conjugate to the Hamiltonian in this problem is $P = i/2 \log \left(\frac{p + iq}{\sqrt{2}}\right)$. This is a relatively complicated operator and substitution into the eq. (8) yields a result, the interpretation of which has not so far been made clear. Work on this point is in progress.

[1] E. Schrödinger, *Ann. der Phys.*, **79**, 734 (1926).

[2] C. Eckart, *Phys. Rev.*, **28**, 711 (1926).

[3] M. Born and N. Wiener, *Zs. f. Phys.*, **36**, 174 (1926).

[4] *Math. Ann.*, **98**, 1 (1927).

[5] See, for example, R. B. Lindsay, *J. Math. and Phys.*, Mass. Inst. of Tech., **3**, 191 (1924). See also *J. Opt. Soc. Amer.*, **11**, 17 (1925).

[6] P. A. M. Dirac, *Proc. Roy. Soc.*, **113**, 625 (1927).

[7] The proof of (23) for matrices was given by Born and Jordan, *Zs. f. Phys.*, **34**, 858 (1925). The proof for operators follows similar lines, i. e., by induction from simple operations performed on the fundamental relation (6).

[8] See the original paper (referred to in footnote 4). Also E. H. Kennard, *Zs. f. Physik*, **44**, 326 (1927).

[9] See R. H. Fowler and L. Nordheim, *Proc. Roy. Soc.*, **119**, 176 (1928).

PHYSICAL PROBLEMS WITH DISCONTINUOUS INITIAL CONDITIONS

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1. These problems are usually treated by the methods developed by the great French mathematician Jean Baptist Joseph Fourier who died 100 years ago.

In the case when the differential equation is

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}, \tag{1}$$

and the initial condition

$$v = f(x) \quad \text{when} \quad \tau = 0. \tag{2}$$

Fourier's solution is (*Oeuvres*, t. 1, p. 401 (1888))—*Théorie Amalyrique de la chaleur*, **1822**—

$$v = \frac{1}{\pi} \int_0^\infty e^{-\lambda^2 \tau} d\lambda \int_{-\infty}^\infty \cos \lambda(x - \xi) f(\xi) d\xi. \tag{3}$$

This expression satisfies the initial condition when it is legitimate to write

$$\operatorname*{Lim}_{\tau\to 0} v = \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^\infty \cos\lambda(x-\xi)f(\xi)d\xi = f(x). \quad (4)$$

For a discussion of these steps reference may be made to Carslaw's *Conduction of Heat,* Chapter 3, 1921.

The theory of the Fourier integral shows that at a point where $f(x)$ has a finite discontinuity, the value which the integral (4) may be expected to take is $\bar{f}(x)$, where

$$\bar{f}(x) = \operatorname*{Lim}_{\epsilon\to 0}\frac{1}{2}[f(x+\epsilon)+f(x-\epsilon)] = \frac{1}{2}[f(x+0)+f(x-0)].$$

This may be different from $f(x)$ and so the solution (3) does not cover all possible cases. The same remark is applicable also to Laplace's solution

$$\begin{aligned} v &= (\tau\pi)^{-1/2}\int_{-\infty}^\infty e^{-\frac{(x-\xi)^2}{4\tau}} f(\xi)d\xi \\ &= (\pi)^{-1/2}\int_{-\infty}^\infty e^{-s^2} f(x+2s\sqrt{\tau})ds, \end{aligned} \quad (6)$$

which is obtained by changing the order of integration in (3). As $\tau\to 0$, $v\to\bar{f}(x)$. It should be mentioned that Fourier (*Oeuvres,* t. 1, p. 421) uses Laplace's formula to find v in the case when $f(x) = 1$ for $+a > x > -a$ and $f(x) = 0$ for $x^2 > a^2$. His expression for v is

$$v = \frac{1}{2}[F(-x-a,\tau) - F(-x+a,\tau)] \quad (7)$$

where

$$F(\sigma,\tau) = (2/\sqrt{\pi})\int_{\sigma/2\sqrt{\tau}}^\infty e^{-s^2}ds. \quad (8)$$

It is easily seen that this makes $v = 1/2$ for $x^2 = a^2$. On the other hand, the solution $v = F(|x - x_0|, \tau - \tau_0)$ corresponds to the following initial condition for $\tau = \tau_0$

$$v = 1 \text{ for } x = x_0, \qquad v = 0 \text{ for } x \neq x_0. \quad (9)$$

Writing

$$S(x-x_0, \tau-\tau_0) = [\pi(\tau-\tau_0)]^{-1/2} e^{-\frac{(x-x_0)^2}{4(\tau-\tau_0)}}, \quad (10)$$

the complete solution for the initial condition (2) is

$$v = \int_{-\infty}^\infty S(x-x_0,\tau)f(x_0)dx_0 + \sum_{n=1}^m F(|x-x_n|,\tau)[f(x_n) - \bar{f}(x_n)], \quad (11)$$

where the summation extends over all the points of discontinuity of $f(x)$.

