ μονάδων συγκείμενον πλῆθος*.

The third definition has no exact parallels; the second definition given by Theon perhaps comes closest to it, "the advance of multitude beginning with the monad and its retreat ending in the monad" (*προποδισμὸς πλήθους ἀπὸ μονάδος ἀρχόμενος καὶ ἀναποδισμὸς εἰς μονάδα καταλήγων*), and this in turn was practically identical with

[1] I. 17 — II. 5, and II. 21 to the end (the latter dealing with proportions).

[2] II. 6–17.

[3] I. 4–5.

[4] *Introduction*, I. 7. 1. Heath, *History*, vol. I, p. 69.

[5] πλῆθος . . . τὸ πεπερασμένον ἀριθμός, *Metaphysica*, IV. 13.

[6] πλῆθος ὡρισμένον, according to Iamblichus, *In Nicomachi Arithmeticam Introductionem*, p. 10, 18 (Pistelli).

[7] P. 18, 3 ff. (Hiller).

[8] Iamblichus, p. 10, 8–10. Euclid's definition is found in the *Elements*, VII, def. 2. Aristoxenus used a definition like Euclid's, according to the fragment of his Περὶ ἀριθμητικῆς in Stobaeus, *Eclogae* I, *Prooem.* 6, vol. I, p. 20, 1 (ed. Wachsmuth-Hense = Diels, *Fragmente der Vorsokratiker*, I³, p. 346. 2).

the expression used by Moderatus of Gades, the Pythagorean.[1] Of the three, this is the most truly Pythagorean, and it evidently has reference to that conception of number as a stream, moving out from the monad, of which more will be said later.

The number which Nicomachus has just defined and which is dealt with in the *Introduction* is the 'scientific' number, and not, as the previous discussion has shown, to be identified with the conceptual number which was the basis of creation.[2] It is a matter of regret that there is nowhere a full explanation of the relation between these two numbers, and furthermore, that Nicomachus left without discussion here two subjects treated at some length by Theon of Smyrna,[3] the monad and its counterpart, one, and the distinction between numbers and numerable things (ἀριθμοί, ἀριθμητά). On these subjects we are reduced almost to conjecture.

As that of which every number is made up, and that into which it can be reduced ultimately by analysis,[4] the monad and the dyad are singled out by Nicomachus as the elements of number. This may seem strange, for the very definition of 'element' demands that it should be something ultimate, incapable of further analysis, and the dyad, which is twice 1, does not, apparently, answer this requirement. Even on Nicomachus's own statement this can be alleged, for in certain passages he clearly enough says that the dyad comes from the doubling of the monad,[5] or that all numbers are made up of monads,[6] while on the other hand he often refers in the *Introduction* to the dyad, as well as to the monad, as an element of number.[7] Yet this must be regarded as only a minor inconsistency, for it is certain that the dyad was elementary according to his system of numbers, and the reason can be determined.

[1] Stobaeus, *Eclogae* I, *Prooem.* 8, vol. I, p. 21 (ed. Wachsmuth-Hense).

[2] See p. 98.

[3] P. 18, 3 — 21, 19 (Hiller).

[4] This is Nicomachus's definition of 'element,' II. 1. 1.

[5] See *Introduction*, II. 17. 1 (p. 109, 6 H.), and the division of 2 into 2 unities, as I. 8. 4–5. Even in the *Theologumena Arithmeticae*, where for the most part the dyad is spoken of as elementary, it is implied that the monad produces it; e.g., p. 6 (Ast): καὶ δυάδος γὰρ παρεκτικὴ διαφορηθεῖσα. Cf. also the passages where the dyad and other numbers are thought of as progressing out of the monad; p. 116, with n. 2.

[6] *Ibid.*, I. 7. 1 (the definition of number); 8, 2; 10 (p. 16, 16 H.); 11, 3; II. 6. 2–3; 7. 3. Cf. also such expressions as διαλυόμενος εἰς μονάδας, II. 8. 1 and the following chapters. I. 11. 3 is particularly instructive.

[7] *Ibid.*, II. 1. 1; 17. 2; 18. 1, 4. The important passages of the *Theologumena Arithmeticae* will be cited in the following discussion.

An explanation of the matter may be sought along two similar lines of argument which finally fuse. According to the first, one may regard the monad and the dyad as the sources, origins or beginnings (ἀρχαί) of number, and in the second place, as its elements (στοιχεῖα).

Nicomachus's method of approach along the first line is graphic and depends somewhat upon the help of geometry and its fundamental conceptions.[1] In geometry we begin with the point, which is indimensional. This is the beginning of the first dimensional form, the line, and by movement the point generates the line. Now Nicomachus had a similar idea of the nature of multitude and number; they form a series, as it were a moving stream, which proceeds out of unity, the monad.[2] Just as the point is not part of the line (for it is indimensional, and the line is defined as that which has one dimension), but is potentially a line, so the monad is not a part of multitude nor of number, though it is the beginning of both, and potentially both.[3] The monad is unity,[4] absence of multitude, potentiality; [5] out of it the dyad first separates itself and 'goes forward' [6] and then in succession follow the other numbers. Now all this concerns the monad far more than the dyad, and in fact it is the former which Nicomachus distinguishes as the ἀρχή par excellence; the dyad is rather 'like a beginning,' ἀρχοειδής.[7]

[1] What follows is substantially the argument of *Introduction*, II. 6. 3 ff.

[2] This, with the kindred idea that 1 contains all the numbers potentially, is why the monad is identified with the chaos of Hesiod, *ἐξ οὗ τὰ λοιπὰ ὡς ἐκ μονάδος*, *Theologumena Arithmeticae*, p. 6 (Ast). Cf. *ibid.*, p. 11, 22 ff.: *ὅτι μήκους πρώτη ἔμφασις ἐν δυάδι, ὡς ἀπὸ σημείου τῆς μονάδος, ἐπ' ἄπειρον δὲ τοῦτο καὶ διαιρεῖται καὶ αὔξεται.*

[3] It was a well-known principle in antiquity that the beginning of a thing is not the thing itself: cf. II. 6. 3. Although here it is distinctly stated that the monad is not *ἀριθμός*, and in *Theologumena Arithmeticae*, p. 13 (cited, n. 6), that it is not *πλῆθος*, still in one passage of this work, p. 18, 1 ff., it is called *ποσόν τι*: *ἔστι γὰρ ποσόν τι ἡ μονὰς καὶ καθ' ἑαυτὸ δὲ θεωρούμενον καὶ μονώτατον περαῖνον καὶ ἀληθῶς ὁρίζον.* But it is at least certain that a special kind of *ποσόν* is here referred to, and the passage would also suggest that to Nicomachus *πλῆθος* connotes always plurality.

[4] *Theologumena Arithmeticae*, p. 8: *τῆς . . . μονάδος ἕνωσιν δηλούσης.* Every thing is called 'one' in accordance with it, *ibid.* See page 103. The monad is indivisible: *Introduction*, I. 8, 4–5; 10, 2.

[5] The monad potential number: *Theologumena Arithmeticae*, p. 6, 9 ff.; p. 13, bottom, p. 17. Potentially all the forms of number; *ibid.*, p. 3; 4, 22; 5, 15, 31, etc. In the *Introduction*, cf. I. 16. 8; 9. 2, 3; 10. 2; 14. 1; 15. 3; 17. 7.

[6] *Theologumena Arithmeticae*, p. 13: *ὅτι ἡ μὲν μονὰς τοῦ παντὸς ἀριθμοῦ λόγον ἀδιατύπωτον, ἔτι καὶ ἀδιάρθρωτον ὡς ἐν σπέρματι ἐν ἑαυτῇ ἔχει, ἡ δυὰς δὲ βραχεῖά τις ἐπ' ἀριθμὸν προχώρησις, οὐκ ἄντικρυς δὲ τοιαύτη διὰ τὸ ἀρχοειδές, ἡ τριὰς δὲ τὴν τῆς μονάδος δύναμιν εἰς ἐνέργειαν καὶ ἐπέκτασιν προχωρεῖν ποιεῖ.* Cf. also the designation of the dyad as 'daring,' a common arithmological topic (see p. 105): *προβέβηκε γὰρ ἤδη ἐπὶ πρᾶξιν* (*Theologumena Arithmeticae*, p. 7).

[7] *Theologumena Arithmeticae*, p. 15: *ἀρχὴ κατ' ἐνέργειαν ἀριθμοῦ ἡ τριὰς μονάδων συστήματι ὁριζομένου· μονὰς μὲν γὰρ τρόπον τινὰ ἡ δυὰς διὰ τὸ ἀρχοειδές, σύστημα δὲ μονάδος καὶ δυάδος ἡ τριὰς πρώτη.* Cf. the preceding note. The monad alone mentioned as *ἀρχή*, *Introduction*, I. 8. 2; II. 6. 3; 7. 3.

For in the progress out of unity into multitude, he states, we do not encounter an actual number until we come to the triad [1] and the dyad is neither one thing nor the other.[2]

The real grounds for this notion are not easy to grasp in the fragmentary state of the evidence. It does not seem to be a conclusive reason that the dyad cannot be divided, like other even numbers, into both equal and unequal parts,[3] nor that whereas 1, as the element and origin, gives a sum greater than its product ($1 + 1 > 1 \times 1$), and true numbers characteristically give a product greater than their sum (e.g., $3 \times 3 > 3 + 3$), the product of 2 by 2 is equal to the sum of $2 + 2$, thus constituting 2 a middle ground between unity and multitude.[4] These are both alleged; but a more fundamental reason in the eyes of Nicomachus seems to be that real numbers must have form (εἶδος) and arrangement (σχῆμα), or, as he otherwise puts it, be a real 'combination' (σύστημα) of monads; and the dyad fails in all these particulars, while the triad satisfies the conditions.

As Nicomachus says, "Each thing in the world is 'one' in accordance with the natural and systematizing monad in it, and again everything is separable so far as it partakes of the dyad, connected with necessity and matter; wherefore first their congress produced the first multitude, the element of things, which would be a triangle, whether of magnitudes or numbers, bodily or bodiless. For as rennet curdles flowing milk by its peculiar creative and active faculty, so the unifying force of the monad advancing upon the dyad, source of easy movement and breaking down, infixed a bound, and a form, that is, number, upon the triad; for this is the beginning of actual number, defined by combinations of monads. But the dyad too is a monad, because of its beginning-like nature." [5]

[1] Cf. the preceding notes.

[2] Cf. *Theologumena Arithmeticae*, p. 13 (cited p. 116); p. 9: ὅτι νοουμένου πλήθους κατὰ τριάδα τοῦ τ' ἀντιθεωμένου τῷ πλήθει κατὰ τὴν μονάδα μεταίχμιον ἡ δυὰς ἂν εἴη.

[3] *Ibid.*, p. 11: οὐκ ἀριθμὸς δὲ ἡ δυὰς οὐδὲ ἄρτιος, ὅτι μὴ ἐνεργείᾳ· ἀμέλει πᾶς ἄρτιος καὶ ἴσα καὶ ἄνισα δύναται ὁ αὐτὸς μερίζεσθαι, μόνη δὲ ἡ δυὰς εἰς ἄνισα οὐκ ἂν μερισθείη, καὶ εἰς ἴσα μερισθεῖσα ἄδηλον, ὁποτέρου γένους αὐτὰ ἕξει, ὡς ἀρχοειδής τις οὖσα. Cf. the argument of *Introduction*, I. 8. 1–2, that the monad is the source because it is half of one number adjacent to it, whereas other numbers are the half of the two adjacent terms.

[4] *Theologumena Arithmeticae*, p. 9. Another argument somewhat of the same character is that of one or two objects we say ἑνικά, δυϊκά, but of three, not τρισσικά, but simply πληθυντικά (*ibid.*, p. 15, 21 ff.).

[5] *Ibid.*, p. 8 (Ast): ἓν μὲν οὖν ἕκαστόν τι ἐν κόσμῳ κατὰ τὴν ἐν αὐτῷ φυσικὴν καὶ συστηματικὴν μονάδα, διαιρετὸν δὲ πάλιν ἕκαστον, καθ' ὅσον ἀναγκαίου καὶ ὑλικῆς δυάδος μετέσχε· διόπερ ἡ πρώτη σύνοδος αὐτῶν πρῶτον ὡρισμένον πλῆθος ἀπετέλεσε, στοιχεῖον τῶν ὄντων, ὃ ἂν εἴη τρίγωνον μεγεθῶν τε καὶ ἀριθμῶν σωματικῶν τε καὶ ἀσωμάτων· ὡς γὰρ ὁ ὀπὸς τὸ κεχυμένον γάλα συστρέφει κατὰ τὸ

The dyad has no form, and the triad has, for the dyad, as 'otherness,' is infinite: "It appears also to be the 'infinite,' since it is the 'other,' and this beginning with the next to one goes on to infinity."[1] It is "without arrangement, for from the triangle and the triad the actual polygonals to an infinite number of sides advance; but no plane figure was ever composed of two straight lines or two angles; so it is in accord with this alone that we have the 'undefined' and 'without arrangement.'"[2] One further factor to be taken into consideration is that the triad is the first to show what Nicomachus calls 'natural sequence,' that is, the possession of beginning, middle, and end. As such, it is the idea of completeness, and the dyad fails to measure up to its standard.[3]

The second mode of demonstration, based on the elementary character of the monad and the dyad, is more satisfactory, and has already been foreshadowed in the discussion of Nicomachus's philosophy. There it was seen that the monad and the dyad were actually identified with 'sameness' and 'otherness,'[4] that is, they are not so much numbers themselves as forms which are impressed upon numbers and things. If we put the matter on the basis of the Aristotelian logic of which Nicomachus is so fond, it becomes evident that they are not true numbers (*ἐνεργείᾳ*, *ἐντελεχείᾳ*) but only forms, and as the sources of numbers, they are themselves potential numbers (*δυνάμει*).

The frequent occurrence of these terms makes it allowable to take this view. Furthermore, the analysis of numbers shows in them the presence of the monad and dyad, 'sameness' and 'otherness,' as elements. The chief instances mentioned by Nicomachus are the odd, square, and cubic numbers, which are characterized by 'sameness,' and the even and heteromecic numbers, which display 'otherness.'

ποιητικόν τε καὶ ἐργαστικὸν ἰδίωμα, οὕτως ἡ ἑνωτικὴ δύναμις τῆς μονάδος, προελθοῦσα τῇ δυάδι, εὐπορίας καὶ λύσεως οὔσῃ πηγῇ, πέρας ἐνεποίησεν, εἶδος δέ, ὅπερ ἐστὶν ἀριθμός, τῇ τριάδι· ἀρχὴ γὰρ κατ' ἐνέργειαν ἀριθμοῦ αὕτη, μονάδων συστήμασιν ὁριζομένου. μονὰς δὲ τρόπον τινὰ καὶ ἡ δυὰς διὰ τὸ ἀρχοειδές.

[1] *Ibid.*, p. 11: ὅτι δὲ καὶ τὸ ἄπειρον φαίνεται, εἴγε καὶ τὸ ἕτερον, τοῦτο δὲ ἀπὸ τοῦ παρ' ἓν ἀρξάμενον εἰς ἄπειρον ἐκπίπτει.

[2] *Ibid.*, p. 11: ὅτι ἡ δυὰς φαίνεται ἀσχημάτιστος, εἴπερ ἀπὸ μὲν τριγώνου καὶ τριάδος τὰ ἐπ' ἄπειρον πολύγωνα ἐνεργείᾳ προχωρεῖ, ἐκ δὲ μονάδος πάνθ' ὁμοῦ κατὰ δύναμιν ὑπάρχει, ὑπὸ δὲ δύο οὔτε εὐθειῶν ποτε οὐδὲ γωνιῶν εὐθύγραμμον συνίσταται σχῆμα· κατὰ μόνην ἄρα αὐτὴν τὸ ἀόριστον καὶ ἀσχημάτιστον.

[3] *Ibid.*, p. 14: ἔτι δὲ καὶ πᾶσα οὐσία διέξοδον ἔχουσα φυσικὴν ὅρους ἔχει τρεῖς, ἀρχήν, ἀκμήν, τελευτήν, οἷον πέρατα καὶ μέσον, διαστήματα δέ, οἷον αὔξησιν καὶ φθίσιν, ὥστε τὴν μὲν δυάδος φύσιν καὶ τὸ ἑκάτερον ἐμφαίνεσθαι τῇ τριάδι τῶν περάτων. See also *ibid.*, p. 15, top. Cf. the explanation of this by Moderatus of Gades, p. 103.

[4] See p. 100.

The reason why the odd and even have these qualities, as stated by Nicomachus,[1] is that 'their species are formed' (εἰδοποιεῖσθαι) by the monad and dyad respectively.[2] The squares and cubes, on the one hand, and the heteromecic numbers, on the other, receive 'sameness' and 'otherness' from the fact that their composition is dependent upon the odd and even numbers, and, furthermore, they show it either in the equality of their sides, or, in the case of the heteromecic numbers, in their inequality.[3] The cubes in a special sense are the product of odd numbers, and partake of 'sameness' to an even greater extent.[4] The elements, then, of these series are the monad and dyad, and, as we have seen, they are practically identified with forms.

But after all the dyad is never quite on a level with the monad as an element; a minor indication of the feeling of Nicomachus is seen in his rather condescending remark that it is 'a monad in a sense' and in his designation 'beginning-like' for it.[5] For at bottom it is the monad that is the real beginning of the number series. It is that which 'remains,' ever the same itself and in mathematical operations conferring the same persistence upon other numbers; [6] the monad too is potentially all numbers.[7] It has already been noted that the numerical series is compared to a moving stream, progressing out of the monad, which occupies the place of a point, and in the first movement going on to the dyad, which introduces 'otherness' after the absolute unity of the monad.[8] The movement then extends to the triad, the first true

[1] See *Introduction*, II. 17. 2, and Iamblichus, *In Nicomachi Arithmeticam Introductionem Liber*, p. 12, 26 ff. Though Nicomachus does not mention the matter, it was an old Pythagorean notion that the 'odd' and 'even' were finite and infinite respectively because the one resists, the other admits, dichotomy: Simplicius, *Physica Auscultatio*, p. 105 a: οὗτοι δὲ τὸν ἄπειρον τὸν ἄρτιον ἀριθμὸν ἔλεγον, διὰ τὸ πᾶν μὲν ἄρτιον, ὥς φασιν οἱ ἐξηγηταί, εἰς ἴσα διαιρεῖσθαι, τὸ δὲ εἰς ἴσα διαιρούμενον ἄπειρον κατὰ τὴν διχοτομίαν. The behavior of 'odd' and 'even' in addition and multiplication was also observed.

[2] *Introduction*, II. 17. 2; 18. 1, 4; 20. 2. The reason why the 'odd' is 'given specific form' by the monad seems to be at least partly accounted for by definition of the 'odd' (I. 7. 2); whereas 2 is always a factor of the 'even.' Furthermore these numbers were first in the two classes, though there was some dispute over this point, and seem to embody 'oddness' and 'evenness' in the same way as 'sameness' and 'otherness.'

[3] Squares: *Introduction*, II. 17. 3; 18. 3; heteromecic numbers, *ibid.*, and II. 18. 1.

[4] *Ibid.*, II. 20. 5.

[5] See p. 116, with notes 6 and 7.

[6] *Introduction*, II. 6. 3; 17. 4; *Theologumena Arithmeticae*, p. 3, 2 ff. It is to be noted that μονή and μένειν were technical terms in Neo-Platonic theology to describe the unchanging existence of God; cf. Plotinus, *Enneades*, I. 7. 1, and the Christian application of the same term in Augustine, *De Genesi ad Litteram*, IV. 18. 34. The Neo-Pythagorean doctrine of the monad probably plays a part in the development of this idea.

[7] See p. 116, n. 5.

[8] *Theologumena Arithmeticae*, p. 8, 11: ἀρχή τε καὶ πυθμὴν ὡσανεὶ τῆς τοῦ ἀριθμοῦ ἑτεροειδείας κατ' εἰκόνα ὕλης; *Introduction*, II. 17. 5: ἑτερότητος καταρκτική.

number, and so on to the decad, which in a sense repeats the monad.[1] That is, it has the functions in the series 10, 20, 30 . . . 100 which the monad has in the series 1, 2, 3 . . . 10, and all the terms between 10 and 100 are made up of components from these two series singly or in combination. The series goes on still further, with 100, 1,000, and 10,000 successively assuming the position of the monad in their respective series or courses, and because of this they were called by the Pythagoreans "monads of the second, third, etc., courses." The Pythagoreans, it may also be remarked, did not recognize that 12, for example, might be made the end of the first series precisely as well as 10; they were convinced that 10 was divinely and naturally constituted as the climax of the series and that no other could vie with it. It was an instance of the operation of 'nature' as opposed to 'human convention' — a contrast which Nicomachus is fond of pointing out[2] — and evidenced by man's possession of ten fingers and toes, by the ten categories and the ten forms of relative number.[3]

In this movement of number from the monad Nicomachus pictures its advance by steps, or places (χῶραι), occupied by the successive terms (ὅροι) or number; for number, it will be remembered, is the property of the discrete. The number system is thus based on the integers, and Nicomachus does not consider zero a part of it.[4] The beginning of the series is 1, for although 1 and 2 are not really numbers at all, they begin the numerical series, and we find Nicomachus constantly using them in exactly the same way as the others.

Furthermore, the numerical series is 'by marvellous and divine nature' a harmony, that is, 'a unification of the diverse and a concord of the disagreeing.'[5] All harmonies have to be constructed out of opposites, and in number these are of course the 'same' and the 'other' as seen especially in the odd and even numbers and in their derivatives. The odd and the even give evidence at the very start of the harmonious construction of the whole system by their occurrence in alternate places in the natural series.[6] The harmony of the numbers, which is of course

[1] For number as a moving stream, cf. p. 116. On the 'monads of the courses' and the Pythagorean decimal system, cf. I. 19. 17 and note *ad loc.*

[2] Cf. *Introduction*, I. 6. 4; 23. 7; II. 3. 2; 17. 2; *Theologumena Arithmeticae*, p. 3, 25 (Ast).

[3] The only reference in the *Introduction* to the sacredness of the decad is II. 22. 1; but compare the *Theologumena Arithmeticae* throughout the chapter on the decad.

[4] The only reference to zero in the *Introduction* is in II. 6. 3, and here it is not a term.

[5] This is a quotation from Philolaus cited in II. 19. 1. Most of the data upon the harmony of the natural series will be found in I. 6. 2 ff.

[6] *Introduction*, I. 6. 4.

to Nicomachus evidence of the harmony of the universe, displays itself in all sorts of regularities, observed in the relations of numbers; those which Nicomachus delights to point out as proofs of the 'good order,' [1] 'friendship' and 'coöperation' [2] of numbers are for the most part the inflexible workings of rules, and the observation that numbers of certain specific characters occur at stated intervals either in the natural series or in some other regularly constituted group derived from it, as, for example, the odd numbers or the doubles.[3] The discovery of other facts must have assisted the Pythagorean mathematicians greatly in their belief in harmony, for instance, the discovery of the musical concords in the first few numbers of the natural series and in the numbers of the sides, edges, and angles of the cube.[4]

Harmony, however, must be founded not on a series of opposites only, but upon opposites which are not irrational to one another.[5] This requirement is met by the numbers of the natural series. They have certain fixed relations to one another, capable of expression and definition, which Nicomachus studies under the head of relative number.[6] The fundamental relation of any two terms is equality with one another; [7] other relations will vary to one side or the other, and numbers will exceed or be exceeded by one another according to the various ratios. We may leave to Nicomachus himself the exposition of this subject, though it may be remarked that he gives an inadequate definition of ratio.[8]

Another characteristic of numbers, which does not concern their mutual relations so much as themselves absolutely, is their ability, according to the ideas of Nicomachus, to conform to geometrical arrangements.[9] Fundamentally this rests upon the definition of number as a combination of monads, and the further assumption that the constituent monads of any term are capable of arrangement (σχῆμα). Granting this, the arrangements may be either linear, plane or solid, and may imitate any of the figures recognized by geometry. An ex-

[1] See εὐταξία, εὔτακτος in Hoche's index.

[2] *Introduction*, II. 19. 1.

[3] Instances of this sort are found in those sections where Nicomachus describes the 'generation' of certain types of numbers, e.g., the multiples, I. 18. 4 ff. The regular comparison of terms between two series gives certain types of ratio in regular order; cf. I. 19. 8 ff.

[4] *Introduction*, II. 15. 4; 26. 2, on the musical concords as discovered in the cube. Arithmological writers point out that 4 includes the chief harmonies.

[5] *Ibid.*, I. 6. 3.

[6] *Ibid.*, I. 17, and following chapters.

[7] *Ibid.*, I. 23. 6; II. 1. 1–2.

[8] *Ibid.*, II. 21. 3; see the note *ad loc.*

[9] *Ibid.*, II. 6 ff., treats of plane and solid numbers.

ample of this sort of thinking has already been met, implicit in the statement that the triad has beginning, middle, and end, which can have no meaning unless it implies that the triad is to be regarded as a linear arrangement of three monads, 1 1 1.[1] Numbers of this sort are called linear (γραμμικοί, II. 6. 1). But these three monads can be as easily arranged in two dimensions as in one, and then they will form a triangle, ∴, and the same can be done with any of the numbers which are summations of the natural series from 1 to any given term. The triangular number is the elementary form of the plane number, just as the triangle is the element of plane figures in general; [2] but the numbers are capable also of arrangements in the form of squares, pentagons and all the regular polygons, as well as in the form of parallelograms of all kinds. Furthermore, if a third dimension be added, and the monads grouped in more than one plane, all sorts of solid numbers can be constructed.

One further point remains to be emphasized, namely, that the Pythagoreans could not regard numbers in the cold, impartial manner of the modern mathematician and that Nicomachus is at one with them in this. Numbers are the sources of form and of energy in the world; they are dynamic, active even on their own fellows; hence they convey to one another qualities and sometimes take on an almost human character in their capabilities for mutual influence. We have already noted that the monad donates 'sameness,' equality, and permanence, the dyad 'otherness' and inequality. In addition, the perfect numbers are compared to good things and are therefore few, while the 'superabundant' and 'deficient,' like vices and bad things, are many.[3] The decad possesses a different type of perfection, but one which distinguishes it even more.

All this is significant of a type of thinking that endows numbers with qualities which are not in any sense mathematical, regarding some as better or worse,[4] younger or older [5] than others, and allowing them

[1] See p. 105. A similar conception is implied in the definition in I. 7. 2. It may be noted that the dyad is called σχήματος ἄμοιρος, *Theologumena Arithmeticae*, p. 8.

[2] *Introduction*, II. 7. 4, 5; 12. 8; *Theologumena Arithmeticae*, pp. 8, 18. For a rather full sketch of the further development of polygonal numbers the reader is referred to the chapter on "Polygonal, Pyramidal and Figurate Numbers," in L. E. Dickson's *History of the Theory of Numbers*, vol. II (Carnegie Institution, Washington, 1920), pp. 1–39.

[3] *Introduction*, I. cc. 14–16.

[4] *Ibid.*, I. 23. 4 implies that equality is 'better' as well as 'older' than inequality.

[5] In *ibid.*, I. 19. 8, 14, the multiple is shown to be 'older' than the superparticular. 'More original' is doubtless the fundamental idea; but by 'older' the Greek also connotes 'more honored.'

to transmit characters, like parents to their progeny.[1] As a consequence, there is found in the *Introduction* a tendency to classify in groups of three, which has been noted by the historians of mathematics; indeed we read in the *Theologumena Arithmeticae* that the number 3 seems to influence the science of arithmetic greatly because of the threefold classifications found there.[2] That such a treatment of numbers is to be found in the *Theologumena Arithmeticae* is not at all surprising, but the *Introduction* itself is not by any means free from it, and though the modern reader may find such procedure unscientific, it lends the charm of quaintness to the book.

[1] *Introduction*, I. 23. 6 (equality is the 'mother and root' of inequality); I. 4. 1 (arithmetic is the 'mother' of geometry, etc.).

[2] *Theologumena Arithmeticae*, p. 15, 23 ff. (Ast).

CHAPTER IX

TRANSLATORS AND COMMENTATORS OF NICOMACHUS

JUDGED by the standards of the mathematician, Nicomachus cannot rank with the leaders of the science even as it was known in antiquity; estimated, however, by the number of his translators, scholiasts, commentators, and imitators, he is undoubtedly one of the most influential. From his own day until the sixteenth century, among the Greeks, the Latins, and the Arabs, there was scarcely a place where he was not honored as an arithmetician, or a time when learned men failed to regard his work as the basis of the science.

It is the *Introduction to Arithmetic* that won him this glory; the *Manual of Harmony* helped, for the scribes of medieval times made many copies of it, but this never became as celebrated as the other. Nor, it should be added, was the *Geometrical Introduction* valued so highly; no copies of it survive, no references are made to it by name,[1] save by Nicomachus himself. In geometry Euclid reigned supreme, and if Nicomachus's name is coupled with his, it is with the understanding that the former is first in geometry, the latter in arithmetic. Perhaps the accident of having lived just when he did is responsible in some measure for Nicomachus's great popularity; had the Middle Ages been so generously endowed as the time of the Alexandrian mathematicians or the Renaissance with the spirit of independent inquiry and the genius to carry it out, he would not have occupied his exalted place. As it was, however, the classic *arithmetica* which he represents ruled almost exclusively.

Nicomachus was introduced to the Roman world not long after his death — perhaps even during his life — by Apuleius of Madaura, who is said by Cassiodorus and by Isidore of Seville to have been the first to translate the *Introduction* into Latin.[2] This translation has

[1] *Introduction*, II. 6. 1.

[2] Cassiodorus, *De Artibus ac Disciplinis Liberalium Litterarum*, Migne, *Patrologia Latina*, vol. LXX, p. 1208 A–B; Isidore of Seville, *Etymologiae*, Migne, *Patrologia Latina*, vol. LXXXII, p. 155.

disappeared without leaving a trace, as far as we know, and the *De Institutione Arithmetica* of Boethius,[1] which is, as the author frankly admits, a version of the Nicomachean *Introduction*, became and remained the source through which the Latin-speaking portion of the world knew Nicomachus. This book is discussed in the later part of the present chapter. The Arabs learned of Nicomachus and his arithmetic through Thâbit ibn Qorah (836–901 A.D.), whose translation of parts of the *Introduction* is extant in manuscript form.[2]

Another sign of the popularity of Nicomachus is the large accumulation of scholia associated with the manuscripts of his *Introduction to Arithmetic*. Most important of these is the collection written by Johannes Philoponus of Caesarea, the grammarian and theologian of the sixth century, who is best known from his Aristotelian commentaries and his treatise on the creation of the world. The scholia on Book I were published by Hoche (Leipzig, 1864), together with a scholium of Theodorus Protocensor, but those on Book II are not as yet edited. This collection is reported by Hoche to exist, in two recensions, in the Göttingen, Hamburg, and Giessen manuscripts, which he consulted; it also exists in an Oxford manuscript and in one preserved at the Escurial.[3] Philoponus was not enough of a mathematician to add anything of value to the subject-matter of Nicomachus, and his scholia are by no means so important as the commentary by Iamblichus, which will be discussed later. Other scholia, by Soterichus and by unnamed writers, have been found in the manuscripts and published;[4] of unpublished material, the scholia of Asclepius of Tralles, reported to exist in a manuscript at the Escurial, promise the greatest interest.[5]

[1] Ed. G. Friedlein, Leipzig, 1867.

[2] Hoche, p. ix; Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke*, in *Abhandlungen zur Geschichte der mathematischen Wissenschaften* (Leipzig, 1900), vol. X, pp. 35, 37; Arabic manuscript in the British Museum, 426, 15°.

[3] Cf. Hoche, p. viii; H. O. Coxe, *Catologus Codicum Manuscriptorum Qui in Collegiis Aulisque Oxoniensibus Hodie Adservantur*, Oxford, 1852, vol. I, under Greek MSS of Lincoln College library, No. XXXIII (a paper MS, of the beginning of the sixteenth century containing Nicomachus's *Introduction* and Philoponus's scholia); MS X. I. 9 of the library of S. Lorenzo del Escurial (cf. Charles Graux, *Essai sur les Origines du Fonds Grec de l'Escurial, Fasc.* 46 of *Bibliothèque de l'École des Hautes Études*, Paris, 1880, p. 268).

[4] Hoche published the scholia of Soterichus, Elberfeld, 1871; Nobbe (Leipzig, 1828 and 1862) those of the Nürnberg and Wolfenbüttel manuscripts, and Hoche (1863) those of the Giessen manuscript.

[5] Cf. Hoche, p. ix; Graux, *op. cit.*, p. 267, on the Escurial MS numbered T. I. 12. The library of New College, Oxford, has a MS of the *Introduction* 'cum glossulis' (cf. Coxe, *op. cit.*, vol. I, under No. CCXCIX of the New College MSS).

Of the commentaries, the book of Iamblichus [1] is the most important, but other works of a similar nature are known to have existed. There is still extant, for example, an anonymous *Prolegomena to the Introduction to Arithmetic of Nicomachus*, which was published by P. Tannery, but contains little to interest either the mathematician or the historian.[2] It is reported that Proclus Procleius of Laodicea in Syria, who is not to be confused with Proclus Diadochus, and a certain Heronas, of the fourth or the fifth century at latest, also wrote upon the *Introduction*.[3]

The list of commentators and scholiasts is, however, not as long as that of the authors influenced by Nicomachus. Beginning with the well-known names of Martianus Capella, Cassiodorus, Isidore of Seville, and Michael Psellus, it includes dozens of writers on arithmetic from the tenth to the sixteenth century who followed him through the medium of Boethius. Of these something will be said in the following chapter; for the present we shall turn our attention first to Iamblichus, and then to Boethius, the two most noteworthy of the followers of Nicomachus.

Iamblichus, philosopher and prolific writer, was a native of Chalcis in Coele Syria, which accounts for the pride expressed for his famous countryman in his *Commentary*. He lived in the fourth century, although his dates cannot be fixed exactly, and seems to have been educated in Rome among Neo-Platonic influences, which after all were not far removed from Neo-Pythagoreanism. At any rate his tastes led him to ascribe great influence in the world to numbers, and his interest in this phase of ancient philosophy is witnessed by the fact that, besides writing on the *Introduction*, he also compiled a *Life of Pythagoras*. He is reputed also to be the author of the compilation edited by Ast under the title *Theologumena Arithmeticae*.

His *Commentary* is really a treatise on arithmetic based on the *Introduction*, that is, a work having the same plan and purpose as the *Introduction* itself; we might perhaps call it a new edition. In general he follows the order set by Nicomachus and reproduces his material, but he has added many things. Some of these amplifications are merely further illustrative material and new observations on the

[1] *In Nicomachi Arithmeticam Introductionem Liber* (ed. H. Pistelli).

[2] In his edition of the works of Diophantus, vol. II, pp. 73–77.

[3] On Proclus, cf. Suidas s.v. Eutocius, *Ad Archimedis de Sphaera et Cylindro*, II, p. 28 (ed. Bas.), mentions an ὑπόμνημα εἰς τὴν ἀριθμητικὴν εἰσαγωγήν by Heronas, who is otherwise unknown.

tables of numbers which Nicomachus used; some of them are discussions of the Pythagorean virtues of numbers; others are propositions taken from earlier mathematicians which did not appear in the *Introduction*. One of the most valuable features of the book is the inclusion of notes on the history of certain theorems and upon terminology. In general, however, Iamblichus is not credited with the discovery of any new propositions of importance. The following summary of his *Commentary* and comparison with Nicomachus will explain its character:

SUMMARY OF IAMBLICHUS'S *COMMENTARY*

(*Page references are to Pistelli's edition*)

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3, 5 — 5, 25: Introduction, chiefly praise of Nicomachus.

5, 26 — 10, 7 = Nicomachus, I. 1–5 in general; Iamblichus even gives indication of the quotations made by Nicomachus. He omits chapter 6, an important one from the standpoint of Nicomachean philosophy.

10, 8–24: Definitions of number, referring to Thales, Pythagoras, Eudoxus, Hippasus and Philolaus.[1]

11, 1–26: A discussion of the monad. This topic is missing in the *Introduction*. But Theon of Smyrna speaks of it at some length.[2]

12, 1 — 13, 10 = Nicomachus, I. 7, in general; the odd and even.

13, 10 — 15, 5: Material probably based on the *Theologumena Arithmeticae*, including the explanation of the 'lambdoid figure' which is there mentioned.[3]

15, 5–15: An account of the genesis of the odd and the even series from 1 and 2 (not in Nicomachus).

15, 16 — 16, 11, takes its start from Nicomachus, I. 8. 1–2, but leads into a discussion of the number 5 as justice (pp. 16, 11 — 20, 6), which parallels *Theologumena Arithmeticae*, pp. 28–30 (Ast).

20, 7 — 35, 10 = Nicomachus, I. 8. 3 — c. 16; the varieties of the even and the odd, the superabundant, deficient, and perfect numbers. Nicomachus is followed substantially. It may be noted that criticism of Euclid for his definitions, a favorite topic of Iamblichus, appears in several places (pp. 20, 10–14; 20, 19 — 21, 3; 23, 18 — 24, 14; 30, 28 — 31, 21). The most interesting is the last of these passages where Iamblichus maintains in opposition to Euclid that 2 is not prime.[4] Another passage (p. 34, 20 — 35, 10) deals with the virtues of 6 in the Pythagorean manner. On p. 27, 3 is found the statement that εὐθυμετρικός and εὐθυγραμμικός are additional names of prime numbers; Theon[5] also uses the former, and Thymaridas is here cited by Iamblichus as authority for the latter.

[1] See p. 114.
[2] Theon of Smyrna, p. 18, 5 ff. (Hiller); see p. 37.
[3] *Theologumena Arithmeticae*, p. 3 (Ast).
[4] Cf. Nesselmann, *op. cit.*, p. 242.
[5] P. 23, 12 (Hiller).

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35, 11 — 43, 12 = Nicomachus, I. 17–21: the discussion of equality and inequality, with no essential additions.

43, 13 — 51, 20: The 'three rules' of Nicomachus, I. 23. 6 ff., are given, out of order, and then Iamblichus returns to the multiple superparticular, etc., as in Nicomachus, I. 22 and 23. 1–3. The reversal of the 'three rules' (Nicomachus, II. 2) is omitted.

51, 21 — 52, 27 = Nicomachus, II. 3–4, but briefly. Nicomachus, II. 5, is omitted.

52, 28 — 56, 17 contains four propositions that are not found in the *Introduction*.

56, 23 — 72, 2 presents the subject of plane numbers, following Nicomachus, II. 6–12, but with the important addition (pp. 62, 18 — 68, 26) of the celebrated 'epanthema' of Thymaridas.[1]

72, 2 — 93, 7 = Nicomachus, II. 17–20. The parallelism exists in a general way, but there are additions, especially that of the 'diaulos' theorem,[2] and Iamblichus brings out, in the discussion of the squares and heteromecic numbers, the Pythagorean aspects of the monad and dyad at greater length than did Nicomachus. There is further criticism of Euclid (p. 74, 24 ff.). On p. 82, 13 ff., he points out that the squares are alternately even and odd [3] and all the heteromecic numbers even, principles which he applies to the exegesis of the Platonic account of the 'marriage number.'[4] On p. 83, 10 ff., we have the proposition that the products of squares by squares are squares; those of heteromecic numbers by heteromecic numbers are heteromecic numbers; and those of μικτοί (promecic numbers) never squares. This principle appears in part later, on p. 90, 20. Iamblichus then sets up the table of squares and heteromecic numbers and discusses the properties that may be discovered therein, sometimes agreeing with Nicomachus and sometimes adding from other sources.

The table is as follows:

Squares,	1	4	9	16	25	36	49	64	81	100,
Heteromecic numbers,	2	6	12	20	30	42	56	72	90	110,

or, in general terms,

Squares, $a^2, (a+1)^2, (a+2)^2, (a+3)^2, (a+4)^2 \ldots.$

Heteromecic numbers, $a, a(a+1), (a+1)(a+2), (a+2)(a+3) \ldots.$

The first set of observations, pp. 83, 27 — 87, 22, is in general agreement with Nicomachus (II. 19. 3–4; 20. 3–4). It is to be noted that Iamblichus omits

[1] See Nesselmann, *op. cit.*, pp. 232 ff.; Heath, *History*, vol. I, pp. 94–96.

[2] The theorem is based on the principle stated in the *Introduction*, II. 12. 2, that the sum of two consecutive triangular numbers is a square: $1 + 2 + \ldots + (a-1) + a + (a-1) + \ldots + 2 + 1 = a^2$. The progress to and from a is compared to the Greek double race-course, δίαυλος. Cantor, *op. cit.*, vol. I, p. 460, implies that the theorem in this form may be the work of Iamblichus. This is very improbable, especially as Iamblichus designates it 'the so-called diaulos' (p. 75, 25). It is found in *Theologumena Arithmeticae*, p. 9 (Ast). Nesselmann, *op. cit.*, p. 237 ff., discusses it fully. See Heath, *History*, vol. I, p. 114.

[3] So Theon of Smyrna, p. 34, 3.

[4] *Republic*, 546 ff.

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the proposition of Nicomachus, II. 20. 1, that $m^2 + m$ is always heteromecic, and he does not put in the same form the observation that the squares may be put into proportions with the heteromecic numbers as means.[1]

The added propositions are these:

84, 24: $a(a+1) + 2a^2 + (a+1)(a+2)$, and the like, give results that may be called ἀνόμοιοι (unlike; that is, the opposite of 'like,' which describes numbers that share in 'sameness'[2] because they are either odd or are summations of the odd series, squares). This is the converse of the proposition (p. 84, 18) that $a^2 + 2a(a+1) + (a+1)^2$, and the like, is always a square; which is stated in passing by Nicomachus (p. 117, 8–9).

84, 27: $a(a+1) + 2(a+1)^2 + (a+1)(a+2)$, and the like, always give a square.

86, 15: $\frac{m(m+1)}{2}$ is always a triangular number. The materials in this section, with the exceptions noted, are found for the most part in the *Introduction*, but Iamblichus has somewhat varied their order and form.

88, 15 — 91, 3: This section contains further observations based upon the same table of squares and heteromecic numbers; to judge from the way in which it is framed in the narrative, it would seem that Iamblichus intended it as an enlargement on the Nicomachean original. As a matter of fact there is nothing of great importance in it that can be called original with Iamblichus; some of the material he seems to have derived from the *Theologumena Arithmeticae* or similar sources, some from the *Introduction*, and for some of the remainder earlier parallels can be pointed out.

The propositions here added are these:

$1 + 2 + 3 \ldots + 9 + 10 + 9 \ldots + 3 + 2 + 1 = 100$; $10 + 20 + 30 \ldots + 90 + 100 + 90 \ldots + 30 + 20 + 10 = 1{,}000$; $100 + 200 + 300 \ldots + 900 + 1{,}000 + 900 \ldots + 300 + 200 + 100 = 10{,}000$. This will be recognized at once as an application of the 'diaulos' theorem, based on material from the *Theologumena Arithmeticae*, which appeared earlier in the commentary.[3] It can certainly not be original with Iamblichus, since, in addition to its probable Nicomachean, or even earlier, origin, it involves the Pythagorean conception of 1, 10, 100, 1,000, . . ., as μονάδες ἁπλαί, δευτερωδούμεναι, τριωδούμεναι, etc., a terminology employed by Nicomachus himself.[4] The passage is, however, considered noteworthy by historians of mathematics in view of its significance in the development of a true decimal system.

The theorem about special properties of squares (already found in Theon, p. 35, 17 ff.):

$m(m+2) + 1$ or $m(m-2) + 1$ is always a square.

$a \times ma = ma^2$.

[1] *Introduction*, II. 19. 4. [2] Cf. p. 118. [3] P. 75, 25 ff. See p. 128, n. 2.

[4] *Introduction*, I. 19. 17 (but spelled δευτεροδουμένη, etc.).

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Any triangular number multiplied by 8 becomes a square when 1 is added. This proposition was known to Plutarch.[1]

$a^2 \times b^2$ is always a square. This is stated by Nicomachus, II. 24. 10.

In a geometrical progression beginning with 1, if the term after 1 is a square, so are the others, i.e., the series is 1, m^2, m^4, m^6, ... This occurs in Euclid, *Elements*, IX. 9.

If $a : b = b : c$, and a is a square, c is a square also (= Euclid, *Elements*, VIII. 22).

If a^2 measures b^2, a measures b (= Euclid, *Elements*, VIII. 14).

ab is the geometrical mean between a^2 and b^2 (cf. Euclid, *Elements*, VIII. 11, and Nicomachus, II. 24. 9).

91, 3 — 93, 7: Pleuric and diametric numbers are discussed. This is a subject not treated by Nicomachus, but already known through Theon.[2]

94, 8 — 95, 15: Solid numbers in general, with their classification; this is based upon the material in Nicomachus, II. 13, 16. 1–2, 3 and 17. 6. P. 94, 15 ff. is an added section upon the monad as potential surface and solid.

95, 15 — 97.7: Pyramids; the basis of the section is Nicomachus, II. 13, 14, but it is condensed. The description of the geometrical pyramid (Nicomachus, II. 13. 2–5) is omitted. There are added, however, some unimportant observations on the table of pyramidal numbers (pp. 96, 6 ff.; 96, 13 ff.).

97, 7 — 98, 13: Cubes. The section is based in general on Nicomachus, II. 20. 5, with material from other contexts (e.g., cf. p. 97, 10 ff. with Nicomachus, II. 15. 2, and p. 98, 2 f. with Nicomachus, II. 24. 10). It is modeled as closely as possible after the section on the square and so contains certain parallels with Euclid (e.g., p. 98, 7 = *Elements*, VIII. 23; p. 98, 8 = *Elements*, VIII. 15; cf. also p. 98, 4 and *Elements*, IX. 3; and p. 97, 21 with *Elements*, IX. 8 and Theon of Smyrna, p. 34, 16 ff.). Of the other propositions there may be mentioned here p. 98, 5: "If, in a geometrical progression beginning with 1, the term after 1 is a cube, the other terms are cubes" (cf. Euclid, IX. 9); and also the observation on p. 97, 23 that $a \times a^3 = (a^2)^2$.

98, 14 ff.: Proportions. Iamblichus departs less from the Nicomachean model in the remaining portion of the book than in the previous parts. The added material is either unessential amplification or in the nature of historical notes and comments on the mathematical terminology; among the latter are several bits of information that are very valuable. He tells us, for example (pp. 100, 15 ff.; 113, 16 ff.), something about the history of the theory of proportions;[3] on p. 100, 1, he remarks that τὸ ἀνάλογον was properly applied to disjunct propositions or proportional series, whereas ἀναλογία meant the continuous proportion; on p. 100, 15, there is inserted a valuable note on the difference between ἀναλογία and μεσότης;[4] the information about the names ἑστηκυῖα and ὑπεναντία as applied to the harmonic proportion (pp. 108, 5 ff.; 110, 18 ff.) is also welcome.

[1] *Platonicae Quaestiones*, v. 2, 1003 F.

[2] See Cantor, *op. cit.*, vol. I, p. 436.

[3] See the notes on *Introduction*, II. 22. 1 and II. 25. 1.

[4] See the note on *Introduction*, II. 21. 2.

He strives for greater accuracy than Nicomachus in the definitions of proportion and ratio, and adds the statement that the interval between terms may vary while the ratio remains the same, or vice versa. The 'ratio of number to number,' which was thought not to fall under any of the specified ten species of ratio and was classed by itself, is mentioned by both Theon (p. 80, 7) and Iamblichus (p. 99, 15), but not by Nicomachus. Of the added matter, a good example is the set of rules for producing the various proportions (pp. 101, 24 ff.; 106, 5 ff.; 112, 1 ff.). The 'three rules' given by Nicomachus (I. 23) will produce from three monads, three dyads, and the like, terms in geometrical progression, as Iamblichus notes. Similarly he constructs rules to make the arithmetical proportion, by taking the first number for the first term of the proportion; the first plus the second for the second term; the first plus the second plus the third for the third term.

The most important additions, besides the historical notes, include the observations of the properties of groups of three consecutive or regularly occurring numbers taken from the natural series (102, 17 ff.). He observes that 1, 2, 3 = 6 (the second triangle); 1, 3, 5 = 9 (the second square); 3, 4, 5 = 12 (the second pentagon); 1, 3, 5 give a square; 1, 4, 7 give a pentagon; and so forth. But since the first group, 1, 2, 3, gives 6, the other groups of three consecutive numbers from the series 'are fashioned after'[1] the hexad; that is, if 10 be taken as a unit (and the substitution made in each sum), all the sums will be 6. Thus 4, 5, 6 = 15, equivalent to 1 + 5; 7, 8, 9 = 24, equivalent to 2 + 4, and so forth. This manipulation of the number 10 is of course of the same character as that already observed in connection with the μονάδες δευτερωδούμεναι (p. 88, 15 ff.). It may be noted also that Iamblichus omits mention of the peculiarity of the arithmetic proportion on which Nicomachus most prides himself (p. 125, 18–23, Hoche) and of the fact that the Pythagoreans called the cube 'geometrical harmony' (Nicomachus, II. 26).

Iamblichus's *Commentary*, as the foregoing synopsis clearly enough shows, consists of a thoroughly Nicomachean framework overlaid with additions of various kinds. In mere extent these are quite considerable, but aside from the one section containing the 'epanthema' of Thymaridas and the few statements of historical interest, their worth is very small. Many of them are merely observations based upon the tables used by both Nicomachus and Iamblichus; a large number seem to depend upon Euclid; and another portion is derived from the *Theologumena Arithmeticae*, or something similar. After sifting out the statements obviously borrowed from these sources, nothing of importance is left to the credit of Iamblichus himself. It is noteworthy that he does not attempt to modify the definitions and classifications of Nicomachus. In general, the Nicomachean original, in

[1] The word εἰδοποιεῖσθαι is used. The proposition occurs also in *Theologumena Arithmeticae*, p. 34 (Ast). See Heath, *History*, vol. I, p. 114.

Iamblichus's hands, lost much of the clearness of statement and simplicity of arrangement which made it so useful and popular.

When we turn to Boethius we meet with another admirer of Nicomachus who allowed himself to take even less liberty with his original.

Boethius and Nicomachus

It has long been recognized that the *De Institutione Arithmetica* of Boethius (here cited in the paging of the edition of G. Friedlein, Leipzig, 1867) has so little claim to originality that it may be called a translation of the *Introduction* of Nicomachus.[1] The judgment is a just one. A comparison of the two books will convince the reader that Boethius follows Nicomachus from first to last, expanding here and condensing there, as he says in his preface that he will do,[2] but never adding anything essential, either original or derived from other sources, that departs from his model. His additions seem to be in general mere development of the material which Nicomachus supplied — numerical examples, or explicit enlargement of statements left by Nicomachus in general form; and similarly his changes impress one as being observations or deductions made by Boethius himself on the basis of the work before him. There is little indication that Boethius used any other sources than the *Introduction* itself. It is a far more important question whether he has omitted anything of an essential nature in his translation, as we may term the *De Institutione Arithmetica*.

As for this problem, a detailed comparison of the two books has brought the writer to the conclusion that the historians of mathematics, especially Cantor, have been unduly severe in dealing with Boethius. Granting that he was not an able mathematician — that he was content to follow Nicomachus so closely, and cannot be said to have improved upon him in any way, shows this clearly enough — it is nevertheless true that he presents, with certain limitations, a fairly adequate Latin version of the original, omitting few things which might have been included. His limitations as a translator will shortly appear, and it will be seen that he omits certain sections. But he is not guilty of leaving out, as Cantor charges, some of the best things, from the mathematical standpoint, in Nicomachus.

[1] Nesselmann, *op. cit.*, p. 221; Cantor, *op. cit.*, vol. I, pp. 573 ff.
[2] P. 4, 30 ff.

Cantor says,[1] specifically, that he omits the propositions that the cubes are derived from the summation of the odd numbers (Nicomachus, II. 20. 5), and that the polygonal number with n angles and side r and the triangular number with side $r - 1$ make together the polygonal with $n + 1$ angles and side r (Nicomachus, II. 12. 7). With regard to the first proposition Cantor is simply mistaken; the principle is amply stated in Boethius, II. 39 (p. 136). He is more nearly right as to the second; Boethius in fact does not, like Nicomachus, sum up in a general statement this principle, but, aside from this, he fully parallels Nicomachus, II. 12. 5–7, giving specific instances of the working of the theorem for several of the polygonal numbers.[2] The fairest criticism to make of Boethius is that as a mathematical writer he displays too little originality, independence and progressiveness, and too much prolixity.

In the composition of his treatise Boethius more often expands than condenses. His method is to intersperse between sections literally translated, or closely paraphrased, others in which the general principles stated by Nicomachus are furnished with exhaustive explanation and copious numerical examples. Nothing is left to the reader to supply. Almost any chapter, compared with the original, will prove to be of this character. Boethius also supplies data in tabular form to a far greater extent than did Nicomachus. The order of the original is preserved for the most part, but occasionally a rearrangement is found.

Inasmuch as these peculiarities are of minor importance and the omissions are rather to be considered, we may now turn to an enumeration of the more important of these. It will be found that Boethius had especial difficulty with both the logical terminology of Nicomachus and those passages in which the Pythagorean elements of the latter's thought come out most strongly.

The former difficulty led to no lengthy omissions, but inasmuch as Nicomachus was in general careful to adopt the terminology of Aristotelian logic and to arrange his materials formally in genera and species, using frequently, for example, such terms as *γένος*, *γενικῶς*, *εἶδος*, *εἰδικῶς*, *εἰδικός*, *ἰδίωμα*, *ἰδικῶς*, *συμβεβηκώς*, *συμβεβηκότως*, *συμβαίνει*, and the like, Boethius, finding himself at a loss to translate

[1] *Op. cit.*, vol. I, p. 580.

[2] Just the reverse of this situation is sometimes seen; Nicomachus enumerates details without making a generalization, and the latter is formulated by Boethius. See the list of additions made by Boethius, p. 136.

them, often omitted a word or two. No one of these omissions is important in itself, but to lose most of the expressions of this type from the version prevents it from truly representing the original in at least this characteristic trait. Examples of such outright omissions are to be found, according to Hoche's edition of Nicomachus, on p. 49, 5, εἰδικῶς; p. 59, 3–6; p. 113, 25, διόπερ . . . ὡρισμένῳ τοιοῦτον; p. 114, 3, καὶ ἑτερότητος . . . ἀοριστίας. The last two share in the involved and difficult Pythagorean logic with which Boethius quite naturally found difficulty.

Of the omissions of Nicomachus made by Boethius the following, of somewhat greater extent than those already mentioned, are important enough to list:

I. 1: The distinction between κυρίως ὄντα and ὁμονύμως ὄντα, so characteristic of Nicomachus.
I. 3: One of the definitions of number (πλῆθος ὡρισμένον, Nicomachus, p. 13, 7).
I. 9: Pp. 16, 1–3; 5–6; 18, 11–15. This can be ascribed to condensation.
I. 10: I. 9. 3.
I. 17: The direction to 'check off the numbers,' p. 33, 8 ff.
I. 19: Part of I. 14. 2, the comparison of inequality to vices, ill health and the like.
I. 20: The second mention of the perfect number 4,128 (p. 43, 17–20) and the statement that 1 is prime per se and not by participation (p. 44, 1–3).
I. 25: I. 19. 7.
I. 32: P. 64, 21–22; I. 23. 5 entire (the reference to the ethical virtues); pp. 66, 16–19, 22 — 67, 2; all of I. 23. 15, which is not unimportant; all of section 17 (this is made up of examples, some of which Boethius rearranges and reports).
II. 1: P. 74, 5–8 (the monad and dyad as elements).
II. 3: II. 5. 1.
II. 9: P. 89, 3–5 (the arrangement of units in the geometrical number).
II. 14: The rule for determining the side of the pentagonal number (p. 92, 12–16).
II. 18: The general theorem stated in II. 12. 7. (See above.)
II. 25: The reference to σφηκίσκοι, p. 107, 15–21. Boethius excuses himself, however, for the omission.

It can hardly be claimed on the basis of this list that the omissions made by Boethius seriously detract from the value of the *De Institutione* as a version.

Boethius never sets himself in direct opposition to Nicomachus, but the statements of the two do not always agree exactly. Sometimes these differences are merely variations in the illustrations, sometimes they seem to be formulations based upon the original; perhaps in a few cases Boethius did not fully comprehend the intention of Nicomachus. The following instances may be considered:

1. Boethius, after defining the even as a number that can be divided into either two equal or two unequal parts, says, I. 5, p. 14, 23 ff.: *praeter solum paritatis principem, binarium numerum, qui in aequalem non recipit sectionem, propterea quod ex duabus unitatibus constat et ex prima duorum quodammodo paritate.*

   Nicomachus, p. 14, 1: πλὴν τῆς ἐν αὐτῷ ἀρχοειδοῦς δυάδος θάτερον τὸ διχοτόμημα μόνον ἐπιδεχομένης τὸ εἰς ἴσα, κτλ.

   Since Boethius is so plainly wrong here, it is reasonable and charitable to suppose that the MSS are at fault and to correct his text (e.g., *in aequalem* to *in inaequalia*).
2. Boethius, I. 9, p. 20, 19–21, says with regard to the even-times even numbers that when corresponding factors are multiplied the product equals the major extreme (i.e., of the even-times even series in question; e.g., in the series 1, 2, 4, 8, 16, 32, $4 \times 8 = 32$). Nicomachus puts it: The product of the means is equal to that of the extremes (I. 8. 14); but from his examples he makes it clear that the product of any two corresponding factors may be used. Boethius has chosen, apparently, to express the matter in a different way.
3. Boethius, I. 11, p. 26, 21, has *duplices* (*sc. disponantur*): Nicomachus had said ἀρτιάκις ἄρτιοι.
4. The following passages are evidently meant to correspond:

   Boethius, I. 11, p. 25, 27–29, *Nam et partes solvuntur et usque ad unitatem sectio illa non pervenit, sed ante unitatem invenitur terminus, quem secare non possis.*[1]

   Nicomachus, I. 10, p. 22, 19–21, πλείονα μὲν γὰρ τοῦ ἑνὸς τμήματα ⟨ἐν τῷ μείζονι μέρει ἔχων ὁρᾶται, πλείονα δὲ τοῦ ἑνὸς ἄτμητα⟩ ἐν τῷ ἐλάττονι.
5. Boethius, I. 15, p. 32, 14: *neque habet quicquam in se principalis intellegentiae;* Nicomachus, p. 28, 12, καὶ ὅτι οὐκ ἀρχοειδής.
6. Boethius, I. 18, p. 37, 25, *datis enim duobus numeris inaequalibus;* Nicomachus specified *odd* numbers.
7. Boethius, I. 30, p. 64, 13–14; Nicomachus, p. 63, 8–9. Boethius in his table directs the reader to compare the third and fifth rows; Nicomachus, the odd rows, beginning with the fifth, to the third.

[1] The text of Nicomachus has here been supplemented by Hoche, and it may be questioned whether it may yet be considered settled.

8. Boethius, *ibid.*, p. 64, 15 ff.; Nicomachus, p. 63, 11 ff. With reference to the same table Nicomachus tells us to compare the second and fifth, third and seventh, fourth and ninth rows, etc.; Boethius, the second row to the fifth, seventh, ninth, etc.
9. Boethius, II. 48, p. 157, 13: *et rursus minor terminus ad medii contra minorem terminum comparati differentiam triplus est;* Nicomachus, p. 135, 3–4: διαφορὰ τῶν ἄκρων πρὸς διαφορὰν τῶν ἐλαττόνων. Both are working with the series 3, 4, 6; Boethius takes $3:(4-3) = 3:1$, and Nicomachus derives the same in a different way, $(6-3):(4-3) = 3:1$.

It is easy to see from the unimportant character of these deviations and from their comparative fewness in such an extended text that Boethius did not do violence to his model.

Consideration of the additions made by Boethius will confirm this view. In this connection no mention need be made of the expansions already referred to, which merely weaken the book in contrast with the greater succinctness of Nicomachus, but do not impair its general accuracy. The following passages of Boethius may, however, be noted:

1. I. 1, p. 9: *nihil enim quod infinitum est vel scientia potest colligi vel mente comprehendi.*
2. *Ibid.*, p. 11. The reference to the *tonus* does not occur in Nicomachus at this point (but cf. II. 29. 4).
3. I. 12. The whole chapter, a description of the table, is apparently added by Boethius, but contains no new facts.
4. I. 14, p. 31, 1–8: an added explanation with definition of the term 'measuring.'
5. I. 23, p. 47, 15: *idem autem dico 'numerat' quod 'metitur.'*
6. *Ibid.*, p. 48, 8: *excepto ternario . . . praetermittit.* An added observation.
7. *Ibid.*, p. 48, 24: *semperque una terminorum intermissione si crescat adiectio, ordinatas te multiplicis numeri vices invenire miraberis.* Nicomachus did not formulate this generalization from the specific facts which he had fully enumerated.
8. I. 25, p. 52, 7–9: *et deinceps . . . succrescunt.* A case like the last.
9. II. 4, p. 89, 22: *omnium intervallorum esse principium et natura insecabile, quod Graeci atomon vocant, id est ita deminutum atque parvissimum ut eius pars inveniri non possit.*
10. II. 6, p. 91, 9: *duae enim lineae rectae spatium non continent.*
11. II. 25, p. 111, 15 (definition of parallelepipeds): *quae sunt, quotiens superficies contra se sunt, et ductae in infinitum nunquam concurrent.*
12. II. 30. Definitions of circle and square are given.
13. II. 33. Nicomachus shows that two successive squares plus twice the included heteromecic numbers produces a square; Boethius in addition points out that two successive heteromecic numbers plus twice the in-

cluded square gives a square, and that the squares thus produced are those of the odd and even numbers respectively.[1]

14. II. 40, p. 137, 13: 'difference' is defined.
15. II. 41, p. 139, 13–19. A note on the origin of categories.
16. II. 45. The whole chapter, a comparison of the three major types of proportion to oligarchy, aristocracy, and democracy, is an addition.

None of these additions can be regarded as original with Boethius, save perhaps the last one, and none of them is of any importance, save the thirteenth. The fact that he was making a translation necessitated some of them; and some consist of the insertion of definitions or details that Nicomachus evidently found it not worth while to include.

The importance of the translation of Boethius lies in the fact that it was the chief medium through which the Roman world and the Middle Ages learned the principles of formal Greek arithmetic; the data adduced above will amply show that in general Boethius faithfully followed his model.

[1] This proposition was given by Iamblichus (p. 84, 27); see p. 129.

CHAPTER X

THE SUCCESSORS OF NICOMACHUS

Even in the time of Nicomachus Greek mathematics was in its decline; the golden age of the science had long passed. For more than a thousand years after Nicomachus we find few noteworthy contributions to the scientific theory of numbers anywhere in the world.

One of these notable exceptions is Diophantus of Alexandria (*c.* 250 A.D.), who stands out as the greatest Greek contributor in the field of analysis. His *Arithmetica* includes material which we treat under algebra, and also subjects related to the theory of numbers. Furthermore Diophantus wrote a treatise, *On Polygonal Numbers*,[1] in which he adds several important theorems relating to polygonal numbers. The proofs are Euclidean in form and the whole work reveals the hand of the master mathematician.

The writers of the early Middle Ages who show the influence of Nicomachus were, like other writers of their day, not inclined to break away from the beaten path. The most prominent of these writers on arithmetic are Martianus Capella (fifth century after Christ), Cassiodorus (*c.* 490–*c.* 580 A.D.) and Isidorus of Seville (*c.* 570–636 A.D.). If any one of these men had mathematical ability, the fact is not revealed by his discussion of arithmetic. We present an epitome of the content of the three works in question, in parallel columns and with references to Nicomachus, taking as the chief exponent of the group Martianus Capella, whose account is the most voluminous:

Capella, Book VII (ed. Eyssenhardt)	Isidorus[2]	Cassiodorus
First comes a discussion of the virtues of the numbers 1–10 (cf. *Theologumena Arithmeticae*).	Enumeration of the four mathematical sciences (cf. *Introduction*, I. 3); notice of ancient authors on mathematics	1204 B–C: The priority of Arithmetic (*Introduction*, I. 4–5)
		1204 C: Discrete quantity (*Introduction*, I. 2. 4)

[1] T. L. Heath, *Diophantus of Alexandria, A Study in the History of Greek Algebra* (second edition, Cambridge, 1910), pp. 247–259.

[2] Migne, *Patrol. Lat.*, vol. LXXXII, pp. 154–161.

CAPELLA	ISIDORUS	CASSIODORUS
P. 265, 4: Number defined (*Introduction*, I. 7)	Chap. 3	1204 D
Ibid., 6: Classification,[1] *a paribus par* *a paribus impar* *ab imparibus par* *ab imparibus impar*	Chap. 5 *pariter par* *pariter impar* *impariter par* *impariter impar*	1205 A (like Isidorus)
Ibid., 9: Prime numbers (*Introduction*, I. 11)		
Ibid., 13: Numbers 1–10 classified		
Ibid., 25: First series of numbers to 9, second to 90, third to 900, fourth to 9,000		
P. 266, 14. The addition of odd numbers gives squares (*Introduction*, II. 9. 3, etc.).		
P. 267, 1: Definition of even and odd (*Introduction*, I. 7)	Chap. 5	
Ibid., 4: Definitions of subclasses (*Introduction*, I. 8 ff.)	*Ibid.*	
P. 268, 8: Another classification, with definitions (of all numbers): *per se incompositi* *per se compositi* *inter se incompositi* *inter se compositi* (*Introduction*, I. 11 ff.; as a classification of odd numbers only; the last type omitted)[2]	*Ibid.* *primus et simplex* *secundus et compositus* *mediocris* (This classification is confined to odd numbers.)	1205 B (agrees with Isidorus)

[1] This is unlike Euclid's classification of even numbers, but agrees with Nicomachus and Theon; the terminology used by Capella, who gives a few Greek terms, is, however, rather that of Euclid, but following the other scheme of classification. Capella does not appear to understand either system of classification.

[2] This is rather Euclidean than Nicomachean. Isidore and Cassiodorus, however, follow Nicomachus.

Capella	Isidorus	Cassiodorus
P. 269, 24: Another classification; *perfecti, ampliores perfectis, imperfecti*[1] (*Introduction*, I. 14 ff., of even numbers only)	*Ibid.*: *superflui, diminuti, perfecti* (confined to even numbers)	1206 A: *superfluus, indigens, perfectus*
P. 270, 15: Plane and solid numbers (*Introduction*, II. 6 ff.) P. 271, 5: Kinds of plane numbers; triangular, square, heteromecic (*Introduction*, II. 7, 3) *Ibid.*, 26: Similar plane numbers[2]	Chap. 7: *discreti, continentes; continentes* = linear, plane and solid; circular and spherical numbers	1207 D (agrees with Isidorus)
P. 272, 8: Relative number (*Introduction*, I. 17. 6 ff.) *multiplicati* (πολλαπλάσιοι) *replicati* (ὑποπολλαπλάσιοι) *membro membrisve antecedentes* (ἐπιμόριοι, ἐπιμερεῖς) *membro membrisve inferiores* (ὑπεπιμόριοι, ὑπεπιμερεῖς) *binis nominibus usi* (πολλαπλασιεπιμόριοι, etc.)	Chap. 6: first division into equal, unequal; the latter into greater, less; the subspecies in agreement with Nicomachus	1205 C–D, 1206 C — 1207 A (agrees with Isidorus)

From this point on, there is no correspondence between the subjects treated by our three authors. Capella takes up, in order, equality (p. 272, 21; cf. *Introduction*, I. 17. 1–5), difference (*distantia*, *ibid.*, 26; διάστημα would represent the idea in Greek, and Nicomachus has no treatment of this subject), and the proposition that there may be the same difference between two terms, but a different ratio, depending upon their order (*ibid.*, 30; cf. *Introduction*, II. 23. 1, although Nicomachus gives this no separate attention). He then takes up ratios (p. 273, 5 ff.) in much the same way as Nicomachus (*Introduction*,

[1] Euclid does not give these classes (except the perfect numbers alone).
[2] An Euclidean topic.

I. 18 ff.). The superparticular, it may be observed, is called *membrorum ratio*, that is, a ratio where the excess is a *membrum* (aliquot part) of the smaller number, as 9 : 6, and the superpartient is called *partium ratio*, for here the excess is either one *pars*, or several *partes* (that is, what Nicomachus calls the 'paronymous part' of every number, always unity, as $\frac{1}{4}$ of 4). Capella even enumerates the πυθμένες, root ratios, using the Greek word.

Toward the end Capella inserts certain material which does not occur in Nicomachus, but which is found, though not in the same order, in Euclid. Capella follows propositions but not the proofs as found in Euclid. Euclid, however, does not give any arithmetical illustration, but confines himself to the logical proof, employing lines as illustrative material, while Capella gives always the numerical illustration. Thus Capella states (p. 278, 17), without proof, that the product of an even number by an odd number is even, and of an odd number by an odd number is odd (p. 278, 18); Euclid gives these statements as propositions, IX. 28 and IX. 29. Similar propositions on the addition of a series of odd numbers or even numbers are given by Capella (p. 278, 24 — 279, 3) and by Euclid in IX. 21, 22, and 23; theorems involving subtraction of odd or even from even or odd numbers appear in Euclid (IX. 24, 25, 26) and in Capella (p. 279, 18–26). Many other similar parallels could be presented, but this detailed analysis is hardly necessary.

In general, we may say that Capella presents the numerical side of the greater part of the propositions of the seventh book of Euclid and of the simpler propositions of the eighth and ninth books, but not in the same order as in Euclid. He adds little that is new, and apparently sometimes does not comprehend the text of Euclid; corruptions due to transcribers must be considered as a possible explanation of the errors in the text as it stands. In passing, it is to be noted that in his discussion of music Capella avoids the mathematical treatment of numerical ratios which is commonly found in the early works in music.

Isidore does not give the numerical illustrations of Euclidean propositions, but follows more closely passages in Nicomachus relating to the limited number of topics which are discussed by him. Capella opens with an extended and mystical discussion of the numbers from one to ten, while Isidore discusses the etymology of the Latin words from one to ten, and for some larger numbers, only touching the mystical element with reference to the Scriptures. Distinctly different in Isidore is the reference to the infinity of numbers (*numeri infiniti*), showing that if you conceive of any number as terminating the series of numbers, adding one would give another larger number. This suggests Euclid's proof that the number of primes is infinite.

Isidore follows closely the text of Cassiodorus, employing frequently the actual phraseology of Cassiodorus. Isidore, however, enlarges occasionally, departing from the material of Boethius, while Cassiodorus does not. Isidore includes a brief discussion of arithmetical, geometrical and harmonical means which is not given by Cassiodorus.

Both Isidore and Cassiodorus were ultimately dependent on Nicomachus, undoubtedly through Boethius, whom they cite; Capella also depended on Nicomachus, possibly through the translation of Apuleius, and used further material, including classifications found in Euclid, but through what mediation, if any, we do not know.

Contemporary with Martianus Capella and with Proclus was the Greek Domninos of Larissa, whose treatise [1] in many points is suggestive of the larger work by Capella. Domninos has been cited [2] as an exponent of a movement away from Nicomachus to return to Euclid. Particularly noteworthy is his rejection of the theory of polygonal numbers, except plane and solid, possibly because this theory had been abused to the extent of attempting to calculate areas of polygons by means of the figurate numbers. His method of exposition is entirely similar to that of Nicomachus and Theon, not including any attempt at proofs. Domninos touches upon the implications of the decimal nature of the number system and refers to a projected elementary treatise on arithmetic, which, if completed, has not survived.

Rabanus Maurus of the ninth century in his *De Clericorum Institutione* [3] follows textually brief portions of Isidore, but his discussion is much more attenuated than those of the three men just mentioned. Hugo of St. Victor of the twelfth century also includes a few lines on arithmetic (from Boethius) in his *Eruditio Didascalica*.[4] Neither of these men could be said to give real instruction in arithmetic.

A treatise in Greek on arithmetic, which compares favorably in poverty of content with the Latin works that have been mentioned,

[1] Paul Tannery, *Le Manuel d'Introduction Arithmétique du Philosophe Domninos de Larissa*, *Revue des Études Grecques*, vol. 19 (1906), pp. 359-382. [2] Tannery, *loc. cit.*

[3] Migne, *Patrologia Latina*, vol. CVII, *De Clericorum Institutione*, chap. xxii, *De Arithmetica*, cols. 399–400.

[4] Migne, *Patrologia Latina*, vol. CLXXVI, *Eruditio Didascalica*, Liber I, chap. viii, col. 755, and chap. xii, col. 756, *De Arithmetica*.

is that commonly ascribed to Michael Psellus (1020–1105 A.D.?).[1] This work, certainly of the eleventh century, presents the Nicomachean arithmetic again in the original tongue, but possibly through the mediation of the Latin translation. The discussion of the classes of numbers and progressions includes nothing new; the only variation from the Nicomachean material appears to be the definition of multiplication as "taking the multiplicand as many times as there are units in the multiplier."

In the eleventh century appeared also treatises on an ancient arithmetical game, *rithmomachia*.[2] The rules of the game are fundamentally dependent upon the Nicomachean classifications of numbers; the game continued to be popular for several centuries and possibly aided to revive for a time the arithmetic which was being supplanted as a discipline by the Hindu-Arabic system of computation.

There is abundant evidence to show that the study of the mystical arithmetic and of the arithmetic as taught by Boethius continued in the church schools for more than a thousand years after Boethius. True contributions to the science of numbers, however, do not appear to have emanated from these institutions. For such contributions at this time to the progress of all science, we must look rather to the Arabs, to whom civilization is greatly indebted for the continued advance of learning during the period when Europe was in darkness.

The Arabic writers on arithmetic, from the eighth century through the fifteenth, were undoubtedly more or less familiar with the general content of Nicomachus and more certainly with Euclid, but their scientific inclinations led them fortunately to stress the new Hindu arithmetic, an instrument essential to scientific progress in many lines.

