ON THE GROUPS OF ORIENTABLE TWO-MANIFOLDS

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1. *Introduction.*—The group of the orientable two-manifold of genus $p > 1$ has been investigated at length by Nielson,[2] but his work rests on an *a priori* construction of the universal covering surface allied with considerations of non-Euclidian geometry. In the present paper, we initiate and present briefly a direct intrinsic investigation of the same question. This method has the evident advantage of all intrinsic methods: it brings out clearly the basic elements of the problem.

2. *Products of Generators of the Group.*—The group of an orientable two-manifold of genus $p > 0$ is generated by $4p$ distinct operations: a_i, b_i, $i = 1, 2, \ldots, p$, and their inverses; with the single *fundamental relation*

$$R \equiv \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} = 1.$$

The cyclic orders represented by R and R^{-1} will be designated by 0_1 and 0_2, respectively. The cyclic order represented by $\prod_{i=1}^{p} a_i b_i^{-1} a_i^{-1} b_i$ will be designated by 0_3.

By a product we mean always a product of generators. The identity standing by itself is not a product. A product of any $2p$ consecutive generators of 0_1 or 0_2 will be called *a direct or an inverse semicycle*, respectively. A direct or an inverse semicycle whose first generator is g will be designated by (G) or $(G)'$, respectively.

A product will be said to be *reduced* if it has the following three properties:

A. No generator immediately follows its inverse.

B. There are no more than $2p$ consecutive generators in the order 0_1 or 0_2.

C. If there is a direct [inverse] semicycle $g_1 g_2 \ldots g_{2p}$, the generator immediately following this semicycle in the product is one of the $2p-1$ generators preceding [succeeding] g_{2p}^{-1} in the order 0_3.

Two products are said to be *equal* if one can be derived from the other by means of the relations:

$$g\, g^{-1} = 1, \quad f R f^{-1} = 1,$$

where g is a generator and f a finite product. A product is said to be *reducible*, if there exists an equal reduced product; *non-reducible* otherwise. We have necessary and sufficient conditions for reducibility of finite and infinite products, and existence theorems for reducible and non-reducible finite and infinite products. We prove also that a reduced finite product has either one equal but not identical reduced finite product or else none; and show what type of reduced finite product has an equal one and to what type this equal product belongs. From the considerations of reduced finite products we can easily obtain similar conclusions for reduced infinite products.

For an even p, the reduced infinite products with the periods (G) or $(G)'$ will be called *direct or inverse products.* For an odd p, the reduced infinite products with the periods

$$(A_i),\ (B_i)(B_{i-1})\ \dots\ (B_{i-p+1}),\ (A_i^{-1}),\ (B_i^{-1})(B_{i+1}^{-1})\dots(B_{i+p-1}^{-1}),$$

or

$$(A_i)'(A_{i+1})'\dots(A_{i+p-1})',\ (B_i)',\ (A_i^{-1})'(A_{i-1}^{-1})'\dots(A_{i-p+1}^{-1})',\ (B_i^{-1})',$$

will be called *direct or inverse products.*[3] Each direct product has one and only one equal but not identical reduced infinite product and this latter is inverse. Any other reduced infinite product has none. We shall say that two infinite products are *equivalent* when they are identical beyond a certain point. Then obviously there are $4p$ non-equivalent direct [inverse] products for an even p and only $2p + 2$ non-equivalent ones for an odd p.

3. *Representation of Reduced Infinite Products by Points of a Circle.*—Divide a circle into $4p$ non-overlapping closed intervals, and then subdivide each interval into a definite number of non-overlapping closed intervals. By dividing repeatedly the intervals and associating them with reduced finite products according to a rule based on the three properties of reduced product, we can associate a sequence of mutually inclusive closed intervals on the circle with each reduced infinite product. The limit point of the sequence will be taken to represent the product. All reduced infinite products are uniquely represented by points of the circle. We show that a point of the circle, not a point of division, represents a reduced infinite product which is neither direct nor inverse, and that a point of division on the circle represents two equal but not identical reduced infinite products, one of which is direct and the other inverse. Hence we have a *one-to-one correspondence between the points of the circle and the non-identical reduced infinite products.*

The points of division are everywhere dense on the circle. Hence the points which represent reduced infinite products equivalent to $4p[2p + 2]$ direct or inverse products for an even [odd] p are everywhere dense on the circle.

4. *Infinite Net. A Closed Two-Cell.*—With the group we can associate an infinite net N of $4p$-sided polygons such that any vertex of N is incident with $4p$ sides and $4p$ polygons. All products, finite and infinite, are

represented uniquely by paths, which are made up of the sides of the net and whose initial points coincide with the same vertex of N, arbitrarily chosen. The terminal points of the paths representing two equal finite products coincide with the same vertex of N.

We can associate with the net N an infinite two-manifold which can be mapped topologically on the interior E_2 of a circle S. The points of S may be taken to represent the reduced infinite products and thus the infinite paths of the transformed net representing these products, and will be called the *ideal elements of the infinite two-manifold.* By a proper definition of continuity on $E_2 + S$, we can prove *that the infinite two-manifold and its ideal elements defined by means of the group is a closed two-cell.*

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[2] Nielson, J., *Acta Mathematica*, **50,** 189–379 (1927). This is the third of his four papers on this subject.

[3] We agree to set $(G_m) \equiv (G_n)$ and $(G_m)' \equiv (G_n)'$, if $m = cp + n$, c being an integer and $1 \leqq n \leqq p$.

INVERSE COMMUTATOR SUBGROUPS

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If s and t represent any two operators of a given group G then the operator $s^{-1}t^{-1}st$ is commonly called the commutator of s and t. In the present article the operator $s^{-1}t^{-1}st^{-1}$ is defined as the *inverse commutator* of s and t and some fundamental properties of these commutators are developed. Since the transform of an inverse commutator by any operator of the group is an inverse commutator of this group it results directly that all the inverse commutators of a group generate an invariant subgroup of this group which will be called the *inverse commutator subgroup.* The corresponding quotient group cannot involve any operator whose order exceeds 2 and hence it must be the abelian group of order 2^m, and of type (1, 1, 1, ...). Since all the inverse commutators of such a group are obviously equal to the identity it results directly that *the inverse commutators of a group generate its smallest invariant subgroup which gives rise to an abelian quotient group of order 2^m and of type* (1, 1, 1, ...), *and every invariant subgroup which gives rise to such an abelian quotient group must involve the inverse commutator subgroup.*