This result may be interpreted to mean that the solution $S(x - x_0, \tau)$ corresponds to a unit source associated with an element of length dx_0 while the solution F corresponds to a unit source associated with a point. When $\tau = \tau_0$ the function $S(x - x_0, \tau - \tau_0)$ is zero at all points except $x = x_0$, where it is infinite, it is in fact a type of function which is now called a Dirac function[1] though functions of this type were used by Fourier 100 years ago.[2]

$S(x - x_0, \tau - \tau_0)$ will be called the first fundamental solution of equation (1) and $F(|x - x_0|, \tau - \tau_0)$ the second fundamental solution. Both of these solutions are defined only for $\tau \geqq \tau_0$. Our aim is to find complete systems of fundamental solutions for the most important partial differential equations of mathematical physics. Only a few preliminary results are given in this paper. The fundamental solutions of the first type have already been studied by several writers[3] and the properties of these solutions are included in the properties of the more general functions that are usually called Green's functions.

2. For the simple wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{12}$$

the fundamental solution of the second type which corresponds to the initial condition for $t = t_0$

$$\begin{aligned} u &= 1 & x &= x_0 \\ u &= 0 & x &\neq x_0 \\ \frac{\partial u}{\partial t} &= 0 & &\text{everywhere} \end{aligned} \tag{13}$$

is

$$\begin{aligned} u &= 0 & x &> x_0 + t - t_0 \\ u &= 1 & x_0 + t - t_0 > x &> x_0 - t + t_0 \\ u &= 0 & x &< x_0 - t + t_0 \end{aligned} \tag{14}$$

and this solution is valid for $t \geqq t_0$.

If, on the other hand, the differential equation is

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \tag{15}$$

which corresponds to a propagation of waves in one direction only, the fundamental solution of the second type corresponding to the initial condition $u = 1$, $x = x_0$, $u = 0$, $x \neq x_0$ (for $t = t_0$) is

$$u = 1 \quad x = x_0 + t - t_0, \quad u = 0 \text{ elsewhere.}$$

3. We now pass on to a consideration of the equations

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \tag{16}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{17}$$

Considering first the wave-equation, let u be required to satisfy the following conditions (for $t = 0$)

$u = f(x, y, z)$ on the sphere S whose equation is $x^2 + y^2 + z^2 = a^2$, $u = 0$ elsewhere, $\frac{\partial u}{\partial t} = 0$ everywhere.

An appropriate solution has been found elsewhere[4] to be

$$u = \frac{a}{r}\bar{f}, \tag{18}$$

where $r^2 = x^2 + y^2 + z^2$ and $\bar{f}$ is the mean value of f along that circle C on S whose points are all at distance t from (x, y, z). If no such circle exists the value of u is zero.

A solution of equation (16) can often be derived from a solution of (17) by making use of the theorem that if $u(x, y, z, t)$ is a solution of (17) answering the requirements that $u = G(x, y, z)$, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then

$$v = (2/\sqrt{\pi}) \int_0^\infty u(x, y, z, 2s\sqrt{\tau})e^{-s^2}\, ds \tag{19}$$

is usually a solution of (16) which satisfies the requirement

$$v = G(x, y, z) \text{ when } \tau = 0. \tag{20}$$

This result may be used to find $v(x, y, z, \tau)$ in the case when G is an assigned function of position for points on the sphere $x^2 + y^2 + z^2 = a^2$ and is zero elsewhere. In the particular case when v is required to be unity over the sphere S at time $\tau = 0$ and zero elsewhere at this time, the solution is found by this method to be

$$v = \frac{a}{r}\,[F\{|a - r|, \tau\} - F\{(a + r, \tau\}]. \tag{21}$$

The general problem of this type for the equation of conduction arises when v has initially an assigned value $f(x, y, z)$ for each point (x, y, z) of a certain surface S and is initially zero elsewhere. The solution of this problem is evidently unique because if there were two different solutions of the problem their difference would be a non-vanishing solution of (16) with an initial value zero everywhere. Such a solution does not exist.