It is not at all probable, as Cantor asserts,[3] that the famous Arabic mathematician, Al-Khowarizmi (*c.* 825 A.D.), wrote a speculative arithmetic along Greek lines. No work of Al-Khowarizmi suggests

[1] *Michaelis Pselli Compendium Mathematicum* (Leyden, 1647), containing the *Quadrivium* ascribed to Michael Psellus and translated by Xylander; see P. Tannery, *Zeitschrift für Mathematik und Physik, Historisch-literarische Abtheilung*, vol. XXXVII, p. 41.

[2] Wappler, *Bemerkungen zur Rhythmomachie*, in *Zeitschrift für Mathematik und Physik, Historisch-literarische Abtheilung*, vol. XXXVII, pp. 1–17; see Smith and Eaton, *Rithmomachia, the Great Medieval Number Game*, in *American Mathematical Monthly*, vol. XVIII, pp. 73–80.

[3] *Geschichte der Mathematik*, vol. I (3d edition), pp. 715–716; the error is due to a misinterpretation of a passage in Al-Khowarizmi's arithmetic. Al-Khowarizmi refers to his 'aljebr w'almuqabala,' as 'another book in arithmetic' (i.e., 'in alio libro arithmetico'), and Cantor bases the speculative arithmetic on this phrase.

more than casual interest in this field. Another Arab, Thâbit ibn Qorah (826–901), refers to the generation of perfect numbers and of 'amicable numbers,' a pair of numbers such that each equals the sum of the factors of the other; he makes also direct reference to Nicomachus. Among his works is included *Extracts from the Two Books of Nicomachus.*[1]

Ibn Khaldun, in his encyclopedic work, takes up briefly the discussion of the *arithmetica*, mentioning the three types of even numbers and thus showing acquaintance with other Greek authorities than Euclid. He states, however, that while some ancient Mohammedans did treat this subject, modern writers were inclined to reject this type of work as serving no practical purpose.[2]

Many treatises touching mystical arithmetic are still preserved in Arabic, Ethiopic, and related languages, although few of these have as yet received the serious attention of scholars.[3]

The revival of learning in Europe, particularly scientific learning, was stimulated by the translation into Latin of Arabic works of science. The schools of the Arabs in Spain were open to the Christians, and the translations came largely from such students, resident in Spain. Not only works of Arabic origin, but also works ultimately Greek or Hindu in origin, were translated into Latin from the Arabic. The twelfth century was notably the period during which the making of such translations was in active progress,[4] while the period from the thirteenth to the fifteenth century witnessed the popularization of the material given in the translations.

Coincident with the study of the Greek geometry and the Hindu-Arabic arithmetic and the algebra of the Arabs was a European revival of interest in the Greek arithmetic. Such writers as Pope Sylvester II of the tenth century [5] and Johannes de Muris [6] of the fourteenth century touched upon the Nicomachean arithmetic. The early

[1] Suter, *Die Mathematiker und Astronomen der Araber*, pp. 35–37; the Arabic version is extant in the British Museum (426, 150).

[2] MacGuckin de Slane, *Prolégomènes Historiques d'Ibn Khaldoun*, in *Notices et Extraits des Manuscrits de la Bibliothèque Nationale*, vol. 21, 1868, pp. 129–132.

[3] Personal communication of my colleague, Professor Worrell, formerly director of the American School of Oriental Studies in Jerusalem.

[4] See Karpinski, *Robert of Chester's Translation of the Algebra of Al-Khowarizmi*, in *Bibliotheca Mathematica*, 3d Series, vol. XI, pp. 125–131.

[5] *Gerberti postea Silvestri II Papae Opera Mathematica* (ed. Bubnov, Berlin, 1899), pp. 29–35.

[6] See Karpinski, *The "Quadripartitum Numerorum" of John of Meurs*, in *Bibliotheca Mathematica*, 3d Series, vol. XIII, pp. 99–114.

printed works on arithmetic were, of course, largely devoted to the new system of arithmetic, popularizing the use of the Hindu-Arabic numerals. However, in the compendiums of arithmetic and in the encyclopedic works we find frequently extensive treatment of the speculative arithmetic, the Greek, as opposed to the practical computations communicated by the Arabs. Thus, the Italian, Lucas de Burgo San Sepulchri, in his *Summa de Arithmetica* of 1494,[1] devoted folios 1–19 (38 pages) to the speculative arithmetic. Similarly the Archbishop of Canterbury, Thomas Bradwardin (*c.* 1290–1349), wrote a treatise, based on Boethius, called *Arithmetica Speculativa*, which was printed many times [2] and which formed the basis of a work on arithmetic [3] by the Spaniard, Pedro Sanchez Ciruelo (*c.* 1470–1560).

Dominican

Jordanus Nemorarius, head of a Teutonic monastic order (who perished in 1236 in a shipwreck), wrote a thoroughly Greek arithmetic in ten books to which Faber Stapulensis added demonstrations. In the 1514 edition [4] are included the *Arithmetica Decem Libris Demonstrata*, a work on music after Greek models, by Faber Stapulensis, also an epitome of the arithmetic of Boethius, and a work on the game *rithmomachia*.

Among the encyclopedias which treated the arithmetica may be mentioned the popular *Margharita Philosophica* of Gregorius Reisch (who died in 1525) and the *Speculum Doctrinale* of Vincent of Beauvais (1190–1264).

This list of writers on the arithmetica could be extended most materially. Briefly it may be said that the works which we have cited showing the continued use throughout Europe of the arithmetic of Nicomachus, as translated by Boethius, are typical of the period from the tenth to the sixteenth century. The extent of time and territory included within the influence of Nicomachus and Boethius amply justifies the study of arithmetic of Nicomachus as a notable document in the history of learning.

[1] Copy in the University of Michigan Library.

[2] See Smith, *Rara Arithmetica* (Boston, 1908), vol. I, p. 61.

[3] *Cursus Quatuor Mathematicarum Artium Liberalium* appeared in numerous editions; the Alcala, 1526 edition, is in the University of Michigan Library. Ciruelo was the author, also, of a work on the Hindu-Arabic art of reckoning.

[4] I am indebted to Mr. John G. White of Cleveland, Ohio, for the use of his copy of the 1514 edition.

CHAPTER XI

THE MANUSCRIPTS AND TEXT OF THE *INTRODUCTION TO ARITHMETIC*

It would be gratifying indeed to print a fresh text of the *Introduction to Arithmetic*, based upon a complete examination of the manuscripts, and to supplement this by an adequate account of the manuscripts and their relationships. Unfortunately, this has proved impossible; the writer has not been able to devote the necessary time and travel to the task. Consequently Hoche's text has been made the basis of the translation and commentary, and the remarks made in this chapter on the manuscripts and text are to be regarded as tentative, based as they are on fragmentary information.

The editions of the *Introduction to Arithmetic* have been as follows:

1. The first edition was that of Wechel (Paris, 1538). Wechel probably used a single manuscript, now lost, referred to as P in the following pages.

2. The second was that of Friedrich Ast (Leipzig, 1817), containing also the *Theologumena Arithmeticae.* Ast used, besides the first edition, the three Munich manuscripts, μ, m, and S, but failed to realize the relative value of his sources and relied too much upon emendation.

3. Gottfried Müller (*Particula Nova Notitiae et Recensionis Codicum Manuscriptorum Qui in Bibliotheca Episcopatae Numburgo-Cizensis Asservantur*, Leipzig, 1818) brought forward the Giessen codex, C, and published a collation of it with Ast's edition.

4. C. F. A. Nobbe (*Specimen Arithmeticae Nicomacheae e Duobus Codicibus Manuscriptis*, Leipzig, 1828) issued a brief notice of the Nürnberg and Wolfenbüttel manuscripts, N and Γ, with a text of the *Introduction to Arithmetic*, I. 1–3, and critical notes.

5. Richard Hoche (*Nicomachi Geraseni Pythagorei Introductionis Arithmeticae Libri II*, Teubner, Leipzig, 1866), using for the first time the Göttingen and Hamburg manuscripts, G and H, and to some extent all those previously mentioned, produced the edition which is

still standard. He had preceded this with an article in a program of the gymnasium at Wetzlar (1862) in which G was for the first time used.

The manuscripts mentioned and used by Hoche are the following:

1. G, Codex Gottingensis philol. 66, a parchment manuscript of the tenth century, iii + 266 leaves, $11\frac{1}{2}$ by 15 cm., with columns $13\frac{1}{2}$ by 10 cm. and 20 lines to the page in the text, 39 in the scholia. This contains the text of the *Introduction*, the commentary of Johannes Philoponus, and a scholium of Theodorus Protocensor. Corrections have been made throughout by a much later hand which Hoche calls G_2.

2. m, Codex Monacensis 238, paper, of the fourteenth century, containing only the first part of the *Introduction* as far as the words πρὸς τῷ ὅλῳ, I. 19. 4, p. 50, 2 Hoche.

3. P, the first edition by Wechel (Paris, 1538). This deserves to be reckoned among the manuscripts, since the manuscript upon which it was based, apparently an ancient and a good one, has disappeared.

4. C, Codex Cizensis, paper, of the last of the fourteenth or the early fifteenth century, containing the *Introduction*, another version of Philoponus's commentary, and the problems printed upon pages 148–154 of Hoche's edition.

5. μ, Codex Monacensis 76, paper, of the sixteenth century, which Hoche thinks a copy of C.

6. S, Codex Monacensis 482, of the thirteenth or fourteenth century.

7. H, Codex Hamburgensis, of the sixteenth century.[1]

8. N, Codex Norimbergensis, of the last of the fourteenth or the beginning of the fifteenth century.

9. Γ, Codex Guelferbytanus, of the early sixteenth century.[2]

The following manuscripts in addition are known to the writer through photographic or photostatic reproductions of a few pages of each, or through collations of typical passages made by friends:

10. A, Codex Atheniensis 1115, of the fifteenth century, in the National Library at Athens.[3] Paper; 138 pages at present, four having been lost at the beginning. It contains anonymous prolegomena

[1] H. Omont, *Manuscrits Grecs des Villes Hanseatiques*, Hamburg, No. 48.

[2] O. von Heinemann, *Handschriften der herzoglichen Bibliothek zu Wolfenbüttel*, Wolfenbüttel, 1913.

[3] Cf. the catalog of the manuscripts of the National Library of Greece by J. and A. I. Sakkeliou, 1892. Through the kindness of Professor F. W. Kelsey the writer has secured photographs of the first 26 pages, carrying the text to p. 29, 13 Hoche.

to the *Introduction to Arithmetic*, then the two books of the *Introduction* itself, and several other treatises.

11. B, Codex Atheniensis 1238, in the National Library at Athens.[1] 136 leaves, on ordinary paper 22 by 17 cm., of about the beginning of the eighteenth century. This contains the text of Book I with the commentary ascribed to Proclus intervening between sections.[2]

12. J, in the Greek Patriarchal Library at Jerusalem, number 5 of those given by the Patriarch Nicodemus, written by the scribe Maximus in 1801.[3] 39 paper leaves, 39 by 23 cm., in single column, 23 lines to a page. J is not a complete text, but a series of excerpts.

13. V, Codex Ms. Gr. Ottoboniani Graec. Bibliothecae Vaticanae 310. 160 leaves, grayish-brown paper about 17 by 10 cm., of the fifteenth century; written in a compact hand in very black ink and with many abbreviations, 27 or 28 lines to a page. The *Introduction* is on foll. 124–160 r.[4]

14. D, Codex Vaticanus Gr. 1709, of perhaps the fifteenth century; fine heavy white paper 29 by 21 cm. with very wide margins, about 26 lines to a page. There are several treatises besides the *Introduction*, which occupies foll. 203–210; the end is not extant.

15. E, Codex Vaticanus Gr. 196, of perhaps the fifteenth century, on very fine heavy paper 29.5 by 21 cm. Nicomachus occupies foll. 1–30.

16. F, Codex Vaticanus Gr. 186, of apparently the sixteenth century; beautifully written on both sides of thick gray paper $6\frac{3}{8}$ by $4\frac{5}{8}$ inches; scholia accompany the text of Book I, but not Book II.

17. L, Linc. Coll. (c) Gr. 33, in the library of Lincoln College, Oxford.[5]

[1] The writer has from the same source photographs as far as p. 11, 5 Hoche.

[2] Hoche, in his edition of the scholia of Johannes Philoponus (Leipzig, 1864), regards the commentary of Proclus as a recension of that by Philoponus. He there makes mention of this Athenian manuscript, which he had not seen, and correctly infers that it contains much the same commentary as the Giessen manuscript, C.

[3] See the catalog of this library by A. Papadopoulos-Kerameus, Petrograd, 1891-99, vol. III, p. 181. A considerable portion of J is known to the writer through photographs secured by Professor Kelsey. Its superscription is: *ἐκ τῶν Νικομάχου Γερασινοῦ δύω βιβλίων ἀριθμητικῆς. ἐκ τοῦ πρώτου.* At the end occurs the following notice: *τέλος τῶν ἐκ τῆς ἀριθμητικῆς Νικομάχου Γερασινοῦ. σὺν θεῷ ἀντεγράφη ἐν ἔτει σωτηρίῳ ͵αωά, φευρ: ι΄, ἐν πόλει τοῦ θεοῦ Ἰλημ διὰ χειρὸς ἀναξίου δούλου τοῦ θεοῦ Μαξίμου.*

[4] Cf. Feron and Battaglini, *Codices Manuscripti Graeci Ottoboniani Bibliothecae Vaticanae*, Rome, 1893, p. 166. Descriptions and partial collations of this and the three following manuscripts have been furnished by the writer's colleague, Professor John G. Winter.

[5] H. O. Coxe, *Catalogus Codicum Manuscriptorum Qui in Collegiis Aulisque Oxoniensibus Hodie Adservantur*, Oxford, 1852, vol. I, Lincoln College, p. 17–18, No. XXXIII.

18. O, New College B 299, at Oxford. Fifteenth century; the text of Nicomachus occurs on foll. 52–60.[1]

19. e_1, at the Escurial, numbered T–I–11. A folio manuscript of 222 leaves in several fifteenth-century hands, from the library of Hurtado de Mendoza. The *Introduction to Arithmetic* occupies foll. 206r–210r, with marginal scholia.[2]

20. e_2, number T–II–6 of the Escurial collection, a paper folio of 114 leaves 30 by 20.5 cm., containing the *Introduction* with scholia; the two books are in different hands. Book I runs to fol. 97v, and Book II occupies foll. 103r–114v. Columns 22.5 by 15.5 cm.[3]

21. e_3, number X–I–9 of the Escurial collection; a paper folio of 266 leaves, 33.5 by 22.5 cm., written in one column 21 by 13.5 cm., 19 lines to the column, in a sixteenth-century hand. Contains the *Introduction* with Philoponus's commentary; from the collection of Hurtado de Mendoza. Book I on foll. 4r–48r, Book II on foll. 49r–91v.[4]

22. e_4, number Σ–II–15 of the Escurial collection; fourteenth century, paper, 179 leaves 28 by 21 cm., with one column of writing 21.5 by 14.5 cm., 36 lines to the column; worm-eaten. The *Introduction* begins with fol. 160r and continues through 171v.[5]

23. e_5, number Υ–III–12 of the Escurial collection; paper quarto of the fourteenth century, 81 leaves 21.8 by 14.2 cm., in one column 17 by 10.5 cm., 31 lines to a column. Book I on foll. 1r–21r, Book II on foll. 21r–42r.[6]

24. p_1, Ancien Fonds Grec, No. 2483. This and the ten following manuscripts are in the Bibliothèque Nationale at Paris. Fourteenth century, 318 leaves, containing the *Introduction* with Philoponus's commentary.[7]

25. p_2, Ancien Fonds Grec, No. 2481. A fifteenth-century quarto of 147 leaves, paper, with the *Introduction* on foll. 1–52v, followed by Euclid.[8]

[1] H. O. Coxe, *op. cit.*, vol. I, New College, p. 107. Dr. Charles Singer, of Oxford, has furnished a note on this manuscript.

[2] E. Miller, *Catalogue des Manuscrits Grecs de la Bibliothèque de l'Escurial*, Paris, 1848, p. 109. The descriptions of this and other Escurial manuscripts are supplemented by notes and photographs of specimen pages furnished by the writer's colleague, Professor Henry Arthur Sanders.

[3] Miller, *op. cit.*, pp. 125–126.

[4] *Op. cit.*, p. 294.

[5] *Op. cit.*, p. 90.

[6] *Op. cit.*, p. 253.

[7] Professor Sanders has also furnished descriptions of the Parisian manuscripts. H. Omont, *Inventaire Sommaire des Manuscrits Grecques de la Bibliothèque Nationale*, vol. III, p. 267.

[8] Omont, *ibid.*

26. p3, Ancien Fonds Grec, No. 2480. Paper quarto of the eighteenth century, viii + 346 + 319 leaves, containing the *Introduction* with Philoponus's commentary.[1]

27. p4, Ancien Fonds Grec, No. 2479, thirteenth-century parchment (mostly sheepskin, but partly goat), 201 leaves; the *Introduction* with marginal scholia of Soterichus.[2]

28. p5, Ancien Fonds Grec, No. 2376. Paper, 251 leaves 32 by 21.5 cm., in one column of 30 lines to a page, copied by Valeriano Albino in 1539; Book I on foll. 57r–78v, Book II on foll. 78v–101v. Figures and tables in the margin.[3]

29. p6, Ancien Fonds Grec, No. 2375. Paper, 45 leaves 31.5 by 20.5 cm., of the sixteenth century, copied by Constantine Palaeocappa; contains Book I, chapters 1–17. 5 with Philoponus's commentary.[4]

30. p7, Ancien Fonds Grec, No. 2374. Paper, of the sixteenth century, copied by Jean d'Otrante; 42 leaves, 32 by 22.8 cm., in single column of 24 lines; Book I on foll. 1r–21v, Book II on foll. 22r–42v.[5]

31. p8, Ancien Fonds Grec, No. 2373. Paper, of the fourteenth century, 124 leaves 25.3 by 17 cm., one column of 35 lines to the page; Book I on foll. 1r–17r, Book II on foll. 17r–36v.[6]

32. p9, Coislin., No. 174. Paper, fifteenth century, 441 leaves 31 by 19.8 cm., in single column of 33 lines 20 by 11.5 cm. Book I occupies foll. 41r–58; Book II foll. 59–79v. Philoponus's commentary accompanies the text.[7]

33. p10, Supplement Grec, No. 450; paper, fifteenth century, 183 leaves 23.2 by 16.8 cm., with column of 21 lines 16.5 by 15 cm. Nicomachus occupies foll. 6r–68v, with no commentary.[8]

34. p11, Ancien Fonds Grec, No. 2372; sheepskin parchment of the fifteenth century; 109 leaves 25 by 18 cm., with single column of 27 lines 18 by 11 cm.; formerly Medic.-Reg. 2657. Nicomachus, without commentary, appears on foll. 1–53, followed by other treatises.[9]

There will next be listed manuscripts known to the writer only from their descriptions in the catalogues of various libraries:

35. At the Escurial, Number 21 of the manuscripts of Cardinal Sirlet is said to contain Euclid followed by the text of the *Introduction to Arithmetic*.[10]

[1] Omont, *op. cit.*, vol. III, p. 267.
[2] *Ibid.*
[3] Omont, vol. II, p. 248.
[4] *Ibid.*
[5] *Ibid.*
[6] *Ibid.*
[7] Omont, *op. cit.*, vol. III, p. 148.
[8] Omont, *op. cit.*, vol. III, p. 264.
[9] Omont, *op. cit.*, vol. II, p. 248.
[10] Miller, *op. cit.*, p. 324.

36. At the Bibliothèque Nationale, Paris, Ancien Fonds Grec, No. 2377, contains Nicomachus with commentary; lines and part lines are in red, with much commentary intervening. Book I on foll. 1r–88v, Book II on foll. 89r–161v.[1]

37. At Hamburg, Mathematici Graeci, in folio, IV a. A paper manuscript of the fourteenth century; 55 leaves 25.2 by 16.5 cm., bound in parchment; the *Introduction* on foll. 1–53r, followed by Aristotle's *Nicomachean Ethics*.[2]

38. At Erlangen, a defective sixteenth-century manuscript.[3]

39. At Leyden, a paper codex of the fourteenth century, 40 leaves in large quarto, which belonged to Meibom and later to P. Burmann the second.[4]

40. At Moscow, a paper codex of 101 leaves, of the fourteenth century, containing the *Introduction* with scholia on foll. 66 ff.[5]

41. Cambridge University Library, Gg. I (ii), foll. 21–22, fifteenth century; apparently a brief series of excerpts from Nicomachus.[6]

42. Oxford, Bodleian Library, Selden Greek 19, of the fourteenth century; "Nicomachi Geraseni Arithmeticae libri duo imperfecti."

43. Cambridge University Library, Kk. V, 28, foll. 1–119v., fifteenth century. Foll. 1–88 contain the *Introduction to Arithmetic* and foll. 89 ff. the *Manuale Harmonicum*.

44. Vaticanus Graec. 199, consisting of one leaf only, fifteenth century.[7]

As the writer has already indicated, the following remarks are to be understood as in no sense constituting a formal attempt to classify the manuscripts of the *Introduction to Arithmetic*. They are based on three sources of information, Hoche's apparatus criticus, photographic reproductions of specimen pages, and partial collations made by friends. The first is avowedly not exhaustive, except for the first

[1] Note furnished by Professor H. A. Sanders.

[2] H. Omont, *Manuscrits Grecs des Villes Hanséatiques*, number 47.

[3] Johann Conrad Irmscher, *Handschriften-Katalog der kgl. Universitäts-Bibliothek zu Erlangen*, Frankfurt-a.-M. and Erlangen, 1852, number 94.

[4] Geel, *Catalogus Librorum Manuscriptorum Qui inde ab Anno 1741 Bibliothecae Lugduno Batavae Accesserunt*, p. 37, number 126.

[5] Chr. F. de Matthaei, *Accurata Codicum Graecorum Manuscriptorum Bibliothecarum Mosquensium Sanctissimae Synodi Notitia et Recensio*, Leipzig, 1805, p. 147, Number CCXXIII.

[6] Dr. Charles Singer, of Oxford, furnished the writer notes on numbers 41, 42, and 43.

[7] Professor John G. Winter's note. Professor Winter informs me also that entries in the *Catalogus Vaticanus Graecus* indicate that the following Vatican codices contain the *Introduction to Arithmetic* in whole or in part: Codd. Vat. Gr., 199, 1040, 1051, 1411, 195, 197, 256, 1026, 198; Book I only, Codd. Vat. Gr. 199, 256; excerpts, Cod. Pal. Gr. 129.

three chapters of Book I, and the information derived from photographs and collations has been in every instance fragmentary. From all three the most that might be claimed is a fairly accurate report of the readings of the majority of the manuscripts in the first three chapters of the *Introduction*. Obviously this is not a sufficient basis for conclusions of permanent value, and the following paragraphs are ventured only because nothing has previously been said of the character of the manuscripts which Hoche did not employ.

It was not especially hard for Hoche, using only nine manuscripts, to see that GmP form one family and $C\mu G_2$SHNΓ another, giving the following stemma:[1]

Archetype

Family I — Family II

G_1m — P — $C\mu$ — G_2SHNΓ

Were these the only manuscripts in existence their relationships would thus be represented with substantial accuracy, and indeed, even with added information, these groupings are in the main correct. When other manuscripts, however, are taken into the reckoning, it becomes evident that the text of Nicomachus was an exceedingly mixed one, and that clear-cut groupings are not so easy to make as perhaps Hoche thought. In particular, manuscripts exist which exhibit to a greater degree than SHNΓG_2, or even than $C\mu$, certain characteristic readings of the group G_1mP, or in other words, the differentiation between Hoche's two main families is rather a gradation than a series of clear-cut steps.

To illustrate what may be said on these points, there follows a selection of the thirty-seven readings in the first three chapters of the text which appear to the writer particularly significant. These are the passages on which the manuscripts seem to divide; the individual peculiarities of single manuscripts, misspellings, and minor changes in the order of words are not included.

Page 1, 10 ἁπλῶς ὁ τέχνης τινὸς, GmPFp_3
ἁπλῶς πᾶς ὁ τέχνης τινὸς, $C\mu$BSHNΓADELOV$e_{1\text{-}5}$ $p_{1\text{-}2,\,4\text{-}11}$

Page 2, 1 συστειλας πάντων τὸ ὄνομα, GmPHVp_7
συστειλας τὸ ὄνομα, $C\mu$BSNΓADEFO$e_{1\text{-}5}$ $p_{1\text{-}2,\,4\text{-}6,\,8\text{-}11}$

[1] G will be used ordinarily to indicate the first hand in G, unless it becomes necessary to distinguish more clearly.

Page 2, 2 τὴν ἐν τούτῳ γνῶσιν, GmPCμBVp$_{7}$
τὴν ἐν τούτῳ πάντων γνῶσιν, SHNΓADEFOe$_{1-5}$ p$_{1-2, 4-6, 8-11}$
4 εἰκότως καὶ τὴν ταύτης ὄρεξιν, GmPCμBAVEFe$_{1, 3-4}$ p$_{3, 7, 11}$
εἰκότως τὴν ταύτης ὄρεξιν, SHNΓDOe$_{2, 5}$ p$_{1-2, 4-6, 9-10}$
5 σοφίας ὄρεξιν, GmPCμBFe$_{5}$ p$_{3, 6-7}$
σοφίας ἔφεσιν, SHNΓAVDEOe$_{1-4}$ p$_{1-2, 4-5, 8-11}$
7 καὶ ταύτην δὲ τὴν σοφίαν, GmPCμBSFe$_{2}$ p$_{3, 6, 7, 9}$
καὶ αὐτὴν δὲ τὴν σοφίαν, HNΓADEOVe$_{1, 3-5}$ p$_{1-2, 4-5, 8, 10-11}$
12 ἐξιστάμενα, GmPCμBHAVEFe$_{1-4}$ p$_{6-7, 11}$
ἀφιστάμενα, SNΓDOe$_{2}$ p$_{1-2, 4-5, 8-10}$
13 ταῦτα (or ταῦτ') ἂν GmPFp$_{3, 6}$
ταῦτα δ' ἄν, CμBSHNΓADEOVe$_{1-5}$ p$_{1-2, 4-5, 7-11}$
13 τὰ ἄυλα καὶ ὧν κατὰ, GmPF (om. καὶ F) p$_{6}$
τὰ ἄυλα καὶ (τὰ) ἀίδια ὧν κατὰ, SHNΓADEOVe$_{1-5}$ p $_{1-5, 8, 10-11}$ (τῶν p$_{3}$)
τὰ ἄυλα καὶ ἀίδια καὶ ὧν κατὰ, CμBp$_{7, 9}$
14 τῶν ὁμονύμως ὄντων καὶ καλουμένων, G$_{1}$mBSHNDFe$_{3, 5}$ p$_{3, 6, 10}$
τῶν ὁμονύμως ὄντων καλουμένων, PCμG$_{2}$ΓAEOVe$_{1-2, 4}$ p$_{1-2, 4-5, 7-9, 11}$
15 λέγεται, G$_{1}$mPSNΓCμBp$_{6}$
λέγεται εἶναι, G$_{2}$HADEFOVe$_{1-5}$ p$_{1-5, 7-11}$
19 περὶ τὰ ὑλικὰ καὶ σωματικὰ, add. SHAFVOe$_{1-3}$ p$_{1-2, 4-5, 7, 10-11}$, om. GmPCμBNΓEe$_{4-5}$ p$_{6, 8-9}$
20 περὶ αὐτὴν, GmPCμBSHNΓDAEOe$_{4-5}$ p$_{1-2, 4-6, 8-10}$
περὶ αὐτὰ, FVe$_{1-3}$ p$_{7-11}$
20 σὺν αὐτῇ, GmPCμBSHNΓDEOVe$_{4-5}$ p$_{1-2, 4-6, 8-10}$
σὺν αὐτοῖς, AFe$_{1-3}$ p$_{7, 11}$ p$_{8}$ (sup. lin.)

Page 3, 1 μικρότητες, GmPCμSHNAVFe$_{1, 3-4}$ p$_{1, 4, 11}$
σμικρότητες, BΓDEOe$_{2, 5}$ p$_{2, 5-10}$
1 ἀνισότητες, om. GmPCμBSe$_{2, 4}$ p$_{4, 10}$. add. HNΓADEFOVe$_{1, 3, 5}$ p$_{1-2, 5-9, 11}$
3 τὰ ἐν ἑκάστῳ σώματι, GmPCμBSHNAFVe$_{1, 3-5}$ p$_{1-2, 4-6, 8}$ (ἃ add. AVe$_{1, 3-4}$ p$_{4}$)
τὰ ἐν ἑκάστῳ σώματα, ΓDEe$_{2}$ p$_{7, 9-11}$ (ἃ add. Ep$_{7, 11}$)
11 ἐπιδιαμένοντα, GmPBAFVe$_{1, 3-4}$ p$_{1, 6-8, 11}$
διαμένοντα, CμSHNΓDEe$_{5}$ p$_{2, 4-5, 9-10}$
12 τὰ δὲ ἐν γενέσει, GmPFVp$_{9}$
ταῦτα δὲ ἐν γενέσει, CμBSHNΓADEe$_{1-5}$ p$_{1-2, 4-8, 10-11}$
18 μεταβαινει, GmPCμNAFe$_{1}$ p$_{7, 11}$
μεταβάλλει, BV
μεταρρεῖ, SHΓDEe$_{4}$ p$_{1-2, 5-6, 8-10}$
μεταρρεῖ καὶ μεταβαίνει, e$_{5}$ Ast.
19 τί τὸ ὂν ἀεὶ, GmPFe$_{4-5}$ p$_{7}$
τί τὸ ὂν μὲν ἀεὶ, CμBSHNΓADEVe$_{1}$ p$_{1-2, 5-6, 8-11}$

Page 4, 9 ἡμῖν, G (ἡμ////) mPAVEFe$_{1, 4-5}$ p$_{2, 5, 7-8, 11}$
ἐστίν, CμBSNΓDp$_{6, 9-10}$
om. H

Page 4, 12 διελεῖν, GmPCμBHND
διεξελθεῖν, S (διελεῖν in marg.) AFVe$_1$ p$_{7, 10-11}$
διελθεῖν, ΓEe$_4$ (εξ in marg.) e$_5$ p$_{2, 5, 8-9}$
16 κόσμος δένδρον, GmPCμBFp$_{2, 7}$
δένδρον κόσμος, SHNΓADEVe$_{4-5}$ p$_{5, 8-11}$
19 σωρός, χορός, GmPEFe$_5$ p$_{2, 7-9}$
χορός, σωρός, CμBSHNΓADVe$_4$ p$_{5, 10-11}$
20 εἰδῶν τούτων, GmPCμBHAFVp$_{7, 11}$
τούτων εἰδῶν, SNΓDEe$_5$ p$_{5, 8-10}$
τούτων δύο εἰδῶν, e$_4$ p$_2$

Page 5, 4 διὰ ταῦτα, GmPBHFp$_8$
διὰ ταύτης, CμSAEVe$_{4-5}$ p$_{7, 10-11}$
δι' αὐτῆς, NΓDp$_{2, 5}$
διὰ αὐτῆς p$_9$
8 ποτε ἐπιστήμη, GmFVEHe$_5$ p$_{2, 5, 7-9, 11}$
ἐπιστήμη ποτὲ, CμBe$_4$
om. ποτε, PSNΓA (add. supra lin.) Dp$_{10}$
15 οἷον ἄρτιον, GmPCμBp$_7$
οἷον τετράγωνον ἄρτιον, SHNΓADEFVe$_4$ p$_{2, 5, 8-11}$
17 μεῖζον ἔλαττον ἥμισυ, GmPμSHAEFVp$_{2, 5, 7-8, 10-11}$
ἥμισυ μεῖζον ἔλαττον, BCNΓDp$_9$
ἥμισυ ἔλαττον, e$_4$
19 ἄρα, GmPFe$_4$ p$_{2, 5, 9-10}$, om. CμBSHNΓADEVp$_{7-8, 11}$
19 ἐπιλήψονται, G$_1$mPBp$_7$
διαλήψονται, CμSHNΓAG$_2$DEFVe$_4$ p$_{2, 5, 8-11}$

Page 6, 18 δοκοῦντι, GH
δοκοῦντι τὸ, PBS
δοκοῦντί τοι, mCμNΓADEFVp$_{2, 8-9}$
δοκοῦντι τὸν, e$_4$

Page 7, 6 περισκοπεῖ καὶ διορίζεται, Gm
διορίζεται, H
σκοπεῖ καὶ διορίζεται, SNΓADEFVe$_4$ p$_{2, 8-9}$
περισκοπεῖ, PCμB
7 χρὴ εἶναι, GmSNΓADEVe$_4$ p$_{2, 8-9}$
δεῖ εἶναι, PCμBH
8 προδιαλεχθέντα καὶ προδιαβεβαιωθέντα, GmBF
προλεχθέντα καὶ βεβαιωθέντα, e$_4$
προλεχθέντα καὶ διαβεβαιωθέντα, SHNΓ
προλεχθέντα καὶ προβεβαιωθέντα, AVEp$_8$
διαλεχθέντα, μ
προβεβαιωθέντα, C
πρὸς διαλεχθέντα καὶ διαβεβαιωθέντα, P
προλεχθέντα, Dp$_{2, 9}$
11 ἀναλογίαν, GmPF
ὁμολογίαν, CμBSHNΓADEVe$_4$ p$_{2, 8-9}$

As far as this limited list of readings is at all significant, it shows, first, that the manuscripts may be roughly grouped. The most important group is that composed of G, m, and P, which are nearly always (30 out of 37 times) associated and are clearly close relatives. G and m are even closer to each other than P is to either or both of them; they agree in 35 of the 37 cases. Other manuscripts, such as F, p_3, p_6, and p_7, sometimes one, sometimes another, from time to time agree with GmP, but the latter combination is the one most often found to have the same reading. Another trio which similarly evidences relationship is C, μ, and B. Among the rest it is not so easy to demonstrate close kinship.

G is the oldest of the manuscripts, and one may safely agree with Hoche in thinking it the best representative of the primitive text, although it is not infallible. Though very like G, m cannot be a copy of it, and P, as Hoche noticed, is characterized by numerous blunders which may be the fault of an illiterate scribe or perhaps of Wechel or his typesetter. P, furthermore, shows a rather noticeable tendency to agree with CμB, possibly because of some contamination in its ancestry with manuscripts of that type.

Although these two groups, GmP and CμB, stand out quite clearly by themselves, the tradition is in general a very mixed one, and it would be folly to attempt, on so meager information, to unravel all these tangled threads. The evidence seems to point to a series of revisions or recensions, more or less general, which has affected the later manuscripts. The readings in pages 2. 13 (first example), 3. 12, 19, and 7. 11, among those cited above, are examples of the changes in the text that found their way into nearly all but GmP. Sometimes these are attempts at correction, often ill-advised and superficial, like the change so commonly introduced into the manuscripts in page 2. 13, the first of the passages just cited; this is an obvious effort to supply a balance that the scribe thought was needed, but it really mars the sense and can hardly have been the original text.

In other cases the variant is the result of the introduction into the text of a marginal or interlinear gloss, or even a caption. Without much doubt the latter mischance has occurred at page 2. 19 in many manuscripts outside of the groups GmP and CμB. Perhaps the variants in pages 2. 5, 13 (second example) and 7. 11 were originally glosses. Still others, and these are very characteristic of the group in which Hoche put SHNΓ, are in the nature of additions to the text, either of words easily understood, as in page 2. 15, or of extra terms in a series of Nicomachus's examples, as in pages 3. 1 (second example) and 5. 15. Simple corruptions have come in and have been perpetuated in many manuscripts; see, for example, pages 4. 12; 7. 6, 8. It is to be observed that GmP generally stand aloof from these apparent tamperings with the text.

There is certainly evidence in plenty that two or more readings were known in very many passages. It is entirely possible that a single manuscript or a whole series of them would retain both the primitive reading and the gloss, and in subsequent copyings from such manuscripts sometimes one and sometimes the other reading would be reproduced. This may explain why the large group embracing SHNΓADEFOLV, the Escurial manuscripts and most of the Parisian ones, is seldom unanimous, and sometimes one, sometimes another, agrees with GmP, though as a whole they are characterized by the presence of the sort of variants that have just been described. Consequently it is difficult to discern clearly defined subfamilies among them, and together with CμB one may venture to designate them as representing the vulgate text of the *Introduction to Arithmetic*.

It must also be observed that manuscripts were frequently compared with one another and variant readings noted down, which doubtless were taken into the text as time went on. Evidence of this is easily discovered. Hoche's apparatus shows clearly that the second hand in G introduced a number of readings from a manuscript of the vulgate type, and in the margin of A, referring to page 10. 13, ἢ ἡμιόλιον, is found the note, *τοῦτο ἐπ' ἐνίων οὐ κεῖται*. H, in fact, omits the words referred to. Possibly comparison with other manuscripts explains the reading of m in page 6. 23, where it parts company with G and P and sides with the vulgate. The same explanation may explain the notable peculiarity of P already mentioned, its tendency to agree with CμB.

C and μ, as Hoche observed, are very much alike, but it is not probable that either is the direct source of the other. Of the additional manuscripts with which the writer is acquainted, B is to be associated with this pair and the three make up a subfamily characterized on the one hand by peculiar readings of their own — as, the addition of ὢ λῶστε BC, λῶστε μ in page 8. 19, and the elaborated reading ἢ διπλάσιον ἢ τριπλάσιον ἢ τετραπλάσιον, page 10. 12 — and on the other by less frequently modifying the text in the manner of the vulgate and hence more nearly approaching the text of GmP. In illustration of the latter point such passages as pages 2. 2, 5, 7, 15, 3. 1 (first example), 18, 4. 12, 16, 20, and 5. 15 may be noted. Yet in many cases CμB are aligned with the vulgate against GmP; for example, pages 1. 10; 2. 1, 13; 3. 12; 4. 9, 19; 5. 8 and 7. 11. Sometimes these three disagree among themselves; B agrees with GmP somewhat more frequently than do the other two.

In the following table will be found the percentages of agreement with G in the selected readings given above, for each of the manuscripts:

m	94	B	51	e_4	32	p_{11}	28	p_5	18
P	91	C	46	e_3	31	S	27	p_9	18
p_3	75	μ	46	e_1	30	N	27	e_2	18
F	62	V	37	A	29	E	27	p_{10}	15
p_6	54	H	35	p_4	29	p_2	24	Γ	13
p_7	53	e_5	35	p_8	29	p_1	23	D	13

As far as these statistics are significant they confirm the observation that GmP and CμB form well-defined subfamilies and suggest that F, p_3, p_6, and p_7 form a class intermediate between GmP and the vulgate. These four, however, do not by any means show the kinship evidenced by GmP and CμB; in this sense they are not a family.

The relations between the many representatives of the vulgate are so complicated that is is hardly possible to consider them seriously on the basis of the available data. Their percentages of agreement with G give a general idea of their interest and value for textual study. The likeness of some of them to one another is quite noticeable. To show a few of these relationships the following percentages of agreement among certain pairs and groups may be observed. They are reckoned on the data furnished by 35 of the selected readings already cited:

SNΓD	62.7	ΓDE	65.6	SH, HA, AVE, HN	62.7
SND	65.6	ΓE	74.1	HV	68.4
SNΓ	68.4	ΓD	88.4	AE	71.2
SΓ	71.2			AV	82.7
SN	77.				
NΓD	79.8				
ND, NΓ	85.5				

Of the new manuscripts, J is in some respects the most interesting. It has not been possible to compare it with the others in the paragraphs above, because it does not contain the first three chapters of Book I. It does, however, agree with P in such a remarkable manner that only two alternatives are open; it must be copied either from the lost source of P (or a manuscript of exactly the same nature), or from the printed first edition. While the former alternative is alluring, the writer is inclined to think the latter the more probable. Its extremely late date and the fact that it is only a collection of excerpts are important points. Furthermore, P seldom differs from the other manuscripts importantly without J differing in the same way; in other words, J shares the peculiarities of P. The most notable instance is the omission by both of page 24. 2–4. Again, J disagrees with P in a fair number of instances, but never radically. The differences are slight omissions or transpositions which could easily be made in copying from the printed book, and variations of spelling or inflectional endings which might represent the efforts of a scribe, well versed in Greek, to correct the text of P, with its many blunders. There are in J practically no *additions* to the text of P, and such as occur may readily be explained in the manner just suggested.[1] Altogether it is easier to believe J copied from, or is descended from, the printed first edition, with a number of arbitrary changes due to the copyist, than an almost miraculously recovered representative of the manuscript text used by Wechel.

A few remarks on certain readings in Hoche's printed text are included at this point, and at the end of the chapter will be found an apparatus criticus for the first three chapters of Book I, based on Hoche's, but enlarged and revised so as to report for this limited portion of the text upon all the manuscripts known in any way to the writer.

P. 2, 13. — Read ταῦτα δ' ἂν εἴη τὰ ἄυλα (Hoche omits δ' with GmP). This may be one of the cases to which Hultsch refers[2] where something has been lost in G which the others have preserved. The word here supplied is found in the other manuscripts and removes the asyndeton.

[1] The only notable one is ἔστω, added by J in page 15. 11, where C alone of the other manuscripts (according to Hoche) agrees with J. This, however, is not a hard interpolation in a mathematical context, and the agreement with C may be only a coincidence, as there is little evidence of a special relationship between C and J.

[2] In his review of Hoche's edition, *Neue Jahrbücher für Philologie und Pädagogik*, vol. XCVII (1868), pp. 762–770, especially p. 763.

P. 5, 5. — Hultsch (*op. cit.*, p. 763) points out that the correct text is **αἱ δὲ πάντως πεπερασμένων εἰσὶν ἐπιστῆμαι**, since the reading of G (**αἱ δὲ αἱ ἐπιστῆμαι κτλ.**) shows clearly that **αἱ ἐπιστῆμαι** is a gloss.

P. 6, 17–18. — Read **καλῶς μοι δοκοῦντί τοι περί.** Hoche, with G (and H?) omits **τοί**; but the unanimous testimony of the other manuscripts makes it probable that a word has dropped out between **δοκοῦντι** and **περί**, for in all the others something is inserted, and **τοί** seems to be best attested. Even m and P oppose G here, and it is easier to suppose that a word dropped out of G only, and was preserved in the rest, than that all the others were interpolated.

P. 7, 1. — Read **μωσικᾶς** (CμBNΓ $p_{2,9}$, Nobbe), instead of Hoche's **μουσικᾶς** (**μουσικῶν**, P) GmSH. The two forms are easily confused, but **μωσικᾶς** is the correct dialectic form and it is improbable that it would be restored by emendation.

P. 9, 19 and p. 15, 1. — These are similar cases, both discussed by Hultsch (*loc. cit.*, p. 764). In both these passages G omits the article, and **ἄνθρωπος** and **καὶ ἀρτιοπέριττον**, respectively, are to be read.

P. 16, 5. — Read **ἑξῆς πάντας**, which is found in G_2SHA (although H in the margin has the other reading, **ἐξ ἑνός**, which does not necessarily occur in all the other manuscripts, for Hoche's apparatus is not exhaustive). **ἐξ ἑνός** can mean only 'from unity,' but Nicomachus regularly says **ἀπὸ μονάδος** in this sense, as in the sentence immediately following. A corruption must have come into the family GmP.

P. 19, 10. — Read **εἰδικῶς** for **ἰδικῶς**. This is simply a matter of spelling; the adverb is to be connected with **εἶδος** (not **ἴδιος**), but in late Greek it is sometimes spelled **ἰδικῶς**. In p. 55, 20 GmP themselves use the better spelling **εἰδικῶς** and to avoid confusion it is better to use that form here also.

P. 19, 8. — Omit, as a gloss, **μόνον μέσου . . . πολλαπλασιαζομένου,** for the reasons stated by Hultsch (*loc. cit.*, p. 763).

P. 20, 1. — Nicomachus is stating that in the class of even-times odd numbers no factor can have a value of the same kind (even or odd) with its name, and to avoid a meaningless text it is necessary to read **τῷ ἑαυτοῦ ὀνόματι** with C, or to suppose that the original had **αὑτοῦ** (cf. the reading of S, **αὐτοῦ**), of which the unintelligible **τῷ αὐτῷ ὀνόματι**, which appears in the others, is a corruption easily made.

P. 20, 4. — We may accept Hultsch's arguments for the doubly reduplicated form **ὠνοματοπεποιημένον** of G (*loc. cit.*, p. 764).

P. 23, 20. — **αὑτῶν**, not **αὐτῶν**, is doubtless the proper reading. It improves the Greek, and may easily have suffered corruption in GmP. CSHA have **κατὰ τὴν ἑαυτῶν τάξιν** here, with order changed, but probably preserving the proper form of the pronoun.

P. 24, 5. — Hultsch (*loc. cit.*, p. 765) suggests that **ἀπογενήσονται**, with GP, be read here instead of **ἀπογεννήσονται**, on the ground that **ἀπογίνεσθαι** should be the passive of **ἀπογεννᾶν** in mathematical language, just as **γίνεσθαι** is the passive of **γεννᾶν** in common speech. Nicomachus, however, is quite free to use the passive of **γεννᾶν**.

P. 25, 9. — In favor of retaining Hoche's **πολυπλασιασμούς**, against **πολυπλασιασμοῦ** PCSH, it can be urged that the word means 'result of multiplication' in p. 133, 7, although Theon of Smyrna, who uses it frequently, seems always to employ it to designate the operation of multiplication itself (e.g., pp. 23, 18; 26, 5; 27, 7, 14; 28, 13; 29, 18, etc., Hiller).

P. 27, 6. — Read **δύναιντο** (omitting **ἄν**) with GP. This is Hultsch's suggestion, who calls attention to the use of the same unusual construction in p. 66, 22, where the manuscripts all agree (*loc. cit.*, p. 766).

P. 31, 5. — Hultsch (*loc. cit.*, p. 765) would omit **ἄν** with GPJ and read **ἐθέλομεν** with P. Hoche's text, however, should probably be retained, since Nicomachus's usage in such constructions, where the subjunctive occurs, seems uniformly to employ **ἄν**. Only in GPJ here, in S in II. 2. 2, and in SH in II. 11. 2, is the word omitted, of the instances collected on p. 173.

P. 33, 19. — Read **ἑαυτοῦ** for **ἑαυτῶν**. This is an excellent opportunity to restore an evidently correct text on the best authority, G_1. The only possible meaning of what Hoche prints is "those that are measured by unity alone in accordance with their own quantity," that is, prime numbers, but this is not the correct sense. Of such numbers disposition has already been made; here he speaks of "those that are measured by one (measure) alone in accordance with its own quantity," that is, the squares of odd prime numbers. The comma after **ποσότητα** (line 19, Hoche) should be deleted.

P. 35, 5. — JPCSH here have kept the better text, **ἀποφαίνου** for **ἀποφαίνουσι**. Cf. p. 36, 3.

P. 35, 10. — Read **προβληθῇ** with PCSH for the syntactical reasons discussed elsewhere (p. 175).

P. 35, 20. — We may agree with Hultsch (*loc. cit.*, p. 766) that **καταλείπεται** is corrupt, **κατα-** being a dittography from the numeral κα; but λείπεται fits better than his favored suggestion, λοιπὰ τά.

P. 42, 1. — Omit τίς ἀριθμός ἐστι with Hultsch (*loc. cit.*, p. 764).

P. 42, 2. — Hultsch's restoration of προκατηχήθη from G_1's προκατήχθη is worthy of adoption (*loc. cit.*, p. 766).

P. 44, 21. — ἡ τῆς ἰσότητος is unnecessary, and, as the variation in the manuscripts shows, is probably a gloss.

P. 54, 17. — Read ἀνίσοις (G_2CS) for ἄνισοι. The expression 'unequal heteromeces' is meaningless, but ἀνίσοις, referring to πλευραῖς, yields good sense. (I have considered the possibility that ἀνισάκις has here been lost from all the manuscripts. The words ἀνισάκις ἴσοι would correctly characterize the 'heteromeces,' and would balance ἰσάκις ἴσοι above; it would also explain the retention of the senseless ἄνισοι in G_1. The change adopted, however, is less violent, and though II. 17. 6 offers similar expressions, Nicomachus nowhere uses ἀνισάκις ἄνισος.)

P. 120, 6. — Perhaps ποιά should be inserted in this line; in the explanatory note on the passage I have stated reasons for thinking that the word may have fallen out.

P. 125, 10–13. — Hoche reads ἴδιον δὲ ὑπάρχει τῆσδε τῆς μεσότητος . . . τὸ κατὰ σύνθεσιν τῶν ἄκρων ὑποδιπλάσιον ἢ ἶσον τὸ μέσον εἶναι ("it is the peculiarity of this proportion . . . that the sum of the mean terms is half or the mean term is equal"), which certainly is corrupt, for the sum of the extremes is never half the mean term. Ast, without using C, emended, reading διπλάσιον τοῦ μέσου ἢ ἴσον τοῖς μέσοις, which is exactly the text of C. This gives good sense, for the sum of the extremes is "double the mean term or equal to the mean terms" (if there are two), and it must be substantially what Nicomachus really said. Unfortunately ὑποδιπλάσιον (ὑποδιπλάσιος, G) is attested by all the manuscripts but C, and we can hardly reject it. The simplest remedy is to read τοῦ κατὰ σύνθεσιν τῶν ἄκρων ὑποδιπλάσιον ἢ ἴσον τὸ μέσον εἶναι "the mean is half of the sum of the extremes, or equal to it," interpreting 'the mean' to refer both to the single mean in the continuous proportion and to the sum of the two means in the disjunct. I am not sure that this is satisfactory, and suspect that the original text was more explicit; but for the present this suggestion is the best that I can make.

P. 110, 5. — Omit τοιοῦτοι, which has the support only of CSH, and can be dispensed with.

P. 137, 22. — ἄρα, found in GP, is preferable to ἆρα.

A Critical Apparatus Covering Chapters 1–3 of Book I of the *Introduction to Arithmetic*

PAGE 1 (GmPCμBSHNΓAVDEFLOe$_{1-5}$ p$_{1-11}$)[1]

Line 1 *Γερασινοῦ* PD

2 *Πυθαγορικοῦ* GmHVOe$_{1-2,4}$ p$_{2-5}$, *Πυθαγορείου* NΓFe$_5$, *Πυθαγορίου* D, *Πυθαγωρίου* E, om. PCμBJLp$_{1-7}$

3 *εἰσαγωγῆς ἀριθμητικῆς* V p$_{3-5}$, *εἰσαγωγὴ ἀριθμητικῆς* E, *εἰσαγωγῆς* om. PN Boethius, *εἰσαγωγὴ τ/// ἀριθμητικῆς* G$_1$, *τ* in *ς* mut. G$_2$, *εἰσαγωγὴ τῆς ἀριθμητικῆς* m, *ἀριθμητικῆς εἰσαγωγή* S, *τοῦ Γερασηνοῦ ἀριθμητικῆς εἰσαγωγῆς πρῶτον βιβλίον* μ

4 *ὧν εἰς δύο* D, *τῶν εἰς τὰ δύο* mHEp$_1$, *πρῶτον εἰς τὰ δύο* F, *τῆς εἰς δύο τὸ πρῶτον* L, *εἰς //// δύο* G

5 *πρώτως* p$_1$

6 *προκατάρξαντος* Ee$_{1-3}$ p$_{1-2,6,8}$ — *ὡρίσαντο* H, *ὁρίζοντο* e$_5$ — *τὴν φιλοσοφίαν* Le$_5$ (*τὴν* supra lin. e$_5$)

7 *φιλίαν φιλοσοφίας* e$_1$

8 *πάντων*] *μόνον* e$_5$ p$_4$

9 *συνκεχυμένῳ* Γ — *καί* om. CL

10 *ἁπλῶς*] *πᾶς* add. CμBSHNΓADELOVe$_{1-5}$ p$_{1-2,4-11}$ (p$_8$ supra lin.) — *καὶ σκυτοτόμος* om. p$_2$

11 *ἢ ἔμπειρος ἢ δημιουργίας* B, *ἢ δημιουργίας ἔμπειρος* A

PAGE 2 (GmPCμBSHNΓAVDEFOe$_{1-5}$ p$_{1-2,4-11}$ p$_3$ [ll. 4–9, 13–17])

Line 1 *συστήλας* A — *πάντων* om. CμBSNΓADEFOe$_{1-5}$ p$_{1-2,4-6,8-11}$

2 *κατάληψιν*] *μετάληψιν* A — *τούτῳ*] *πάντων* add. SHNΓADEFOe$_{1-5}$ p$_{1-2,4-6,8-11}$

3 *γνῶσιν* supra lin. p$_1$

4 *καί* post *εἰκότως* om. SHNΓDOe$_{2,5}$ p$_{1-2,4-6,9-10}$

5 *προσηγόρευσεν*] *προσηγόρασεν* E, *ἐκάλεσεν* NΓDe$_5$ — *σοφίας ὄρεξιν*] *σοφίας ἔφεσιν* SHNΓAVDEOe$_{1-4}$ p$_{1-2,4-5,8-11}$ (*ὄρεξιν* Iamblichus p. 5, *ἔφεσιν* *ibid.* p. 10)

6 *ἐστί* supra lin. p$_1$ — *ἄλλως*] *ἄλλων* p$_3$ — *παρόσον* codd. — *συνεσταλμένον* add. Γp$_{9-10}$ — *ὅσον*] *ὅσων* e$_2$

7 *αὔτην* HNΓADEOVe$_{1,3-5}$ p$_{1-2,4-5,8,10-11}$

8 *ὁρίζετο* A — *εἶναι* post *ἐπιστήμην* add. Ae$_{2-4}$ p$_{10-11}$ — *τῆς*] *τοῖς* e$_3$

9 *ἀληθείας* om. e$_1$

10 *ἄπταιστον κατάληψιν* p$_2$

11 *τὰ κατὰ*] *τά* om. e$_3$, *κατὰ αὐτά* Γ, *κατὰ ταῦτα* p$_{10}$ — *καὶ ὡσαύτως καὶ ὡς*

ου

αὐτῶς e$_2$ — *διατελεῖντα* G

12 *ἐξιστάμενα*] *ἀφιστάμενα* SNΓDOe$_5$ p$_{1-2,4-5,8-10}$

[1] The manuscripts reported upon for a given page are thus noted. Page and line references are to Hoche's edition.

13 ταῦτ᾽ ἄν Pm, ταῦτα δ᾽ ἄν CμBSHNΓADEOVe_{1-5} $p_{1-2, 4-5, 7-11}$ — εἴη] ᾖ Hp_{4} — καὶ ὧν] καὶ (τά AH) ἀίδια, ὧν SHNΓADEOVe_{1-5} $p_{1-5, 8-11}$, καὶ ἀίδια καὶ ὧν CμB$p_{7, 9}$ — καί om. F, cf. Iamblichus p. 5 — ὧν] τῶν p_{3}

14 λοιπὸν ἕκαστον NΓDp_{9}, λοιπῶν p_{7} — ὁμωνύμως] ὁμωνύμων m B, ὁμωνύμως οὕτω καλουμένων HBp_{4} — καὶ om. PCμG_{2}ΓAEOV$e_{1-2, 4}$ $p_{1-2, 4-5, 7-9, 11}$

15 καί post τι add. O$e_{5}$$p_{1-2, 4, 8}$ — εἶναι post λέγεται add. G_{2}HADEFOVe_{1-5}, $p_{1-5, 7-11}$

16 δήπου καὶ ὑλικά in marg. e_{5} — καί post ῥύσει om. p_{5}

17 ἐστι διὰ παντός AV$e_{2, 4}$, p_{11} — ἐστι] εἰσι Γ — τήν eras. p_{2}

18 ἀρχῆς supra lin. p_{8} — ἰδιότητα] ἀιδιότητα e_{4}

19 ἦν supra lin. e_{5} — ἀλλοιοτή B, ἀληωτή A — περὶ τὰ ὑλικὰ καὶ σωματικά add. SHAFVOe_{1-3} $p_{1-2, 4-5, 7, 10-11}$

20 περὶ] πε p_{4} — αὐτὴν] αὐτά FVe_{1-3} $p_{7, 11}$ — καί om. NΓADEVe_{5} — αὐτῇ] αὐτοῖς AFe_{1-3} $p_{7, 11}$ (-οις supra lin. p_{8})

21 ποσότητες ποιότητες V$e_{1, 3}$ p_{11} — μέγεθοι p_{7} — σχηματισμοὶ . . . ἰσότητες om. p_{4} (add. supra lin. manus secunda)

PAGE 3 (GmPCμB SNΓAVEDFe_{1} e_{2} [to line 3] e_{3} [to line 13] e_{4-5} p_{1-2} p_{4} [to line 17] p_{5-11})

Line 1 σμικρότητες BΓDEO$e_{2, 5}$ $p_{2, 5-10}$, Io. Philoponus ι — ἰσότητες] in marg. e_{3}. ἀνισότητες add. HNΓADEFOV$e_{1, 3, 5}$ $p_{1-2, 5-9, 11}$, Io. Philoponus ι

2 ἁπλῶς] εἰπεῖν add. D — περιέχονται E

3 σώματι] σώματα ΓDEe_{2} $p_{7, 9-11}$, ἅ add. AEV$e_{1, 3-4}$ $p_{4, 7, 11}$ — καθ᾽ αὑτά CAD, δή add. e_{4}

4, 7 συμβεβηκότως G_{2} bis corr. ex συμβεβηκότα; συμβεβηκότων B — παραπολάβει e_{5}

7 συμβεβηκότως δὲ] μετέχει καὶ τῶν μετεχόντων κτλ. p_{2}, συμβεβηκότος e_{5} — ὅ] ὅσα PBp_{6}

7–8 om. E, add. supra lin.; om. p_{8}, add. in marg.

8 σώματα p_{7}

9 μὲν] καί add. NΓDe_{4} $p_{2, 4-6}$ — ἄυλα ἀίδια C

10 ἀπαράλακτα A$p_{1, 10}$

11 ἑαυτῶν CμSΓD$p_{1-2, 4, 7, 9}$ — οὐσίᾳ] -αν supra lin. e_{5} (manus secunda) — διαμένοντα CμSHNΓDEe_{5} $p_{2, 4-5, 9-10}$

12 αὐτῶν] αὐτοῦ εἴδους τούτων μ — τὰ] ταῦτα CμBSHNΓADE $e_{1, 3-5}$ $p_{1-2, 4-8, 10-11}$

13 τε om. HNΓAV

14 φένεται A

15 μὲν] G_{2} corr. ex μὴ — καθόσον codd.

16 τῆς ἑαυτῶν φύσεως P — ἑαυτῶν] αὐτῶν μS$e_{4}$$p_{10}$ — ὄντως] οὕτως e_{1}

17 ἐπὶ τοῦ αὐτοῦ CμSp_{7} — διαμένῃ m

18 μεταβαίνει] μεταβάλλει BV, μεταρρεῖ SHΓDEe_{4-5} $p_{1-2, 5-6, 8-10}$, μεταρρεῖ καὶ μεταβαίνει e_{5} Ast — παντοίως G_{2} corr. ex παντοία — ἀλλασόμενα Ap_{11}, ἀλλασσόμενον S — καὶ κατά A p_{11}

19 Πλατωνα S — ὂν] μὲν add. CμBSHNΓADEVe_{1} $p_{1-2, 5-6, 8-11}$

PAGE 4 (GmPCμBSHNΓAVDEFe_1 [to line 12] $e_{4\text{-}5}$ p_1 [to line 3] $p_{2,5,7\text{-}11}$ p_6 [to line 11])

Line 1 οὐκ] μή SNΓ, οὐκ in marg. Γ — γενόμενον m — μέν om. Be_4, ἀεί add. p_1

2 δέ om. p_7

3 αἰεί Fp_8 — καί om. SDEVp_1 — κατὰ] μετά B — τ' αὐτά P, ταῦτα D — δοξητή e_1

4 γιγνόμενον Gm — τε om. HNΓDE, supra lin. e_5 (manus secunda)

5 ὄντως] ὄν e_5 (corr. supra lin. man. secunda) — ὄν] ἔλεγε ταῦτα ὁ Τίμαιος παρὰ Πλάτωνι add. μ

7 ἐφιέμεθα τέλους B — αὐτή μ

8 ὑφ' om. SAV$e_{1,5}$ $p_{10\text{-}11}$, add. supra lin. e_5 (manus secunda)

9 ἡμῖν] ημ̂///G, om. H, ἐστιν CμBSNΓD $p_{9\text{-}10}$ — ὡς ἔφην] ὤφθη A p_{11}, ὡς φι e_1 — σοφίας] ἐστίν add. e_5

10 ὀρέξεις] ἔφεσις H, Γ in marg., $p_{2,6}$ — τῆς] τοῖς A $p_{7,11}$ (της in marg. p_{11})

11 ὄντα δὲ] δέ om. p_7

12 διελεῖν] διεξελθεῖν S (διελεῖν in marg.) AFVe_1 $p_{7,10\text{-}11}$, διελθεῖν ΓEe_4 (εξ in marg. e_4) e_5 $p_{2,5,8\text{-}9}$, τε add. Γ, τε καί add. $p_{6,9}$ — ἀρθρῶσαι P — τὰ om. p_7 — συμβεβηκότως m

13 ὧν τοίνυν Γ — καὶ καθ' Γ — τῶν τε κυρίως om. p_5

14 ὅπερ ἐστὶ] οὐκ ὄντως κυρίως καὶ ὄντων κυρίως add. e_4

15 ἀλληλοχούμενα A

16 δένδρον κόσμος SHNΓADEV$e_{4\text{-}5}$ $p_{5,8\text{-}11}$ — καὶ om. Se_4 — ἅπερ] ἃ Γ p_9 — ἰδίως καί κυρίως B

17 τε om. H

18 καλεῖται] καί add. Gm

19 σωρός] om. p_2, σορός A p_{11}, χορός, σωρός CμBSHNΓADVe_4 $p_{5,10\text{-}11}$, Boethius, στρατός add. S p_{10} — καὶ om. e_4, supra lin. e_5

20 δύο ἄρα C — τούτων εἰδῶν SNΓDEe_5 $p_{5,8\text{-}10}$, τούτων δύο εἰδῶν e_4 p_2

21 νομιστέον] νοητέον SNΓe_5 $p_{7,10}$ — ἐπειδή A p_{11} — καὶ πᾶν μέγεθος . . . 5, 1 πλῆθος om. D

22 ἄπειρον Γ — αὐτῶν G_1P, ἑαυτῶν μSNΓEVe_5 $p_{9\text{-}10}$

PAGE 5 (GmPCμBSHNΓAVDEFe_4 e_5 [through line 8] $p_{2,5,7\text{-}11}$)

Line 1 εἰσι BC — γάρ om. C — ὁρισμένης A $p_{5,11}$

2 παύσεται E p_8 — προκόπτων A

3 ὁρισμένης $p_{5,11}$

4 διὰ ταύτης CμSAEV$e_{4,5}$ $p_{7,10\text{-}11}$, δι' αὐτῆς NΓD $p_{2,5}$, διὰ αὐτῆς p_9 προχωρεῖ] χωρεῖ V, προχωρεῖν p_7, προχθορεῖ p_{10}

5 αἱ δὲ αἱ ἐπιστῆμαι G, αἱ δὲ ἐπιστῆμαι ἀπείρων δὲ οὐδέποτε φαίνεται, ἀλλὰ πάντως τῶν πεπερασμένων εἰσίν· δῆλον δὲ ὅτι μ — περὶ πεπερασμένων p_8 — εἰσὶν] ἐστίν P

6 φαίνετε corr. in φαίνεται A, φαίνονται e_4 — δὴ om. p_2

7 οὔτε ad init.] οὐ V p_{11}, — τε add. supra lin. A

8 ποτε om. PSNΓA (supra lin.) D p_{10} — ἐπιστήμη ποτέ CμBe_4 — ἑκάτερον] καί add. mPCμS, erasum in G

9 καθ' ἑαυτὸ ἑκάτερον Γ — ἐστιν A ἔσται p_2 — πλέον HNΓDF $p_{9,11}$

10 ἀπ'] ἐπ' μ $p_{2,11}$, om. B — ἀφωρισμένον] ὡρισμένον E, ἀφορισμένον p_5
11 μὲν] γάρ add. S $p_{7,10}$ — πόσον G_2
12 πηλίκον] ποικίλον p_{11} (πηλίκον in marg.)
14 ὁρᾶτε A — αὐτό PDA p_2
15 οἷον] τετράγωνον add. SHNΓADEFVe_4 $p_{2,5,8-11}$, Boethius — περισσόν NΓ — καὶ τά G_2 — ἐοικώτως m
16 ἔχον ἤδη CB
17 ἥμισυ μεῖζον ἔλαττον CBNΓD p_9, ἥμισυ ἔλαττον e_4 — ἔλασσον H, ἔλλατον A
18 ὕμισυ A — καὶ τά G_2 — δηλονότι codd.
19 ἄρα om. CμBSHNΓADEV $p_{7-8,11}$ — διαλήψονται CμG_2SHNΓADEFV e_4 $p_{2,5,8-11}$ — ἐπιστημονικαὶ μέθοδοι

PAGE 6 (GmPCμBSHNΓAVDEFe_4 p_2 p_5 [through line 12] p_7 [through line 8] p_{8-9} p_{10} [through line 5] p_{11} [through line 17])

Line 1 διευκρινήσασι D — τοῦ om. p_7 — πόσου G_2
2 αὐτό CNΓADEV $p_{2,11}$, αὗτό p_9 — τό post δέ om. Γ
3 ἐπεὶ] ἐπί PS
4 μονῇ] ἐπιμονῇ Ae_4 p_{11}
5 περιφορᾷ] φορᾷ A $p_{2,5,11}$ — τὰ αὐτὰ] τά om. PV, ταυτά BS p_5, ταῦτα A p_{10-11} — διακριβώσουσι HEV, διακρίβωσι A
6 οὖν post μέν add. EV — ἠρεμοῦν PH, ὀρεμοῦν G_1
7 περιφερόμενον AEV p_8
8 ὄντως B
9 δυνατὸν ἀκριβῶσαι PCμBE, ἀκριβῶσαι δυνατόν AV $p_{8,11}$ — ἀκριβώσασθαι H
10 ἡ σοφία HAE $p_{8,11}$
11–15 ζωγραφία . . . θεωρίας . . . συνεργίαν S
12 βαναύσαις p_{11} (corr. supra lin. man. sec.) — πρὸς] εἰς E $p_{8,11}$ — θεορίης A, θεωρίας e_4 — τοι om. m, αἱ F, ται A, καί add. H, τε p_{11}
14 λόγον PA
15 συνεργείην B
16 ἀλλά om. A — καὶ] δή E
17 τὸ αὐτό om. H — πως om. $p_{2,9}$
18 δοκοῦντι] τό add. PSB, τοι mCμNΓADEFV $p_{2,8-9}$, cf. Io. Philoponus κα, τόν e_4 — μαθηματικά P
19 ὀρθῶς om. Cμ, in marg. e_4, ὀρθῶς αὐτοὺς ἔντι περί V, ὀρθῶς αὐτούς E — οἷα] οι supra lin. add. A — ἑκάστω H — φρονεῖν A e_4 p_8
20 τᾶς Gm, τάς P, τῆς cett. — ὅλων] ἄλλων m, ////λλων G, ὄντων D (om. τῶν) — φήσιος B
22 τε] τι G_1P, τοι E p_8 — γεωμετρίας G_2CμHNΓAe_4 — τὰς γεωμετρίας καὶ μουσικᾶς καὶ ἀριθμητικᾶς S — καί om. F
23 καὶ σφαιρικᾶς om. G_1P — παρέδοσαν p_2 — ἄμμι H, ἄμμην B, ἡμῖν AVe_4 p_8 (αμ supra lin. A, manus secunda) — σαφᾶ A

PAGE 7 (GmPCμBSHNΓAVD [to line 20] E [to line 21] F [to line 15]$e_4$$p_{2,8-9}$)

Line 1 καί om. Fp_2 — μουσικῶν P, μωσικᾶς CμBNΓ$p_{2,9}$, μωσικῆς A — ταῦτα δέ P
2 δοκοῦντ' εἶμεν A (ναι supra lin. man. 2) p_8 — περὶ γὰρ ἀδελφεά om. G_1

3 τὰν] καί e_4
4 τρεῖς καὶ δεκάτου H, τρισκεδεκάτου A, βιβλίου add. Γ
5 ὄνπερ CNΓADEVe_4 $p_{2,9}$
6 περισκοπεῖ] om. H, σκοπεῖ SNΓADEFVe_4 $p_{2,8-9}$ — καὶ διορίζεται om. PCμB, διωρίζεται A
7 χρὴ] δεῖ PCμBH, εἶναι post χρὴ add. ADEVp_{8-9} — ὄντως om. PCB
8 τά G_2 ex το//// (τοὺς?) — προλεχθέντα καὶ βεβαιωθέντα e_4, προλεχθέντα καὶ διαβεβαιωθέντα SHNΓ, προλεχθέντα καὶ προβεβαιωθέντα AVEp_8 (ὑπὸ πλειόνων προλεχθέντα καὶ add. man. 2 supra lin. A), διαλεχθέντα μ, προβεβαιωθέντα C, πρὸς διαλεχθέντα καὶ διαβεβαιωθέντα P, προλεχθέντα (om. καὶ προδιαβεβαιωθέντα) D$p_{2,9}$
11 τὴν om. A — ἀναλογίαν] ὁμολογίαν CμBSHNΓADEV$e_4$$p_{2,8-9}$, Io. Philoponus c. κε — μίαν om. H, εἶναι μίαν καὶ ἀναφανῆναι μ
12 κατὰ τὸν τρόπον τοῦτον A — τρόπον] τὸ προσῆκον p_8
13 ἂν om. De_4 $p_{2,9}$ — ὁ λεγόμενος μ — ὀρθῶς] ἔχον add. A, ἔχειν add. e_4 p_8 — τὰ πάντα SHAEVp_8, ἅπαντα NΓD
14 μανθάνῃ NΓ, μανθάνοι Dp_9 — τούτων ἁπάντων SHNADEVp_9, τὸ τούτ. ἀπ Γ
15 ἄλλος mCA — ἐπιχειριεῖται p_2 — τὴν φιλοσοφίαν NΓD$p_{2,9}$, σοφίαν F
17 ἀλλ' om. V — ταῦτα /////τὰ G_1, ταῦτα γὰρ τά m
19 πάντα ταῦτα AV p_9
20 τοῦτον δή SNΓD p_9 — τὸν σοφώτατον HAEVe_4
21 τε] ἅμα S — δῆλον δὲ ὅτι καί VE
22 γὰρ] δέ p_8 — ὅτι] ὡς H — τισιν ἤ SND, τισιν ἢ καὶ Γ — ἐοίκασι NΓDp_9 — ταῦτα τὰ] τὰ τοιαῦτα p_2

PAGE 8 (GmPCμBSHNΓAe_4 $p_{2,8-9}$)

Line 1 ἡμῶν τὴν διάνοιαν S
3 συντρόφων] συνήθων p_2 — ἡμῖν] ἡμῶν e_4 — βρεφῶν] ἔτι add. SNΓAe_4 $p_{2,8-9}$
4 ὄντων ἔτι H, ὄντων om. A
5 τε] om. NΓ, μὲν CμSA$p_{2,8-9}$
7 πρῶτον P — τῶν . . . νοητικῶν P, τῷ . . . νοητῷ NΓ, τῷ ἐν (αυ supra lin.) ταῖς νοητικῷ A, διανοητικῷ B — τῷ] τό e_4
10 τοῖς μαθήμασι δοκοῦντος Np_2
12 διανομὰς καὶ λογισμοὺς NΓp_9
13 δὲ om. p_8 — στρατοπ ///δευσ G (ss) — καὶ add. CμS
14 γεωμοιρίας A — συγκήσεις P, συγκίσεις SNΓ
16 δὲ] ἢ add. p_9 — καὶ ἀστρονομία om. S
17 γεωρ///γίας G, -αν SNΓp_{8-9} — ναυτιλίας Cp_2
18 εὐχερείας Gm, εὐχρείας P (in marg., γρ. εὐκαιρίας), εὐκαιρίας cett. — ἐπιτηδειότητος P — προδηλοῦσαι G_2, -ούσας SNΓ
19 εἴ] ὦ λῶστε add. CB, λῶστε μ, ὦ ἑταῖρε p_9 — ἐπιπλήττων om. e_4 (πλήττων supra lin.) — ἔοικε δεδίεται P
20 ἄχρηστον P — ταῦτα om. μ — τὰ μαθήματα ταῦτα NΓ$p_{2,9}$
22 τῆς om. C

CHAPTER XII

THE LANGUAGE AND STYLE OF NICOMACHUS

An elementary treatise on arithmetic is hardly the vehicle best fitted to exhibit the talents of a stylist. The requisites are the simple, natural excellences of clarity, accuracy, and purity of diction; rhetorical ornaments are out of place. Such a style is what one would expect, and what, in general, one finds, in the *Arithmetical Introduction*. Nicomachus maintains throughout a tone of earnestness, which he does not lose in attempted 'fine writing' even in those portions, like the first six chapters and a few others, where he is not for the moment engaged in actual mathematical demonstration. Even here there are few rhetorical figures, although these are not entirely lacking even in the strictly mathematical sections. The rhetorical questions [1] are few but effective; the similes [2] not far-fetched and usually of real value for the exposition of the subject in hand.

That Nicomachus was no mere amateur writer can be easily seen when the structure of his sentences and paragraphs is taken into account. Clarity is usually present, and on the whole there are few obscure passages. This is undoubtedly due to the fact that the author had a clear idea of what he wanted to say and a decided knack for logical exposition, so that his materials meet the eye in the order in which they are most easily understood. And it must also be allowed that he knew how to turn a good period; [3] for even in the most technical sections the periodic structure is frequently found, joined, to be sure, to sentences more loosely put together, since the subject is not one that easily is adapted to the periodic form.

