WEEKLY PROBLEM PAPERS

MILNE

MACMILLAN & Co

WEEKLY PROBLEM PAPERS.

WEEKLY PROBLEM PAPERS

WITH NOTES

INTENDED FOR THE USE OF STUDENTS PREPARING FOR MATHEMATICAL SCHOLARSHIPS

AND

FOR THE JUNIOR MEMBERS OF THE UNIVERSITIES WHO ARE READING FOR MATHEMATICAL HONOURS

BY THE

REV. JOHN J. MILNE, M.A.

PRIVATE TUTOR

MEMBER OF THE LONDON MATHEMATICAL SOCIETY

LATE SCHOLAR OF ST. JOHN'S COLLEGE, CAMBRIDGE

London

MACMILLAN AND CO.

AND NEW YORK

1891

Richard Clay and Sons, Limited,
LONDON AND BUNGAY.

First Edition printed 1885
Reprinted 1886. *Reprinted with corrections* 1891

PREFACE.

The following collection of Problems is intended to supply a decided want, viz. a series of Problem Papers in elementary subjects, which, without being too easy, shall not at the same time be utterly out of the reach of students in the highest forms of our public schools. Of mathematical problems there is no lack, but the general complaint about them is, that they are far too difficult for any but very advanced pupils, and as they are as a rule unaccompanied by any hint as to the particular manner in which they are to be attempted, they usually fail to serve any educational purpose. The following Problems have all been set to pupils who were reading for scholarship examinations, and have been selected chiefly from scholarship papers, and those set in the elementary subjects in the Tripos in recent years, so that an intimate acquaintance might be formed with the style of questions which have been already proposed in these examinations. In making the selection, the greatest care has been taken to exclude all that were found on trial to be unsuitable either from being too easy or too difficult. Of course it is not always easy to hit the happy medium, especially with Problems, many of which, though apparently difficult, are in reality easy when once the right method of attempting them is discovered. In order that facility in attacking Problems may be acquired, it is necessary that a student should not only have constant practice in them, but also that he should be able to find a solution of those which he himself may be unable to solve, for there is no doubt that whilst some benefit is derived even from unsuccessful attempts, the benefit is very greatly increased in those cases by a solution, as methods are thus acquired which can be applied with success to similar Problems.

A second volume is therefore in course of preparation, containing solutions for the use of private students and of mathematical teachers, who have not always the time to spare for them.

Questions on the following subjects only have been admitted: Algebra, Arithmetic, Euclid, Trigonometry, Geometrical Conics, the Elementary parts of Analytical Conics, Statics, and in a few of the later papers, Dynamics. It will be seen from the solutions that the Problems which occur relating to Algebraical and Geometrical maxima and minima can easily be solved from elementary considerations without employing Differential Calculus.

It is recommended that one paper should be set each week so that each Problem may be fairly tried, and as they will be found to be graduated in point of difficulty, the order given in the book should be adhered to. Together with each paper the author has found it of the greatest service to give alternately five riders on Geometrical Conics and five on Euclid. Those given in Mr. Besant's *Conics* and Mr. Todhunter's *Euclid*, beginning at No. 441, were used not only for their intrinsic merit, but also because solutions to the riders in both books have been recently published.

PREFACE TO THE THIRD EDITION.

In this edition the following alterations have been made. Articles 25, 26 on the Summation of Series have been expanded.

Instead of Prof. Purser's proof of Feuerbach's Theorem formerly given on pp. 33, 34, of which a complete proof is now to be found in most advanced text-books on geometry, I have given a new proof by Mr. R. F. Davis.

In the preface to the *Solutions* I stated that "some of the questions in the *Problem Papers* were taken from the Triposes of 1875 and 1878, solutions of which have been published by Messrs. Macmillan, and for the use of those who did not possess these solutions I gave as an Appendix an equal number of alternative questions which are to a great extent similar in character to the corresponding problems." These have now been added as an Appendix to the present volume, and they will be found solved at the end of the *Solutions of the Weekly Problem Papers.*

The errata which were previously printed in a separate list have now been corrected in the text.

I take this opportunity of stating that improved solutions of several problems which were sent to me from time to time by various mathematical friends will be found on pp. 269—285 of the *Companion to the Weekly Problem Papers, or Supplementary Chapters on Elementary Mathematics.* For any further corrections or solutions I shall at all times be very grateful.

JOHN J. MILNE.

INVERMARK, ALLEYN PARK, DULWICH, S.E.
September, 1891.

WEEKLY PROBLEM PAPERS.

I. ALGEBRA.

1. (1) If the two expressions $ax^2 + bx + c$ and $a'x^2 + b'x + c'$ have a common factor, to shew that $(a'b - ab')(b'c - bc') = (ac' - a'c)^2$.

Let $x - \alpha$ be this common factor.

$$\text{Then} \quad a\alpha^2 + b\alpha + c = 0 \quad . \; . \; . \; . \; . \; . \quad (1)$$

$$\text{and} \quad a'\alpha^2 + b'\alpha + c' = 0 \quad . \; . \; . \; . \; . \; . \quad (2)$$

Multiply (1) by a' and (2) by a and subtract

$$\therefore \; \alpha(a'b - ab') = ac' - a'c \; . \; . \; . \; . \; (A).$$

Again, multiply (1) by c' and (2) by c. Then subtract and divide the result by α.

$$\therefore \; \alpha(ac' - a'c) = b'c - bc' \; . \; . \; . \; . \; (B)$$

$\therefore$ from A and B

$$(a'b - ab')(b'c - bc') = (ac' - a'c)^2.$$

(2) Or more briefly thus. From (1) and (2)

$$\frac{\alpha^2}{b'c - bc'} = \frac{\alpha}{ac' - a'c} = \frac{1}{a'b - ab'},$$

$$\therefore \; (a'b - ab')(b'c - bc') = (ac' - a'c)^2.$$

---

2. If the two expressions $x^2 + ax + b$ and $x^2 + a'x + b'$ have a common multiple of the form $x^3 + px + q$, then $ab = a'b' = -aa'(a + a')$.

It is obvious that

$$x^3 + px + q \equiv (x^2 + ax + b)(x - a)$$
$$\equiv (x^2 + a'x + b')(x - a')$$
$$\therefore\ x(b - a^2) - ab = x(b' - a'^2) - a'b'$$
$$\therefore\ ab = a'b'$$
$$\text{and } b - a^2 = b' - a'^2$$
$$\therefore\ b - b' = a^2 - a'^2$$
$$\therefore\ a'(a^2 - a'^2) = a'b - a'b' = a'b - ab = b(a' - a)$$
$$\therefore\ - a'(a + a') = b$$
$$\therefore\ - aa'(a + a') = ab = a'b'$$

Binomial Theorem.

3. To find the greatest term in the expansion of $(a+x)^{\pm n}$.

Note.—In following proof expressions on the left of the vertical refer to the $+^{\text{ve}}$ index, those on the right to the $-^{\text{ve}}$ index.

The $(r + 1)^{\text{th}}$ term is

$$\frac{n\ldots\ldots(n-r+1)}{\lfloor r}x^r \cdot a^{n-r} \quad \Bigg| \quad \frac{n\ldots\ldots(n+r-1)}{\lfloor r}x^r \cdot a^{n-r},$$

The r^{th} term is

$$\frac{n\ldots\ldots(n-r+2)}{\lfloor r-1}x^{r-1}a^{n-r+1}, \quad \Bigg| \quad \frac{n\ldots\ldots(n+r-2)}{\lfloor r-1}x^{r-1}a^{n-r+1}.$$

The $(r + 1)^{\text{th}}$ term is obtained from the r^{th} by the multiplier

$$\frac{n-r+1}{r} \cdot \frac{x}{a}, \quad \Bigg| \quad \frac{n+r-1}{r} \cdot \frac{x}{a},$$

∴ the $(r+1)^{\text{th}}$ term is $>$ r^{th} term as long as this multiplier is > 1.

And since this multiplier is

$$\left(\frac{n+1}{r}-1\right)\frac{x}{a}, \qquad \Bigg| \qquad \left(\frac{n-1}{r}+1\right)\frac{x}{a},$$

we see that it continually decreases as r increases.

∴ the r^{th} term is the greatest, or equal to the greatest when this multiplier is *first* less than, or equal to 1.

i.e. $\left(\frac{n+1}{r}-1\right)\frac{x}{a} <$ or $=1$, | $\left(\frac{n-1}{r}+1\right)\frac{x}{a} <$ or $=1$,

i.e. $\frac{n+1}{r}-1 <$ or $=\frac{a}{x}$, | $\frac{n-1}{r}+1 <$ or $=\frac{a}{x}$,

i.e. $n+1 <$ or $=\frac{r(a+x)}{x}$, | $n-1 <$ or $=\frac{r(a-x)}{x}$,

i.e. $\frac{(n+1)x}{a+x} <$ or $= r$, | $\frac{(n-1)x}{a-x} <$ or $= r$,

∴ the r^{th} term is the greatest, or equal to the greatest when r is first

$$> \frac{(n+1)x}{a+x}, \qquad \Bigg| \qquad > \frac{(n-1)x}{a-x},$$

---

4. If a_r be coefficient of x^r in expansion of $(1+x)^n$, to prove that

$$\frac{a_1}{a_0}+2\frac{a_2}{a_1}+3\frac{a_3}{a_2}+\ldots\ldots+n\frac{a_n}{a_{n-1}}=\frac{n(n+1)}{2}.$$

For

$$a_r = \frac{n(n-1)\ldots.(n-r+1)}{\underline{|r}},\ a_{r-1} = \frac{n(n-1)\ldots.(n-r+2)}{\underline{|r-1}}.$$

$$\therefore\ r\,.\,\frac{a_r}{a_{r-1}} = r\,.\,\frac{n-r+1}{r} = n-r+1.$$

Putting for r in succession 1, 2, 3, ... we get

$$\frac{a_1}{a_0} + 2\frac{a_2}{a_1} + 3\frac{a_3}{a_2} + \ldots. = n + (n-1) + (n-2) + \ldots. + 1$$

$$= \frac{n(n+1)}{2}.$$

---

5. Also

$$(a_0+a_1)(a_1+a_2)\ldots.(a_{n-1}+a_n) = a_0a_1\ldots.a_{n-1}\,.\,\frac{(n+1)^n}{\underline{|n}}.$$

For $$\frac{a_r}{a_{r-1}} + 1 = \frac{n-r+1}{r} + 1 = \frac{n+1}{r},$$

$$\therefore\ \frac{a_0+a_1}{a_0}\,.\,\frac{a_1+a_2}{a_1}\,.\,\frac{a_2+a_3}{a_2}\ldots.\frac{a_{n-1}+a_n}{a_{n-1}}$$

$$= \frac{n+1}{1}\,.\,\frac{n+1}{2}\ldots.\frac{n+1}{n}.$$

$$\therefore\ (a_0+a_1)(a_1+a_2)\ldots.(a_{n-1}+a_n) = a_0a_1\ldots.a_{n-1}\,.\,\frac{(n+1)^n}{\underline{|n}}.$$

---

6. Also $a_0^2 + a_1^2 + a_2^2 + \dots + a_n^2 = \dfrac{|\underline{2n}}{\{|\underline{n}\}^2}$.

For $$(1+x)^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$\left(1+\frac{1}{x}\right)^n = a_0 + a_1\frac{1}{x} + a_2\frac{1}{x^2} + \dots$$

$$\therefore\ a_0^2 + a_1^2 + a_2^2 + \dots = \text{absolute term in } (1+x)^n \cdot \frac{(1+x)^n}{x^n}.$$

$$= \text{coef. } x^n \text{ in } (1+x)^{2n}.$$

$$= \frac{|\underline{2n}}{\{|\underline{n}\}^2}.$$

---

7. To find the value of $\Sigma(a_n)$, $\Sigma(na_n)$, $\Sigma(n^2 a_n)$ ......

where $\Sigma(n^r a_n) = 1^r a_1 + 2^r a_2 + 3^r a_3 + \dots + n^r a_n$

$$(2+x)^n = \{1 + (1+x)\}^n$$

$$= a_0 + a_1(1+x) + a_2(1+x)^2 + \dots + a_n(1+x)^n$$

$\therefore$ equating coefficients of like powers of x

$$2^n = a_0 + a_1 + a_2 + \dots + a_n = \Sigma(a_n)$$

$$n2^{n-1} = 1\,.\,a_1 + 2\,.\,a_2 + \dots + n\,.\,a_n = \Sigma(na_n) \quad (1)$$

$$n(n-1)2^{n-2} = 1\,.\,2a_2 + \dots + n(n-1)a_n\,.$$

$$= \Sigma(n^2 a_n) - \Sigma(na_n) \quad \dots\dots \quad (2)$$

$\therefore$ Adding (1) and (2)

$$n(n+1)2^{n-2} = 1^2 a_1 + 2^2 a_2 + \dots + n^2 a_n = \Sigma(n^2 a_n) \quad (3)$$

Again we have

$$n(n-1)(n-2)2^{n-3} = 1.2.3a_3 + \ldots. + n(n-1)(n-2)a_n$$
$$= \Sigma(n^3a_n) - 3\Sigma(n^2a_n) + 2\Sigma(na_n) \quad . \quad . \quad . \quad (4)$$

$\therefore$ From (1), (3) and (4)

$$n^2(n+3)2^{n-3} = \Sigma(n^3a_n)$$
$$= 1^3a_1 + 2^3a_2 + \ldots. + n^3a_n.$$

By proceeding in the same manner we can find the value of $\Sigma(n^ra_n)$ where r is any integer $< n$.

---

8. From these results we can find the sum of the binomial coefficients when combined with factors consisting of the natural numbers connected by any given law.

Ex. To find the value of

$$1 . 3^2a_1 + 2 . 4^2a_2 + \ldots. + n(n+2)^2a_n.$$

The n^{th} term is $(n^3 + 4n^2 + 4n)a_n$

$\therefore$ the required sum is $\Sigma(n^3a_n) + 4\Sigma(n^2a_n) + 4\Sigma(na_n)$, and is $\therefore$ known.

---

9. Also $a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \ldots. = \dfrac{2^{n+1}-1}{n+1}$.

For

$$\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + x + n . \frac{x^2}{2} + \frac{n(n-1)}{\underline{|2}} . \frac{x^3}{3} + \ldots.$$

$$\therefore \frac{(1+x)^{n+1} - 1}{n+1} = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \ldots.$$

Put $x = 1$ $\therefore a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots = \frac{2^{n+1} - 1}{n+1}$.

---

10. Again

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} + \frac{x}{n+1} + \frac{x^2}{2} + n \cdot \frac{x^3}{2 \cdot 3}$$
$$+ \frac{n(n-1)}{2} \cdot \frac{x^4}{3 \cdot 4} + \dots$$

$\therefore$ Putting $x = 1$ and transposing

$$\frac{2^{n+2} - (n+3)}{(n+1)(n+2)} = \frac{1}{1 \cdot 2} \cdot a_0 + \frac{1}{2 \cdot 3} a_1 + \frac{1}{3 \cdot 4} a_2 + \dots$$

---

11. To find the sum of the first n coefficients in $\frac{1}{(1-x)^p}$.

$$\frac{1}{(1-x)^p} = 1 + px + \frac{p(p+1)}{\lfloor 2} x^2 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$\therefore \frac{1}{(1-x)^{p+1}} = 1 + (1+p)x + \left(1 + p + \frac{p(p+1)}{\lfloor 2}\right)x^2 + \dots$$

$\therefore$ required sum is coefficient of x^{n-1} in $\frac{1}{(1-x)^{p+1}}$,

$$\text{and } \therefore = \frac{(p+1)(p+2)\dots(p+n-1)}{\lfloor n-1}.$$

---

12. The coefficient of x^{n+r-1} in $\dfrac{(1+x)^n}{(1-x)^2}$ is $2^{n-1}(n+2r)$,

$$\frac{(1+x)^n}{(1-x)^2} = (1-x)^{-2}\{2-(1-x)\}^n,$$

$$= (1-x)^{-2}.\{2^n - n.2^{n-1}(1-x) + \frac{n.(n-1)}{\lfloor 2}\cdot 2^{n-2}(1-x)^2 \ldots .\}$$

$$= 2^n(1-x)^{-2} - n.2^{n-1}(1-x)^{-1} + \frac{n(n-1)}{\lfloor 2}.2^{n-2}$$

+ terms which need not be considered.