To determine the value of v it is convenient to consider the definite integral

$$w = \int_0^\infty v(x, y, z, \tau)e^{-\lambda^2\tau}\, d\tau. \qquad (22)$$

If the differentiations under the integral sign are permissible this integral satisfies the differential equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - \lambda^2 w = -v(x, y, z, 0). \qquad (23)$$

Let us consider the case when $v(x, y, z, 0) = Y_n(\theta, \phi)$ where Y is a spherical harmonic of degree n. Taking the surface S to be the sphere $x^2 + y^2 + z^2 = a^2$ and assuming that $w = R(r) . Y_n(\theta, \phi)$, we find that at points not on S, the function $W(r) = rR(r)$ must satisfy the equation

$$\frac{d^2W}{dr^2} = \left[\frac{n(n+1)}{r^2} + \lambda^2\right]W. \qquad (24)$$

Let W_0 be a solution suitable for the space outside S, W_1 a solution suitable for the space inside S and let the arbitrary constants in W_0 and W_1 be adjusted so that

$$W_0 \frac{dW_1}{dr} - W_1 \frac{dW_0}{dr} = 1. \qquad (25)$$

We now replace S by a thin shell bounded by the surfaces $r = a - \epsilon \equiv \alpha$ and $r = a + \epsilon \equiv \beta$. Assuming that

$$rw = AW_1(r)Y_n(\theta, \phi) \qquad r < \alpha$$
$$rw = [PW_0(r) + QW_1(r) + H(r)]Y_n(\theta, \phi) \qquad \alpha < r < \beta$$
$$rw = BW_0(r)Y_n(\theta, \phi) \qquad r > \beta$$

the continuity of rw and $\frac{\partial}{\partial r}(rw)$ require that

$$AW_1(\alpha) = PW_0(\alpha) + QW_1(\alpha) + H(\alpha)$$
$$AW_1'(\alpha) = PW_0'(\alpha) + QW_1'(\alpha) + H'(\alpha)$$
$$BW_0(\beta) = PW_0(\beta) + QW_1(\beta) + H(\beta)$$
$$BW_0'(\beta) = PW_0'(\beta) + QW_1'(\beta) + H'(\beta).$$

These equations give

$$A = Q + W_0(\alpha)H'(\alpha) - W_0'(\alpha)H(\alpha)$$
$$O = Q + W_0(\beta)H'(\beta) - W_0'(\beta)H(\beta)$$
$$A = \int_\alpha^\beta [W_0''(r)H(r) - W_0(r)H''(r)]dr$$

i.e., $$A = \int_\alpha^\beta W_0(r)\left[\left\{\lambda^2 + \frac{n(n+1)}{r^2}\right\} H(r) - H''(r)\right]dr.$$

Similarly

$$B = \int_{\alpha}^{\beta} W_1(r) \left[\left\{ \lambda^2 + \frac{n(n+1)}{r^2} \right\} H(r) - H''(r) \right] dr.$$

To ascertain how $H(r)$ must be chosen we consider the case $n = 0$ the solution for which may be derived from (21) by means of equation (22). In this case we find that

$$\begin{aligned} \lambda W_0(r) &= e^{-\lambda r} & \lambda W_1(r) &= \sinh \lambda r \\ \lambda A &= 2ae^{-\lambda a} & \lambda B &= 2a \sinh \lambda a \end{aligned} \qquad (26)$$

and the correct result is obtained by writing

$$\lambda^2 H(r) - H''(r) = aX[\lambda(r - a)], \qquad (27)$$

where $X(0) = 1$ and $X(s)$ is a function such that for any continuous function $f(s)$

$$\lim_{\epsilon \to 0} \int_{-\lambda\epsilon}^{\lambda\epsilon} f(s)X(s)ds = 2f(0). \qquad (28)$$