Join to what has already been said the observation that Nicomachus throughout preserves a decided enthusiasm for his subject,[4] and that he constantly puts himself, by little touches, sometimes attractively naïve, into a personal relation with his readers, but without the sacri-

[1] I. 4. 1.
[2] I. 14. 3; 15. 1; 16. 1; 23. 4, 6; II. 27. 1.
[3] Cf. I. 6. 1; II. 21. 1, etc.; for a technical section, see II. 19. 1.
[4] Note the zest in I. 23. 4; 19. 20.

fice of dignity,[1] and one has a fair idea of his merits as a writer. The faults into which he could easily be betrayed — dryness and excessive pedanticism — he has thus in large measure avoided. We may not claim for him superlative excellence, nor assert that his book has the liveliness or interest of a novel or a history, for the extended consideration of such a subject as arithmetic is apt to suffer from moments of aridity; but it is just, I believe, to assert that Nicomachus, as a scientific writer, possessed assets which not all could claim, and that the *Arithmetical Introduction* on its own merits will engage the reader's interest to a degree that is decidedly creditable for a book of its class.

To the student of antiquity, however, Nicomachus's language will be a matter of greater interest than his style. Contemporary with, or only a generation later than, the latest books of the New Testament, his writings may be expected to manifest certain of the non-classical peculiarities that are observed both in the New Testament, in the papyri, and in Greco-Roman writers generally. It is most just to compare Nicomachus with such writers as Lucian, for, like him, Nicomachus adopted the literary, not the contemporary spoken, style; that is, both are Atticists. But it must be recognized that his subject strongly influences Nicomachus's language. He does not, like Lucian, employ the dialogue, but confines himself to sober exposition, so that his syntax of necessity shows less variety; there is, for example, little occasion for *oratio obliqua*. Still no late writer succeeded in suppressing entirely the marks of Hellenistic idiom, and there are many traces in the *Introduction*, which we shall proceed to review.

One of the surest tests of Hellenistic Greek is its tendency to use μή instead of οὐ.[2] Μή came to be used with participles indiscriminately,[3] and in the New Testament the older usage is so far abandoned that οὐ is quite regularly found in protases with the indicative.[4] This last usage we cannot attribute to Nicomachus; he always uses μή in protases and in conditional relative clauses, but with participles he

[1] E.g., II. 14. 5; 28. 1, 6.

[2] On this in general, cf. Moulton, *Prolegomena to a Grammar of New Testament Greek* (Edinburgh, 1908), Index, s.v. *Negatives;* Burton, *Moods and Tenses of New Testament Greek* (Chicago, 1903), pp. 178 ff.; Gildersleeve's fundamental article in *American Journal of Philology*, I (1880); and E. L. Green, in *Studies in Honor of B. L. Gildersleeve*, pp. 471 ff.

[3] Burton, sec. 485: "In the New Testament participles in all relations usually take μή as the negative." Cf. Gildersleeve, *American Journal of Philology*, I, 55. Examples are given by W. Schmid, *Der Atticismus* (Stuttgart, 1887), from Dio Chrysostom (vol. I, p. 100), Lucian (I, p. 246), Aristides (II, p. 61), etc.

[4] Burton, 469; Moulton, p. 171.

sometimes follows, sometimes abandons, the syntactical standards of classical Greek. By far the greatest number of times when μή is found with the participle it is correctly used;[1] in several more it will probably pass muster;[2] but in a few cases it is undoubtedly incorrect on classical standards. Such are I. 3. 1: ἐπεὶ τοῦ ποσοῦ τὸ μὲν ὁρᾶται καθ' ἑαυτό, μηδεμίαν πρὸς ἄλλο σχέσιν ἔχον κτλ.; I. 23. 8: οἷον νόμους . . . οἷς πᾶσα ἡ προλεχθεῖσα πρόβασις . . . εὐοδώσει μὴ λειποτακτουμένη;[3] II. 22. 1: αἱ δὲ ταύταις ὑπεναντίαι ἄλλαι τρεῖς, ἰδίων μὴ τετευχυῖαι ὀνομάτων, κτλ.

Sometimes μή is joined in this way to a genitive absolute, as in I. 16. 4: παραπλησίως πάντας . . . ἀπογεννήσεις . . . μηδενὸς παραλειπομένου; I. 23. 8: γένοιτο γὰρ . . . τὰ τοῦ πολλαπλασίου ἅπαντα εἴδη . . . σοῦ μηδὲν ἐπιτηδεύοντος μηδὲ συλλαμβάνοντος; II. 22. 3: μηδενὸς παραλειπομένου μηδ' ὑπεξαιρουμένου; cf. also I. 11. 1. The use of μή in I. 7. 2, ἔστι δὲ ἄρτιον μέν, ἃ οἷόν τε εἰς δύο ἴσα διαιρεθῆναι μονάδος μέσον μὴ παρεμπιπτούσης, may be regarded as justified in view of the generalizing and characterizing nature of the clause in which it stands, and there are many instances where Nicomachus negatives a participle with μή because it is equivalent to such a clause. But to sum up, there are only seven instances of the loose use of μή in the *Introduction*, over against about twenty-four where the participle is correctly used, and about eleven where οὐ is correctly used with the participle.[4] With other constructions, too — the infinitive,[5] ὥστε,[6] εἰ and ἐάν[7] clauses — his usage ordinarily agrees with the classic standard. But I note τὸ μὲν γὰρ πρώτιστον αὐτῶν . . . συμβαίνει τοὺς μὲν ὑπολόγους ἔχειν τοὺς ἀρτίους, ἄλλον δὲ οὐδαμῶς οὐδένα κτλ. (I. 19. 2), where μηδαμῶς would be naturally called for; and a similar instance where οὐδέποτε, not μηδέποτε, is used with an accusative

[1] I. 4. 2 *bis*, 4 *ter*; 6. 3; 7. 2, 3; 10. 2, 5 *bis*; 13. 1, 8 *bis*; 16. 4; 23. 8, 11, 12; II. 11. 1; 16. 1; 22. 3; 27. 1.

[2] A peculiar case in II. 6. 1: τὰ γὰρ . . . ὑπόλοιπα προσπληρωθήσεται διαλιπόντων . . . ἡμῶν καὶ προτεχνολογησάντων ἕτερά τινα προὐργιαιτέραν τὴν σκέψιν ἔχοντα . . . τῷ καθ' αὑτὸ ποσῷ καὶ μὴ τῷ πρὸς ἕτερόν πως ἔχοντι. The participle ἔχοντι can doubtless be regarded as characterizing. In I. 10. 2; 13. 1 (cited in preceding note), the participles are in ὅταν clauses.

[3] Cf. I. 16. 4: γένεσις δὲ αὐτῶν γλαφυρά τε καὶ ἀσφαλὴς οὔτε παραλείπουσά τινα τῶν τελείων, κτλ., a very similar sentence where οὐ appears.

[4] I. 16. 4; 19. 16; II. 6. 1 *bis*; 22. 3 *bis*; 28. 1, 6; in the genitive absolute, add I. 19. 8; II. 9. 3; 17. 2. It may be added that adjectives that are equivalent to conditional relatives are correctly negatived with μή by Nicomachus: e.g., I. 16. 4; II. 3. 2 *bis*.

[5] μή with infinitive: II. 2. 4, 14. 5, I. 8. 2, 6, 11; 9. 2; 10. 3; ὥστε μή with infinitive, I. 8. 8, οὐ with infinitive in direct discourse, II. 20. 5.

[6] ὥστε οὐ with present indicative, II. 17. 1; ὥστε μή with infinitive, I. 8. 8.

[7] These always have μή; e.g., II. 14. 5.

and infinitive with *συμβαίνει* (I. 12. 1). There is one instance of *μή* in a relative clause which is apparently not generalizing or characterizing: *ἴδιον δὲ ἔχει ἡ γεωμετρικὴ μεσότης, ὃ μηδεμία τῶν λοιπῶν, κτλ.* (II. 24. 3). This is a Hellenistic trait.[1] The construction of *μή* with the indicative in cautious assertions, however, is classical; it occurs in *Introduction*, I. 8. 7 and I. 9. 3.[2]

To come next to the question of the use of the moods, it is most important to inquire whether in Nicomachus, as in Hellenistic Greek in general,[3] the optative shows signs of obsolescence. My count shows twenty-six examples of the potential optative (twenty-five with *ἄν*) and twenty-four of the optative in dependent constructions.[4] Perhaps the nature of the subject matter precludes the occurrence of the real optative in wishes; but there are a few additional circumstances that betray the fact that Nicomachus, too, was beginning to forsake this mood. For one thing, in most cases where a protasis has the optative, its apodosis will show some construction other than the normal potential optative, either a future or a present indicative, or an imperative, or there may be no conclusion at all. Out of twenty-one optative protases there are only three followed by the potential optative in the apodosis.[5] It is also noteworthy that the optative occurs only three times in any other construction than these two, the potential and the optative protasis; it is found twice in final clauses and once after a verb of fearing.

Among these examples of the optative there are a few that are noteworthy for their violation of ordinary classical syntax, although in all these cases the manuscript tradition must be taken into account, and at best we cannot be absolutely sure what Nicomachus wrote. One,

[1] See Gildersleeve, *American Journal of Philology*, I, 54. Many examples given by Schmid, *op. cit.*, from Philostratus (vol. IV, p. 92), Dio Chrysostom (I, 100), Lucian (I, 245), and others.

[2] Nicomachus uses the perfect each time. Cf. Moulton, *op. cit.*, pp. 192 f. (with citations of papyri).

[3] Moulton, *op. cit.*, 194 ff.

[4] Potential optative with *ἄν*, I. 1. 2; 2. 5; 4. 2; 9. 4; 11. 3; 12. 2; 14. 2; 15. 1; 19. 20; 23. 3; II. 5. 5; 6. 2 *bis;* 7. 4; 12. 1; 13. 1; 18. 3; 21. 1; 22. 2; 23. 3; 24. 6; 27. 2; 27. 7 (with *ὡς*); 29. 1; in indirect question, II. 27. 2; without *ἄν*, I. 23. 8 (perhaps here should be added the optatives in protases with *ἐάν* or *εἰ . . . ἄν*, I. 8. 9; 12. 2; II. 12. 3; 24. 1).

Optatives in dependent constructions: with *εἰ*, I. 9. 6; 10. 10 *bis;* 13. 9 *bis*, 11, 13; 14. 3 *bis;* II. 4. 3; 6. 3; 12. 4; 17. 5, 7; 23. 3; II. 23. 2; I. 15. 1 *bis*. With *ἵνα*, II. 27. 1; 29. 1. With *ἐάν*, *εἰ . . . ἄν*, as above.

[5] Of the conditions cited in the previous note, only three, I. 15. 1 *bis*, II. 23. 2, have the optative in both clauses. Moulton, *op. cit.*, p. 196, states, "Neither in LXX nor in NT is there an example of *εἰ* with the optative answered with the optative with *ἄν*, nor has one been quoted from the papyri."

and perhaps two, passages contain potential optatives without ἄν.[1] This is a phenomenon attested for Hellenistic Greek, although it was not unknown earlier; there are many examples in Homer and cases continually occur in the poets, but probably it was not felt to be a good prose use in classical times.[2] Another peculiar usage, of which there are four examples, is of the use of ἐάν or εἰ . . . ἄν with the optative, II. 12. 3: κἂν τοῖς πενταγώνοις οἱ τρίγωνοι προστιθοῖντο . . . γεννήσουσιν, κτλ. (εἰ $\overset{\text{ον}}{\text{τρίγων}}$ προστεθοῖντο S, εἰ τρίγωνοι προστίθενται H); I. 8. 9: εἰ γένοιτο ἂν οὕτως; I. 12. 2: ὅτι διαλυθείη ἂν εἰς ἐκείνους ἐξ ὧν συνέστηκεν εἴπερ καὶ μετρηθείη ἂν ὑπ' αὐτῶν (ἄν *om.* CSH); II. 24. 1: ἐὰν δὲ πλείονες ὅροι εἶεν (εἰ CSH).

It is very possible that these cases are all due to a late scribe, for there are no instances of such a construction in the New Testament and Moulton can find but one, in a Cyzicene inscription, elsewhere.[3] There are a number of instances in Homer and other classical literature where ἐάν (αἴ κε) with the optative represents in indirect discourse an original subjunctive, but some of them have received special explanation and in the great majority of cases the editors have emended the texts.[4] In general there is at present not sufficient evidence to fix upon this as a common prose construction of Hellenistic Greek. Perhaps one of the foregoing instances, I. 12. 2, may be explained on the ground that the clause with εἴπερ is after all rather causal than conditional. Before leaving the optative, it may be remarked that, with his evident fondness for the potential construction, Nicomachus often employs it where an indicative would serve as well.

In non-Atticizing Hellenistic Greek, consecutive clauses with ὥστε tend to show the infinitive more than the indicative, and when the indicative is used, the clause is usually coördinated with the main clause of the sentence, ὥστε meaning 'and so,' 'therefore'; the true consecutive clause with the indicative is rare.[5] Furthermore, the dis-

[1] γένοιτο, I. 23. 8. Here Ast added ἄν, but the MSS, followed by Hoche, agree in omitting it. In I. 11. 3 (p. 27, 6 Hoche), ἄν is to be omitted; see p. 160.

[2] See Smyth, *Greek Grammar*, 1821–1822; Kühner-Gerth, I, 225 f.; Moulton, 198; Robertson, *Grammar of the Greek New Testament*, 937–938.

[3] Cited p. 239, third paragraph. Schmid, *op. cit.*, cites examples from Lucian (vol. I. 244), Aristides (II. 59), Philostratus (one case, vol. IV, 90).

[4] See the list in Kühner-Gerth (ed. 1898), II. 549, 4.

[5] This is illustrated by Moulton's statistics on the New Testament (p. 209; cf. Burton, 235). He finds eighty-three examples of ὥστε; in fifty-one the infinitive is used; of the remaining thirty-two which have the indicative, only a very few (he cites but two clear cases) are true subordinate consecutive clauses and in the others the particle means 'and so,' 'therefore.'

tinction between the infinitive, which normally expresses the result that the action of the main verb tends to produce, and the indicative, expressing the actual result, seems to be more and more ignored, and the infinitive comes to be used where the indicative would be more natural.[1]

In Nicomachus the use of *ὥστε* conforms much more nearly to the classical usage than does that of the New Testament, where the indicative has all but disappeared from the real subordinate clause, for there are at least five cases where the indicative is clearly consecutive;[2] but the infinitive is used nine times, and in most of these it would be difficult to insist that the result expressed is the tendency and not the actuality;[3] in fact, in most of them the latter interpretation is the natural one. This is the only mark of looseness in Nicomachus's usage, unless we include the fact that all the passages with the real subordinate indicative (save one where the quasi-future *πρόεισι* occurs) employ the future tense.[4] As noted above, *ὥστε*, meaning 'and so,' 'therefore,' occurs several times.[5]

As one would expect of an author of his period, Nicomachus almost always uses the subjunctive with *ἵνα* in final clauses. I have noted but two cases where the optative is found, strangely enough both after verbs in primary tenses, II. 27. 1: *ὥσπερ . . . ἄλλον ἐξ ἄλλου τρόπον ἀποτελεῖσθαι δύνανται αἱ προλεχθεῖσαι μεσότητες . . . ἵνα εἰκότως καὶ ἐτυμώτατα καλοῖντο . . . οὕτως κτλ.*; II. 29. 1: *κυρίως γὰρ αὕτη . . . ἁρμονία ἂν λεχθείη μόνη παρὰ τὰς ἄλλας, εἴπερ μὴ ἐπίπεδος μηδὲ μιᾷ μόνῃ μεσότητι συνδεομένη, ἀλλὰ δυσίν, ἵν' οὕτω τριχῇ διιστάνοιτο, ὡς ὁ κύβος, κτλ.* To this may be joined the instance, referred to above, of an optative after a verb of fearing, I. 3. 7: *ὡς ἡδὺς εἶ, ὅτι ἔοικας δεδιέναι, μὴ ἄρα ἄχρηστα ταῦτα τὰ μαθήματα προστάττοιμι.*[6]

[1] Smyth, 2260 ff., and Goodwin, *Moods and Tenses*, 582 ff., leave no chance for the infinitive to express the idea normally conveyed by the indicative. But cf. Burton, 235: "Since, however, an actual result may always be conceived of as that which the cause in question is calculated or adapted to produce, the infinitive may be used when the result is obviously actual."

[2] I. 8. 12; 13. 6; 19. 2 (*πρόεισι*); II. 15. 3; 22. 3.

[3] I. 8. 8, 10; 18. 6; 19. 17, 19; 20. 2; 22. 2; II. 8. 3; 29. 2. Save perhaps the first case, these may all be regarded as expressing real results. A good illustration of this type is I. 18. 6.

[4] See the examples cited in note 2. Gildersleeve, *American Journal of Philology*, VII, 173, remarks that the future indicative is common enough in this construction.

[5] I. 10. 10; 19. 20; 20. 1; II. 17. 1, 7; 21. 3; 24. 11.

[6] This is an inaccurate quotation of Plato, *Republic*, 527 D: *ἡδὺς εἶ, ἦν δ' ἐγώ, ὅτι ἔοικας δεδιότι τοὺς πολλούς, μὴ δοκῇς ἄχρηστα μαθήματα προστάττειν.* It is noteworthy that Theon of Smyrna, p. 3, 8, quotes it in almost the same form as does Nicomachus: *ἡδὺς εἶ, ὅτι ἔοικας δεδιέναι μὴ ἄχρηστα τὰ μαθήματα προστάττοιμι.*

These final clauses can be explained, to be sure, as presenting the purpose as a mere conception of the mind, without regard for its fulfilment, a notion which sometimes justifies the use of an optative after a primary tense in classical Greek;[1] but Moulton[2] has very acutely observed that such an employment of the optative is a sign of a desire to Atticize, and they are perhaps best understood in that light, along with the object clause in I. 3. 7.

After all, the most significant thing about Nicomachus's use of the final clause is that no examples of the so-called 'ecbatic ἵνα'[3] — introducing clauses that are not purposive at all, but are used as substantives where the classical Greek would employ the infinitive — are to be found. In this he is quite classical. But sometimes his final clauses seem to express an idea very close to that of result, instead of purpose, as this example may suffice to illustrate: *ὅταν τοίνυν δύο ὅρων ἄκρων τριχῇ διαστατῶν ἀμφοτέρων, εἴτε ἰσάκις ἴσων ἰσάκις, ἵνα κύβος ᾖ, κτλ.*, II. 29. 2. It is worth noting that in one case, II. 2. 1, *μόνον ἵνα* with the subjunctive means 'provided only that' and introduces a proviso.

In clauses meaning 'as long as,' 'as far as,' there are many combinations: with the present indicative, *μέχρις οὗ*, 12 cases;[4] *μέχρις ὅσων*, 1 case;[5] *ἐφ' ὅσον*, 1 case;[6] *ἕως*, 1 case;[7] *μέχρι*, 1 case;[8] with the subjunctive, *μέχρις ἄν*, 9 cases;[9] *μέχρις οὗ ἄν*, 1 case;[10] *ἐφ' ὅσον ἄν*, 1 case;[11] *ἕως ἄν*, 1 case.[12]

This variation between the indicative and the subjunctive might

[1] See the list of optatives after primary verbs in Kühner-Gerth, II, 383 b, "wenn die Handlung des Finalsatzes, ohne Rücksicht auf ihre Verwirklichung, als bloss gedacht, als reine Vorstellung erscheinen soll." Goodwin, *Greek Moods and Tenses*, 322–23, regards the optative in final clauses after a primary verb as very rare and to be viewed as a mere irregularity of construction, unless the leading verb implies a reference to the past as well as the present; but he has only the classical writers in mind.

[2] P. 197. Cf. also Robertson, p. 983. The optative with ἵνα after a secondary tense is not found in the New Testament. There are two examples of it after a primary verb (*Ephesians*, i. 17; *II Thessalonians*, ii. 25), but the text is uncertain in both places and the former can be regarded as a volitive optative (Robertson, *loc. cit.*).

[3] See Moulton, p. 206.

[4] I. 9. 4; 10. 7, 8, 9; 16. 4; 18. 1; 19. 6; 23. 7; II. 8. 1; 9. 1; 12. 6; 13. 5; 20. 5.

[5] II. 2. 3: μέχρις ὅσων.

[6] I. 18. 5: ἐφ' ὅσον βούλεταί τις παρακολουθεῖν.

[7] II. 5. 5: ἕως προχωρεῖν θέλεις (ἕως ἂν ··· θέλῃς, SH)

[8] II. 14. 5: μέχρι βούλει.

[9] I. 8. 4, 10, 11; 9. 4; 10. 8; 16. 7; 23. 7; II. 2. 2; 11. 1.

[10] I. 13. 3: μέχρις οὗ ἂν προχωρεῖν ἐθέλωμεν.

[11] I. 18. 6: ἐφ' ὅσον ἂν εὐτονῇ τις ἕπεσθαι.

[12] II. 13. 9: ἕως ἂν ἡ μονὰς . . . φανῇ.

give rise to the suspicion that Nicomachus is less rigidly bound by the usual canons than a classical writer would be, and to a limited extent this is possibly so. But if closer attention is paid to the clauses themselves, his usage will be seen to conform fairly well to the recognized standards, which demand an indicative if the action is marked as a fact referring to a definite present or past occasion and the subjunctive to refer to future or indefinite present time.[1] It may be observed that every one of the examples with the indicative is of the type *μέχρι οὗ βούλει* 'as far as (up to the point that) you desire,'[2] and the period of time during which this desire lasts is, in these clauses, considered definite, so that their construction is normal. But the same interval may be also regarded as of indefinite extent, so that in a few clauses of a very similar nature the subjunctive is used.[3]

This inconsistency, which is easily explicable and after all very slight, is really the only one present; for most of the examples are of a different kind and do not introduce the notion of desire at all. 'Until (it) reaches the monad'[4] may be taken as typical of them; the subjunctive is obviously proper. It may be remarked that *μέχρι* (*μέχρις*) is Nicomachus's favorite word of this group,[5] and that he uses it freely as a preposition in phrases like *μέχρι παντός*, *μέχρις ἀπείρου*, in conformity with the general Hellenistic liking for the prepositional use of this and similar adverbs. In one phrase, thrice repeated, *μέχρις ἀεί*, he shows an interesting parallel to the *ἕως πότε* of *Revelation*, vi. 10, which may be taken as additional evidence for Moulton's contention that this is not a Hebraism.[6]

A few expressions also show peculiarity in the use of *ἄν*, namely: I. 13. 1, *ὅταν μετρεῖται* (*μετρῆται*, C, *μετρούμενος*, H); I. 13. 12, *ἐὰν προεβλήθη* (*ἐάν*, *om.* H, *προβληθῇ*, PCSH); I. 8. 7, *ὃ ἐὰν ἔχῃ* (so GP, *ἄν cett.*); I. 9. 2, *ὃ ἐὰν εὑρεθῇ μέρος ἔχων* (*ἄν*, CSH). Although all these expressions have the support of the oldest and best manuscript

[1] See Smyth, *Greek Grammar*, 1943, 2383. In the New Testament there are three examples of *μέχρι* (*μέχρις οὗ* in two of them) followed by the subjunctive without *ἄν*. See Robertson, 975. As a preposition it is common.

[2] I. 10. 8 has *μέχρι οὗ ἔχεις*, 'as far as you can'; cf. also II. 2. 3; but all the others are of the type indicated.

[3] E.g., I. 9. 4: *μέχρις ἂν προχωρεῖν ἐθέλῃς*; II. 11. 1: *μέχρις ἄν τις θέλῃ* and cf. I. 13. 3; 18. 6 as cited above.

[4] I. 8. 4: *μέχρις ἂν εἰς τὴν . . . μονάδα καταντήσῃ, κτλ.*

[5] It occurs in 23 of the 27 cases cited above, besides many where it is prepositional; *ἄχρι* is attested by but one MS (H) twice; *ἕως* twice; *πρίν* not at all.

[6] *Introduction*, I. 23. 8; II. 4. 3; 12. 5. Cf. Moulton, 107 n. Nicomachus also uses *ἐς ἀεί*, I. 10. 6 *bis*, *et passim*.

authorities, there must remain some question whether Nicomachus actually wrote them. But those which show a confusion of ἐάν and ἄν are types of a very common Hellenistic phenomenon,[1] and must have come into the manuscript tradition very early. It is even possible that Nicomachus so wrote them. But the other two cases are more complicated. The instance in I. 13. 1 is generalizing; in I. 13. 12, particular.

The former is easy to parallel in the papyri[2] and the New Testament,[3] and similar clauses even admit the past tenses of the indicative with ἄν.[4] It is but a single instance of the "weakening of the connexion between compounds of ἄν and the subjunctive" which Moulton discusses on pp. 167 ff. The second example, if genuine, would have to be explained in a similar way, but though there are instances of ἐάν with the present indicative in the New Testament expressing, as Burton thinks,[5] simple present suppositions, and many of ἐάν with the future indicative,[6] those where the past tenses are found with ἐάν are so feebly attested that such usages, conceivable though they may be, are probably best regarded as confined to the more illiterate.[7] Nicomachus is much more likely to have written προβληθῇ, which is the reading of four MSS.

The use of periphrastic forms of the verb, which is more frequent in late than in classical Greek, is not very common in Nicomachus. I have noted only a few instances.[8] It may be remarked that Nicomachus is fond of employing the perfect tense in the normal way, to indicate an action completed but continuing in force in the present.

There are also marks of his post-classical usages in Nicomachus's selection of words. Some, of course, which are purely mathematical terms, might well fail to appear elsewhere, even though they were known to earlier writers; but there are a considerable number among the less strictly arithmetical words which are not credited by the lexica

[1] Cf. Moulton, especially pp. 42 f.

[2] See Moulton, p. 168, for examples.

[3] See Burton, 309.

[4] Burton, 315. There are no instances of this common New Testament construction in Nicomachus.

[5] Burton, 247.

[6] Moulton, p. 168; Burton, 254.

[7] Moulton, p. 168.

[8] I. 5. 1, εἰσὶν ὠνομασμέναι as a present; I. 14. 1, εἰσὶν . . . διαιρούμενοι; II. 6. 3, ἔσται ἐπέχουσα; II. 6. 7, εἶναι λειπόμενον. But it may be noted, in addition, that Nicomachus is very apt to use with εἰμί such words as γεννητικός, καταρκτικός, or ἐπιδεκτικός in the sense of a simple verb; this is virtual periphrasis.

to other writers, and still more used by Nicomachus that are found only in post-Aristotelian literature. As a test of this, Nicomachus's vocabulary may first be compared with two of the lists printed at the end of Thayer's lexicon of New Testament Greek, one giving New Testament words found in post-classical writers generally, and the other, his 'Biblical' list, consisting of words that do not occur until 150 B.C. In common with the first group Nicomachus has *θρησκεία, νεωτερικός, ποταπός* (for *ποῖος*), *προκοπή*, and *σημειοῦν*; with the second he has *ἀνταποκρίνεσθαι, βαθμός, ἐπισυναγωγή, ἐπισωρεύειν, καθεξῆς, καταντᾶν*. The lists as given by Thayer overlap, and some of the words last cited are to be found in both of them.

Most of the late words introduced by Nicomachus may be classed, however, as new or uncommon compounds, not a few of which are possibly innovations of his own; many of them too are doubly or trebly compounded. Words assigned to Nicomachus only, or to the group embracing him together with the *Theologumena Arithmeticae* and Iamblichus's *Commentary*, include *ἀντιπαρώνυμος*, a double compound; also *ἀντιπεπόνθησις, ἀρτιακῶς, ἀρτιοδύναμος, ἀρτιοταγής, δευτεροδεῖσθαι, δευτεροταγής, διπλασίασις, ἐμπλέγδην, ἐναντιοπαθεῖν, ἐναντιωνυμεῖν, ἐναντιώνυμος, ἐπικορύφωσις, κορύφωσις, μειζονάκις, μοναδιστί, προσσώρευσις* and *σήμανσις*.

Other post-classical words that may be mentioned are the following: *ἀναμφιλέκτως, ἀντιδιαστέλλειν, ἀντιδιαστολή, ἀντιπαρωνυμεῖσθαι, ἀντισυγκρίνειν, ἀπαράβατος, ἀπαράλλακτος, ἀπαρέγκλιτος, ἀπαρεμπόδιστον, ἀπίσωσις, ἀποίκιλος, ἀποκαταστατικός, ἀπομειουρίζειν, ἀρτιοπέριττος, ἀυλία, γαμμοειδῶς, διαγνωστικός, διχοτόμημα, εἰδοποιεῖσθαι, εἰδοποίησις, ἐμφαντάζεσθαι, ἐμφαντικός, ἔναρθρος, ἐνατενίζειν, ἐνοπτρίζειν, ἐξάπλωσις, ἐξελιγμός, ἐπιδιαμένειν, ἐπισυντιθέναι, ἐπισωρεία, ἐπισώρευσις, ἐπιτροχάζειν, ἑτερογενῶς, ἑτερώνυμος, εὐπερίληπτος, ἡμίσευμα, θυμηδία, ἰδικῶς, κατάληψις* (as a Stoic term), *καταλληλία, καταρκτικός, μεγεθύνειν, μεσεμβολεῖν, μεσιτεία, μεταδίωξις, ὀλιγοεξία, ὁμωνυμεῖν, παρακολούθημα, παραπολαύειν, προεπισκοπεῖν, προσσωρεύειν, προυποδεικνύναι, προυποκεῖσθαι, προχάραγμα, συναναγνώσις, τεχνολογία, ὑπερεκπίπτειν, ὑπερέκπτωσις, φιλαλληλία, φιλάλληλος, χρησιμεύειν, ψυχογονία*.

To these may be added a few which are cited by W. Schmid[1] as late formations used by the Atticists and found also in Nicomachus: *ἀνάγνωσμα, ἀνταφαιρεῖν, ἀντιστρέφειν, ἐπεξευρίσκειν, ἐπιφάνεια,*

[1] *Op. cit.*, vol. IV, pp. 635 ff.

ἐνυπάρχειν, παράθεσις, περίστασις, περικυκλεῖν, προσσωρεύειν, προεκτιθέναι, συννεύειν, φυσιολογεῖν, multiples in *-ιων* (*διπλασίων*, etc.) and word-groups written together with but a single accent, like *τουτέστι, καθά, καθεξῆς, ὡσανεί*, and *οἱονεί*.

In matters of orthography and accidence Nicomachus does not depart so far from the literary standard; yet there are a few instances worthy of noticing, e.g., the non-Attic forms *γίνεσθαι* and *γινώσκειν*, with their compounds, are usual with him instead of the classical spellings, and we may mention the forms *ἀποδεικνύειν*, II. 1. 2; *διιστάνειν* (*διιστάνοιτο*), II. 29. 1; *ὀνοματοπεποιημένον*, I. 9. 2,[1] and *παρέδωκαν*, I. 3. 4. (This, however, is in a quotation of Archytas.) He uses the forms *πλείονες* (I. 10. 3) and *πλείονα* (I. 10. 4; 20. 1) instead of the contracted forms which were more common in Attic Greek, and varies between the forms *πλεῖον* and *πλέον*.[2] The spelling *σσ* usually appears instead of *ττ*,[3] contrary to the Attic rule, but in a few words, notably *ἀρτιπέριττος, διττῶς, κατορυττόμενον, ἐλάττων, κρεῖττον*, and *τάττειν*, *ττ* is found, sometimes alternating with *σσ*. Among the words which occur more than once in the *Introduction*, *ττ* is used exclusively only in *ἐλάττων*.

[1] See, however, the critical note, p. 159.

[2] See W. Schmid, *op. cit.*, vol. III, p. 24. *πλέον* was usual in the neuter.

[3] See W. Schmid, *op. cit.*, vol. IV, p. 579. Some of the words cited above are quoted by Nicomachus, and it is perfectly evident that he preferred *σσ*.

PART II

TRANSLATION OF THE *INTRODUCTION TO ARITHMETIC* OF NICOMACHUS OF GERASA, THE PYTHAGOREAN

BOOK I

CHAPTER I

The ancients, who under the leadership of Pythagoras first made science systematic, defined philosophy[1] as the love of wisdom. Indeed the name itself means this, and before Pythagoras all who had knowledge were called 'wise' indiscriminately — a carpenter, for example, a cobbler, a helmsman, and in a word anyone who was versed in any art or handicraft. Pythagoras, however, restricting the title so as to apply to the knowledge and comprehension of reality, and calling the knowledge of the truth in this the only wisdom, naturally designated the desire and pursuit of this knowledge philosophy, as being desire for wisdom. 1

He is more worthy of credence than those who have given other definitions, since he makes clear the sense of the term and the thing defined. This 'wisdom' he defined as the knowledge, or science, of the truth in real things, conceiving 'science' to be a steadfast and firm apprehension of the underlying substance, and 'real things' to be those which continue uniformly and the same in the universe and never depart even briefly from their existence; these real things[2] would be things immaterial, by sharing in the substance of which everything else that exists under the same name and is so called is said to be 'this particular thing,'[3] and exists. 2

[1] In his introductory statements Nicomachus does not run counter to widespread beliefs of ancient times. The origin of the names 'philosophy,' 'philosopher' (*φιλοσοφία, φιλόσοφος*) was commonly ascribed to Pythagoras; compare the citations given by Ritter and Preller, *Hist. Phil. Graec.*, 3, and (Plut.) *Epit.*, I. 3. 8 (= Diels, *Doxographi Graec.*, 280–281). As to the belief that Pythagoras corrected a wrong use of the terms, compare the following parallel with Nicomachus's statements furnished by Ammonius (*In Porphyrii Isagogen Prooem.*, p. 9, 7): "Pythagoras, however, says, 'Philosophy is the love of wisdom,' and he was the first to assail the error found among the ancients; for whereas they would call 'wise' a man who pursued any art whatsoever . . . he shifted this epithet to God, so as to call him alone wise (God, I mean) and endowed with wisdom and knowledge of those things that are eternal" (*ὁ μέντοι Πυθαγόρας φησί, Φιλοσοφία ἐστὶ φιλία σοφίας, πρῶτος τῷ παρὰ τοῖς παλαιοτέροις ἐπιστὰς ἁμαρτήματι. ἐπειδὴ γὰρ ἐκεῖνοι σοφὸν ὠνόμαζον τὸν ἡντιναοῦν μετιόντα τέχνην . . . μεθίστησι τὴν προσηγορίαν ταύτην ἐπὶ τὸν θεὸν ὡς μόνον ἐκεῖνον καλεῖσθαι σοφόν, τὸν θεὸν φημι, σοφίαν τε καὶ τὴν τῶν ὄντων ἀιδίων ἔχοντα γνῶσιν*).

[2] See Part I, p. 92.

[3] *τόδε τι*: in Aristotle the technical expression for the particular thing of which being is predicated. The principles which Nicomachus calls Pythagorean are here expressed in Platonic and

3 For bodily, material things are, to be sure, forever involved in continuous flow and change — in imitation of the nature and peculiar quality of that eternal matter and substance which has been from the beginning, and which was all changeable and variable[1] throughout. The bodiless things, however, of which we conceive in connection with or together with matter, such as qualities, quantities,[2] configurations, largeness, smallness, equality, relations, actualities, dispositions, places, times, all those things, in a word, whereby the qualities found in each body are comprehended — all these are of themselves immovable and unchangeable, but accidentally they share in and partake of the affections of the body to which they belong.
4 Now it is with such things that 'wisdom' is particularly concerned, but accidentally also with things that share in them, that is, bodies.

CHAPTER II

1 Those things, however, are immaterial, eternal, without end, and it is their nature to persist ever the same and unchanging, abiding by their own essential being, and each one of them is called real in the proper sense. But what are involved in birth and destruction, growth and diminution, all kinds of change and participation, are seen to vary continually, and while they are called real things, by the same term as the former, so far as they partake of them, they are not actually real by their own nature; for they do not abide for even the shortest moment in the same condition, but are always passing over in all
2 sorts of changes. To quote the words of Timaeus, in Plato,[3] "What is that which always is, and has no birth, and what is that which is always becoming but never is? The one is apprehended by the men-

Aristotelian terminology. Late Pythagoreans thus ascribed to their founder much that he never could have said.

[1] Philoponus (*scholia in Nic.*, ed. Hoche) on this passage says that Ammonius criticized Nicomachus for saying that matter is τρεπτὴ καὶ ἀλλοιωτή. "He ought to have said τρεπτικὴ καὶ ἀλλοιωτική, for the changes and variations take place about it; it itself does not change nor vary; for if it itself changed, there would have to be still another matter wherein it would vary and change. And so it is itself unchanging and unvarying, but its forms vary; I mean quantities, qualities, . . ." Philoponus retorts that when change and variation take place, it really is the substrate which we say changes; the forms (qualities, etc.) predicated of it do not change; they pass away and come into being. It is to be noted that, as in the case of Plato, the question of 'primary' and 'secondary' matter can be raised in connection with Nicomachus's doctrines; see p. 93.

[2] See Part I, p. 94.

[3] *Timaeus*, 27 D. Nicomachus closely follows the original, with only minor variations. His quotations of Plato are not usually so exact; cf. I. 3. 5, 7.

tal processes, with reasoning, and is ever the same; the other can be guessed at by opinion in company with unreasoning sense, a thing which becomes and passes away, but never really is."

Therefore, if we crave for the goal that is worthy and fitting for man, namely, happiness of life[1] — and this is accomplished by philosophy alone and by nothing else, and philosophy, as I said, means for us desire for wisdom, and wisdom the science of the truth in things, and of things some are properly so called, others merely share the name — it is reasonable and most necessary to distinguish and systematize the accidental qualities of things. 3

Things, then, both those properly so called and those that simply have the name, are some of them unified and continuous, for example, an animal, the universe, a tree, and the like, which are properly and peculiarly called 'magnitudes';[2] others are discontinuous, in a side-by-side arrangement, and, as it were, in heaps, which are called 'multitudes,' a flock, for instance, a people, a heap, a chorus, and the like. 4

Wisdom, then, must be considered to be the knowledge of these two forms. Since, however, all multitude and magnitude[3] are by their own nature of necessity infinite — for multitude starts from a definite root and never ceases increasing; and magnitude, when division beginning with a limited whole is carried on, cannot bring the dividing process to an end, but proceeds therefore to infinity — and since sciences[4] are always sciences of limited things, and never of infinites, it is accordingly evident that a science dealing either with 5

[1] The word used by Nicomachus, *εὐζωΐα*, is once employed by Aristotle in the *Ethica Nicomachea*, I. 8. 1098 b 20 ff.: *συνᾴδει δὲ τῷ λόγῳ καὶ τὸ εὖ ζῆν καὶ τὸ εὖ πράττειν τὸν εὐδαίμονα· σχεδὸν γὰρ εὐζωΐα τις εἴρηται καὶ εὐπραξία.* The 'happy life,' in a certain definite sense, is the goal that is becoming a man, according to Aristotle. See *ibid.*, I. 10. 1101 a 14 ff., I. 6. 1097 b 25 — 1098 a 20.

[2] On this and the following definitions, and their parallels in Aristotle, see p. 112 and notes.

[3] That is, multitude and magnitude per se and unqualified. This point is noted by Proclus, *In Prim. Eucl. Elem. Lib. Comm.*, p. 6, 15 Friedl.: *ὅ τε γὰρ ἀριθμὸς ἀπὸ μονάδος ἀρξάμενος ἄπαυστον ἔχει τὴν αὔξησιν, ἀεὶ δὲ ὁ ληφθεὶς πεπέρασται, καὶ ἡ τῶν μεγεθῶν διαίρεσις ἐπ' ἄπειρον χωρεῖ, τὰ δὲ διαιρούμενα πάντα ὥρισται, καὶ κατ' ἐνεργείαν πεπέρασται τὰ μόρια τοῦ ὅλου.* In the *Theologumena Arithmeticae*, p. 3 Ast, also there is reference to this matter in the same terminology: "And it [*sc.* the monad] is evidently beginning, middle, and end of all, since it bounds the infinite division of the continuous in the direction of the smaller than itself, and in the direction of the greater it cuts off a similar increase in the discrete, and this not by our decree, but by that divine nature." This passage is probably Nicomachean. Hero of Alexandria (*Definition* 119, in Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, Berlin, 1864, p. 33) speaks of magnitude as "that which is increased and divided to infinity" (*μέγεθός ἐστι τὸ αὐξανόμενον καὶ τεμνόμενον εἰς ἄπειρον*).

[4] The matter included in the rest of this section is touched on by Proclus, *op. cit.*, p. 36. 3 Friedl.: *ἐπισκοπεῖν δ' αὖ τὸ πηλίκον καὶ ποσὸν οὔτε μέγεθος ἁπλῶς οὔτε πλῆθος ἀλλὰ τὸ καθ*

magnitude, per se, or with multitude, per se, could never be formulated, for each of them is limitless in itself, multitude in the direction of the more, and magnitude in the direction of the less. A science, however, would arise to deal with something separated from each of them, with quantity, set off from multitude, and size, set off from magnitude.

CHAPTER III

1 Again, to start afresh, since of quantity one kind is viewed by itself, having no relation to anything else, as 'even,' 'odd,' 'perfect,' and the like, and the other is relative to something else and is conceived of together with its relationship to another thing, like 'double,' 'greater,' 'smaller,' 'half,' 'one and one-half times,' 'one and one-third times,' and so forth, it is clear that two scientific methods will lay hold of and deal with the whole investigation [1] of quantity; arithmetic, absolute quantity, and music, relative quantity.[2]

2 And once more, inasmuch as part of 'size' is in a state of rest and stability, and another part in motion and revolution, two other sciences in the same way will accurately treat of 'size,' geometry the part that abides and is at rest, astronomy that which moves and revolves.

3 Without the aid of these, then, it is not possible to deal accurately with the forms of being nor to discover the truth in things, knowledge of which is wisdom, and evidently not even to philosophize properly, for "just as painting contributes to the menial arts toward correctness of theory, so in truth lines, numbers, harmonic intervals, and the revolutions of circles bear aid to the learning of the doctrines of wis-

ἑκάτερον ὡρισμένον· τοῦτο γὰρ ἀφελούσας τῶν ἀπείρων τὰς ἐπιστήμας κατανοεῖν, ὡς οὐκ ἐνὸν τὴν καθ' ἑκάτερον ἀπειρίαν γνώσει περιλαβεῖν. This he states with regard to Pythagorean mathematics, and in view of the agreement of terminology to be seen here and in the passage cited in the preceding note, it is probable that he used the work of Nicomachus as his authority.

[1] Nicomachus thus subdivides the subject matter and assigns the special fields of the four mathematical sciences: I, treating number (*τὸ ποσόν*) (1) as such, absolutely (*καθ' ἑαυτό*), Arithmetic; and (2) relative number (*πρὸς ἄλλο*), Music; II, treating quantity (*τὸ πηλίκον*) (1) at rest, Geometry; (2) in motion, Astronomy (*σφαιρική*). Proclus, *op. cit.*, *Prol.*, p. 35. 21 ff., Friedl., gives the same division of the field of the mathematical sciences, using the same terms, in his report of the Pythagorean mathematics, probably drawing upon this work. It is to be noted that Nicomachus does not in fact adhere strictly to his classification, for he treats in this work of relative number, which falls in the domain of Music, and in the discussion of linear, plane and solid numbers he comes close to Geometry. The classification of Theon of Smyrna (cf. Part I, p. 113, n. 4) includes Music (i.e., the mathematical consideration of harmony) under Arithmetic and avoids this inconsistency.

[2] To illustrate what is meant by relative things Aristotle uses the example of double and half (*Met.*, IV. 15. 1020 b 26).

dom," says the Pythagorean Androcydes.[1] Likewise Archytas of 4
Tarentum,[2] at the beginning of his treatise *On Harmony*, says the same thing, in about these words: "It seems to me that they do well to study mathematics, and it is not at all strange that they have correct knowledge about each thing, what it is. For if they knew rightly the nature of the whole, they were also likely to see well what is the nature of the parts. About geometry, indeed, and arithmetic and astronomy, they have handed down to us a clear understanding, and not least also about music. For these seem to be sister sciences; for they deal with sister subjects, the first two forms of being."

Plato, too, at the end of the thirteenth book of the *Laws*,[3] to which 5

[1] He is mentioned as a writer *On Symbols* (περὶ συμβόλων) in the *Theol. Arith.*, p. 40 Ast.

[2] The passage here quoted is found also, at greater length, in Porphyry, *In Ptolem. Harm.*, p. 236; a few words of the same passage appear in Iamblichus, *In Nicomachi Arithmeticam Introductionem Liber*, p. 9, 1 Pistelli; cf. Diels, *Die Fragmente der Vorsokratiker*, vol. I[3], pp. 330 ff., and Blass, in *Mélanges Graux*, pp. 574 ff. On Archytas, and the problem of the title of this work, see pp. 20 ff.

[3] The *Epinomis*, from which Nicomachus here quotes 991 D ff., is now recognized as not genuinely Platonic. Nicomachus doubtless cited the passage from memory, for he does not give it exactly; but he can hardly have distorted it as Hoche would make him. The portion of Hoche's text that seems to demand correction is φανήσεται δ' ἂν ὃ λέγομεν ὀρθῶς, εἴ τις εἰς ἓν βλέπων πάντα μανθάνει. This reading introduces a future indicative with ἄν, a construction not elsewhere found in Nicomachus, and one which a writer of his class would not be likely to use, although it occurs in late Greek. Certainly he would not thrust such a palpable modernism into a quotation of Plato. Hoche perhaps has not reported the manuscripts correctly on this passage; his apparatus sometimes is faulty (see Hultsch, in *Neue Jahrbücher für Philologie und Pädagogik*, vol. XCVII, 1868, pp. 762 ff.). I have not had access to the manuscripts, but I note that C. F. A. Nobbe (*Specimen Arithmeticae Nicomacheae*, Leipzig, 1828, p. 19), who used the Nürnburg and Wolfenbüttel manuscripts (N and Γ), omits εἰ here, and that Hoche does not report the fact. Now this word εἰ is the cause of all the difficulty with the text. It did not occur in the original, but it could easily have been inserted by a scribe who misunderstood the preceding ἄν. The familiarity of the phrase εἴ τις would be another motive; ὀρθῶς τις, on the other hand, is unusual. The introduction of εἰ, also, might bring in μανθάνει for μανθάνῃ. I therefore read φανήσεται δ' ἄν, ὃ λέγομεν, ὀρθῶς τις εἰς ἓν βλέπων πάντα μανθάνῃ. The change of the last word is supported by Nobbe's manuscripts, but is not so necessary as the omission of εἰ.

The original passage reads: "Every geometrical figure, system of numbers, composition of harmony and the regularity in the revolution of the stars must appear to one, who properly learns, to be one in principle in all cases; and it will so appear, if, as we say, one learns correctly, looking to one thing. For it will appear to them on consideration that there naturally exists one bond of all these things; and if any one will pursue these matters in another way, he must call to his aid Fortune, as we say. For without these sciences there will never arise a fortunate being in the cities; but this is the mode, this the nurture, these the lessons, whether hard or easy; this way must one proceed, and to neglect them is impious before the gods. . . . And the man that has apprehended all this in this way, this man I call most truly wise, and so I also maintain both in jest and in earnest" (πᾶν διάγραμμα ἀριθμοῦ τε σύστημα καὶ ἁρμονίας σύστασιν ἅπασαν τῆς τε τῶν ἀστρῶν περιφορᾶς τὴν ὁμολογίαν οὖσαν μίαν ἁπάντων ἀναφανῆναι δεῖ τῷ κατὰ τρόπον μανθάνοντι, ἀναφανήσεται δέ, ἄν, ὃ λέγομεν, ὀρθῶς τις εἰς ἓν βλέπων μανθάνῃ· δεσμὸς γὰρ πεφυκὼς πάντων τούτων εἷς ἀναφανήσεται διανοουμένοις· εἰ δ' ἄλλως πως ταῦτα μεταχειριεῖταί τις, τύχην δεῖ καλεῖν, ὥσπερ καὶ λέγομεν. οὐ γὰρ ἄνευ γε τούτων μήποτέ τις ἐν πόλεσιν εὐδαίμων γένηται φύσις,

some give the title *The Philosopher*, because he investigates and defines in it what sort of man the real philosopher should be, in the course of his summary of what had previously been fully set forth and established, adds: "Every diagram, system of numbers, every scheme of harmony, and every law of the movement of the stars, ought to appear one to him who studies rightly; and what we say will properly appear if one studies all things looking to one principle, for there will be seen to be one bond for all these things, and if any one attempts philosophy in any other way he must call on Fortune to assist him. For there is never a path without these; this is the way, these the studies, be they hard or easy; by this course must one go, and not neglect it. The one who has attained all these things in the way I describe, him I for my part call wisest, and this I maintain through
6 thick and thin." For it is clear that these studies are like ladders and bridges that carry our minds from things apprehended by sense and opinion to those comprehended by the mind and understanding, and from those material, physical things, our foster-brethren known to us from childhood, to the things with which we are unacquainted, foreign to our senses, but in their immateriality and eternity more akin to our souls, and above all to the reason [1] which is in our souls.

7 And likewise in Plato's *Republic*, when the interlocutor of Socrates appears to bring certain plausible reasons to bear upon the mathematical sciences, to show that they are useful to human life, arithmetic for reckoning, distributions, contributions, exchanges, and partnerships, geometry for sieges, the founding of cities and sanctuaries, and the partition of land, music for festivals, entertainment, and the worship of the gods, and the doctrine of the spheres, or astronomy, for farming, navigation and other undertakings, revealing beforehand the proper procedure and suitable season, Socrates, reproaching him, says: "You amuse me, because you seem to fear that these are useless studies that I recommend; but that is very difficult, nay, impossible. For the eye of the soul, blinded and buried by other pursuits, is rekindled and aroused again by these and these alone, and it is

ἀλλ' οὗτος ὁ τρόπος, αὕτη ἡ τροφή, ταῦτα τὰ μαθήματα, εἴτε χαλεπὰ εἴτε ῥᾴδια, ταύτῃ πορευτέον· ἀμελῆσαι δὲ οὐ θεμιτόν ἐστι θεῶν . . . τὸν δὲ ξύμπαντα ταῦτα οὕτως εἰληφότα, τοῦτον λέγω τὸν ἀληθέστατα σοφώτατον, ὃν καὶ διισχυρίζομαι παίζων καὶ σπουδάζων ἅμα . . .).

Theon of Smyrna quotes the first part of this passage, p. 84, 7 ff. Hiller, and the last part, p. 2, 16 ff.

[1] A reference to the νοῦς as the highest part of the soul in accordance with the ancient view that the soul is made up of parts.

better that this be saved than thousands of bodily eyes, for by it alone is the truth of the universe beheld."[1]

CHAPTER IV

Which then of these four methods[2] must we first learn? Evidently, the one which naturally exists before them all, is superior and takes the place of origin and root and, as it were, of mother to the others. 1

And this is arithmetic,[3] not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan, relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends; but also because it is naturally prior in birth, inasmuch as it abolishes other sciences with itself,[4] but is not abolished together with them. 2

[1] The original passage (*Republic*, 527 D ff.) reads: "'You amuse me,' said I, 'because you are like one who fears the crowd lest you seem to enjoin useless studies. It is, however, not at all a trifling matter, but a difficult one to believe that in these studies some instrument of every man's soul is cleansed and rekindled, which was being destroyed and blinded by his other pursuits, a thing more worthy to save than countless eyes: for by it alone is truth beheld.'" (ἡδὺς εἶ, ἦν δ' ἐγώ, ὅτι ἔοικας δεδιότι τοὺς πολλούς, μὴ δοκῇς ἄχρηστα μαθήματα προστάττειν. τὸ δ' ἔστιν οὐ πάνυ φαῦλον ἀλλὰ χαλεπὸν πιστεῦσαι ὅτι ἐν τούτοις τοῖς μαθήμασιν ἑκάστου ὄργανόν τι ψυχῆς ἐκκαθαίρεταί τε καὶ ἀναζωπυρεῖται ἀπολλύμενον καὶ τυφλούμενον ὑπὸ τῶν ἄλλων ἐπιτηδευμάτων, κρεῖττον ὂν σωθῆναι μυρίων ὀμμάτων· μόνῳ γὰρ αὐτῷ ἀλήθεια ὁρᾶται). Theon of Smyrna (p. 3, 8 ff. Hiller) quotes this passage.

[2] This group of studies, music, arithmetic, geometry, and astronomy, make up the 'quadrivium,' or, as Boethius, who apparently first used the term, called it, 'quadruvium.' 'Trivium,' to designate the study of grammar, rhetoric, and dialectic, may also go back to his time. See Gow, *op. cit.*, p. 72, note.

[3] Plato also said that arithmetic should be first learned and that it is the basis of all other arts. *Rep.*, 522 C: οἷον τοῦτο τὸ κοινόν, ᾧ πᾶσαι προσχρῶνται τέχναι τε καὶ διάνοιαι καὶ ἐπιστῆμαι, ὃ καὶ παντὶ ἐν πρώτοις ἀνάγκη μανθάνειν. ποῖον; ἔφη. τὸ φαῦλον τοῦτο, ἦν δ' ἐγώ, τὸ ἕν τε καὶ τὰ δύο καὶ τὰ τρία διαγιγνώσκειν. λέγω δὲ αὐτὸ ἐν κεφαλαίῳ ἀριθμόν τε καὶ λογισμόν. ἢ οὐχ οὕτω περὶ τούτων ἔχει, ὡς πᾶσα τέχνη τε καὶ ἐπιστήμη ἀναγκάζεται αὐτῶν μέτοχος γίγνεσθαι; It is interesting to observe parallels to many of the topics of this chapter in Caxton's *Mirrour of the World* (*Publications of the Early English Text Society*, extra series, vol. CX, pp. 36–37): "The fourth scyence is called arsmetrique. This science cometh after rethoryque, ande is sette in the myddle of the vii sciences. And without her may none of the vii sciences parfyghtly ne weel and entierly be knowen. Wherfor it is expedyent that it be weel knowen and conned; ffor alle the sciences take of it their substaunce in suche wise that without her they may not be. And for this reson was she sette in the myddle of the vii sciences, and there holdeth her nombre; ffor fro her procede alle maners of nombres, and in alle thynges renne, come and goo. And no thyng is without nombre. But fewe perceyue how this may be, but yf he haue be maistre of the vii artes so longe that he can truly save the troughe."

[4] Cf. below II. 22. 3. Nicomachus of course refers merely to abolishment in thought. Arithmetic, since it treats of numbers and numerical relations fundamental to the other sciences, is logically prior to them, and if it did not exist they could not exist. Nicomachus uses both προ-

For example, 'animal' is naturally antecedent to 'man,' for abolish 'animal' and 'man' is abolished; but if 'man' be abolished, it no longer follows that 'animal' is abolished at the same time. And again, 'man' is antecedent to 'schoolteacher'; for if 'man' does not exist, neither does 'schoolteacher,' but if 'schoolteacher' is nonexistent, it is still possible for 'man' to be. Thus since it has the property of abolishing the other ideas with itself, it is likewise the older.

3 Conversely, that is called younger and posterior which implies the other thing with itself,[1] but is not implied by it, like 'musician,' for this always implies 'man.' Again, take 'horse'; 'animal' is always implied along with 'horse,' but not the reverse; for if 'animal' exists, it is not necessary that 'horse' should exist, nor if 'man' exists, must 'musician' also be implied.

4 So it is with the foregoing sciences; if geometry exists, arithmetic must also needs be implied, for it is with the help of this latter that we can speak of triangle, quadrilateral, octahedron, icosahedron, double, eightfold, or one and one-half times, or anything else of the sort which is used as a term by geometry, and such things cannot be conceived of without the numbers that are implied with each one. For how can 'triple' exist, or be spoken of, unless the number 3 exists beforehand, or 'eightfold' without 8? But on the contrary 3, 4, and the rest might be
5 without the figures existing to which they give names. Hence arithmetic abolishes geometry along with itself, but is not abolished by it, and while it is implied by geometry, it does not itself imply geometry.

CHAPTER V

1 And once more is this true in the case of music; not only because the absolute is prior to the relative, as 'great' to 'greater' and 'rich'

γενέστερος and *πρότερος* in this passage with the meaning 'prior.' Aristotle uses the term *πρότερος* and his logic forms the basis for Nicomachus's present argument. For instance, in *Met.*, 1019 a 1 ff., after discussing several forms of *πρότερα* and *ὕστερα*, he has the following: *τὰ μὲν δὴ οὕτω λέγεται πρότερα καὶ ὕστερα, τὰ δὲ κατὰ φύσιν καὶ οὐσίαν, ὅσα ἐνδέχεται εἶναι ἄνευ ἄλλων, ἐκεῖνα δὲ ἄνευ ἐκείνων μή· ᾗ διαιρέσει ἐχρῆτο Πλάτων.* He also uses the verb *συναναιρεῖσθαι* ('abolish') as does Nicomachus: e.g., *Met.*, 1059 b 39: *ᾗ δὲ συναναιρεῖται τοῖς γένεσι τὰ εἴδη, τὰ γένη ταῖς ἀρχαῖς ἔοικε μᾶλλον· ἀρχὴ γὰρ τὸ συναναιροῦν.* Cf. also Iamblichus *In Nicom.*, p. 10, 2 Pistelli.

[1] This form of argument also is Aristotelian, e.g., *Top.*, VI. 6. 17: *ἐπιφέρει γὰρ ἑκάστη τῶν διαφορῶν τὸ οἰκεῖον γένος, καθάπερ τὸ πεζὸν καὶ τὸ δίπουν τὸ ζῷον συνεπιφέρει.* Logically exhausted it stands in *Top.*, II. 4. 111 a 25 ff. thus: *οὐ γὰρ ἀναγκαῖον ὅσα τῷ γένει ὑπάρχει καὶ τῷ εἴδει ὑπάρχειν· ὅσα δὲ τῷ εἴδει ὑπάρχει ἀναγκαῖον καὶ τῷ γένει· ὅσα γὰρ τῷ γένει οὐχ ὑπάρχει, οὐδὲ τῷ εἴδει· ὅσα δὲ τῷ εἴδει μὴ ὑπάρχει, οὐκ ἀναγκαῖον τῷ γένει μὴ ὑπάρχειν.*

to 'richer' and 'man' to 'father,' but also because the musical harmonies, diatessaron, diapente, and diapason, are named for numbers; similarly all of their harmonic ratios are arithmetical ones, for the diatessaron is the ratio of 4 : 3, the diapente that of 3 : 2, and the diapason the double ratio; and the most perfect, the di-diapason, is the quadruple ratio.

More evidently still astronomy attains through arithmetic the in- 2
vestigations that pertain to it, not alone because it is later than geometry[1] in origin — for motion naturally comes after rest — nor because the motions of the stars have a perfectly melodious harmony,[2] but also because risings, settings, progressions, retrogressions, increases, and all sorts of phases are governed by numerical cycles and quantities.

So then we have rightly undertaken first the systematic treatment 3
of this, as the science naturally prior, more honorable, and more venerable, and, as it were, mother and nurse of the rest; and here we will take our start for the sake of clearness.

CHAPTER VI

All that has by nature with systematic method been arranged in 1
the universe[3] seems both in part and as a whole to have been determined and ordered in accordance with number, by the forethought and the mind of him that created all things; for the pattern was fixed, like a preliminary sketch, by the domination of number preëxistent[4] in the mind of the world-creating God, number conceptual only and immaterial in every way, but at the same time the true and the eternal essence, so that with reference to it, as to an artistic plan, should be created all these things, time, motion, the heavens, the stars, all sorts of revolutions.

It must needs be, then, that scientific number, being set over such 2
things as these, should be harmoniously constituted, in accordance with itself; not by any other but by itself. Everything that is har- 3

[1] Plato in *Rep.*, 528 A—B points out that it is a mistake to study bodies (στερεά) in motion before studying them per se (αὐτὸ καθ' αὑτό).

[2] The music of the spheres. Boethius, I. 1, paraphrases: *quod armonicis modulationibus motus ipse celebratur astrorum.*

[3] This chapter, with I. 4. 2, gives the fullest information we have about Nicomachus's theories of cosmogony. See Part I, p. 107.

[4] This is the eternal number, to be distinguished from the 'scientific number' mentioned in the next section. Cf. Part I, p. 98.

moniously constituted is knit together out of opposites [1] and, of course, out of real things; for neither can non-existent things be set in harmony, nor can things that exist, but are like one another, nor yet things that are different, but have no relation one to another. It remains, accordingly, that those things out of which a harmony is made are both real, different, and things with some relation to one another.

4 Of such things, therefore, scientific number consists; for the most fundamental species in it are two, embracing the essence of quantity,[2] different from one another and not of a wholly different genus, odd and even, and they are reciprocally [3] woven into harmony with each other, inseparably and uniformly, by a wonderful and divine Nature, as straightway we shall see.

CHAPTER VII

1 Number [4] is limited multitude or a combination of units or a flow [5] of quantity made up of units; and the first division of number is even and odd.

2 The even [6] is that which can be divided into two equal parts without a unit intervening in the middle; [7] and the odd is that which cannot be divided into two equal parts because of the aforesaid intervention of a unit.

3 Now this is the definition after the ordinary conception; by the Pythagorean doctrine, however, the even number is that which admits of division into the greatest and the smallest parts at the same operation, greatest in size and smallest in quantity, in accordance

[1] Note that the definition of harmony quoted from Philolaus in II. 19. 1 implies as much, and compare Part I, pp. 100, 120, on the general subject of harmony in the numerical system.

[2] That is, they are elementary, for they are formed by the two elements of number, the monad and dyad respectively, and embody by reason of this origin 'sameness' and 'otherness,' the fundamental cosmic forces. Cf. Part I, p. 99.

[3] Cf. Theon of Smyrna, p. 23. 3 Hiller: "And the even and the odd numbers alternate, being observed in alternate places" (ἐναλλὰξ δ' εἰσὶν ἀλλήλοις οἵ τε ἄρτιοι καὶ οἱ περιττοὶ παρ' ἕνα θεωρούμενοι).

[4] Cf. Part I, p. 114, on these definitions.

[5] The translation 'flow' for χύμα is that adopted by Heath (*Euclid*, II. 280), whose note on definitions of number may be consulted. χύμα elsewhere in Nicomachus is used to mean 'series' (see the Glossary), but probably the metaphorical idea is present even in those cases.

[6] Theon of Smyrna, p. 21, 22 Hiller: καὶ ἄρτιοι μὲν εἰσιν οἱ ἐπιδεχόμενοι τὴν εἰς ἴσα διαίρεσιν . . . περισσοὶ δὲ οἱ εἰς ἄνισα διαιρούμενοι, κτλ. Euclid defines thus: "An even number is one that is halved" (ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος), *Elements*, VII, Def. 6.

[7] See Part I, p. 122 with note 1.

with the natural contrariety [1] of these two genera; and the odd is that which does not allow this to be done to it, but is divided into two unequal parts.

In still another way, by the ancient definition, the even is that which 4
can be divided [2] alike into two equal and two unequal parts, except that the dyad,[3] which is its elementary form, admits but one division, that into equal parts; and in any division whatsoever it brings to light only one species of number, however it may be divided, independent of the other. The odd [4] is a number which in any division whatsoever, which necessarily is a division into unequal parts, shows both the two species of number together, never without intermixture one with another, but always in one another's company.

By the definition in terms of each other, the odd is that which 5
differs by a unit from the even in either direction, that is, toward the greater or the less, and the even is that which differs by a unit in either direction from the odd, that is, is greater by a unit or less by a unit.

[1] That is, halves are the greatest possible parts of a term in magnitude; and there is a smaller number of them than of any other fractional part. Thus greater magnitude of factors is associated with a smaller number of them; this is the 'natural contrariety' of magnitude and quantity. Cf. Boethius, I. 4, for a discussion of this notion, and Iamblichus, p. 12, 3 ff. Pistelli. This principle may be illustrated by what was called the "lambdoid diagram" (see Part I, p. 127) from its likeness to the Greek lambda, Λ. This diagram sets forth in the form of the letter lambda, converging at unity, the series of natural numbers and the series of fractions, thus:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & \frac{1}{2} & & & \\ & & 3 & & & & \frac{1}{3} & & \\ & 4 & & & & & & \frac{1}{4} & \\ 5 & & & & & & & & \frac{1}{5} \end{array}$$

and so on. It will be noted that the corresponding integers and fractions show the 'natural contrariety' referred to. The diagram occurs in Iamblichus's commentary (p. 14, 3 ff. Pistelli) following on the discussion of these definitions, and it is referred to in *Theol. Arith.*, p. 3 (bottom) Ast.

[2] When an even number is divided into two parts, whether equal or unequal, these parts are always either both odd or both even ('only one species of number,' as Nicomachus says). Iamblichus, p. 12, 14 ff. Pistelli. See Heath, *History*, vol. I, p. 70.

[3] Iamblichus (p. 13, 7 ff. Pistelli) notes that the monad is distinguished from all the odd numbers by not even admitting division into unequal parts, and the dyad from the even numbers by admitting division into equal parts only. Theon does not notice this property of the dyad, but discusses at some length the question whether the monad is odd or even (p. 21, 24 ff. Hiller). On Nicomachus's doctrine, that the monad and dyad are both elements of number and not themselves numbers but its 'beginnings,' compare Part I, pp. 116 ff.

[4] If an odd number is divided into two parts these will always be unequal and one odd, the other even ('the two species of number').

CHAPTER VIII

1 Every number is at once half the sum of the two on either side of itself,[1] and similarly half the sum of those next but one in either direction, and of those next beyond them, and so on as far as it is possible
2 to go. Unity alone, because it does not have two numbers on either side of it, is half merely of the adjoining number; hence unity is the natural starting point of all number.

3 By subdivision of the even,[2] there are the even-times even, the odd-times even, and the even-times odd. The even-times even and the even-times odd are opposite to one another, like extremes, and the odd-times even is common to them both like a mean term.

4 Now the even-times even [3] is a number which is itself capable of being divided into two equal parts, in accordance with the properties of its genus,[4] and with each of its parts similarly capable of division, and again in the same way each of their parts divisible into two equals until the division of the successive subdivisions reaches the
5 naturally indivisible unit. Take for example 64; one half of this is 32, and of this 16, and of this the half is 8, and of this 4, and of this 2, and then finally unity is half of the latter, and this is naturally indivisible and will not admit of a half.

6 It is a property of the even-times even that, whatever part of it be taken, it is always even-times even in designation, and at the same time, by the quantity of the units in it, even-times even in value;

[1] Thus 5 is half the sum of $4 + 6$, $3 + 7$, $2 + 8$, etc. For a typically Pythagorean application of this principle cf. *Theol. Arith.*, p. 28 f. Ast.

[2] Euclid, among the definitions of *Elem.*, VII, defines the even-times even, even-times odd, odd-times even and odd-times odd (the latter is "one which is measured by an odd number an odd number of times"). Nicomachus confines himself to a tripartite division of the even only; Euclid's classification applies to all numbers. The 'odd-times odd' of Euclid is not found in Nicomachus's *Introduction* at all, and in defining the three classes given by both Nicomachus and Euclid the former uses somewhat different formulas, which are consistently praised by Iamblichus in his commentary. (See the notes on I. 8. 7, above p. 127, and Nesselmann, p. 192.) Theon (p. 25, 5 ff. Hiller) gives the same classification as Nicomachus here, and like him refers to even numbers alone. His definitions are compared to those of Nicomachus in the following notes. It may be noted that 'odd-times odd' occurs in Theon as another name for the prime number (p. 23, 14 Hiller). See Heath, *History*, vol. I, pp. 70 ff., on the classification of numbers.

[3] Theon of Smyrna, p. 25, 7 ff. Hiller, gives the definition of the even-times even substantially as follows: It is a number that has three characteristics: (1) It is produced by the multiplication of two even numbers; (2) all its parts are even, down to unity; (3) none of its parts has its designation in terms of an odd number. Euclid's definition is: "The even-times even number is that which is measured by an even number an even number of times" (**ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν**), *Elements*, VII, *Def.* 8.

[4] 'Its genus' is the even; cf. I. 7. 2. So too Philoponus notes (ed. Hoche, p. 15).

and that neither of these [1] two things will ever share in the other class. Doubtless it is because of this that it is called even-times even, be- 7
cause it is itself even and always has its parts,[2] and the parts of its parts down to unity, even both in name and in value; in other words, every part that it has is even-times even in name and even-times even in value.[3]

[1] The specific things meant by 'neither of these' (ἑκάτερον τούτων) are the 'name' (προσηγορίαν, p. 15, 17 Hoche, or better ὄνομα implied in ἀρτιάκις ἀρτιώνυμον, *ibid.*) and the 'value' (δύναμις implied in p. 15, 18, ἀρτιάκις ἀρτιοδύναμον) of any part of the even-times even number. These "never share in another variety"; i.e., another variety of number, or of even numbers, than the even-times even. They are called halves, fourths, eighths, etc. (even-times even names), and their values are always 2, 4, 8, 16, etc. (even-times even numbers). On the use of δύναμις and ἀρτιοδύναμος here, cf. on I. 8. 7.

[2] Philoponus writes the following scholium upon this: "Here then Euclid is convicted of making a poor definition of the even-times even number in his Seventh Book; for he says that an even-times even number is one that is measured by an even number an even number of times (ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν). For by this definition the merely even numbers also that are not even-times even, will be found to be even-times even; e.g., 24 is not even-times even, for it is not subdivided to the monad; but according to Euclid it will be found to be even-times even; for, look you, it is measured by 4, an even number, an even number of times, 6; for $4 \times 6 = 24$. So his was a bad definition."

[3] Cf. the note on I. 8. 6 for the meaning of this statement. The word translated 'value' is δύναμις, which as a mathematical term usually means 'square' or 'square root,' but in non-technical Greek may bear the meaning assigned (e.g., Thuc., VI, 46, 3). The word is again similarly employed in section 10 of this chapter and in I. 9. 2; 10. 5; besides which the phrase ἀρτιάκις ἀρτιοδύναμος seems to have the corresponding sense of "even-times even in value" (I. 8. 6). This interpretation has the support of Boethius, while Philoponus understands the passage differently. Boethius, I. 9, says, *Sed ideo mihi videtur hic numerus pariter par vocatus, quod eius omnes partes et nomine et quantitate pares pariter inveniantur.* 'Quantity,' however, does not represent δύναμις as well as 'value,' since it is strictly a case of number rather than of quantity. T. L. Heath understands δύναμις in the sense proposed, and commenting on this passage of Nicomachus (*Euclid*, II, 282) says: "He says . . . that any part, i.e., any submultiple, of an even-times even number is called by an even-times even designation, while it also has an even-times even value . . . when expressed as so many actual units. That is, the $\frac{1}{2^m}$th part of 2^n (where m is less than n) is called after the even-times even number 2^m, while its actual value (δύναμις) in units is 2^{n-m}, which is also an even-times even number."

On the other hand Philoponus (*schol.* 56 on I. 8. 6, p. 15 Hoche) says: "Since he does not employ the ordinary language of the usage of most people, I think it reasonable first to impart the meaning of the various terms and then to interpret the whole sense of what is said here. Now he calls δυνάμεις the numbers from which the parts of any number take their names, e.g., the half from 2, etc. . . . Now he says that all the parts of the even-times even are themselves even-times even in name (ἀρτιάκις ἀρτιώνυμα), and the δυνάμεις, from which their names are taken, are even-times even powered (ἀρτιάκις ἀρτιοδύναμοι). For example, the even-times even number 16 has as its second part 8, 4 as its fourth, 2 as its eighth; each of these parts is even-times even. . . . So reasonably the parts are called even-times even in name because they take their names from even-times even numbers. . . . And the 'powers,' from which the parts are names, are even-times even powered (ἀρτιάκις ἀρτιοδύναμοι). . . . Similarly the expression 'and in the number of their monads ἀρτιάκις ἀρτιοδύναμον'; for $\frac{1}{4}$ of 16, named from 4, has the number of monads of 4, from which it is named, ἀρτιάκις ἀρτιοδύναμον. So the expression 'in the number of monads in it' is to be taken either as applying to the designation of the fraction (μόριον) or as applying to the part (μέρος) itself" (but he immediately states that he prefers the former explanation). What Philo-

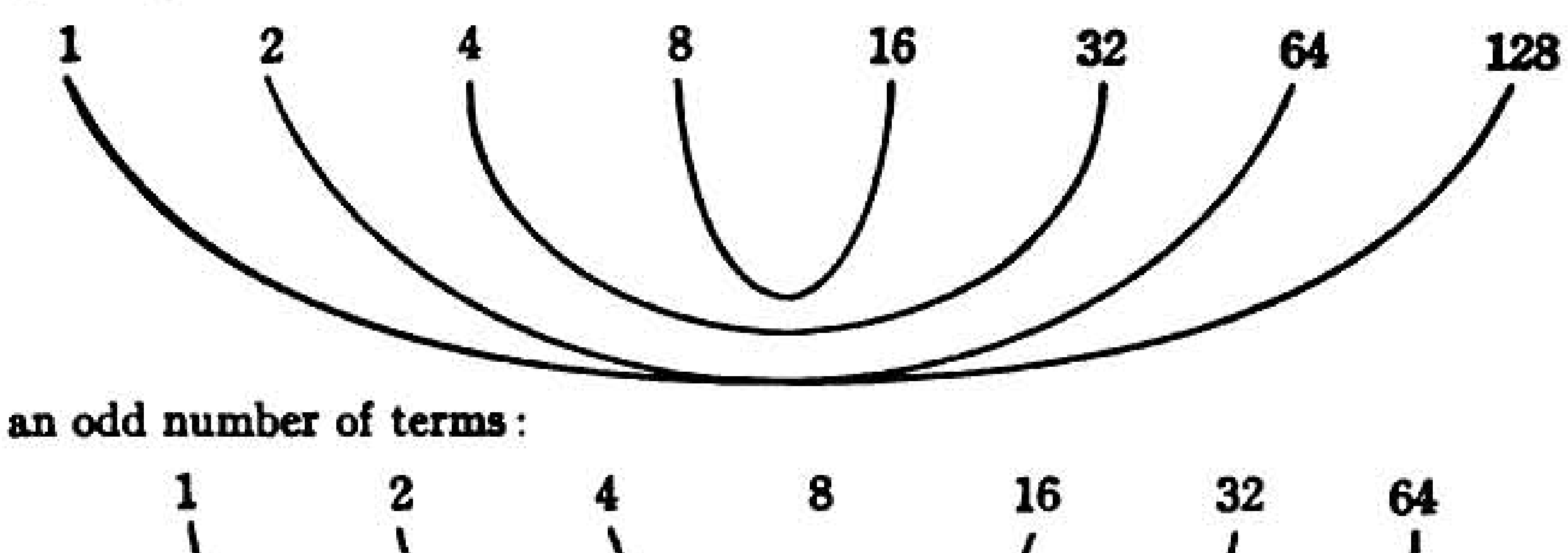
8 There is a method of producing the even-times even, so that none will escape, but all successively [1] fall under it, if you do as follows:
9 As you proceed from unity, as from a root, by the double ratio to infinity, as many terms as there are will all be even-times even, and it is impossible to find others besides these; for instance, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512. . . .
10 Now each of the numbers set forth was produced by the double ratio, beginning with unity, and is in every respect even-times even, and every part [2] that it may be found to have is always named from

ponus understands by ἀρτιάκις ἀρτιοδύναμος may be seen from his statement in this same scholium: "And 16 would be ἀρτιάκις ἀρτιοδύναμος if from it there should be named some part of an even-times even number."