$\therefore$ coefficient . $x^{n+r-1} = 2^n(n+r) - 2^{n-1}.n$,

$$= 2^{n-1}(n+2r).$$

---

13. To find the sum of the first n coefficients in $\dfrac{(1+x)^n}{(1-x)^2}$,

Let $\dfrac{(1+x)^n}{(1-x)^2} = a_0 + a_1x + a_2x^2 + \ldots .$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots .$$

$$\therefore \quad \frac{(1+x)^n}{(1-x)^3} = a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + \ldots .$$

$\therefore$ the sum of the first n coefficients in $\dfrac{(1+x)^n}{(1-x)^2}$ is evidently the coefficient of x^{n-1} in $\dfrac{(1+x)^n}{(1-x)^3}$, which is easily shewn by the method of the preceding article to be $2^{n-3}n(n+3)$.

---

14. If $a_r = \dfrac{1 \,.\, 3 \,.\, 5 \,\ldots\ldots 2r - 1}{2 \,.\, 4 \,.\, 6 \,\ldots\ldots 2r}$, prove that

$$a_{2n+1} + a_1 a_{2n} + \ldots\ldots + a_{n-1} a_{n+2} + a_n a_{n+1} = \tfrac{1}{2}.$$

$$(1-x)^{-\frac{1}{2}} = 1 + a_1 x + a_2 x^2 + \ldots\ldots + a_{2n} x^{2n} + a_{2n+1} x^{2n+1} + A.$$

$$(1-x)^{-\frac{1}{2}} = A + a_{2n+1} x^{2n+1} + a_{2n} x^{2n} + \ldots\ldots + a_1 x + 1$$

$$\therefore\ (1-x)^{-1} = B + x^{2n+1}\{a_{2n+1} + a_1 a_{2n} + \ldots\ldots + a_{2n} a_1 + a_{2n+1}\}$$

Thus the given expression is $\frac{1}{2}$ the coefficient x^{2n+1} in expansion of $(1 - x)^{-1}$, and is $\therefore = \frac{1}{2}$.

---

15. If $a_0, a_1, a_2 \ldots\ldots a_n$ represent the terms in order in the expansion of $(a + x)^n$, shew that

$$\{a_0 - a_2 + a_4 \ldots\ldots\}^2 + \{a_1 - a_3 + a_5 \ldots\ldots\}^2 = (a^2 + x^2)^n$$

$$(a + x)^n = a_0 + a_1 + a_2 + a_3 + \ldots\ldots$$

Change x into $x\sqrt{-1}$

$$\therefore\ (a + x\sqrt{-1})^n = a_0 + a_1\sqrt{-1} - a_2 - a_3\sqrt{-1} + a_4 + \ldots\ldots$$

$$= \{a_0 - a_2 + a_4 \ldots\ldots\} + \sqrt{-1}\,\{a_1 - a_3 + \ldots\ldots\}$$

Change the sign of x

$$\therefore\ (a - x\sqrt{-1})^n = a_0 - a_1\sqrt{-1} - a_2 + a_3\sqrt{-1} + a_4 + \ldots\ldots$$

$$= \{a_0 - a_2 + a_4 - \} - \sqrt{-1}\,\{a_1 - a_3 + \ldots\ldots\}$$

$\therefore$ By multiplication

$$(a^2 + x^2)^n = \{a_0 - a_2 + a_4 \ldots\ldots\}^2 + \{a_1 - a_3 + a_5 \ldots\ldots\}^2.$$

---

16. To shew that all the coefficients in the expansion of $(1 + x)^{-n}$ are integers if n be an integer.

The coefficient of x^r

$$= \frac{n(n+1) \ldots . (n+r-1)}{\lfloor r} = \frac{\lfloor n+r-1}{\lfloor r \lfloor n-1}.$$

Now this is the number of combinations of $n + r - 1$ things taken r together, and is $\therefore$ an integer. And since this is the general expression for any coefficient we infer that every coefficient is an integer.

---

17. To prove that difference of coefficients of x^{r+1} and x^r in $(1 + x)^{n+1}$ = difference of coefficients of x^{r-1} and x^{r+1} in $(1 + x)^n$.

Let a_{r-1} and a_{r+1} be coefficients of x^{r-1} and x^{r+1} in $(1 + x)^n$.

Then coefficient of x^r in $(1 + x)^{n+1}$ is $a_r + a_{r-1}$ and coefficient of x^{r+1} in $(1 + x)^{n+1}$ is $a_r + a_{r+1}$.

$\therefore$ difference of coefficients in each case $= a_{r-1} \sim a_{r+1}$.

---

18. To prove that

$$a_1 - \frac{1}{2}a_2 + \frac{1}{3}a_3 \ldots . (-1)^{n-1} . \frac{a_n}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots . + \frac{1}{n}.$$

Let

$$f(n) = a_1 - \frac{1}{2}a_2 + \frac{1}{3}a_3 \ldots .$$

$$= n - \frac{1}{2} . \frac{n(n-1)}{\lfloor 2} + \frac{1}{3} . \frac{n(n-1)(n-2)}{\lfloor 3} \ldots .$$

$$= (n-1) - \frac{1}{2} \cdot \frac{(n-1)(n-2)}{\underline{|2}} + \frac{1}{3} \cdot \frac{(n-1)(n-2)(n-3)}{\underline{|3}} \cdots\cdots$$

$$+ 1 - \frac{n-1}{\underline{|2}} + \frac{(n-1)(n-2)}{\underline{|3}} + \cdots\cdots$$

$$= f(n-1) + \frac{1}{n}\left\{n - \frac{n(n-1)}{\underline{|2}} + \frac{n(n-1)(n-2)}{\underline{|3}} \cdots\cdots\right\}$$

$$= f(n-1) + \frac{1}{n}\left\{1 - (1-1)^n\right\},$$

$$= f(n-1) + \frac{1}{n}.$$

Similarly $f(n-1) = f(n-2) + \dfrac{1}{n-1}$,

" "

$$f(3) = f(2) + \frac{1}{3},$$

$$f(2) = f(1) + \frac{1}{2},$$

$$f(1) = f(0) + \frac{1}{1},$$

$\therefore$ By addition $f(n) = f(0) + \dfrac{1}{1} + \dfrac{1}{2} + \cdots\cdots + \dfrac{1}{n}$

and $f(0) = 0$.

---

19. To shew that

$$\frac{n(n+1)\cdots\cdots(n+m-1)}{\underline{|m}} - n \cdot \frac{n(n+1)\cdots\cdots(n+m-4)}{\underline{|m-3}} +$$

$$\frac{n \cdot \overline{n-1}}{\underline{|2}} \cdot \frac{n(n+1)\cdots\cdots(n+m-7)}{\underline{|m-6}} - \cdots\cdots = 0 \text{ if } m > 2n$$

and $= 1$ if $m = 2n$.

$$1 + x + x^2 = \frac{1 - x^3}{1 - x} = (1 - x)^{-1}(1 - x^3).$$

$$\therefore (1 + x + x^2)^n = (1 - x)^{-n}(1 - x^3)^n.$$

Now

$$(1 - x)^{-n} = 1 + \ldots. + \frac{n(n+1)\ldots.(n+m-1)}{\lfloor m}x^m + \ldots.$$

$$(1 - x^3)^n = 1 - nx^3 + \frac{n(n-1)}{\lfloor 2}x^6 + \ldots.$$

$$\therefore (1 + x + x^2)^n = S + x^m\Big\{\frac{n(n+1)\ldots.(n+m-1)}{\lfloor m}$$

$$- n\,.\,\frac{n(n+1)\ldots.(n+m-4)}{\lfloor m-3} + \ldots.\Big\} +$$

$\therefore$ the given expression is the coefficient of x^m in

$(1 + x + x^2)^n$, and

$\therefore\ = 0$ if $m > 2n$,

since highest power of x in $(1 + x + x^2)^n$ is x^{2n}

and $= 1$ if $m = 2n$.

---

20. If $f(r) = \dfrac{\lfloor n}{\lfloor r\lfloor n-r} + n\,.\,\dfrac{\lfloor n}{\lfloor r+1\lfloor n-r-1}$

$$+ \frac{n\,.\,\overline{n-1}}{\lfloor 2}\,\frac{\lfloor n}{\lfloor r+2\lfloor n-r-2} + \ldots.$$

Then $f(0) + nf(1) + \dfrac{n(n-1)}{\lfloor 2}f(2) + \ldots. = \dfrac{\lfloor 3n}{\lfloor n\lfloor 2n}.$

$$(1 + x)^n = 1 + \ldots. + \frac{\lfloor n}{\lfloor r\lfloor n-r}x^r + \ldots.$$

$$(1 + x)^n = 1 + nx + \ldots. + nx^{n-1} + x^n.$$

By multiplication we find that the coefficient of x^{n+r} in $(1+x)^{2n}$ is the series $f(r)$. Thus we see that the series $f(0)$, $f(1)$, ... are the coefficients of succeeding powers of x beginning with x^n in expansion of $(1+x)^{2n}$.

$$\therefore (1+x)^{2n} = 1 + \ldots\ldots + f(0) \,.\, x^n + f(1) \,.\, x^{n+1} + \ldots\ldots$$

and $(1+x)^n = 1 + nx + \ldots\ldots + nx^{n-1} + x^n$.

$$\therefore (1+x)^{3n} = 1 + \ldots\ldots + x^{2n}\Big\{f(0) + nf(1) + \frac{n \,.\, \overline{n-1}}{\lfloor 2}f(2) + \ldots\ldots + f(n)\Big\} + \ldots\ldots$$

$$= 1 + \ldots\ldots + x^{2n} \,.\, \frac{\lfloor 3n}{\lfloor n \lfloor 2n} + \ldots\ldots$$

$$\therefore f(0) + nf(1) + \frac{n \,.\, \overline{n-1}}{\lfloor 2}f(2) + \ldots\ldots = \frac{\lfloor 3n}{\lfloor n \lfloor 2n}.$$

---

21. To find remainder after n terms of expansion of $(1-x)^{-2}$.

Let $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots + R$.

$\therefore$ by multiplication

$$\begin{aligned} 1 = 1 &+ 2x + 3x^2 + \ldots\ldots + R. \\ &- 2x - 4x^2 - \ldots\ldots - 2nx^n - 2x \,.\, R. \\ &+ x^2 + \ldots\ldots + (n-1)x^n + nx^{n+1} + x^2 \,.\, R. \end{aligned}$$

$$\therefore 0 = -(n+1)x^n + nx^{n+1} + R(1-x)^2$$

$$\therefore R = \frac{(n+1-nx)x^n}{(1-x)^2}.$$

---

22. If a_r be the coefficient of x^r in expansion of $\dfrac{1}{1-2nx-x^2}$, to shew that

$$a_r - 2(1+2n^2)a_{r-2} + a_{r-4} = 0.$$

$$(1-2nx-x^2)^{-1} = 1 + \dots + a_r x^r + \dots$$

Multiply by $1 - 2nx - x^2$.

$$\begin{aligned}\therefore 1 = 1 + \dots + {} & a_r x^r + \dots \\ & - 2na_{r-1}x^r + \dots \\ & - a_{r-2}x^r + \dots\end{aligned}$$

$$\therefore a_r - 2na_{r-1} - a_{r-2} = 0,$$

$$a_{r-1} - 2na_{r-2} - a_{r-3} = 0,$$

$$a_{r-2} - 2na_{r-3} - a_{r-4} = 0,$$

$$\therefore \frac{a_r - a_{r-2}}{2n} - 2na_{r-2} - \frac{a_{r-2} - a_{r-4}}{2n} = 0.$$

$$\therefore a_r - 2(1+2n^2)a_{r-2} + a_{r-4} = 0.$$

---

23. To shew that the coefficient of x^r in

$$\left(x+\frac{1}{x}\right)^n \text{ is } \frac{\lfloor n}{\lfloor \tfrac{1}{2}(n-r) \lfloor \tfrac{1}{2}(n+r)},$$

$$\left(x+\frac{1}{x}\right)^n = x^n + a_1 x^{2\left(\frac{n}{2}-1\right)} + a_2 x^{2\left(\frac{n}{2}-2\right)} + \dots$$

$$+ a_{\frac{n-r}{2}} \cdot x^{2\left(\frac{n}{2} - \frac{n-r}{2}\right)} + \dots$$

$\therefore$ the term involving x^r is the $\left(\frac{n-r}{2}+1\right)^{\text{th}}$, of which the coefficient is

$$\frac{n(n-1)\ldots\left\{n-\left(\frac{n-r}{2}+1\right)+2\right\}}{\underline{|\tfrac{1}{2}(n-r)}} = \frac{n(n-1)\ldots\left(\frac{n+r}{2}+1\right)}{\underline{|\tfrac{1}{2}(n-r)}}$$

$$= \frac{\underline{|n}}{\underline{|\tfrac{1}{2}(n-r)}\,\underline{|\tfrac{1}{2}(n+r)}}.$$

---

24. The summation and factors of the series

$$1^r + 2^r + 3^r + \ldots + n^r.$$

Denote the series by S^r_n. By the ordinary methods,

$$S^0_n = 1 + 1 + \ldots = n;\quad S^1_n = 1 + 2 + 3 + \ldots = \frac{n(n+1)}{2}$$

$$S^2_n = 1^2 + 2^2 + \ldots = \frac{n(n+1)(2n+1)}{6};$$

$$S^3_n = 1^3 + 2^3 + \ldots = \frac{n^2(n+1)^2}{4}.$$

There are two general formulæ, which may readily be obtained as follows.

$$n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5n + 1$$

$$(n-1)^5 - (n-2)^5$$
$$= 5(n-1)^4 - 10(n-1)^3 + 10(n-1)^2 - 5(n-1) + 1$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$2^5 - 1^5 = 5.2^4 - 10.2^3 + 10.2^2 - 5.2 + 1$$

$$1^5 - 0^5 = 5.1^4 - 10.1^3 + 10.1^2 - 5.1 + 1$$

$\therefore$ by addition,

$$n^5 = 5S^4_n - 10S^3_n + 10S^2_n - 5S^1_n + S^0_n.$$

Now from the law of formation we see that the multipliers of S_n^4 &c. are binomial coefficients. $\therefore$ generally,

$$n^{r+1} = (r+1)S_n^r - \frac{(r+1)r}{\lfloor 2} S_n^{r-1} + \ldots\ldots + (-1)^r S_n^0 \ldots (A)$$

Again,

$$(n+1)^5 - n^5 = 5n^4 + 10n^3 + 10n^2 + 5n + 1$$

$$n^5 - (n-1)^5$$
$$= 5(n-1)^4 + 10(n-1)^3 + 10(n-1)^2 + 5(n-1) + 1$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$2^5 - 1^5 = 5 \,.\, 1^4 + 10 \,.\, 1^3 + 10 \,.\, 1^2 + 5 \,.\, 1 + 1$$

$\therefore$ by addition,

$$(n+1)^5 - 1 = 5S_n^4 + 10S_n^3 + 10S_n^2 + 5S_n^1 + S_n^0$$

$\therefore$ as before, in the general case we have

$$(n+1)^{r+1} - 1 = (r+1)S_n^r + \frac{(r+1)r}{\lfloor 2} S_n^{r-1} + \ldots\ldots + S_n^0 \; (B)$$

25. If we wish to find the value of S_n^r for any given value of r by means of (A) or (B), it is evident that we must first find the values of S_n^{r-1}, S_n^{r-2}, &c. We will now shew that S_n^r can be made to depend only upon the sums of the previous even powers if r is even, and upon the odd powers, if r is odd.