This function $X(s)$ must, like a Dirac function, become infinite for one or more values of s in the range $(-\lambda\epsilon, \lambda\epsilon)$. In the general case we write

$$\left\{ \lambda^2 + \frac{n(n+1)}{r^2} \right\} H(r) - H''(r) = aX[\lambda(r - a)] \qquad (29)$$

and the analysis gives simply

$$\lambda A = 2aW_0(a) \qquad \lambda B = 2aW_1(a). \qquad (30)$$

The solution of our problem is thus

$$\begin{aligned} \lambda rw &= 2aW_0(a)W_1(r)Y_n(\theta, \phi) & r &< a \\ \lambda rw &= 2aW_1(a)W_0(r)Y_n(\theta, \phi) & r &> a. \end{aligned} \qquad (31)$$

In the case when $n = 0$ the solution is

$$\begin{aligned} \lambda^2 rw &= 2ae^{-\lambda a} \sinh \lambda r & r &< a \\ \lambda^2 rw &= 2ae^{-\lambda r} \sinh \lambda a & r &> a. \end{aligned} \qquad (32)$$

The corresponding solution for the plane $x = x_0$ is

$$\lambda^2 w = e^{-\lambda|x - x_0|}. \qquad (33)$$

The solution of the conduction problem for the sphere S may be obtained directly by means of equation (19) and the solution of the wave-equation

$$u = (a/r)P_n(\cos \omega)Y_n(\theta, \phi), \qquad (34)$$

where $t^2 = r^2 + a^2 - 2ar \cos \omega$, when this equation gives a real value of ω, u being otherwise zero. The expression for v is thus

$$v = (2/\sqrt{\pi}) \int_{\frac{|r-a|^1}{2\sqrt{\tau}}}^{\frac{r+a}{2\sqrt{\tau}}} (a/r) P_n \left[\frac{r^2 + a^2 - s^2\tau}{2ar}\right] e^{-s^2} ds \cdot Y_n(\theta \cdot \phi)$$

$$= \frac{a^2 Y_n(\theta, \phi)}{\sqrt{(\pi\tau)}} \int_0^{\pi} P_n(\cos \omega) \sin \omega . d\omega \frac{1}{\rho} e^{-(\rho^2/4\tau)} \tag{35}$$

where $\rho^2 = r^2 + a^2 - 2ar \cos \omega$. We have also the expressions

$$v = 2a^2(-)^n Y_n(\theta, \phi) \int_0^{\infty} e^{-k^2\tau} J_{n+1/2}(kr_1) J_{-n-1/2}(kr_2) dk/(ar)^{1/2}$$

$$r_1 = r, \; r_2 = a \text{ if } r < a$$
$$r_1 = a, \; r_2 = r \text{ if } r > a$$

and this, combined with (31) gives the relation

$$W_0(r_2) W_1(r_1) = (-)^n \int_0^{\infty} \frac{\lambda dk}{\lambda^2 + k^2} J_{n+1/2}(kr_1) J_{-n-1/2}(kr_2)/(r_1 r_2)^{1/2} \quad r_1 < r_2$$

which is closely related to an integral given by Sonine.[5]

[1] After P. A. M. Dirac, *Proc. Roy. Soc. Lond.*, **113** (1927), 621.

[2] Oeuvres, t. 1, p. 234.

[3] I. Fredholm, *Acta. Math.*, **23** (1900), 1; *Compt. Rend.*, **129** (1899), 32; *Rend. Palermo*, **25** (1908), 346. J. Le Roux, *Compt. Rend.*, **137** (1903), 1230. N. Zeilon, *Arkiv. Mat. Astr. och Fysik.*, **6** (1911), **9** (1914); *Nov. Act. Soc. Sc. Upsaliensis*, **4**, (1919) 5, J. Hadamard, *Lectures on Cauchy's Problem*, New Haven, **1923**, Ch. 3.

[4] *Ann. Math.*, **31** (1930), 158.

[5] N. Sonine, *Math. Ann.*, **16** (1880), 59. See also H. M. Macdonald, *Proc. Lond. Math. Soc.*, **1**, 35 (1902), 428.

ON THE POSSIBLE INFLUENCE OF THE MOSAIC STRUCTURE OF CRYSTALS ON THE DETERMINATION OF AVOGADRO'S NUMBER

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(*A*) *The Experimental Situation.*—The experimental determination of Avogadro's number is very closely related to the determination of the charge of the electron on the one hand and to the absolute determination of the wave-length of x-rays on the other hand. Millikan, as is well known, measured the charge of the electron directly by the oil drop method, long before absolute measurements of the wave-lengths of x-rays were attempted. Values for these wave-lengths, therefore, were obtained from the charge e of the electron in the following indirect way. From e and Faraday's constant F, one derives immediately Avogadro's number,