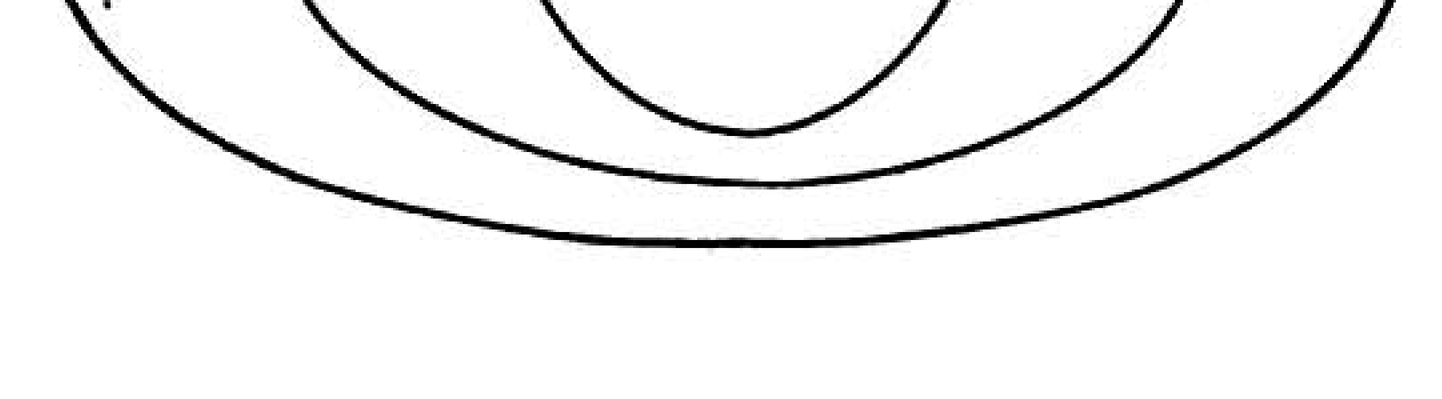
It is quite tempting to consider δύναμις the quotient of a factor when the latter is divided into the number under consideration, but the rendering adopted seems to fit the context better, while on the other hand Philoponus himself virtually confesses that his interpretation of δύναμις does not hold in I. 9. 2 (*To take a single example, the number 18*, etc.), when he says, "If he meant δύναμις to be the number from which the part gets its name, and the half gets its name from 2, how is it that he says here that the half is odd in δύναμις? We must therefore read with hyperbaton, i.e., 'the half, 9, evenly named in δύναμις (that is, from 2, for the half is the second part), is odd in its number, 9.'" That is, in the text, τὸ μὲν ἥμισυ ἀρτιακῶς ὠνομασμένον ὑπάρχει θ′ περισσὸν τῇ δυνάμει, he proposes to connect τῇ δυνάμει with ὠνομασμένον, a hyperbaton too violent to be probable. In I. 9. 2 δύναμις seems certainly to mean 'value,' and it is hardly possible that it would be used in different senses in such similar contexts. A further question suggested by this passage is whether Nicomachus would class the monad as an even number, as, strictly speaking, is implied here. In the *Theol. Arith.*, p. 3 Ast, we find the statement that the monad "embraces all things in potentiality," and among the specific statements following it is said that "it is even and odd and even-times odd." The monad was called Male-Female, and this designation is perhaps connected with the notion that it is both even and odd. See Part I and on section 13 below.

[1] Reading ἑξῆς (found in several MSS) for Hoche's ἐξ ἑνός.

[2] Boethius, I. 9: *Illud autem non minima consideratione dignum est, quod eius omnis pars ab una parte quacumque, quae intra ipsum numerum est, denominatur tantamque summam quantitatis includit quota pars est alter numerus pariter paris illius, qui eum continet, quantitatis. Itaque fit, ut sibi partes ipsae respondeant, ut quota pars una est, tantam habeat altera quantitatem, et quota pars ista est, tantam in priore summa necesse sit multitudinis inveniri.* This correspondence of factors is thus graphically shown in one of the MSS:



and with an odd number of terms:



some one of the numbers before it in the series,[1] and the sum of units in this part is the same as one of the numbers before it, by a system of mutual correspondence, indeed, and interchange. If there is an even number of terms of the double ratio from unity, not one mean term can be found, but always two, from which the correspondence and interchange of factors and values, values and factors, will proceed in order, going first to the two on either side of the means, then to the next on either side, until it comes to the extreme terms, so that the whole will correspond in value to unity and unity to the whole. For example, if we set down 128 as the largest term, the number of terms will be even, for there are eight in all up to this number; and they will not have one mean term, for this is impossible with an even number, but of necessity two, 8 and 16. These will correspond to each other as factors; for of the whole, 128, 16 is one eighth and conversely 8 is one sixteenth. Thence proceeding in either direction, we find that 32 is one fourth, and 4 one thirty-second, and again 64 is one half, and 2 one sixty-fourth, and finally at the extremes unity is one one-hundred-twenty-eighth, and conversely 128 is the whole, to correspond with unity.

If, however, the series consists of an odd number of terms, seven 11
for example, and we deal with 64, there will be of necessity one mean term in accordance with the nature of the odd;[2] the mean term will correspond to itself because it has no partner; and those on either side of it in turn will correspond to one another until this correspondence ends in the extremes. Unity, for example, will be one sixty-fourth, and 64 the whole, corresponding to unity; 32 is one half, and 2 one thirty-second; 16 is one fourth, and 4 one sixteenth; and 8 the eighth part, with nothing else to correspond to it.

It is the property of all these terms when they are added together 12
successively to be equal to the next in the series, lacking one unit, so

[1] παρώνυμον . . . ἑνὸς τῶν ἐντὸς αὐτοῦ. ἐντὸς in this passage means 'up to,' as, e.g., also in I. 16. 3, and reference is made to the series of even-times even numbers given above. If any number of this series be selected (e.g., 64), its factors will all be numbers that have occurred in the series before 64; each factor contains as many units as one of these numbers (e.g., the fourth part contains 16 units), receives a name, as a factor, from one of them (e.g., 16 is called *fourth* from 4) and reciprocally gives a name to another factor (e.g., 16 gives the name *sixteenth* to 4), κατὰ ἀντιπερίστασιν καὶ ἀμοιβήν. It will facilitate the understanding of this and the following sections to note that when Nicomachus selects for his demonstration any specific even-times even number he considers it as the end of a series; e.g., 64 must be thought of as the series 1, 2, 4, 8, 16, 32, 64.

[2] That is, in accordance with the definition of 'odd.' A unit always 'intervenes' to prevent the division of the odd number into halves; see I. 7. 2.

that of necessity their summation in any way whatsoever will be an odd number,[1] for that which fails by a unit of being equal to an even
13 number is odd. This observation will be of use to us very shortly in the construction of perfect numbers.[2] But to take an example, the terms from unity preceding 256 in the series, when added together, are within 1 of equaling 256, and all the terms before 128, the term immediately preceding, are similarly equal to 128 save for one unit; and to the next terms the sums of those below them are similarly related. Thus unity itself[3] is within one unit of equaling the next term, which is 2, and these two together fail by 1 of equaling the next, and the three together are within 1 of the next in order, and you will find that this goes on without interruption to infinity.

14 This too it is very needful to recall: If the number of terms of the even-times even series dealt with is even, the product of the extremes will always be equal to the product of the means; if there is an odd number of terms, the product of the extremes will be equal to the square of the mean. For, in the case of an even number of terms, 1 times 128 is equal to 8 times 16 and further to 2 times 64 and again to 4 times 32, and this is so in every case; and with an odd number of terms, 1 times 64 equals 2 times 32, and this equals 4 times 16, and this again equals 8 times 8, the mean term alone multiplied by itself.

CHAPTER IX

1 The even-times odd[4] number is one which is by its genus itself even, but is specifically[5] opposed to the aforesaid even-times even. It is a number of which, though it admits of the division into two equal

[1] Substituting in the formula $S = \frac{rl - a}{r - 1}$, the sum of n terms of this series is $\frac{2l - 1}{1}$. The $(n + 1)$st term is $2l$, and $2l - 1$ is one less, as Nicomachus says.

[2] See chapter 16.

[3] Here treated as a member of the even-times even series. Cf. on I. 8. 7.

[4] Theon of Smyrna, p. 25, 19 ff. Hiller, thus defines: "The even-times odd numbers are measured by the dyad and some odd number; upon division into equal halves, they always have odd halves" (*ἀρτιοπέρισσοι δέ εἰσιν οἱ ὑπὸ δυάδος καὶ περισσοῦ οὑτινοσοῦν μετρούμενοι, οἵτινες ἐκ παντὸς περισσὰ μέρη ἔχουσι τὰ ἡμίσεα κατὰ τὴν εἰς ἴσα διαίρεσιν*). Euclid (*Elements*, VII, *Def.* 9) has "An even-times odd number is one that is measured by an even number an odd number of times" (*ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν*).

[5] Read *εἰδικῶς*, with Codd. Cizensis, Monacensis 482 and Hamburgensis, instead of *ἰδικῶς* (Hoche, following G). The word is evidently to be contrasted with *τῷ γένει* and both are logical terms.

halves, after the fashion of the genus common to it and the even-times even, the halves are not immediately divisible into two equals, for example, 6, 10, 14, 18, 22, 26, and the like; for after these have been divided their halves are found to be indivisible.

It is the property of the even-times odd that whatever factor it may be discovered to have is opposite in name to its value,[1] and that the quantity of every part is opposite in value to its name, and that the numerical value of its part never by any means is of the same genus as its name.[2] To take a single example, the number 18, its half, with an even name, is 9, odd in value; its third part, again, with an odd designation, is 6, even in value; conversely, the sixth part is 3 and the ninth part 2; and in other numbers the same peculiarity will be found. 2

It is possibly for this reason that it received such a name, that is, because, although it is even, its halves are at once odd. 3

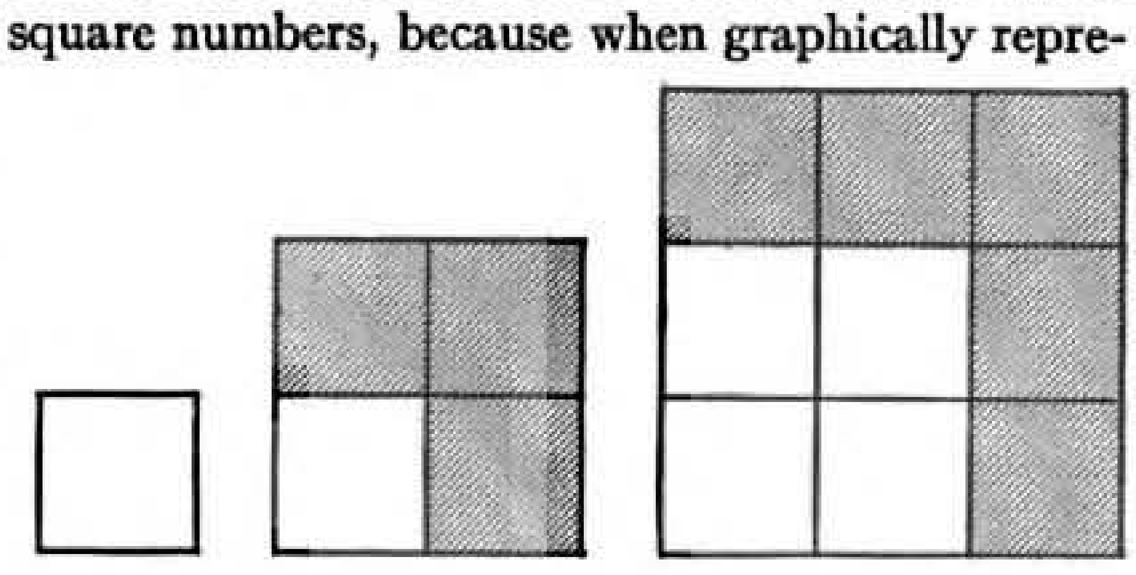
This number is produced from the series beginning with unity, with a difference of 2, namely, the odd numbers, set forth in proper order as far as you like and then multiplied by 2. The numbers produced would be, in order, these: 6, 10, 14, 18, 22, 26, 30, and so on, as far as you care to proceed. The greater terms always differ by 4 from the next smaller ones, the reason for which is that their original basic forms,[3] the odd numbers, exceed one another by 2 and were multiplied by 2 to make this series, and 2 times 2 makes 4. 4

Accordingly, in the natural series of numbers the even-times odd 5

[1] For *δύναμις* in the sense of 'numerical value' (i.e., here, odd or even), cf. I. 8. 7 and the note, as well as Boethius, I. 10: *Accidit autem his quod omnes partes contrarie denominatas habent, quam sunt quantitates ipsarum partium quae denominantur. Neque unquam fieri potest ut quaelibet pars huius numeri eiusdem generis denominationem quantitatemque suscipiat.*

[2] It seems necessary to take the dative (*τῷ . . . ὀνόματι*) with *ὁμογενῆ*, and in that case the reading of C (*ἑαυτοῦ*) is to be preferred to the *αὐτῷ* of G, read by Hoche. Nicomachus is pointing out that in this class of numbers no even factor can have an even name. Hoche's text would be translated "by virtue of the same name," which is not clear.

[3] *γνώμονες*: This term is frequently used in arithmetic with reference to the odd numbers which are added together to make the series of square numbers, because when graphically represented they may be successively affixed to the previous figure in the characteristic form of the gnomon (see the Figure). Nicomachus, however, also employs the term with reference to any set or series of numbers that may be successively added to form a second set or series. Thus the gnomons of the hexagonal numbers are the terms from the natural series from 1 with a common difference of 4. (See II. 11. 1 ff. and II. 13. 6.) We also find the word used in this wider sense by Theon of Smyrna (e.g., p. 37, 11 Hiller) and by Hero of Alexandria (*Def.* 59, p. 21 Hultsch). For a discussion of the origin and use of the term see Cantor, *op. cit.*, vol. I, pp. 161 ff.



numbers will be found fifth[1] from one another, exceeding one another by a difference of 4, passing over three terms, and produced by the multiplication of the odd numbers by 2.

6 They are said to be opposite in properties[2] to the even-times even, because of these the greatest extreme term alone is divisible, while of these former the smallest only proved to be indivisible; and in particular because in the former case the reciprocal arrangement of parts[3] from extremes to mean term or terms makes the product of the former equal to the square or product of the latter; but in this case by the same correspondence and comparison the mean term is one half the sum of the extremes,[4] or if there should be two means, their sum equals that of the two extremes.

CHAPTER X

1 The odd-times even number is the one which displays the third form of the even, belonging in common to both the previously mentioned species like a single mean between two extremes, for in one respect it resembles the even-times even, and in another the even-times odd, and that property wherein it varies from the one it shares with the other, and by that property which it shares with the one it differs from the other.

2 The odd-times even number[5] is an even number which can be

[1] That is, in the Greek manner counting in both the term from which one starts and the last. So the Olympic games, which we would say came every fourth year, were to the Greeks a 'fifth-year festival.'

[2] E.g., 10 can be divided into two parts (5, 5), but neither is divisible by 2; while in the even-times even series each number can be divided, and its parts divided, down to the 'least part,' 1. 'The greater extreme,' speaking of the even-times odd, would be the even-times odd number itself under consideration.

[3] Cf. I. 8. 10, where the reciprocal relation of the factors of the even-times even numbers was treated. Each factor of such a number is itself a term in the even-times even series, of which 1 and the given number are regarded as the extremes (ἀκρότητες). For the relation between the product of the extremes and the square of the mean (or the product of two means), cf. I. 8. 14.

[4] Thus in the even-times odd series

6, 10, 14, 18, 22,

the mean term (14) is $\frac{1}{2}$ the sum of the extremes ($6 + 22 = 28$); and in the series

6, 10, 14, 18, 22, 26,

the sum of the means ($14 + 18 = 32$) equals the sum of the extremes ($6 + 26 = 32$).

[5] Theon of Smyrna, p. 26, 5 Hiller, defines the odd-times even as a number produced by the multiplication of an odd by an even number, which has even halves when it is divided by 2, but on further division has some parts odd, others even (περισσάκις δὲ ἄρτιοί εἰσιν, ὧν ὁ πολλαπλασιασμὸς ἐκ δυεῖν ὡντινωνοῦν περισσοῦ καὶ ἀρτίου γίνεται, καὶ πολλαπλασιασθέντες εἰς ἴσα μὲν ἄρτια

divided into two equal parts, whose parts also can so be divided, and sometimes even the parts of its parts, but it cannot carry the division of its parts as far as unity. Such numbers are 24, 28, 40; for each of these has its own half and indeed the half of its half, and sometimes one is found among them that will allow the halving to be carried even farther among its parts. There is none, however, that will have its parts divisible into halves as far as the naturally indivisible unit.

Now in admitting more than one division, the odd-times even is like the even-times even and unlike the even-times odd; but in that its subdivision never ends with unity, it is like the even-times odd and unlike the even-times even. 3

It alone has at once the proper qualities of each of the former two,[1] and then again properties which belong to neither of them; for of them one had only the highest term divisible, and the other only the smallest indivisible, but this neither; for it is observed to have more divisions than one in the greater term, and more than one indivisible in the lesser. 4

Furthermore, there are in it certain parts whose names are not opposed to their values nor of the opposite genus,[2] after the fashion of the even-times even; and there are also always other parts of a name opposite and contrary in kind to their values, after the fashion of the even-times odd. For example, in 24, there are parts not opposed in name to their values, the fourth part, 6, the half, 12, the sixth, 4, and the twelfth, 2; but the third part, 8, the eighth, 3, and the twenty-fourth, 1, are opposed; and so it is with the rest. 5

This number is produced by a somewhat complicated method, and shows, after a fashion, even in its manner of production, that it is a mixture of both other kinds. For whereas the even-times even is made from even numbers, the doubles from unity to infinity, and the even-times odd from the odd numbers from 3, progressing to infinity, this must be woven together out of both classes, as being common to 6

μέρη δίχα διαιροῦνται, κατὰ δὲ τὰς πλείους διαιρέσεις ἃ μὲν ἄρτια μέρη, ἃ δὲ περισσὰ ἔχουσιν). Euclid's definition (*Elements*, VII, *Def.* 10) is: "The odd-times even number is one that is measured by an odd number an even number of times" (*περισσάκις ἀρτιός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν*).

[1] Cf. I. 9. 6. It is to be observed that Nicomachus in speaking of these numbers conceives of them serially; e.g., to him the even-times even number 16 carries with it the series 1, 2, 4, 8, 16; and so of the others; as 3, 6 (even-times odd), 3, 6, 12, 24 (odd-times even).

[2] Cf. I. 8. 7; 9. 2. 'Of the opposite genus' refers to even and odd. The name is 'contrary to its value' if, e.g., the denominator of the fraction is odd and its value, or amount, even.

7 both. Let us then set forth the odd numbers from 3 by themselves in due order in one series:

3, 5, 7, 9, 11, 13, 15, 17, 19, . . .

and the even-times even, beginning with 4, again one after another in a second series after their own order:

4, 8, 16, 32, 64, 128, 256, . . .

8 as far as you please. Now multiply by the first number of either series — it makes no difference which — from the beginning and in order all those in the remaining series and note down the resulting numbers; then again multiply by the second number of the same series the same numbers once more, as far as you can, and write down the results; then with the third number again multiply the same terms anew, and however far you go you will get nothing but the odd-times even numbers.

9 For the sake of illustration let us use the first term of the series of odd numbers and multiply by it all the terms in the second series in order, thus: 3×4, 3×8, 3×16, 3×32, and so on to infinity. The results will be 12, 24, 48, 96, which we must note down in one line. Then taking a new start do the same thing with the second number, 5×4, 5×8, 5×16, 5×32. The results will be 20, 40, 80, 160. Then do the same thing once more with 7, the third number, 7×4, 7×8, 7×16, 7×32. The results are 28, 56, 112, 224; and in the same way as far as you care to go, you will get similar results.

Odd numbers		3	5	7	9	11	13	15
Even-times even		4	8	16	32	64	128	256
Odd-times even numbers	Breadth	16	24	48	96	192	384	768
		20	40	80	160	320	640	1280
		28	56	112	224	448	896	1792
		36	72	144	288	576	1152	2304
		44	88	176	352	704	1408	2816
		Length						

10 Now when you arrange the products of multiplication by each term in its proper line, making the lines parallel, in marvelous fashion there will appear along the breadth of the table the peculiar property of the

even-times odd, that the mean term is always half the sum of the extremes, if there should be one mean, and the sum of the means equals the sum of the extremes if two. But along the length of the table the property of the even-times even will appear; for the product of the extremes is equal to the square of the mean, should there be one mean term, or their product, should there be two. Thus this one species has the peculiar properties of them both, because it is a natural mixture of them both.

CHAPTER XI

Again, while the odd is distinguished over against the even in classification and has nothing in common with it, since the latter is divisible into equal halves and the former is not thus divisible, nevertheless there are found three species of the odd,[1] differing from one another, of which the first is called the prime and incomposite,[2] that which is 1

[1] There is great disagreement among the ancient authorities upon this classification. In the first place, Nicomachus confines these species to odd numbers, thus securing a threefold classification to balance that of the even numbers (see above). He is followed in this by Iamblichus (p. 26, 18), but Euclid (*Elem.*, VII, *Deff.*, 11–14) and Theon (p. 23, 6 ff. Hiller) make it a classification of both the even and the odd. Nicomachus then divides into (*a*) prime and incomposite; (*b*) secondary and composite; (*c*) that which is absolutely composite but relatively prime. Nesselmann, *op. cit.*, p. 194, points out that the latter two classes are not mutually exclusive, for *b* includes *c*. The difficulty is overcome by Iamblichus, who thus classifies: (*a*) the absolutely prime, which is a priori relatively prime as well; (*b*) the absolutely secondary, which includes as sub-classes the relatively prime and relatively secondary; the two sub-classes are dependent upon the association of terms in specific instances. Euclid (*loc. cit.*) gives definitions of primes, relative primes, composite, and relatively composite numbers. This need not of course imply a strict classification along these lines. Theon, however, seems to understand it as such, and to establish his classification after this model, making his definitions agree with those of Euclid: (*a*) absolutely prime; (*b*) relatively prime; (*c*) absolutely composite; (*d*) relatively composite. The last class does not correspond to any set up by Nicomachus; it consists of numbers like 8 and 9 taken in connection with 6. Cf. T. L. Heath on Euclid, *loc. cit.*, for an extended discussion; also his *History*, vol. I, pp. 72 ff.

[2] Euclid defines a prime number as 'one measured by unity alone' (ὁ μονάδι μόνῃ μετρούμενος), *Elem.*, VII, *Def.* 11. The number 2 satisfies his definition and is also called prime by Aristotle (*Top.*, VIII. 2. 157 a 39). But in Nicomachus prime numbers are a class of *odd* numbers, not of number in general. See Heath on the matter, *Euclid*, II, 284–85. Theon of Smyrna (p. 23, 9 Hiller) defines the 'absolutely prime and incomposite' number as one 'measured by no number but by unity alone' (οἱ ὑπὸ μηδενὸς μὲν ἀριθμοῦ, ὑπὸ μόνης δὲ μονάδος μετρούμενοι). He states that these numbers were sometimes called 'linear' and 'rectilinear' (γραμμικοί, εὐθυμετρικοί) 'because lengths and lines are viewed in one dimension,' and that they are also called 'odd-times odd' (περισσάκις περισσοί). Theon leaves it vague whether he regards the dyad as prime; for after stating that the even numbers are not prime because they are measured by other numbers than unity alone, he says that the dyad is an exception and is therefore called 'odd-like' (καὶ οἱ λοιποὶ ἄρτιοι κατὰ τὰ αὐτὰ ὑπό τινων μειζόνων τῆς μονάδος ἀριθμῶν καταμετροῦνται πλὴν τῆς δυάδος· ταύτῃ γὰρ μόνῃ συμβέβηκεν ὅπερ καὶ ἐνίοις τῶν περισσῶν, τὸ ὑπὸ μονάδος μετρεῖσθαι μόνον. ἅπαξ γὰρ β′, β′· διὸ καὶ περισσοειδὴς εἴρηται ταὐτὸ τοῖς περισσοῖς πεπονθυῖα).

opposed to it the secondary and composite, and that which is midway between both of these and is viewed as a mean among extremes, namely, the variety which, in itself, is secondary and composite, but relatively is prime and incomposite.

2 Now the first species, the prime and incomposite, is found whenever an odd number admits of no other factor save the one with the number itself as denominator,[1] which is always unity; for example, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. None of these numbers will by any chance be found to have a fractional part with a denominator different from the number itself, but only the one with this as denominator, and this part will be unity in each case; for 3 has only a third part, which has the same denominator as the number and is of course unity, 5 a fifth, 7 a seventh, and 11 only an eleventh part, and in all of them these parts are unity.

3 It has received this name because it can be measured only by the number which is first and common to all, unity, and by no other; moreover, because it is produced by no other number combined with itself save unity alone; for 5 is 5×1, and 7 is 7×1, and the others in accordance with their own quantity. To be sure, when they are combined with themselves, other numbers might be produced, originating from them as from a fountain and a root, wherefore they are called 'prime,' because they exist beforehand as the beginnings of the others. For every origin[2] is elementary and incomposite, into which everything is resolved and out of which everything is made, but the origin itself cannot be resolved into anything or constituted out of anything.

CHAPTER XII

1 The secondary, composite number[3] is an odd number, indeed, because it is distinguished as a member of this same class, but it has no

[1] As $\frac{1}{3}$ in the case of 3.

[2] ἀρχή. Cf. the discussion of element (στοιχεῖον), II. 1.

[3] Nicomachus does not admit even numbers into the class of composites, doubtless because he has already exhausted their classification. Theon of Smyrna, however, makes the composite a division of number in general and gives even numbers among his examples. As noted above, he distinguishes the 'absolute composite' numbers that can be measured by some smaller numbers and 'relative composites,' those which are measured by some measure, but are prime to each other, as 8, 6, 9, with the measures 2 and 3. In this connection 1 is not considered a common measure, for, as he states, it is not itself a number but the beginning of number. Euclid, *Elements*, VII, *Def.* 14, defines a composite number as 'one measured by some number' (σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος).

elementary quality, for it gets its origin by the combination of something else. For this reason it is characteristic of the secondary number to have, in addition to the fractional part with the number itself as denominator, yet another part or parts with different denominators, the former always, as in all cases, unity, the latter never unity, but always either that number or those numbers by the combination of which it was produced. For example, 9, 15, 21, 25, 27, 33, 35, 39; each one of these is measured by unity, as other numbers are,[1] and like them has a fractional part with the same denominator as the number itself, by the nature of the class common to them all; but by exception and more peculiarly they also employ a part, or parts, with a different denominator; 9, in addition to the ninth part, has a third part besides; 15 a third and a fifth besides a fifteenth; 21 a seventh and a third besides a twenty-first, and 25, in addition to the twenty-fifth, which has as a denominator 25 itself, also a fifth, with a different denominator.

It is called secondary, then, because it can employ yet another meas- 2
ure along with unity, and because it is not elementary, but is produced by some other number combined with itself or with something else; in the case of 9, 3; in the case of 15, 5 or, by Zeus, 3; and those following in the same fashion. And it is called composite for this, or some such, reason: that it may be resolved into those numbers out of which it was made, since it can also be measured by them. For nothing that can be broken down is incomposite, but by all means composite.

CHAPTER XIII

Now while these two species of the odd are opposed to each other a 1
third one[2] is conceived of between them, deriving, as it were, its specific form from them both, namely the number which is in itself secondary and composite, but relatively to another number is prime and incomposite. This exists when a number, in addition to the common measure, unity, is measured by some other number and is therefore able to admit of a fractional part, or parts, with denominator other than the number itself, as well as the one with itself as denominator.

[1] That is, the primes.

[2] Theon of Smyrna (p. 24, 8 ff.) has this class, which he calls 'prime to one another and not absolutely prime,' and he points out that 8, 9, and 10 are prime to the 'absolute primes.' Euclid defines relatively prime numbers as 'those that are measured only by unity as the common measure' (πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ), *Elements*, VII, *Def.* 13.

When this is compared with another number of similar properties, it is found that it cannot be measured by a measure common to the other, nor does it have a fractional part with the same denominator as those in the other. As an illustration, let 9 be compared with 25. Each in itself is secondary and composite, but relatively to each other they have only unity as a common measure, and no factors in them have the same denominator, for the third part in the former does not exist in the latter nor is the fifth part in the latter found in the former.

2 The production of these numbers is called by Eratosthenes the 'sieve,' because we take the odd numbers mingled together and indiscriminate and out of them by this method of production separate, as by a kind of instrument or sieve, the prime and incomposite by themselves, and the secondary and composite by themselves, and find the mixed class by themselves.

3 The method of the 'sieve' is as follows. I set forth all the odd numbers in order, beginning with 3, in as long a series as possible, and then starting with the first I observe what ones it can measure, and I find that it can measure the terms two places apart, as far as we care to proceed. And I find that it measures not as it chances and at random, but that it will measure the first one, that is, the one two places removed, by the quantity of the one that stands first in the series, that is, by its own quantity, for it measures it 3 times; and the one two places from this by the quantity of the second in order, for this it will measure 5 times; and again the one two places further on by the quantity of the third in order, or 7 times, and the one two places still farther on by the quantity of the fourth in order, or 9 times, and so *ad infinitum* in the same way.

4 Then taking a fresh start I come to the second number and observe what it can measure, and find that it measures all the terms four places apart, the first by the quantity of the first in order, or 3 times; the second by that of the second, or 5 times; the third by that of the third, or 7 times; and in this order *ad infinitum.*

5 Again, as before, the third term 7, taking over the measuring function, will measure terms six places apart, and the first by the quantity of 3, the first of the series, the second by that of 5, for this is the second number, and the third by that of 7, for this has the third position in the series.

6 And analogously throughout, this process will go on without in-

terruption, so that the numbers [1] will succeed to the measuring function in accordance with their fixed position in the series; the interval separating terms measured is determined by the orderly progress of the even numbers from 2 to infinity, or by the doubling of the position in the series occupied by the measuring term, and the number of times a term is measured is fixed by the orderly advance of the odd numbers in series from 3.

Now if you mark the numbers with certain signs, you will find that 7
the terms which succeed one another in the measuring function neither measure all the same number — and sometimes not even two will measure the same one — nor do absolutely all of the numbers set forth submit themselves to a measure, but some entirely avoid being measured by any number whatsoever, some are measured by one only, and some by two or even more. Now these that are not measured at 8
all, but avoid it, are primes and incomposites, sifted out as it were by a sieve; those measured by only one measure in accordance with its own quantity [2] will have but one fractional part with denominator different from the number itself, in addition to the part with the same denominator; and those which are measured by one measure only, but in accordance with the quantity of some other number than the measure and not its own, or are measured by two measures at the same time, will have several fractional parts [3] with other denominators besides the one with the same as the number itself; these will be secondary and composite.

The third division,[4] the one common to both the former, which is 9
in itself secondary and composite but primary and incomposite in relation to another, will consist of the numbers produced when some prime and incomposite number measures them in accordance with its

[1] It is generally assumed (as by Heath, *History*, vol. I, p. 100) that in the 'sieve of Eratosthenes' only the odd prime numbers take on successively the measuring function, and indeed this is all that is necessary, for, e.g., 9 is a multiple of 3 and all its multiples are likewise multiples of 3. The text, however, seems to imply that all the odd numbers should be used, although perhaps Nicomachus did not intend that he should be so strictly interpreted.

[2] Reading ἑαυτοῦ (instead of ἑαυτῶν) with G_1. The numbers referred to are the squares of odd prime numbers. 9, e.g., has ninths, of course, and as it is measured by 3, it will also have thirds, the denomination of which is derived from 3.

[3] Thus, if a, b, m are odd numbers greater than unity, and $m = ab$, m is measured by a in the quantity of b, and vice versa, and m will have the factors $\frac{m}{a}$ and $\frac{m}{b}$ named respectively from b and a.

[4] Nicomachus evidently contemplates admitting into this division only the squares of prime odd numbers, though numbers like 15 (3×5) and 77 (7×11), when compared, would satisfy his requirements equally well.

own quantity, if one thus produced be compared to another of similar origin. For example, if 9, which was produced by 3 measuring by its own quantity, for it is 3 times 3, be compared with 25, which was produced from 5 measuring by its own quantity, for it is 5 times 5, these numbers have no common measure except unity.

10 We shall now investigate how we may have a method[1] of discerning whether numbers are prime and incomposite, or secondary and composite, relatively to each other, since of the former unity is the common measure, but of the latter some other number also besides unity; and what this number is.

11 Suppose there be given us two odd numbers and some one sets the problem and directs us to determine whether they are prime and incomposite relatively to each other or secondary and composite, and if they are secondary and composite what number is their common measure. We must compare the given numbers and subtract the smaller from the larger as many times as possible; then after this subtraction subtract in turn from the other, as many times as possible; for this changing about and subtraction from one and the other in turn will necessarily end either in unity or in some one and the same
12 number, which will necessarily be odd. Now when the subtractions terminate in unity they show that the numbers are prime and incomposite relatively to each other; and when they end in some other number, odd in quantity and twice produced,[2] then say that they are secondary and composite relatively to each other, and that their common measure is that very number which twice appears.

For example, if the given numbers were 23 and 45, subtract 23 from 45, and 22 will be the remainder; subtracting this from 23, the remainder is 1, subtracting this from 22 as many times as possible you will end with unity. Hence they are prime and incomposite to one another, and unity, which is the remainder, is their common measure.

13 But if one should propose other numbers, 21 and 49, I subtract the smaller from the larger and 28 is the remainder. Then again I sub-

[1] This mode of determining common factors is found in Euclid (VII. 1; X. 2) and is commonly termed the Euclidean method of finding the greatest common divisor of numbers.

[2] Reference to the second example following will show that the term 'produced twice' (διφορούμενον) by this process is the final subtrahend, which is equal to the final remainder. Boethius, I. 18, is somewhat more explicit in describing the operation: *Datis enim duobus numeris inaequalibus, auferre de maiore minorem oportebit, et qui relictus fuerit, si maior est, auferre ex eo rursus minorem, si vero minor fuerit, eum ex reliquo maiore detrahere atque hoc eo usque faciendum, quoad unitas ultima vicem retractionis impediat, aut aliquis numerus, impar necessario, si utrique numeri impares proponantur; sed eum, qui relinquitur, numerum sibi ipsi videbis aequalem.*

tract the same 21 from this, for it can be done, and the remainder is 7. This I subtract in turn from 21 and 14 remains; from which I subtract 7 again, for it is possible, and 7 will remain. But it is not possible to subtract 7 from 7; hence the termination of the process with a repeated 7 has been brought about, and you may declare the original numbers 21 and 49 secondary and composite relatively to each other, and 7 their common measure in addition to the universal unit.

CHAPTER XIV

To make again a fresh start, of the simple even numbers, some are superabundant, some deficient, like extremes set over against each other, and some are intermediary between them and are called perfect. 1
Those which are said to be opposites to one another, the superabundant and deficient, are distinguished from one another in the relation of inequality[1] in the directions of the greater and the less; for apart from these no other form of inequality could be conceived, nor could evil,[2] disease, disproportion, unseemliness, nor any such thing, save in terms of excess or deficiency. For in the realm of the greater[3] there arise excesses, overreaching, and superabundance, and in the less need, deficiency, privation, and lack; but in that which lies between the greater and the less, namely, the equal, are virtues, wealth, moderation, propriety, beauty, and the like, to which the aforesaid form of number, the perfect, is most akin. 2

Now the superabundant number[4] is one which has, over and above the factors which belong to it and fall to its share, others in addition, just as if an animal should be created with too many parts or limbs, 3

[1] Nicomachus refers here to the general relation of inequality which is opposed to equality (I. 17. 2) as one of the primary divisions of relative number. Technically equality and inequality are *σχέσεις*, 'relations,' the term applied here to inequality as it is to equality in I. 17. 4; subclasses of the unequal, first the greater and the less (cf. I. 17. 6) and then the specific ratios (*αἱ δέκα ἀριθμητικαὶ σχέσεις*, I. 23. 4) are also called 'relations.' For the notion *σχέσις* and the kindred term *λόγος*, 'ratio,' cf. on II. 21. 2.

[2] According to Aristotle virtue is the mean, and vice is excess or deficiency. Cf. *Eth. Nic.*, II. 6. 1106 b 33, *καὶ διὰ ταῦτ' οὖν τῆς μὲν κακίας ἡ ὑπερβολὴ καὶ ἡ ἔλλειψις, τῆς δ' ἀρετῆς ἡ μεσότης.* These are just the varieties that Nicomachus assigns to inequality.

[3] Cf. Arist., *Eth. Nic.*, as quoted in the preceding note and 1106 b 24: *ἡ δ' ἀρετὴ περὶ πάθη καὶ πράξεις ἐστίν, ἐν οἷς ἡ μὲν ὑπερβολὴ ἁμαρτάνεται καὶ ἡ ἔλλειψις [ψέγεται], τὸ δὲ μέσον ἐπαινεῖται καὶ κατορθοῦται· ταῦτα δ' ἄμφω τῆς ἀρετῆς. μεσότης τις ἄρα ἐστὶν ἡ ἀρετή, στοχαστική γε οὖσα τοῦ μέσου.*

[4] *ὑπερτελὴς ἀριθμός.* Theon of Smyrna, pp. 45, 10 ff., 46, 4, includes this class, but calls them *ὑπερτέλειοι.*

with ten tongues, as the poet says,[1] and ten mouths, or with nine lips, or three rows of teeth, or a hundred hands, or too many fingers on one hand. Similarly if, when all the factors in a number are examined and added together in one sum, it proves upon investigation that the number's own factors exceed the number itself, this is called a superabundant number, for it oversteps the symmetry which exists between the perfect and its own parts. Such are 12, 24, and certain others, for 12 has a half, 6, a third, 4, a fourth, 3, a sixth, 2, and a twelfth, 1, which added together make 16, which is more than the original 12; its
4 parts, therefore, are greater than the whole itself. And 24 has a half, a third, fourth, sixth, eighth, twelfth, and twenty-fourth, which are 12, 8, 6, 4, 3, 2, 1. Added together they make 36, which, compared to the original number, 24, is found to be greater than it, although made up solely of its factors. Hence in this case also the parts are in excess of the whole.[2]

CHAPTER XV

1 The deficient number[3] is one which has qualities the opposite of those pointed out, and whose factors added together are less in comparison than the number itself. It is as if some animal should fall short of the natural number of limbs or parts, or as if a man should have but one eye, as in the poem, "And one round orb was fixed in his brow";[4] or as though one should be one-handed, or have fewer than five fingers on one hand, or lack a tongue, or some such member. Such a one would be called deficient and so to speak maimed, after the peculiar fashion of the number whose factors are less than itself, such as 8 or 14. For 8 has the factors half, fourth, and eighth, which are 4, 2, and 1, and added together they make 7, and less than the original number. The parts, therefore, fall short of making up the
2 whole. Again, 14 has a half, a seventh, a fourteenth, 7, 2, and 1, respectively; and all together they make 10, less than the original number. So this number also is deficient in its parts, with respect to making up the whole out of them.

[1] The reference is to Homer's description of Scylla, *Odyssey*, XII. 85 ff.
[2] See also L. E. Dickson, *History of the Theory of Numbers*, Washington, 1909, vol. I, chapter I.
[3] Cf. Theon, p. 46, 9 ff.
[4] The Cyclops. Cf. Hesiod, *Theogony*, 145.

CHAPTER XVI

While these two varieties are opposed after the manner of extremes, 1
the so-called perfect number [1] appears as a mean, which is discovered to be in the realm of equality, and neither makes its parts greater than itself, added together, nor shows itself greater than its parts, but is always equal to its own parts. For the equal is always conceived of as in the mid-ground between greater and less, and is, as it were, moderation between excess and deficiency, and that which is in tune, between pitches too high and too low.

Now when a number, comparing with itself the sum and combina- 2
tion of all the factors whose presence it will admit, neither exceeds them in multitude nor is exceeded by them, then such a number is properly said to be perfect, as one which is equal to its own parts. Such numbers are 6 and 28; for 6 has the factors half, third, and sixth, 3, 2, and 1, respectively, and these added together make 6 and are equal to the original number, and neither more nor less. Twenty-eight has the factors half, fourth, seventh, fourteenth, and twenty-eighth, which are 14, 7, 4, 2 and 1; these added together make 28, and so neither are the parts greater than the whole nor the whole greater than the parts, but their comparison is in equality, which is the peculiar quality of the perfect number.

It comes about that even as fair and excellent things are few 3
and easily enumerated, while ugly and evil ones are widespread, so also the superabundant and deficient numbers are found in great multitude and irregularly placed — for the method of their discovery is irregular — but the perfect numbers are easily enumerated and arranged with suitable order; for only one is found among the units, 6, only one other among the tens, 28, and a third in the rank of the hundreds, 496 alone, and a fourth within the limits of the thousands, that is, below ten thousand, 8,128.[2] And it is their

[1] Euclid's definition, *Elem.*, VII. 22, is: "A perfect number is one that is equal to its own parts." Similarly Theon of Smyrna defines it, p. 45, 10. See Heath, *History*, vol. I, p. 74.

[2] Subsequent investigation has determined seven more perfect numbers. T. L. Heath, *History*, vol. I, pp. 74–75, and L. E. Dickson, *op. cit.*, vol. I, chapter I.

I.	$2(2^2-1)=6$.	VII.	$2^{18}(2^{19}-1)$.
II.	$2^2(2^3-1)=28$.	VIII.	$2^{30}(2^{31}-1)$.
III.	$2^4(2^5-1)=496$.	IX.	$2^{60}(2^{61}-1)$, with 37 digits.
IV.	$2^6(2^7-1)=8{,}128$.	X.	$2^{88}(2^{89}-1)$.
V.	$2^{12}(2^{13}-1)=33{,}550{,}336$.	XI.	$2^{126}(2^{127}-1)$.
VI.	$2^{16}(2^{17}-1)=8{,}568{,}910{,}416$.		

Theon, in his notice of the perfect numbers, mentions only 6 and 28.

accompanying characteristic[1] to end alternately in 6 or 8, and always to be even.

4 There is a method of producing them,[2] neat and unfailing, which neither passes by any of the perfect numbers nor fails to differentiate any of those that are not such, which is carried out in the following way.

You must set forth the even-times even numbers from unity, advancing in order in one line, as far as you please: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024, 2,048, 4,096. . . . Then you must add them together, one at a time, and each time you make a summation observe the result to see what it is. If you find that it is a prime, incomposite number, multiply it by the quantity of the last number added, and the result will always be a perfect number. If, however, the result is secondary and composite, do not multiply, but add the next and observe again what the resulting number is; if it is secondary and composite, again pass it by and do not multiply; but add the next; but if it is prime and incomposite, multiply it by the last term added, and the result will be a perfect number; and so on to infinity. In similar fashion you will produce all the perfect numbers in succession, overlooking none.

For example, to 1 I add 2, and observe the sum, and find that it is 3, a prime and incomposite number in accordance with our previous demonstrations; for it has no factor with denominator different from the number itself,[3] but only that with denominator agreeing. Therefore I multiply it by the last number to be taken into the sum, that is, 2; I get 6, and this I declare to be the first perfect number in actuality,[4] and to have those parts which are beheld in the numbers of which it is composed. For it will have unity[5] as the factor with denominator the same as itself, that is, its sixth part; and 3 as the half, which is seen in 2, and conversely 2 as its third part.

5 Twenty-eight likewise is produced by the same method when another number, 4, is added to the previous ones. For the sum of the three, 1,

[1] Cf. Boethius, I. 20: *Et semper hi numeri duobus paribus terminantur, 6 et 8, et semper alternatim in hos numeros summarum fine provenient.* See p. 52.

[2] Euclid (IX. 36) and Theon of Smyrna (p. 45, 14 ff. Hiller) report this method of discovering perfect numbers.

[3] That is, $\frac{1}{3}$.

[4] 'Actual' (ἐνεργείᾳ) is here specified because in sect. 8 Nicomachus designates 1 as a potential (δυνάμει) perfect number. On his use of Aristotelian terminology, as here, see p. 35.

[5] Boethius, I. 20, speaking of this matter says of 6: *habet unam quidem a se denominatam partem, id est, sextam, 3 vero medietatem secundum dualitatem, at vero 2 secundum coacervationem, id est, secundum ternarium, quoniam coacervati 3 multiplicati sunt.*

2, and 4, is 7, and is found to be prime and incomposite, for it admits only the factor with denominator like itself, the seventh part. Therefore I multiply it by the quantity of the term last taken into the summation, and my result is 28, equal to its own parts, and having its factors derived from the numbers already adduced, a half corresponding to 2; a fourth, to 7; a seventh, to 4; a fourteenth to offset the half; and a twenty-eighth, in accordance with its own nomenclature, which is 1 in all numbers.

When these have been discovered, 6 among the units and 28 in the 6
tens, you must do the same to fashion the next. Again add the next 7
number, 8, and the sum is 15. Observing this, I find that we no longer have a prime and incomposite number, but in addition to the factor with denominator like the number itself,[1] it has also a fifth and a third, with unlike denominators. Hence I do not multiply it by 8, but add the next number, 16, and 31 results. As this is a prime, incomposite number, of necessity it will be multiplied, in accordance with the general rule of the process, by the last number added, 16, and the result is 496, in the hundreds; and then comes 8,128 in the thousands, and so on,[2] as far as it is convenient for one to follow.

Now unity is potentially a perfect number, but not actually; for 8
taking it from the series as the very first I observe what sort it is, according to the rule, and find it prime and incomposite; for it is so in very truth,[3] not by participation like the rest, but it is the primary

[1] $\frac{1}{15}$.

[2] Thomas Taylor (*Theoretic Arithmetic*, p. 33) gives the following table, showing how the perfect numbers may be formed by Nicomachus's method:

Evenly even numbers:
1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024, 2,048, 4,096;
Odd numbers produced by adding the above:
1, 3, 7, 15′, 31, 63′, 127, 255′, 1,023′, 2,047′, 4,095′, 8,191′;
Perfect numbers:
1, 6, 28, 496, 8,128.

The odd numbers which are not prime, and hence cannot be used to make perfect numbers, are marked with an accent.

[3] This statement is to be understood in the light of Nicomachus's essentially Pythagorean view of arithmetic, and with it should be compared II. 17. 2 and incidental remarks elsewhere, e.g., II. 17. 4, 5; 18. 1, 4 (end); 20. 2; 1. 1 (end), etc. In II. 17. 2 it is stated that 'sameness' is found fundamentally in the monad, and most of the other passages cited bring out the principle that the odd numbers and the squares participate in 'sameness' indirectly, through the monad, the monad being that which 'determines the specific form' of the odd numbers (εἰδοποιεῖν), and the latter in turn acting as the bases (γνώμονες) of the squares. Likewise 'otherness' inheres fundamentally in the dyad, and is hence conveyed secondarily, as it were, into the even numbers and the heteromecic numbers. A similar construction may be placed on the passage at hand; the only difference is that 'primeness,' instead of 'sameness,' is in question. The monad is per se prime,

9 number of all, and alone incomposite. I multiply it, therefore, by the last term taken into the summation, that is, by itself, and my re-
10 sult is 1; for 1 times 1 equals 1. Thus unity is perfect potentially; for it is potentially equal to its own parts, the others actually.

CHAPTER XVII

1 Now that we have given a preliminary systematic account of absolute quantity we come in turn to relative quantity.[1]

2 Of relative quantity, then, the highest generic divisions are two, equality and inequality; for everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these.

3 Now the equal is seen, when of the things compared one neither exceeds nor falls short in comparison with the other, for example, 100 compared with 100, 10 with 10, 2 with 2, a mina with a mina, a talent with a talent, a cubit with a cubit, and the like, either in bulk, length,
4 weight, or any kind of quantity. And as a peculiar characteristic, also, this relation[2] is of itself not to be divided or separated, as being most elementary, for it admits of no difference. For there is no such thing as this kind of equality and that kind, but the equal exists in
5 one and the same manner. And that which corresponds to an equal thing, to be sure, does not have a different name from it, but the same;

and other prime numbers are secondarily or by participation prime, for they are combinations of the monad (cf. I. 11. 3), though of no other numbers. The monad on the other hand can be broken up into no smaller components and is therefore elementary. It is further to be noted that the word for 'prime' (πρῶτος) is slightly ambiguous, and even in this specialized use of the word there must be a suggestion of the original sense, 'first.' The monad is obviously 'first' in a higher degree than any prime number.

[1] According to I. 3. 1, this subject belongs to music rather than to arithmetic. Cf. p. 114.

[2] That is, a thing of one class can never be said to be equal to a thing of a different class. Nicomachus does not state this principle in its broadest form, namely, that it is impossible to establish any ratio between objects of entirely different classes. The latter is the form in which Theon of Smyrna, p. 73, 16 ff., puts the matter, following, as he says, Adrastus. (See above, p. 41.) Nicomachus, however, demonstrates elaborately in I. 23. 6 ff. and II. 1 and 2 the proposition that the relation of equality is the element of all ratio, so that, if the connecting link be supplied for him, it may be said that he implies that only homogeneous things may have a ratio. Theon's statement is as follows: "The ratio of analogy between two homogeneous terms is their definite relation (ποιὰ σχέσις) to one another; e.g., double or triple. For as to the relation between unlike things, Adrastus says that it cannot be known; e.g., a cubit and a mina, a *choenix* and a *kotyle*, 'white' and 'sweet' or 'warm,' these things cannot be brought together and compared. But homogeneous things may be, e.g., lengths with lengths, surfaces with surfaces, solids with solids, weights with weights, . . . and whatever things are of the same genus or species and therefore have some mutual relation."

like 'friend,' 'neighbor,' 'comrade,' so also 'equal'; for it is equal to an equal.

The unequal, on the other hand, is split up by subdivisions, and one part of it is the greater, the other the less, which have opposite names and are antithetical to one another in their quantity and relation. For the greater is greater than some other thing, and the less again is less than another thing in comparison, and their names are not the same, but they each have different ones, for example, 'father' and 'son,' 'striker' and 'struck,' 'teacher' and 'pupil,' and the like. 6

Moreover, of the greater, separated by a second subdivision into five species,[1] one kind is the multiple, another the superparticular, another the superpartient, another the multiple superparticular, and another the multiple superpartient. And of its opposite, the less, there arise similarly by subdivision five species, opposed to the foregoing five varieties of the greater, the submultiple, subsuperparticular, subsuperpartient, submultiple-superparticular, and submultiple-superpartient; for as whole answers to whole, smaller to greater, so also the varieties correspond, each to each, in the aforesaid order, with the prefix sub-. 7 8

[1] The terms here used are adapted from Boethius's translations and are employed in Thomas Taylor's *Theoretic Arithmetic*. The present classification was no doubt the ordinary scientific one. Theon (pp. 74, 20 ff.; 76, 1 ff.), however, gives two different classifications, of which the latter is like that of Nicomachus, except that Theon adds the unnecessary class οἱ οὐδέτεροι. In p. 74, 20 ff. he divides ratios first into greater, less, and equal, and then the greater into multiples, superparticulars and 'those of neither class' (οὐδέτεροι), the less into submultiples, subsuperparticulars and 'those of neither class.' It may be noted that in this context οὐδέτεροι is properly used, and it might cover ratios of all the classes mentioned by Nicomachus other than those which Theon specifically includes. He proceeds (p. 74, 23 ff.) to enumerate the members of these classes which are also 'concords,' συμφωνίαι, in music, citing as οὐδέτεροι the ratios 9:8 (i.e., the 'tone') and 256:243 (i.e., the 'limma'), which, he says, are the 'beginnings' of concord and are therefore neither themselves concords nor yet outside of concord. Next (p. 75, 17) he goes on to say that there are, however, certain other ratios spoken of *in arithmetic*, with which he will deal in due time; besides the ones given, 'also superpartients, multiple superpartients and still others.' These he enumerates, as has been said, in his final statement (p. 76, 1 ff.) which agrees with Nicomachus save for the inclusion of the οὐδέτεροι; he takes pains to tell us that this is '*the arithmetical tradition*' (κατὰ τὴν ἀριθμητικὴν παράδοσιν) and is the classification of Adrastus. It would seem fair to conclude, then, that the former classification is the *musical* tradition, and was not taken from Adrastus. Perhaps he has carried over from the 'musical' list the οὐδέτεροι; it is not suitable in the second list, as Hiller notes (see his critical note). He uses this class to cover 'the ratio of number to number' (a direct reference to *Timaeus*, 36 B) in p. 80, 7, i.e., the 'limma,' as before. Compare with Nicomachus's list also Johannes Pediasimus, *Geometria*, in *Neue Jahrb. f. Phil. u. Paed.*, vol. XCII, pp. 366 ff. (f. 43 b of the Munich MS there cited).

CHAPTER XVIII

1 Once more, then; the multiple[1] is the species of the greater first and most original by nature, as straightway we shall see, and it is a number which, when it is observed in comparison with another, contains the whole of that number more than once. For example, compared with unity, all the successive numbers beginning with 2 generate in their proper order the regular forms of the multiple; for 2, in the first place, is and is called the double, 3 triple, 4 quadruple, and so on; for 'more than once' means twice, or three times, and so on in succession as far as you like.

2 Answering to this is the submultiple, which is itself primary in the smaller division of inequality. It is the number which, when it is compared with a larger, is able to measure it completely more than once, and 'more than once' starts with twice and goes on to infinity.

3 If then it measures the larger number that is being compared twice only, it is properly called the subdouble,[2] as 1 is of 2; if thrice, subtriple, as 1 of 3; if four times, subquadruple, as 1 of 4, and so on in succession.

4 While each of these, the multiple and the submultiple, is generically infinite, the varieties by subdivision and the species also are observed naturally to make an infinite series. For the double, beginning with 2, goes on through all the even numbers, as we select alternate numbers out of the natural series; and these will be called doubles in comparison with the even and odd numbers successively placed beginning

5 with unity. All the numbers[3] from the beginning two places apart, and third in order, are triples, for example, 3, 6, 9, 12, 15, 18, 21, 24. It is their property to be alternately odd and even, and they themselves in the regular series from unity are triples of all the numbers in succession as far as one wishes to go on with the process.

6 The quadruples are those in the fourth places, three apart, for instance, 4, 8, 12, 16, 20, 24, 28, 32, and so on. These are the quadruples of the regular series of numbers from unity going on as far as

[1] Theon's definition (p. 76, 8 Hiller) is: "It is the multiple ratio when the greater term contains the smaller more than once, i.e., when the greater term is exactly measured by the smaller with no remainder." Euclid (VII, *Def.* 5) has: "A greater number is multiple of the less when it is measured by the less" (πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάττονος ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος); the same definition as Euclid's is given by Hero of Alexandria, *Definition* 121, ed. Hultsch.

[2] Nicomachus more often uses the terms half, third, etc., for these fractions.

[3] That is, from the natural series.

one finds it convenient to follow. It belongs to them all to be even; for one needs only to take the alternate terms out of the even numbers already selected. Thus necessarily it is true that the even numbers, with no further designation,[1] are all doubles, the alternate ones quadruples, those two places apart sextuples, and those three places apart octuples, and this series will go on, on this same analogy, indefinitely.

The quintuples will be seen to be those four places apart, placed 7
fifth from one another,[2] and themselves the quintuples of the successive numbers beginning with unity. Alternately they are odd and even, like the triples.

CHAPTER XIX

The superparticular,[3] the second species of the greater both naturally 1
and in order, is a number that contains within itself the whole of the number compared with it, and some one factor of it besides.

If this factor is a half, the greater of the terms compared is called 2
specifically[4] sesquialter, and the smaller subsesquialter; if it is a third, sesquitertian and subsesquitertian; and as you go on throughout it will always thus agree, so that these species also will progress to infinity, even though they are species of an unlimited genus.

For it comes about that the first species, the sesquialter ratio, has as its consequents[5] the even numbers in succession from 2, and no other at all, and as antecedents the triples in succession from 3, and no other.
These must be joined together regularly, first to first, second to second, 3
third to third — 3 : 2, 6 : 4, 9 : 6, 12 : 8 — and the analogous numbers to the ones corresponding to them in position.

[1] τοῖς ἁπλῶς ἀρτίοις: that is, numbers not otherwise characterized than as even (as opposed, e.g., to the *alternately occurring* even numbers, or to even-times even numbers).

[2] The reference is now to the 'natural series' 1, 2, 3, 4, etc., of which the author was speaking in sections 4 and 5 before he digressed in section 6 to point out how the even multiples are placed in the series of even numbers.

[3] Theon's definition (p. 76, 21 ff.) is: "It is the superparticular ratio, when the greater term contains the less once and some one part of the less, i.e., when the difference between the greater and the less is such as to be a factor of the less."

[4] καλεῖται . . . εἰδικῶς: It is to be noted that this is a technical expression of logic. The genus in this case is the 'superparticular,' and the 'sesquialter' is the species. The superparticular was itself treated above (I. 17. 7) as a species of the genus 'greater inequality.' Boethius here, and generally, misses the technical force of γενικῶς and εἰδικῶς in Nicomachus, and simply omits them in his translation.

[5] The words translated 'antecedent' and 'consequent' (πρόλογος, ὑπόλογος) mean respectively the larger and the smaller terms in a ratio between unequal quantities. Boethius, I. 24, adopts the translations *duces* and *comites* (*Voco autem maiores numeros duces, minores comites*).

4 If we care to investigate the second species of the superparticular, the sesquitertian (for the fraction naturally following after the half is the third), we shall have this definition of it — a number which contains the whole of the number compared, and a third of it in addition to the whole. We may have examples of it, in the proper order, in the successive quadruples beginning with 4 joined to the triples from 3, each term with the one in the corresponding position in the series,
5 for example, 4 : 3, 8 : 6, 12 : 9, and so on to infinity. It is plain that that which corresponds to the sesquitertian but is called, with the prefix sub-, subsesquitertian, is the number, the whole of which is contained and a third part in addition, for example, 3 : 4, 6 : 8, 9 : 12, and the similar pairs of numbers in the same position in the series.

6 And we must observe the never-failing corollary of all this, that the first forms in each series, the so-called root numbers,[1] are next to one another in the natural series; the next after the root-forms show an interval of only one number; the third two; the fourth three;
7 the fifth four; and so on, as far as you like. Furthermore, that the fraction after which each of the superparticulars[2] is named is seen in the lesser of the root numbers, never in the greater.

8 That by nature and by no disposition of ours the multiple is a more elementary and an older form than the superparticular we shall shortly learn, through a somewhat intricate process. And here, for a simple

[1] πυθμένες: properly 'stock' but here translated 'root.' πυθμήν is the technical designation for that one in a series of equal ratios which is expressed in the lowest terms; in the words of Theon of Smyrna, p. 80, 15 ff.: "Of all the ratios grouped in one species (e.g., double sesquialter, etc.) those that are expressed in the smallest numbers and numbers prime to one another are called primary among those bearing the same ratio and roots (πυθμένες) of those of the same species." πυθμήν is so used by Plato in the famous passage on the marriage number (*Rep.*, 546 B–C). Apollonius of Pergae used the term πυθμήν in a somewhat similar way, to designate the units which serve as the 'stock' in numbers consisting of those units multiplied by 10 or its powers; thus 5 is the πυθμήν of 50, 500, 5,000, etc. See Cantor, *op. cit.*, vol. I, p. 347. Another use of the word πυθμήν is described by Heath, *History*, vol. I, p. 116. In the present case, if the sesquialters are derived from the double and the triple series,

2 4 6 8 10, etc.,
3 6 9 12 15, etc.,

as described above, the 'root sesquialter' is the ratio 3 : 2, the 'second from the root numbers' is the ratio 6 : 4, etc. Boethius, I. 25, observes that the number of the 'intervening numbers,' of which Nicomachus is here speaking, is always one less than the number designating the order of the ratio in the series. E.g., in the third ratio of the series above, 9 : 6, there are two intervening numbers between 9 and 6 (8 and 7).

[2] The number giving the name to each variety of superparticular (e.g., in the case of the sesqui*tertian*, one and one *third*, ἐπί τ ρ ι τ ο ς) is to be observed always in the first instance of that ratio, that is, in the 'root numbers' explained in the preceding note; and more specifically this number is always the smaller of the two in each 'root ratio,' e.g., 2 in 3 : 2 (whence sesqui*alter*, ἡ μι όλιος).

demonstration, we must prepare in regular and parallel lines the multiples specified above, according to their varieties, first the double in one line, then in a second the triple, then the quadruple in a third, and so on as far as the tenfold multiples, so that we may detect their order and variety, their regulated progress, and which of them is naturally prior, and indeed other corollaries delightful in their exactness. Let the diagram be as follows: 9

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Let there be set forth in the first row the natural series from unity, 10
and then in order those species of the multiple which we were bidden to insert.

Now then in comparison with the first rows beginning with unity, 11
if we read both across and up and down in the form of the letter gamma,[1] the next rows both ways, themselves in the form of a gamma, beginning with 4, are multiples according to the first form of the multiple, for they are doubles. The first differs by unity from the first, the second from the second by 2, the third from the third by 3, the next by 4, those following by 5, and you will find that this follows throughout.

The third rows in both directions from 9, their common origin, will be the triples of the terms in that same first row according to the second form of the multiple; the cross-lines like the letter chi,[2] ending in the

[1] The top row and the left-hand column of the table, which we are directed to use, meeting at 1 in a right angle, present the form of the Greek capital letter gamma, Γ. With the terms in these series we compare those of row 2 and column 2, which meet in the term 4 and make the same figure, a Γ; but these second rows are regarded, not as ending with 4, but as continuing to the terms 2 in the first row and the first column, as Nicomachus's immediately following observation shows and as Boethius (I. 27) interpreted him. The first term of the first series, 1, is surpassed by the first term of the second, 2, by 1, and so on; cf. section 6 above.

[2] Row 3 and column 3, which we now compare with row 1 and column 1, meet at the term 9 at right angles and run beyond 9 to 3 in both the first row and the first column. They therefore present the appearance of the Greek capital letter chi (X), the two lines of which in inscriptional forms often make right angles.

12 term 3 in either direction, are to be taken into consideration. The difference, for these numbers, will progress after the series of the even numbers, being 2 for the first, 4 for the next, 6 for the third; and this difference nature has of her own accord [1] interpolated for us between these rows that are being examined, as is evident in the diagram.

13 The fourth row, whose common origin in both directions is 16, and whose cross-lines [2] end with the terms 4, exhibits the third species of multiple, the quadruple, when it is compared with that same first row according to corresponding positions, first term with first, second with second, third with third, and so on. Again, the differences of these numbers are 3, 6, then 9, then 12, and the quantities that progress by steps of 3. These numbers are detected [3] in the structure of the diagram in places just above the quadruples, and in the subsequent forms of the multiple the analogy will hold throughout.

14 In comparison with the second line reading either way, which begins with the common origin 4 and runs over in cross-lines to the term 2 in each row, the lines which are next in order beneath display the first species of the superparticular, that is, the sesquialter, between terms occupying corresponding places. Thus by divine nature,[4] not by our convention or agreement, the superparticulars are of later origin than the multiples. For illustration, 3 is the sesquialter of 2, 6 of 4, 9 of 6, 12 of 8, 15 of 10, and throughout thus. They have as a difference [5] the successive numbers from unity, like those before them.

15 The sesquitertians, the second species of superparticular, proceed with a regular, even advance from 4 : 3, 8 : 6, 12 : 9, 16 : 12, and so
16 on; having also a regular increase [6] of their differences. And in the other multiple and superparticular relations you will see that the results are in harmony and not by any means inconsistent as you go on to infinity.

[1] The point is again that it is nature, and not we ourselves, that is responsible for this regularity. Cf. section 8 above and the note on section 14.

[2] χιασμοί: That is, as before (see on section 11), lines in the form of the letter chi. 16 is their common origin, as their meeting point, but they run beyond it in all four directions.

[3] These successive differences, that is, are nothing but the series of triples and therefore of course appear in the line above the quadruples in the table.

[4] Nicomachus here contrasts φύσις, 'nature,' and νόμος, 'law' or 'convention,' a common topic of philosophy since the sophistic period. Cf. Burnet, *Greek Philosophy, Part I, Thales to Plato* (London, 1914), pp. 105 ff. See p. 120.

[5] That is, $3 - 2 = 1$; $6 - 4 = 2$; $9 - 6 = 3$; etc. The differences in order are the natural series, which was the case with the differences in the series of doubles (cf. section 11).

[6] In the series of sesquitertian ratios, 4 : 3, 8 : 6, 12 : 9, 16 : 12, etc., the differences are, as before, 1, 2, 3, 4, etc., so that 'equal,' ἴσοι, as applied here, means 'regular,' i.e., 'regularly increasing.'

The following feature of the diagram, moreover, is of no less exact- 17
ness. The terms at the corners are units; the one at the beginning a simple unit, that at the end the unit of the third course,[1] and the other two units of the second course appearing twice; so that the product (of the first two) is equal to the square (of the last). Further- 18
more, in reading either way there is an even progress from unity to the tens, and again on the opposite sides two other progressions from 10 to 100.

The terms on the diagonal from 1 to 100 are all square numbers, 19
the products of equals by equals, and those flanking them on either side are all heteromecic, unequal, and the products of sides of which one is greater[2] than the other by unity; and so the sum[3] of two successive squares and twice the heteromecic numbers between them is always a square, and conversely a square is always produced from the two heteromecic numbers on the sides and twice the square between them.

An ambitious person might find many other pleasing things dis- 20
played in this diagram, upon which it is not now the time to dwell,

[1] Ast, *Theol. Arith.*, pp. 254–55, has in his note on this passage: "Unitates sunt tres, prima 1, secunda 10, tertia 100 . . . ἐν διαφορήσει in distractione, h.e., oppositione vel decussatione." As Iamblichus explains (*In Nicom.*, p. 88, 24 ff. Pistelli), 'monads of the second and third courses' (μονὰς δευτερωδουμένη, τριωδουμένη) are Pythagorean terms for 10 and 100. This designation depends on their belief that the first decade epitomizes all number, and the following numbers simply repeat, in a sense, the first 10. So in the *Theol. Arith.* (p. 59 Ast) we are informed that they called 10 Pan "because no number is naturally greater, but if any is so conceived it somehow circles about to it again in repetition; for the hundred is 10 decads, the thousand is 10 hundreds, and each of the others will come, taking the return path either to it or to one of the numbers up to it." This notion is of necessity linked with the doctrine that 10 is the perfect number, and as the author (probably Nicomachus himself) says just before the words quoted, 10 exists as the epitome of all numbers in itself in order to offset and control unlimited multitude, to act as "a measure for the whole and as it were a gnomon and a straight edge" in the hands of the creating deity. The units then form the first course, the tens the second, the hundreds the third, and so on, circling around 10 and its powers as the turning points of a race-course. Cf. also Nesselmann, *Geschichte der Algebra*, I. Th., p. 239 (Berlin, 1842). The 'product of the first two,' then, is $1 \times 100 = 100$, and the 'square of the other' is 10^2 or 100.

[2] Hoche here (p. 54, 17 of the text) reads ἄνισοι. It is hard to see what ἄνισοι ἑτερομήκεις would be. It is better to read ἀνίσοις with G_2 and two other MSS referring to πλευραῖς (or perhaps ἀνισάκις has dropped out of the text; ἀνισάκις ἄνισοι would be easily explicable and would balance ἰσάκις ἴσοι). To illustrate the meaning of the passage it may be observed that the numbers 'flanking' 4 are 2 and 6, which are respectively 1×2 and 2×3; 9 is flanked by 6, and 12 (2×3, 3×4). In general m^2 is flanked by $(m-1)m$ and $(m+1)m$. On heteromecic numbers, see II. 17.

[3] Thus 6 is the heteromecic number between 4 and 9 and $4 + 9 + (2 \times 6) = 25 = 5^2$. The general formula for this proposition would be $m^2 + (m+1)^2 + 2\,m(m+1) = (2\,m+1)^2$. Again, 6 and 12 flank 9; now $6 + 12 + (2 \times 9) = 36 = 6^2$, or, in general, $(m-1)m + (m+1)m + 2\,m^2 = 4\,m^2$.

for we have not yet gained recognition of them from our *Introduction*, and so we must turn to the next subject. For after these two generic relations of the multiple and the superparticular and the other two, opposite to them, with the prefix sub-, the submultiple and the subsuperparticular, there are in the greater division of inequality the superpartient, and in the less its opposite, the subsuperpartient.

CHAPTER XX

1 It is the superpartient[1] relation when a number contains within itself the whole of the number compared and in addition more than one part of it; and 'more than one' starts with 2 and goes on to all the numbers in succession. Thus the root-form of the superpartient is naturally the one which has in addition to the whole two parts of the number compared, and as a species[2] will be called superbipartient; after this the one with three parts besides the whole will be called supertripartient as a species; then comes the superquadripartient, the superquintipartient, and so forth.

2 The parts have their root and origin with the third, for it is impossible in this case to begin with the half. For if we assume that any number contains two halves of the compared number, besides the whole of it, we shall inadvertently be setting up a multiple instead of a superpartient, because each whole, plus two halves of it, added together makes double the original number. Thus it is most necessary to start with two thirds, then two fifths, two sevenths, and after these two ninths, following the advance of the odd numbers; for two quarters, for example, again are a half, two sixths a third, and thus again superparticulars will be produced instead of superpartients, which is not the problem laid before us nor in accord with the systematic construction of our science.

3 After the superpartient the subsuperpartient immediately[3] is produced, whenever a number is completely contained in the one compared with it, and in addition several parts of it, 2, 3, 4, or 5, and so on.

[1] Defined by Theon of Smyrna, p. 78, 6 ff. Hiller.

[2] See on I. 19. 2 (p. 215).

[3] That is to say, given a superpartient the existence of a subsuperpartient naturally follows. For if 9 is a superpartient of 7, being $1\frac{2}{7}$ of it, then 7 is contained in 9 $1\frac{2}{7}$ times and is a subsuperpartient of 9.

CHAPTER XXI

The regular arrangement and orderly production of both species[1] 1
are discovered when we set forth the successive even and odd numbers, beginning with 3, and compare with them simple series of odd numbers only,[2] from 5 in succession, first to first — that is, 5 to 3, — second to second — that is, 7 to 4, — third to third — that is, 9 to 5, — fourth to fourth — that is, 11 to 6, — and so on in the same order as far as you like. In this way the forms of the superpartient and the subsuperpartient, in due order, will be disclosed through the root-forms of each species, the superbipartient first, then the supertripartient, superquadripartient, and superquintipartient, and further in succession in similar manner; for after the root-forms of each species the ones which follow them will be produced by doubling, or tripling, both the terms, and in general by multiplying after the regular forms of the multiple.

TABLE OF THE SUPERPARTIENTS

Root-forms	5	3	7	4	9	5	11	6	13	7
	10	6	14	8	18	10	22	12	26	14
	15	9	21	12	27	15	33	18	39	21
	20	12	28	16	36	20	44	24	52	28
	25	15	35	20	45	25	55	30	65	35
	30	18	42	24	54	30	66	36	78	42
	35	21	49	28	63	35	77	42	91	49
	40	24	56	32	72	40	88	48	104	56
	45	27	63	36	81	45	99	54	117	63

It must be observed that from the two parts in addition to the 2
whole which are contained in the greater term, we are to understand 'third,'[3] in the case of three parts, 'fourth,'[4] with four parts, 'fifth,'

[1] The superpartient and subsuperpartient.

[2] καθαροὺς . . . περισσοὺς μόνους. Here καθαρὸς means 'pure' in the sense of 'with no admixture from another class of terms,' as in I. 22. 3, 4; II. 27. 4.

[3] That is, when a superpartient contains, besides the lesser number, two parts of the lesser number, it is understood that those parts are thirds, etc. Cf. Boethius, I. 28: *Hoc quoque videndum est, quoniam, cum duae partes ex minore plus in maioribus sunt, tertii semper vocabulum subauditur, ut superbipartiens . . . . dicatur superbipartiens tertias. . . .*

[4] These terms represent the ratios, respectively, of $1\frac{2}{3}$, $1\frac{3}{4}$, $1\frac{4}{5}$, $1\frac{5}{6}$ to 1.

with five, 'sixth,' and so on, so that the order of nomenclature is something like this: superbipartient, supertripartient, superquadripartient, then superquintipartient, and similarly with the rest.

3 Now the simple, uncompounded relations of relative quantity are these which have been enumerated. Those which are compounded of them and as it were woven out of two into one are the following, of which the antecedents [1] are the multiple superparticular and multiple superpartient, and the consequents the ones that immediately arise in connection with each of the former, named with the prefix sub-; together with the multiple superparticular the submultiple superparticular, and with the multiple superpartient the submultiple superpartient. In the subdivision [2] of the genera the species of the one will correspond to those of the other, these also having names with the prefix sub-.