I. Let r be even.

Then from (A) and (B) by addition,

$$(n+1)^{r+1} + n^{r+1} - 1$$
$$= 2\left\{(r+1)S_n^r + \frac{(r+1)r(r-1)}{\lfloor 3} S_n^{r-2} + \ldots\ldots \right.$$
$$\left. + \frac{(r+1)r}{\lfloor 2} S_n^2 + S_n^0 \right\}$$

Put for S_n^0 its value n, and transpose.

$$\therefore\ (n+1)^{r+1} + n^{r+1} - 2n - 1$$
$$= 2\left\{(r+1)S_n^r + \frac{(r+1)r(r-1)}{\lfloor 3}S_n^{r-2} + \ldots\right.$$
$$\left. + \frac{(r+1)r}{\lfloor 2}S_n^2\right\} \quad . \quad . \quad . \quad . \quad (C)$$

Rearranging the left-hand side of (C) in the form $\left\{(n+1)^{r+1} + n^{r+1}\right\} - (2n+1)$, we see that it contains the factor $2n+1$. The expression also vanishes when we put $n = 0$, and $n = -1$. $\therefore$ it contains $n(n+1)$ as a factor. $\therefore$ it contains as a factor $n(n+1)(2n+1)$. In (C) putting $r = 4$, we see that $2(5S_n^4 + 10S_n^2)$ contains as a factor $n(n+1)(2n+1)$. $\therefore$ S_n^4 contains as a factor $n(n+1)(2n+1)$.

By putting r in succession $= 6, 8, \ldots.$ we see that S_n^r contains as a factor $n(n+1)(2n+1)$. Now this expression is a multiple of 6.

$\therefore$ when r is even, S_n^r contains as a factor the expression $\dfrac{n(n+1)(2n+1)}{6}$, *i.e.* S_n^2.

II. Suppose r odd.

Adding (A) and (B) we have

$$(n+1)^{r+1} + n^{r+1} - 1$$
$$= 2\left\{(r+1)S_n^r + \frac{(r+1)r(r-1)}{\lfloor 3}S_n^{r-2} + \ldots + (r+1)S_n^1\right\}$$

Putting for S_n^1 its value $\dfrac{n(n+1)}{2}$ and transposing,

$$(n+1)^{r+1} + n^{r+1} - (r+1)n(n+1) - 1$$
$$= 2\left\{(r+1)S_n^r + \frac{(r+1)r(r-1)}{\lfloor 3}S_n^{r-2} + \ldots\right.$$
$$\left. + \frac{(r+1)r(r-1)}{\lfloor 3}S_n^3\right\} \quad . \quad . \quad . \quad . \quad (D)$$

We will now shew that the left-hand side of (D) contains $n^2(n+1)^2$ as a factor.

Expanding, we have

$$n^{r+1} + (r+1)n^r + \ldots . + \frac{(r+1)r}{\underline{|2}} n^2 + (r+1)n + 1$$
$$+ n^{r+1} - (r+1)n^2 - (r+1)n - 1,$$

which is obviously a multiple of n^2.

Again, the expression on the left of (D)

$$= (n+1)\left\{(n+1)^r - (r+1)n\right\} + n^{r+1} - 1$$
$$= (n+1)\left\{(n+1)^r - (r+1)n + n^r - n^{r-1} + \ldots . - 1\right\}$$

If we put $n = -1$ in the expression in $\{\ \ \}$, we obtain

$$\left\{0 + (r+1) - (r+1)\right\}, \text{ which } = 0.$$

$\therefore$ the expression on the left-hand side of (D) contains $n^2(n+1)^2$ as a factor.

$\therefore$ as before, we see that when r is odd, S^r_n contains as a factor $n^2(n+1)^2$. And this is a multiple of 2^2.

$\therefore$ S^r_n contains as a factor the expression

$$\frac{n^2(n+1)^2}{4}, \text{ i.e. } S^3_n.$$

---

26. If we write S^2_n for $\dfrac{n(n+1)(2n+1)}{6}$, S^3_n for $\dfrac{n^2(n+1)^2}{4}$, and A for $n^2 + n - 1$, we find

$$5\,S^4_n = S^2_n\,(3A+2); \qquad 3\,S^5_n = S^3_n\,(2A+1);$$
$$7\,S^6_n = S^2_n\,(3A^2+3A+1); \quad 6\,S^7_n = S^3_n\,(3A^2+2A+1);$$
$$15\,S^8_n = S^2_n\,(5A^3+5A^2+4A+1);$$
$$5\,S^9_n = S^3_n\,A(2A^2+A+2);$$
$$11\,S^{10}_n = S^2_n\,A(3A^3+2A^2+5A+1);$$
$$6\,S^{11}_n = S^3_n\,(2A^4+5A^2-2A+1).$$

27 I. When the law of the formation of a series is known, the usual mode of summing the series is to split up the n^{th} term into the difference of two other terms $u_n - u_{n-1}$, where u_n is the same function of n that u_{n-1} is of $n - 1$.

Ex. $\dfrac{2}{1\,.\,3} + \dfrac{2\,.\,4}{1\,.\,3\,.\,5} + \dfrac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5\,.\,7} + \ldots.$

Here the n^{th} term is

$$\frac{2\,.\,4\,.\,6\,\ldots\,.\,2n}{1\,.\,3\,.\,5\,\ldots\,.\,(2n+1)}.$$

This can be written

$$\frac{2\,.\,4\,.\,6\,\ldots\,.\,2n}{1\,.\,3\,.\,5\,\ldots\,.\,(2n+1)}\{2n+2-(2n+1)\},$$

$$\text{which} = \frac{2\,.\,4\,.\,6\,\ldots\,.\,(2n+2)}{1\,.\,3\,.\,5\,\ldots\,.\,(2n+1)} - \frac{2\,.\,4\,.\,6\,\ldots\,.\,2n}{1\,.\,3\,.\,5\,\ldots\,.\,(2n-1)},$$

$$= u_n - u_{n-1}.$$

And the first term is $\dfrac{2\,.\,4}{1\,.\,3} - \dfrac{2}{1}$.

$$\therefore S_n = \frac{2\,.\,4\,.\,6\,\ldots\,.\,(2n+2)}{1\,.\,3\,.\,5\,\ldots\,.\,(2n+1)} - 2.$$

II. Sometimes, however, we have to discover the law of the series before we can write down the n^{th} term.

Ex. $-15 - 13 - 6 + 9 + 35 + 75 + \ldots.$

Form a new series by subtracting each term from the term which follows it, and repeat the process until we get a series of equal terms.

$$\begin{aligned} \text{Thus } S &= -15 - 13 - 6 + 9 + 35 + 75 + \ldots. \\ S_1 &= 2 + 7 + 15 + 26 + 40 + \ldots. \\ S_2 &= 5 + 8 + 11 + 14 + \ldots. \\ S_3 &= 3 + 3 + 3 + \ldots. \end{aligned}$$

$\therefore$ the n^{th} term is of the form $A + Bn + Cn^2 + Dn^3$.

To determine A, B, C, D put n successively $= 1, 2, 3, 4$.

$$\therefore \left.\begin{aligned} A + B + C + D &= -15 \\ A + 2B + 4C + 8D &= -13 \\ A + 3B + 9C + 27D &= -6 \\ A + 4B + 16C + 64D &= 9 \end{aligned}\right\}.$$

To solve these equations it will be found best in practice to eliminate the successive quantities by subtracting each equation from the following one. Thus eliminating A we get

$$\left.\begin{aligned} B + 3C + 7D &= 2 \\ B + 5C + 19D &= 7 \\ B + 7C + 37D &= 15 \end{aligned}\right\}.$$

Eliminating B in a similar manner we get

$$\left.\begin{aligned} 2C + 12D &= 5 \\ 2C + 18D &= 8 \end{aligned}\right\}.$$

Thus we get $D = \frac{1}{2}$, $C = -\frac{1}{2}$, $B = 0$, $A = -15$.

$$\therefore \text{ the } n^{\text{th}} \text{ term is } -15 - \tfrac{1}{2}n^2 + \tfrac{1}{2}n^3.$$

Thus we see that the given series is the modified sum of three other series in which the n^{th} terms are respectively $1, n^2, n^3$.

$\therefore$ applying the results obtained on page 18, we find for the sum of n terms

$$S_n = -15n - \frac{n(n+1)(2n+1)}{12} + \frac{n^2(n+1)^2}{8}.$$

III. Sometimes we find that instead of arriving at a series of equal terms, we finally get a geometric series

$$\begin{aligned} \textit{Ex.}\text{—}S &= 1 - 8 - 18 - 20 + 4 + 90 + 310 + \ldots \\ S_1 &= -9 - 10 - 2 + 24 + 86 + 220 + \ldots \\ S_2 &= -1 + 8 + 26 + 62 + 134 + \ldots \\ S_3 &= 9 + 18 + 36 + 72 + \ldots \end{aligned}$$

$\therefore$ The n^{th} term is of the form

$$A + Bn + Cn^2 + D \,.\, 2^{n-1}.$$

Determine A, B, C, D as before by giving n the successive values 1, 2, 3, 4, and we find $A = 0$, $B = -3$, $C = -5$, $D = 9$.

$\therefore$ the n^{th} term is $9 \,.\, 2^{n-1} - 3n - 5n^2$.

$\therefore$ the sum of n terms

$$S_n = 9(2^n - 1) - \frac{3n(n+1)}{2} - \frac{5n(n+1)(2n+1)}{6}\,.$$

IV. Now consider the series

$$2 + 1 + 1 + 7 + 37 + 151 + 541 + \ldots.$$

Here we find on trial that the preceding methods give us no information concerning the n^{th} term. It is best, therefore, to introduce x and treat the series as a recurring series.

Thus

$$\begin{aligned}
S &= 2 + x + x^2 + 7x^3 + 37x^4 + 151x^5 + 541x^6 + \ldots \\
pxS &= \quad 2px + px^2 + px^3 + 7px^4 + 37px^5 + 151px^6 + \ldots \\
qx^2S &= \qquad 2qx^2 + qx^3 + qx^4 + 7qx^5 + 37qx^6 + \ldots \\
rx^3S &= \qquad\quad 2rx^3 + rx^4 + rx^5 + 7rx^6 + \ldots
\end{aligned}$$

Add, and assume that the coefficients of x^3, x^4, and x^5 vanish

$$\therefore \left.\begin{aligned}
2r + q + p + 7 &= 0 \\
r + q + 7p + 37 &= 0 \\
r + 7q + 37p + 151 &= 0
\end{aligned}\right\}.$$

From these equations we get $p = -6$, $q = 11$, $r = -6$.

By trial we find that these values of p, q, r make the coefficient of x^6 vanish. Therefore we know that

$$S = \frac{2 + x(1 + 2p) + x^2(1 + p + 2q)}{1 + px + qx^2 + rx^3}.$$

$$= \frac{2 - 11x + 17x^2}{1 - 6x + 11x^2 - 6x^3} = \frac{2 - 11x + 17x^2}{(1 - x)(1 - 2x)(1 - 3x)}.$$

Resolving this expression into partial fractions, we find

$$S = \frac{4}{1 - x} - \frac{3}{1 - 2x} + \frac{1}{1 - 3x}.$$

$\therefore$ the n^{th} term of the given series is

$$4 - 3 \cdot 2^{n-1} + 3^{n-1},$$

$\therefore$ the sum of n terms

$$S_n = 4n + \frac{3^n - 1}{3 - 1} - 3 \cdot \frac{2^n - 1}{2 - 1},$$

$$= \frac{5}{2} + 4n + \frac{3^n}{2} - 3 \cdot 2^n.$$

It is easy to see that II. and III. might be treated as recurring series, but in practice we should often find a difficulty in resolving the scale of relation into its simple factors. The cases in which the methods given in II. and III. do not help us are those in which the n^{th} term involves the sum of the n^{th} terms of two or more geometric series.

V. Consider the series

$$\frac{14}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} + \frac{22}{2 \cdot 3 \cdot 4} \cdot \frac{1}{2^2} + \frac{32}{3 \cdot 4 \cdot 5} \cdot \frac{1}{2^3} + \frac{44}{4 \cdot 5 \cdot 6} \cdot \frac{1}{2^4}$$
$$+ \frac{58}{5 \cdot 6 \cdot 7} \cdot \frac{1}{2^5} + \cdots$$

By methods employed in II. and III. we find the law of formation of the numerators to be

$$n^2 + 5n + 8.$$

$\therefore$ the n^{th} term is

$$\frac{n^2 + 5n + 8}{n(n+1)(n+2)} \cdot \frac{1}{2^n}.$$

Assume this to be of the form $u_n - u_{n-1}$

$$\text{and} = \frac{An + B}{(n+1)(n+2)} \cdot \frac{1}{2^n} - \frac{A(n-1) + B}{n(n+1)} \cdot \frac{1}{2^{n-1}}.$$

$$\begin{aligned}\therefore\ n^2 + 5n + 8 &\equiv n(An + B) - 2(n+2).\{A(n-1) + B\} \\ &\equiv -An^2 - (2A + B)n + 4A - 4B,\end{aligned}$$

$\therefore$ equating coefficients of like powers of n, we find that the values $A = -1$, $B = -3$ satisfy this identity.

$\therefore$ the n^{th} term

$$= -\frac{n+3}{(n+1)(n+2)} \cdot \frac{1}{2^n} + \frac{n+2}{n(n+1)} \cdot \frac{1}{2^{n-1}},$$

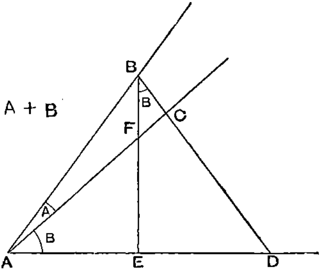
and the 1^{st} term

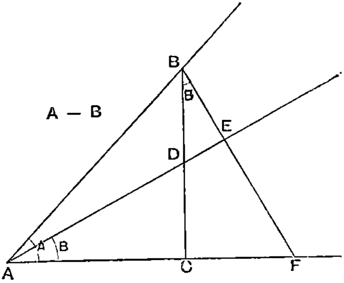
$$= -\frac{4}{2\,.\,3} \cdot \frac{1}{2} + \frac{3}{1\,.\,2} \cdot \frac{1}{2^0},$$

$\therefore$ the sum of the series to n terms is

$$-\frac{n+3}{(n+1)(n+2)} \cdot \frac{1}{2^n} + \frac{3}{2}.$$

II. Trigonometry.





1. BC is perpendicular to AC and BE to AD.

$$\sin(A \pm B) = \frac{BE}{AB} = \frac{BE \,.\, AD}{AB \,.\, AD} \overset{(1)}{=} \frac{BD \,.\, AC}{AB \,.\, AD} = \frac{(BC \pm CD)AC}{AB \,.\, AD}$$

$$= \frac{BC}{AB} \cdot \frac{AC}{AD} \pm \frac{AC}{AB} \cdot \frac{CD}{AD},$$

$$= \sin A \cos B \pm \cos A \sin B,$$

$$\cos(A \pm B) = \frac{AE}{AB} = \frac{AE \,.\, AF}{AB \,.\, AF} = \frac{AE(AC \mp CF)}{AB \,.\, AF}$$

$$\overset{(2)}{=} \frac{AC}{AB} \cdot \frac{AE}{AF} \mp \frac{BC \,.\, EF}{AB \,.\, AF},$$

$$= \cos A \cos B \mp \sin A \sin B,$$

$$\tan(A \pm B) = \frac{BE}{AE} \overset{(3)}{=} \frac{BD}{AF} = \frac{BC \pm CD}{AC \mp CF} = \frac{\dfrac{BC}{AC} \pm \dfrac{CD}{AC}}{1 \mp \dfrac{BC}{AC} \cdot \dfrac{CF}{BC}},$$

$$= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

$$\cot(A \pm B) = \frac{AE}{BE} = \frac{AF}{BD} = \frac{AC \mp CF}{BC \pm CD}$$

$$= \frac{\frac{AC}{CB} \cdot \frac{CB}{CF} \mp 1}{\frac{BC}{CF} \pm \frac{CD}{CF}\left(= \frac{CA}{CB}\right)} = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}.$$

(1) From similar triangles

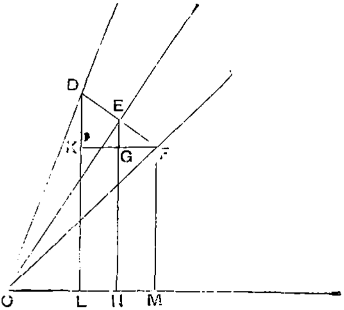
BED, ACD, $BE : BD :: AC : AD \therefore BE \,.\, AD = BD \,.\, AC.$

(2) From similar triangles

AEF, BCF, $AE : EF :: BC : CF \therefore AE \,.\, CF = BC \,.\, EF.$

(3) From similar triangles

BED, AEF, $BE : BD :: AE : AF.$



2. Denote MOD by A and MOF by B, and bisect DOF by OE.

Then $MOE = \frac{1}{2}(A + B)$, $EOD = \frac{1}{2}(A - B)$

DEF is perpendicular to OE. Then $OD = OF$ and $DE = EF$.