CHAPTER XXII

1 Now the multiple superparticular is a relation [3] in which the greater of the compared terms contains within itself the lesser term more than once and in addition some one part of it, whatever this may be.

2 As a compound, such a number is doubly diversified after the peculiarities of nomenclature of its components on either side; for inasmuch as the multiple superparticular is composed of the multiple and superparticular generically, it will have in its subdivisions according to species a sort of diversification and change of names proper both to the first part of the name and to the second. For instance, in the first part, that is, the multiple, it will have double, triple, quadruple, quintuple, and so forth, and in the second part, generically from the superparticular, its specific forms in due order, the sesquialter, sesquitertian, sesquiquartan, sesquiquintan, and so on, so that the combination will proceed in somewhat this order:

Double sesquialter, double sesquitertian, double sesquiquartan, double sesquiquintan, double sesquisextan, and analogously.

Beginning once more: triple sesquialter, triple sesquitertian, triple sesquiquartan, triple sesquiquintan.

[1] See I. 19. 2 and the note, on the terms 'antecedent' and 'consequent.'

[2] That is, just as submultiple superpartient corresponds to multiple superpartient, so submultiple superbipartient (a subclass) answers to multiple superbipartient, etc.

[3] Theon of Smyrna, p. 78, 23 ff. Hiller, defines this ratio.

Again: quadruple sesquialter, quadruple sesquitertian, quadruple sesquiquartan, quadruple sesquiquintan.

Again: quintuple sesquialter, quintuple sesquitertian, quintuple sesquiquartan, quintuple sesquiquintan, and the forms analogous to these *ad infinitum.* Whatever number of times the greater contains the whole of the smaller, by this quantity the first part of the ratio of the terms joined together in the multiple superparticular is named; and whatever may be the factor, in addition to the whole several times contained, that is, in the greater term, from this is named the second kind of ratio of which the multiple superparticular is compounded.

Examples of it are these: 5 is the double sesquialter[1] of 2; 7 the 3
double sesquitertian of 3; 9 the double sesquiquartan of 4; 11 the double sesquiquintan of 5. You will furthermore always produce them in regular order, in this fashion, by comparing with the successive even and odd numbers from 2 the odd numbers, exclusively, from 5, first with first, second with second, third with third, and the others each with the one in the same position in the series. The successive terms beginning with 5 and differing by 5 will be without exception double sesquialters of all the successive even numbers from 2 on, when terms in the same position in the series are compared; and beginning with 3, if all those with a difference of 3 be set forth, as 3, 6, 9, 12, 15, 18, 21, and in another series there be set forth those that differ by 7, to infinity, as 7, 14, 21, 28, 35, 42, 49, and the greater be compared with the smaller, first to first, second to second, third to third, fourth to fourth, and so on, the second species will appear, the double sesquitertian, disposed in its proper order.

Then again, to take a fresh start, if the simple series of quadruples 4
be set forth, 4, 8, 12, 16, 20, 24, 28, 32, and then there be placed beside it in another series the successive numbers beginning with 9, and increasing by 9, as 9, 18, 27, 36, 45, 54, we shall have revealed once more the multiple superparticular in a specific form, that is, the double sesquiquartan in its proper order; and any one who desires can contrive this to an unlimited extent.

The second kind begins with the triple sesquialter, such as 7 : 2, 14 : 4, 5
and in general the numbers that advance by steps of 7 compared with
the even numbers in order from 2. Then once more, 10 : 3 is the first 6
triple sesquitertian, 20 : 6 the second, and, in a word, the multiples of 10 in succession, compared with the successive triples. This indeed

[1] Because it contains 2 twice, plus 1; i.e., is $2\frac{1}{2} \times 2$.

we can observe with greater exactitude and clearness in the table studied above, for in comparison with the first row the succeeding rows in order,[1] compared as whole rows, display the forms of the multiple in regular order up to infinity when they are all compared in each case to the same first row; and when each row is compared to all those above it, in succession, the second row being taken as our starting point, all the forms of the superparticular are produced in their proper order; and if we start with the third row,[2] all of those beginning with the fifth that are odd in the series when they are compared with this same third row, and those following it, will show all the forms of the superpartient in proper order. In the case of the multiple superparticular, the comparisons will have a natural order of their own if we start with the second row and compare the terms from the fifth, first to first, second to second, third to third, and so on, and then the terms of the seventh row to the third, those of the ninth to the fourth, and follow the corresponding order as far as we are able to go.

7 It is plain that here too the smaller terms have names corresponding to the larger ones, with the prefix sub-, according to the nomenclature given them all.

CHAPTER XXIII

1 The multiple superpartient [3] is the remaining relation of number. This, and the relation called by a corresponding name with the prefix sub-, exist when a number contains the whole of the number compared more than once (that is, twice, thrice, or any number of times) and certain parts of it, more than one, either two, three, or four, and so on,
2 besides. These parts [4] are not halves, for the reasons mentioned above, but either thirds, fourths, or fifths, and so on.

3 From what has already been said it is not hard to conceive of the

[1] Referring to the table in chapter 19, the successive rows of which are multiples of the first (since this is simply the multiplication table).

[2] That is, the comparisons are to be 5th row with the 3d, 7th with the 4th, 9th with the 5th, etc. Hence we will have:

$$\frac{5}{3} = \frac{10}{6} = \frac{15}{9} = \text{etc.} = \frac{1\frac{2}{3}}{1}, \text{ superbipartient;}$$

$$\frac{7}{4} = \frac{14}{8} = \frac{21}{12} = \text{etc.} = \frac{1\frac{3}{4}}{1}, \text{ supertripartient;}$$

$$\frac{9}{5} = \frac{18}{10} = \frac{27}{15} = \text{etc.} = \frac{1\frac{4}{5}}{1}, \text{ superquadripartient, etc.}$$

[3] Theon's definition is found p. 79, 15 ff. Hiller.

[4] See 20. 2 above.

varieties of this relation, for they are differentiated in the same way as, and consistently with, those that precede, double superbipartient, double supertripartient, double superquadripartient, and so on. For example, 8 is the double superbipartient of 3, 16 of 6, and in general the numbers beginning with 8 and differing by 8 are double superbipartients of those beginning with 3 and differing by 3, when those in corresponding places in the series are compared, and in the case of the other varieties one could ascertain their proper sequence by following out what has already been said. In this case, too, we must conceive that the nomenclature of the number compared goes along and suffers corresponding changes, with the addition of the prefix sub-.

Thus we come to the end of our speculation upon the ten arithmetical 4
relations for a first Introduction. There is, however, a method[1] very exact and necessary for all discussion of the nature of the universe which very clearly and indisputably presents to us the fact that that which is fair and limited, and which subjects itself to knowledge,[2] is naturally prior to the unlimited, incomprehensible, and ugly, and furthermore that the parts and varieties of the infinite

[1] The principle about to be stated is that of the 'three rules' (Cantor, *op. cit.*, vol. I, p. 431; Nesselmann, *op. cit.*, p. 198), by following which, starting from three equal terms, other sets of three in different ratios may be derived, and by the reversal of which any proportion in three terms may be reduced to the original equality. The present purpose is to show that equality is more elementary than any form of inequality as measured by ratios (cf. II. 1. 1), and it follows for Nicomachus as a Pythagorean that what is true of numbers is also true of the universe, and that 'equality' and 'sameness' are therefore elements and principles. The proposition was undoubtedly not original with Nicomachus, for its history can be traced back several centuries. In Theon of Smyrna (p. 107, 24 Hiller) it is given on the authority of Adrastus, a Peripatetic, whose date is stated in the Pauly-Wissowa encyclopædia to be the middle of the second century A.D. E. Hiller (*Rhein. Mus.*, vol. XXVI, pp. 582 ff.) has shown that the book of Adrastus which Theon is probably quoting is his commentary on Plato's *Timaeus*. It is further probable from the context of Theon that Eratosthenes (*ca.* 276–194 B.C.) knew the 'three rules.' He is there cited in these words: "So we shall take three magnitudes and the proportion residing in them and change the terms, and we shall show that all mathematics is made up of the proportion of quantities and that their source and element are the principle of the proportion" (λαβόντες δὴ τρία μεγέθη καὶ τὴν ἐν τούτοις ἀναλογίαν κινήσομεν τοὺς ὅρους καὶ δείξομεν ὅτι πάντα τὰ ἐν τοῖς μαθήμασιν ἐξ ἀναλογίας ποσῶν τινων συγκεῖται καὶ ἔστιν αὐτῶν ἀρχὴ καὶ στοιχεῖον ἡ τῆς ἀναλογίας φύσις). Another citation of Eratosthenes (Theon, p. 82, 22 ff.) informs us that the 'principle (φύσις) of the proportion' is ratio, which should be taken into consideration in connection with the statements above. Theon immediately adds, after the passage first cited, 'But Eratosthenes says that he will omit the demonstrations' (τὰς δὲ ἀποδείξεις ὁ μὲν Ἐρατοσθένης φησὶ παραλείψειν), and proceeds to give the 'three rules' as stated by Adrastus. Eratosthenes's reference to 'three magnitudes' and 'changing the terms,' however, seems, especially in view of the context of Theon, to apply to nothing else than the 'three rules,' and it must be inferred from his own statement that he would 'omit the demonstrations,' that the latter were familiar to him. E. Hiller (*Philologus*, vol. XXX, pp. 60 ff.) has shown that this quotation of Eratosthenes is probably taken from his Πλατωνικός, and that this, like the book of Adrastus, was a commentary on the *Timaeus*.

[2] Cf. I. 2. 5.

and unlimited are given shape and boundaries by the former, and through it attain to their fitting order and sequence, and like objects brought beneath some seal [1] or measure all gain a share of likeness to it and similarity of name when they fall under its influence. For thus it is reasonable that the rational part of the soul will be the agent which puts in order the irrational part, and passion and appetite, which find their places in the two forms of inequality, will be regulated by the reasoning faculty as though by a kind of equality and sameness.

5 And from this equalizing process there will properly result for us the so-called ethical virtues,[2] sobriety, courage, gentleness, self-control, fortitude, and the like.

6 Let us then consider the nature of the principle that pertains to these universal matters. It is capable of proving that all the complex species of inequality and the varieties of these species are produced out of equality, first and alone, as from a mother and root.

7 Let there be given us equal numbers in three terms, first, units, then two's in another group of three, then three's, next four's, five's, and so on as far as you like. For them, as the setting forth of these terms has come about by a divine, and not human, contrivance, nay, by Nature herself, multiples will first be produced, and among these the double will lead the way, the triple after the double, the quadruple next, and then the quintuple, and, following the order we have previously recognized, *ad infinitum;* second, the superparticular, and here again the first form, the sesquialter, will lead, and the next after it, the sesquitertian, will follow, and after them the next in order, the sesquiquartan, the sesquiquintan, the sesquisextan, and so on *ad infinitum;* third, the superpartient, which once more the superbipartient will lead, the supertripartient will follow immediately upon it,

[1] Whatever is absolutely indeterminate can never remain the same or even retain the same name even for an instant; for then it would be determined. Objects of this sort are to be looked for among the material things mentioned in I. 1. 3. When they are impressed with form, they are no longer indeterminate but determined thereby; they remain like themselves and like their pattern, the ideal form, from one moment to the next, and can be called by the same name from time to time ('partake of likeness and similarity of name'). As Nicomachus points out above (I. 1. 2 — 2. 1) it is only by virtue of the form with which they are impressed and not of themselves that such things have any existence and appellation; both their being and their name are those of the forms and not their own.

[2] Aristotle gives sobriety (*σωφροσύνη*) as the mean in matters concerning pleasure and pain with licentiousness (*ἀκολασία*) as the excess and no extreme to match it on the side of deficiency (*Eth. Nic.*, 1107 b 4 ff.). Courage (*ἀνδρεία*) he calls the mean between fear and recklessness, *φόβος* and *θάρρη* (*ibid.*, 1107 a 33). Gentleness (*πραότης*; the extremes *ὀργιλότης*, *ἀοργησία*): see *ibid.*, 1108 a 4 ff.; self-control and patience (*ἐγκράτεια*, *καρτερία*) are discussed together, *ibid.*, 1145 b 8 ff.

and then will come the superquadripartient, the superquintipartient,[1] and according to the foregoing as far as one may proceed.

Now you must have certain rules, like invariable and inviolable 8
natural laws, following which the whole aforesaid advance and progress from equality may go on without failure. These are the directions:[2] Make the first equal to the first, the second equal to the sum of the first and second, and the third to the sum of the first, twice the second, and the third. For if you fashion according to these rules you would get first all the forms of the multiple in order out of the three given terms of the equality, as it were, sprouting and growing without your paying any heed or offering any aid. From equality you will first[3] get the double; from the double the triple, from the triple successively the quadruple, and from this the quintuple in due order, and so on. From these same multiples in their regular order, reversed, there 9
are immediately produced by a sort of natural necessity through the agency of the same three rules the superparticulars, and these not as it chances and irregularly but in their proper sequence; for from the first,[4] the double, reversed, comes the first, the sesquialter, and from the second, the triple, the second in this class, the sesquitertian; then the sesquiquartan from the quadruple, and in general each one from the one of similar name. And with a fresh start, if the superparticulars 10
are set forth in the order of their production, but with terms reversed, the superpartients, which naturally follow them, are brought to light,

[1] καὶ εὐθὺς τὸ ἐπιπενταμερές (p. 66, 14 Hoche) is omitted by Codex G.

[2] As stated by Theon of Smyrna, p. 107, 24 ff., Adrastus thus formulated the rule: "Given three terms in any proportion, if three others be taken formed from these, the first equal to the first, the second equal to the sum of the first and second, and the third the sum of the first, twice the second and the third, those thus taken will again be proportional." Algebraically this method obtains from a, ar, ar^2 the series, a, $a(1 + r)$, $a(1 + r)^2$. All of the remaining results of this chapter are included in this formula. The examples given by Theon start with three equal terms, as here.

[3] The results thus produced will be:

1	1	1	equality
1	2	4	doubles
1	3	9	triples
1	4	16	quadruples
1	5	25	quintuples, etc.

Theon gives like results.

[4] Theon includes this process in his discussion; its results are as follows:

4	2	1, doubles reversed, giving	4	6	9, sesquialters
9	3	1, triples reversed, giving	9	12	16, sesquitertians
16	4	1, quadruples reversed, giving	16	20	25, sesquiquartans
25	5	1, quintuples reversed, giving	25	30	36, sesquiquintans, etc.

the superbipartient from the sesquialter,[1] the supertripartient from the sesquitertian, the superquadripartient from the sesquiquartan, and
11 so on *ad infinitum*. If, however, the superparticulars [2] are set forth with terms not in reverse but in direct order, there are produced through the three rules the multiple superparticulars, the double sesquialter out of the first, the sesquialter; the double sesquitertian from the second, the sesquitertian, the double sesquiquartan from the
12 third, the sesquiquartan, and so on. From those produced by the reversal of the superparticular, that is, the superpartients, and from those produced without such reversal, the multiple superparticulars, there are once more produced, in the same way and by the same rules, both when the terms are in direct or reverse order, the numbers that show the remaining numerical relations.

13 The following must suffice as illustrations of all that has been said hitherto, the production of these numbers and their sequence, and the
14 use of direct and of reverse order. From the relation and proportion in terms of the sesquialter, reversed so as to begin with the largest term, there arises a relation in superpartient ratios, the superbipartient; and from it in direct order, beginning with the smallest term, a multiple superparticular relation, the double sesquialter. For example, from 9, 6, 4, we get either 9, 15, 25 or 4, 10, 25. From the relation in terms of sesquitertians, beginning with the greatest term, is derived a superpartient, the supertripartient; beginning with the smallest term, a double sesquitertian. For example, from 16, 12, 9 comes either 16, 28, 49 or 9, 21, 49. And from the relation in terms of sesquiquartans, when it is arranged to begin with the largest term, is derived a superpartient, the superquadripartient; when it starts with the smallest term, a multiple superparticular, the double sesquiquintan; for instance, from 25, 20, 16 comes either 25, 45, 81 or 16, 36, 81.

15 In the case of all these relations that are thus differentiated,[3] and

[1] Theon reports this matter as well. The results:

Reversed superparticulars:			Resulting superpartients:		
9	6	4 (sesquialter)	9	15	25 (superbipartient)
16	12	9 (sesquitertian)	16	28	49 (supertripartient)
25	20	16 (sesquiquartan)	25	45	81 (superquadripartient), etc.

[2] This gives the following results:

Superparticulars:			Multiple superparticulars:		
4	6	9 (sesquialter)	4	10	25 (double sesquialter)
9	12	16 (sesquitertian)	9	21	49 (double sesquitertian)
16	20	25 (sesquiquartan)	16	36	81 (double sesquiquartan).

[3] What Nicomachus meant by *πασῶν . . . τῶν διαζευχθεισῶν*, 'the contrasted ratios,' is shown by Iamblichus's commentary, which here has *τῶν πλασσομένων σχέσεων*. They are the pairs of

of the one from which both of the differentiated ones are derived, the last term is always the same and a square; the first term becomes the smallest, and invariably the extremes are squares.

Moreover the multiple superpartients and superpartients of other 16
kinds are made to appear in yet another way out of the superpartients;
for example, from the superbipartient relation arranged so as to be-
gin with the smallest term comes the double superbipartient, but,
arranged so as to start with the greatest, the superpartient ratio of
8 : 5. Thus from 9, 15, 25 comes either 9, 24, 64 or 25, 40, 64. From
the supertripartient, beginning with the smallest term, we have the
double supertripartient, and, beginning with the largest, the ratio of
11 : 7. Thus, from 16, 28, 49 comes either 16, 44, 121 or 49, 77, 121.
Again, from the superquintipartient, as, for example, 25, 45, 81, be- 17
ginning with the lesser term we derive the double superquintipartient
in the terms 25, 70, 196, but beginning with the greater a superpartient
again, the ratio of 14 : 9, in the terms 81, 126, 196. And you will
find the results analogous and in agreement with the foregoing in all
successive cases to infinity.[1]

ratios that may be derived from any given ratio by the application of the rules under discussion to the given ratio taken in direct and reversed order in turn, and it is because of the latter circumstance that they are called 'contrasted' (so Ast, *Theol. Arith.*, p. 268, *disiunctis et inter se oppositis, una nimirum recta, altera conversa*). In further illustration of the meaning the ratios mentioned by the author may be examined:

Original ratios:

Direct order,	4	6	9	9	12	16	16	20	25;
Reverse order,	9	6	4	16	12	9	25	20	16;
Derived forms,	9	15	25	16	28	49	25	45	81;
	4	10	25	9	21	49	16	36	81

Now whenever these derivative ratios are produced, (1) the last term, a square, is the same in each (25, 49, 81 in the scheme above); (2) the first term in the first derivative is the larger square of the original ratio, but in the second it is the smaller ('it changes from the larger to the smaller'); (3) all the extreme terms are squares.

[1] Certain of the MSS (See the critical note, p. 70, 15 Hoche) here add: "Moreover in all the given series the extremes are always squares; and the mean terms are derived from their sides multiplied together; and the first term of the generating ratio becomes the smaller term of the ratio generated. And in both the ratios generated the last and greater square is the same." This material was used by Ast to reconstruct the text of section 15, which would then read much like the addition to the text just translated. In comparing the ratios given in the preceding note it may be observed that in 4, 6, 9, for example, the mean, 6, is 2 × 3 (the product of the sides of the squares 4 and 9) and the same is true of the rest. Then again the first term of 9, 6, 4 is the smallest of the series 9, 15, 25 derived from it, while the first term of 4, 6, 9 is the smallest of the derivative series 4, 10, 25, and so with the rest.

BOOK II

CHAPTER I

1 An element is said to be, and is, the smallest thing which enters into the composition of an object and the least thing into which it can be analyzed. Letters, for example, are called the elements of literate speech, for out of them all articulate speech is composed and into them finally it is resolved. Sounds are the elements of all melody; for they are the beginning of its composition and into them it is resolved. The so-called four elements of the universe in general are simple bodies, fire, water, air, and earth;[1] for out of them in the first instance we account for the constitution of the universe, and into them finally we conceive of it as being resolved.

We wish also to prove that equality is the elementary principle[2] of relative number; for of absolute number, number per se, unity and the dyad[3] are the most primitive elements, the least things out of which it is constructed, even to infinity, by which it has its growth,
2 and with which its analysis into smaller terms comes to an end. We have, however, demonstrated that in the realm of inequality advance and increase have their origin in equality and go on to absolutely all the relations with a certain regularity through the operation of the three rules.[4] It remains, then, in order to make it an element in very truth, to prove that analyses also finally come to an end in equality. Let this then be considered our procedure.

CHAPTER II

1 Suppose then you are given three terms, in any relation whatsoever and in any ratio, whether multiple, superparticular, superpartient, or a compound of these, multiple superparticular or multiple superpar-

[1] The ordinary list of elements for practically all Greek philosophy. These four were distinguished as primitive bodies in immemorial antiquity, but the more scientific idea of them as elements seems to have originated with Empedocles. On the matter see Burnet's summary, *Greek Philosophy, Part I, Thales to Plato*, p. 26.

[2] See on I. 23. 4. [3] Cf. Part I, pp. 99 ff. [4] That is, those given in I. 23. 8.

tient, provided only that the mean term is seen to be in the same ratio to the lesser as the greater to the mean, and vice versa. Subtract always from the mean the lesser term, whether it be first or last in order, and set down the lesser term itself as the first term of your new series; then put as your second term what remains from the second after the subtraction; then after having subtracted the sum of the new first term and twice the new second term from the remaining number — that is, the greater of the numbers originally given you — make the remainder your third term, and the resulting numbers will be in some other ratio, naturally more primitive.[1] And if again in the same 2
way you subtract the remainder from these same terms,[2] it will be found that your three terms have passed back into three others more primitive, and you will find that this always takes place as a consequence, until they are reduced to equality, whence by every necessity it appears evident that equality is the elementary principle of relative quantity.

There follows upon this speculation a most elegant principle, ex- 3
tremely useful in its application to the Platonic psychogony[3] and the problem of all harmonic intervals; for in the Platonic passage we are frequently bidden, for the sake of the argument, to set up series of intervals of two, three, four, five, or an infinite number of sesquialter ratios, or two sesquitertians, sesquiquartans, sesquioctaves, or superparticulars of any kind whatsoever, and in each case three, four, or five of them, or as many as may be directed. It is reasonable that 4
we should do this not in an unscientific, unintelligent fashion, it may be even blunderingly, but artistically, surely, and quickly, by the following procedure.

CHAPTER III

Every multiple will stand at the head[4] of as many superparticular 1
ratios corresponding in name with itself as it itself chances to be removed from unity,[5] and no more nor less under any circumstances.

[1] This is because the process is the reverse of the former. Theon of Smyrna, p. 110, 19 ff., gives this rule, taking it from Adrastus.

[2] For example, take 8, 32, 128 (quadruple series). The first term of the new series will be 8; the second will be $32 - 8 = 24$; the third will be $128 - [(2 \times 24) + 8]$, or 72. This gives a triple series. Then similarly from 8, 24, 72 will be derived 8, 16, 32, the double series, and from the latter 8, 8, 8, a series of equal terms.

[3] See Plato, *Timaeus*, 35 A ff.

[4] ἡγήσεται: That is, with reference to the table in section 4; 'will head a column.'

[5] That is, in the list of doubles (see the table).

2 The doubles, then, will produce[1] sesquialters, the first one, the second two, the third three, the fourth four, the fifth five, the sixth six, and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number agrees with the multiples that have generated them, at that point by a divine device, as it were, there is found the number which terminates them all because it naturally is not divisible by that factor whereby the progression of the superparticular ratios went on.

From the triples all the sesquitertians will proceed, likewise equal in number to the number of the generating terms, and coming to an end, after the independence of their advance is lost, in numbers not divisible by 3. Similarly the sesquiquartans come from the quadruples, reaching a culmination after their independent progression in a number that is not divisible by 4.

3 As an example, since doubles generate sesquialters corresponding to them in number,[2] the first row of multiples[3] will be 1, 2, 4, 8, 16, 32, 64. Now since 2 is the first after unity, this will be the origin of one sesquialter only, 3, which number is not divisible by 2, so that another sesquialter might arise out of it. The first double, therefore, is productive of but one sesquialter, and the second, 4, of two. For it produces its own sesquialter, 6, and that of 6, 9, but there is none for 9 because it has no half. Eight, which is the third double, is father to three sesquialters; one its own, 12; the second, 18, the sesquialter of 12; and third, 27, that of 18; there is no fourth one, however, because of the general rule, for 27 is not divisible by 2. Sixteen, the fourth double, will stand at the head of four sesquialters, 24, 36, 54, and finally 81, so that they may of necessity be equal in number to what generated them; for 81 by its nature is not divisible by 2. And this, as you go on, you will find holds true in similar fashion to infinity.

4 For the sake of illustration let there be set down the table of the doubles, thus:

[1] φύσουσι: In the same sense that the even numbers 'produced' sesquialters by the process of I. 19. 2; but each double is here regarded as the source or producer not only of its own sesquialter, but also that of this sesquialter itself, and so on, as far as the ratio can be carried on in integers.

[2] The number of the multiple is of course that of its order in the series of doubles, or triples, etc.

[3] That is, doubles, the simplest subclass.

The double ratio in the breadth of the table

	1	2	4	8	16	32	64	
		3	6	12	24	48	96	
The triple ratio along			9	18	36	72	144	The sesquialter ra-
the hypotenuse				27	54	108	216	tio in the depth
					81	162	324	of the table
						243	486	
							729	

CHAPTER IV

We must make a similar table in illustration of the triple: 1

The triple ratio in the breadth

	1	3	9	27	81	243	729	
		4	12	36	108	324	972	
The quadruple			16	48	144	432	1296	The sesquiter-
ratio on the				64	192	576	1728	tian ratio in
hypotenuse					256	768	2304	the depth
						1024	3072	
							4096	

In the foregoing table we shall observe that in the same way the first triple, 3, stands at the head of but one sesquitertian ratio, 4, its own sesquitertian, which immediately shuts off the development of another like it; for 4 is not divisible by 3, and hence will not have a sesquitertian. The second triple is 9, and hence will begin a series of only two sesquitertian ratios, 12, its own, and 16, that of 12; but 16 cuts off further progress, for it is not divisible by 3 and hence will not have a sesquitertian. Next in order is the triple 27, three times removed 2
from 1, for the triples progress thus: 1, 3, 9, 27. Therefore this number will stand at the head of three sesquitertian ratios and no more. The first is its own, 36; the second the sesquitertian of 36, 48; the third that of the last, 64, and this no longer has a third part and therefore will not admit of a sesquitertian. The fourth leads a series of four sesquitertians and the fifth, of course, five.

Such, then, is the illustration; and for the other multiples let the 3
manner of your tables be the same. Observe that likewise here, as we found to be true in our previous discussion, Nature shows us that the doubles are more nearly original than the triples, the triples than

the quadruples, these latter than the quintuples, and so on throughout. For the highest rows of figures, across the breadth of the tables, if they are doubles, will have doubles lying parallel to them, and the numbers lying diagonally, on the hypotenuse, will be of the next succeeding variety, greater by 1, that is, triples, seen also in a series of parallel lines. If, however, there are triples across the breadth, the diagonals will by all means be quadruples; if the former are quadruples, then the latter are quintuples, and so forth.

CHAPTER V

1 It remains, after we have explained what other ratios are produced by combination of ratios, to pass on to the succeeding topics of the *Introduction*.

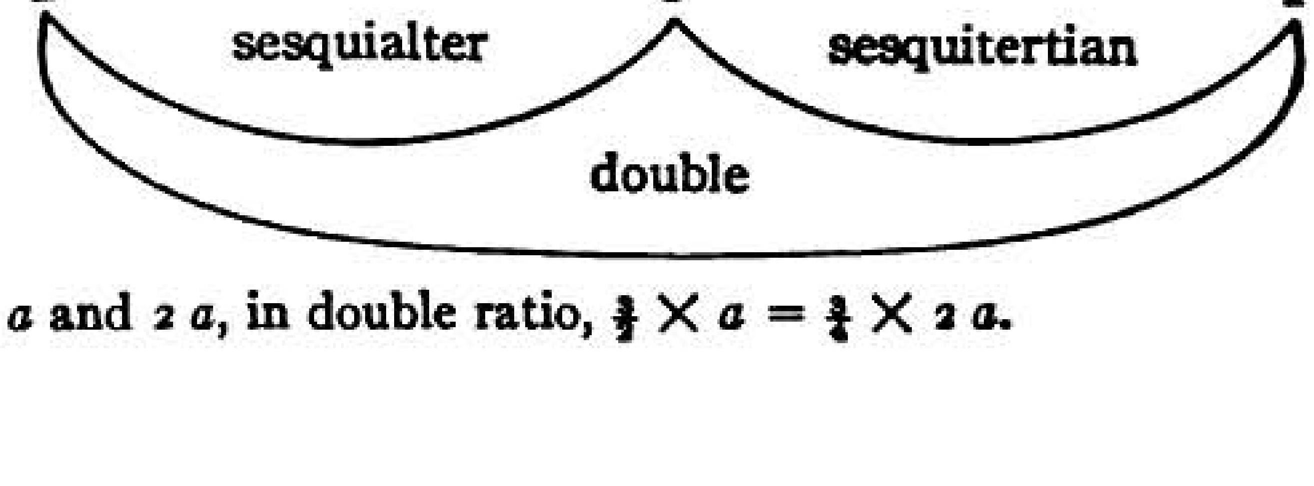
2 Now the first two ratios of the superparticular, combined, produce the first ratio of the multiple, namely, the double; for every double is a combination of sesquialter and sesquitertian, and every sesquialter and sesquitertian [1] combined will invariably produce a double.

For example,[2] since 3 is the sesquialter of 2, and 4 the sesquitertian of 3, 4 will be the double of 2, and is a combination of sesquialter and sesquitertian. Again, as 6 is the double of 3, we shall find between them some number [3] that will of necessity preserve the sesquitertian ratio to the one and the sesquialter to the other; and indeed 4, lying between 6 and 3, gives the sesquitertian ratio to 3 and the sesquialter to 6.

3 It was rightly said, then, that the double, when resolved, is resolved into the sesquialter and the sesquitertian, and that when sesquialter and sesquitertian are combined there arises the double, and that the first two forms of the superparticular combined make the first form of the multiple.

[1] That is, when the last term of the first is the same as the first term of the second ratio; for given the general formula for the sesquialter, $a + \frac{a}{2}$, then the sesquitertian of the second term, $a + \frac{a}{2} + \frac{a}{3} + \frac{a}{6} = 2\,a$, is the double of the first term; or, more simply, $\frac{3}{2} \cdot \frac{4}{3} = 2$.

[2] Some of the MSS diagrammatically illustrate thus:



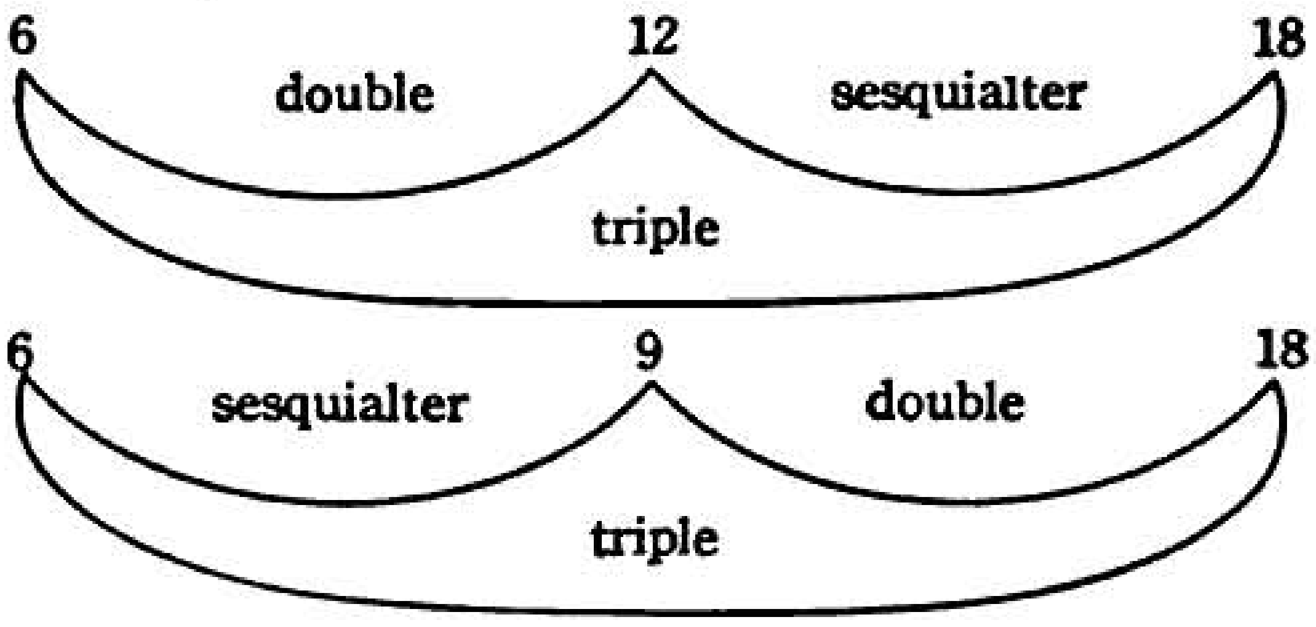
[3] That is, given a and $2\,a$, in double ratio, $\frac{4}{3} \times a = \frac{2}{3} \times 2\,a$.

But again, to take another start, this first form of the multiple which 4
has thus been produced, together with the first form of the superparticular,[1] will produce the next form of the same class, that is, the second multiple, the triple; for from every multiple and sesquialter combined a triple of necessity arises. For example,[2] as the double of 6 is 12, and the sesquialter of this is 18, then immediately 18 is the triple of 6; and to take another method, if I do not care to make 12 the mean term, but rather 9, the sesquialter of 6, the same result will come about, without deviation and harmoniously; for while 18 is the double of 9 it will preserve the triple ratio to 6. Hence from the sesquialter and the double, the first forms of the superparticular and the multiple, there arises by combination the second form of the multiple, the triple, and into them it is always resolved. For look you; 6, which is the 5
triple of 2, will have a mean term 3, which will exhibit two ratios, the sesquialter with regard to 2, and the double ratio of 6 to itself.

But if this triple ratio,[3] likewise, the second form of the multiple, is

[1] The sesquialter.

[2] Diagrams given in the MSS:



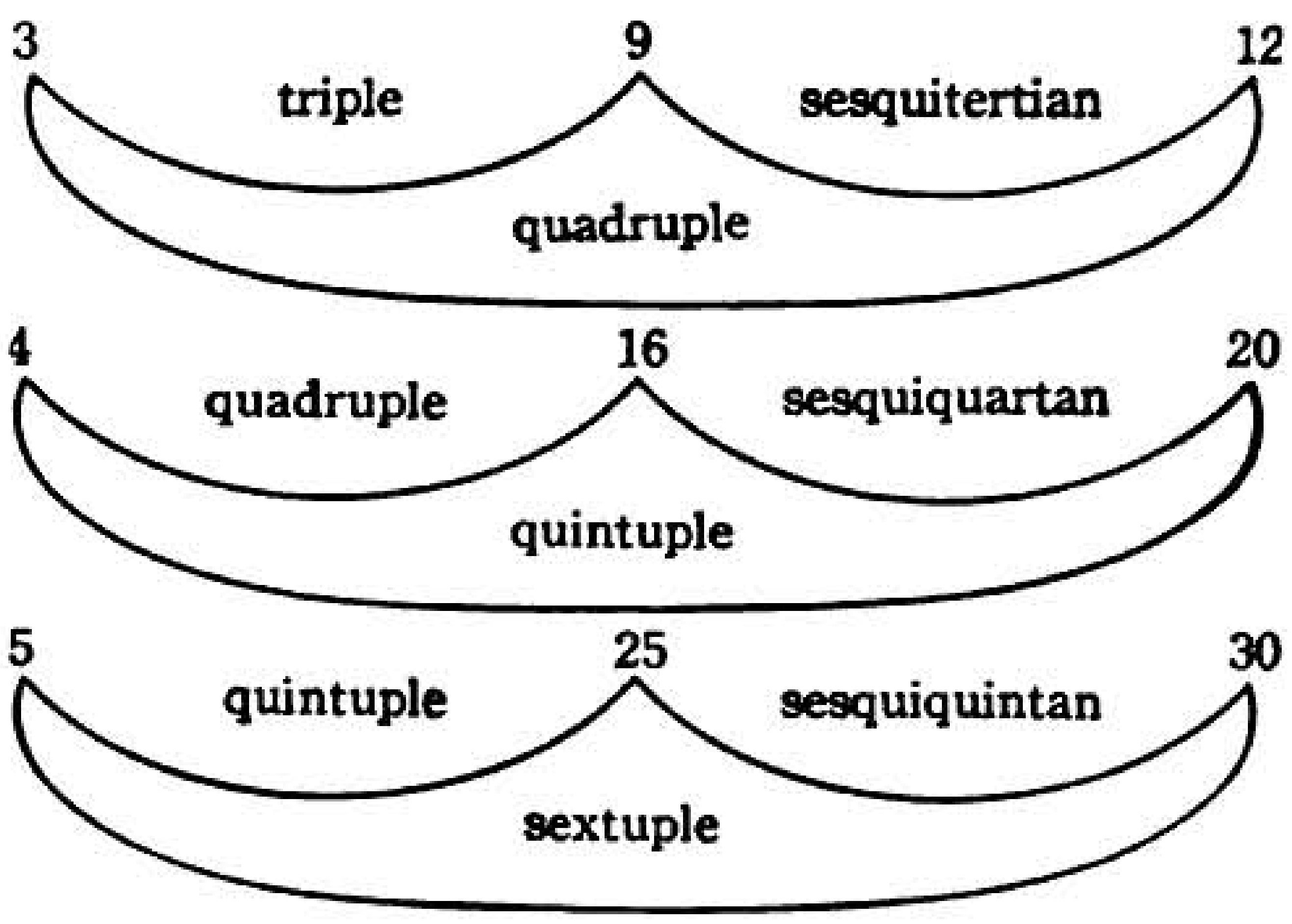
These principles may also be demonstrated in general terms:

$$m;\ 2m;\ \left(2m + \frac{2m}{2}\right) = 3m,$$

or $m;\ m + \frac{m}{2};\ 2\left(m + \frac{m}{2}\right) = 3m,$

or, arithmetically, $\frac{3}{2} \cdot \frac{2}{1} = 3.$

[3] Diagram from the MSS:



(or, 3, 4, 12)

(or, 4, 5, 20)

(or, 5, 6, 30)

Algebraic statements of the matter above:

(*a*) $m;\ 3m;\ \left(3m + \frac{3m}{3}\right) = 4m,$

combined with the sesquitertian, which is the second form of the superparticular, there would be produced from them the next form of the multiple, namely, the quadruple, and this also will of necessity be resolved into them after the same fashion as the cases previously set forth; and the quadruple, taking into combination the sesquiquartan, will make the quintuple, and, once more, the latter with the sesquiquintan will make the sextuple, and so on to the end. Thus the multiples in regular order from the beginning with the superparticulars in regular order from the beginning will be found to produce the next larger multiples. For the double with the sesquialter makes the triple, the triple with the sesquitertian the quadruple, the quadruple with the sesquiquartan the quintuple, and as far as you wish to proceed no contrary result will appear.

CHAPTER VI

1 Up to this point then we have sufficiently discussed relative number, by a process of selection measuring out what is easily comprehended and appropriate to the nature of the matters thus far introduced. Whatever remains to be said on this topic will be filled in after we have put it aside and have first discussed certain subjects which involve a more serviceable inquiry, having to do with the properties of absolute number, not relative. For mathematical speculations[1] are always to be interlocked and to be explained one by means of another. The subjects which we must first survey and observe are concerned with linear, plane, and solid numbers, cubical and spherical, equilateral and scalene, 'bricks,' 'beams,' 'wedges,' and the like, the tradition concerning which, to be sure, since they are more closely related to magnitude, is properly given in the *Geometrical Introduction*.[2] Yet

$$\text{or, } m;\ m+\frac{m}{3};\ \left(3m+\frac{3m}{3}\right)=4m.$$

$$(b)\qquad m;\ 4m;\ \left(4m+\frac{4m}{4}\right)=5m,$$

$$\text{or, } m;\ m+\frac{m}{4};\ \left(4m+\frac{4m}{4}\right)=5m.$$

$$(c)\qquad m;\ 5m;\ \left(5m+\frac{5m}{5}\right)=6m,$$

$$\text{or, } m;\ m+\frac{m}{5};\ \left(5m+\frac{5m}{5}\right)=6m.$$

[1] Boethius, II. 4: *Amat enim quodammodo matheseos speculatio alterna probationem ratione constitui.*

[2] Cf. p. 79.

the germs of these ideas are taken over into arithmetic, as the science which is the mother of geometry and more elementary than it. For we recall that a short time ago we saw that arithmetic abolishes the other sciences with itself,[1] but is not abolished by them, and conversely is of necessity implied by them but does not itself imply them.

First, however, we must recognize that each letter by which we indicate a number, such as iota, the sign for 10, kappa for 20, and omega for 800, designates that number by man's convention and agreement, not by nature. On the other hand, the natural, unartificial, and therefore simplest indication of numbers would be the setting forth one beside the other of the units contained in each. For example, the writing of one unit by means of one alpha will be the sign for 1; two units side by side, that is, a series of two alphas, will be the sign for 2; when three are put in a line it will be the character for 3, four in a line for 4, five for 5, and so on. For by means of such a notation and indication alone could the schematic arrangement of the plane and solid numbers mentioned be made clear and evident, thus: 2

The number 1, α
The number 2, α α
The number 3, α α α
The number 4, α α α α
The number 5, α α α α α

and further in similar fashion.

Unity, then, occupying the place and character of a point, will be the beginning of intervals and of numbers, but not itself an interval or a number,[2] just as the point is the beginning of a line, or an interval, but is not itself line or interval.[3] Indeed, when a point is added to a point, it makes no increase, for when an non-dimensional thing is added to another non-dimensional thing, it will not thereby have dimension; just as if one should examine the sum of nothing added to nothing, 3

[1] Cf. I. 4. 2–5. [2] See p. 116.

[3] With this passage should be compared Theon of Smyrna, p. 81, 6 ff., where 'interval' (διάστημα) is defined: "'Interval' and 'ratio' (λόγος) are different; for 'interval' is that which is between homogeneous unequal terms, 'ratio' merely the relation of homogeneous terms to one another. Wherefore there is in the case of equal terms no interval between, but there is one and the same ratio, that of equality; whereas in the case of unequals, there is one and the same interval from each to each, but a different and opposite ratio of each to each. For example, there is one and the same interval from 2 to 1 and from 1 to 2, but a different ratio; 2 : 1 is a double ratio and 1 : 2 is one half." He then quotes Eratosthenes on the subject. This will explain what is said below as to intervals in connection with the relation of equality.

which makes nothing. We saw[1] a similar thing also in the case of equality among the relatives; for a proportion is preserved — as the first is to the second, so the second is to the third — but no interval is generated in the relation of the extremes to each other, as there is in all the other relations with the exception of equality. In exactly the same way[2] unity alone out of all number, when it multiplies itself, produces nothing greater than itself.

Unity, therefore, is non-dimensional and elementary, and dimension first is found and seen in 2, then in 3, then in 4, and in succession in the following numbers; for 'dimension' is that which is conceived of as between two limits.

4 The first dimension is called 'line,' for 'line' is that which is extended in one direction. Two dimensions are called 'surface,' for a 'surface' is that which is extended in two directions. Three dimensions are called 'solid,' for a 'solid' is that which is extended in three directions, and it is by no means possible to conceive[3] of a solid which has more than three dimensions, depth, breadth, and length. By these are defined the six directions which are said to exist in connection with every body and by which motions in space are distinguished, forward, backward,[4] up, down, right and left; for of necessity two directions opposite to each other follow upon each dimension, up and down upon one, forward and backward upon the second, and right and left upon the third.

[1] The reference is the series of equal numbers employed in I. 23. 7 ff. In the series 1, 1, 1; 2, 2, 2, etc., the ratio is the same between any pair of terms; the extremes have the same ratio as the means; that is, they are all equal, so there is no interval between the extremes.

[2] The Neo-Pythagoreans commonly used this fact to substantiate their identification of the monad with God. Like God the monad is immutable and eternal (e.g., Chalcidius, *Comm. in Tim.*, c. 39: *sola inconcusso iure est atque in statu suo perseverat; semper eadem . . . immutabilis, et singularitas semper*). The name monad they derived from 'remain' (*μονάς, μένειν*) because the monad 'remains' the same under these conditions (cf. *Theol. Arith.*, p. 3 Ast; Iamblichus *In Nic.*, p. 11, 24 f.; Theon, p. 19, 7). See also II. 17. 4 below.

[3] Philo Judaeus, *De Decalogo*, 7, also states that there can be only three dimensions (*πλείους γὰρ τριῶν διαστάσεις οὐκ ἐγέννησεν*).

[4] The six categories of relative position (and motion) also were frequently cited in Neo-Pythagorean arguments; the topic was, moreover, invested with greater significance from the fact that Plato employed it, in close connection with the varieties of motion, in *Timaeus*, 43 B. Adding rotation, Plato mentions seven varieties of motion, *ibid.*, 34 A (cf. 40 A–B), and 10 (not all spatial however) in *Laws*, 894 C. The Neo-Pythagoreans regarded it significant of the peculiar virtues of 6, therefore, that there should be six 'so-called spatial positions' (*Theol. Arith.*, p. 36 Ast, *αἱ λεγόμεναι σωματικαὶ περιστάσεις*; cf. also Philo, *Leg. Alleg.*, I. 2; (Plut.) *Epit.*, III. 15, 10 = *Doxog. Graec.*, 380, 24; M. Capella, VII, 736, who adds that the seventh, circular motion, is eternal). Many of them similarly used the group of seven motions in praise of the number 7 (e.g., Anatolius, *ap. Theol. Arith.*, p. 42 Ast; Lydus, *De Mens.*, II, 11; Philo, *De Mund. Op.*, 41, and *Leg. Alleg.*, I. 4; Macrobius, *Comm. in Somn. Scip.*, I. 6. 81). Nicomachus, then, is using a topic very frequently employed.

The statement, also, as it happens, can be made conversely thus: 5
If a thing is solid, it has by all means three dimensions, length, depth and breadth; and conversely, if it has the three dimensions, it is always a solid, and nothing else.

That which has but two dimensions, therefore, will not be a solid, 6
but a surface, for the latter admits of but two dimensions. Here too it is possible similarly to reverse the statement; directly stated, a surface is that which has two dimensions, and conversely, that which has two dimensions is always a surface.

The surface, then, is exceeded by the solid by one dimension, and the 7
line is exceeded by the surface by one, for the line is that which is extended[1] in but one direction and has only one dimension, and it falls short of the solid by two dimensions. The point falls short of the latter by one dimension, and hence it has already been stated that it is non-dimensional, since it falls short of the solid by three dimensions, of the surface by two, and of the line by one.

CHAPTER VII

The point, then, is the beginning of dimension, but not itself a 1
dimension, and likewise the beginning of a line, but not itself a line; the line is the beginning of surface, but not surface; and the beginning of the two-dimensional, but not itself extended in two directions.
Naturally, too, surface is the beginning[2] of body, but not itself body, 2
and likewise the beginning of the three-dimensional, but not itself extended in three directions.

Exactly the same in numbers, unity is the beginning of all number 3
that advances unit by unit in one direction; linear number is the beginning of plane number, which spreads out like a plane in one more dimension; and plane number is the beginning of solid number, which possesses a depth[3] in the third dimension, besides the original ones. To illustrate and classify, linear numbers are all those which begin with 2 and advance by the addition of 1 in one and the same

[1] τὸ διαστατόν is translated 'that which is extended'; διάστημα is here translated by 'dimension,' though in a general sense it might be rendered 'extension.' διάστασις is used as a synonym for διάστημα.

[2] These statements are paralleled in Photius's report of a *Life of Pythagoras* (*Codex* 249, p. 249 a, 19 Bekk.).

[3] Cf. Plato, *Timaeus*, 53 C: τὸ δὲ τοῦ σώματος εἶδος πᾶν καὶ βάθος ἔχει.

dimension; and plane numbers are those[1] that begin with 3 as their most elementary root and proceed through the next succeeding numbers. They receive their names also in the same order; for there are first the triangles, then the squares, the pentagons after these, then the hexagons, the heptagons, and so on indefinitely, and, as we said, they are named after the successive numbers beginning with 3.

4 The triangle, therefore, is found to be the most original and elementary form of the plane number. This we can see from the fact that, among plane figures,[2] graphically represented, if lines are drawn from the angles to the centers each rectilinear figure will by all means be resolved into as many triangles as it has sides; but the triangle itself, if treated like the rest, will not change into anything else but itself.

[1] Nicomachus, here and in the following chapters, adopts the broadest view of what constitutes the class of plane numbers. Not all the ancients agreed with him; Euclid, in *Elements*, VII, *Def.* 17, defines the plane number as we should, as that which is produced when two numbers multiply each other, the multiplier and multiplicand being its sides (*ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί*), and Theon of Smyrna twice defines them similarly (p. 31, 9 Hiller, *ὅσοι ὑπὸ δύο ἀριθμῶν πολλα πλασιάζονται*; p. 36, 5, *οἱ διχῶς πολλαπλασιαζόμενοι ἐπίπεδοι*). Th. Martin clearly explains the difference between this application of the term and its more comprehensive use by Nicomachus: "En effet, les nombres rectangles et carrés expriment la mesure des surfaces, et les nombres parallélipipèdes rectangles et cubiques expriment la mesure des solides. Au contraire, les nombres triangles, pentagones, hexagones, etc., de même que les nombres tétraèdres, pentaèdres, hexaèdres, etc., n'expriment rien qu'une disposition imaginaire des unités dans l'espace" (*Chapitres IX^e et XX^e du Livre Second de l'Introduction Arithmétique de Nicomaque de Gérase*, Rome, 1858, p. 7).

In spite of his definition, Theon of Smyrna lists triangular and other polygonal numbers, like Nicomachus, and consequently must have known and shared to a certain extent Nicomachus's conception of them, whether or not he was aware of any inconsistency; and that this conception was somewhat generally current is shown by its appearance in the works of Philo Judaeus (see p. 32). Further, it may be noted that this notion of the polygonals is found in Diophantus when (*De Polygonis Numeris*, vol. I, p. 450, 3 Tannery) he remarks that "each of the numbers beginning with the triad and increasing by unity is a polygonal number in the first degree from the monad, and has as many angles as the number of units in it, and its side is the next number after the monad, 2" (*ἕκαστος τῶν ἀπὸ τῆς τριάδος ἀριθμῶν αὐξομένων μονάδι πολύγωνός ἐστι πρῶτον ἀπὸ τῆς μονάδος, καὶ ἔχει γωνίας τοσαύτας ὅσον ἐστὶν τὸ πλῆθος τῶν ἐν αὐτῷ μονάδων· πλευρά τε αὐτοῦ ἐστιν ὁ ἑξῆς τῆς μονάδος ἀριθμός, ὁ β′*). From Nicomachus's point of view, evidently, the same number could be called linear, plane, or solid, according to the assumed arrangement of its component monads.

[2] Nicomachus here agrees with Plato, *Timaeus*, 53 C ff., in declaring the triangle to be the fundamental form of the plane surface. Plato in the passage cited uses the principle further to explain the forms of the minutest particles of the four elements. He agrees with Nicomachus in stating that all plane surfaces may be reduced to triangles (*Timaeus*, 53 C, *ἡ δὲ ὀρθὴ τῆς ἐπιπέδου βάσεως ἐκ τριγώνων συνέστηκε*), but with reference to the subdivision of the triangle itself, he points out that each may be reduced by dropping a perpendicular from the apex (instead of drawing lines to the center, according to Nicomachus) to two elementary forms, the right-angled scalene or the right-angled isosceles. Cf. also *Theologumena Arithmeticae*, p. 18, Ast, and II. 12. 8 below.

Hence the triangle is elementary among these figures; for everything else is resolved into it, but it into nothing else. From it the others likewise would be constituted, but it from no other. It is therefore the element of the others, and has itself no element. Likewise, as 5
the argument proceeds in the realm of numerical forms, it will confirm this statement.

CHAPTER VIII

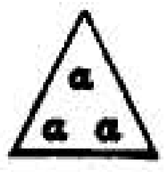
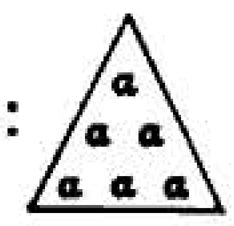
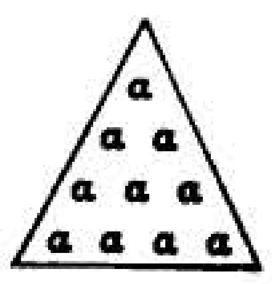
Now a triangular number is one which, when it is analyzed into 1
units, shapes into triangular form the equilateral placement of its parts in a plane. 3, 6, 10, 15, 21, 28, and so on, are examples of it; for their regular formations, expressed graphically, will be at once triangular and equilateral. As you advance you will find that such a numerical series as far as you like takes the triangular form, if you put as the most elementary form the one that arises from unity, so that unity may appear to be potentially a triangle,[1] and 3 the first actually.

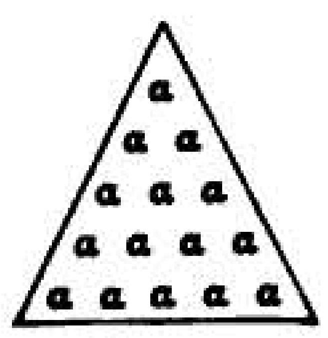
Their sides will increase by the successive numbers, for the side of 2
the one potentially first is unity; that of the one actually first, that is, 3, is 2; that of 6, which is actually second, 3; that of the third, 4; the fourth, 5; the fifth, 6; and so on.

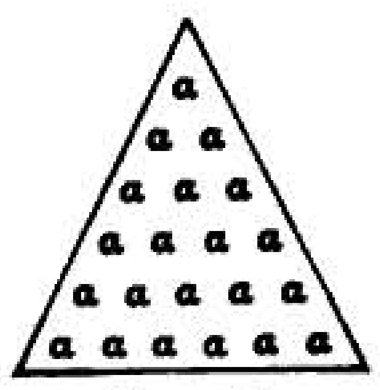
The triangular number is produced [2] from the natural series of num- 3
ber set forth in a line, and by the continued addition of successive terms, one by one, from the beginning; for by the successive combinations and additions of another term to the sum, the triangular numbers in regular order are completed. For example, from this natural series, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, I take the first term and have the triangular number which is potentially first, 1, ◬; then adding the next term I get the triangle actually first, for 2 plus 1 equals 3. In its graphic representation it is thus made up: Two units, side by side, are set beneath one unit, and the number three is made

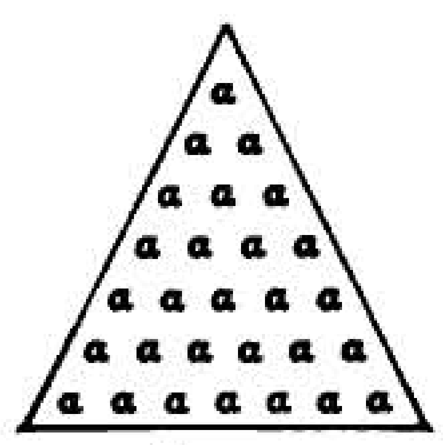
[1] This is again the distinction between potential and actual, and according also to Theon, p. 33, 5, the monad is the first potentially triangular number. On what potentiality might be conceived to mean in this case, cf. Boethius, II. 8: *Nam si cunctorum mater est numerorum* (*sc. unitas*), *quicquid in his quae ab ea nascuntur numeris invenitur necesse est ut ipsa naturali quadam potestate contineat.*

[2] Theon of Smyrna, p. 32, 22 ff., notes this method of generating triangular numbers. Cf. also Johannes Pediasimus, *Geometria*, in *Neue Jahrb. f. Phil. u. Paed.*, vol. XCII, 1865, pp. 366 ff. (f. 40 a of the Munich MS there cited).

a triangle:  Then when next after these the following number, 3, is added, simplified into units, and joined to the former, it gives 6, the second triangle in actuality, and furthermore, it graphically represents this number:  Again, the number that naturally follows, 4, added in and set down below the former, reduced to units, gives the one in order next after the aforesaid, 10, and takes a triangular form:  5, after this, then 6, then 7, and all the numbers in order, are added, so that regularly the sides of each triangle will consist of as many numbers[1] as have been added from the natural series to produce it:

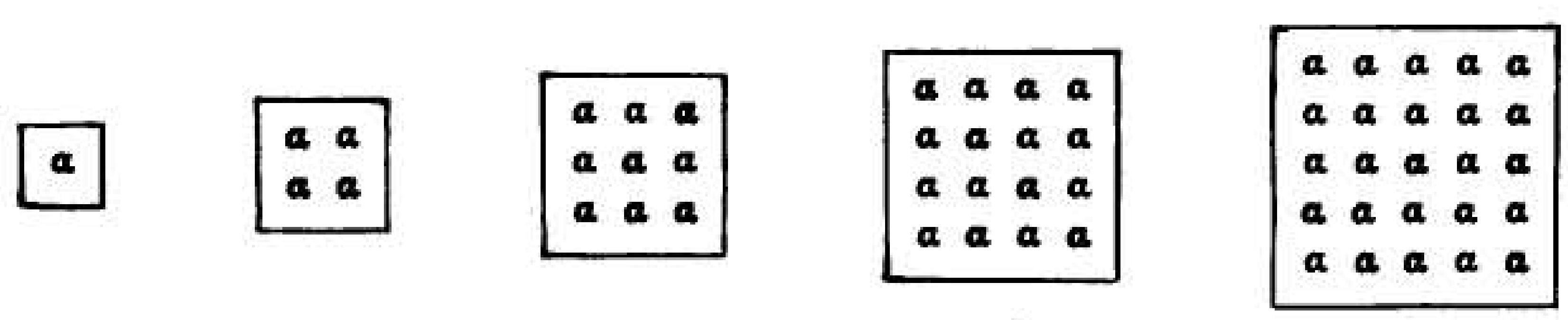

Side 5


Side 6


Side 7

CHAPTER IX

1 The square is the next number[2] after this, which shows us no longer 3, like the former, but 4, angles in its graphic representation, but is none the less equilateral. Take, for example, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100; for the representations of these numbers are equilateral, square figures, as here shown; and it will be similar as far as you wish to go:


1 4 9 16 25

2 It is true of these numbers, as it was also of the preceding, that the advance in their sides progresses with the natural series. The side

[1] Theon of Smyrna, p. 37, 13 ff., states that the units in the sides will equal the last number added.

[2] This number is treated by Theon of Smyrna (pp. 26, 14; 28, 3; 34, 1; 39, 10), who repeats himself several times.

of the square potentially first, 1, is 1; that of 4, the first in actuality, 2; that of 9, actually the second, 3; that of 16, the next, actually the third, 4; that of the fourth, 5; of the fifth, 6, and so on in general 3
with all that follow.

This number also is produced[1] if the natural series is extended in a line, increasing by 1, and no longer the successive numbers are added to the numbers in order, as was shown before, but rather all those in alternate places, that is, the odd numbers. For the first, 1, is potentially the first square; the second, 1 plus 3, is the first in actuality; the third, 1 plus 3 plus 5, is the second in actuality; the fourth, 1 plus 3 plus 5 plus 7, is the third in actuality; the next is produced by adding 9 to the former numbers, the next by the addition of 11, and so on.

In these cases, also, it is a fact that the side of each consists of as 4
many units as there are numbers taken into the sum to produce it.[2]

CHAPTER X

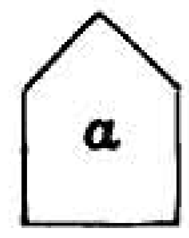
The pentagonal number is one which likewise upon its resolution 1
into units and depiction as a plane figure assumes the form of an equilateral pentagon. 1, 5, 12, 22, 35, 51, 70, and analogous numbers are examples. Each side of the first actual pentagon, 5, is 2, for 1 is the 2
side of the pentagon potentially first, 1; 3 is the side of 12, the second of those listed; 4, that of the next, 22; 5, that of the next in order, 35, and 6 of the succeeding one, 51, and so on. In general the side contains as many units as are the numbers that have been added together to produce the pentagon, chosen out of the natural arithmetical series set forth in a row. For in a like and similar manner, there are added together to produce the pentagonal numbers[3] the terms beginning with 1 to any extent whatever that are two places apart, that is, those that have a difference of 3.

[1] Cf. Theon of Smyrna, *ll. cc.* He adds the obvious generation of squares by multiplying numbers by themselves (implied by Nicomachus, II. 18. 3), and adds that the squares are alternately odd and even (p. 34, 3). The method of Nicomachus was known to the old Pythagoreans; cf. Aristotle, *Phys.*, III. 4, and Cantor, *op. cit.*, vol. I, p. 160.

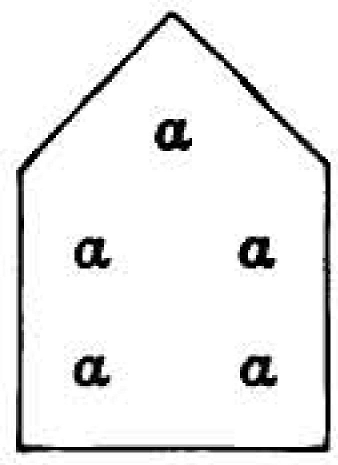
[2] So in the first square, 1, the side is 1 and only one term is taken to produce it. In the second, 4, the side is 2 and two terms are taken to produce it (1 + 3). Generally, the algebraic sum of 1, 3, 5 . . . to n terms is n^2.

[3] Cf. Theon of Smyrna, pp. 34, 11 and 39, 14, on the derivation of pentagonals.

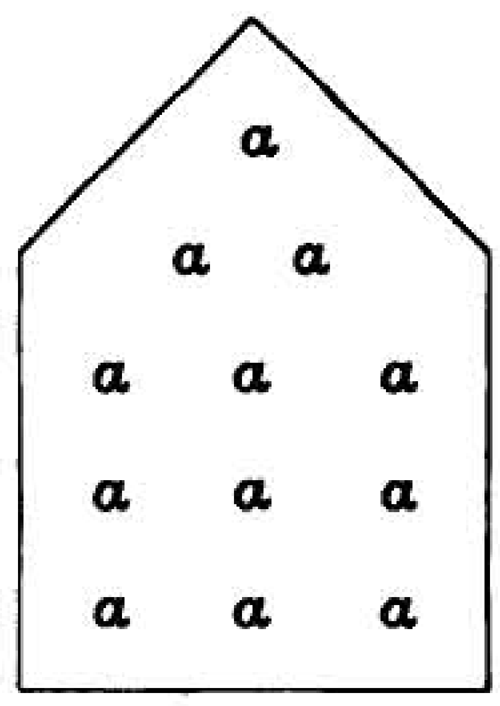
Unity is the first pentagon,[1] potentially, and is thus depicted:



5, made up of 1 plus 4, is the second, similarly represented:

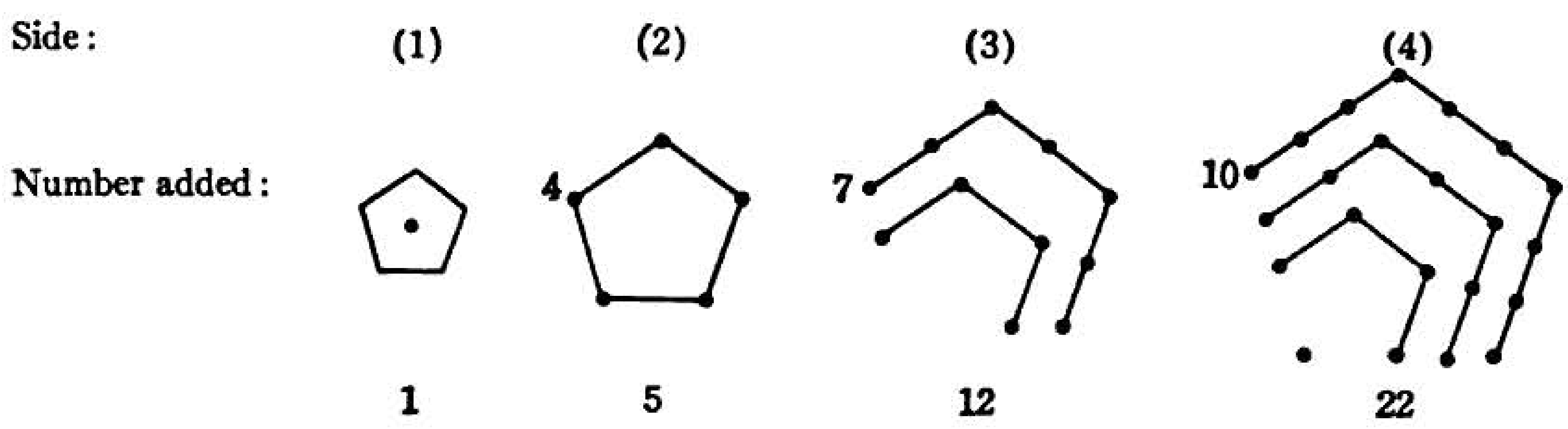


12, the third, is made up out of the two former numbers with 7 added to them, so that it may have 3 as a side, as three numbers have been added to make it. Similarly the preceding pentagon, 5, was the combination of two numbers and had 2 as its side. The graphic representation of 12 is this:



The other pentagonal numbers will be produced by adding together one after another in due order the terms after 7 that have the difference 3, as, for example, 10, 13, 16, 19, 22, 25, and so on. The pentagons will be 22, 35, 51, 70, 92, 117, and so forth.

[1] The figures given are those found in MS G. The regular pentagonal arrangement is given by M. Martin (*op. cit.*) in a way to show the numbers added in each instance. He takes these from editions of Theon and Iamblichus, but cf. Hoche, p. 87, critical notes. On the other hand the statements of II. 12. 2 seem to favor the schemes given by G.

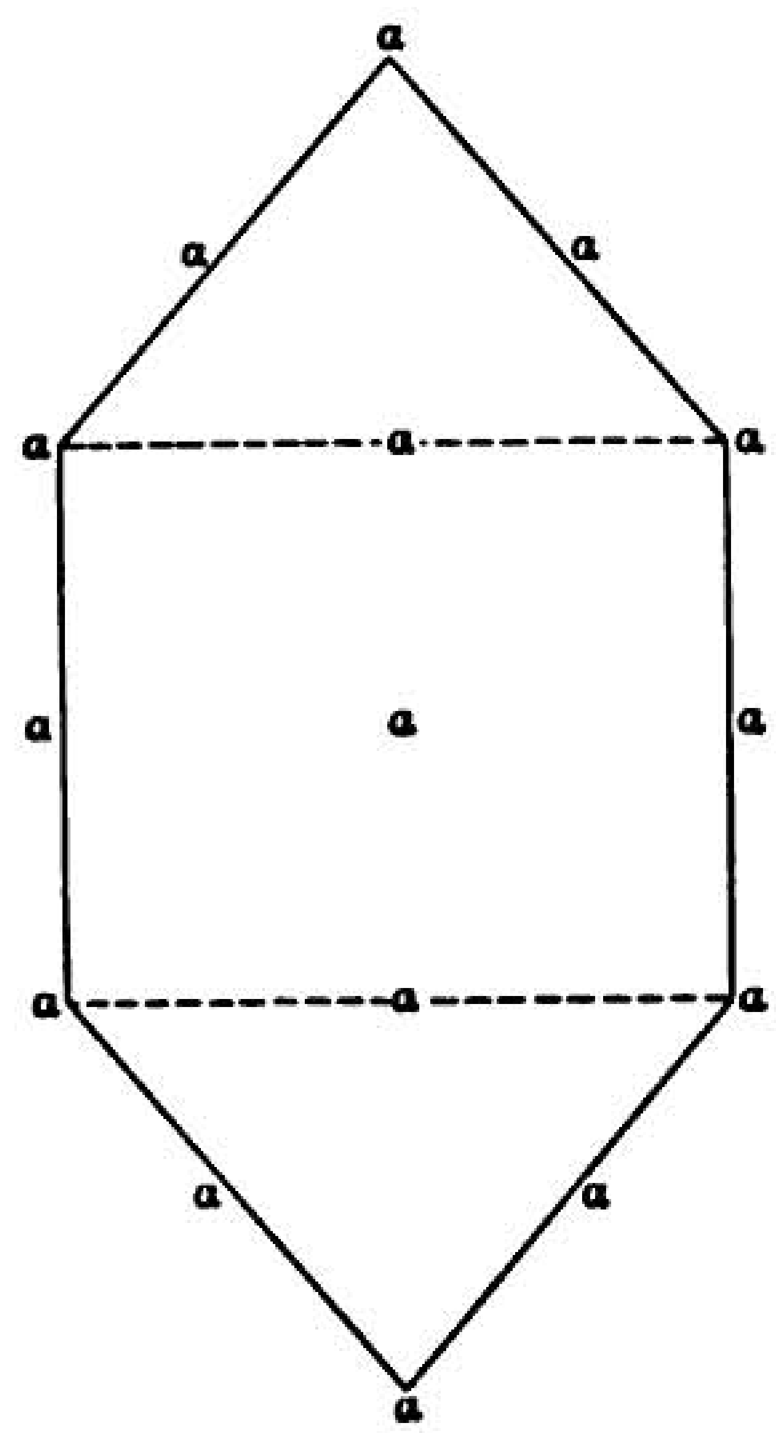


CHAPTER XI

The hexagonal, heptagonal, and succeeding numbers will be set forth in their series by following the same process, if from the natural series of number there be set forth series with their differences increasing by 1. For as the triangular number was produced by admitting into the summation the terms that differ by 1 and do not pass over any in the series; as the square was made by adding the terms that differ by 2 and are one place apart, and the pentagon similarly by adding terms with a difference of 3 and two places apart (and we have demonstrated these, by setting forth examples both of them and of the polygonal numbers made from them), so likewise the hexagons will have as their root-numbers[1] those which differ by 4 and are three places apart in the series, which added together in succession will produce[2] the hexagons. For example, 1, 5, 9, 13, 17, 21, and so on; so that the hexagonal numbers produced will be 1, 6, 15, 28, 45, 66, and so on, as far as one wishes to go. 1

The heptagonals, which follow these, have as their root-numbers terms differing by 5 and four places apart in the series, like 1, 6, 11, 16, 21, 26, 31, 36, and so on. The heptagons that thus arise are 1, 7, 18, 34, 55, 81, 112, 148, and so forth. 2

[1] That is, gnomons; the term being used in the broader sense. See on I. 9. 4, and cf. II. 9. 3.
[2] MS G gives the following diagram of the hexagonal number 15:



3 The octagonals[1] increase after the same fashion, with a difference of 6 in their root-numbers and corresponding variation in their total constitution.

4 In order that, as you survey all cases, you may have a rule generally applicable,[2] note that the root-numbers of any polygonal differ by 2 less than the number of the angles shown by the name of the polygonal — that is, by 1 in the triangle, 2 in the square, 3 in the pentagon, 4 in the hexagon, 5 in the heptagon, and so on, with similar increase.

CHAPTER XII

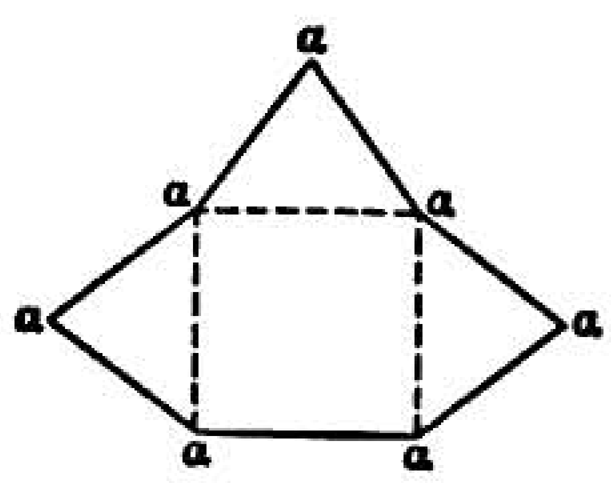
1 Concerning the nature of plane polygonals this is sufficient for a first *Introduction*. That, however, the doctrine of these numbers is to the highest degree in accord with their geometrical representation, and not out of harmony with it, would be evident, not only from the graphic representation in each case, but also from the following:

[1] The following illustrations are from the same MS:

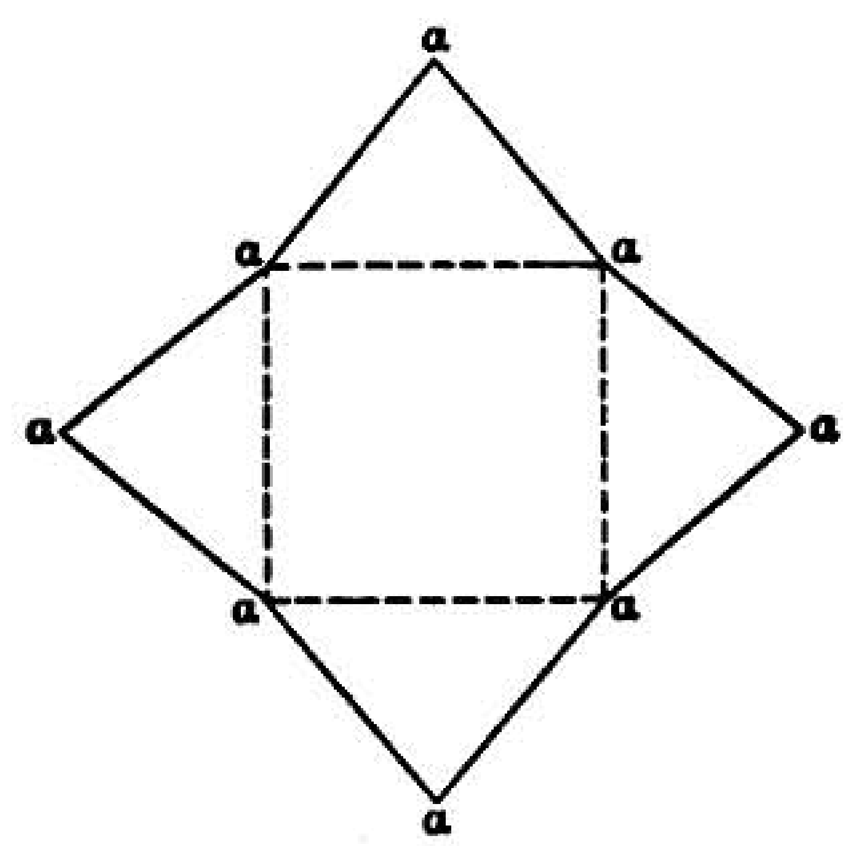
Derivation of heptagonals:

$\underbrace{1, 2, 3, 4, 5, 6, 7}_{7}$, $\underbrace{8, 9, 10, 11}_{18}$, $\underbrace{12, 13, 14, 15, 16}_{34}$, $\underbrace{17, 18, 19, 20, 21}_{55}$

Heptagonal



Octagonal



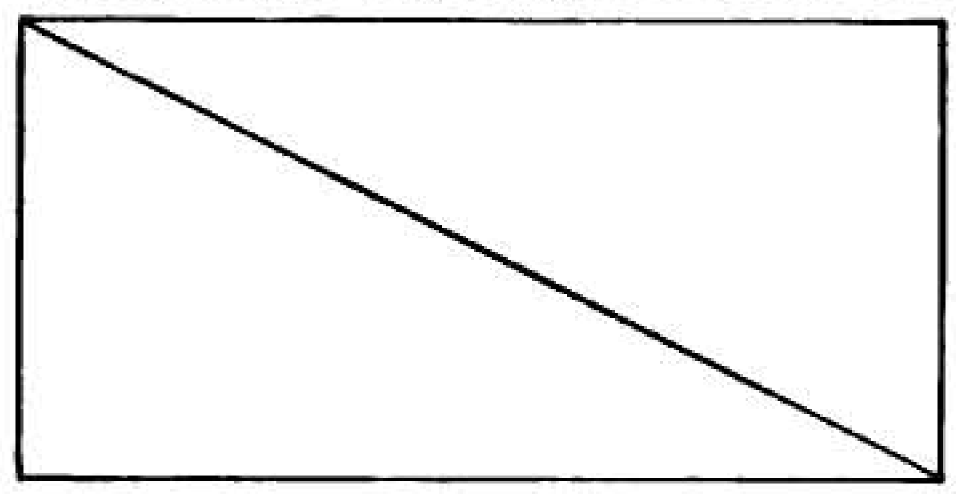
[2] Cf. also Theon, pp. 34, 6 and p. 40, 11 ff. The principle here stated by Nicomachus had already been given by Hypsicles (*ca.* 180 B.C.), whose theorem is cited by Diophantus (*De Polygonis Numeris, Prop.* IV) as follows: "If as many numbers as you please be set out at equal interval from 1, and the interval is 1, their sum is a triangular number; if the interval is 2, a square; if 3, a pentagonal; and generally the number of angles is greater by 2 than the interval." Diophantus gives this as a theorem of 'Hypsicles ἐν ὅρῳ,' which may mean either that it occurred 'in a definition' which he made somewhere in his writings, or that it was in a book called Ὅρος. Cf. Nesselmann, *op. cit.*, p. 466; Gow, *op. cit.*, p. 87.

Every square figure[1] diagonally divided is resolved into two triangles and every square number is resolved into two consecutive triangular numbers, and hence is made up of two successive triangular numbers. For example, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, and so on, are triangular numbers and 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, squares. If you add 2
any two consecutive triangles that you please, you will always make a square, and hence, whatever square you resolve, you will be able to make two triangles of it.

Again, any triangle[2] joined to any square figure makes a pentagon, for example, the triangle 1 joined with the square 4 makes the pentagon 5; the next triangle, 3 of course, with 9, the next square, makes the pentagon 12; the next, 6, with the next square, 16, gives the next pentagon, 22; 10 and 25 give 35; and so on.

Similarly,[3] if the triangles are added to the pentagons, following 3

[1] MS G gives the following figure as an illustration. The principle may be proved from the formulas of arithmetic progression,



$$S = \frac{n}{2}(a + l),\ l = a + (n - 1)d.$$

Two successive triangular numbers, formed according to definition by the summation of n and $n + 1$ terms respectively, will therefore be $\frac{n^2 + n}{2}$ and $\frac{n^2 + 3n + 2}{2}$, and their sum is $n^2 + 2n + 1$, which is $(n + 1)^2$, a perfect square.

The Neo-Pythagoreans employed an interesting development of this principle to display the relative characters of the monad and the dyad (cf. *Theol. Arith.*, p. 9 Ast, and Iamblichus *In Nic.*, p. 75, 20 ff.). The matter is stated in the *Theol. Arith.*, *l. c.*, as follows: The monad is the cause of squares not only because the odd numbers successively arranged about it give squares, but also "because each side, as the turning point (*sc.* of a double race course) from the monad as starting point to the monad as finish line has as the sum of its going forth and of its return its own square" (ἀλλ' ὅτι καὶ ἑκάστη πλευρά, ὥσπερ καμπτὴρ ἀπὸ ὕσπληγγος μονάδος εἰς δὲ νύσσαν μονάδα, πάλιν εἶχε τῆς προόδου καὶ ἐπανόδου τὴν σύνθεσιν ἀφ' ἑαυτῆς αὐτὸν τὸν τετράγωνον). That is, to take the side 5, when the successive numbers up to 5 are set out as one side of the race-track, 5 is made the turning point and the other side is made up of the descending numbers to 1, e.g.,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ & & & & 5 \\ 1 & 2 & 3 & 4 & \end{array}$$

the sum of the whole series is 25, or 5^2. The series 1 . . . 5, of course, is one triangular number, and the descending series 4 . . . 1 the immediately preceding one. From its resemblance to the double race course of the Greek games this proposition was apparently recognized under the name 'diaulos' (cf. Iamblichus, p. 75, 25). Its further application to the heteromecic numbers is not pertinent to the present subject.

[2] This may be seen by comparing the figure of the pentagon as shown in the diagrams accompanying Chapter X; and it is an argument in favor of representing them as does MS G.

[3] This proposition and the preceding are special cases of the theorem that the polygonal number of r sides with side n, plus the triangular number with side $n - 1$, makes the polygonal number with $r + 1$ sides and side n. Algebraically $\frac{n+1}{2}(2 + nd) + \frac{n(n+1)}{2} = \frac{n+1}{2}[2 + n(d + 1)]$.

the same order, they will produce the hexagonals in due order, and again the same triangles with the latter will make the heptagonals in order, the octagonals after the heptagonals, and so on to infinity.

4 To remind us, let us set forth rows of the polygonals, written in parallel lines, as follows: The first row, triangles, the next squares, after them pentagonals, then hexagonals, then heptagonals, then if one wishes the succeeding polygonals.

Triangles	1	3	6	10	15	21	28	36	45	55
Squares	1	4	9	16	25	36	49	64	81	100
Pentagonals	1	5	12	22	35	51	70	92	117	145
Hexagonals	1	6	15	28	45	66	91	120	153	190
Heptagonals	1	7	18	34	55	81	112	148	189	235

You can also set forth the succeeding polygonals in similar parallel lines.

5 In general, you will find that the squares are the sum of the triangles above those that occupy the same place in the series, plus the numbers of that same class in the next place back;[1] for example, 4 equals 3 plus 1, 9 equals 6 plus 3, 16 equals 10 plus 6, 25 equals 15 plus 10, 36 equals 21 plus 15, and so on.

The pentagons are the sum of the squares above them in the same place in the series, plus the elementary triangles that are one place further back in the series; for example, 5 equals 4 plus 1, 12 equals 9 plus 3, 22 equals 16 plus 6, 35 equals 25 plus 10, and so on.

6 Again, the hexagonals are similarly the sums of the pentagons above them in the same place in the series plus the triangles one place back; for instance, 6 equals 5 plus 1, 15 equals 12 plus 3, 28 equals 22 plus 6, 45 equals 35 plus 10, and as far as you like.

7 The same applies to the heptagonals, for 7 is the sum of 6 and 1, 18 equals 15 plus 3, 34 equals 28 plus 6, and so on. Thus each polygonal number is the sum of the polygonal in the same place in the series with one less angle, plus the triangle, in the highest row, one place back in the series.

8 Naturally, then, the triangle is the element of the polygon[2] both in figures and in numbers, and we say this because in the table, reading

[1] That is, in the column next to the left.

[2] Cf. II. 7. 4. *Theol. Arith.*, p. 8 Ast, states that the triangle is the element of both magnitudes and numbers and is made by the congress of the monad and the dyad.

either up and down[1] or across, the successive numbers in the rows are discovered to have as differences the triangles in regular order.

CHAPTER XIII

From this it is easy to see what the solid number is and how its series advances with equal sides; for the number which, in addition to the two dimensions contemplated in graphic representation in a plane, length, and breadth, has a third dimension, which some call depth, others thickness, and some height, that number would be a solid number, extended in three directions and having length, depth, and breadth. 1

This first makes its appearance in the so-called pyramids. These 2
are produced from rather wide bases narrowing to a sharp apex, first after the triangular form[2] from a triangular base, second after the form of the square from a square base, and succeeding these after the pentagonal form from a pentagonal base, then similarly from the hexagon, heptagon, octagon, and so on indefinitely.

Exactly so among the geometrical solid figures; if one imagines three 3
lines from the three angles of an equilateral triangle, equal in length to the sides of the triangle, converging in the dimension height to one and the same point, a pyramid would be produced, bounded by four triangles, equilateral and equal one to the other, one the original triangle, and the other three bounded by the aforesaid three lines. And 4
again, if one conceives of four lines starting from a square, equal in length to the sides of the square, each to each, and again converging in the dimension height to one and the same point, a pyramid would be completed with a square base and diminishing in square form, bounded by four equilateral triangles and one square, the original

[1] Ast, *Theol. Arith.*, p. 288, declares that καὶ πλάτος (the reading of the Paris MS for Hoche's καὶ κατὰ πλάτος, p. 99, 5) is an interpolation, but Hoche retains the words on the authority of Philoponus. The triangular numbers are the differences in the table taken 'in depth' (κατὰ βάθος); for in reading down the second column the common difference is 1, that of the third column is 3, of the fourth 6, and so on, the differences agreeing in turn with each of the triangular numbers. This observation is omitted by Boethius, who devotes II. 19 to showing that the triangular numbers furnish the differences taken across the breadth. When the numbers of the table are compared with those of the same column but in the row next above, and the comparisons are carried across the whole table, the differences are found to be the triangular numbers. Algebraically the corresponding equation is the same as that given in the note above to II. 12. 3.

[2] That is, successive sections parallel to the base are triangular. On pyramids, cf. Theon, p. 42, 3 ff.

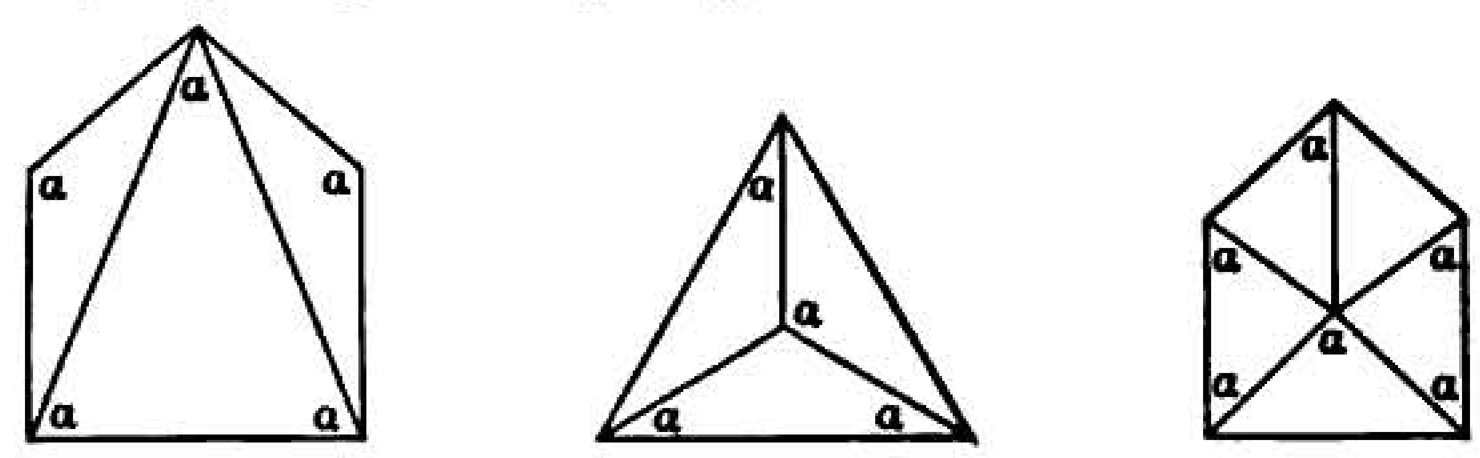
5 one. And starting from a pentagon,[1] hexagon, heptagon, and however far you care to go, lines equal in number to the angles, erected in the same fashion from the angles and converging to one and the same point, will complete a pyramid named from its pentagonal, hexagonal, or heptagonal base, or similarly.

6 So likewise among numbers, each linear number increases from unity, as from a point, as for example, 1, 2, 3, 4, 5, and successive numbers to infinity; and from these same numbers, which are linear and extended in one direction, combined in no random manner, the polygonal and plane numbers are fashioned — the triangles by the combination of root-numbers [2] immediately adjacent, the square by adding
7 every other term, the pentagons every third term, and so on. In exactly the same way, if the plane polygonal numbers are piled one upon the other and as it were built up, the pyramids that are akin to each of them are produced, the triangular pyramid from the triangles, the square pyramid from the squares, the pentagonal from the pentagons, the hexagonal from the hexagons, and so on throughout.

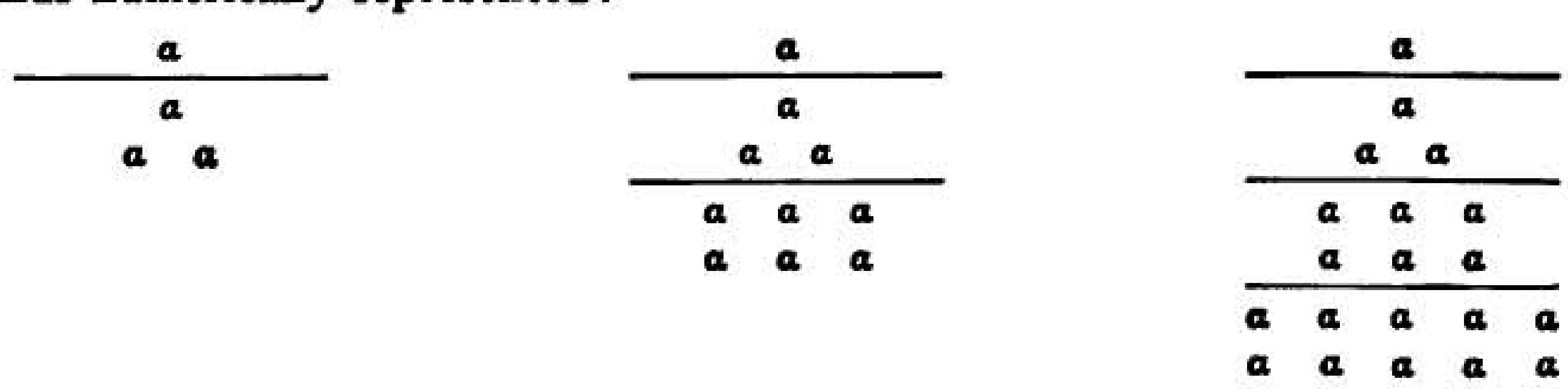
8 The pyramids with a triangular base, then, in their proper order, are these: 1, 4, 10, 20, 35, 56, 84, and so on; and their origin is the piling up of the triangular numbers one upon the other, first 1, then 1, 3, then 1, 3, 6, then 10 in addition to these, and next 15 together with the foregoing, then 21 besides these, next 28, and so on to infinity.

9 It is clear that the greatest number is conceived of as being lowest,

[1] The following diagrams are from Codex G:
Pyramids on square, triangular and pentagonal bases:



Pyramids numerically represented:



First Pyramid *Second Pyramid* *Third Pyramid*

They are built up in layers as it were (cf. sections 7, 9 *infra*), like piles of shot or spheres of any kind, and the layers are the triangular numbers in order. If all were put in triangular form, it would be clearer.

[2] That is, gnomons; see on I. 9. 4. In this case the gnomons are the natural series.

for it is discovered to be the base; the next succeeding one is on top of it, and the next on top of that; until unity appears at the apex and, so to speak, tapers off the completed pyramid into a point.

CHAPTER XIV

The next pyramids in order are those with a square base which rise 1
in this shape to one and the same point. These are formed in the same way as the triangular pyramids of which we have just spoken. For if I extend in series the square numbers in order beginning with unity, thus, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, and again set the successive terms, as in a pile, one upon the other in the dimension height, when I put 1 on top of 4, the first actual pyramid with square base, 5 is pro- 2
duced, for here again unity is potentially the first. Once more, I put this same pyramid entire, composed of 5 units, just as it is, upon the square 9, and there is made up for me the pyramid 14, with square base and side 3 — for the former pyramid had the side 2, and the one potentially first 1 as a side. For here too each side of any pyramid whatsoever must consist of as many units as there are polygonal numbers piled together to create it.

Again, I place [1] the whole pyramid 14, with the square 9 as its base, 3
upon the square 16 and I have 30, the third actual pyramid of those that have a square base, and by the same order and procedure from a pentagonal, hexagonal, or heptagonal base, and even going on farther, we shall produce pyramids by piling upon one another the corresponding polygonal numbers, starting with unity as the smallest and going on to infinity in each case.

[1] The square pyramids might be represented thus:

```
  a             a              a

 a a           a a            a a
 a a           a a            a a

              a a a          a a a
              a a a          a a a
              a a a          a a a

                            a a a a
                            a a a a
                            a a a a
                            a a a a
```

These layers are to be piled one above the other in space, and thus the edges will contain as many units as there are layers, or, in other words, as many as the numbers of square numbers taken in addition.

4 From this too it becomes evident that triangles are the most elementary; for absolutely all of the pyramids that are exhibited and shown, with the various polygonal bases, are bounded by triangles up to the apex.

5 But lest we be heedless of truncated, bi-truncated, and tri-truncated pyramids, the names of which we are sure to encounter in scientific writings, you may know that if a pyramid with any sort of polygon as its base, triangle, square, pentagon, or any of the succeeding polygons of the kind, when it increases by this process of piling up does not taper off into unity, it is called simply truncated when it is left without the natural apex that belongs to all pyramids; for it does not terminate in the potential polygon, unity, as in some one point, but in another polygon, and an actual one, and unity is not its apex, but its upper boundary becomes a plane figure with the same number of angles as the base. If, however, in addition to the failure to terminate in unity it does not even terminate in the polygon next to unity and the first in actuality, such a pyramid is called bi-truncated, and if, still further, it does not have the second actual polygon at its upper limit, but only the one next beneath, it will be called tri-truncated, yes, even four times truncated, if it does not have the next one as its limit, or five times truncated at the next step, and so on as far as you care to carry the nomenclature.

CHAPTER XV

1 While the origin, advance, increase, and nature of the equilateral solid numbers of pyramidal appearance is the foregoing, with its seed and root in the polygonal numbers and the piling up of them in their regular order, there is another series of solid numbers of a different kind, consisting of the so-called cubes, 'beams,' 'bricks,' 'wedges,' spheres and parallelepipedons, which has the order of its progress somewhat as follows:

2 The foregoing squares 1, 4, 9, 16, 25, 36, 49, 64, and so on, which are extended in two directions and in their graphic representation in a plane have only length and breadth, will take on yet a third dimension and be solids and extended in three directions if each is multiplied by its own side; 4, which is 2 times 2, is again multiplied by 2, to make 8; 9, which is 3 times 3, is again increased by 3 in another dimension and

gives 27; 16, which is 4 times 4, is multiplied by its own side, 4, and 64 results; and so on with the succeeding squares throughout.

Here, too, the sides will be composed of as many units as were in the sides of the squares from which they arose, in each case; the sides of 8 will be 2, like those of 4; those of 27, 3, like those of 9; those of 64, 4, like those of 16; and so on, so that likewise the side of unity, the potential cube, will be 1, which is the side of the potential square, 1. 3

In general, each square is a single plane, and has four angles and four sides, while each several cube, having increased out of some one square multiplied by its own side, will have always six plane surfaces, each equal to the original square, and twelve edges, each equal to and containing exactly the same number of units as each side of the original square, and eight solid angles, each of which is bounded by three edges like in each case to the sides of the original square.

CHAPTER XVI

Now since the cube is a solid figure with equal sides in all dimensions, 1
in length, depth, and breadth, and is equally extended in all the six so-called directions,[1] it follows that there is opposed to it that which has its dimensions in no case equal to one another, but its depth unequal to its breadth and its length unequal to either of these, for example 2 times 3 times 4, or 2 times 4 times 8, or 3 times 5 times 12, or a figure which follows some other scheme of inequality.

Such solid figures,[2] in which the dimensions are everywhere unequal 2
one to another, are called scalene in general. Some, however, using other names, call them 'wedges,' for carpenters', house-builders' and blacksmiths' wedges and those used in other crafts, having unequal sides in every direction, are fashioned so as to penetrate; they begin with a sharp end and continually broaden out unequally in all the dimensions. Some also call them *sphekiskoi*, 'wasps,' because wasps' bodies also are very like them, compressed in the middle and showing the resemblance mentioned. From this also the *sphekoma*,[3] 'point of the helmet,' must derive its name, for where it is compressed it imitates the waist of the wasp. Others call the same numbers 'altars,' using

[1] Cf. II. 6. 4 and the note.

[2] Cf. Theon's brief account of the solid numbers, p. 41, 8 ff. He has only the name 'little altars' (cf. below) for scalene numbers.

[3] The point of the helmet where the plume was affixed.

their own metaphor, for the altars of ancient style, particularly the Ionic, do not have the breadth equal to the depth, nor either of these equal to the length, nor the base equal to the top, but are of varied dimensions everywhere.

3 Now whereas the two kinds of numbers, cube and scalene, are extremes, the one equally extended in every dimension, the other unequally, the so-called parallelepipedons are solid numbers like means between them. The plane surfaces of these are heteromecic numbers,[1] just as in the case of the cubes the faces were squares, as has been shown.

CHAPTER XVII

1 Again, then, to take a fresh start, a number is called heteromecic[2] if its representation, when graphically described in a plane, is quadrilateral and quadrangular, to be sure, but the sides are not equal one to another, nor is the length equal to the breadth, but they differ by 1. Examples are 2, 6, 12, 20, 30, 42, and so on, for if one represents them graphically he will always construct them thus: 1 times 2 equals 2, 2 times 3 equals 6, 3 times 4 equals 12, and the succeeding ones similarly, 4 times 5, 5 times 6, 6 times 7, 7 times 8, and thus indefinitely, provided only that one side is greater than the other by 1 and by no other number. If, however, the sides differ otherwise than by 1, for instance, by 2, 3, 4 or succeeding numbers, as in 2 times 4, 3 times 6, 4 times 8, or however else they may differ, then no longer will such a number be properly called a heteromecic, but an oblong number. For the ancients of the school of Pythagoras and his successors saw 'the other' [3] and 'otherness' primarily in 2, and 'the same' and 'sameness'

[1] See the following chapter.

[2] There is no good English equivalent for ἑτερομήκης. Boethius calls this number *altera parte longior*. To this class belong numbers of the type $n(n + 1)$. The definition is repeated in II. 18. 2; cf. Theon, p. 26, 21 ff.

[3] 'The other,' 'difference,' 'the same,' and 'sameness' are Platonic terms, rather than early Pythagorean. They could have been included as opposites in the lists of such (the συστοιχίαι), such as that preserved by Aristotle in *Met.*, I. 5; but they do not occur there. On the other hand we are informed by Simplicius (*Phys.*, 181, 7 D), quoting Eudorus, that the Pythagoreans made the ἀρχή primarily 'the one' (τὸ ἕν), secondarily 'the one' and its opposite, under which were classified respectively 'elegant things' (ἀστεῖα) and 'trivial things' (φαῦλα). This second ἀρχή, Eudorus further says, was called the 'indefinite dyad' (ἀόριστος δυάς). This latter again is a Platonic term. 'The same' and 'the other' (ταὐτόν, θάτερον) may be seen in a Platonic context in the famous account of the making of the world-soul, *Timaeus*, 35 A ff. (See on II. 18. 4), and are generally considered to be Pythagorean at least in ultimate origin. Plato, however, was

in 1, as the two beginnings of all things, and these two[1] are found to differ from each other only by 1. Thus 'the other' is fundamentally 'other' by 1, and by no other number, and for this reason customarily 'other'[2] is used, among those who speak correctly, of two things and not of more than two.

Moreover, it was shown that all odd number is given its specific 2
form[3] by unity, and all even number by 2. Hence we shall naturally say that the odd partakes of the nature of 'the same,' and the even of that of 'the other'; for indeed there are produced by the successive additions of each of these — naturally, and not by our decree — by the addition of the odd numbers from 1 to infinity the class of the squares, and by the addition of the evens from 2 to infinity, that of the heteromecic numbers.[4]

There is, accordingly, every reason to think that the square once 3
more shares in the nature of the same; for its sides display the same ratio, alike, unchanging and firmly fixed in equality, to themselves; while the heteromecic number partakes of the nature of the other; for just as 1 is differentiated from 2, differing by 1 alone, thus also the

undoubtedly the one who contributed most to the vogue of these particular terms. Nicomachus's present statements, then, may reasonably be regarded as in accord with later Pythagoreanism which was strongly influenced by Plato. Cf. also Theophrastus, *Met.*, 33, p. 322, 14 Br. Theon of Smyrna describes the heteromecic numbers in a manner that agrees in the main with Nicomachus. He briefly defines them (p. 26, 21) as "those with one side greater than the other by a unit," and notes two methods of producing them in series, (*a*) by adding together in succession the terms in the series of even numbers, and (*b*) by multiplying together successive pairs of terms in the natural series. Both methods are mentioned by Nicomachus (sections 1, 2).

[1] Cf. the picturesque personification of Theon (p. 27, 1): "For the beginning of numbers, the monad, which is odd, seeking 'otherness,' made the dyad heteromecic by its own doubling" (ἡ γὰρ ἀρχὴ τῶν ἀριθμῶν, τουτέστιν ἡ μονάς, περιττὴ οὖσα τὴν ἑτερότητα ζητοῦσα τὴν δυάδα ἑτερομήκη τῷ αὐτῆς διπλασιασμῷ ἐποίησε).

[2] A somewhat similar distinction in terms was adopted by the arithmologists (see p. 117, n. 4) as a topic in praise of the number 3 (See *Theol. Arith.*, p. 14 Ast; Lydus, *De Mensibus*, IV. 64 Wünsch; Anatolius, p. 31, 8 ff. Heiberg; Chalcidius, *In Timaeum*, c. XXXVIII; Theon of Smyrna, p. 100, 13 ff. Hiller). The purport of these passages is that of 3 we can first use the term 'all,' for of one thing or two things we say 'one' or 'both.' The *Theologumena Arithmeticae* adds that, in expressions like 'thrice ten thousand,' 3 is used as a symbol of plurality. The notion that 3 was called 'all' as the first possessor of beginning, middle, and end is coupled with the statement above in some of the sources cited. These passages have a bearing on the present utterance of Nicomachus so far as they illustrate the Pythagorean idea that 'otherness,' represented by 2, and 'plurality' are not identical. Duality and 'otherness,' first seen in and typified by 2, are elementary; plurality is derived.

[3] Boethius, 2. 27, gives the following explanation why the odd is founded (*perfici* is his expression) on unity and the even on the dyad: *Nam cuiuscunque medietas unus est, ille impar est; cuius vero duo, hic paritate recepta in gemina aequa disiungitur.* Cf. I. 7. 2.

[4] This method of deriving the heteromecic series is given below in II. 18. 2 and 20. 3, and by Theon (p. 27, 8 ff., 31, 14 ff.).

sides of every heteromecic number differ from one another, one differing from the other by 1 alone.

To illustrate, if I have set out before me the successive numbers in series beginning with 1, and select and arrange by themselves the odd numbers in the line and the even by themselves in another, there are obtained these two series:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28

4 Now, then, the beginning of the odd series is unity, which is of the same class as the series and possesses the nature of 'the same,' and so whether it multiplies itself in two dimensions or in three [1] it is not made different, nor yet does it make any other number depart from what it was originally,[2] but keeps it just as it was. Such a property it is impossible to
5 find in any other number. Of the other series the beginning is 2, which is similar in kind to this series and imitates 'otherness'; for whether it multiplies itself or another number, it causes a change,[3] for example, 2 times 2, 2 times 3.

6 But in cases like 8 times 8 times 2, or 8 times 8 times 3, such solid forms are called 'bricks,' [4] the product of a number by itself and then by a smaller number; if, however, a greater height is joined to the square, as in 3 times 3 times 7, 3 times 3 times 8, or 3 times 3 times 9, or however many times the square be taken, provided only it be a greater number of times than the square itself, then the number is a 'beam,' the product of a number by itself and then by a larger number. The

[1] ἐπιπέδως ἢ στερεῶς: 'as a surface or as a solid.'

[2] That is, when it multiplies any other number. Boethius, II. 28, says of unity: . . . *in tantum eiusdem nec mutabilis substantiae est, ut, cum vel se ipsa multiplicaverit vel in planitudine vel in profunditate, vel si alium quemlibet numerum per se ipsa multiplicet, a prioris quantitatis forma non discrepet.* Cf. II. 6. 3.

[3] ἔκστασις: literally 'a standing out of' (*sc.* its former state or, as here, number), hence, 'change.' Cf. Aristotle, *De Anima*, 406 b 13, πᾶσα κίνησις ἔκστασίς ἐστι τοῦ κινουμένου ᾗ κινεῖται. When 2 is the multiplier, the result is always different from the multiplicand; for Nicomachus's number system, consisting of positive integers only, $2x$ is always different from x.

[4] Such a definition as this suits well certain kinds of Roman bricks which were square in their broadest aspect and relatively thin. The Romans introduced baked brick into Greek lands, and Nicomachus would doubtless be acquainted with this variety. Theon, p. 41, 8 ff., gives the same name and definition. Theon also similarly names and defines 'beams' and cubes, but for the 'wedges' he has only the name 'little altars' (cf. II, 16. 2) of the several that Nicomachus uses. Hero of Alexandria (Definition 113, in Hultsch, *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, p. 31) defines 'bricks' as solids with the length less than the breadth and depth, the two latter being sometimes equal (on the 'bricks,' cf. also Theon, p. 113, 5); and the 'beam' (*ibid.*, *Definition* 112) he defines as a solid having a length greater than the breadth or thickness, the two latter being sometimes equal.

'wedges,' to be sure, were the products of three unequal numbers, and cubes of three equal ones.

Among the cubes, some of them, in addition to being the product of 7
three equal numbers, have the further property of ending at every multiplication in the same number as that from which they began; these are called spherical, and also recurrent.[1] Such indeed are those with sides 5 or 6; for however many times I increase each one of these, it will by all means end each time in the same figure, the derivative of 6 in 6 and that of 5 in 5. For example, the product of 5 times 5 will end in 5, and so will 5 times this product and if necessary, 5 times this again, and to infinity no other concluding term will be found except 5. From 6, too, in the same fashion 6 and no other will be the concluding term; and so 1 likewise is potentially spherical and recurrent, for as is reasonable it has the same property as the spheres and circles. For each one of them, circling and turning around, ends where it begins. And so these numbers aforesaid are the only ones of the products of equal factors to return to the same starting point from which they began, in the course of all their increases. If they increase in the manner of planes, in two dimensions, they are called circular, like 1, 25, and 36, derived from 1 times 1, 5 times 5, and 6 times 6; but if they have three dimensions, or are multiplied still further than this, they are called spherical solid numbers, for example 1, 125, 216, or, again, 1, 625, 1,296.

CHAPTER XVIII

Regarding the solid numbers this is for the present sufficient. The 1
physical philosophers, however, and those that take their start with mathematics, call 'the same' and 'the other' the principles of the universe, and it has been shown that 'the same' inheres in unity and the odd numbers, to which unity gives specific form, and to an even greater degree in the squares, made by the continued addition of odd numbers, because in their sides they share in equality; while 'the other' inheres in 2 and the whole even series, which is given specific form by 2, and partic-

[1] ἀποκαταστατικοί: So Theon of Smyrna, p. 38, 16 ff. Hiller, citing 5 and 6 as examples. Lydus also (*De Mensibus*, IV. 76 Wünsch) calls 5 a σφαῖρα for the same reason. This property of 5 is mentioned also by Anatolius, p. 33, 2 ff. Heiberg; and by Capella, *De Nuptiis Phil. et Merc.*, VII, 735 (who calls it *apocatastaticus*). Anatolius remarks on the similar property of 6; cf. also *Theol. Arith.*, p. 35 Ast. In fact these propositions were regular topics of arithmology.

ularly in the heteromecic numbers, which are made by the continued addition of the even numbers, because of the share of the original inequality[1] and 'otherness' which they have in the difference between their sides. Therefore it is most necessary further to demonstrate how in these two, as in origins and seeds, there are potentially existent all the peculiar properties of number, of its forms and subdivisions, of all its relations, of polygonals, and the like.

2 First, however, we must make the distinction whereby the oblong (promecic) number[2] differs from the heteromecic. The heteromecic is, as was stated above,[3] the product of a number multiplied by another larger than the first by 1, for example, 6, which is 2 times 3, or 12, which is 3 times 4. But the oblong is similarly the product of two differing numbers, differing, however, not by 1 but by some larger number, as 2 times 4, 3 times 6, 4 times 8, and similar numbers, which in a way exceed in length and overstep the difference of 1.

3 Therefore, since squares are produced from the multiplication of numbers by their own length, and have their length the same as their breadth, properly speaking they would be called 'idiomecic' or 'tautomecic';[4] for example, 2 times 2, 3 times 3, 4 times 4, and the rest. And if this is true, they will admit in every way of sameness and equality, and for this reason are limited and come to an end; for 'the equal' and 'the same' are so in one definite way. But since the heteromecic numbers are produced by the multiplication of a number by not its own, but another number's length, they are therefore called 'heteromecic,' and admit of infinity and boundlessness.

4 In this way, then, all numbers and the objects in the universe which have been created with reference to them are divided and classified and are seen to be opposite one to another, and well do the ancients at the very beginning of their account of Nature make the first subdivision

[1] That is, a unit, for it was shown in II. 17. 2 that 'the other is fundamentally other by the unit,' and the difference between the sides of a heteromecic number is by definition a unit.

[2] προμήκης: Theon of Smyrna, p. 27, 23 ff., gives a similar definition of this class of numbers, though he calls them παραλληλόγραμμοι ἀριθμοί, but in p. 30, 8 ff. he defines προμήκεις as the products of two unequal terms, which differ by 1, 2, or any other number, thus including the heteromecic among the oblong numbers. Theon gives the following classification of oblongs in connection with the definition just cited: (*a*) the heteromecic numbers are oblongs in a sense; (*b*) numbers that by one factoring are heteromecic, by another oblong, as 12, which is either 3×4 or 2×6; (*c*) numbers that are oblong by all possible factorings, e.g., 40, which is 2×20, 4×10, or 5×8.

[3] Cf. II. 17. 1.

[4] ἰδιομήκεις . . . καὶ ταυτομήκεις, as opposed to ἑτερομήκεις or προμήκεις.

in their cosmogony on this principle. Thus Plato[1] mentions the distinction between the natures of 'the same' and 'the other,' and again, that between the essence which is indivisible and always the same and the one which is divided; and Philolaus[2] says that existent things must all be either limitless or limited, or limited and limitless at the same time, by which it is generally agreed that he means that the universe is made up out of limited and limitless things at the same time, obviously after the image of number, for all number is composed of unity and the dyad, even and odd, and these in truth display equality and inequality, sameness and otherness, the bounded and the boundless, the defined and the undefined.

CHAPTER XIX

That we may be clearly persuaded of what is being said, namely, 1
that things are made up of warring and opposite elements[3] and have

[1] Cf. Plato, *Timaeus*, 35 A (Archer-Hind's translation): "From the undivided and ever changeless substance and that which becomes divided in material bodies, of both these he mingled in the third place the form of Essence, in the midst between the Same and the Other; and this he composed on such wise between the undivided and that which is in material bodies divided; and taking them, three in number, he blended them into one form, forcing the nature of the Other, hard as it was to mingle, into union with the Same," etc.

[2] Philolaus, the Pythagorean, was a native of Croton or of Tarentum. Ritter and Preller (*Hist. Phil. Gr.*) give 440 B.C. as his *floruit*. This fragment (1 b Chaignet, 3 Mullach) is found in much fuller form in Stobaeus, *Ecl. Phys.*, I. 21. 7 (vol. I, p. 187, Wachsmuth-Hense).

[3] It is a question whether Nicomachus here has in mind strictly Pythagorean ideas of the origin and constitution of the universe, or the Platonic account in the *Timaeus*, which is in fact strongly Pythagorean in tone. Elsewhere he refers to the *Timaeus* (I. 2. 1; II. 2. 3; 18. 4; 24. 6) and emphasizes the fact that he hopes to make his work useful for the interpretation of Plato (II. 24. 11) and of the ancient texts read in the schools, among which the *Timaeus* was certainly included (II. 21. 1; 28. 1). There is so much in common between Plato and the Pythagoreans that probably Nicomachus would think of both in making this statement. For the Pythagorean doctrine that chaotic matter was ordered on harmonic principles cf. Philolaus, in Stobaeus, *op. cit.*, p. 189 (fr. 4 Chaignet, 3 Mullach): ἐπεὶ δὲ ταὶ ἀρχαὶ ὑπᾶρχον οὐχ ὁμοῖαι οὐδ' ὁμόφυλοι ἔσσαι, ἤδη ἀδύνατον ἦς κα αὐταῖς κοσμηθῆμεν, αἰ μὴ ἁρμονία ἐπεγένετο ᾡτινιῶν τρόπῳ ἐγένετο.

Plato gives a clearer picture of the 'warring and opposite' constituents of the universe (Nicomachus does not call them στοιχεῖα, 'elements') in *Timaeus*, 30 A: βουληθεὶς γὰρ ὁ θεὸς ἀγαθὰ μὲν πάντα, φλαῦρον δὲ μηδὲν εἶναι κατὰ δύναμιν, οὕτω δὴ πᾶν ὅσον ἦν ὁρατὸν παραλαβὼν οὐχ ἡσυχίαν ἄγον ἀλλὰ κινούμενον πλημμελῶς καὶ ἀτάκτως, εἰς τάξιν αὐτὸ ἤγαγεν ἐκ τῆς ἀταξίας. . . .

That this τάξις is a harmony, and furthermore that it is a sort of mathematical harmony, Plato makes clear by showing that it is secured by the interweaving of the world-soul into the whole extent of the universe (36 E) and that the world-soul is constituted on harmonic principles (34 C ff.). We may further compare 53 B: ὅτε δ' ἐπεχειρεῖτο κοσμεῖσθαι τὸ πᾶν, πῦρ πρῶτον καὶ ὕδωρ καὶ γῆν καὶ ἀέρα, ἴχνη μὲν ἔχοντα αὑτῶν ἄττα, παντάπασί γε μὴν διακείμενα ὥσπερ εἰκὸς ἔχειν ἅπαν, ὅταν ἀπῇ τινὸς θεός, οὕτω δὴ τότε πεφυκότα ταῦτα πρῶτον διεσχηματίσατο εἴδεσί τε καὶ ἀριθμοῖς.

The idea of a chaos of warring elements (*frigida pugnabant calidis, umentia siccis*, Ovid, *Met.*, I. 19) is a commonplace in ancient literature after Hesiod (Cf. *Classical Philology*, VIII, p. 405 with note 4). Cf. with this passage I. 6. 3.

in all likelihood taken on harmony — and harmony[1] always arises from opposites; for harmony is the unification of the diverse and the reconciliation of the contrary-minded — let us set forth in two parallel lines no longer, as just previously, the even numbers from 2 by themselves and the odd numbers from 1, but the numbers that are produced from these by adding them successively together, the squares from the odd numbers, and the heteromecic from the even. For if we give careful attention to their setting forth, we shall admire their mutual friendship and their coöperation to produce and perfect the remaining forms, to the end that we may with probability conceive that also in the nature of the universe from some such source as this a similar thing was brought about by universal providence.[2]

2 Let the two series then be as follows: That of the squares, from unity, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, and that of the heteromecic numbers, beginning with 2 and proceeding thus, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240.

3 In the first place, then, the first square is the fundamental multiple[3] of the first heteromecic number; the second, compared to the second, is its sesquialter; the third, sesquitertian of the third; the fourth, sesquiquartan of the fourth; then sesquiquintan, sesquisextan, and so on similarly *ad infinitum*. Their differences,[4] too, will increase according to the successive numbers from 1; the difference of the first terms is 1, of the second 2, of the third 3, and so on. Next, if first the second term of the squares be compared with the first heteromecic number, the third with the second, the fourth with the third, and the rest similarly, they will keep unchanged the same ratios as before, but their differences will begin to progress no longer from 1, but from 2, remaining the same as before, and according to the advance observed in the former comparison, the first to the first will be the first, or root-form, multiple, the second to the second the second sesquialter from the root-

[1] Cf. the remainder of the fragment of Philolaus quoted in the preceding note: *τὰ μὲν ὦν ὁμοῖα καὶ ὁμόφυλα ἁρμονίας οὐδὲν ἐπεδέοντο, τὰ δὲ ἀνόμοια μηδὲ ὁμόφυλα μηδὲ ἰσολαχῆ ἀνάγκα τᾷ τοιαύτᾳ ἁρμονίᾳ συγκεκλεῖσθαι, αἱ μέλλοντι ἐν κόσμῳ κατέχεσθαι.* The words are also quoted by Theon of Smyrna, p. 12, 10, and Ast on the passage cites further Iamblichus, *In Nicom.*, p. 73, 1 Pistelli, and Asclepius, *In Nicom.*

[2] *πρόνοια*, 'forethought,' may be also translated 'providence' and has reference to teleology. Even before Plato's time the teleological idea was in the air, but it was Plato who first made it an essential part of the theory of the constitution of the universe. This is another indication that Nicomachus is decidedly a Platonizing Pythagorean.

[3] That is, the double ($2 = 2 \times 1$). For the use of the term *πυθμήν*, cf. on I. 19. 6.

[4] That is, comparing homologous terms of the two series. The difference between 1 and 2 is 1; between 4 and 6, 2; between 9 and 12, 3; and so on.

form, the third to the third the third sesquitertian from the root-form, and the succeeding terms will go on in similar fashion.

Furthermore, the squares[1] among themselves will have only the odd 4
numbers as differences, the heteromecic, even numbers. And if we put[2] the first heteromecic number as a mean term between the first two squares, the second between the next two, the third between the two following, and the fourth between the two next succeeding, therein will be seen still more regularly the numerical relations[3] in groups of three terms. For as 4 is to 2, so is 2 to 1; and as 9 is sesquialter to 6, so is 6 to 4; and as 16 to 12, so is 12 to 9, and so on, with both numbers and ratios regularly advancing. As the greater is to the mean, so will the mean be to the lesser, and not in the same ratio, but always a different one, by an increase. In all the groupings, too, the product of the extremes is equal to the square of the mean; and the extremes, plus twice the mean, by exchange will always give a square.[4] What is neatest of all, from the addition of both[5] there comes about the production of the triangles in due order, showing that the nature of these is more ancient[6]

[1] *Squares:* 1 4 9 16 25 36 49, etc.
3 5 7 9 11 13, etc.; odd differences.
Heteromecic: 2 6 12 20 30 42 56, etc.
4 6 8 10 12 14, etc.; even differences.

[2] Cf. Theon, p. 28, 16 ff. Hiller.

[3] That is, m^2, $m(m+1)$, and $(m+1)^2$ always constitute the terms of a geometrical proportion. Theon notes this (c. 16), adding that the successive heteromecic numbers do not make a proportion with the included square; i.e., $m(m-1)$, m^2, and $m(m+1)$ are not proportional.

[4] The ratios formed as directed and the additional properties of the series may be seen in this table:

Ratios	*Sum of extremes*	*Plus*	$2 \times$ *mean term*		*Sum*
$1:2 = 2:4$	5	+	4	=	9
$4:6 = 6:9$	13	+	12	=	25
$9:12 = 12:16$	25	+	24	=	49
$16:20 = 20:25$	41	+	40	=	81

Generally, in the ratio $m^2 : m(m+1) = m(m+1) : (m+1)^2$, the sum of the extremes plus twice the mean will be $4m^2 + 4m + 1$, which is a perfect square.

[5] This may best be seen by setting the squares and heteromecic numbers alternately and combining them in pairs thus:

1 2 4 6 9 12 16 20 25 30
3 6 10 15 21 28 36 45 55 = triangular numbers.

[6] This is significant to Nicomachus because he believed that the 'same' and the 'other' were the ultimate elements, and that they resided par excellence in the squares and the heteromecic numbers respectively (Cf. II. 17. 3; 18. 1). Now that into which all plane figures are ultimately analyzed is the triangle (II. 7. 4; cf. 12. 8); as Nicomachus himself shows; and it is to be remembered that in the *Timaeus* the triangle is made the ultimate basis of the corpuscles of the elements, a theory which he doubtless has in mind. Therefore the present proposition confirms his position in holding 'sameness' and 'otherness' to be the most elementary things, prior even to the elementary triangle. An interesting confirmation of the interpretation above is found in

than the origin of all things, thus: 1 plus 2, 2 plus 4, 4 plus 6, 6 plus 9, 9 plus 12, 12 plus 16, 16 plus 20, and by this process the triangles, which give rise to the polygons, come forth in order.

CHAPTER XX

1 Still further, every square plus its own side becomes heteromecic, or by Zeus, if its side is subtracted from it. Thus, 'the other' is conceived of as being both greater and smaller than 'the same,' since it is produced, both by addition and by subtraction, in the same way that the two kinds of inequality[1] also, the greater and the less, have their origin from
2 the application of addition or subtraction to equality. This also is sufficient evidence that the two forms partake of sameness and otherness, of otherness in an indefinite fashion, but of sameness definitely, 1 and 2 generically,[2] but the odd of sameness after the manner of a subordinate species because it belongs to the same class as 1, and the even of otherness because it is homogeneous with 2.
3 There is also a still clearer reason why the square, since it is the product of the addition of odd numbers, is akin to sameness, and the heteromecic numbers to otherness because it is made up by adding even numbers; for as though they were friends of one another, these two forms share in their two rows the same differences when they do not have the same ratios, and conversely the same ratios when they do not have the same differences. For the difference[3] between 4 and 2 in the

Theol. Arith., 8 Ast (on the dyad) in a context certainly Nicomachean: "Wherefore the first congress of these (*sc.* the monad and the dyad) first perfected defined multitude, the element of things, which would be a triangle both of magnitudes and of numbers somatic and bodiless; for as the rennet coagulates the running milk by its active and effective property, so the unifying force of the monad approaching the dyad, which is the source of facile movement and of dissolution, gave a bound and a form that is a number to the triad; for this is the beginning of actuality in number as that is defined by the combination of monads." In the present text 'their nature' refers to the squares and heteromecic numbers and 'the origin of all things' to the triangle, as the elementary figure.

[1] Cf. I. 17. 6. The results obtained by adding to or subtracting their sides from the square numbers are as follows:

$4 + 2 = 6 = 2 \times 3$	$4 - 2 = 2 = 1 \times 2$
$9 + 3 = 12 = 3 \times 4$	$9 - 3 = 6 = 2 \times 3$
$16 + 4 = 20 = 4 \times 5$	$16 - 4 = 12 = 3 \times 4$
$25 + 5 = 30 = 5 \times 6$	$25 - 5 = 20 = 4 \times 5$
or	or
$m^2 + m = (m + 1)m$	$m^2 - m = (m - 1)m$

[2] See p. 101 for comment in this passage.

[3] The examples in the text show sameness of difference coupled with difference of ratio. For the converse, compare 4, 6, and 6, 9. The ratio is sesquialter in both cases, but the difference varies.

double ratio is found between 6 and 4 as a superparticular; and again the difference between 9 and 6, as a sesquialter, is found between 12 and 9 as a sesquitertian, and so on. What is the same in quality[1] is different in quantity, and just the opposite, what is the same in quantity is different in quality. Again, it is clear that in all their relations[2] 4
the same difference between two terms will necessarily be called fractions with names that differ by 1, and be the half of one and the third of the other, or the third of one and the quarter of the other, or the fourth of one and the fifth of the other, and so on.

But what will most of all confirm the fact that the odd, and never 5
the even, is preëminently the cause of sameness, is to be demonstrated in every series beginning with 1 following some ratio, for example, the double ratio, 1, 2, 4, 8, 16, 32, 64, 128, 256, or the triple, 1, 3, 9, 27, 81, 243, 729, 2,187, and as far as you like. You will find[3] that of necessity all the terms in the odd places in the series are squares, and no others by any device whatsoever, and that no square is to be found in an even place.

But all the products of a number multiplied twice into itself, that is, the cubes, which are extended in three dimensions and seen to share in sameness to an even greater extent, are the product of the odd numbers, not the even,[4] 1, 8, 27, 64, 125, and 216, and those that go on analogously, in a simple, unvaried progression as well. For when the successive odd numbers are set forth indefinitely beginning with 1, observe this: The first one makes the potential cube; the next two,

[1] This is substantially a repetition of the previous statement. In the terminology of Nicomachus two pairs of numbers have a relation (σχέσις) qualitatively (ποιότητι) alike if they exhibit the same ratio; numerically (ποσότητι), if they have the same arithmetical difference. This terminology appears again in the discussion of the proportions; cf. 23, 4 below.

[2] σχέσεις: That is, the comparisons of the pairs of terms from the two series; see on 21. 2 below for a further discussion of the meaning of σχέσις 'relation.' For illustration of the meaning, we may take pairs of homologous terms from the two series, as 1, 2; 4, 6; 9, 12; 16, 20; etc., and their differences 1, 2, 3, 4, etc. 1 is the whole of 1 and the half of 2; 2 is the half of 4 and the third of 6; 3 is the third part of 9 and the fourth of 12, etc. Or, compare 4, 2 with 4, 6; 9, 6 with 9, 12; etc.

[3] Philo, *De Mundi Opificio*, 36, points out further properties of the table of doubles: "If one doubles he will find that the third from unity (i.e., counting both ends in the Greek fashion) is a square, the fourth a cube, and the seventh, arising from both the third and the fourth, a cube and square together." Similarly Theon, p. 34, 16 ff., points out that the terms in every other place (he does not specifically say 'odd') of the series of multiples are squares, those in every third place cubes, and those in every fifth place both cubes and squares. He adds furthermore that squares are always divisible by 3 and 4, either as they stand or when 1 is subtracted, that those divisible by 3 when 1 is subtracted are always divisible by 4 if they are even, and vice versa, etc.

[4] See Part I, p. 133.

added together, the second; the next three, the third; the four next following, the fourth; the succeeding five, the fifth; the next six, the sixth; and so on.

CHAPTER XXI

1 After this it would be the proper time to incorporate the nature of proportions, a thing most essential for speculation about the nature of the universe and for the propositions of music, astronomy, and geometry, and not least for the study of the works of the ancients,[1] and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting.

2 A proportion,[2] then, is in the proper sense, the combination of two or more *ratios*, but by the more general definition the combination of

[1] εἰς τὰς τῶν παλαιῶν συναναγνώσεις: Boethius, II. 40: *ad veterum lectionum intelligentiam.*

[2] Two Greek words, ἀναλογία and μεσότης, the former of which is used here, may be translated 'proportion,' and Nicomachus points out that an ἀναλογία is, strictly speaking, a combination of ratios (like 1, 2, 4; i.e., in his classification only the 'geometrical proportions,' γεωμετρικαὶ ἀναλογίαι or μεσότητες). Properly then an arithmetical progression of three or four terms (e.g., 1, 2, 3, or 1, 2, 3, 4) should not be called an ἀναλογία, but in practice it is so called. It was formerly thought until Nesselmann cleared up the matter (*Geschichte der Algebra*, p. 210, note) that ἀναλογία meant, properly, a proportion of four terms (i.e., a disjunct proportion; see section 6), and μεσότης a continued proportion (of three terms; cf. section 5). Arguing from the statements of Iamblichus (pp. 100, 15 ff. and 104, 19 ff. Pistelli; the latter passage runs ἡ δὲ δευτέρα μεσότης ἡ γεωμετρικὴ κυρίως ἀναλογία κέκληται, διότι λόγον τὸν αὐτὸν οἱ ὅροι περιέχουσιν, ἀνὰ τὸν αὐτὸν λόγον διεστῶτες) and Nicomachus himself (see II. 24. 1), Nesselmann states that originally ἀναλογία was applied only to geometrical proportions and μέσοτης to the other two, the harmonic and the arithmetic, but that in later usage the distinction of terms vanished. It is certain that Nicomachus uses them indiscriminately of all three types. But the present passage makes the matter perfectly clear if the proper stress is laid on the words λόγος and σχέσις. Nicomachus here definitely states that a proportion (ἀναλογία) is properly or strictly (κυρίως) the combination of two or more *ratios* (λόγων), but *in a more general sense* (κοινότερον) a combination of *relations* (σχέσεων). Now λόγος is the term which properly means ratio, the measurement of one number in terms of another, and it is not used by Nicomachus with reference to the relations between numbers in any other sense (he does, to be sure, use λόγος with other meanings, but not with reference to the relations between numbers; see the Glossary).

Nicomachus is undoubtedly woefully lacking in precision when he defines λόγος (section 3), but in his usage he is consistent. σχέσις on the other hand means simply *relation* and can refer to any kind of relation, including λόγοι proper and mere numerical excess or deficiency as well; it is therefore a more general term and sometimes, but not always, synonymous with λόγος. Here, however, Nicomachus uses it in the general sense, so that it includes the relation of exceeding and being exceeded. He admits, then, that ἀναλογία is used of arithmetic and harmonic progressions as well as of the true proportions, the geometric. It may be noticed that in discussing arithmetic proportions Nicomachus does not use the term λόγος to describe their mutual relation; in fact he says that they are not in the same λόγος (II. 23. 1: ὅταν . . . μὴ μέντοι λόγος ὁ αὐτὸς ἐν τοῖς ὅροις πρὸς ἀλλήλους γίνηται). In II. 22. 3 (ἐν γὰρ τῇ τοῦ ἁπλοῦ ἀριθμοῦ φυσικῇ ἀπὸ μονάδος ἐκθέσει . . . σώζεται ὁ ταύτης μόνης λόγος, where ταύτης = τῆς ἀριθμητικῆς) two or more *relations*, even if they are not brought under the same ratio, but rather a difference, or something else.

Now a ratio[1] is the relation of two terms to one another, and the 3
combination of such is a proportion, so that three is the smallest number of terms of which the latter is composed, although it can be a series of more, subject to the same ratio or the same difference. For example, 1 : 2 is one ratio, where there are two terms; but 2 : 4 is another similar ratio; hence 1, 2, 4 is a proportion, for it is a combination of ratios, or of three terms which are observed to be in the same ratio to
one another. The same thing may be observed also in greater numbers 4
and longer series of terms; for let a fourth term, 8, be joined to the former after 4, again in a similar relation, the double, and then 16 after 8 and so on.

Now if the same term, one and unchanging, is compared to those on 5

λόγος does not mean 'ratio,' but 'scheme,' 'plan,' or, practically, 'definition.' In I. 5. 1 he speaks of ἁρμονικοὶ λόγοι, meaning the intervals of harmony, but only to point out that these are really ratios in the strict sense.

A good illustration of the proper use of the words λόγος and σχέσις is found in Theon, p. 22, 19 Hiller: αὐξόμενοι δὲ τοὺς λόγους τῆς πρὸς ἀλλήλους σχέσεως αὐτῶν μειοῦσιν ("as they increase they diminish the ratios of their relation to one another"). He is speaking of the fact that in the natural series of numbers the ratios between the terms decrease as you ascend in the series, e.g., 2 : 1 > 3 : 2 > 4 : 3, etc. λόγος is the ratio; σχέσις means the general relation existing between pairs of terms, that is, the simple pairing and comparing of the terms. It is further to be noted with regard to the terminology of proportions that, according to Iamblichus (p. 99, 25 ff.), ἀναλογία properly refers to the continued proportion (that in three terms), as opposed to τὸ ἀνάλογον, which means rather a geometrical progression and is the more proper term to apply to the disjunct proportion (in four terms). Needless to say, the distinction is not observed by Nicomachus.

[1] In view of what has been said in the preceding note this definition is a poor one, for it merely asserts that a ratio (λόγος) is a relation (σχέσις). Nicomachus is either guilty of carelessness, or, as is very probable, the word ποιά has fallen out before σχέσις, leaving no trace in the MSS. The addition of this one word would make the definition fairly satisfactory, although it would still lack the precision of Euclid's, or Theon's. In mathematical language ποιὰ σχέσις 'a relation of some kind,' 'a qualitative relation,' means one that can be described as 'of some kind,' that is, double, triple, sesquialter, or the like, in other words, a ratio proper, and it would be contrasted with ποσὴ σχέσις, 'a relation of a certain amount,' which would mean a relation where a mere arithmetical difference between the terms is in question. Euclid uses this terminology in his definitions of ratio and proportion in Book V, *init.*: "Ratio is the qualitative relation with reference to size between two homologous magnitudes. Proportion is the likeness of ratios" (λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις. ἀναλογία δ' ἐστὶν ἡ τῶν λόγων ὁμοιότης); cf. also Hero of Alexandria, *Definition* 127, ed. Hultsch, p. 36. As Nesselmann (*Gesch. d. Alg.*, 212) showed, the inclusion of ὁμογενῶν, πηλικότητα and ὁμοιότης brings out points overlooked by Nicomachus, but even more important is ποιά. On the necessity of terms in a ratio being homogeneous, see on I. 17. 4; Nicomachus neglects this matter. Theon's definitions of ratio and proportion are more like Euclid's, but that of proportion is either poorly stated or wrongly transmitted: "Ratio is the qualitative relation in analogy (κατ' ἀνάλογον) existing between two terms of the same genus" (p. 73, 16); "Proportion is the qualitative relation of ratios to one another" (p. 74, 12).

either side of it, to the greater as consequent[1] and to the lesser as antecedent, such a proportion is called continued; for example, 1, 2, 4 is a continued proportion as regards quality,[2] for 4 : 2 equals 2 : 1, and conversely 1 : 2 equals 2 : 4. In quantity, 1, 2, 3, for example, is a continued proportion,[3] for as 3 exceeds 2, so 2 exceeds 1, and conversely, as 1 is less than 2, by so much 2 is less than 3.

6 If, however, one term answers to the lesser term, and becomes its antecedent and a greater term, and another, not the same, takes the place of consequent and lesser term with reference to the greater, such a mean and such a proportion is called no longer continued, but disjunct; for example, as regards quality, 1, 2, 4, 8, for 2 : 1 equals 8 : 4, and conversely 1 : 2 equals 4 : 8, and again 1 : 4 equals 2 : 8 or 4 : 1 equals 8 : 2; and in quantity, 1, 2, 3, 4, for as 1 is exceeded by 2, by so much 3 is exceeded by 4, or as 4 exceeds 3, so 2 exceeds 1, and by interchange, as 3 exceeds 1, so 4 exceeds 2, or as 1 is exceeded by 3, by so much 2 is exceeded by 4.

CHAPTER XXII

1 The first three proportions,[4] then, which are acknowledged by all the ancients, Pythagoras, Plato, and Aristotle, are the arithmetic, geometric, and harmonic; and there are three others subcontrary to

[1] On the meaning of 'antecedent' and 'consequent,' see the note on I. 19. 2.

[2] On the meaning of 'quality' referring to ratios, compare on II. 20. 3. With reference to proportions, as here, the meaning is consistent with the former usage. A proportion as regards quality (κατὰ ποιότητα, κατὰ τὸ ποιόν, ποιότητι) is a series of terms exhibiting similar ratios, and a proportion in quantity (κατὰ ποσότητα, κατὰ τὸ ποσόν, ποσότητι) is an arithmetical progression, with a common difference. Cf. also II. 22. 2; 23. 4 below.

[3] συνημμένη: Theon, p. 82, 10, uses the term συνεχής and διῃρημένη for 'disjunct' (in Nicomachus, διεζευγμένη, sect. 6).

[4] Iamblichus adds concerning the history of the proportions (p. 100, 19): "But of old there were but three means in the days of Pythagoras and the mathematicians of his time, the arithmetic, the geometric, and the third in order, which was once called the subcontrary, but had its name forthwith changed to harmonic by the schools of Archytas and Hippasus, because it seemed to embrace the ratios that govern the harmonized and tuneful. And it was formerly called subcontrary because its character was somehow subcontrary to the arithmetic. . . . After this name had been changed, those who came later, Eudoxus and his school, invented three more means, and called the fourth properly subcontrary because its properties were subcontrary to the harmonic . . . and the other two they named simply from their order, fifth and sixth. The ancients and their successors thought that this number, i.e., six, of means could be set up; but the moderns have found four more in addition, devising their formation from the terms and the intervals." Cf. also p. 113, 16 ff. He adds (p. 116, 1 ff.) that the first six were in use from Plato's time to Eratosthenes, and that the other four were devised by Myonides and Euphranor, both Pythagoreans, who lived later. Apparently Moderatus of Gades, as well as Nicomachus, employed all ten forms (see Proclus, *In Tim.*, II. 18. 29 ff. Diehl).

them, which do not have names of their own, but are called in more general terms the fourth, fifth, and sixth forms of mean; after which the moderns discover four others as well, making up the number ten,[1] which, according to the Pythagorean view, is the most perfect possible. It was in accordance with this number indeed that not long ago the ten relations[2] were observed to take their proper number, the so-called ten categories,[3] the divisions and forms of the extremities of our hands and feet, and countless other things which we shall notice in the proper place.[4]

Now, however, we must treat from the beginning, first, that form of 2
proportion which by quantity[5] reconciles and binds together the comparison of the terms, which is a quantitative equality as regards the difference of the several terms to one another. This would be the arithmetic proportion, for it was previously reported that quantity is its peculiar belonging. What, then, is the reason that we shall treat 3
of this first, and not another? Is it not clear that Nature shows it forth before the rest? For in the natural series of simple numbers, beginning with 1, with no term passed over or omitted, the definition of this proportion[6] alone is preserved; moreover, in our previous statements,[7] we demonstrated that the *Arithmetical Introduction* itself is antecedent to all the others, because it abolishes them together with itself, but is not abolished together with them, and because it is implied by them, but does not imply them. Thus it is natural that the mean which shares the name of arithmetic will not unreasonably take

[1] The sacredness of the number 10 was a favorite theme of the Pythagoreans. 10 symbolized for them the universe, and by the *tetraktys* ($1 + 2 + 3 + 4 = 10$) their most sacred oath was taken. It is the all-inclusive nature which they discovered in the decad that gained it its peculiar reverence from them, and Nicomachus here cites evidence of the type accepted by the Pythagorean school to substantiate that property of the decad. It is well to note that in two other Nicomachean sources similar statements are found. Photius's report of the *Theologumena Arithmeticae* represents the decad as the universe because there are 10 fingers, 10 toes, 10 categories and 10 parts of speech, and because it comprehends all solid and plane figures, all kinds of number and of numerical relations; and there is a close parallel passage in Ast's *Theol. Arith.*, p. 59. Another instance of the reverence paid to the decad and its supposed universal character among numbers was seen in I. 19. 17 (Cf. the note). On the Pythagorean decad in general, cf. Philolaus, in Stobaeus, *Ecl.*, I (Wachsmuth-Hense, vol. I, p. 16); Aristotle, *Met.*, I. 5. 986 a, 8.

[2] The reference is to I. 17. 7–8 and the following discussion.

[3] The Aristotelian categories; cf. Part I, p. 95, notes 1 and 2. Boethius, II. 42, says that Archytas the Pythagorean first distinguished the categories, *licet quibusdam sit ambiguum*, and that Plato followed his distinction. There seems to have been a book on the categories (falsely) attributed to Archytas. See Part I, *ibid.*

[4] This seems to be a reference to the work called *Theologumena Arithmeticae.*

[5] κατὰ τὸ ποσόν: Cf. the notes on II. 20. 3; 21. 5.

[6] ὁ ταύτης μόνης λόγος; cf. on II. 21. 2.

[7] Cf. I. 4. 2 ff. and the note.

precedence of the means which are named for the other sciences, the geometric and harmonic; for it is plain that all the more will it take precedence over the subcontraries,[1] over which the first three hold the
4 leadership. As the first and original, therefore, since it is most deserving of the honor, let the arithmetic proportion have its discussion at our hands before the others.

CHAPTER XXIII

1 It is an arithmetic proportion,[2] then, whenever three or more terms are set forth in succession, or are so conceived, and the same quantitative difference is found to exist between the successive numbers, but not the same ratio among the terms, one to another. For example, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13; for in this natural series of numbers, examined consecutively and without any omissions, every term whatsoever is discovered to be placed between two and to preserve the arithmetic proportion to them. For its differences as compared with those ranged on either side of it are equal; the same ratio, however, is not preserved among them.

2 And we understand that in such a series there comes about both a continued and a disjunct proportion; for if the same middle term answers to those on either side as both antecedent and consequent, it would be a continued proportion, but if there is another mean along with it, a disjunct proportion comes about.

3 Now if we separate out of this series any three consecutive terms whatsoever, after the form of the continued proportion, or four or more terms after the disjunct form, and consider them, the difference of them all would be 1, but their ratios would be different throughout. If, however, again we select three or more terms, not adjacent, but separated, separated nevertheless by a constant interval, if one term was omitted in setting down each term, the difference in every case will be 2; and once more with three terms it will be a continued proportion; with more, disjunct. If two terms are omitted, the difference will always be 3 in all of them, continued or disjunct; if three, 4; if four, 5; and so on.

4 Such a proportion,[3] therefore, partakes in equal quantity in its

[1] That is, the 'fourth, fifth and sixth' forms. Cf. section 1. The first three are the 'leaders' of the subcontraries because the latter are based on them.

[2] Cf. the definition in Theon, p. 113, 18 ff.

[3] Cf. on II. 20. 3; 21. 5; 22. 2.

differences, but of unequal quality; for this reason it is arithmetic. If on the contrary it partook of similar quality, but not quantity, it would be geometric instead of arithmetic.

A thing is peculiar[1] to this proportion that does not belong to any 5
other, namely, the mean is either half of, or equal to, the sum of the extremes, whether the proportion be viewed as continuous or disjunct or by alternation; for either the mean term with itself, or the mean terms with one another, are equal to the sum of the extremes.

It has still another peculiarity; what ratio each term has to itself,[2] 6
this the differences have to the differences; that is, they are equal.

Again, the thing which is most exact, and which has escaped the notice of the majority,[3] the product of the extremes when compared to the square of the mean is found to be smaller than it by the product of the differences, whether they be 1, 2, 3, 4, or any number whatever.

In the fourth place,[4] a thing which all previous writers also have noted, the ratios between the smaller terms are larger, as compared to those between the greater terms. It will be shown that in the harmonic proportion, on the contrary, the ratios between the greater terms are greater than those between the smaller; for this reason the harmonic proportion is subcontrary to the arithmetic, and the

[1] The translation here follows the reading of Ast (τὸ κατὰ σύνθεσιν τῶν ἄκρων διπλάσιον τοῦ μέσου ἢ ἴσον τοῖς μέσοις εἶναι). Stated in algebraic form, and letting $a > b > c > d$, the propositions in this section are:

I. (1) $a - b = c - d$, *or* (2) $a - b = b - c$ (*typical forms of the progression*),
then II. (1) $a + d = b + c$, *or* (2) $a + c = 2b$.

But if I is given, it is evident that the following also are true:

III. $a - c = b - d$ (*by alternation*),
then IV. $a + d = b + c$.

The proposition is noticed by Theon, *loc. cit.*, and by Nicomachus himself in his *Handbook of Music* (c. 8, p. 251, 13 Von Jan).

[2] Boethius, II. 43: *Namque omnis terminus sibi aequalis est et differentiae differentiis sunt aequales.*

[3] Boethius, II. 43, says that this was discovered by Nicomachus. In general this proposition may be stated: If $a - b = b - c$, $b^2 - ac = (a - b)(b - c) = (a - b)^2 = (b - c)^2$. Nicomachus in his *Handbook of Music* (c. 8, p. 251, 15, Von Jan) again mentions this property of the proportion.

[4] Thus in the series 1, 2, 3, comparing the ratio of the lesser terms (1, 2) and that of the greater (2, 3),

$$2 : 1 > 3 : 2.$$

But in a similar comparison of the terms of the harmonic progression 3, 4, 6,

$$4 : 3 < 6 : 4.$$

In a geometric progression, as 1, 2, 4,

$$2 : 1 = 4 : 2,$$

as Nicomachus proceeds to say. Nicomachus states that this fact had been noted by all previous writers. The statement is borne out by the fact that it occurs as early as Archytas (fragment 2, Diels, *Die Fragmente der Vorsokratiker*, I[3], p. 334, in Porphyry, *In Ptol. Harm.*, p. 267). For its enunciation by Archytas, see p. 21, where he is quoted.

geometric is midway between them, as it were, between extremes, for this proportion has the ratios between the greater terms and those between the smaller equal, and we have seen that the equal is in the middle ground between the greater and the less. So much, then, about the arithmetic proportion.

CHAPTER XXIV

1 The next proportion[1] after this one, the geometric, is the only one in the strict sense of the word to be called a proportion, because its terms are seen to be in the same ratio. It exists whenever, of three or more terms, as the greatest is to the next greatest, so the latter is to the one following, and if there are more terms, as this again is to the one following it, but they do not, however, differ from one another by the same quantity, but rather by the same quality of ratio, the opposite of what was seen to be the case with the arithmetic proportion.

2 For an example, set forth the numbers beginning with 1 that advance by the double ratio, 1, 2, 4, 8, 16, 32, 64, and so on, or by the triple ratio, 1, 3, 9, 27, 81, 243, and so on, or by the quadruple, or in some similar way. In each one of these series three adjacent terms, or four, or any number whatever that may be taken, will give the geometric proportion to one another; as the first is to the next smaller, so is that to the next smaller, and again that to the next smaller, and so on as far as you care to go, and also by alternation. For instance, 2, 4, 8; the ratio which 8 bears to 4, that 4 bears to 2, and conversely; they do not, however, have the same quantitative difference. Again, 2, 4, 8, 16; for not only does 16 have the same ratio to 8 as before, though not the same difference, but also by alternation it preserves a similar relation — as 16 is to 4, so 8 is to 2, and conversely, as 2 is to 8, so 4 is to 16; and disjunctly, as 2 is to 4, so 8 is to 16; and conversely and in disjunct form, as 16 is to 8 so 4 is to 2; for it has the double ratio.

3 The geometric proportion has a peculiar property shared by none of the rest, that the differences of the terms[2] are in the same ratio to

[1] ἀναλογία: Cf. Theon, pp. 107, 5 and 114, 1 ff., on this proportion. Euclid defines numbers in proportion as follows: "Numbers are in proportion when the first is the same multiple of the second as the third of the fourth, or the same part of it, or the same parts" (ἀριθμοὶ ἀνάλογόν εἰσιν ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν), *Elements*, VII, *Def.* 21.

[2] Thus in the series 1, 2, 4, 8, 16, the ratio is double and the ratio between the successive differences (1, 2, 4, 8) is also double.

each other as the terms to those adjacent to them, the greater to the less, and vice versa. Still another property is that the greater terms have as a difference, with respect to the lesser, the lesser terms themselves, and similarly difference differs from difference, by the smaller difference itself, if the terms are set forth in the double ratio;[1] in the triple ratio both terms and differences will have as a difference twice the next smaller, in the quadruple ratio thrice, in the quintuple four times, and so on.

Geometric proportions come about not only among the multiples, 4
but also among all the superparticular, superpartient, and mixed forms, and the peculiar property of this proportion in all cases is preserved, that in the continued proportions the product of the extremes is equal to the square of the mean, but in disjunct proportions, or those with a greater number of terms,[2] even if they are not continued, but with an even number of terms, that the product of the extremes equals that of the means.

As an illustration of the fact that in all the relations, all kinds of mul- 5
tiples, superparticulars, superpartients, and mixed ratios the peculiar property of this proportion is preserved, let that suffice[3] and be sufficient for us wherein we fashioned, beginning with equality, by the three rules all the kinds of inequality out of one another, when they were in both direct and reverse order; for each act of fashioning and each series set forth is a geometric proportion with all the aforesaid properties as well as a fourth, namely, that they keep the same ratio[4] in both the greater and the smaller terms. Moreover, if we set forth the series shared by both heteromecic and square numbers, one by one,

[1]
Doubles	1	2	4	8	16	32	64	128
Differences	1	2	4	8	16	32	64	
Differences of differences	1	2	4	8	16	32		

In general, $2^{n+1} - 2^n = 2^n$

Triples	1	3	9	27	81	243	729
Differences	2	6	18	54	162	486	
Differences of differences	4	12	36	108	324		

In general, $3^{n+1} - 3^n = 3^n(3 - 1) = 2 \times 3^n$

Quadruples	1	4	16	64	256	1,024	4,096
Differences	3	12	48	192	768	3,072	
Differences of differences	9	36	144	576	2,304		

In general, $4^{n+1} - 4^n = 4^n(4 - 1) = 3 \times 4^n$

[2] Thus in the series 2, 4, 8, 16, 32, 64

$$2 \times 64 = 4 \times 32 = 8 \times 16 \ (= 128).$$

[3] Cf. I. 23. 7 ff.