Then $DL + FM = 2EN$,

$$\therefore \frac{DL}{DO} + \frac{FM}{FO} = 2\frac{EN}{EO} \cdot \frac{EO}{OD},$$

$$\therefore \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}.$$

$$OL + OM = 2\,ON,$$

$$\therefore \frac{OL}{OD} + \frac{OM}{OF} = 2\frac{ON}{OE} \cdot \frac{OE}{OD},$$

$$\therefore \cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}.$$

$$DL - FM = KD = 2\,EG,$$

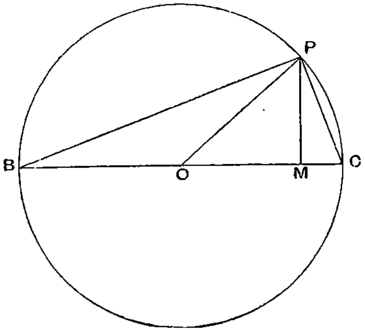
$$\therefore \frac{DL}{OD} - \frac{FM}{OF} = 2\frac{EG}{EF} \cdot \frac{ED}{OD},$$

$$\therefore \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}.$$

$$OM - OL = LM = 2\,MN = 2\,FG.$$

$$\therefore \frac{OM}{OF} - \frac{OL}{OD} = 2\frac{FG}{EF} \cdot \frac{ED}{OD},$$

$$\therefore \cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$



3. Let O be the centre of a circle, and let PBC be denoted by A. Then $POC = 2A$ and $CPM = A$.

$$\sin 2A = \frac{PM}{OP} = \frac{2PM}{BC} = 2\frac{PM}{PB}\cdot\frac{PB}{BC}$$
$$= 2\sin A\cos A.$$

$$\cos 2A = \frac{OM}{OP} = \frac{BM - BO}{OB} = 2\frac{BM}{BP}\cdot\frac{BP}{BC} - 1$$
$$= 2\cos^2 A - 1.$$

$$\cos 2A = \frac{OM}{OP} = \frac{OC - MC}{OP} = 1 - 2\frac{MC}{PC}\cdot\frac{PC}{BC}$$
$$= 1 - 2\sin^2 A.$$

$$\cos 2A = \frac{2OM}{2OP} = \frac{OM + OC - CM}{BC} = \frac{BM - MC}{BC}$$
$$= \frac{BM}{BP}\cdot\frac{BP}{BC} - \frac{MC}{PC}\cdot\frac{PC}{BC}$$
$$= \cos^2 A - \sin^2 A.$$

$$\tan 2A = \frac{2PM}{2OM} = \frac{2PM}{BM - MC} = \frac{2\dfrac{PM}{BM}}{1 - \dfrac{MC}{MP}\cdot\dfrac{MP}{MB}}$$
$$= \frac{2\tan A}{1 - \tan^2 A}.$$

$$\cot 2A = \frac{2OM}{2PM} = \frac{BM - CM}{2PM} = \frac{\dfrac{BM}{PM}\cdot\dfrac{PM}{CM} - 1}{2\dfrac{PM}{CM}}$$
$$= \frac{\cot^2 A - 1}{2\cot A}.$$

$$\tan A = \frac{PM}{BM} = \frac{PM}{BO + OM} = \frac{\frac{PM}{OP}}{1 + \frac{OM}{OP}}$$

$$= \frac{\sin 2A}{1 + \cos 2A}.$$

$$\cot A = \frac{PM}{CM} = \frac{PM}{OC - OM} = \frac{\frac{PM}{OP}}{1 - \frac{OM}{OP}}$$

$$= \frac{\sin 2A}{1 - \cos 2A}.$$

$$\sin 2A = \frac{2PM}{2OP} = \frac{2PM}{BM + MC} = \frac{2\frac{PM}{BM}}{1 + \frac{MC}{MP} \cdot \frac{MP}{MB}}$$

$$= \frac{2 \tan A}{1 + \tan^2 A}.$$

$$\cos 2A = \frac{2OM}{2OP} = \frac{BM - CM}{BM + CM} = \frac{1 - \frac{CM}{PM} \cdot \frac{PM}{MB}}{1 + \frac{CM}{PM} \cdot \frac{PM}{MB}}$$

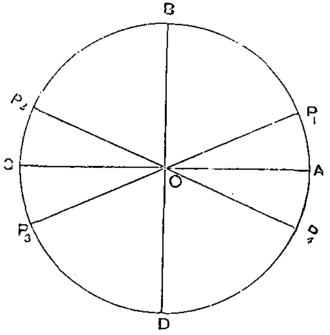
$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

$$(\sin A + \cos A)^2 = \frac{(PM + BM)^2}{PB^2} = 1 + 2\frac{PM}{PB} \cdot \frac{BM}{PB}$$

$$= 1 + 2\frac{PM}{PB} \cdot \frac{PB}{BC} = 1 + \frac{2PM}{2OP}$$

$$= 1 + \sin 2A.$$

So $(\sin A - \cos A)^2 = 1 - \sin 2A.$



4. To prove geometrically that in finding $\sin A$ and $\cos A$ from $\cos 2A$ two values, and from $\sin 2A$ four values must be found.

Let $2a$ be the least positive angle which has its cosine equal to the given value of $\cos 2A$.

Then $$2A = 2n\pi \pm 2a$$

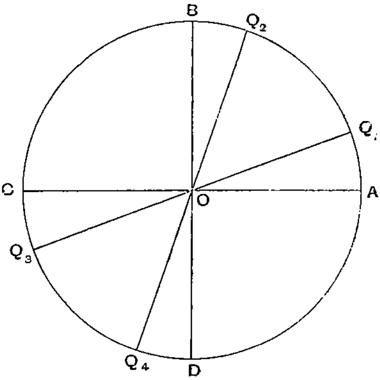
$$\therefore A = n\pi \pm a$$

$\therefore$ if AOP_1 be the angle a, and if we take $AOP_4 = AOP_1$, and produce OP_1 and OP_4 to OP_3 and OP_2, we see that all the angles which are included in the formula $n\pi \pm a$ are bounded by the lines OP_1, OP_2, OP_3, OP_4. And obviously

$$\sin AOP_1 = \sin AOP_2 = -\sin AOP_3 = -\sin AOP_4;$$

$$\cos AOP_1 = -\cos AOP_2 = -\cos AOP_3 = \cos AOP_4.$$

Thus we see that $\sin A$ and $\cos A$ have each two values, which are equal in magnitude and opposite in sign.



5. Again, let 2β be the least positive angle which has its sine equal to the given value of $\sin 2A$.

Then $$2A = n\pi + (-1)^n 2\beta$$

$$\therefore A = n \cdot \frac{\pi}{2} + (-1)^n \beta.$$

If AOQ_1 be the angle β, and if we take $AOQ_2 = \frac{\pi}{2} - \beta$, and produce Q_1O and Q_2O to OQ_3 and OQ_4, we see that all the angles which are included in the formula $n\frac{\pi}{2} + (-1)^n\beta$ are bounded by the lines OQ_1, OQ_2, OQ_3, OQ_4, which lie by pairs in opposite quadrants. And obviously

$$\sin AOQ_1 = -\sin AOQ_3; \ \sin AOQ_2 = -\sin AOQ_4$$
$$\cos AOQ_1 = -\cos AOQ_3; \ \cos AOQ_2 = -\cos AOQ_4.$$

Thus we see that $\sin A$ and $\cos A$ have each four values, which by pairs are equal in magnitude and opposite in sign. We also see that when $a = \frac{\pi}{4}$, OQ_1 coincides with

OQ_2, and OQ_3 with OQ_4, and we then get only two values, equal in magnitude and opposite in sign.

---

6. To resolve $x^n - 2\cos na + \dfrac{1}{x^n}$ into factors without employing imaginary quantities.

The relation between the successive values of $x^m + \dfrac{1}{x^m}$ corresponding to successive integral values of m is

$$x^{m+1} + \frac{1}{x^{m+1}} = \left(x + \frac{1}{x}\right)\left(x^m + \frac{1}{x^m}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right).$$

When $m = 1$, this becomes

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) - 2.$$

An exactly similar relation holds good between the successive values of $2\cos m\theta$, thus

$$2\cos(m+1)\theta = (2\cos\theta)(2\cos m\theta) - 2\cos(m-1)\theta.$$

When $m = 1$, this becomes

$$2\cos 2\theta = (2\cos\theta)(2\cos\theta) - 2.$$

Now let $r_0, r_1, r_2 \ldots . r_n$, be a series of quantities, the successive terms of which are connected by the above relation,

$$\text{viz. } r_{n+1} = r_1 r_n - r_{n-1}.$$

Also, as in the above cases, let $r_0 = 2$, but let r_1 be any quantity whatever.

Then we have

$$r_2 = r_1 r_1 - 2 = r_1^2 - 2,$$

$$r_3 = r_1 r_2 - r_1 = r_1^3 - 3r_1, \&c.$$

Then we see that

(1) r_n is a definite integral function of r_1 of n dimensions, and the coefficient of r_1^n in it is unity.

(2) if $r_1 = x + \frac{1}{x}$, $r_n = x^n + \frac{1}{x^n}$.

(3) if $r_1 = 2\cos\theta$, $r_n = 2\cos n\theta$.

Hence $r_n - 2\cos na$ will vanish when $r_1 =$ any of the n quantities

$$2\cos a, \quad 2\cos\left(a + \frac{2\pi}{n}\right),$$

$$2\cos\left(a + 2 \cdot \frac{2\pi}{n}\right) \dots . 2\cos\left(a + \overline{n-1} \cdot \frac{2\pi}{n}\right),$$

$$\therefore r_n - 2\cos na = \left\{r_1 - 2\cos a\right\}\left\{r_1 - 2\cos\left(a + \frac{2\pi}{n}\right)\right\}$$

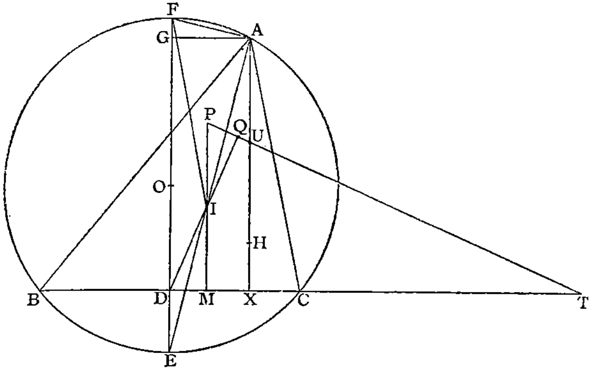
$$\dots . \left\{r_1 - 2\cos\left(a + \overline{n-1} \cdot \frac{2\pi}{n}\right)\right\}$$

for all values whatever of r_1. Let $r_1 = x + \frac{1}{x}$. Then $r_n = x^n + \frac{1}{x^n}$

$$\therefore x^n - 2\cos na + \frac{1}{x^n}$$

$$= \left\{x - 2\cos a + \frac{1}{x}\right\}\left\{x - 2\cos\left(a + \frac{2\pi}{n}\right) + \frac{1}{x}\right\} \dots .$$

7. To prove geometrically that the nine-point circle of a triangle touches the inscribed circle. (Feuerbach's Theorem.)



Let $AUHX$ be the perpendicular from A on BC, H the orthocentre, O the circumcentre, I the incentre, U the mid-point of HA, and D the mid-point of BC.

Then DU is a diameter of the nine-point circle, and if UQ be drawn perpendicular to DI, meeting DI, IM, BC respectively in Q, P, T, then Q is also evidently a point on the N.P. circle.

Since DI, UT are at right angles, the triangles DIM, UTX are similar.

$$\therefore\ DM \,.\, XT = IM \,.\, UX = r \,.\, OG \quad . \quad . \quad . \quad (1)$$

Since $EI^2 = EC^2 = ED \,.\, EF$, the angles EDI, EIF are equal, and

$$IM : DM = AI : AF = MX : FG$$

$$\therefore\ DM \,.\, MX = r \,.\, FG \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$\therefore$ by addition of results (1) and (2),

$$R \,.\, r = DM \,.\, MT = IM \,.\, PM = r \,.\, PM,$$

$$\therefore\ PM = R.$$

$$\therefore\ DI \,.\, IQ = PI \,.\, IM = r(R - r),$$

i.e. in the nine-point circle the rectangle of the segments of a chord through I, the centre of the incircle, is equal to $r(R - r)$.

Now the radius of the nine-point circle is equal to $\frac{1}{2}R$.

$\therefore$ if N be its centre, we have

$$(\tfrac{1}{2}R)^2 - NI^2 = r(R - r),$$

$$\therefore\ NI^2 = (\tfrac{1}{2}R - r)^2,$$

$$\therefore\ NI = \tfrac{1}{2}R - r.$$

i.e. the distance between the centres of the N.P. and in-circles is equal to the difference between their radii, and $\therefore$ the circles touch.

To expand $\sin^{-1} x$ in ascending powers of x.

Assume $\sin^{-1}x = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \ldots .$

when $x = 0$, $\sin^{-1} x = 0$, $\therefore\ A_0 = 0$.