[4] That is, in the series of doubles (1, 2, 4, 8, 16, 32, etc.) the ratio between 32 and 16 is the same as that between 2 and 1.

containing the terms in both series, and then selecting the terms by groups of three beginning with 1, examine them, in each case setting down the last of the former group as the starting point of the next, we shall find that from the multiple relation — that is, the double — all the kinds of superparticulars [1] appear one after the other, the sesquialter, sesquitertian, sesquiquartan, and so on.

6 It would be most seasonable, now that we have reached this point, to mention a corollary that is of use to us for a certain Platonic theorem: [2] for plane numbers are bound together always by a single mean, solids by two, in the form of a proportion. For with two consecutive

[1] The series of squares and heteromecic numbers is 1, 2, 4, 6, 9, 12, 16, 20, 25, etc. Taking them in groups of three as directed the following ratios appear:

1, 2, 4	(double)
4, 6, 9	(sesquialter)
9, 12, 16	(sesquitertian)
16, 20, 25	(sesquiquartan)

[2] The reference is to *Timaeus*, 32 A–B, and Nicomachus endeavors to elucidate a real difficulty in the Platonic text. In stating the case as he does at first briefly and summarily ("planes are always joined by one mean, solid numbers by two"), he doubtless quotes from memory, for he does not report Plato precisely. Plato does not say that planes can have but one mean, but that one suffices (*εἰ οὖν ἐπίπεδον . . . ἔδει γίγνεσθαι τὸ τοῦ παντὸς σῶμα, μία μεσότης ἂν ἐξήρκει τά τε μεθ' ἑαυτῆς ξυνδεῖν καὶ ἑαυτήν· νῦν δέ — στερεοειδῆ γὰρ αὐτὸν προσῆκεν εἶναι, τὰ δὲ στερεὰ μία μὲν οὐδέποτε, δύο δὲ ἀεὶ μεσότητες ξυναρμόττουσιν . . .*). But Nicomachus, going on to restrict the application of these two principles to *consecutive squares* and *consecutive cubes* would seem to be trying to impose upon the Platonic passage an interpretation which would stand mathematical scrutiny.

The words used by Plato, *ἐπίπεδον* and *στερεόν*, are capable of a very broad interpretation and difficulties would then arise. For example, a plane number could be any number of the form ab, and supposing a, b, c, and d to be prime integers, it would be impossible to find one rational mean between the plane numbers ab and cd, for $\sqrt{abcd}$ would be irrational. However, it would always be possible to find a single mean between two successive squares, for if the squares are a^2 and $(a+1)^2$, $a(a+1)$ will be a geometrical mean between them. Furthermore, Nicomachus's statement about the cubes helps to dismiss a real difficulty in the second part of the Platonic theorem, for there are certain solid numbers that can be put into a geometrical proportion with but one mean (e.g., Archer-Hind, *ad loc.*, cites 8, which is 2^3, and 512, which is 8^3, and the proportion $8:64 = 64:512$). But if by solid numbers Plato meant consecutive cubes, as Nicomachus says, then it will be found that no single rational geometrical mean can be inserted between two such. For if the cubes are a^3 and $(a+1)^3$, the geometrical mean would be $a(a+1)\sqrt{a(a+1)}$ and would be irrational.

At the hands of modern commentators the Platonic passage has been subjected to somewhat similar restriction. Archer-Hind in his note follows Martin for the most part, and declares his belief that Plato meant *ἐπίπεδον* and *στερεόν* in the strictest possible sense, the former a number of two factors only, the latter of three, all the factors being prime integers, and that in the case of the solid numbers he restricts himself to cubes. Then it would be possible always to find one geometrical mean between two squares (as $a^2 : ab = ab : b^2$), though in other plane numbers two means might be possible; and the possibility of two cubes with but one rational geometrical mean will be excluded, for if x is the mean between a^3 and b^3, it will have the irrational value $ab\sqrt{ab}$, a and b being prime integers.

squares[1] only one mean term is discovered which preserves the geometric proportion, as antecedent to the smaller and consequent to the greater term, and never more than one. Hence we conceive of two intervals[2] between the mean term and each extreme, in the relation of similar ratios. Again, with two consecutive cubes[3] only two middle 7
terms in proper ratio are found, in accordance with the geometric proportion, never more; hence there are three intervals, one, that between the mean terms compared to one another, and two between the extremes and the means on either side. Thus the solid forms are called 8
three-dimensional and the plane ones two-dimensional; for example, 1 and 4 are planes, and 2 a middle term in proportion, or again 4 and 9, two squares, and their middle term 6, held by the greater and holding the lesser term in the same ratio[4] as that in which one difference holds the other. The reason[5] for this is that the sides of the two squares, one 9
belonging peculiarly to each, both together produced this very number 6. In cubes, however, for example 8 and 27, no longer one but two mean terms are found, 12 and 18, which put themselves and the terms

[1] Cf. Euclid, *Elements*, VIII, 11: "There is one mean term in proportion between two square numbers, and the square has to the square double the ratio of side to side" (δύο τετραγώνων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν). Theon of Smyrna does not include this proposition, nor the following one, concerning cubes; this is strange, since he is professedly offering helps to the study of Plato.

[2] διαστήματα: The Greek word may also be translated 'intervals.' On the meaning of the word in this connection, cf. on II. 6. 3. These differences will bear to each other the ratio of the terms (cf. sect. 3 above).

[3] Cf. Euclid, *Elements*, VIII, 12: "There are two mean terms in proportion between two cubes, and cube has to cube thrice the ratio of side to side" (δύο κύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ κύβος πρὸς τὸν κύβον τριπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν).

[4] In general a proportion between successive squares would be $a^2 : a(a+1) = a(a+1) : (a+1)^2$. The ratio of the sides would be $a : a+1$. The differences on either side would be $a^2 + a - a^2 = a$, and $a^2 + 2a + 1 - a^2 - a = a + 1$. So the differences have the same ratio as the terms themselves. In this kind of proportion the only 'intervals' are those between the first and middle terms and between the middle and last terms, whereas in any proportion with cubes as extremes, as $a^3 : m = n : b^3$, there are three, between a^3 and m, m and n, and n and b^3. It is to be remarked that the same word, διάστημα, can be translated 'interval' and 'dimension' in speaking of geometric squares or cubes. To Pythagoreans such a coincidence would mean much.

[5] If for example the cubes are a^3 and b^3, the proportion may be of the form $a^3 : a^2b = a^2b : ab^2 = ab^2 : b^3$; the constant ratio is $a : b$, and the differences will be $a^3 - a^2b$, $a^2b - ab^2$, and $ab^2 - b^3$. But $(a^3 - a^2b)(ab^2 - b^3) = (a^2b - ab^2)^2$. The differences therefore may be put in continued proportion

$$\frac{a^3 - a^2b}{a^2b - ab^2} = \frac{a^2b - ab^2}{ab^2 - b^3}, \text{ which reduces to } \frac{a}{b}.$$

That is, the ratio between the differences is the same as that between the terms. In the case of cubes of prime numbers there would be eight further possible forms of the proportion, all of which obey this law, as may be readily tested. If, however, the original numbers were not prime, the number of forms increases with the number of factors.

in the same ratio as that which the differences bear to one another; and the reason of this is that the two mean terms are the products of the sides of the cubes commingled, 2 times 2 times 3 and 3 times 3 times 2.
10 In general, then, if a square takes a square, that is, multiplies it, it
always makes a square; but if a square multiplies a heteromecic number,
or vice versa, it never makes a square; and if cube multiplies cube, a
cube will always result, but if a heteromecic number multiplies a cube,
or vice versa, never is the result a cube. In precisely the same way if an
even number multiplies an even number, the product is always even
and if odd multiplies odd always odd; but if odd multiplies even or
11 even odd, the result will always be even and never odd.[1] These mat-
ters will receive their proper elucidation in the commentary on Plato,[2]
with reference to the passage on the so-called marriage number in the
Republic introduced in the person of the Muses. So then let us pass
over to the third proportion, the so-called harmonic, and analyze it.

CHAPTER XXV

1 The proportion that is placed in the third order is one called the harmonic,[3] which exists whenever among three terms the mean on examination is observed to be neither in the same ratio to the extremes, antecedent of one and consequent of the other, as in the geometric proportion, nor with equal intervals, but an inequality of ratios, as in the

[1] The propositions stated here are:

(1) m^2n^2 is always a square;
(2) $m^2n(n + 1)$ is never a square;
(3) m^3n^3 is always a cube;
(4) $m^3n(n + 1)$ is never a cube;
(5) $2m \times 2n$ is always even;
(6) $(2m \pm 1)(2n \pm 1)$ is always odd;
(7) $2m(2n \pm 1)$ is always even.

[2] The formula for the 'marriage number' occurs in the *Republic*, 546 A ff. The meaning of the passage is still disputed. Nicomachus may perhaps refer to some work of his in which he commented on the *Republic*.

[3] Iamblichus (p. 100, 19 ff.) names this among the three kinds of proportion known to Pythagoras and his school, by whom it was called ὑπεναντία because it was considered to be subcontrary to the arithmetic; the name, however, was changed to ἁρμονική, harmonic, by the schools of Archytas and Hippasus, because they found in it the harmonic ratios. Iamblichus adds (p. 108, 3 ff.) that the fundamental forms (πυθμένες) of this proportion are 2, 3, 6 and 3, 4, 6, the multiples or superparticulars of which terms give other examples of it. On account of this limitation the name 'fixed,' 'established' (ἑστηκυῖα) was given to the harmonic proportion by some. It will be noted that Nicomachus uses the examples mentioned, although he does not speak of such a limitation.

arithmetic, but on the contrary, as the greatest term is to the smallest, so the difference between greatest and mean terms is to the difference between mean and smallest term.[1] For example, take 3, 4, 6, or 2, 3, 6. For 6 exceeds 4 by one third of itself, since 2 is one third of 6, and 3 falls short of 4 by one third of itself, for 1 is one third of 3. In the first example, the extremes are in double ratio and their differences with the mean term are again in the same double ratio to one another; but in the second they are each in the triple ratio.

It has a peculiar property, opposite, as we have said,[2] to that of the 2
arithmetic proportion; for in the latter[3] the ratios were greater among the smaller terms, and smaller among the greater terms. Here, however, on the contrary, those among the greater terms are greater and those among the smaller terms smaller, so that in the geometric proportion, like a mean between them, there may be observed the equality of ratios on either side, a midground between greater and smaller.

Furthermore, in the arithmetic proportion[4] the mean term is seen 3
to be greater and smaller than those on either side by the same fraction of itself, but by different fractions of the terms that flank it; in the harmonic, however, it is the opposite, for the middle term is greater and less than the terms on either side by different fractions of itself, but always the same fraction of those terms at its sides, a half of them or a third; but the geometric,[5] as if in the midground between them,

[1] The general formula then is $a : c = a - b : b - c$. For Theon's discussion see p. 114, 14 ff., Hiller.

[2] Cf. II. 23. 6. Iamblichus (p. 110, 18 ff.) says that this was the opinion of the Pythagoreans but that some considered the harmonic proportion contrary to both the arithmetic and geometric. He then argues elaborately for the view expressed in the text, that it is subcontrary to the arithmetic only.

[3] Compare the ratios of the terms in the harmonic series 3, 4, 6 and the arithmetic series 3, 4, 5:

$\frac{4}{3} = 1\frac{1}{3}$ (less) $\quad$ $\frac{4}{3} = 1\frac{1}{3}$ (greater)

$\frac{6}{4} = 1\frac{1}{2}$ (greater) $\quad$ $\frac{5}{4} = 1\frac{1}{4}$ (less)

[4] The following examples will illustrate Nicomachus's meaning.

	Arithmetic	Harmonic	Geometric
Series	3, 4, 5	3, 4, 6	4, 6, 9
Differences	1, 1	1, 2	2, 3
Which are the following fractions of the mean	$\frac{1}{4}$ $\frac{1}{4}$	$\frac{1}{4}$ $\frac{1}{2}$	$\frac{1}{3}$ $\frac{1}{2}$
And the following fractions of the extremes	$\frac{1}{3}$ $\frac{1}{5}$	$\frac{1}{3}$ $\frac{1}{3}$	$\frac{1}{2}$ $\frac{1}{3}$

In each case there are two differences. These are (*a*) in the arithmetic, the same fraction of the middle term but different fractions of the extremes; (*b*) in the harmonic, different fractions of the mean, but the same fraction of the extremes; (*c*) in the geometric, different fractions of both mean and extremes.

[5] What is meant by the elliptic statement of the text, "the geometric . . . neither in the mean alone, nor in the extremes alone, but in both mean and extreme," may be seen from the example in the preceding note. The middle term does not differ from both extremes by the same

shows this property neither in the mean term exclusively nor in the extremes, but in both mean and extreme.

4 Once more, the harmonic proportion has as a peculiar property the fact that when the extremes are added together[1] and multiplied by the mean, it makes twice the product of themselves multiplied by one another.

5 The harmonic proportion was so called because the arithmetic proportion was distinguished by quantity, showing an equality in this respect with the intervals from one term to another, and the geometric by quality, giving similar qualitative relations between one term and another, but this form, with reference to relativity, appears now in one form, now in another, neither in its terms exclusively nor in its differences exclusively, but partly in the terms and partly in the differences; for as the greatest term is to the smallest, so also is the difference between the greatest and the next greatest, or middle, term to the difference between the least term and the middle term, and vice versa.

CHAPTER XXVI

1 In the classification of Being previously set forth we recognized the relative[2] as a thing peculiar to harmonic theory; but the musical ratios of the harmonic intervals are also rather to be found in this proportion. The most elementary is the diatessaron, in the sesquitertian ratio, 4 : 3, which is the ratio of term to term[3] in the example in the double ratio, or of difference to difference in that which follows, the triple, for these differences are of 6 to 2 or again of 6 to 3. Immediately following is the diapente, which is the sesquialter, 3 : 2 or again, 6 : 4, the ratio of

part of itself nor by the same part of the extremes; but from the first term by the same fraction of itself as the fraction of the last term by which it differs from the last term; and conversely, sameness and difference of the fraction involved in comparing the terms do not lie, in this type of proportion, exclusively in the relations to the middle term, or in those of the middle term to the extremes.

[1] Thus in 2, 3, 6, $(2 + 6) \times 3 = 24$ and $2 \times 2 \times 6$.

In general $\frac{a}{c} = \frac{a - b}{b - c}$, whence $b = \frac{2ac}{a + c}$ and $b(a + c) = 2ac$.

[2] See I. 3. 1.

[3] The examples referred to are the harmonic proportions cited in II. 25. 1. The proportion in double ratio is 3, 4, 6, and that in triple ratio is 2, 3, 6. The first two terms are in sesquitertian ratio in the former (4 : 3) and the differences of 6 to 2 and of 3 respectively (4, 3) give the same ratio in the latter.

term to term.[1] Then the combination[2] of both of these, sesquialter and sesquitertian, the diapason, which comes next, is in the double ratio, 6 : 3 in both of the examples, the ratio of term to term.[3] The following interval, that of the diapason and diapente together,[4] which preserves the triple ratio of the two of them together, since it is the combination of double and sesquialter, is as 6 : 2, the ratio of term to term in the example in the triple ratio, and likewise of difference to difference in the same, and in the proportion with double ratio it is the ratio of the greatest term to the difference between that term and the mean term, or of the difference between the extremes to the difference between the smaller terms. The last and greatest interval, the so-called di-diapason, as it were twice the double, which is in the quadruple ratio,[5] is as the middle term in the proportion in the double ratio to the difference between the lesser terms, or as the difference between the extremes, in the example in the triple ratio, to the difference between the lesser terms.

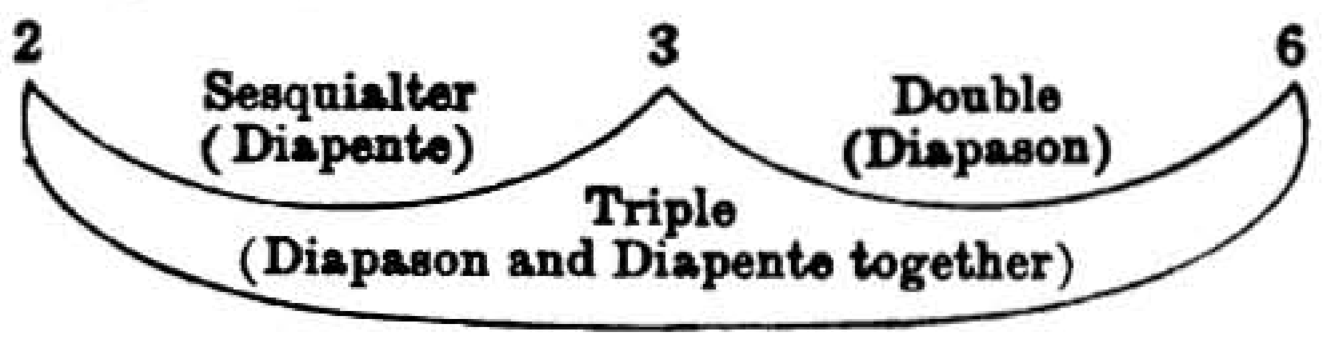
Some, however, agreeing with Philolaus, believe that the proportion 2
is called harmonic because it attends upon all geometric harmony, and they say that 'geometric harmony' is the cube because it is harmonized in all three dimensions, being the product of a number thrice multiplied together. For in every cube[6] this proportion is mirrored; there are in every cube 12 sides, 8 angles and 6 faces; hence 8, the mean between 6 and 12, is according to harmonic proportion, for as the extremes are to each other, so is the difference between greatest and

[1] That is, the sesquialter ratio may be derived from the terms of the two harmonic proportions cited without calling in differences. 6 : 4 comes from the double ratio; 3 : 2, from the triple.

[2] Cf. II. 5. 2.

[3] In both series (2, 3, 6 and 3, 4, 6) the diapason occurs among the terms (3 : 6).

[4] This, a triple ratio, is seen in the triple harmonic series 2, 3, 6 (*a*) in the terms 6 and 2; (*b*) in the differences ($6 - 3 = 3$ and $3 - 2 = 1$); and in the double series 3, 4, 6 (*a*) in the ratio of 6 to ($6 - 4$), i.e., 2, and (*b*) in the ratio between the difference of 6 and 3 and that between 4 and 3. On the triple ratio as the combination of the double and the sesquialter, cf. II. 5. 4. The harmonic series 2, 3, 6 is of the type that Nicomachus used in illustration in II. 5, as may be seen from the following arrangement:



[5] In the double series 3, 4, 6, it is the ratio of 4 to ($4 - 3 =$) 1; in the triple, 2, 3, 6, it is the ratio of ($6 - 2 =$) 4 to ($3 - 2 =$) 1.

[6] Nicomachus shows first that this series, 6, 8, 12, conforms to all the tests of the harmonic proportion just stated, and then that all the ratios of the harmonic intervals are to be found among its terms and differences.

middle term to that between the middle and smallest terms, and, again, the middle term is greater than the smallest by one fraction of itself and by another is less than the greater term, but is greater and smaller by one and the same fraction of the extremes. And again, the sum of the extremes multiplied by the mean makes double the product of the extremes multiplied together. The diatessaron is found in the ratio 8 : 6, which is sesquitertian, the diapente in 12 : 8, which is sesquialter; the diapason, the combination of these two, in 12 : 6, the double ratio; the diapason and diapente combined, which is triple, in the ratio of the difference of the extremes to that of the smaller terms, and the di-diapason is the ratio of the middle term to the difference between itself and the lesser term. Most properly, then, has it been called harmonic.

CHAPTER XXVII

1 Just as in the division of the musical canon,[1] when a single string is stretched or one length of a pipe is used, with immovable ends, and the mid-point shifts in the pipe by means of the finger-holes, in the string by means of the bridge, and as in one way after another the aforesaid proportions, arithmetic, geometric, and harmonic, can be produced, so that the fact becomes apparent that they are logically[2] and very properly named, since they are brought about through changing and shifting the middle term in different ways, so too it is both reasonable and possible to insert the mean term that fits each of the three proportions between two arithmetic terms, which stay fixed and do not change, whether they are both even or odd. In the arithmetic proportion this mean term is one that exceeds and is exceeded by an equal amount; in the geometric proportion it is differentiated from the

[1] The μουσικὸς κανών was a measuring rod corresponding to, and placed by the side of, the monochord, on which by means of a movable bridge experiments were made to determine the musical intervals precisely, by the use of mathematics instead of by ear. The procedure in such experiments may be gathered from Boethius, *Inst. Mus.*, IV. 5: *Sit chorda intensa AB. Huic aequa sit regula, quae propositis partitionibus dividatur, ut ea regula chordae apposita eaedem divisiones in nervi longitudine signentur quas antea signaveramus in regula. Nos vero nunc ita dividimus quasi ipsam chordam et non regulam partiamur.* He then describes the actual division. Apparently similar experiments could be made on one pipe of the flute. The word used by Nicomachus for 'bridge,' ὑπαγωγεύς, is properly referred to a movable, as opposed to a fixed, bridge (μαγάς); Ast in his note on the present passage cites a scholium on Ptolemy, *Harm.*, I. 8: μαγὰς ἡ μὴ ἀγομένη, ὑπαγωγεὺς δὲ ὁ ἀγόμενος· καταχρηστικῶς δὲ καὶ μαγὰς λέγεται ὑπαγωγεύς. The expression κάνωνος κατατομή is the title of a work of Euclid.

[2] That is, they are properly called μεσότητες ('means') because the mean term, μέσος ὅρος, determines their character.

extremes by the same ratio, and in the harmonic it is greater and smaller than the extremes by the same fraction of those same extremes.

Let there be given then, first, two even terms, between which we must find how the three means would be inserted, and what they are. Let them be 10 and 40. 2

First, then, I fit to them the arithmetic mean. It is 25, and the attendant properties of the arithmetic proportion are all preserved; for as each term is to itself, so also is difference to difference; they are in equality, therefore. And as much as the greater exceeds the means by so much the latter exceeds the lesser term; the sum of the extremes[1] is twice the mean; the ratio of the lesser terms[2] is greater than that of the greater; the product of the extremes[3] is less than the square of the mean by the amount of the square of the differences; and the middle term[4] is greater and less than the extremes by the same fraction of itself, but by different fractions regarded as parts of the extremes. 3

If, however, I insert 20 as a mean between the given even terms, the properties of the geometrical proportion come into view and those of the arithmetic are done away with. For as the greater term is to the middle term, so is the middle term to the lesser; the product of the extremes is equal to the square of the mean; the differences are observed to be in the same ratio to one another as that of the terms;[5] neither in the extremes[6] alone nor in the middle term alone does there reside the sameness of the fraction concerned in the relative excess and deficiency of the terms, but in the middle term and one of the extremes by turns; and both between greater and smaller terms there is the same ratio. 4

But if I select 16 as the mean, again the properties of the two former proportions disappear and those of the harmonic are seen to remain 5

[1] $40 + 10 = 2 \times 25$.

[2] $\frac{25}{10} > \frac{40}{25}$.

[3] $25^2 - (10 \times 40) = 225 = 15^2$.

[4] As the numerical difference is constant, it follows that the mean is both exceeded and exceeds by the same fraction of itself (i.e., $\frac{15}{25}$ or $\frac{3}{5}$). But 15 compared with the extremes is $\frac{3}{2}$ of 10 and $\frac{3}{8}$ of 40.

[5] The differences in the series 10, 20, 40 are 10 and 20. Both terms and differences are in the double ratio.

[6] The reference is to the peculiarity of the geometrical proportion noted in II. 25. 3. In this case the difference between 10 and 20 is the whole of 10 and half of 20; that between 20 and 40 is the whole of 20 and half of 40. If both the differences are viewed in relation either to the mean or to the respective extremes, the fraction is not constant; but if one difference be regarded as a fraction of the mean, while the other is regarded as a fraction of the extreme, there will be "identity of the fraction of excess and defect."

fixed, with respect to the two even terms. For as the greatest term[1] is to the least, so is the difference of the greater terms to that of the lesser; by what fractions, seen as fractions of the greater term, the mean is smaller than the greater term, by these the same mean term is greater than the smallest term when they are looked upon as fractions of the smallest term; the ratio between the greater terms is greater, and that of the smaller terms, smaller, a thing which is not true of any other proportion; and the sum of the extremes multiplied by the mean is double the product of the extremes.

6 If, however, the two terms that are given are not even but odd, like 5, 45, the same number, 25, will make the arithmetic proportion; and the reason for this is that the terms on either side overpass it and fail to come up to it by an equal number, keeping the same quantitative difference with respect to it. 15 substituted makes the geometric proportion, as it is the triple and subtriple of each respectively; and if 9 takes over the function of mean term it gives the harmonic; for by those parts of the smaller term by which it exceeds, namely, four fifths of the smaller, it is also less than the greater, if they be regarded as parts of the greater term, for this too is four fifths, and if you try all the previously mentioned properties of the harmonic ratio you will find that they will fit.

7 And let this be your method whereby you might scientifically fashion the mean terms that are illustrated in the three proportions. For the two terms given you, whether odd or even, you will find the arithmetic[2] mean by adding the extremes and putting down half of them as the mean, or if you divide by 2 the excess of the greater over the smaller, and add this to the smaller, you will have the mean. As for the geometric mean, if you find the square root of the product[3] of the extremes, you will produce it, or, observing the ratio[4] of the terms to one

[1] In the series 10, 16, 40

(a) $\frac{40}{10} = \frac{40 - 16}{16 - 10} = \frac{24}{6}$.

(b) The difference between 40 and 16, 24, is $\frac{3}{5}$ of 40, and the difference between 16 and 10, 6, is $\frac{3}{5}$ of 10.

(c) $\frac{40}{16} > \frac{16}{10}$.

(d) $(40 + 10) \times 16 = 2 \times (40 \times 10) = 800$.

[2] In general terms, the arithmetic mean between a and b is $\frac{a + b}{2}$, or, if $a < b$, it is $a + \frac{b - a}{2}$.

[3] In general terms the geometric mean is $\sqrt{ab}$.

[4] Nicomachus is stating the following proposition put into general terms: given a and ar, $\frac{ar}{2}$ will form a geometric proportion with them and $a : \frac{ar}{2} = \frac{ar}{2} : ar$. This will hold good only for

another, divide this by 2 and make the mean, for example, the double, in the case of a quadruple ratio. For the harmonic mean,[1] you must multiply the difference of the extremes by the lesser term and divide the product by the sum of the extremes, then add the quotient to the lesser term, and the result will be the harmonic mean.

CHAPTER XXVIII

So much, then, concerning the three proportions celebrated by the 1
ancients, which we have discussed the more clearly and at length for just this reason, that they are to be met with frequently and in various forms in the writings of those authors.[2] The succeeding forms,[3] however, we must only epitomize, since they do not occur frequently in the ancient writings, but are included merely for the sake of our own

proportions of the type cited in Nicomachus's example, those where the ratio is the quadruple. For if $a : \frac{ar}{2} = \frac{ar}{2} : ar$, then $a^2r = \frac{ar^2}{4}$ and $4r - r^2 = 0$. Only two values of r satisfy this condition, 4 and 0, of which the latter need not be taken into consideration when dealing with the number system of Nicomachus. Nicomachus should have directed that the square root of r be taken, not its half; then, provided that r were a perfect square, he could set up the proportion, $a : a\sqrt{r} = a\sqrt{r} : ar$. The number 4 is the only one of which the half and the square root are identical (among the positive integers).

[1] Given a and c (if $a > c$), the harmonic mean is $\frac{(a-c)c}{a+c} + c$. This is equal to $\frac{2ac}{a+c}$, which was given above as the general formula for the harmonic mean (see on II. 25. 4) and would be a simpler one to follow.

[2] That is, in the sections of the works of ancient authors read and interpreted in the schools (ἀναγνώσματα).

[3] Theon of Smyrna also enumerated other proportions in addition to the first and principal three. In addition he says (p. 106, 13) there are the 'subcontrary' (*sc.* to the harmonic; cf. p. 115, 5 ff.), fifth and sixth, and then six more, subcontraries to the first six. This second set of six, however, he leaves without explanation or illustration. For the sake of the conspectus, the general formulae for the proportions as the two authors give them may be set down, letting $a > b > c$ in each:

	NICOMACHUS		THEON	
1. Arithmetic	$a - b = b - c$	(II. 23. 5)	$a - b = b - c$	(p. 113, 18).
2. Geometric	$\frac{a}{b} = \frac{b}{c}$	(II. 24)	$\frac{a}{b} = \frac{b}{c}$	(p. 114, 1).
3. Harmonic	$\frac{a}{c} = \frac{a-b}{b-c}$	(II. 25)	$\frac{a}{c} = \frac{a-b}{b-c}$	(p. 114, 14).
4. Subcontrary to harmonic	$\frac{a}{c} = \frac{b-c}{a-b}$	(II. 28. 3)	$\frac{c}{a} = \frac{a-b}{b-c}$	(p. 115, 5).
5. Fifth form	$\frac{b}{c} = \frac{b-c}{a-b}$	(II. 28. 4)	$\frac{c}{b} = \frac{a-b}{b-c}$	(p. 115, 12).
6. Sixth form	$\frac{a}{b} = \frac{b-c}{a-b}$	(II. 28. 5)	$\frac{b}{a} = \frac{a-b}{b-c}$	(p. 115, 20).

In the last three forms Theon consistently reverses the formulae of Nicomachus.

acquaintance with them and, so to speak, for the completeness of our
2 reckoning. They are set forth by us in an order based on their opposition to the three archetypes already described, since they are fashioned out of them and have the same order.

3 The fourth, and the one called subcontrary,[1] because it is opposite to, and has opposite properties to, the harmonic proportion, exists when, in three terms, as the greatest is to the smallest, so the difference of the smaller terms is to that of the greater, for example 3, 5, 6. For the terms compared are seen to be in the double ratio, and it is plain wherein it is opposite to the harmonic proportion; for whereas they both have the same extreme terms, and in double ratio, in the former the difference of the greater terms as compared to that of the lesser preserved the same ratio as that of the extremes, but in this proportion just the reverse, the difference of the smaller compared with that of the greater. You must know that its peculiar property is this. The product of the greater[2] and the mean terms is twice the product of the mean and the smaller; for 6 times 5 is twice 5 times 3.

4 The two proportions, fifth and sixth, were both fashioned after the geometrical, and they differ from each other thus.

The fifth form exists,[3] whenever, among three terms, as the middle term is to the lesser, so their difference is to the difference between the greater and the mean, as in 2, 4, 5, for 4 is the middle term, the double of 2, the lesser, and 2 is the double of 1 — the difference of the smallest terms as compared with that of the largest. That which makes it contrary to the geometric proportion is that in the former,[4] as the middle term is to the lesser, so the excess of the greater over the mean is to the excess of the mean over the lesser term, whereas in this proportion, on the contrary, it is the difference of the lesser compared to that of the

[1] The harmonic proportion is $\frac{a}{c} = \frac{a-b}{b-c}$. The one now under discussion is of the form, $\frac{a}{c} = \frac{b-c}{a-b}$. In the example given the 'elements compared' are the extremes and differences respectively.

[2] The statement is not correct; it should be that the product of the greater by the middle equals the product of the middle by the less multiplied by the ratio. For if $\frac{a}{c} = \frac{b-c}{a-b}$ where $a > b > c$, let r be the ratio. Then bcr will equal $bc \times \frac{a}{c}$, or ab. Or the mean multiplied by the sum of the extremes equals the sum of the squares of the extremes, $b(a+c) = a^2 + c^2$.

[3] That is, if $a > b > c$, the proportion is $\frac{b}{c} = \frac{b-c}{a-b}$.

[4] That is, $\frac{b}{c} = \frac{a-b}{b-c}\left(\text{derived from } \frac{a}{b} = \frac{b}{c}\right)$; cf. II. 24.

greater. Nevertheless it is peculiar to this proportion[1] that the product of the greatest by the middle term is double that of the greatest by the smallest, for 5 times 4 is twice 5 times 2.

The sixth form[2] comes about when, in a group of three terms, as the 5
greatest is to the mean, so the excess of the mean over the lesser is to the excess of the greater over the mean, for example 1, 4, 6, for both are in the sesquialter ratio. There is in this case also a reasonable cause for its opposition to the geometrical; for here, too, the likeness of the ratios reverses, as in the fifth form.

These are the six proportions commonly spoken of among previous 6
writers, the three prototypes[3] having lasted from the times of Pythagoras down to Aristotle and Plato, and the three others, opposites of the former, coming into use among the commentators and sectarians who succeeded these men. But certain men have devised in addition, by shifting the terms and differences of the former, four more which do not much appear in the writings of the ancients, but have been sparingly touched upon as an over-nice detail. These, however, we must run over in the following fashion, lest we seem ignorant.

The first of them,[4] and the seventh in the list of them all, exists when, 7
as the greatest term is to the least, so their difference is to the difference of the lesser terms, as 6, 8, 9, for on comparison the ratio of each is seen to be the sesquialter.

[1] This statement again should be corrected in the same way as the former one. It can be stated that if $\frac{b}{c} = \frac{b-c}{a-b}$, where $a > b > c$, and r is the constant ratio, $ab = acr$. For $r = \frac{b}{c}$ and $ac \times \frac{b}{c} = ab$. Nicomachus's proposition applies only to cases where $r = 2$.

[2] General form, if $a > b > c$, $\frac{a}{b} = \frac{b-c}{a-b}$. This gives from the series, 1, 4, 6, $\frac{6}{4} = \frac{4-1}{6-4}$, the constant ratio being $\frac{3}{2}$, sesquialter. From the geometric proportion $\frac{a}{b} = \frac{b}{c}$ is derived $\frac{b}{c}\left(\text{or } \frac{a}{b}\right) = \frac{a-b}{b-c}$ (See note on section 4). Here the ratio of differences is inverted. (Cf. Euclid, *Elem.*, V., *Def.* 16: *ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου.* That is, as Heath puts it, conversion is taking instead of the ratio of a to b the ratio of a to $a-b$. This is evidently not the sort of conversion here intended.)

[3] Boethius, II. 52, says: *et hae quidem sunt sex medietates, quarum tres usque a Pythagora ad Platonem Aristotelemque manserunt. Post vero qui insecuti sunt has tres alias, de quibus supra disseruimus, suis commentariis addiderunt.* See note on II. 22. 1.

[4] The general form is $\frac{a}{c} = \frac{a-c}{b-c}$, if $a > b > c$. In the given series 6, 8, 9, we have

$$\frac{9}{6} = \frac{9-6}{8-6} = \frac{3}{2}.$$

8 The eighth proportion,[1] which is the second of this group, comes about when, as the greatest is to the least term, so the difference of the extremes is to the difference of the greater terms, as 6, 7, 9; for this also has sesquialters for the two ratios.

9 The ninth[2] in the complete list, and third in the number of those subsequently invented, exists when there are three terms and whatever ratio the mean bears to the least, that also the difference of the extremes has in comparison with that of the smallest terms, as 4, 6, 7.

10 The tenth,[3] in the full list, which concludes them all, and the fourth in the series presented by the moderns, is seen when, among three terms, as the mean is to the lesser, so the difference of the extremes is to the difference of the greater terms, as 3, 5, 8, for it is the superbipartient ratio in each pair.

11 To sum up, then, let the terms of the ten proportions be set forth[4] in one illustration, for the sake of easy comprehension:

First: 1, 2, 3	Sixth: 1, 4, 6
Second: 1, 2, 4	Seventh: 6, 8, 9
Third: 3, 4, 6	Eighth: 6, 7, 9
Fourth: 3, 5, 6	Ninth: 4, 6, 7
Fifth: 2, 4, 5	Tenth: 3, 5, 8

CHAPTER XXIX

1 It remains[5] for me to discuss briefly the most perfect proportion, that which is three-dimensional and embraces them all, and which is most useful for all progress in music and in the theory of the nature of the universe. This alone would properly and truly be called harmony[6]

[1] The general form, if $a > b > c$, is $\frac{a}{c} = \frac{a-c}{a-b}$, and in the given series 6, 7, 9, we have
$$\frac{9}{6} = \frac{9-6}{9-7} = \frac{3}{2}.$$

[2] The general form, if $a > b > c$, is $\frac{b}{c} = \frac{a-c}{b-c}$, and in the given series 4, 6, 7, we have $\frac{6}{4} = \frac{7-4}{6-4} = \frac{3}{2}$.

[3] That is, if $a > b > c$, $\frac{b}{c} = \frac{a-c}{a-b}$, and in the given series 3, 5, 8, $\frac{5}{3} = \frac{8-3}{8-5} = \frac{5}{3}$.

[4] It is to be noticed that all the proportions can be formed from numbers in the first decade.

[5] Iamblichus calls this proportion *μουσική*, and says of it (p. 118, 23 Pistelli): *εὕρημα δ' αὐτήν φασιν εἶναι Βαβυλωνίων, καὶ διὰ Πυθαγόρου πρώτου εἰς Ἕλληνας ἐλθεῖν* ("they say it was a discovery of the Babylonians, and that it was by Pythagoras first introduced among the Greeks").

[6] Cf. II. 26. 2.

rather than the others, since it is not a plane, nor bound together by only one mean term, but with two, so as thus to be extended in three dimensions,[1] just as a while ago it was explained that the cube is harmony.

When, therefore, there are two extreme terms, both of three dimen- 2
sions, either numbers multiplied thrice by themselves so as to be a cube, or numbers multiplied twice by themselves and once by another number so as to be either 'beams' or 'bricks,' or the products of three unequal numbers, so as to be scalene, and between them there are found two other terms which preserve the same ratios to the extremes alternately and together,[2] in such a manner that, while one of them preserves the harmonic proportion, the other completes the arithmetic, it is necessary that in such a disposition of the four the geometric proportion appear,[3] on examination, commingled with both mean terms — as the greatest is[4] to the third removed from it, so is the second from

[1] Or, 'three intervals'; for the sense, cf. II. 24. 6.

[2] Ast thus comments on the words translated (ἐναλλάξ, ἀναμίξ): "ἐναλλάξ est permutato s. inverso ordine, et ἀναμίξ promiscue s. inter se; ἀναμίξ medii termini 8 et 9 [referring to the example 6, 8, 9, 12] ad extremos 6 et 12 ita se habent, ut aequalem inter se servent proportionem; 8 enim ad 6 sesquitertiam habet rationem, ut 12 ad 9; inverso autem ordine 12 ad 8 ita se habet, ut 9 ad 6; utriusque enim ratio est sesquialtera." Ast apparently means 'alternately' or 'by alternation' by 'permutato sive inverso ordine' (= ἐναλλάξ), as his illustration shows. ἐναλλάξ is so used in II. 21. 6, but in the same section ἀναμίξ is used in precisely the same way, both meaning 'by alternation.' So we must assume either that the terms are here used as synonyms, as in II. 21. 6, or that ἐναλλάξ means 'alternately' and ἀναμίξ something else, 'directly,' perhaps, as Ast would imply. It is quite certain that ἐναλλάξ means alternately. Cf. T. L. Heath on Euclid, V. Def. 12: "The word ἐναλλάξ is of course a common term which has no exclusive reference to mathematics. But this same use of it with reference to proportions already occurs in Aristotle, *Anal. Post.*, I. 5. 74 a 18, καὶ τὸ ἀνάλογον ὅτι ἐναλλάξ, 'and that a proportion (is true) alternately, or *alternando*.' Used with λόγοι as here, the adverb ἐναλλάξ has the sense of an adjective, 'alternate'; we have already had it similarly used of 'alternate angles' (αἱ ἐναλλὰξ γωνίαι) in the theory of parallels." It is also clear that τοὺς αὐτοὺς λόγους refers to geometric proportion, not to the harmonic and arithmetic referred to after ὥστε, for λόγος is not used of the relation existing between the terms in arithmetic proportion. I have translated ἀναμίξ 'together' but with some diffidence, taking the sense to be approximately that given by Ast.

[3] Such a proportion is of the form $a, \frac{2ab}{a+b}, \frac{a+b}{2}, b$. These will form a geometric proportion, for the product of the extremes equals that of the means. Nicomachus further specifies that both a and b are to be of the general forms m^3, m^2n or lmn. In giving his example he considers unity a factor. Boethius, II. 54, thus describes this proportion: *Haec autem huiusmodi invenietur, si duobus terminis constitutis, qui ipsi tribus creverint intervallis, longitudine latitudine et profunditate, duo huiusmodi termini medii fuerint constituti et ipsi tribus intervallis notati, qui vel ab aequalibus per aequales aequaliter sint producti vel ab inaequalibus ad inaequalia aequaliter, vel ab inaequalibus ad aequalia aequaliter vel quolibet alio modo, atque ita, cum armonicam proportionem custodiant, alio tamen modo comparati faciant arithmeticam medietatem, hisque geometrica medietas, quae inter utrasque versatur, deesse non possit.*

[4] The Greek fashion of counting, that is, reckoning in both ends, is used in the specification of these terms. 'By alternation' here is ἐμπλέγδην 'in interlocking or interwoven fashion'; practically equivalent to ἐναλλάξ.

it to the fourth; for such a situation makes the product of the means equal to the product of the extremes. And again, if the greatest term be shown to differ from the one next beneath it by the amount whereby this latter differs from the least term, such an array becomes an arithmetic proportion and the sum of the extremes is twice the mean. But if the third term from the greatest exceeds and is exceeded by the same fraction of the extremes, it is harmonic and the product of the mean by the sum of the extremes is double the product of the extremes.

3 Let this be an example of this proportion, 6, 8, 9, 12. 6 is a scalene number, derived from 1 times 2 times 3, and 12 comes from the successive multiplication of 2 times 2 times 3; of the mean terms the lesser is from 1 times 2 times 4, and the greater from 1 times 3 times 3. The extremes are both solid and three-dimensional, and the means are of the same class. According to the geometric proportion, as 12 is to 8, so 9 is to 6; according to the arithmetic, as 12 exceeds 9, by so much does 9 exceed 6; and by the harmonic, by the fraction by which 8 exceeds 6,[1] viewed as a fraction of 6, 8 is also exceeded by 12, viewed as a fraction of 12.

4 Moreover 8 : 6 or 12 : 9 is the diatessaron, in sesquitertian ratio; 9 : 6 or 12 : 8 is the diapente in the sesquialter; 12 : 6 is the diapason in the double. Finally, 9 : 8 is the interval of a tone, in the superoctave ratio, which is the common measure of all the ratios in music, since it is also the more familiar, because it is likewise the difference[2] between the first and most elementary intervals.

5 And let this be sufficient concerning the phenomena and properties of number, for a first *Introduction.*

[1] 8 exceeds 6 by 2, or by $\frac{1}{3}$ of 6; 12 exceeds 8 by 4, or by $\frac{1}{3}$ of 12; so 6, 8, 12 is a harmonic series.

[2] Boethius has the following explanation: *unde notum est, quod inter diatessaron et diapente consonantiarum tonus differentia est, sicut inter sesquitertiam et sesquialteram proportionem sola est epogdous differentia* (II. 54).

PART III

SUPPLEMENTARY AIDS TO THE INTERPRETATION OF THE *INTRODUCTION TO ARITHMETIC*

I. EXTENSIONS OF A THEOREM OF NICOMACHUS

LONG before the time of Nicomachus the fact that the sum of the first n odd numbers gives n^2 was known. The theorem of Nicomachus states [1] that if you take the nth sequence of n odd numbers, calling the first 1, the second 3, 5, the third 7, 9, 11, and so on, you obtain n^3.

In considering this theorem it seemed to me that there should be analogous theorems for the fifth and seventh powers, and possibly for higher prime powers. The theorem for the fifth powers may be given as follows:

In the series of odd numbers, the first gives 1^5; omit the next one, and the following 4 give 2^5; omit the following 3, and the next 9 give 3^5; omit 6, and the following 16 give 4^5; omit 10, and the following 25 give 5^5; in general, omit $\frac{n(n+1)}{2}$ terms or a number equal to the nth triangular number and take the $(n+1)^2$ following to obtain $(n+1)^5$.

1   5, 7, 9, 11   19, 21, ... 35
[3]   4 terms   [13, 15, 17]   9 terms   [37, 39, ... 47]

49, 51, ... 79   101, 103, 105, ... 149
16 terms   [81, 83, ... 99]   25 terms

Each group of upper numbers consists of n^2 terms and the sum is n^5; each group of lower numbers consists of $\frac{n(n+1)}{2}$ terms.

The general proof consists of the summation of the n^2 terms of the given arithmetical series and follows without difficulty.

For the seventh powers the corresponding theorem is as follows:

In the series of odd numbers, the first one gives 1^7; omit the next 3 and the following 8 give 2^7; omit the next 15 and the following 27 give 3^7; omit 42 terms, take 64 and obtain 4^7; omit 90 terms, take 125 terms and obtain 5^7; omit 165 terms, take 216 terms and obtain 6^7; . . . in general, omit a number of terms given by the product of the nth triangular number, multiplied by the $(n+1)^{\text{st}}$ odd number which

[1] 2.20.5. For the translation see page 263 of this volume.

is $2n+1$, and the following sequence of $(n+1)^3$ terms will give $(n+1)^7$.

The algebraic proof is again by summation of an arithmetical series and follows without difficulty.

Both of these theorems and that of Nicomachus are special cases of the following more inclusive theorem:

In the series of odd numbers the sequence of n^k terms, following upon the sequence of $\frac{1}{2}(n-1)n^k$ terms which begin with 1, has as sum n^{2k+1}. For $k = 1$ this gives the theorem of Nicomachus and for $k = 2$ and 3 the theorems stated above.

My colleague, Mr. Norman Anning, further extends this generalization in the two following tables which give such sums to arrive at even as well as odd powers of k.

In the series of odd numbers, if we omit the first p, then the sum of the next q is n^{2k+1}; the possible values of p and q are exhibited in the following table:

p	q	p	q
$\frac{1}{2}(n^1-1)n^k$	n^k	. . . . .	.
$\frac{1}{2}(n^3-1)n^{k-1}$	n^{k-1}	$\frac{1}{2}(n^{2k-3}-1)n^2$	n^2
$\frac{1}{2}(n^5-1)n^{k-2}$	n^{k-2}	$\frac{1}{2}(n^{2k-1}-1)n^1$	n^1

Similarly for even powers of n, if we omit the first p, then the sum of the next q terms is n^{2k}, values of p and q being given by the following table:

p	q	p	q
$\frac{1}{2}(n^2-1)n^{k-1}$	n^{k-1}	$\frac{1}{2}(n^{2k-2}-1)n^1$	n^1
$\frac{1}{2}(n^4-1)n^{k-2}$	n^{k-2}	. . . . .	.
. . . . .		And, of course,	
		0	n^k

kth powers can be so exhibited in $E\left(\frac{k}{2}\right)$ different ways; where $E\left(\frac{k}{2}\right)$ denotes the number of integers less than $\frac{k}{2}$.

My former colleague, Mr. E. B. Escott, generalized my theorem as follows: In the series of odd numbers if the first $\frac{1}{2}\,n^k\,(n^{l-2k}-1)$ terms be omitted, the sum of the next n^k terms is n^l. This gives my theorem for $l=2k+1$, and Mr. Anning's results for other values of l.

These theorems are Greek in their nature and would have delighted the heart of Nicomachus.

II. GLOSSARY OF GREEK TERMS

ἄγω, (1) *carry on* an operation, I. 10. 2. λύσιν ἄγειν = *disiunctio progreditur*, Boethius.

(2) *draw* lines, II. 7. 4. Boethius, *producere, ducere*.

ἀγωγή, *procedure*, II. 14. 3. So Diophantus, *Arith.*, 6. 20.

ἀδιαίρετος, *incapable of being divided* (often = *incapable of being halved*), I. 17. 4, 9. 6, 11. 1. Boethius, I. 13, *ne secari queat*; I. 21, *indivisa*; I. 11, *quem secare non possis*.

ἀδιάκριτος, *indiscriminate, not assorted*, I. 13. 2.

ἀδιάστατος, *not having extension, non-dimensional*, II. 6. 3, 7. Boethius, II. 4, *sine intervalli demensione*; *ibid.*, *intervallo carere, sine intervallo esse*.

ἀδιαφορέω, *fail to distinguish as different* (in methods of selection), with acc., I. 16. 4.

ἀδίχαστος, *incapable of being cut in two* or *halved*, I. 9. 1. Boethius, *indivisibilis et insecabilis*, I. 10.

ἀδύνατος, *impossible*, passim.

ἄθροισμα, *a sum* in addition, I. 14. 3.

ἀκολουθέω, *follow out, observe principles*, I. 23. 3. So Diophantus, *Arith.*, 6. 4. 15.

ἀκολουθία, *order, sequence*, I. 18. 1, 23. 2, 9.

ἀκόλουθος, (1) *following, next in order*, I. 19. 5, 13; II. 5. 1, 6. 3, 8. 3, 11. 2, 12. 2.

(2) of a principle, *holding true, consistent*, I. 19. 11; II, 2. 2; hence ἀ. ἐστιν = *it follows*; II. 16. 1. Boethius, II. 25, *oportet*.

(3) *corresponding*, I. 21. 1.

ἀκολούθως, (1) *similarly*, II. 11. 1.

(2) with dat. = *in accordance with, according to*, I. 23. 7; II. 26. 2.

ἄκοσμος, *not governed by rule* or *method*, I. 16. 3. Boethius, *a nullo certo fine generatos*, I. 20.

ἄκρος, (1) masc., *extreme term of any series*, I. 23. 15 (so Euclid, Diophantus); neut., I. 8. 11, 9. 6 (esp. *extremes of proportions*, II. 24. 4, etc.). Syn., ἀκρότατος (ὅρος), ἀκρότης. Boethius, *extremitas, extremus terminus, extremus*.

(2) *edge* of a wedge, II. 16. 2.

ἀκρότατος (*sc.* ὅρος), *extreme term*, I. 8. 10, etc.

ἀκρότης, *extreme, extreme term of a series*, I. 8. 3, 10, 9. 6, 10. 1, 16. 1; II. 16. 3, etc. (Nicomachus prefers ἄκρος, ἄκρον for the extremes in proportion in II. 21 ff.). Boethius *extremus*, etc., *summitas*.

ἀλληλουχούμενος, *continuous*, I. 2. 4. Boethius, *continua et suis partibus iuncta*.

ἄλογος, (1) *unreasoning*, of part of the soul, I. 23. 4. So ἀλόγως, *without reason*, II. 22. 3.

(2) *having no ratio* (*with* = πρός with acc.), I. 6. 3. Euclid uses ἄλογος in the sense of *irrational*.

ἄλφα, *alpha*, the sign for 1, II. 6. 2.

ἀμέριστος, *indivisible* (Platonic), II. 18. 4.

ἀμέτοχος, *not sharing in*, I. 7. 4.

ἀμοιβή, *exchange*; κατ' ἀμοιβήν, *by exchange*, I. 8. 10, 9. 6. Boethius, *invicem*.

ἀνάγω, *reduce*, referring to the reversal of an operation, II. 2. 2. Boethius, *reduco*. Cf. ἀναποδίζομαι.

ἀνακόπτω, *cut off, preclude* the continuance of an operation, II. 4. 1. Cf. ἀποκλείω.

ἀναλογέω, *be analogous to, correspond to*, I. 22. 2; II. 14. 3.

ἀναλογία, (1) *analogy, correspondence*, I. 13. 6.

(2) *a proportion* (so Euclid, Diophantus, Archimedes; Boethius, *proportio, proportionalitas, medietas*); properly, *combination of ratios*, II. 21. 3, i.e., including only the 'geometric proportion,' II. 24. 1; but in practice extended to include 'arithmetic' and 'harmonic' proportions, as II. 22. 1. Syn., μεσότης (cf. II. 21. 6). Kinds of proportion, II. 22. 1.

(3) loosely used in sense of *relation, ratio*; ἡ ἐν ἡμιολίοις σχέσις καὶ ἀ., I. 23. 14; II. 6. 3, 2. 1.

ἀνάλογος, *corresponding*, I. 23. 17; τὸ ἀ., *correspondence*, I. 19. 13; neut. as adv., *in similar manner*, I. 18. 6.

ἀνάλυσις, *a breaking down into components, analysis*, II. 1. 2. Boethius, *solvi, resolvi*.

ἀναλύω, *break down into components, resolve*, I. 11. 3; II. 1. 1, 5. 4–5. v. διαλύω. Boethius, *solvere, resolvere*.

ἀναμίξ, *alternately* (i.e., in proportions if $a:b = c:d$, then $a:c = b:d$), II. 21. 6, 24. 2. But see note on II. 29. 2. Boethius, *permutatim*, II, 40, *permixtim, permixte*.

ἀνάπαλιν, (1) *vice versa*, I. 19. 19; II. 2. 1.
(2) of proportions, *by inversion, inversely* (cf. Euclid, V. 13, defined; so Archimedes), II. 21. 6. Boethius, *conversim*, II. 40. Syn., ἀναστρόφως.

ἀναπλάσσω, *fashion, make, derive*, II. 28. 2.

ἀναποδίζομαι, pass., *be reduced* by reversing a process, II. 2. 2. Cf. ἀνάγω.

ἀναστρέφω, (1) *reverse the order of terms* (opp., ὀρθῶς τιθέναι, κεῖμαι), I. 23. 9; II. 24. 5.
(2) *invert a ratio*, II. 28. 5. See note *ad loc.* for Euclid's usage.
(3) *invert a proposition* (syn., ἀντιστρέφω), II. 6. 5.

ἀναστροφή, *reversal of order*, I. 23. 13. Euclid, conversion of a ratio.

ἀναστρόφως, *in inverted order*, I. 23. 10; of proportions, *inversely* (i.e., if $a:b = c:d$, then $b:a = d:c$; syn., ἀνάπαλιν), II. 24. 2.

ἀνεγείρω, *arouse*, I. 3. 7; *erect lines*, II. 13. 5. Boethius, *surgere*.

ἀνεπίδεκτος, *incapable of receiving* (e.g., division by a given measure), II. 3. 2.

ἀνθυπακούω, *correspond to* as equals, I. 17. 5 (cf. Boethius, *quae ei quantitas comparatur*); as reciprocals, e.g., superparticulars to subsuperparticulars, etc., I. 19. 5, 21. 3.

ἀνισάκις, adv., *an unequal number of times*; in the phrases ἀ. ἄνισοι ἀ. (numbers that are the product of three unequal factors, as *mny*), and ἰσάκις ἴσοι ἀ. (the products of two equal factors by a different one, as m^2n), II. 17, 6, 29. 2. Cf. Boethius, *ex aequalibus (inaeq.) aequaliter (inaeq.) per aequalia (inaeq.) producti*.

ἄνισος, *unequal*, I. 19. 19, etc. Diophantus.

ἀνισότης, *inequality*, I. 14. 2, 23. 6. Boethius, *inaequalitas*. Its two forms, I. 14. 2; the dyad, πρώτη ἀ., II. 18. 1.

ἀνίσταμαι, *be erected* from a base, as pyramids, II. 14. 1. Boethius, *proficiscor*.

ἀνταποκρίνομαι, *correspond to*, used of corresponding factors, as 8 and 2 in 16 ($8 = \frac{1}{2}$ and $2 = \frac{1}{8}$), I. 8. 10, 11. Boethius, *respondere*.

ἀνταπόκρισις, *correspondence*, I. 8. 11.

ἀνταφαίρεσις, *subtraction in turn*, used of the continued subtractions to find the common divisor, I. 13. 11. Cf. Boethius, *vicissim ista subtractio*, I, 18; *reciproca deminutio*, *ibid*.

ἀνταφαιρέω, *subtract in turn* (cf. ἀνταφαίρεσις), I. 13, 11–12.

ἀντεξετάζω, *examine in comparison with, compare*, I. 13. 1, 14. 3. Boethius, *comparati . . . ad se invicem*, I. 16.

ἀντιδιαστέλλω, *distinguish from, as contrary*, with dat., I. 9. 1, 17. 8, 18. 2. Boethius, *obponere*.

ἀντιδιαστολή, *correspondence*, used of paired factors of numbers (cf. ἀνταποκρίνομαι), I. 8. 11, 16. 5.

ἀντίθετος, *opposite*, I. 17. 6, 8, 19. 20; II. 6. 4; *compared*, I. 20. 2.

ἀντιθέτως, *correspondingly*, I. 19. 18.

ἀντίκειμαι, *be opposed, be the opposite of*, with dat., I. 11. 1, 13. 1, 16. 1; II. 16. 1, etc. Boethius, *contra se positi*. Diophantus uses the word of corresponding factors (v. ἀνταποκρίνομαι).

ἀντιλαμβάνω, *take in substitution* (cf. ἀντιτίθημι), II. 27. 5.

ἀντιπαρωνυμέομαι, *be named from reciprocally*, I. 8. 10.

ἀντιπαρώνυμος, *having corresponding, but opposite, name* (as double vs. sesquialter), II. 3. 1. Boethius, *similis*.

ἀντιπερίστασις, *reciprocal opposition* or *correspondence* of paired factors (v. ἀνταποκρίνομαι), I. 8. 10 (see note *ad loc.*), 9. 6. Used also with reference to the operation of finding the common divisor in sense of *shifting, exchange*, I. 13. 11.

ἀντιστρέφω, *invert*; ἀ. τὸν λόγον, *state the converse*, II. 6. 6.

ἀντιστροφή, ἐξ ἀ., *conversely*, I. 9. 2. Euclid, *conversio*.

ἀντιστρόφως, *conversely*, I. 16. 4.

ἀντισυγκρίνω, *compare over against*, I. 13. 11, 20. 1.

ἀντιτίθημι, *substitute for*, II. 27. 6 (cf. ἀντιλαμβάνω).

ἀντονομάζω, *give a corresponding, but opposite name*, I. 22. 7, 23. 1.

ἀντονομασία, *corresponding opposite nomenclature*, I. 23. 3.

ἀντωνυμέω, *have a corresponding opposite name*, I. 17. 6.

ἀνυπερβάτως, *without skipping, without omitting* terms in a series, II. 23. 1.

ἄνω, (1) *above*, one of the varieties of relative position (v. περίστασις), II. 6. 4.
(2) *top* line of a table or diagram, II. 3. 3.

ἄνωθεν, *anew, again*, I. 10. 8, etc.

ἀνωτάτω, *highest* in the classification of genera and species, I. 17. 2; *topmost* line of a table, II. 4. 3.

ἀνωτέρω, *above, before* in the course of the treatise, II. 26. 1.

ἅπαξ, *once, taken once as a factor*, I. 8. 14, etc. Diophantus.

ἀπαράλλακτος, *unchanging* (λόγος), II. 17. 3, 19. 3 (ὅρος), II. 21. 5. Neut. as adv., *without fail, without deviation*, II. 5. 4; also ἀπαραλλάκτως, I. 23. 3.

ἀπαρεμπόδιστος, *without hindrance*, of a regularly proceeding operation, I. 13. 6.

ἄπειρος, *limitless, infinite*, I. 2. 5, 18. 4, 19. 1. μέχρις ἀπείρου, ἐπ' ἄπειρον, *ad infinitum*, I. 2. 5, 8. 9, 10. 9, 13. 3, etc. Diophantus, Euclid.

ἁπλοῦς, *simple*; hence, *clear and easy*, I. 19. 8; II. 6. 2, 20. 5; *incomposite*, as elements, II. 1. 1; *not complicated, mere, without the restrictions of classification and definition*, II. 22. 3 (ἁ. ἀριθμός), I. 21. 3, the simple superparticular vs. the multiple superparticular, etc. Euclid; Boethius, *simplex*.

ἁπλῶς, *simply, in a word*, I. 22. 5; II. 1. 1, 3, etc.; frequently qualifying nouns, signifying that they are to be taken absolutely, without such limitation as that of genus by a difference; e.g., I. 14. 1 (ἁπλῶς ἄρτιοι, the *merely* even as opposed to ἀρτιάκις ἄρτιοι), I. 2. 5 (ἁ. μέγεθος, *magnitude per se, absolute*); cf. I. 18. 6; II. 14. 5, 16. 2.

ἀπό, *beginning with*, II. 19. 2, etc. (so Diophantus); εἶναι ἀ., *to be removed from*, II. 3. 1; of cubes, *constructed on* a given side, II. 17. 7 (so Diophantus); τὸ ἀπό, *the square of* (so Diophantus), I. 8. 14, 19. 17; II. 23. 6, 24. 4, 27. 3, 4, etc. So used by Archimedes of geometrical squares.

ἀποβαίνω, *result*, I. 23. 5; of addition, I. 16. 5.

ἀπογεννάω, *produce* by an operation, I. 10. 8, 16. 4, 18. 1, etc.

ἀποδείκνυμι, *demonstrate, prove*, II. 1. 1–2, 17. 2, 18. 1, 22. 3. Diophantus; Euclid, ἀπόδειξις.

ἀποδεικτικός, *capable of proving*, I. 23. 6.

ἀποδίδωμι, *give as a result, produce*, II. 5. 2, 8. 3, 12. 2; *to show* a ratio, II. 5. 2; of a square showing four angles, II. 9. 1.

ἀποδοτικός, *productive of*, with gen., II. 5. 1 ff.

ἀποκαταστατικός, *recurrent* numbers, II. 17. 7. Boethius, *cyclici vel spherici;* Euclid, v. σφαιρικός.

ἀποκλείω, *lock off, preclude the continuance* of an operation, II. 4. 1.

ἀποκορυφόω, *complete*, II. 13. 5.

ἀποκρίνομαι, *answer to, be compared with, have a ratio toward* terms in a proportion, II. 21. 5. Boethius, *se communicare.*

ἀπολαμβάνω, *acquire, reach* a certain quantity, said of an ascending series, II. 3. 2.

ἀπολείπω, *leave* after subtraction, I. 13. 12.

ἀπολήγω, *end*, of the end of a process (freq. with εἰς μονάδα), I. 10. 3, etc.

ἀπομειουρίζω, *taper off*, said of pyramids, II. 13. 9.

ἀπόστασις, *interval*, the number of terms regularly omitted in choosing from a series, II, 11. 1.

ἀποτέλεσμα, *completed thing*, I. 4. 2; *the result* of an operation, I. 19. 16.

ἀποτελέω, *complete, make*, I. 6. 1, 13. 9; II. 11. 1, etc.; *give* as the result of addition, I. 16. 1, 2, 4; II. 14. 3, etc.; *give* as the result of multiplication, I. 10. 8, 9, 12. 1; II. 15. 2, etc.

ἀποτέμνω, *take* (*cut*) *away* from a series for separate consideration, II. 23. 3, 24. 5.

ἀριθμητικός, (1) *numerical, having to do with numbers, arithmetic*, I. 23. 4 (σχέσεις); 5. 1 (λόγοι).

(2) ἡ ἀριθμητική, *the science of number, arithmetic*, I. 3. 1, 4, 7; II. 6. 1, etc. Boethius, *arithmetica*.

(3) ἀ. ἀναλογία, μεσότης, *arithmetic proportion* (i.e., *progression*) of the type $a - b = c - d$, II. 23. 1 (defined). Called also *numerical* (ἡ κατὰ τὸ ποσὸν τὴν τῶν ὅρων σύγκρισιν οἰκειοῦσα, II. 22. 2; cf. 21. 5–6, 23. 4).

ἀριθμός, (1) *a number* (Diophantus), I. 7. 1 (defined), 8. 1, etc.

(2) collectively, of series of numbers, esp. the natural series (freq. ὁ εὔτακτος ἀριθμός), I. 18. 5, etc. Used of other series (e.g., the multiples of 9), I. 22. 4. Boethius, *numerus*.

ἁρμόζω, *fit*, *agree* with a principle, I. 22. 6; *be suitable*, II. 21. 1; pass., *be fitted together*, *constituted*, I. 6. 2–3; II. 26. 2.

ἁρμονία, *harmony*, II. 19. 1 (defined); of the spheres, I. 5. 2. Usually, *musical harmony*, I. 3. 5; II. 26. 1. Used as name for the cube, γεωμετρικὴ ἁ., II. 26. 2, 29. 1; of the 'most perfect' proportion, II. 29. 1.

ἁρμονικός, *harmonic* intervals, I. 3. 3; II. 2. 3; of ratios, I. 5. 1; of a theory, II. 26. 1; of the title of a book of Archytas (ἁρμονικὸς [λόγος]), I. 3. 4. ἁ. ἀναλογία or μεσότης, *the harmonic proportion*, II. 22. 1, 23. 6.

ἀρτιάκις, adv., *an even number of times*. See ἄρτιος, ἀρτιοδύναμος, ἀρτιώνυμος.

ἀρτιακῶς, *in an even fashion*; ἀ. ὠνομασμένον, *with an even name*, I. 9. 2. Opp. περισσῶς.

ἀρτιογενής, *even by genus*, I. 8. 10.

ἀρτιοδύναμος, *even in value* or *amount*; ἀρτιάκις ἀ., *even-times even in value*, I. 8. 6.

ἀρτιοπέριττος, *even times odd*, I. 9. 1 (defined); a species of the even, I. 8. 3, I. 9. 5, 10. 1, etc. Boethius, *pariter impar*.

ἄρτιος, *even*, I. 7. 2–4 (defined). Varieties, ἀρτιάκις ἄρτιον, περισσάρτιον, ἀρτιοπέριττον, I. 8. 3, 9. 3, 10. 2, 11. 1, etc. ἁπλῶς ἀ., *the even as such*, distinguished from its species, I. 14. 1. ἀρτιάκις ἄ., *the even times even* number of the type 2^n, I. 8. 4 (defined), etc. Boethius, *par*; *pariter par* = ἀρτιάκις ἄ.; Euclid.

ἀρτιοταγής, *with an even number of terms*, II. 24. 4.

ἀρχέγονος, *of primitive origin*, I. 19. 8; II. 6. 1.

ἀρχέτυπος, adj., *original*, *archetypal*, I. 4. 2; II. 28. 2.

ἀρχή, (1) *beginning*, *starting point*, passim; ἐξ ἀρχῆς as adj., *original*, I. 9. 4 (of *given* numbers, I. 13. 13); *origin* of lines, esp. of rows in a diagram, as I. 19. 13, 17.

(2) *source*, *origin* of things in general, I. 11. 3; II. 17. 1, 19. 4. Boethius, *principium*, *caput*, *initium*.

ἀρχικός, *original*, *primitive*, I. 17. 4; II. 1. 1, 7. 3, 19. 4.

ἀρχοειδής, *original*, *primal*, I. 7. 4; II. 6. 3.

ἄρχω, act., *begin*, *be the first* term in a series, II. 17. 5; mid., with ἀπό, *take beginning*, *start from*, I. 8. 10, 11. 3, 18. 2, etc. ἀρχόμενος, with gen., is frequent adverbially.

ἀστρονομία, *astronomy*, I. 3. 7. v. σφαιρική.

ἀσύνθετος, *incomposite*, *not composed of factors* (of prime numbers, v. πρῶτος), I. 11. 1 ff., 21. 3, 12. 2. Euclid.

ἀσφαλής, *unfailing*, of a process, I. 16. 4.

ἄσχιστος, *incapable of separation*, *differentiation*, used of a σχέσις, I. 17. 4.

ἄτακτος, *disarranged*, *not in due* or *natural order*, *not subject to regularity*, I. 16, 3. Opp. εὔτακτος. ἀτάκτως, I. 23. 9.

ἄτμητος, *not capable of being divided*, I. 10. 4, 9. 1.

ἄτομος, *indivisible*, *without factors*, of the monad, I. 8. 4–5, 10. 2, etc.

αὐξέω, pass., *be increased*, II. 13. 6, 15. 4, etc.; *be multiplied*, II. 15. 2, 17. 7. Used by Diophantus of numbers increasing from unity to infinity.

αὔξησις, *increase*, I. 2. 1; *augment*, I. 19. 15; increase, by multiplication, II. 17. 7.

ἀφαίρεσις, *subtraction*, II. 20. 1 (Diophantus); *an operation of subtraction*, I. 13. 12; II. 2. 1, 20. 1.

ἀφαιρέω, *subtract*, I. 13. 11, 12, 13; II. 2. 1 (so Diophantus, Archimedes). Boethius, *auferre*, *detrahere*, *demere*.

ἀφίστημι, *set aside*, *set down* terms, II. 2. 3; mid., *differ from* in attributes, I. 10. 3.

βαθμός, *stage*, *degree*, *rank*, *class*, I. 16. 3.

βάθος, *depth*, *a dimension*, II. 13. 1; hence, referring to a table, *the direction up and* *down* the columns, I. 19. 11; II. 3. 4, 4. 1. Boethius, *altitudo*, *crassitudo*, *profunditas*; Euclid.

βάσις, *base*, II. 13. 2, 14. 1 ff., 16. 2. Diophantus, of triangles; Boethius, *basis*; Euclid.

βλαστάνω, *grow*; hence, *be produced*, I. 23. 8.

βωμίσκος, *little altar*, the name of a kind of solid called also σφηνίσκος, σκαληνός, σφηκίσκος, *q. v.* Boethius, *bomiscus*, *arula*.

βωμός, same as βωμίσκος, *q. v.*

γαμμοειδῶς, *after the form of Gamma* (Γ), i.e., at right angles, I. 19. 11.

γένεσις, *origin*, *production*, or *mode of production*, I. 8. 8, 12. 1, 13. 2, etc.

γενικός, *pertaining to a genus*, *generic*, I. 17. 2 (διαίρεσις), 19. 20 (σχέσις). γενικώτατος, *most typical of a class*, I. 14. 2.

γενικῶς, *as genera*, *generically*, I. 18. 4, 22. 2; II. 20. 2. Opp. ὑποβεβηκότως.

γεννάω, *produce*, *make*, *create*, I. 10. 6, 22. 3, etc.

γεννητικός, *productive*, II. 3. 3, 19. 4.

γένος, *genus*, in the Aristotelian sense, I. 9, 1, and *passim*. See εἶδος.

γεωμετρία, *geometry*, I. 3. 2, 4. 5, 5. 2, etc. Boethius, *geometria*.

γεωμετρικός, *geometrical*. γ. ἀναλογία, μεσότης, *geometric proportion*, type $a:b=c:d$, II. 22. 1, etc. (so Diophantus). γ. εἰσαγωγή, *introductory treatise on geometry*, II. 6. 1. Boethius, *geometricus*.

γίγνομαι, *be generated* or *produced* by a process or principle, I. 9. 5; of multiplication, II. 18. 2. So δὶς γενόμενος = *multiplied by* 2, II. 15. 2. The participles are used of the products of multiplication, I. 10. 8; II. 27. 5, 7 (cf. Diophantus, p. 170, 16 T), or sums in addition, I. 16. 4 (cf. Diophantus, p. 322, 7) in masc. and neut.

γλαφυρός, *neat*, *nice*, *exact*, of rules, principles, I. 19. 6, 8.

γνώμων, the *gnomon* of the sun dial; the carpenter's *square*. Used by Nicomachus of the numbers of a series which added together produce successively numbers of a certain type, II. 11. 1 (v. note *ad loc.*); esp. the odd numbers, I. 9. 4 (v. note *ad loc.*). Boethius, *radix et fundamentum*.

γράμμα, *a letter*, used arbitrarily as a numerical sign, II. 6, 3. Boethius, *compendium*, *signum numeri*, *notula*.

γραμμή, *a line*, II. 6. 2 (defined, *ibid.*, 4). Boethius, *linea*; Euclid.

γραμμικός, *pertaining to lines*; of numbers, *linear*, II. 7. 3 (defined), 6. 1 (Boethius, *lineares numeri*); of figures, *geometric* as opposed to numerical (ἀριθμητικός), II. 7. 4. Euclid.

γωνία, *angle*, II. 9. 1; 15. 4. Diophantus. στερεὰ γ., *solid angle*, II. 15. 4.

δείκνυμι, *show*, *display* as having, I. 15. 1, 19. 13; II. 14. 4; *demonstrate*, *prove*, II. 9. 3, 16. 3, 29. 2. Diophantus.

δεκαπλάσιος, *tenfold*, I. 19. 8, 22. 6, etc.

δεκάς, *the decad*, *the number ten* (δέκατος ἀριθμός, II. 22. 1). Plur., *the ten's*, i.e., 10–99, I. 16. 3.

δεκτικός, *capable of admitting* an ἐπίτριτος, II. 4. 2.

δευτεροδέω, *be of the second course* (v. τριοδέω), I. 19. 17.

δεύτερος, *second*; hence, *secondary* (opp. to *prime*), I. 11. 1 ff. Euclid; Boethius, *secundus et compositus*.

δευτεροταγής, *arranged in the second place* in a series, I. 13. 5.

διὰ πασῶν, *diapason*, interval with ratio 2 : 1, *octave*, II. 26. 1. Boethius, *diapason*.

διὰ πασῶν ἅμα καὶ διὰ πέντε, *diapason and diapente together*, interval with ratio 3 : 1, II. 26. 1. Boethius, (*simul*) *diapente et diapason*.

διὰ πέντε, *diapente*, interval with ratio 3 : 2, II. 26. 1. Boethius, *diapente*.

διὰ τεσσάρων, *diatessaron*, the harmonic interval 4 : 3, II. 26. 1. Boethius, *diatessaron*.

διάγραμμα, *table*, *diagram*, I. 19. 9.

διαγράφω, *make a table*, II. 4. 1.

διαγώνιος, *placed on the diagonal*, I. 19. 19. διαγωνίως, *diagonally*, *by a diagonal*, II. 12. 1.

διαζεύγνυμι, *unyoke*; hence, *separate* and *oppose*, I. 23. 15. διεζευγμένη ἀναλογία, *disjunct proportion*, having four terms (opp. συνημμένη), II. 21. 6. Boethius, *disiuncta*; Euclid.

διάζευξις, *disjunction*, of proportions, II. 24. 4.

διαίρεσις, *separation* or *division into parts*. Diophantus. *A class* or *division made by dividing*, I. 17. 2.

διαιρετός, *capable of division*, I. 9. 6, 11. 1. Boethius, *divisionem recipere; sectione solvi*.