And, since $\sin^{-1}(-x) = -\sin^{-1} x$,

$$\therefore\ - A_1 x + A_2 x^2 - A_3 x^3 + A_4{}^4 - \ldots .$$
$$\equiv - A_1 x - A_2 x^2 - A_3 x^3 - A_4 x^4 - \ldots .$$

$\therefore$ all the coefficients with even suffixes vanish

$\therefore \sin^{-1} x \equiv A_1 x + A_3 x^3 + A_5 x^5 + \ldots.$

$$\therefore x \equiv \sin\left\{A_1 x + A_3 x^3 + A_5 x^5 + \ldots.\right\}$$

$$\equiv A_1 x + A_3 x^3 + \ldots. - \frac{1}{\underline{|3}}\left\{A_1 x + A_3 x^3 + \ldots.\right\}^3$$

$$+ \frac{1}{\underline{|5}}\left\{A_1 x + \ldots.\right\}^5 - \ldots.$$

$$\equiv A_1 x + \left(A_3 - \frac{A_1^3}{\underline{|3}}\right)x^3 + \left(A_5 - \frac{3}{\underline{|3}}A_1^2 A_3 + \frac{A_1^5}{\underline{|5}}\right)x^5 + \ldots.$$

$\therefore$ equating coefficients of like powers of x, we get

$$A_1 = 1,\ A_3 - \frac{A_1^3}{\underline{|3}} = 0, \qquad \therefore A_3 = \frac{1}{2\,.\,3}$$

$$A_5 = \frac{3}{\underline{|3}}\,.\,A_1^2 A_3 - \frac{A_1^5}{\underline{|5}}, \qquad \therefore A_5 = \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5}\ \&c.$$

$$\therefore \sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{x^7}{7} + \ldots.$$

To expand $(\sin^{-1} x)^2$ we have

$$(\sin^{-1} x)^2 = \left\{x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{x^5}{5} + \ldots.\right\}^2$$

$$= x^2 + \frac{x^4}{3} + x^6\left(\frac{1}{36} + \frac{3}{20}\right) + x^8\left(\frac{1.3.5}{4.6.7} + \frac{1}{2.4.5}\right) + \ldots$$

$$= x^2 + \frac{4}{3}\cdot\frac{x^4}{4} + \frac{16}{3\,.\,5}\cdot\frac{x^6}{6} + \frac{32}{5\,.\,7}\cdot\frac{x^8}{8} + \ldots.$$

$$= 2\left\{\frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2\,.\,4}{3\,.\,5}\cdot\frac{x^6}{6} + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\cdot\frac{x^8}{8} + \ldots.\right\}$$

Thus we see that writing the expansion of $\sin^{-1}x$ in the form

$$1\left\{\frac{x^1}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\cdots\right\}$$

if we increase by unity every number which occurs, we get the expansion of $(\sin^{-1}x)^2$.

Paper I.

1. The debts of a bankrupt amount to £2,134 10*s.* 6*d.* and his assets consist of property worth £916 15*s.* 4*d.* and an undiscounted bill of £513 due 4 months hence, simple interest being reckoned at 4 per cent. How much in the pound can he pay his creditors?

2. If $s = a + b + c$, prove that

$$(as + bc)(bs + ca)(cs + ab) = (b + c)^2 (c + a)^2 (a + b)^2.$$

3. Divide $2x^3 - 6x + 5$ by $\sqrt[3]{2}\,.\,x + \sqrt[3]{4} + 1$.

4. Shew that

$$\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta)$$
$$= 1 + 2\cos(\beta - \gamma)\cos(\gamma - \alpha)\cos(\alpha - \beta).$$

5. If p, q, r be the bisectors of the angles of a triangle, prove that

$$\frac{\cos\frac{A}{2}}{p} + \frac{\cos\frac{B}{2}}{q} + \frac{\cos\frac{C}{2}}{r} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

6. Two circles touch at C, and a point D is taken without them such that the radii AC, CB subtend equal angles at D. If DE, DF be tangents, prove geometrically that $DE\,.\,DF = DC^2$.

7. At any point P of an ellipse the tangent PT and the normal are drawn, S and H are the foci, and through the

centre C a line is drawn parallel to SP cutting the normal in R and the tangent in T. Shew that $RT = HP$.

PAPER II.

1. Find the G.C.M. of

$$nx^{n+1} - (n+1)x^n + 1 \text{ and } x^n - nx + n - 1.$$

2. Solve the equations

(1) $\dfrac{ax+b}{cx+b} + \dfrac{bx+a}{cx+a} = \dfrac{(a+b)(x+2)}{cx+a+b}$,

(2) $2x\sqrt{x^2+a^2} + 2x\sqrt{x^2+b^2} = a^2 - b^2$,

(3) $ax^2 + bxy + cy^2 = bx^2 + cxy + ay^2 = d$.

3. In any triangle prove that

$$\frac{ab - r_1 r_2}{r_3} = \frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2}$$

where a, b, c are the sides, r_1, r_2, r_3 the radii of the escribed circles.

4. If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in A.P., prove that

$$4(1 - \cos\theta)(1 - \cos\phi) = \cos\theta + \cos\phi.$$

5. Inscribe in a circle geometrically a triangle whose sides shall be parallel to three given straight lines.

6. Any three tangents to a parabola, the tangents of whose inclinations to the axis are consecutive terms of a fixed H.P. will form a triangle of constant area.

7. A square rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a string whose length is equal to a side of the square. Shew that the distances of 3 of its angular points from the wall are as 1 : 3 : 4.

PAPER III.

1. In the expansion of $(1 - x)^{-\frac{1}{n}}$ prove that the sum of the coefficients of the first r terms bears to the coefficient of the r^{th} term the ratio of $1 + n(r - 1)$ to 1.

2. From the formula

$$\log_e\left(\frac{1}{1 - x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdot \;\cdot\cdot$$

calculate $\log_{10} 5$ to 5 places of decimals, given $\log_{10} e =$ ·43429.

3. Eliminate θ between

$$m = \operatorname{cosec}\theta - \sin\theta \text{ and } n = \sec\theta - \cos\theta,$$

and shew that $m^{\frac{2}{3}} + n^{\frac{2}{3}} = (mn)^{-\frac{2}{3}}$.

4. Shew that the sum of n terms of the series

$$1 + \frac{\cos\alpha}{\cos\alpha} + \frac{\cos 2\alpha}{\cos^2\alpha} + \frac{\cos 3\alpha}{\cos^3\alpha} + \cdots\cdot$$

is equal to zero if $n\alpha = \pi$.

5. $ABCD$ is a quadrilateral figure inscribed in a circle. AB, DC meet in E, BC, AD meet in F. Shew that the circle on EF as diameter cuts the circle $ABCD$ orthogonally.

6. A, B, C are three points on the circumference of a circle. Forces act along AB, BC, inversely proportional to these straight lines. Shew that their resultant acts along the tangent at B.

7. OQ, OQ' are two tangents to a parabola. The diameter through O meets the parabola in P, and the tangent at P meets OQ, OQ' in R, R' respectively. Shew that QR', $Q'R$ are divided by the parabola in the ratio of 8 to 1.

PAPER IV.

1. If x, y, z be unequal, and if

$$2a - 3y = \frac{(z-x)^2}{y} \text{ and } 2a - 3z = \frac{(x-y)^2}{z},$$

then will

$$2a - 3x = \frac{(y-z)^2}{x}, \text{ and } x + y + z = a.$$

2. If α, β, γ be the angles of a triangle, prove that

$$\cos\left(\frac{3\beta}{2} + \gamma - 2\alpha\right) + \cos\left(\frac{3\gamma}{2} + \alpha - 2\beta\right) + \cos\left(\frac{3\alpha}{2} + \beta - 2\gamma\right)$$
$$= 4\cos\frac{5\alpha - 2\beta - \gamma}{4}\cos\frac{5\beta - 2\gamma - \alpha}{4}\cos\frac{5\gamma - 2\alpha - \beta}{4}.$$

3. If a line join the points where an escribed circle touches the produced sides of a triangle, and corresponding lines be drawn for the other escribed circles so as to form an outer triangle; and if from the outer triangle another triangle be formed in the same way, and so on, prove that these triangles tend to become equiangular.

4. Forces act at the middle points of the sides of a triangle at right angles to the sides and respectively proportional to them. Shew that if they all act inwards or outwards, they are in equilibrium.

5. Explain the fallacy in the following reasoning.

Since $e^{2n\pi\sqrt{-1}} = \cos 2n\pi + \sqrt{-1}\sin 2n\pi$,

$$\therefore e^{2\pi\sqrt{-1}} = e^{4\pi\sqrt{-1}} = e^{6\pi\sqrt{-1}} = \dots.$$

Raise each to the power $\sqrt{-1}$. $\therefore e^{-2\pi} = e^{-4\pi} = e^{-6\pi}$, which is not true.

6. Two circles whose radii are a, b, cut one another at an angle α. Prove that the length of the common chord is

$$\frac{2ab\sin\alpha}{\sqrt{a^2 + 2ab\cos\alpha + b^2}}.$$

7. If CR, SY, HZ be perpendiculars upon the tangent to an ellipse at a point P such that $CR = CS$, shew that

(1) R lies on the tangent at B,

(2) the perpendicular from R on SH will divide it into two parts equal to SY, HZ respectively,

(3) $SP : HP :: SR^2 : HR^2$.

Paper V.

1. Find to two places of decimals the cube root of 1037.

2. If $a + \dfrac{bc - a^2}{a^2 + b^2 + c^2}$ be not altered in value by interchanging a pair of the letters a, b, c not equal to each other, it will not be altered by interchanging any other pair; and it will vanish if $a + b + c = 1$.

3. Form the equations whose roots are respectively the squares and the square roots of the equation

$$ax^2 + bx + c = 0.$$

4. Solve the equation $\sec 4\theta - \sec 2\theta = 2$.

5. $ABCD$ is the rectangular floor of a room whose length AB is a feet. Find its height, which at C subtends at A an angle α, and at B an angle β. If $a = 48$ ft., $\alpha = 18°$, $\beta = 30°$, prove that the height is nearly 18 ft. 10 in.

6. Find the equation to the two parabolas having their focus at any given point P of the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and passing through the foci of the conic. Also, shew that the angle between the directrices is twice the eccentric angle of P.

7. ACB is a diameter of a circle of which C is the centre, and in AC a point D is taken, such that the rectangle $AC \cdot AD$ is equal to the square on CD. If the circle described with centre B and radius BD cuts the given circle in E, prove that AE is one-fifth of the circumference.

PAPER VI.

1. If the equations

$$\left.\begin{aligned} ax + by &= 1 \\ cx^2 + dy^2 &= 1 \end{aligned}\right\}$$

have only one solution, prove that

$$\frac{a^2}{c} + \frac{b^2}{d} = 1, \text{ and } x = \frac{a}{c}, \ y = \frac{b}{d}.$$

2. Given $\log_8 9 = a$, $\log_3 5 = b$, find the logarithms to the base 10 of the first 4 digits.

3. Shew that if $A + B + C$ be an odd multiple of π,

$$\sin^2 B + \sin^2 C = \sin^2 A + 2 \cos A \sin B \sin C.$$

4. D, E, F are the feet of the perpendiculars from the angular points on the sides of the triangle ABC. Shew that the radius of the circle inscribed in the triangle DEF is $2R \cos A \cos B \cos C$, where R is the radius of the circle circumscribing ABC.

5. A uniform heavy rod is placed across a smooth horizontal rail, and rests with one end against a smooth vertical wall, the distance of which from the rail is $\frac{1}{16}$ of the length of the rod. Find the angle which the rod makes with the horizon when there is equilibrium.

6. A, B, C, D are four points in space: the straight lines AB and DC are divided in the same ratio in the points E, F. AD and BC are divided in another the same ratio in G and H. Prove that the straight lines EF and GH lie in one plane.

7. If two chords of a rectangular hyperbola be at right angles, each of their four extremities is the orthocentre of the triangle formed by the other three.

PAPER VII.

1. If the equation

$$\frac{a}{x+a}+\frac{b}{x+b}=\frac{c}{x+c}+\frac{d}{x+d}$$

have a pair of equal roots, then either one of the quantities a or b is equal to one of the quantities c or d, or else

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}.$$

Prove also that the roots are then

$$-a,\ -a,\ 0;\ -b,\ -b,\ 0;\ \text{or } 0,\ 0,\ -\frac{2ab}{a+b}.$$

2. Employ the Binomial Theorem to shew that

$$\frac{23}{24}-\frac{2}{3}\sqrt{2}=\frac{1}{2^3\underline{|3}}-\frac{1\cdot 3}{2^4\underline{|4}}+\frac{1\cdot 3\cdot 5}{2^5\underline{|5}}-\cdots.$$

3. Given that

$$x=y\cos R+z\cos Q$$
$$y=z\cos P+x\cos R$$

and that $P+Q+R$ is an odd multiple of π, prove that

$$z=x\cos Q+y\cos P.$$

Hence also prove that $\cos P=\dfrac{y^2+z^2-x^2}{2yz}$.

4. Shew that the length of the perpendicular from the centre of the nine point circle on BC is $\frac{1}{2}R\cos\overline{C-B}$.

5. A', B', C' are the middle points of the sides of the triangle ABC, and through A, B, C are drawn three parallel straight lines meeting $B'C'$, $C'A'$, $A'B'$ in a, b, c respectively. Prove that the triangle abc is half the triangle ABC, and that bc passes through A, ca through B, and ab through C.

6. Two points (ξ, η), (x, y) are connected by the relation

$$u = \frac{az + b}{cz + d} \text{ where } u \equiv \xi + \eta\sqrt{-1},\ z = x + y\sqrt{-1},$$

($x, y, \xi, \eta, a, b, c, d$ being all real). Shew that when (x, y) describes a circle, (ξ, η) describes another circle.

7. From a point O in a parabola OQ is drawn perpendicular to the diameter at P. Prove that the straight line drawn from Q perpendicular to the tangent at P will meet the normal at O on the axis of the parabola.

Paper VIII.

1. Given $\log_8 9 = a$, $\log_3 5 = b$, $\log_5 7 = c$, find the logarithms to the base 10 of the digits 5, 6, 7, 8, 9.

2. Shew that $\frac{1}{55}\{3 \cdot 10^x - 25(-1)^x\}$ is a positive integer when x is a positive integer.

3. A circle, centre O and radius r, is inscribed in a triangle ABC, and touches the sides in D, E, F. Circles are inscribed in the quadrilaterals $AEOF$, $BFOD$, $CDOE$. If r_1, r_2, r_3 be their radii, prove that

$$\frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1}{r - r_1} \cdot \frac{r_2}{r - r_2} \cdot \frac{r_3}{r - r_3}.$$

4. $ABCD$ is a quadrilateral. Four circles, centres M, N, P, Q, are described so as to touch respectively the consecutive sides CD, DA, AB; DA, AB, BC; AB, BC, CD; BC, CD, DA. Shew that a circle can be described about the figure $MNPQ$.

5. From the centre of an ellipse a perpendicular is drawn to the tangent at any point P, meeting it in Y. From Y another tangent is drawn meeting the ellipse in P'. If $P'Q$ be a diameter shew that PQ is normal at P.

6. $ABCD$ is a quadrilateral. Forces act along the sides AB, BC, CD, DA, measured by α, β, γ, δ times those sides respectively. Shew that if there is equilibrium, $\alpha\gamma = \beta\delta$.

Shew also that $\dfrac{\Delta ABD}{\Delta ABC} = \dfrac{\alpha}{\delta} \cdot \dfrac{\gamma - \beta}{\beta - \alpha}$.

7. Shew that the straight lines which bisect the angles between the two lines

$$(h - a)y = k(x - a)\,;\ (h + a)y = k(x + a)$$

have for their equation

$$\left\{hk(x - h)^2 - (y - k)^2\right\} = (x - h)\,(y - k)\,(h^2 - k^2 - a^2).$$

Paper IX.

1. Prove that $cx^2 - ax + b$ will be a common divisor of $ax^3 - bx^2 + c$, and $bx^3 - cx + a$ if it will divide either of them.

2. Shew that the sum to n terms of the series

$$\frac{2^2}{1^2(1^2+2^2)} + \frac{3^2}{(1^2+2^2)(1^2+2^2+3^2)}$$
$$+ \frac{4^2}{(1^2 + 2^2 + 3^2)\,(1^2 + 2^2 + 3^2 + 4^2)} + \dots$$
$$\text{is } 1 - \frac{6}{(n + 1)\,(n + 2)\,(2n + 3)}.$$

3. If

$$\sin A = p\sin B,\ \cos A = q\cos B,\ \sin A + \cos A = r(\sin B + \cos B)$$

prove that

$$(p - r)^2\,(1 - q^2) + (q - r)^2\,(1 - p^2) = 0.$$

4. If p, q, r be the perpendiculars on the sides of a triangle ABC from the centre of the circumscribed circle, prove that

$$\frac{qr}{bc} + \frac{rp}{ca} + \frac{pq}{ab} = \frac{1}{4}.$$

5. $ABCD$ is a quadrilateral. The sides DA, CB; AB, DC are produced to meet in F and E. If the two bisectors of the angles at F are parallel to the two bisectors of the angles at E, prove that a circle will go round $ABCD$.