διαιρέω, *divide* a geometrical figure, II. 12. 1. Diophantus; Archimedes; Boethius, *dividere*.

διακρίνω, pass., with πρός, *be distinguished from*, I. 11. 1; *be separated* or *sifted out*, I. 13. 8.

διαλείπω, *leave* as an interval *between*; used in locating terms in a series, I. 13. 3. v. παραλείπω, ὑπερβαίνω.

διαλλάττω, *vary from*, II. 17. 3.

διαλυτός, *separable*, *capable of resolution*, I. 12. 2.

διαλύω, *resolve*, I. 12. 2; II. 5. 3. Diophantus, *solve* a problem.

διαμένω, *hold true*, II. 27. 5.

διάστασις, *dimension* (= διάστημα), II. 6. 5-6. Euclid.

διαστατός, *extended*, *having dimension*, II. 6. 4. Boethius, *distentus*; Euclid.

διαστέλλω, *distinguish*, *make a distinction*, II. 18. 2. Diophantus.

διάστημα, (1) *interval*, II. 6. 3, end (defined). *Interval between numerical terms* (= difference), II. 6. 3 (p. 84, 19 ff.); II. 21. 3 (*difference* in arithmetic progression). Usually = *dimension* (*extension*), II. 13. 1, 15. 2, etc. The three dimensions are enumerated in II. 6. 4.

(2) *the harmonic intervals*, II. 2. 3. Euclid; Boethius, *intervallum*.

διασώζω, *preserve* a ratio *through* a series of terms, II. 23. 1, 24. 2.

διατηρέω, *preserve* a constant difference, II. 27. 6.

διαφέρω, (1) *differ* in attributes, I. 10. 1 (cf. ἀφίστημι); in quantity, I. 9. 4; II. 17. 1. Diophantus.

(2) *be greater than*, I. 19. 11. Boethius, *transcendere; superare*, etc.

διαφορά, *difference* (arithmetical), I. 19. 12 (Diophantus); *variation*, I. 23. 6. Boethius, *differentia, discrepantia*.

δίδωμι, *give* as the terms of a problem, II. 2. 1. Diophantus.

διεχής, *not consecutive*, *separated* terms in a series, II. 23. 3.

διϊστάνω, pass., *be extended*. τριχῇ δ., *be extended in three dimensions*, *be a solid*, II. 29. 1.

δικόλουρος, *doubly truncated*, of pyramidal numbers, II. 14. 5. Boethius, *bis curtus*.

διπλασίασις, *doubling*, I. 13. 6.

διπλασιάζω, *double*, I. 21. 1. Diophantus.

διπλάσιος, *double*, *the relation* (σχέσις) *of doubles*, I. 3. 1, etc. Diophantus; Boethius, *duplus*.

διπλασίων, *double* (= διπλάσιος), I. 8. 10, etc. Diophantus.

δὶς διὰ πασῶν, *di-diapason*, harmonic interval with the ratio 4 : 1, II. 26. 1. Boethius, *bis diapason*.

διφορέω, pass., *be twice produced* by an operation, I. 13. 12, 13.

διφόρησις, *separation*, *opposition*, I. 19. 17.

δίχα, adv., *in two parts*, I. 8. 4, etc. Diophantus.

διχάζω, *divide into two parts*, I. 7. 4, 8. 4, 9. 1.

διχασμός, *division into two parts*, I. 10. 2.

διχῆ, adv., *into two parts*, I. 11. 1; *in two directions* or *dimensions*, II. 6. 4.

διχοστατέω, *stand apart*, *be opposed*, II. 18. 4.

διχοτόμημα, *a division into two parts*, I. 7. 4. Diophantus, διχοτομία.

δοκίς, *a beam*. Applied to numbers of the type a^2b where $b > a$, II. 6. 1, 15. 1, 17. 6 (defined), 29. 2. Boethius, *tignulum*, *docis*, *asser*.

δυάς, *the dyad*, *the number two*, I. 7. 4, 13. 6; II. 19. 3, etc.

δυασμός, *pairing by two's*, II. 19. 3.

δύναμις, (1) *value*, *amount*, I. 8. 7, 10, 10. 5, 9. 2. Boethius, *quantitas*.

(2) δυνάμει, *potentially* (opp. ἐνεργείᾳ), I. 16. 8; II. 8. 2, 3, 9. 2, 3, etc.

Also κατὰ δύναμιν, in same sense, I. 16. 10. Boethius, *virtute atque potentia*, *vi sua*, *potestate*.

δωδεκάς, *the number twelve*, I. 19. 13.

ἑβδομάς, *the number seven*, I. 13, 13, 22. 3, 5.

ἐγγύς, *next*, esp. of terms next in a series, I. 9. 4, 19. 6, etc.

ἔγκειμαι, *reside in*, *inhere in*, of regular order inherent in a series, I. 13. 6; *be given to*, of names given, I. 22. 7.

εἰδικός, *specific, belonging to* or *pertaining to* a species (εἶδος). Opp., γενικός. Hence εἰδικὴ σχέσις, *specific relation, a species of relation* (equiv. to εἶδος) contained in a γένος, I. 21. 3, 22. 2 (ε. ὑποδιαίρεσις); τὰ εἰδικά = εἴδη, *species*, I. 22. 2. So ε. πολλαπλασιεπιμόριον = *multiple superparticular of a specific kind*, I. 22. 4. Adv. εἰδικῶς (opp., γενικῶς), *in the manner of a species, as a species*, as opposed to genus, I. 19. 8, 20. 1.

εἰδοποιέω, *create a species by furnishing its peculiar attribute*, II. 17. 2, 18. 1. Boethius, *perficere*.

εἰδοποίησις, *creation as a species*, II. 18. 1.

εἶδος, *kind, species*, both in the Aristotelian sense (opp., γένος), I. 18. 4, 19. 2; II. 4, 3, 5. 4, etc., and generally, to mean *variety, kind*. Boethius, *pars, species*.

εἰκῆ, *at random*, I. 13. 3.

εἰκοσάεδρον, *icosahedron*, I. 4. 4.

εἷς, *one, passim*.

εἰσαγωγή, *introduction, introductory treatise*, I. 23. 4; II. 12. 1, 22. 3, 29. 5; referring to the *Introductio Arithmetica*, I. 19. 20; II. 5. 1, 21. 1; to the *Introduction to Geometry*, II. 6. 1.

ἑκατέρωθεν, *in either direction* on a diagram or table, I. 19. 11.

ἑκατέρωθι, *on either side* of a proportion, II 25. 2–3.

ἑκατοντάς, plur., *the hundred's* (100–999; cf. δεκάς), I. 16. 3, 7, 19. 18.

ἔκθεσις, *a series of terms*, I. 8. 11, 23. 7, etc.; also of the terms in a series, I. 8. 10. Boethius, *dispositio*.

ἔκκειμαι, *be set forth as terms* in a problem or operation, I. 13. 7, 19. 10, 23. 8; II. 10. 2, 17. 3, etc. (so Diophantus). Also, *be placed, located*, I. 19. 13, 22. 3.

ἔκστασις, *a change* (cf. ἐξίστημι), II. 17. 5.

ἐκτίθημι, *set forth terms in series*, I. 10. 7, etc. Diophantus; Boethius, *disponere*.

ἐλαττονάκις, *a less number of times* (v. ἰσάκις), II. 17. 6.

ἐλαττόνως, *in the way of being less*; διαφέρειν ἐ., *to be smaller*, II. 11. 4.

ἐλαττόομαι, *fall short of, be less than*, I. 15. 1; II. 6. 7, 21. 5.

ἐλάττων, *smaller*, I. 9. 4, etc. Diophantus.

ἐλλείπω, *be deficient, fall short in quantity*, I. 15. 1; II. 17. 3.

ἔλλειψις, *deficiency*, II. 27. 4.

ἐλλιπής, *deficient*, as applied to a number the sum of whose factors is less than its own quantity, I. 14. 1; 15. 1; 16. 3. Boethius, *deminutus*.

ἐμβάλλω, *insert* a mean term, II. 27. 4. Cf. ἐναρμόζω.

ἐμπεριέχω, *contain*, I. 19. 5.

ἐμπλέγδην, *in interlocking fashion*; of proportions, *alternando*, II. 29. 2.

ἐμφαίνω, *make clear*, I. 1. 1; *display to view* as having, I. 7. 4, 10. 1; II. 22. 3.

ἔμφασις, *demonstration*, I. 19, 8.

ἐναλλάξ, *alternating, one after the other*, I. 6. 4; *by interchange*, referring to proportions; i.e., if $a:b=c:d$, then $a:c=b:d$ (cf. ἀναμίξ), II. 21. 6. ἐ. ποιεῖν, *to make as a result* in addition, II. 19. 4. Boethius, *permutatim*.

ἐναντιοδύναμος, *of opposite value*, e.g., to its name, as when the *fourth* part of a number is *odd* in amount, I. 9. 2.

ἐναντιόομαι, *be opposed, be contradictory*, I. 19. 16.

ἐναντιοπαθέω, *have qualities opposite to* (with dat.), I. 9. 6, 10. 5.

ἐναντιωνυμέω, *have an opposite name* (v. ἐναντιώνυμος), I. 10. 5. Boethius, *contrariam denominationem sumere*.

ἐναντιώνυμος, *with opposite name*, e.g., to its amount (cf. ἐναντιοδύναμος), I. 9. 2.

ἐναρμόζω, *fit in, insert* a mean term, II. 27. 3. Cf. ἐμβάλλω.

ἐνάς, *the number nine*, I. 19. 11.

ἐνδέχομαι, *admit* factors, I. 16. 2.

ἔνειμι, *be contained in*, of factors in a whole, I. 10. 5, 21. 2.

ἐνέργεια, *actuality*, I. 1. 3; esp. ἐνεργείᾳ, *actually, in actuality*, I. 16. 4, 8; II. 8. 2, etc. (opp., δυνάμει); also κατ' ἐνέργειαν, I. 16. 10. Boethius, *actu vel opere*.

ἐνθεωρέω, pass., *be beheld in, conceived of in*, hence *implied*, I. 16. 4. See note *ad loc.*

ἐννοέω, *imagine, conceive*, II. 13. 3.

ἑνοειδῶς, *so as to be unitary*, I. 6. 4.

ἑνόω, *make unitary*; ἡνωμένα, *unitary*, I. 2. 4.

ἐντάττω, *insert* means, II. 27. 1. Cf. τάττω.

ἐντός, *within*, esp. in sense of *up to, in the series terminated by*, I. 8. 10, 13, 16. 3.

ἐνυπάρχω, *be in, contained in*, of parts, I. 22. 2.

ἑξάγωνος, *hexagonal* numbers, II. 7. 3, 11. 1, 12. 3. Boethius, *exagonus*.

ἐξαλλάττω, pass. part., *changing, varying, different*, II. 16. 2; ἐξηλλαγμένως, *in a peculiar way*, I. 12. 1; *in different ways*, II. 20. 4.

ἑξαπλάσιος, *sextuple*, I. 18. 6.

ἐξαπλόω, *reduce* a number to units, II. 8. 3.

ἐξάπλωσις, *simplification, reduction*, II. 10. 1.

ἑξάς, *the number six, hexad*, I. 16. 3.

ἐξετάζω, *examine, consider*, I. 19. 12; II. 23. 1, 25. 1.

ἐξέτασις, *examination, test*, I. 9. 6; II. 4. 3; *arrangement*, II. 29. 2.

ἑξῆς, *in order, in succession*, I. 10. 8. Boethius, *naturaliter constituti*.

ἕξις, *state, condition*, II. 6. 1.

ἐξίστημι, *force out, change* (with gen.), II. 17. 4; mid., *change, depart from*, I. 1. 2.

ἐπαύξησις, *increase*, II. 1. 2, 15. 1.

ἐπεξευρίσκω, *devise in addition*, II. 28. 6.

ἐπιγώνιος, *placed at an angle* or *corner*, I. 19. 17.

ἐπιδεκτικός, *capable of admitting* (v. ἐπιδέχομαι), II. 4. 1.

ἐπιδέχομαι, *be capable of*, e.g., division, *admit of* a factor, I. 9. 1, 11. 2; II. 4. 1.

ἐπικορύφωσις, *culmination, terminal number* of a series, II. 3. 2.

ἐπιμερής, *a superpartient number*; τὸ ἐ., *the superpartient, a ratio* or *relation*, I. 17. 7, etc. Boethius, *superpartiens*.

Varieties:

- **ἐπιδίτριτος**, $1\frac{2}{3}$, I. 23. 14; synonymous are δισεπίτριτος, I. 23. 16, ἐπιδιμερής, I. 20. 1. Boethius, *superbitertius*.
- **ἐπιτριτέταρτος**, $1\frac{3}{4}$, I. 21. 2; synonymous are τρισεπιτέταρτος, *ibid.*, ἐπιτριμερής, I. 20. 1. Boethius, *supertriquartus*.
- **ἐπιτετράπεμπτος**, $1\frac{4}{5}$, I. 21. 2; synonymous are τετρακισεπίπεμπτος, *ibid.*, ἐπιτετραμερής, I. 20. 1. Boethius, *superquadriquintus*.
- **ἐπιπένθεκτος**, $1\frac{5}{6}$, I. 21. 2.
- **τρισεπίπεμπτος**, $1\frac{3}{5}$, I. 23. 16.
- **τετρακισεφέβδομος**, $1\frac{4}{7}$, I. 23. 16.
- **πεντακισεπένατος**, $1\frac{5}{9}$, I. 23. 17.

ἐπιμόριος, *the superparticular*, a ratio or relation, and variety of inequality, I. 17. 7; defined, I. 19. 1.

Varieties:

- **ἡμιόλιος**, I. 19. 2, or ἐφημιόλιος, I. 22. 2, *sesquialter* ($1\frac{1}{2}$). Boethius, *sesqualter, hemiolius*.
- **ἐπίτριτος**, I. 19. 2, sesquitertian ($1\frac{1}{3}$). Boethius, *sesquitertius*.
- **ἐπιτέταρτος**, I. 23. 14, sesquiquartan ($1\frac{1}{4}$). Boethius, *sesquiquartus*.
- **ἐπίπεμπτος**, I. 22. 2, sesquiquintan ($1\frac{1}{5}$). Boethius, *sesquiquintus*.
- **ἔφεκτος**, I. 23. 7, sesquisextan ($1\frac{1}{6}$). Boethius, *sesquisextus*.
- **ἐπόγδοος**, II. 2. 3, sesquioctave ($1\frac{1}{8}$). Boethius, *sesquioctavus*.

ἐπίπεδος, *plane* (usually neut.); applied to numbers (ἀριθμητικὸν ἐ., II. 7, 5), *plane number*; defined, II. 7. 3; 6. 1; *geometrical planes* (γραμμικὸν ἐ.), II, 7. 5; *plane surface*, II. 15. 4; *face of a solid, ibid.* Boethius, *planae numerorum figurae*.

Varieties: **τρίγωνος, τετράγωνος, πεντάγωνος**, ἑξάγωνος, ἑπτάγωνος, II. 7. 3–4.

ἐπισκοπέω, *observe* (in connection with selecting terms from series, to conform to certain requirements), I. 13. 3, etc.

ἐπιστήμη, *science, knowledge*, I. 1. 2, etc.

ἐπιστημονικός, *scientific*; ἐ. ἀριθμός, *the number dealt with by science*, I. 6. 1, 4; ἐ. μέθοδος, *scientific method*.

ἐπιστίζω, *check off*, I. 13. 7.

ἐπισυντίθημι, *add to* a previous sum, I. 16. 7; II. 12. 2.

ἐπισωρεία, *summation, cumulative addition*, II. 15. 1.

ἐπισωρεύω, *add the terms of a series together*, II. 8. 3 (cf. προσσωρεύω). Boethius, *adgregare*.

ἐπισώρευσις, *the addition of a term to a previous sum*, in ἐπισωρεία, I. 16. 4.

ἐπιτίθημι, *add*, II. 14. 3.

ἐπιφάνεια, *a surface, superficies, that which has two dimensions*, II. 6. 4, 7. 1–2. Boethius, *superficies*.

ἑπτάγωνος, *heptagonal* number, II. 7. 3, 11. 1, 12. 3. Boethius, *eptagonus*.

ἑπτάς, *the heptad, the number seven*, I. 16. 5.

ἔσχατος, *last*, I. 16. 4, 23. 15; *ultimate*, II. 1. 1.

ἑτερογενέω, *be of a different genus*, I. 10. 5.

ἑτερογενής, *of a different genus* or *kind*, I. 6. 4, 23. 16; II. 15. 1.

ἑτερομήκης, *heteromecic*, a number of the type $m\ (m+1)$, I. 19, 19; II. 17. 1 (defined), 18. 2, 16. 3, 20. 1, 24. 5. Boethius, *parte altera longior*, *longilaterus*.

ἕτερον, *other*, passim; τὸ ἕ., '*otherness*,' *principle of difference and variation*, II. 17. 2, 18.1; τὸ πρὸς ἕτερόν πως ἔχον, *that which has some relation to another*, *the relative*, II. 6. 1.

ἑτερότης, *difference*, '*otherness*,' *the principle of difference* (v. ἕτερος), II. 17. 1, 18. 1. Boethius, *alteritas*.

ἑτερώνυμος, *having a different name*, *heteronymous*, applied to factors not named from the number factored itself, as 5, which is a *third* of 15, as opposed to 1, which is the *fifteenth* (and is called παρώνυμος), I. 11. 2, 12. 1. Boethius, *pars alieni vocabuli*.

ἑτερωνυμέω, *have a different name*, I. 17. 5.

εὐθεῖα, *straight line* (*sc.* γραμμή), II. 6. 2, 7. 4, 13. 3. Boethius, *linea recte stans*.

εὐθύγραμμος, *rectilinear*, II. 7. 4. So Archimedes.

εὐοδέω, *proceed well*, I. 23. 8.

εὕρεσις, *discovery*, *method of discovery*, I. 16. 3.

εὑρίσκω, *find*, *discover*, passim; pass., *be found* to exist, I. 16. 3.

εὔτακτος, *orderly*, *in natural* or *regular order*, I. 13. 6, 18. 5, etc.

εὐταξία, *due*, *proper* or *natural order*, II. 1. 2. Boethius, *ordinabilis compositio*.

ἐφαρμόζω, *fit*, *test* to see if requirements fit, II. 27. 6.

ἐφεξῆς, *in order*, I. 10. 7; καὶ ἐ., *and so on*, *ibid.*, etc.

ἔφοδος, *principle*, *method*, I. 10. 6; II. 20. 5.

ἡγεμών, *leader*, *head*, in the sense of heading a list, II. 4. 2; in sense of source as opposed to derivation, II. 22. 3.

ἡγέομαι, *be first*, *lead*, I. 23. 7; *be head of* a series (and its starting point), II. 3. 1; *be prior to*, II. 22. 3.

ἡμίσεια, *half*, I. 8. 5.

ἡμίσευμα, *half*, I. 9. 3.

ἥμισυς, adj., *half*, I. 8. 1; neut., substantively, I. 10. 2, etc. Boethius, *medietas*, *secunda pars*.

ἥττων, *smaller term* of ratio, I. 19. 7.

θέσις, *position*, II. 8. 1.

θεωρέω, *behold*, *view*, I. 13. 1, 16. 4, 17. 2, 3, etc. Boethius, *considerare*.

θεώρημα, *principle*, I. 23. 6; II. 6. 1.

ἰδικῶς, *properly*, *peculiarly*, I. 9. 1 (but see note *ad loc.*).

ἰδιομήκης, *with own length*, *proper length*, epithet of squares, II. 18. 3.

ἴδιος, *own*, *proper*, passim; ἰδίᾳ, *apart*, *in one place*, *separately*, I. 13. 2, etc.

ἰδιότης, *peculiar quality*, I. 15. 1.

ἰδίωμα, *peculiar quality*, I. 10. 10. Boethius, *proprietas*.

ἰσάκις, *an equal number of times*; ἰσ. ἴσος, the same number taken the same number of times, i.e., of the type a^2, *a square*, I. 19. 19; ἰ. ἴσος ἰ., a number taken thrice as a factor, *a cube*, II. 17. 6. Cf. also ἀνισάκις.

ἰσάριθμος, *equal in number to* (with dat.) II. 3. 2.

ἰσογώνιος, *having an equal number of angles*, II. 14. 5.

ἰσοδιάστατος, *having the dimensions equal*, of cubes, II. 16. 1.

ἰσόπλευρος, *having the sides equal*, II. 8. 1; of numbers, II. 6. 1. Opp. σκαληνός.

ἴσος, (1) *equal*, passim, often equiv. to the sign =; τὸ ἴ., *equality*, I. 17. 3.
(2) *regular*, I. 19. 15, 18.
Boethius, *aequalis*, *aequus*.

ἰσότης, *equality*, a σχέσις, I. 17. 2. Boethius, *aequalitas*.

καθαρός, *pure*; hence, *of one kind only*, *exclusively*, I. 22. 3; II. 27. 4.

καθεξῆς, *in succession*, *in order*, II. 10. 3.

καθολικός, *general*, I. 16. 7; of the monad as the universal measure, I. 13. 13.

κανών, *rod*; μουσικὸς κ., *the measuring rod* laid beside the monochord in experiments in harmony, II. 27. 1.

κατά, *according to*, *by*; *in*, *at*; καθ' ἑαυτό, *absolute*, I. 11. 1, 13. 1, 17. 1.

καταγραφή, *representation* of numbers *in a figure*, II. 9. 1.

καταλείπω, pass., *be left* in subtraction, I. 13. 13.

καταλήγω, *end in* (with εἰς, referring to the last digit of a number), I. 16. 3; II. 17. 7; cf. τελευτάω. Also used of the termination of a process, I. 13. 11, 12.

κατάληξις, *termination* of an operation, I. 13. 13.

καταντάω, *come to* the end of an operation, I. 8. 4.

καταρκτικός, *beginning*, *initiating*, II. 17. 5.

κατάρχω, *be the beginning of*, *give rise to*, II. 3. 3, 4. 1.

καταστρέφω, *return*, of cyclic numbers ending in the same digit, II. 17. 7.

κατατομή, *division*, esp. the division of the monochord (cf. κανών), II. 27. 1.

κάτω, *below*, one of the περιστάσεις, II. 6. 4, 13. 9.

κεῖμαι, *be placed*, *occupy a position* in a series (defined by such expressions as παρ' ἕνα, ὑπὲρ ἕνα ἑκατέρωθεν, ὑπὲρ δύο), I. 8. 1, 13. 3; II. 9. 4, etc.

κίνησις, *motion*, *spatial motion* (κατὰ τόπον) of six varieties, II. 6. 4.

κοινός, *common*, passim.

κοινωνέω, *be in community with*, *share*, I. 11. 1, 8. 6, 10. 1.

κόλουρος, *truncated*; of pyramidal numbers (terminating in the polygonal number of the series homogeneous to its base next to unity in that series), II. 14. 5. Boethius, *curtus*.

κορυφή, *apex of a pyramid*, II. 13. 2; more generally, *top*, II. 16. 2. Boethius, *vertex*, *cacumen*.

κορύφωσις, *termination in an apex*, II. 14. 5.

κόσκινον, *sieve* of Eratosthenes, his method of discovering prime numbers, I. 13. 2 ff. Boethius, *cribrum*.

κτάομαι, *have* factors, I. 12. 1, 15. 1.

κυβικός, *cubic* numbers, II. 6. 1; *cubic* sides, II. 24. 9.

κύβος, *cube*, geometrical, II. 29. 1; numerical, II. 15. 1, 17. 6 (defined). Boethius, *cybus*.

κυκλικός, *cyclic* number, one the square of which ends in the same digit as the original number, II. 17. 7. Boethius, *cyclicus*.

κύκλος, *circle*, geometrical, II. 17. 7; used of celestial orbits, I. 3. 3.

λαμβάνω, *take*, *employ* in an operation, I. 16. 4, etc.; *reach*, *come to*, of a series reaching a certain point, II. 3. 2.

λείπω, pass., *be left*, in subtraction, I. 13. 12 (Archimedes). So τὸ λειπόμενον, λειφθέν, *remainder*, II. 2. 1, etc. v. καταλείπω. Boethius, *quod relinquitur*.

λογίζομαι, *conceive*, *imagine*, II. 13. 4.

λόγος, (1) *account*, *narrative*, II. 7. 5; *definition*, *statement*, *proposition*, II. 6. 5, 6; so, *scheme*, *plan* of the world, I. 6. 1; cf. I. 23. 7; and μητρὸς λόγον ἐπέχουσαν, I. 4. 1.

(2) *ratio*, I. 8. 10, etc. (Archimedes, etc.); II. 21. 2, 3 (defined); μουσικοὶ λ., II. 26. 1, 29. 4; ἁρμονικὸς λ., *musical interval*, I. 5. 1.

Boethius, *proportio*.

λοιπός, *remaining*, *other*, I. 13. 11; *further*, *more*, II. 4. 1; *the remainder* in subtraction, I. 13. 12.

λύσις, *resolution*, *division*, I. 10. 2.

λύω, *resolve*, II. 12. 1, etc. Cf. ἀναλύω, διαλύω.

μαθήματα, *mathematics*, II. 6. 1.

μέγας, *great*, *large*; τὸ μέγα, *greatness*, I. 5. 1.

μέγεθος, *magnitude*, I. 1. 3, 2. 4, 5.

μεγεθύνω, *make greater*, I. 21. 1; of multiplication, II. 15. 2.

μέθοδος, *method*, *process*, I. 13. 2.

μειζονάκις, *a greater number of times*, II. 17. 6. v. ἰσάκις, ἀνισάκις.

μειουρίζω, pass., *be diminished*, *taper*, of pyramids, II. 13. 2, 14. 5.

μένω, with ὁ αὐτός, *remain the same*, I. 23. 15; ἐν τῷ αὐτῷ, i.e., *quantity*, II. 27. 1.

μερίζω, *divide into parts*, I. 7. 2, 10. 2; II. 18. 4.

μέρος, *part*, I. 8. 4, 7, 9. 2, 12. 1, 14. 3, 15. 1 ff. Boethius, *pars*, *particula*.

μεσεμβολέω, *interpose* a difference between two terms, I. 19. 12.

μεσιτεία, *interposition in the middle*, I. 7. 2.

μέσος, *middle*, passim; ἀνὰ μέσον, *between*, I. 13. 1, etc.; τὸ μ., *middle term* of a series, I. 8. 14. etc.

μεσότης, (1) *mean*, *middle term* of a series (general use), I. 8. 10; *middle*, *middle space*, II. 23. 6; *middle* of a monochord, II. 27. 1; *middle term* of a proportion, *ibid.*

(2) *a proportion* (= ἀναλογία, *q.v.*, in Nicomachus's usage); arithmetic, II. 29. 1; geometric, II. 24. 1, 3, 4; harmonic, II. 25. 1.

Boethius, *medietas*.

μεταβαίνω, *go over to*, *change to*, I. 23. 15.

μεταβιβάζω, *make to change, transpose*, II. 27. 1.

μεταγενέστερος, *later, posterior in origin*, I. 5. 2, 19. 14.

μεταδίδωμι, *share* (τῶν αὐτῶν διαφορῶν), II. 20. 3.

μεταίχμιος, *middle ground*, I. 11. 1; 16, 1; II. 25. 3, etc.

μεταλαμβάνω, *take a share in* the measuring function, I. 13. 7; cf. II. 27. 6.

μεταξύ, *between*, II. 6. 3, etc.

μεταπίπτω, *change*, II. 7. 4.

μετάστασις, *shifting, changing*, II. 27. 1.

μετοχή, *participation*, I. 16. 8.

μετρέω, *measure*, I. 13. 1 (the number of times the measure measures is expressed by κατά with acc., cf. *ibid.*, 3); so τὸ μετρεῖν, *the measuring function*, I. 13. 6. Absolutely, *act as measure*, e.g., I. 13. 9, a number is produced κατὰ τὴν ἑαυτοῦ ποσότητα . . . μετρήσαντός τινος, 'by something acting as measure in accordance with its own amount' (i.e., multiplying itself by itself). Pass., *be measured by*, hence, *have the measure as a factor*, I. 11. 3, 13. 1 (Archimedes); μετρηθῆναι κοινῷ μέτρῳ πρός τινα, be commensurable, I. 13. 1. Boethius, *metiri, numerare*.

μέτριος, *moderate*, I. 16. 1.

μέτρον, *measure*, I. 12. 2, 23. 4; *the function of measure*, I. 13. 5; κοινὸν μ., *common divisor*, I. 13. 10. Boethius, *communis mensura*.

μῆκος, *length*, I. 17. 3; dimension of a plane (with πλάτος), II. 15. 2; of a solid, II. 6. 4. In a diagram, *the long way, horizontally*, I. 10. 10. Boethius, *longitudo*.

μηκύνω, *make long, lengthen*, I. 19. 20; *multiply*, I. 9. 4, 19. 19; II. 29. 3. Boethius, *multiplicare*.

μίγδην, *promiscuous, mingling* (adv. as adj.), II. 24. 9.

μῖγμα, *compound*, I. 10. 10.

μίγνυμι, *mingle, use together*, II. 5. 4.

μικτός, *mixed*, sharing qualities of more than one variety, I. 13. 2; *compound*, as multiple superparticulars, II. 24. 4.

μικρός, τὸ μ., *the lesser* or *least term* of a proportion, II. 27. 4; also μικρότερος = *lesser term*, I. 23. 16.

μονάς, *one, monad, unity*, passim. So ἁπλὴ μονάς, *one*, I. 19. 17; μ. δευτεροδουμένη, *ten; the first* (or *unit*) *of the ten's*, regarded as a course, *ibid.*; μ. τριοδουμένη, *one hundred, ibid.* See note *ad loc.*

Plur., *units*, passim; also, *the numbers 1–9*, I. 16. 3.

μοναδιστί, adv., *in units*, II. 8. 3.

μόριον, *factor, aliquot part*, I. 11. 2, 13. 1; *part* of the units in a number, II. 8. 1.

μυριάς, plur., *the ten thousands* (*10,000–99,999*), I. 16. 3.

ὀγδοάς, *the ogdoad, octad, the number eight*, I. 16. 3.

ὄγκος, *bulk*, I. 17. 3; τῶν σφηκῶν, *body*, II. 16. 2.

οἰκεῖος, *own, proper, belonging properly to*, I. 22. 3; II. 22. 1, etc.

οἰκειόω, *make friendly, reconcile*, in the sense of bringing into mathematical agreement and equality, II. 22. 2.

ὀκτάγωνος, *octagonal* number, II. 11. 3, 12. 3.

ὀκτάεδρον, *octahedron*, I. 4. 4.

ὀκταπλάσιος, *eightfold*, I. 4. 4, 18. 6.

ὅλος, *whole*, passim; τὸ ὅλον, *the whole* of a number dealt with, I. 8. 10; plur., *all things*, the universe, II. 17. 1.

ὁλότης, *a whole*, I. 2. 5.

ὁμογενής, *of the same genus with* (with dat.), II. 20. 2.

ὁμογώνιος, *with the same number of angles*, II. 12. 7.

ὅμοιος, *like*, I. 6. 3, 10. 1, etc.; hence, *consistent, not becoming unlike*, I. 2. 1; II. 17. 3; οἱ ὅμοιοι = *and so on*, I. 9. 1.

ὁμοιοσχημόνως, *after a similar scheme* or *figure*, II. 14. 1.

ὁμοιότης, *likeness, similarity*, I. 23. 4; II. 16. 2, 28. 5.

ὁμοιοτρόπως, *in similar fashion*, II. 6. 6, 10. 2.

ὁμοιόω, *make like*; pass., *be like*, I. 10. 3.

ὁμοταγής, *occupying a corresponding place* in a series, I. 19. 3; ὁμοταγῶς, I. 19. 13.

ὁμωνυμέω, *have the same name*, of factors, I. 13. 1.

ὁμωνυμία, *possession of the same name*, esp. of sensible things, named after ideal things, I. 2. 4, 23. 4.

ὁμώνυμος, *having the same name* or *denomination, homonymous*, of factors, (e.g., 3, one third of 9, and 4 as the third of 12), I. 13. 1; of the arithmetic proportion, compared with the science of arithmetic, II. 22. 3. Boethius, *aequivoca pars*.

ὄνομα, *name*, I. 1. 1, etc.; *designation* of factors (may be odd or even), I. 8. 7, 9. 2; of polygonal numbers (contains numerical element, II. 11. 4.

ὀνομάζω, *name, give a designation to*, I. 8. 7, etc.; the special name expressed by an adv., ἀρτιακῶς ὀ., I. 9. 2, or adv. phrase, μετὰ τῆς ὑπὸ προθέσεως ὀ., I. 21. 3.

ὀνομασία, *nomenclature, system of names*, I. 21. 2.

ὀνοματοποιέω, pass., *be named*, I. 9. 2.

ὀξύς, *sharp, pointed*, II. 13. 2, 16. 2; *high-pitched* tones, I. 16. 1.

ὀπίσω, *backward*, one of the types of motion, II. 6. 4. Boethius, *retro*.

ὁπόσος, *whatever* in number or order, II. 3. 1.

ὀρθότης, *directness* of order, I. 23. 13.

ὀρθῶς, *in direct order*, not reversed, I. 23. 11; II. 6. 6.

ὁρίζω, *define, state* a definition, I. 1. 1, 2; cf. II. 6. 4; *limit, give a definite form*, I. 23. 4; *restrict*, in sense of setting forth as terms to be dealt with in a problem, I. 13. 11.

ὅρος, (1) *bound, limit*, II. 6. 3; hence, *realm, bounds*, I. 16. 3.

(2) *definition*, I. 19. 4, 7. 3, 5. Freq., *term*, I. 8. 10, 23. 7; II. 21. 3, 23. 6, etc. Boethius, *definitio, terminus*.

ὁσάκις, *as many times as*, I. 13. 12.

ὁσακισοῦν, *however many times*, I. 23. 1.

οὐδείς, neut., *nothing*, i.e., *zero*, II. 6. 3.

πάθος, *property*, II. 17. 7.

πανταχῇ, *in every direction, in each dimension*, II. 16. 2.

πανταχόθι, *in every dimension*, II. 15. 3.

πάντοθι, *everywhere, on every side* of a solid, II. 14. 2.

παρά, freq. *except for, lacking*, as παρὰ μονάδα, *lacking one*, I. 8. 12, etc.; *compared with, in relation to* (to express a ratio, = πρός), II. 25. 1.

παραβάλλω, *divide* (orig., *to compare with*), II. 27. 7. Boethius, *committere iuxta*.

παραβολή, *comparison*, I. 17. 3; *division* (cf. παραβάλλω), II. 27. 7.

παράδειγμα, *example, evidence, proof*, II. 24. 5; *display, example (diagram)*, II. 28. 11; the *cosmic plan*, I. 6. 1 (4. 2).

παραδείκνυμι, with πρός, *show* a ratio to, II. 5. 5.

παράθεσις, *a placing* or *setting forth side by side*; ἐν παραθέσει, of discrete objects, vs. continuous, I. 2. 4 (cf. σωρεία).

παράκειμαι, *occupy an adjoining position* in series, I. 8. 2, 19. 19.

παρακολουθέω, *carry out* a process, I. 18. 5; *follow, be corollary*, I. 8. 6.

παρακολούθημα, *a consequence* or *corollary*, I. 19. 8; II. 24. 6; *concomitant property*, II. 27. 3.

παραλαμβάνω, *employ*, I. 16. 4; *take over from another* a function, I. 13. 5.

παραλείπω, *pass by* in selecting terms, II. 23. 3; applied, in act., to the terms themselves as a way of designating those used in a series; τοὺς δύο μέσους παραλείποντας, I. 13. 3 = *every third term*. Boethius, *praeterire*.

παράλειψις, *an omission*, II. 23. 3.

παραλλάττομαι, *vary, differ*, II. 17. 3.

παραλληλεπίπεδον, *parallelepipedon*, II. 15. 1, 16. 3. Boethius, *parallelepipedus*.

παράλληλος, *parallel*, of lines, I. 10. 10; of terms, etc., II. 4. 3; *side by side, consecutive*, of terms in series, II. 6. 2, 23. 3.

παραστίζω, *stand by the side of* in a table, I. 19. 19.

παραυξέω, pass., *increase along with*, II. 8. 2.

παραύξησις, *continued increase*, II. 11. 4.

παρεκτίθημι, *set forth in a parallel series*, I. 22. 4.

παρεμπίπτω, *intervene*, I. 7. 2. Boethius, *intercedere*.

παρεμφαίνω, *show forth* in the course of an operation, I. 19. 14.

παρέπομαι, *be concomitant*, of qualities, I. 16. 3, 19. 6; II. 6. 4; *carry out* a process, I. 22. 6.

παρονομάζω, *designate from* or *with reference to*, I. 22. 2.

παρωνυμία, *designation with reference to*, I. 16. 5.

παρώνυμος, *named from*; *having the same name as*, or *a derivative name*, I. 19. 7; esp., of factors, *that which takes its denominator from the quantity of the factored number*, as one third of 3, I. 11. 2, etc. Cf. ἐναντιώνυμος, ἑτερώνυμος.

πᾶς, *all, every*. Esp. in phrases διὰ παντός, μέχρι παντός, *throughout*; ἐπὶ πάντων, *in* *each case*. Cf. διὰ πασῶν, δὶς διὰ πασῶν, διὰ πασῶν ἅμα καὶ διὰ πέντε.

πάσχω, *have a quality*, I. 15. 1; II. 17. 7. Boethius, *accidere*.

πάχος, *thickness* (= βάθος), a dimension of solids (not of surfaces), II. 13. 1.

πενταγωνικός, *pentagonal*, II. 10. 1.

πενταγωνισμός, *formation into pentagonal shape*, II. 13. 2.

πεντάγωνος, *pentagonal number*, II. 7. 3, 10. 1 (defined), etc. Boethius, *pentagonus*.

πεντακόλουρος, *five times truncated*, of pyramids, II. 14. 5. Cf. κόλουρος, δικόλουρος.

πενταπλάσιος, *fivefold*, I. 18. 7. Boethius, *quintuplus*.

πενταπλασιότης, *the quintuple relation*, II. 5. 5.

πέντε, *five*; cf. διὰ πέντε.

περαίνω, intr., *come to an end, be limited*, II. 18. 3; trans., *define, limit*, I. 23. 4.

περαιόω, pass., *be ended*, of operations, I. 13. 12–13; *come to an end, terminate* (with εἰς; of a series), I. 19. 13.

πέρας, *bound, limit, limiting surface*, of the upper surface of a truncated pyramid, II. 14. 5.

περιγράφω, pass., *be bounded*, II. 13. 3.

περιέχω, (1) *embrace, contain*, II. 24. 5.

(2) pass., *be bounded, contained*, used of solids, bounded by surfaces, II. 13. 3, 14. 4.

Boethius, *continere*.

περικυκλόομαι, *move about in a circular motion*, II. 17. 7.

περισσάρτιος, *the odd-times even number*, I. 10. 1 (defined). Boethius, *impariter par*.

περισσεύω, *be more than, exceed* (with gen.), II. 21. 6.

περισσός, *odd*; I. 7. 1 ff. (defined). Boethius, *impar*.

περισσοταγής, *in an odd place*, in a series, I. 22. 6.

περίστασις, *relative position*, II. 6. 4, 16. 1. The six varieties πρόσω, ὀπίσω, ἄνω, κάτω, δεξιά, ἀριστερά, II. 6. 4; these are opposite by pairs (ἀντίθετος) and one pair is attached to each dimension (*ibid.*). Boethius, *motus*; varieties, *ante, retro, sinistra, dextera, sursum, deorsum*.

πηλίκος, *how great*; τὸ π., *quantity*, as opp. to number, I. 2. 5 (defined), II. 6. 1.

πηλικότης, *magnitude, size*, I. 7. 3. Boethius, I. 4, *spatium*.

πλάσις, *fashioning, creation*, II. 24. 5. v. πλάσσω.

πλάσσω, *fashion*, referring to the creation of numerical forms by rule, I. 23. 8; II. 24. 5.

πλάτος, *breadth, a dimension* of a solid, II. 6. 4; of a surface, with μῆκος, II. 15. 2; in a diagram, the breadth across the page is κατὰ π., II. 3. 4, 4. 1, but up and down (v. μῆκος) in I. 10. 10; *quotient*, II. 27. 7. Boethius, *latitudo*.

πλατύνω, *broaden, acquire* πλάτος, II. 7. 3. Boethius, *extendere*.

πλεονάκις (ἢ ἅπαξ), *more times than once*, I. 18. 1, 22. 1, 23. 1, etc.

πλευρά, *side* of a plane figure, II. 15. 4; of a number, *a factor*, I. 19. 19; *edge* of a solid, II. 15. 4. Boethius, *latus*.

πλῆθος, *multitude*, I. 2. 4, 5, 16. 2, 7. 1.

πληρούντως, *completely, without remainder*, of measuring, I. 18. 2.

πλινθίς, *brick; a kind of solid number*, the product of the square of a number by a smaller term (e.g., a^2b if $a > b$), II. 6. 1, 17. 6 (defined). Boethius, *laterculus*.

ποιέω, *multiply* by or into (ἐπί with acc.), II. 27. 7; *make, give as a result* of addition, II. 6. 3 (cf. 12. 2).

ποιητικός, *capable of making, productive of* (with gen.), II. 5. 5.

ποικιλία, *variety*, I. 19. 8, 22. 2.

ποικίλλομαι, *have varieties*, I. 22. 2.

ποικίλος, *complicated*, I. 10. 6; *various*, I. 23. 6.

ποιός, *of what sort*; τὸ π. = ποιότης, *q.v.*; applied to proportions, II. 21. 6, 23. 4.

ποιότης, *quality*, esp. as applied to the relations of number, i.e., ratios; so that a proportion κατὰ π. is one composed of true ratios vs. proportions κατὰ ποσότητα (arithmetic proportions, where the relation concerned is arithmetic difference), II. 21. 5, 24. 1, 25. 5. Boethius, *qualitas*.

πολλάκις, *a plural number of times* (opp. ἅπαξ), I. 22. 2.

πολλαπλασιάζω, *multiply*, I. 8. 14, etc. (Archimedes); *by* = dat., as I. 10. 8, 16. 4 (so Archimedes); multiplied by itself = act. with reflexive rather than pass., II. 6. 3. Boethius, *multiplicare, ducere (per)*.

πολλαπλασιεπιμερής, *the multiple superpartient*, a σχέσις or ratio, and a species of inequality, I. 17. 7, 23. 16.

Its varieties as mentioned by Nicomachus are:

διπλασιεπιδιμερής ($2\frac{2}{3} : 1$) *double superbipartient*. Boethius, *duplex superbipartiens*.

διπλασιεπιδίτριτος, another form of the preceding word.

διπλασιεπιπενταμερής ($2\frac{5}{6} : 1$), *double superquintipartient*.

διπλασιεπιτετραμερής ($2\frac{4}{5} : 1$), *double superquadripartient*. Boethius, *duplex superquadripartiens*.

διπλασιεπιτετράπεμπτος, another form of the preceding word.

διπλασιεπιτριμερής ($2\frac{3}{4} : 1$), *double supertripartient*. Boethius, *duplex supertripartiens*.

διπλασιεπιτριτέταρτος, another form of the preceding word.

Boethius, *multiplex superpartiens*.

πολλαπλασιεπιμόριος, *the multiple superparticular*, a σχέσις, and a form of inequality, I. 17. 7, 22; 23. 14.

The varieties mentioned by Nicomachus are as follows (v. I. 22. 2, 23. 14):

διπλασιεφημιόλιος ($2\frac{1}{2} : 1$), *double sesquialter*.

διπλασιεφήμισυς, another form of the preceding word.

διπλασιεπίτριτος ($2\frac{1}{3} : 1$), *double sesquitertian*.

διπλασιεπιτέταρτος ($2\frac{1}{4} : 1$), *double sesquiquartan*.

διπλασιεπίπεμπτος ($2\frac{1}{5} : 1$), *double sesquiquintan*.

τριπλασιεφήμισυς ($3\frac{1}{2} : 1$), *triple sesquialter*.

τριπλασιεπίτριτος ($3\frac{1}{3} : 1$), *triple sesquitertian*.

τριπλασιεπιτέταρτος ($3\frac{1}{4} : 1$), *triple sesquiquartan*.

τριπλασιεπίπεμπτος ($3\frac{1}{5} : 1$), *triple sesquiquintan*.

τετραπλασιεφήμισυς ($4\frac{1}{2} : 1$), *quadruple sesquialter*.

τετραπλασιεπίτριτος ($4\frac{1}{3} : 1$), *quadruple sesquitertian*.

τετραπλασιεπιτέταρτος ($4\frac{1}{4} : 1$), *quadruple sesquiquartan*.

τετραπλασιεπίπεμπτος ($4\frac{1}{5} : 1$), *quadruple sesquiquintan*.

πενταπλασιεφήμισυς ($5\frac{1}{2} : 1$), *quintuple sesquialter*.

πενταπλασιεπίτριτος ($5^{1} : 1$), *quintuple sesquitertian*.

πενταπλασιεπιτέταρτος ($5\frac{1}{4} : 1$), *quintuple sesquiquartan*.

πενταπλασιεπίπεμπτος ($5\frac{1}{5} : 1$), *quintuple sesquiquintan*.

Boethius, *multiplex superparticularis*.

πολλαπλάσιος, *the multiple*, a σχέσις, and a form of inequality, I. 17. 7, 18. 1 (defined); neut., freq. as name of the relation.

The varieties mentioned by Nicomachus are as follows (v. I. 18. 1, 19. 5, etc.):

διπλάσιος (2 : 1), *double*.

τριπλάσιος (3 : 1), *triple*.

τετραπλάσιος (4 : 1), *quadruple*.

πενταπλάσιος (5 : 1), *quintuple*.

ἑξαπλάσιος (6 : 1), *sextuple*.

ὀκταπλάσιος (8 : 1), *octuple*.

δεκαπλάσιος (10 : 1), *decuple*.

Boethius, *multiplex* (*multiplicitas*).

πολύγωνος, *polygonal number*, II. 12. 1, etc.

The following varieties are mentioned by Nicomachus:

τρίγωνος, *triangular*.

τετράγωνος, *square*.

πεντάγωνος, *pentagonal*.

ἑξάγωνος, *hexagonal*.

ἑπτάγωνος, *heptagonal*.

ὀκτάγωνος, *octagonal*.

πολυπλασιάζω, same as πολλαπλασιάζω, *q.v.*, II. 15. 4.

πολυπλασίασις, *a multiplication* (operation), II. 17. 7.

πολυπλασιασμός, *multiplication*, *product of multiplication*, I. 10. 10.

πολυπλάσιος, *multiple* (another form of the word πολλαπλάσιος), II. 4. 3, 24. 5, etc.

πολύς, *much*, *many*. Comp., τὸ πλέον, *the greater* (opp. τὸ ἔλαττον), a σχέσις and species of the σχέσις of inequality (= τὸ μεῖζον, the more usual term), I. 14. 2.

ποσάκις, *how many times*; τὸ ποσ. = *the number of times*, I. 13. 6.

ποσός, *of some amount*; τὸ ποσόν, *number* (equiv. to ποσότης), I. 17. 1, 2, etc.; τὸ π., as limited multitude, I. 2. 5. Its classification, absolute and relative, I. 3. 1, etc.;

with reference to proportions κατὰ τὸ ποσόν = κατὰ ποσότητα (v. ποσότης), II. 21. 6.

ποσότης, *number, numerical quantity* (vs. πηλικότης, *quantity*), I. 11. 3. The same as τὸ ποσόν, *q.v.*; with reference to proportions, ἀναλογία κατὰ π. is a 'numerical' proportion, i.e., one distinguished by equality of numerical differences, arithmetic proportion (opp. ἀναλογία, κατὰ ποιότητα; cf. ποιότης), II. 21. 5, 24. 1. Boethius, *quantitas*.

πρεσβύτερος, *older, more primitive in origin*, of mathematical forms, I. 19. 8.

προαποδείκνυμι, *demonstrate previously*, I. 16. 4.

προαποδίδωμι, *assign previously*, II. 22. 2.

προβαίνω, *go on, proceed* to infinity, I. 19. 16.

προβάλλω, pass., *be given*, of terms in a problem, I. 13. 12.

πρόβασις, *progression*, I. 19. 8, 18, 21. 2. etc.; in the sense of a progressing series (of the odd numbers), I. 20. 2; cf. II. 9. 2.

προβιβάζω, act., with personal subject, *carry forward, set forth a series*, I. 16. 4; pass., *be carried forward, progress, go on*, II. 7. 3, 11. 1.

προγενέστερος, *more original*, I. 18. 1; cf. πρεσβύτερος. Logically *prior*, I. 4. 2, 5. 1, 3, etc.

πρόγονος, *ancestor*; that from which a thing is derived; hence the square is π. of the cube, II. 15. 4.

προδείκνυμι, pass., *be designated, given* as terms of a problem, II. 27. 7.

προεκτίθημι, *extend in series previously*, II. 12. 6.

προηγέομαι, *come before, rank before*, II. 22. 3; pass. part., *already mentioned*, I. 16. 5.

πρόθεσις, *prefix* (the prefix *sub*, ὑπο-), I. 17. 8, etc.

πρόειμι, *go on, continue*, I. 18. 2, 4, 20. 1, etc.

πρόκειμαι, *be set before*, as a thing to be done, I. 20. 2; *be given*, as terms in a problem, II. 27. 2.

προκοπή, *increase, amount of increase* in a series, I. 19. 13; *progression with increase*, of a series, I. 13. 6.

προκόπτω, intrans., *increase, be increased*, II. 3. 2, etc.; δυσὶ διαστήμασι, *be squared*, II. 17. 7. Boethius, *praecedere*.

πρόλογος, *the antecedent*, the greater term in a ratio between unequal terms, I. 19. 2. Boethius, *dux*.

προμήκης, *oblong*, a number that is the product of unequal factors differing by more than unity, *promecic*, II. 17. 1, 18. 2; *product*, II. 27. 7. Boethius, *antelongior, promeces, anteriore parte longior*.

προπίπτω, intrans., *go forward, be extended*, II. 18. 2.

προσάπτω, *join, add*, II. 21. 4.

προσγίγνομαι, *happen to, be applied to*, of a process applied to a number, II. 20. 1; *be joined to*, II. 17. 6.

προσεπισωρεύω, *add into the summation of a series*, I. 16. 5.

προσζεύγνυμι, *combine with*, of combining geometrical figures, II. 12. 2.

προσηγορία, *name, designation*, I. 9. 3, 8. 6, 22. 7.

πρόσθεσις, *addition*, II. 7. 3, 20. 1, etc.

προσκτάομαι, *acquire in addition*, II. 7. 3.

προσλαμβάνω, *take, employ* as a term *into an operation*, I. 16. 4; *take with*, II. 5. 5; of addition, *to have something added*, II. 20. 1 (so Archimedes); *receive, acquire* another dimension, II. 13. 1, 15. 2.

προσπλέκω, *join with, add*, II. 12. 3.

προσσωρεύω, *add in the summation of a series*, II. 8. 3. v. ἐπισωρεύω.

προσσώρευσις, *an addition* in the summation of a series, II. 8. 3.

πρόσταγμα, *rule for procedure*, I. 16. 7, 23. 8.

προστίθημι, *add*, of the operation of addition (so Archimedes), II. 27. 7, 12. 2, 3; *combine* in multiplication, I. 12. 2.

πρόσω, *forward*, one of the six varieties of motion, II. 6. 4; opp. ὀπίσω.

προτείνω, *propose* a problem, I. 13. 11.

προτίθημι, *set before, give* terms in a problem, I. 13. 13; pass., *be given* (terms), I. 13. 11.

προϋπόκειμαι, *be presupposed, preëxist* as a basis, I. 4. 4, 11. 3; II. 18. 1.

προχωρέω, *go on, continue*, of a process, I. 8. 13; with personal subject, I. 10. 9, etc. Boethius, *sese supergredi*.

προχώρησις, *advance, progression* in series, I. 13. 6.

πρωτογενής, *first-born* (i.e., of an elementary, primary nature); of the triangle, II. 12. 5.

πρῶτος, *prime*, a kind of the odd, I. 11. 2 (defined). Boethius, *primus et incompositus*.

πυθμενικός, *fundamental, belonging to the simplest form* (v. πυθμήν), II. 2. 2. πυθμενικῶς, *fundamentally, in simplest form*, II. 17. 1.

πυθμήν, *stock, fundamental form*; the simplest form of a numerical relation (e.g. the double in simplest form is seen in 2 : 1), I. 19. 6, 7, 21. 1; II. 19. 3.

πυραμίς, *pyramid, pyramidal number*, II. 13. 2 ff. *Geometrical pyramid*, II. 13. 3. Boethius, *pyramidus*.

πυραμοειδής, *pyramidal* number, II. 15. 1.

ῥίζα, *root*; as source, that on which other things are based, I. 4. 1 (of arithmetic), etc.; *starting point*, I. 20. 2 (the fraction ½ is ῥ. of the superpartient), 2. 5, *root* of multitude. Boethius, *radix*.

σημεῖον, *point*, II. 6. 3, 13. 3 ff.; *a mark* used to check off terms, I. 13. 7; an arbitrary *sign* for a number, II. 6. 2. Boethius, *punctum*.

σημειόω, *note down*, I. 10. 8, 9; *represent by an arbitrary sign*, II. 6. 2.

σημείωσις, *designation, representation*, II. 6. 2.

σκαληνός, *scalene, having unequal sides*; a kind of solid number, II. 6. 1, 16. 2. Opp. ἰσόπλευρος. Cf. βωμίσκος, σφηκίσκος, σφηνίσκος. Boethius, *scalenos gradatus*.

σκοπέω, *observe*, I. 13. 4. v. ἐπισκοπέω.

σπέρμα, *seed*; hence, *origin*, with the implication of the potential existence of the completed thing, as in a seed, II. 15. 1, 18. 1.

σπερματικῶς, *like a seed*; originally, *fundamentally*, II. 17. 1.

στάσις, *rest*, I. 3. 2.

στερεός, *solid*; applied to numbers, *solid numbers*, the products of three factors, II. 6. 1, 7. 3. τὸ σ., *solid body*, II. 6. 4. στερεῶς, *like a solid*, i.e., in three dimensions, II. 17. 4. Boethius, *solidus* (*soliditas*).

στίχος, *row, line* in a table, I. 10. 7. *Series of terms*; φυσικὸς σ., *the natural series*, II. 8. 3.

στοιχεῖον, *element*, II. 1 (defined); of the universe, *ibid.*; used of equality, the element of relative number, II. 2. 2; also of the triangle. Boethius, *elementum*.

στοιχειώδης, *elementary*, I. 11. 3; applied to a triangle, II. 7. 4, 8. 1, 14. 4; the diapente, II. 26. 1; cf. also II. 29. 4.

στοιχηδόν, *in series, in a row*, II. 8. 3, etc.

συγγενής, *of the same genus with*, II. 20. 3. v. ὁμογενής.

σύγκειμαι, *be composed of* (ἐκ), I. 22. 2; II. 18. 4, etc.

συγκεφαλαιόω, *add together*, I. 8. 13; with εἰς τὸ αὐτό, *together*, 14. 4, 15. 1, 16. 2.

συγκεφαλαίωμα, *sum*, in addition, I. 16. 5.

συγκεφαλαίωσις, *sum*, in addition, I. 8. 12.

συγκρίνω, *compare*, I. 21. 1, etc. Boethius, *comparare, aptare*.

σύγκρισις, *comparison* of numbers, I. 15. 1, 22. 6, etc.

συζεύγνυμι, *join together* numbers to form ratios, I. 19. 3.

συζυγία, *a pairing, a pair*, II. 19. 4; used of the ratios, II. 28. 10, 19. 4.

σύζυγον, *a term paired with another* in some relation (e.g., corresponding factors, as 16 and 8, of 128, are σύζυγα), I. 8. 11.

συλλαμβάνω, *take in conjunction with*, II. 5. 2; *help*, I. 23. 8.

σύλληψις, *a combining, combination*, II. 21. 2.

συμβαίνω, *happen, come to light, result*, I. 10. 10; *be a fact*, I. 16. 3; impers., *it is an attribute of, characteristic of* (with *dat.*), I. 9. 2, 10. 10, etc.

συμμετρία, *agreement in measure with* (πρός), I. 14. 3.

σύμμετρος, *agreeing in measure with*, II. 3. 2.

συμμίγνυμι, *combine* terms to make a proportion, II. 21. 3.

σύμπας, *all together*, as an expression for 'sum,' I. 15. 2.

συμπέρασμα, *termination, bound*, of surface of truncated pyramid, II. 14. 5; last digit of a number, II. 17. 7.

συμπλέκω, *involve* in the composition or something, I. 21. 3, 22. 2.

συμπληρόω, *fill out to a certain quantity*, I. 15. 2; II. 22. 1.

συμπλήρωσις, *a filling out to equality with*, I. 15. 1.

συμπροκόπτω, *increase together with*, e.g. the differences of a series increase as the terms increase, I. 19. 12.

συμφωνέω, *give consistent results*, of operations, I. 10. 9.

συμφωνία, *concord*, in harmony, II. 26. 1. Boethius, *symphonia*.

σύμφωνος, *agreeing, in agreement*, I. 19. 16; II. 5. 4. τὸ σ. = συμφωνία, II. 26. 1.

συνάγω, *produce, give as a result*, in addition, I. 14. 4; *bring together, combine*, in addition, I. 16. 2.

συναθροίζω, with ὑφ' ἕν, *add*, I. 16. 2.

συναμφότερος, in plur., *both together* (= the sum of two terms), I. 8. 13; similarly, τὸ σ., I. 16. 4. So Archimedes.

συνάπτω, *join together*; συνημμένη ἀναλογία, *continued proportion*, II. 21. 5–6 (v. διαζεύγνυμι). Boethius, *continua*.

συνδέω, *bind together, connect*, II. 22. 2; by proportional mean terms, II. 29. 1.

συνεξετάζω, *examine together, compare*, I. 19. 11.

συνέχεια, *succession*; ἐν σ. μηκύνειν, *multiply in succession*, II. 29. 3.

συνεχής, *next, following*, of terms in series, I. 8. 13; τῷ σ. ἀριθμῷ, by *regularly progressing number*, II. 8. 2. συνεχῶς, *in succession, without omission*, II. 23. 1. Boethius, *continens*.

συνέχω, *join* by mean terms to make a proportion, II. 24. 6.

σύνθεσις, *addition*; ἴσοι κατὰ σ., *equal by addition* to (i.e., their sum equals) something else, I. 10. 10; *sum*, in addition, II. 29. 2, 8. 3, 27. 3; τὸ κατὰ σ., *sum*, II. 23. 5; *combination*, I. 22. 2; II. 5. 1, 21. 3, Boethius, *compositio*.

σύνθετος, *composite, composed of factors* (opp. πρῶτος, *prime*), I. 11. 1 ff.; *composed of* (ἐκ or ἀπό), I. 21. 3; II. 2. 1; τὸ σ., *sum*, II. 27. 7.

σύνθημα, *convention*, as opposed to natural ordinance, I. 19. 14; II. 6. 2.

συνίστημι, pass., *be combined by addition*, II. 11. 2; with ἐκ, *be composed of*, I. 11. 3; II. 5. 3, 4, etc.; mid., *arise, be formed*, I. 23. 14; II. 14. 2.

συννεύω, *converge*, of the edges of the pyramid, II. 13. 3. Boethius, *inclinari*.

συντελέω, *make, complete, produce*, II. 20. 1, 27. 1; τὸ συντελούμενον, *product*, I. 8. 14.

συντίθημι, *combine* by simple association, II. 5. 2, 3, 4, 5; *add*, I. 8. 1. 15. 1; II. 6. 3, 26. 2, 27. 7 (so Archimedes); *take as a factor*, in passive, with numeral adverb, μονάδος πεντάκις συντεθείσης ὁ ε, I. 11. 3. Boethius, *redigere in unum, addere, iungere*.

σύντρεις, *three together, the sum of three terms*, I. 8. 13.

συνυφαίνω, *weave together*; hence, *constitute* (*of* = ἐκ), I. 10. 6. Cf. ὕφος.

συνυφίστημι, *arise along with* (said of a thing of which the existence is implied in another), I. 20. 3, 21. 3.

συνωνυμέω, *have the same name, agree in name*, I. 17. 5.

συσσωρεύω, *add a series together, make a summation*, II. 10. 2.

σύστασις, *composition*, that of which a thing is made up, II. 1. 1; *composition, construction* in a technical manner, according to rule, I. 8. 13, 3. 5; II. 8. 3, 10. 2, 14. 2.

σύστημα, *a system, systematic arrangement*, I. 3. 5 (= Plato, *Epin.*, 991 D); *a combination* of things making up another thing, I. 7. 1, 8. 10; II. 5. 2, 26. 1; hence, *a combination by addition, sum*, II. 11. 3, 12. 5, 7; *a combination of factors multiplied together*, II. 24. 9.

σφαῖρα, *a sphere*, II. 17. 7. Boethius, *sphera*.

σφαιρικός, *spherical* numbers, II. 6. 1, 17. 7 (defined). σφαιρική, *astronomy*, I. 3. 2. Boethius, *sphericus*.

σφηκίσκος, *a little wasp*, applied to a kind of solid number (same as σκαληνός, *q.v.*, or σφηνίσκος), II. 16. 2.

σφήν, *a wedge*, II. 16. 2.

σφηνίσκος, *a little wedge*, a variety of solid number (= σκαληνός, *q.v.*, or σφηκίσκος), II. 16. 2, 17. 6 (defined). Boethius, *cuneus, cuneolus, spheniscos*.

σφήξ, *a wasp*, II. 16. 2.

σχέσις, (1) a non-mathematical term, *state, condition, habitude*, II. 22. 1 (p. 123. 1 H).
(2) mathematical term, *relation*. In the most general sense, any relation between two terms, including equality (II. 6. 3) and inequality (I. 14. 2); hence, *a relation of excess* or *deficiency*, measured by an arithmetical difference, but often applied to relations which are strictly λόγοι, *ratios*, I. 17. 4, 6, 19. 16, 21. 3; II. 6. 3, 19. 4, 21. 3, 4, etc. Especially

of the ten ratios of relative number, II. 22. 1 (p. 122. 20 H), 24. 5; I. 23. 4. Boethius, *habitudo*.

σχῆμα, *form*, *figure*, II. 7. 4; of a geometrical figure as opposed to arithmetical, II. 12. 1, 2; στερεὸν σ., *solid figure*, II. 13. 3, 16. 2.

σχηματίζω, *represent by a figure*, II. 8. 3.

σχηματισμός, *figuration*, II. 9. 1.

σχηματογραφέω, *represent by a figure*; act., intrans., *admit of graphic representation*, II. 8. 1; pass., II. 10. 2. Boethius, *describere*.

σχηματογραφία, *representation by a figure*, of numbers, II. 6. 2, 10. 2; ἐπίπεδος σ., *plane figuration*, II. 13. 1.

σχίζω, *divide* (non-mathematical), I. 17. 6. Boethius, *secare*.

σώζω, *preserve*, *hold true* a ratio, II. 5. 2; of a principle, II. 22. 3, 24. 5.

σῶμα, *body*, as opp. to surface and line, II. 7. 2.

σωρεία, *summation of a series*, I. 16. 4, etc. *A heaping up* of one thing with another; τὰ κατὰ σ., *discrete things*, I. 2. 4 (cf. παράθεσις).

σωρεύω, *add to a summation of a series*, II. 8. 3.

σωρηδόν, *by summation* or *cumulative addition*, II. 11. 1.

τάξις, *order*, *proper order*, passim; τάξει, *in due order*, I. 10. 9; *place*, *proper place in order*, I. 13. 5.

τάττω, pass., *place*, *give a position in series*, I. 13. 4; *insert* a mean to make a proportion, II. 27. 2; *arrange*, I. 10. 10; II. 17. 3; *regulate*, I. 13. 6.

ταυτομήκης, *of the same length*, *having both dimensions the same* (epithet of squares, similar to ἰσομήκης, *q.v.*), II. 18. 3.

ταὐτόν, *the same* (principle of sameness; = ταυτότης; opp. ἑτερότης, τὸ ἕτερον), II. 17. 1, 18. 3. Boethius, *eadem natura*.

ταυτότης, *sameness* (= ταὐτόν), II. 17. 1; *identity*, II. 27. 4.

τέλειος, *perfect*, of numbers equal in amount to the sum of their factors, I. 14. 1, 16. 1; of perfection of the monad, I. 16. 10; the decad, II. 22. 1; the di-diapason, I. 5. 1. Boethius, *perfectus*.

τελείωσις, *completion*, II. 13. 9.

τελευτάω, *end*, *terminate*, I. 8. 11; of pyramids terminating in a vertex, II. 14. 5; of numbers ending in same digit, II. 17. 7. Cf. καταλήγω.

τελευτή, *termination*, of the digit ending a number, II. 17. 7.

τέλος, *end* of a table, I. 19. 17.

τέμνω, *divide*, esp. διχῇ τ., *bisect*, II. 27. 7.

τέσσαρες, *four*. Cf. διὰ τεσσάρων.

τετραγωνικός, *square*; τ. πλευρά, *square root*, II. 27. 7.

τετραγώνιος, *four-angled*, *quadrangular*, II. 17. 1.

τετραγωνισμός, *formation like a square*, II. 9. 1.

τετράγωνος, *square number*, I. 19. 19; II. 9 (defined); τ. σχῆμα (geometrical), *square figure*, II. 12. 1. Boethius, *quadratus*.

τετρακόλουρος, *four-times truncated* (v. κόλουρος, δικόλουρος), II. 14. 5.

τετραπλασιότης, *the fourfold relation*, II. 5. 5.

τετράπλευρος, *four-sided*, *quadrilateral*, II. 17. 1.

τετράς, *the number four*, I. 9. 4, 10. 7.

τίθημι, *assume*, I. 19. 8, 8. 10; II. 17. 2; *make*, as result of an operation, I. 20. 2; *set down*, as a term, II. 2. 1, 24. 5.

τιθήνη, *nurse*, epithet of arithmetic, I. 5. 3.

τμῆμα, *division*, the operation, I. 10. 4.

τμητός, *capable of being divided*, I. 10. 4.

τομή, *a dividing*, I. 7. 4, 10. 3; *division*, *classification*, I. 7. 1. Boethius, *divisio*.

τονιαῖος, *tone*, a musical interval with the ratio 9 : 8, II. 29. 4. Boethius, *tonus*.

τριακοστόδυος, *thirty-second*, I. 8. 10.

τριάς, *the triad*, *the number three*, I. 10. 6, 13. 3, etc.

τριγωνίζω, *arrange in triangular form*, II. 8. 1, 3.

τριγωνικός, *triangular* (pyramid), II. 14. 1.

τριγωνισμός, *formation like a triangle*, II. 8. 1.

τριγωνιστί, *in triangular form*, II. 8. 3.

τρίγωνος, *triangular number*, II. 8. 1. Boethius, *triangularis*, *triangulus*.

τρικόλουρος, *triply truncated* (v. κόλουρος, δικόλουρος), II. 14. 5. Boethius, *ter curtus*.

τριοδέω, *be of the third course* (cf. δευτεροδέω), I. 19. 17.

τριπλασιάζω, *multiply by three*, I. 21. 1.

τριπλασιότης, *the threefold relation*, II. 5. 5.

τριχῆ, *in three dimensions* or *directions* (with **διαστατός**), II. 6. 4.

ὑπακούω, *correspond to*, *have a relation to*, of terms in proportions, II. 21. 6, 23. 2; in pass., *be implied*, I. 21. 2.

ὑπεναντίος, *subcontrary* proportion, II. 22. 1, etc. Boethius, *oppositus*, *contrarius*.

ὑπεναντιότης, *the state* or *relation of being subcontrary*, II. 28. 5.

ὑπεναντίωσις, *subcontrary relation*, II. 28. 2.

ὑπεξαιρέω, *remove* terms from a series, II. 22. 3.

ὑπεπιμερής, *subsuperpartient*, reciprocal ratio of the superpartient, I. 17. 8. Boethius, *subsuperpartiens*.

ὑπεπιμόριος, *subsuperparticular*, reciprocal ratio of the superparticular, I. 17. 8, 19. 20. Boethius, *subsuperparticularis*.

ὑπεπίτριτος, *subsesquitertian* (3 : 4), a subsuperparticular, reciprocal of the sesquitertian, I. 19. 2. Boethius, *subsesquitertius*.

ὑπερβαίνω, *pass over*, used in locating terms in a series, e.g., I. 13. 3, **τὸν δύο μέσους** ὑπερβαίνοντα, "the term occurring after the omission of two numbers between"; *go beyond*, *exceed*, I. 14. 3; II. 18. 2; hence, *be greater*, II. 27. 6.

ὑπερβάλλω, *exceed*, I. 16. 1, 2; ὑπερβαλλόντως, *with excess*, II. 17. 6.

ὑπερεκπίπτω, *pass by a point*, *be continued beyond*, I. 19. 14.

ὑπερέχω, *exceed*, I. 17. 3. So Archimedes.

ὑπέρκειμαι, *be located beyond* in the next place in a series, II. 12. 5.

ὑπεροχή, *excess*, II. 27. 4, 7. So Archimedes.

ὑπερτελής, *superabundant*, a number the sum of the factors of which is greater than its own amount, I. 14. 1, 3. Boethius, *superfluous*.

ὑπερφέρω, *exceed*, *be greater than*, I. 9. 4; II. 27. 3.

ὑπό, *by*, of multiplication by, I. 8. 14, etc.

τὸ ὑπό, *the product*, I. 8. 14, 19. 17; II. 27. 3, 4.

as a prefix, *sub*, I. 19. 20.

ὑποβαίνω, *come underneath* in the order of rows in a table, I. 19. 14; *be less than*, II. 27. 6; ὑποβεβηκότως, *in subordinate manner*, as of a species to a genus, II. 20. 2.

ὑπόδειγμα, *illustration*, *example*, I. 8. 13, etc.

ὑποδείκνυμι, *show*, *exhibit*, I. 22. 6.

ὑπόδειξις, *illustration*, II. 3. 4.

ὑποδιαίρεσις, *subdivision*, I. 11. 1, 8. 3.

ὑποδιπλάσιος, *subdouble*, i.e., *half*, a σχέσις (species of ὑποπολλαπλάσιος), I. 10. 10, 18. 3 (defined) = ἥμισυς; but the latter usually refers to the term *qua* fraction or part, ὑποδιπλάσιος to it in its relation to the greater number. Boethius, *subduplus*.

ὑπόλογος, *the consequent*, *lesser term* of a ratio, I. 19. 2. v. πρόλογος. Boethius, *comes*.

ὑπομερισμός, *a division into parts*, I. 8. 4.

ὑποπίπτω, *fall under*, *be subject to* (a measure, μέτρῳ), I. 13. 7; (a process) 8. 8.

ὑποπολλαπλασιεπιμερής, *submultiple superpartient*, a σχέσις, reciprocal of the πολλαπλασιεπιμερής (*q.v.*), I. 17. 8. Boethius, *submultiplex superpartiens*.

ὑποπολλαπλασιεπιμόριος, *submultiple superparticular*, a σχέσις, reciprocal of the πολλαπλασιεπιμόριος (*q.v.*), I. 17. 8. Boethius, *submultiplex superparticularis*.

ὑποπολλαπλάσιος, *the submultiple*, a σχέσις, reciprocal of the πολλαπλάσιος (*q.v.*), I. 17. 8, 18. 2 (usually applied to numbers that measure a larger number with especial reference to their ratio to the latter; μόριον, μέρος are applied to them *qua* factors, parts). Boethius, *submultiplex*.

ὑποτάττω, *bring under* a ratio, II. 21. 2.

ὑποτείνω, *subtend* an angle, II. 4. 3.

ὑποτετραπλάσιος, *the subquadruple* (ratio 1 : 4), *reciprocal of the quadruple*, I. 18. 3. Boethius, *subquadruplus*.

ὑποτίθημι, *suppose*, I. 20. 2; *place below*, II. 8. 3.

ὑποτριπλάσιος, *the subtriple* (ratio 1 : 3), *reciprocal of the triple*, I. 18. 3; II. 27. 6. Boethius, *subtriplus*.

ὑφημιόλιος, *the subsesquialter* (ratio $1 : 1\frac{1}{2}$ or 2 : 3), *reciprocal of the sesquialter*, I. 19. 2. Boethius, *subsesqualter*.

ὑφίστημι, mid., *arise from*, I. 10. 6.

ὕφος, *a web*, hence, *structure*, *disposition* of a table, I. 19. 13; φυσικὸν ὕ. τοῦ ἀριθμοῦ, *natural arrangement of number*, the natural series, I. 9. 5.

ὕψος, *height*, a dimension of solids only (same as βάθος), II. 13. 1.

φιλαλληλία, *mutual benevolence*, applied to numbers, II. 19. 1.

φιλάλληλος, *mutually benevolent*, II. 20. 3.

φυσικός *natural*, esp. natural series of number (ἀριθμός, στίχος, ἔκθεσις, χύμα), I. 19. 10; II. 8. 3, etc.

φύω, *produce*, *generate*, II. 3. 2; pass. *be produced*, II. 5. 4.

χαρακτήρ, *character*, *symbol* in notation, II. 6. 2.

χιασμός, *cross-lines* which make the form of the letter chi, I. 19. 11, 14. In the old form of the letter the lines cross in the manner of our sign for plus.

χιλιάς, *a thousand*; plur., *the thousands* (1000–9999), I. 16. 3.

χράομαι, *use*, *employ*, I. 12. 2; μέρεσι χρᾶσθαι, *have factors*, I. 12. 1.

χύμα, (1) *a pouring*, *flow*; of number, I. 7. 1 (as a definition of 'number').

(2) *a series*; *the natural series* (φυσικὸν χ.), I. 18. 4, 19. 6, etc. (ἀριθμητικὸν χ.), II. 10. 2.

χώρα, *place* in a series, II. 20. 5; transferred to the number indicating the place a term occupies in a series, I. 13. 6.

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