Also, if the diagonals BD, AC intersect in G, shew that the two bisectors of the angles at G are parallel to the bisectors of the angles at E.

6. If the chord of contact of two tangents to a parabola be normal at one end, the tangent at the other is bisected by the perpendicular through the focus to the line joining the focus to the intersection of the tangents.

7. A series of circles is described touching two given straight lines. Shew that the polars of any point with respect to the circles will envelop a parabola.

Paper X.

1. A number consisting of three digits is doubled by reversing the digits. Prove that the same will hold for the number formed by the first and last digits; and also that such a number can be found in only one scale of notation out of three.

2. If m be a positive integer, shew that

$$3m(3m+1)^2 > 4\sqrt[m]{(\underline{|3m})}.$$

3. If n be a multiple of 6, prove that

$$n - \frac{n(n-1)(n-2)}{\underline{|3}} \cdot 3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}} \cdot 3^2 - \ldots$$

and

$$n - \frac{n(n-1)(n-2)}{\underline{|3}} \cdot \frac{1}{3} + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}} \cdot \frac{1}{3^2} - \ldots$$

will both vanish.

4. O and O' are the centres of two circles which cut in A, A', and two points B, B' are taken one on each circumference. If C, C' be the centres of the circles round BAB', $BA'B'$, prove that the angle CBC' = angle OAO'.

5. If from the point in which a normal to an ellipse meets the major axis a straight line be drawn perpendicular to the normal, the part of either focal distance between this line and the curve will be an harmonic mean between the focal distances.

6. A pack of cards, equal or unequal, stands on the edge of a horizontal table, each card projecting beyond the one just below it. If the highest card project as far as possible from the table, shew that each card is on the point of moving independently of the rest.

7. An isosceles triangle is described, having each of the angles at its base double of the third angle; with the vertex and one extremity of the base as foci, an ellipse is described, passing through the other extremity of the base. Prove that the distance of the vertex of this ellipse from its directrix is equal to half the latus rectum.

Paper XI.

1. If 1000 lbs. can be carried 1000 miles for £1, and the rate of conveyance is the same abroad, find to 2 places of decimals how many kilograms can be carried 100 kilometres for 20 francs.

Given £1 = 25·2 francs; 1 kilom. = 0·6214 miles; 1 kilog. = 2·2046 lbs.

2. If
$$\left.\begin{aligned} xy + \tfrac{1}{2}(x+y)(a+b) + ab &= 0 \\ xy + \tfrac{1}{2}(x+y)(c+d) + cd &= 0 \end{aligned}\right\}$$
prove that
$$\frac{x-y}{2} = \frac{\sqrt{(a-c)(a-d)(b-c)(b-d)}}{a+b-c-d}.$$

3. Shew that

$$\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{4^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{4^3} + - \ldots$$
$$= \frac{4}{3}(2 - \sqrt{3})\sqrt{3}.$$

4. Shew that

$$\operatorname{cosec} A + \operatorname{cosec}\left(A + \frac{2\pi}{3}\right) + \operatorname{cosec}\left(A + \frac{4\pi}{3}\right) = 3 \operatorname{cosec} 3A.$$

5. If

$$y = \tan^{-1} \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}},$$

shew that $x^2 = \sin 2y$.

6. A glass rod is balanced partly in and partly out of a cylindrical tumbler with the lower end resting against the vertical side of the tumbler. If α and β be the greatest and least angles which the rod can make with the vertical, prove that the angle of friction is

$$\tfrac{1}{2} \tan^{-1} \frac{\sin^3\alpha - \sin^3\beta}{\sin^2\alpha \cos\alpha + \sin^2\beta \cos\beta}.$$

7. Prove that the locus of the intersection of tangents to an ellipse which make equal angles with the major and minor axes respectively, and are not at right angles, is a rectangular hyperbola whose vertices are the foci of the ellipse.

PAPER XII.

1. Solve the equations

$$(z + x)(x + y) = a^2$$
$$(x + y)(y + z) = b^2$$
$$(y + z)(z + x) = c^2.$$

2. Shew that the coefficient of x^n in the expansion of

$$\frac{x}{(x-a)(x-b)} \text{ is } \frac{a^n - b^n}{a-b} \cdot \frac{1}{a^n b^n}.$$

3. Prove that if α, β, γ be any three plane angles

$$(\cos\alpha + \cos\beta + \cos\gamma)\{\cos 2\alpha + \cos 2\beta + \cos 2\gamma$$
$$- \cos(\beta+\gamma) - \cos(\gamma+\alpha) - \cos(\alpha+\beta)\}$$
$$- (\sin\alpha + \sin\beta + \sin\gamma)\{\sin 2\alpha + \sin 2\beta + \sin 2\gamma$$
$$- \sin(\beta+\gamma) - \sin(\gamma+\alpha) - \sin(\alpha+\beta)\}$$
$$\equiv \cos 3\alpha + \cos 3\beta + \cos 3\gamma - 3\cos(\alpha+\beta+\gamma).$$

4. The distance between the orthocentre and the centre of the circumscribing circle is

$$R\sqrt{1 - 8\cos A \cos B \cos C}.$$

5. The straight lines *EAB*, *EDC*, and *FDA*, *FCB* form four triangles in one plane, and *O* is the common point of intersection of the circles circumscribing these triangles. Prove that the rectangle $OA \cdot OC$ = rectangle $OE \cdot OF$.

6. A parabolic wire slides through two small rings *A*, *B*. Shew that the vertex corresponding to the chord *AB* will describe the curve $\rho = \mu \sin^2\theta$, the origin being the middle point of *AB*, and μ a constant.

7. A uniform rod hangs by two strings of lengths l, l', fastened to its ends, and to two points in the same horizontal line, distance a apart, the strings crossing each other. Find the position of equilibrium, and shew that if α, α' be the angles which l and l' make with the horizontal,

$$(l'\cos\alpha' - l\cos\alpha)\sin(\alpha+\alpha') = a\sin(\alpha-\alpha').$$

Paper XIII.

1. Shew that

$$a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2$$

is divisible by $a \pm b \pm c$.

2. Solve the equations

(1) $a^4(a^4-1)^4(x^2+14x+1)^3 = (a^8+14a^4+1)^3x(x-1)^4$.

(2) $$\begin{cases} x + y + z = a + b + c \\ \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3 \\ ax + by + cz = bc + ca + ab. \end{cases}$$

3. Shew that

$$\cos(A+B+C)\cos(A+B-C)\cos(B+C-A)\cos(C+A-B)$$
$$+\sin(A+B+C)\sin(A+B-C)\sin(B+C-A)\sin(C+A-B)$$
$$=\cos 2A \cos 2B \cos 2C.$$

4. With the usual notation, shew that

$$a = (r_2 + r_3)\sqrt{\frac{r \cdot r_1}{r_2 \cdot r_3}}.$$

5. If ABC be an isosceles triangle having each of the angles at the base BC double the third angle, and if the bisectors of the angles ACB, ABC meet AB in E, and the circle round the triangle ABC in D, shew that $ADCE$ is a parallelogram.

6. Prove that the angles subtended at the vertices of a rectangular hyperbola by any chord parallel to the conjugate axis are supplementary.

7. A square table stands upon 4 equal legs which are placed at the middle points of the sides. Shew that no weight less than its own when placed upon it can upset it.

Paper XIV.

1. Shew that

$$(x^2 + xy + y^2)(a^2 + ab + b^2)$$
$$= (ax - by)^2 + (ax - by)(ay + bx + by) + (ay + bx + by)^2.$$

Hence shew that the product of any number of factors of the form $x^2 + xy + y^2$ can be put into the form $X^2 + XY + Y^2$.

2. If

$$\frac{a - ny + mz}{l'} = \frac{b - lz + nx}{m'} = \frac{c - mx + ly}{n'},$$

prove that

$$\frac{x - \dfrac{m'c - n'b}{ll' + mm' + nn'}}{l} = \frac{y - \dfrac{n'a - l'c}{ll' + mm' + nn'}}{m}$$
$$= \frac{z - \dfrac{l'b - m'a}{ll' + mm' + nn'}}{n}.$$

3. A quadrilateral $ABCD$ circumscribes a circle radius r. If a, b, c, d, be the lengths of the tangents from A, B, C, D to the circle, prove that

$$r^2 = \frac{bcd + cda + dab + abc}{a + b + c + d}.$$

4. Shew that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2^n - 1)\theta,$$
$$\equiv 2^{n-1}\cos\theta\cos 2\theta\cos 4\theta \ldots \cos 2^{n-1}\theta,$$
$$\equiv \frac{\sin 2^n\theta}{2\sin\theta}.$$

5. A heavy uniform rod is supported in a horizontal position by 3 equal forces, one acting at one end, and the other two at the other. Shew that the angle between the directions of the two latter must be 120°.

6. Prove that a straight line drawn through a focus of an ellipse, at right angles to a chord, intersects the diameter, which is conjugate to the chord, at a point in the directrix.

7. ABC is an equilateral triangle, and AB is produced to D so that BD is twice AB. Shew that the square on CD is seven times the square on AB.

Paper XV.

1. If the number of births and deaths each year be respectively $\frac{1}{60}$th and $\frac{1}{90}$th of the population of a village at the beginning of a year, in how many years will its population be doubled?

Given

$\log 2 = \cdot 3010300$; $\log 180 = 2\cdot 255272$; $\log 181 = 2\cdot 257679$.

2. Eliminate x and y between the equations

$$ax^2 + bxy + cy^2 = 0$$
$$a'x^2 + b'xy + c'y^2 = 0.$$

3. Find approximately the value of x from the equation

$$\left(\tfrac{10}{3}\right)^{x+2} = 9^{2x-1}, \text{ given } \log 3 = \cdot 4771213.$$

4. From the angular points of a triangle ABC lines are drawn through O the centre of the inscribed circle to meet the circumscribed circle in P, Q, R. Prove that the product of the radii of the circles described about the triangles BOP, COQ, AOR is equal to the product of the radii of the circles described about the triangles COP, AOQ, BOR; and that each of the products is

$$\frac{a^3b^3c^3}{4r^2(a+b+c)^4},$$

where a, b, c are the sides, and r the radius of the inscribed circle of the triangle ABC.

5. If $ABCD$, $AB'C'D'$ be two parallelograms, shew that it is possible to form a triangle with its sides equal and parallel to BB', CC', DD'.

6. If two tangents to a parabola intersect in T, and S be the focus, and from any point in ST perpendiculars be drawn to the tangents, shew that the line joining the feet of these perpendiculars is parallel to the directrix.

7. Prove that the general equation to an ellipse having double contact with the circle $x^2 + y^2 = a^2$, and touching the axis of x at the origin, is

$$c^2x^2 + (a^2 + c^2)y^2 - 2a^2cy = 0.$$

Paper XVI.

1. A and B, starting at the same moment, walk at uniform rates, the former in u hours from Oxford to Cambridge, the latter in v hours from Cambridge to Oxford. They meet on the road α hours before A's arrival at Cambridge, and β hours before B's arrival at Oxford. Prove that

$$u^2 : v^2 :: \alpha : \beta.$$

2. Obtain all the values of x and y from the simultaneous equations

$$x^2 + y^2 = 2a^2,\ \frac{x^2}{a + x} + \frac{y^2}{a + y} = a.$$

3. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51°, and when he retires 30 feet from the edge, he finds the elevation to be 46°. Find the breadth of the river, having given

$$\log 1{\cdot}357 = {\cdot}132630;\ \log 3 = {\cdot}47712$$

$$L\sin 46^\circ = 9{\cdot}856934;\ L\sin 39^\circ = 9{\cdot}798872;\ L\sin 5^\circ = 8{\cdot}940296.$$

4. If $f(\theta)$ be a function of θ given by the equation $f(2\theta) = (1 - \tan^2\theta)f(\theta)$, and if $f(o) = m$, shew that $f(\theta) = m\theta \cot \theta$.

5. In a given circle inscribe a triangle so that two of the sides may pass through given points, and the third be of given length.

6. If a sphere can be drawn to touch all the edges of a tetrahedron, the three straight lines joining the points of contact of the sphere with opposite edges will meet in a point.

7. An ellipse is described touching the asymptotes of an hyperbola, and meeting the hyperbola in $PP'QQ'$. Shew that PP' is parallel to QQ'.

Paper XVII.

1. Divide

$$1 + x + x^2 + x^3 + x^4 + x^6 + x^7 + x^8 + x^9 + x^{15} \text{ by } 1 - x^5 + x^6.$$

2. If

$$\left.\begin{aligned} \frac{x}{a+\alpha} + \frac{y}{a+\beta} + \frac{z}{a+\gamma} &= 1 \\ \frac{x}{b+\alpha} + \frac{y}{b+\beta} + \frac{z}{b+\gamma} &= 1 \\ \frac{x}{c+\alpha} + \frac{y}{c+\beta} + \frac{z}{c+\gamma} &= 1 \end{aligned}\right\}$$

prove that

$$x = + \frac{(a+\alpha)(b+\alpha)(c+\alpha)}{(\alpha-\beta)(\alpha-\gamma)},$$

and similarly for y and z.

3. If

$$\sin a + \sin \beta + \sin \gamma = 0 = \cos a + \cos \beta + \cos \gamma,$$

shew that

$$\cos 3a + \cos 3\beta + \cos 3\gamma = 3 \cos (a + \beta + \gamma)$$

and

$$\sin 3a + \sin 3\beta + \sin 3\gamma = 3 \sin (a + \beta + \gamma).$$

4. If P be the centre of the inscribed, and Q the centre of the circumscribed circle of a triangle, and PQ be produced both ways to meet the circumscribing circle in A and B, shew that $PA \,.\, PB = 2R \,.\, r$.

5. Prove that the straight lines joining the middle points of the opposite sides of a quadrilateral mutually bisect each other, and hence shew that it is possible to describe about a given parallelogram an infinite number of quadrilaterals whose sides shall be bisected by the angular points of the given parallelogram, and that these quadrilaterals are each equal in area to twice the parallelogram.

6. An ellipse is cut from a cone. Shew that the sum of the distances of the extremities of any diameter of the ellipse from the vertex of the cone is constant.

7. Prove that the resultant of forces 7, 1, 1, 3 acting from one angle of a regular pentagon towards the other angles taken in order is $\sqrt{71}$.

Paper XVIII.

1. Shew that

$$\frac{(a+p)(a+q)}{(a-b)(a-c)(a+x)} + \frac{(b+p)(b+q)}{(b-c)(b-a)(b+x)} + \frac{(c+p)(c+q)}{(c-a)(c-b)(c+x)}$$

$$= \frac{(x-p)(x-q)}{(a+x)(b+x)(c+x)}.$$

2. If the letters all denote positive quantities, prove that

$$\frac{(a+b)xy}{ay+bx} \text{ is never greater than } \frac{ax+by}{a+b}.$$

3. Find all the solutions of $\sin 3\theta - \cos\theta = 0$.

Which of them will satisfy the equation

$$1 + \sin^2\theta = 3\sin\theta\cos\theta\,?$$

4. In any triangle prove that

$$\frac{(b-c)(b+c-a)}{b+c}\cdot\tan\left(\frac{A}{2}+C\right)$$
$$= \frac{(c-a)(c+a-b)}{c+a}\cdot\tan\left(\frac{B}{2}+A\right)$$
$$= \frac{(a-b)(a+b-c)}{a+b}\cdot\tan\left(\frac{C}{2}+B\right).$$

5. If from each of the angular points of a quadrilateral perpendiculars be let fall upon the diagonals, the feet of these perpendiculars are the angular points of a similar quadrilateral.

6. AB is a diameter of a circle, and a parabola is described passing through A and B, and having for its directrix a tangent to a concentric circle. Shew that the locus of its focus is an ellipse.

7. If six forces acting on a body be completely represented, three by the sides of a triangle taken in order, and three by the sides of the triangle formed by joining the middle points of the sides of the original triangle, prove that they will be in equilibrium if the parallel forces act in the same direction, and if the scale on which the first three forces are represented be 4 times as large as that on which the latter are represented.

Paper XIX.

1. Prove that

$$(a+b)^3(a^5+b^5)+5ab(a+b)^2(a^4+b^4)$$
$$+15a^2b^2(a+b)(a^3+b^3)+35a^3b^3(a^2+b^2)+70a^4b^4=(a+b)^8.$$

2. Solve the equations

(1) $(c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0$,

(2) $ax+yz=ay+zx=az+xy=b^2$.

3. If $a+b+c=0$, prove that

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}.$$

Hence shew that if

$$\sin\alpha+\sin\beta+\sin\gamma=0=\cos\alpha+\cos\beta+\cos\gamma$$

$$\frac{\cos 5\alpha+\cos 5\beta+\cos 5\gamma}{5}$$

$$=\frac{\cos 3\alpha+\cos 3\beta+\cos 3\gamma}{3}\cdot\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma}{2}$$

$$-\frac{\sin 3\alpha+\sin 3\beta+\sin 3\gamma}{3}\cdot\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{2}.$$

4. If $A+B+C=90^\circ$, prove that

$$\operatorname{cosec}A\operatorname{cosec}B\operatorname{cosec}C-\cot B\tan C-\cot C\tan B-\cot C\tan A$$
$$-\cot A\tan C-\cot A\tan B-\cot B\tan A=2.$$

5. If the diagonals AC, BD of the quadrilateral $ABCD$ inscribed in a circle, centre O, intersect at right angles in a fixed point P, prove that the feet of the perpendiculars from O and P to the sides of the quadrilateral lie on a fixed circle, the centre of which is at the middle point of OP.

6. A heavy scalene triangle ABC lies on a horizontal plane, and a vertical force P, when applied at A is just able to lift the triangle. Would P be able to do this if applied at B, or C, instead of A?

7. If SY be the perpendicular from the focus on the tangent to an hyperbola at P, and CY meet the normal at P in R, shew that $PR = SY$.

Paper XX.

1. Calculate the value of $\sqrt{5}$ to ten places of decimals from the formula

$$\frac{3-\sqrt{5}}{2} = \frac{1}{3} + \frac{1}{3 \, . \, 7} + \frac{1}{3 \, . \, 7 \, . \, 47} + \frac{1}{3 \, . \, 7 \, . \, 47 \, . \, 2207} + \ldots$$

each of the factors in the denominators being equal to the square of the preceding factor diminished by 2.

2. Prove that $3^{2n+2} - 8n - 9$ is a multiple of 64.

3. If

$$\frac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma} = \frac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta}, \text{ and } \beta, \gamma \text{ are unequal,}$$

prove that each expression

$$= \frac{\cos(\beta+\gamma+\theta)}{\sin(\beta+\gamma)\cos^2\alpha}.$$

4. If $A + B + C + D = 2\pi$, shew that

$$\cos\frac{A}{2}\cos\frac{D}{2}\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{B}{2}\cos\frac{C}{2}\sin\frac{A}{2}\sin\frac{D}{2}$$
$$= \sin\frac{A+B}{2}\sin\frac{C+A}{2}\cos\frac{A+D}{2}.$$

5. $ABCD$ is a quadrilateral figure, and two points P, Q are taken in AD, BC, such that $AP : PD :: CQ : QB$. From P and Q straight lines PP', QQ' are drawn equal to, parallel to, and in the same directions as BC and DA respectively. Shew that forces represented by AB, CD, PP', QQ' are in equilibrium.

6. In a given triangle inscribe a parallelogram equal to half the triangle, so that one side is in the same straight line with one side of the triangle, and has one extremity at a given point of that side.

7. Given a focus, the length of the transverse axis, and that the second focus lies on a fixed straight line, prove that the conic will touch two parabolas having the given focus for focus.

Paper XXI.

1. Obtain the value of π correct to 6 places of decimals from the series

$$\pi = \frac{14}{5}\left\{1+\frac{2}{3}\left(\frac{1}{50}\right)+\frac{2.4}{3.5}\left(\frac{1}{50}\right)^2+\frac{2.4.6}{3.5.7}\left(\frac{1}{50}\right)^3+\dots\right\}$$
$$+\frac{948}{3125}\left\{1+\frac{2}{3}\left(\frac{9}{6250}\right)+\frac{2.4}{3.5}\left(\frac{9}{6250}\right)^2\right.$$
$$\left.+\frac{2.4.6}{3.5.7}\left(\frac{9}{6250}\right)^3+\dots\right\}.$$

2. Find the cube root of the expression

$$b^2(a-b)(c-b)\left\{(a-b)^2+(c-b)^2\right\}-ab^2c(a^2+c^2)$$
$$+b^5(a-b+c).$$

3. Shew that if the squares of the sides of a triangle are in A.P. the tangents of the angles are in H.P.

4. If

$$\frac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma}=\frac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta},$$ see Paper XX., No. 3,

shew that

$$\cot\theta=\frac{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}{\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)+\sin^2(\alpha+\beta+\gamma)}.$$

5. A body consists of two parts, and one of them is moved into any other position. Shew that the line joining the two positions of the centre of gravity of the whole body is parallel, and bears a fixed ratio, to the line joining

the two positions of the centre of gravity of the part moved.

6. Considering the 4 circles which touch the sides of a triangle, shew that the square on the distance between the centres of any two together with the square on the distance between the centres of the other two is equal to the square on the diameter of the circle passing through the centres of *any* three.

7. Construct a parabola having given three tangents and the direction of the axis.

Paper XXII.

1. If $x + y + z = xyz$, shew that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}.$$

2. If

$$\phi(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}, \text{ shew that } \phi(x+y) = \frac{\phi(x) + \phi(y)}{1 + \phi(x)\phi(y)}.$$

3. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, shew that

$$\frac{\cos 7\alpha + \cos 7\beta + \cos 7\gamma}{7}$$

$$= \frac{\cos 5\alpha + \cos 5\beta + \cos 5\gamma}{5} \cdot \frac{\cos 2\alpha + \cos 2\beta + \cos 2\gamma}{2}$$

$$- \frac{\sin 5\alpha + \sin 5\beta + \sin 5\gamma}{5} \cdot \frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{2}$$

4. With the usual notation, prove that

$$r_1 + r_2 + r_3 - r = 4R.$$

5. Shew that it is impossible to arrange 6 forces along the edges of a tetrahedron so as to form a system in equilibrium.

6. If a quadrilateral be inscribed in a circle, and the middle points of the arcs subtended by its sides be joined to make another quadrilateral, and so on, shew that these figures tend to become squares.

7. A system of parallelograms is inscribed in an ellipse having their sides parallel to the equi-conjugate diameters. Prove that the sum of the squares on the sides is constant.

Paper XXIII.

1. If

$$a = \frac{1}{3}\left\{\left(\frac{23 + \sqrt{513}}{4}\right)^{\frac{1}{3}} + \left(\frac{23 - \sqrt{513}}{4}\right)^{\frac{1}{3}} - 1\right\}$$

$$b = \frac{2}{81}\left\{\left(\frac{23 + \sqrt{513}}{4}\right)^{\frac{1}{3}} + \left(\frac{23 - \sqrt{513}}{4}\right)^{\frac{1}{3}} - 1\right\}^4.$$

shew that $$a - b = \frac{b}{a}.$$

2. If $\phi(x) = \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$, and $F(x) = \dfrac{2}{a^x + a^{-x}}$

shew that $$F(x + y) = \frac{F(x)F(y)}{1 + \phi(x)\phi(y)}.$$

3. A uniform rod has its lower end fixed to a hinge, and its other end attached to a string which is tied to a point in the same horizontal plane as the hinge, the distance between the point and hinge being equal to the length of the rod. If the tension of the string be equal to the weight of the rod, prove that the inclination of the rod to the horizon is $2\cos^{-1}(\frac{3}{4})^{\frac{1}{4}}$.

4. If $A + B + C = 180^\circ$, prove that

(1) $\sin 6A + \sin 6B + \sin 6C = 4 \sin 3A \sin 3B \sin 3C$

(2) $$\frac{\left(1-\tan\frac{A}{4}\right)\left(1-\tan\frac{B}{4}\right)\left(1-\tan\frac{C}{4}\right)}{\left(1+\tan\frac{A}{4}\right)\left(1+\tan\frac{B}{4}\right)\left(1+\tan\frac{C}{4}\right)} = \frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}$$

5. With the usual notation for the sides and angles of a triangle, prove that

$$\frac{b\cos A - a\cos B}{c} + \frac{b^2\cos 2A - a^2\cos 2B}{2c^2} + \frac{b^3\cos 3A - a^3\cos 3B}{3c^3} + \ldots . = \log\frac{b}{a}.$$

6. AB, CD, EF are given parallel chords of a circle. EC, FD produced meet AB produced in G, H respectively. If from any point P on the circumference PC, PF be drawn meeting AB, or AB produced in Q, R respectively, the rectangle contained by GQ and HR is constant.

7. If two straight lines represented by the equation

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{xy}{c} = 0,$$

be at right angles to each other, and ω be the angle between the axes, shew that

$$\cos\omega = c\left(\frac{1}{a} + \frac{1}{b}\right)$$

Paper XXIV.

1. If α, β be the roots of the equation

$$x^2 + ax + \tfrac{1}{4}(a^2 - b^2) = 0$$

prove that $\alpha + \beta$, $\alpha - \beta$ are the roots of the equation

$$x^2 + (a \pm b)x \pm ab = 0.$$

2. If the sum of the numbers of feet in the margins of two square carpets be known, and if the value of a square foot of one carpet be α and of the other β, shew that, if the sum of the values of the carpets be the least possible, their areas are respectively in the ratio of β^2 to α^2.

3. Prove that if α, β, γ be any plane angles

$$\sin(\beta+\gamma-\alpha)\sin(\beta-\gamma)\cos(\beta-\gamma)+\sin(\gamma+\alpha-\beta)\sin(\gamma-\alpha)\cos(\gamma-\alpha)+\sin(\alpha+\beta-\gamma)\sin(\alpha-\beta)\cos(\alpha-\beta)=0.$$

4. On two of the sides of a given scalene triangle as bases similar isosceles triangles are described externally. Determine the magnitude of the vertical angle of these triangles when the vertices are equidistant from the middle point of the third side of the given triangle.

5. There are three given straight lines, any two of which are together greater than the third. Describe a square which shall be equal to the difference between the sum of the squares on any two of the given lines and the square on the third.

6. *ABCDEF* is a regular hexagon, and at *A* forces act represented in magnitude and direction by *AB*, 2*AC*, 3*AD*, 4*AE*, 5*AF*. Shew that the length of the line which represents their resultant is $\sqrt{351}\,AB$.

7. In a parabola, if the part of the normal included between the curve and the axis be bisected, prove that the locus of the point of bisection is another parabola whose vertex coincides with the focus of the original parabola.

Paper XXV.

1. If $x^2 + ax + b$ and $x^2 + a'x + b'$ have a common measure, then

$$(ab' - a'b)(a - a') + (b - b')^2 = 0.$$

2. If $1, x\ x^3$, and $1, y^2, y^3$ be each in H.P. and if $x + y$ be not equal to zero, and neither x nor y equal to unity, shew that $-y^2, y, x, x^2$ will be in A.P. and that their sum will equal $x^3 + y^3$.

3. Shew that in any triangle $\cos A + \cos B + \cos C$

$$= \sin \frac{A}{2} \cos \frac{B-C}{2} + \sin \frac{B}{2} \cos \frac{C-A}{2} + \sin \frac{C}{2} \cos \frac{A-B}{2}.$$

4. Prove that

$$\frac{1}{2} \tan \frac{\theta}{2} + \frac{1}{4} \tan \frac{\theta}{4} = \frac{1}{4} \cot \frac{\theta}{4} - \cot \theta.$$

Hence shew that

$$\cot \theta + \frac{1}{2} \tan \frac{\theta}{2} + \frac{1}{4} \tan \frac{\theta}{4} + \ldots\ldots = \frac{1}{\theta}.$$

5. From any point P on a given circle tangents PQ, PQ' are drawn to a second circle whose centre is on the circumference of the first. Shew that the chord joining the points where these tangents cut the first circle is fixed in direction, and intersects QQ' on the line of centres.

6. In a weighing machine constructed on the principle of the common steelyard the pounds are read off by graduations reaching from 0 to 14, and the stones by weights hung at the end of the arm. If the weight corresponding to one stone be 7 oz., the moveable weight $\frac{1}{2}$ lb., and the length of the arm one foot, prove that the distances between the graduations are $\frac{3}{4}$ in.

7. If the lines represented by $y = x \tan \alpha$ and $y = x \tan \beta$, where $\alpha = \frac{11\pi}{24}$, $\beta = \frac{19\pi}{24}$ be perpendicular to each other, shew that the angle between the coordinate axes is $\frac{\pi}{4}$.

Paper XXVI.

1. Convert $\frac{1}{19}$, $\frac{2}{19}$, ... $\frac{18}{19}$ into circulating decimals, explaining any methods for deriving one case from another, and for shortening the work.

2. If $x + y + z + w = 0$, prove that

$$wx(w + x)^2 + yz(w - x)^2 + wy(w + y)^2 + zx(w - y)^2 + wz(w + z)^2 + xy(w - z)^2 + 4xyzw = 0.$$

3. Shew that

$$\tan\frac{x+y}{2}\tan\frac{x-y}{2} = \frac{\operatorname{cosec} 2x \operatorname{cosec} y - \operatorname{cosec} 2y \operatorname{cosec} x}{\operatorname{cosec} 2x \operatorname{cosec} y + \operatorname{cosec} 2y \operatorname{cosec} x}.$$

4. If an arc of 10 ft. on a circle of 8 ft. in diameter subtend at the centre an angle of $143° 14' 22''$, find the value of π to 4 places of decimals.

5. If C and D be the centres of the spheres inscribed in a cone, and touching a given section, the sphere described on CD as diameter will intersect the plane in the auxiliary circle of the section.

6. From the centre of the circle circumscribing a triangle ABC a perpendicular to its plane is drawn of length equal to the side of the square inscribed in that circle. Shew that the radius of the sphere which passes through A, B, C, and the extremity of the perpendicular is $\frac{3}{4}$ the perpendicular.

7. If from a point P of an ellipse perpendiculars PM, PN be drawn to the equi-conjugate diameters, prove that the normal at P bisects MN.

Paper XXVII.

1. If a, b, c be the p^{th}, q^{th}, and r^{th} terms respectively both of an A.P. and a G.P. prove that

$$a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1.$$

2. Eliminate x and y between the equations

$$ax^3 + bx^2y + cxy^2 + dy^3 = 0$$
$$a'x^3 + b'x^2y + c'xy^2 + d'y^3 = 0.$$

3. Express $\sec\theta \sec 2\theta$ as the sum of 3 partial fractions.

4. The angles of a triangle ABC are such that the number of degrees in A, the number of grades in B, and the circular measure of C are all equal. Find the angles.

5. If a triangle be described about a parabola, prove geometrically that its orthocentre lies on the directrix.

6. A rectangular sheet of stiff paper, whose length is to its breadth as $\sqrt{2} : 1$, lies on a horizontal table with its longer sides perpendicular to the edge, and projecting over it. The corners on the table are then doubled over symmetrically, so that the creases pass through the middle point of the side joining the corners, and make angles of 45° with it. The paper is on the point of falling over. Shew that it had originally $\frac{25}{48}$ of its length on the table.

7. Prove that the sum of the coordinates of any point of an ellipse referred to a pair of conjugate diameters as axes, cannot be greater than the distance of an end of the major axis from an end of the minor axis.

Paper XXVIII.

1. An A.P. a G.P. and an H.P. have each a and b for their first two terms. Shew that the $(n+2)^{\text{th}}$ terms will be in G.P. if

$$\frac{b^{2n+2} - a^{2n+2}}{ab(b^{2n} - a^{2n})} = \frac{n+1}{n}.$$

2. If $a_1, a_2, a_3, \ldots . a_n$ be n real quantities, and if

$$(a_1^2 + a_2^2 + \ldots . + a_{n-1}^2)(a_2^2 + a_3^2 + \ldots . + a_n^2)$$
$$= (a_1a_2 + a_2a_3 + \ldots . + a_{n-1}a_n)^2.$$

then $a_1, a_2, \ldots .$ are in G.P.

3. If A, B, C be the angles of a triangle, shew that

$$\cos A + \cos B + \cos C > 1, \text{ and } \not> \tfrac{3}{2}.$$

4. If

$$x \sin^2 A \cos B - y \sin^2 B \cos A + z(\cos^2 A - \cos^2 B) = 0$$

and

$$z \sin^2 C \cos A - x \sin^2 A \cos C + y(\cos^2 C - \cos^2 A) = 0$$

where A, B, C are the angles of a triangle whose sides are a, b, c, shew that

$$ax = by = cz.$$

5. From D and E, points on the circumference of the circle circumscribing the triangle ABC, perpendiculars are drawn to the sides, and the straight lines which respectively pass through the feet of the perpendiculars intersect in P. Shew that the locus of P is a circle when A moves on the circumference of the circumscribing circle, and B, C, D, E are fixed.

6. Prove that the locus of the intersection of tangents to a parabola which are inclined at angles of $\frac{\pi}{4}$ is a rectangular hyperbola having one focus and the corresponding directrix coincident with the focus and directrix of the parabola.

7. The equation to a curve referred to coordinate axes which are inclined at an angle ω is $x^2 + y^2 = c^2$. Prove that the locus of the intersection of two tangents to the curve which are at right angles is a circle, the equation to which is

$$x^2 + y^2 + 2xy \cos \omega = 2c^2.$$

Paper XXIX.

1. If

$$x_3 = \log_{x_1} x_2 \,;\ x_4 = \log_{x_2} x_3 \,;\ \ldots\ldots x_n = \log_{x_{n-2}} x_{n-1} \,;$$
$$x_1 = \log_{x_{n-1}} x_n \,;\ x_2 = \log_{x_n} x_1 \,;$$

shew that $x_1 \,.\, x_2 \ldots\ldots x_n = 1$.

2. If

$$a^2x^2 + b^2y^2 + c^2z^2 = 0\,;\ a^2x^3 + b^2y^3 + c^2z^3 = 0\,;$$
$$\frac{1}{x} - a^2 = \frac{1}{y} - b^2 = \frac{1}{z} - c^2\,;$$

prove that

(1) $a^4x^3 + b^4y^3 + c^4z^3 = 0$;

(2) $a^6x^3 + b^6y^3 + c^6z^3 = a^4x^2 + b^4y^2 + c^4z^2$.

3. In any triangle shew that

$$a^2 \cos 2(B - C) = b^2 \cos 2B + 2bc \cos (B - C) + c^2 \cos 2C.$$

4. Prove that

$$\tfrac{1}{2} \log \sec x = \sin^2 x - \frac{\sin^2 2x}{2} + \frac{\sin^2 3x}{3} - \ldots\ldots$$

5. If the three plane angles at the vertex of a tetrahedron be bisected, and the points in which the bisecting lines meet the sides of the base be joined with its opposite angles, the three lines so drawn will meet in a point.

6. Apply a property of the parabola to prove that if four intersecting straight lines be taken three together so as to form four triangles, the orthocentres of these three triangles will be collinear.

7. From a point in the circumference of an ellipse, the semi-axes of which are a and b, two tangents are drawn to a concentric circle. A straight line through the points of

contact intersects the axes of the ellipse at points the distances of which from the centre are α, β. If c be the radius of the circle, prove that

$$\frac{1}{(a\alpha)^2} + \frac{1}{(b\beta)^2} = \frac{1}{c^4}.$$

Paper XXX.

1. Shew that every cube number is the difference of two square numbers, and that if the cube contains an uneven factor a^3, each of the squares is divisible by a^2.

2. Solve the equations

(1) $x^3 + y^3 = b^3$; $xy + a(x + y) = ab$;

(2) $x + y + z = x^2 + y^2 + z^2 = \frac{1}{2}(x^3 + y^3 + z^3) = 3$.

3. Find all the values of θ when

$$\frac{2\cos\theta}{\sqrt{3}} + \frac{\sqrt{3}}{2\cos\theta}$$

has its least value.

4. If a, b, c be the radii of three circles which touch one another externally and r the radius of the circle inscribed in the triangle formed by joining their centres, prove that

$$\frac{1}{r^2} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}.$$

5. If an equifacial tetrahedron be cut by a plane parallel to two edges which do not meet, the perimeter of the parallelogram in which it is cut is double of either edge of the tetrahedron to which it is parallel.

6. AA' is the major axis, and S one of the foci of an ellipse. With S as focus, a parabola is described passing through the extremities of the minor axis. Shew that its vertex bisects SA or SA'.

7. A string 9 feet long has one end attached to the extremity of a smooth uniform heavy rod 2 ft. long; and at the other end carries a ring without weight which slides upon the rod. The rod is suspended by means of the string from a smooth peg. Shew that θ, the angle which the rod makes with the horizon is given by the equation

$$9 \tan^3\theta + 9 \tan\theta = 2.$$

Shew also that one of the roots of this is

$$\tan\theta = 3^{-\frac{1}{3}} - 3^{-\frac{2}{3}}.$$

Paper XXXI.

1. If

$$a = \frac{1}{3}\left\{\left(\frac{23 + \sqrt{513}}{4}\right)^{\frac{1}{3}} + \left(\frac{23 - \sqrt{513}}{4}\right)^{\frac{1}{3}} - 1\right\}$$

$$b = \frac{2}{81}\left\{\left(\frac{23 + \sqrt{513}}{4}\right)^{\frac{1}{3}} + \left(\frac{23 - \sqrt{513}}{4}\right)^{\frac{1}{3}} - 1\right\}^4$$

shew that $$a^2 + b^2 = a - b = \frac{b}{a}.$$

See Paper XXIII., 1.

2. Given

$$a^2 + b^2 = 1;\ \log 2 = \cdot3010300;\ \log(1 + a) = \cdot1928998;$$
$$\log(1 + b) = \cdot2622226;$$

shew that

$$\log(1 + a + b) = \cdot3780762.$$

3. Prove the following formulæ for a plane triangle

(1) $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0.$

(2) $$\frac{a^2 - b^2}{\cos A + \cos B} + \frac{b^2 - c^2}{\cos B + \cos C} + \frac{c^2 - a^2}{\cos C + \cos A} = 0.$$

4. From a point within a regular polygon perpendiculars are let fall on all the sides. Find the sum of the squares on all these perpendiculars.

5. Through a fixed point O any straight line OPQ is drawn cutting a fixed circle in P and Q. On OP and OQ as chords are described circles touching the fixed circle at P and Q. Prove that the two circles so described will intersect on another fixed circle.

6. If the tangent and ordinate at any point P of an ellipse meet the axis major in T and N, and any circle be drawn through N and T, shew that it is cut orthogonally by the auxiliary circle of the ellipse.

7. Investigate the conditions in order that two conic sections, represented by the equations

$$(x - a)^2 + (y - b)^2 = (\alpha x + \beta y + \gamma)^2$$
$$(x - a')^2 + (y - b')^2 = (\alpha' x + \beta' y + \gamma')^2$$

may be identical in magnitude and form.

PAPER XXXII.

1. If $x + y + z = 0$, shew that

$$\left\{\frac{y - z}{x} + \frac{z - x}{y} + \frac{x - y}{z}\right\}\left\{\frac{x}{y - z} + \frac{y}{z - x} + \frac{z}{x - y}\right\} = 9.$$

2. If $x^2 = px + q$, shew that

$$x^n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.\, x + q\,.\,\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta},$$

where $\alpha + \beta = p$, $\alpha\beta = -q$.

3. If $\alpha + \beta = \frac{\pi}{2}$, prove that

$$\frac{\left(1 - \tan\frac{\alpha}{2}\right)\left(1 - \tan\frac{\beta}{2}\right)}{\left(1 + \tan\frac{\alpha}{2}\right)\left(1 + \tan\frac{\beta}{2}\right)} = \frac{\sin\alpha + \sin\beta - 1}{\sin\alpha + \sin\beta + 1}$$

4. P, Q, R are points in the sides BC, CA, AB of a triangle, such that

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = x.$$

Shew that

$$PQ^2 + QR^2 + RP^2 = \tfrac{1}{4}(a^2 + b^2 + c^2) + 3(x - \tfrac{1}{2})^2(a^2 + b^2 + c^2).$$

5. If P be a point equidistant from the angles ABC of a right-angled triangle, A being the right angle, and D the middle point of BC, prove that PD is at right angles to the plane of ABC.

Prove also that the angle between the planes PAC, PBC, and the angle between the planes PAB, PBC are together equal to the angle between the planes PAC, PAB.

6. A chord of a conic section subtends an angle of given magnitude at one of the foci. Find the locus of the point of intersection of the tangents drawn at the extremities of the chord.

7. A frustum of a cone is such that its height is half that of the complete cone. Shew that the centre of gravity of the frustum divides its height in the ratio of 17 to 11.

Paper XXXIII.

1. A selection of c things is to be made, part from a group of a things, and the remainder from a group of b things. Prove that the number of ways in which such a selection may be made will never be greater than when the number of things taken from the group of a things is the integer next less than

$$\frac{(a+1)(c+1)}{a+b+2}.$$

2. Prove that

$$a^{-1} = 1 + 2(1 - a) + 3(1 - a)(1 - 2a) + \dots \dots + n(1 - a)(1 - 2a) \dots \{1 - (n - 1)a\} + a^{-1}(1 - a)(1 - 2a) \dots (1 - na).$$

3. Shew that

$$2\sin\theta + 3\sin 2\theta + \dots + n\sin(n - 1)\theta = \frac{(n + 1)\sin(n - 1)\theta + \sin\theta - n\sin n\theta}{2(1 - \cos\theta)}.$$

4. P is a point within a triangle ABC; a, b, c are the sides BC, CA, AB; α, β, γ are PA, PB, PC. r is the radius of the circle inscribed in ABC, and r_1, r_2, r_3 are the radii of the circles inscribed in PBC, PCA, PAB. Prove that

$$(r_2+r_3)\alpha+(r_3+r_1)\beta+(r_1+r_2)\gamma=(r-r_1)a+(r-r_2)b+(r-r_3)c.$$

5. On the sides of a triangle ABC as bases are described three equilateral triangles aBC, bCA, cAB, all upon the same side of their bases as ABC. Prove that Aa, Bb, Cc are all equal, and pass through a point which lies on all the three circles circumscribing the equilateral triangles.

6. Prove that the distance of any point of an equilateral hyperbola from the centre is a mean proportional between its distances from the foci.

7. If CP be a semi-diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and if $(x_1 y_1)$, $(x_2 y_2)$ be the coordinates of the extremities of a chord EF parallel to CP, shew that

$$\frac{EF^2}{2CP^2} = 1 - \frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2}.$$

PAPER XXXIV.

1. Find the sum of the cubes of n consecutive terms of an A.P. and shew that it is divisible by the sum of the corresponding n terms of the A.P.

2. Shew that one solution of the equations

$$\frac{x - 2\dfrac{xy - z^2}{x + y}}{a} = \frac{z}{c} = \frac{y - 2\dfrac{xy - z^2}{x + y}}{b} = \frac{z^2 - xy}{c^2 - ab}.$$

is

$$\frac{x}{a(a-b)+2c^2} = \frac{y}{b(b-a)+2c^2} = \frac{z}{c(a+b)} = -\frac{a+b}{(a-b)^2+4c^2}.$$

3. Shew that

$$\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}.$$

4. Prove that one of the values of

$$\log(1 + \cos 2\theta + \sqrt{-1}\sin 2\theta) \text{ is } \log(2\cos\theta) + \theta\sqrt{-1},$$

when θ lies between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$. From this deduce Gregory's Series.

5. A quadrilateral is inscribed in a circle. From the centre of the circle perpendiculars are drawn on the sides, and a second quadrilateral is formed by joining the feet of the perpendiculars. Shew that the area of the first quadrilateral is double that of the second.

6. Two confocal ellipses have parallel tangents at the points P, Q; shew that PQ subtends equal angles at the foci.

7. Tangents are drawn to an ellipse, axes $(2a, 2b)$ from an external point (h, k). If $2c$ be the length of the chord of contact, shew that

$$c^2 = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) \frac{\dfrac{b^2h^2}{a^2} + \dfrac{a^2k^2}{b^2}}{\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}\right)^2}.$$

Paper XXXV.

1. A ladies' school consists of 15 pupils who walk out in 5 rows of 3 abreast. They are arranged so that no two pupils should walk twice abreast. Shew that they can only walk out 7 times subject to this condition; and write down the orders on the different days.

2. Solve the equations

(1) $x^2 - (2a - b - c)x + a^2 + b^2 + c^2 - bc - ca - ab = 0$,

(2) $$\left.\begin{aligned} x^2 + 2xy - y^2 &= ax + by \\ x^2 - 2xy - y^2 &= bx - ay \end{aligned}\right\}.$$

3. Prove that one of the values of

$$\sin^{-1}(\cos\theta + \sqrt{-1}\sin\theta)$$

is $\cos^{-1}\sqrt{\sin\theta} + \sqrt{-1}\log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$

when θ is between 0 and $\frac{\pi}{2}$.

4. Through a given point straight lines are drawn parallel to the sides of a regular polygon; and from another given point perpendiculars are drawn to these straight lines. Find the sum of the squares of the perpendiculars.

5. Given a fixed ellipse, shew that the locus of the vertices of all right cones out of which this ellipse can be cut is an hyperbola passing through the foci of the ellipse.

6. From the ends A, B, of a diameter of a circle, the centre of which is C, are drawn any two chords AH, BK, such that the radii CH, CK include a constant angle 2α. Prove that, a being the length of the radius of the circle, the locus of the intersection of the two chords is also a circle, the radius of which is equal to $a \sec \alpha$; and that the distance between the centres of the two circles is $a \tan \alpha$.

7. A frustum of a cone is such that its height is half that of the complete cone. If the frustum be placed with its curved surface on a horizontal plane, shew that it will not topple over if the vertical angle of the cone is less than

$$2 \sin^{-1} \sqrt{\frac{17}{45}}.$$

Paper XXXVI.

1. The number 142857 when multiplied by the digits 1, 2, 3 . . . . 6 gives the same figures in the same cyclical order; but when multiplied by 7 it gives a series of 9s. Explain fully the reason of this.

2. Shew that the sum of all the terms of the series

$$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \ldots .$$

which on division by 7 leave an odd remainder, and of which $n(n - 1)$ is the greatest, is

$$\frac{1}{21}(n + 3)(n^2 + 6n - 4).$$

3. If $\tan \frac{1}{2}\alpha = \tan^3 \frac{1}{2}\beta$, and $\tan \beta = 2 \tan \phi$, shew that $\alpha + \beta = 2\phi$.

4. If x, y, z be the distances of the centre of the nine point circle from the angular points of a triangle, and p its distance from the orthocentre, and R the radius of the circumscribing circle, prove that

$$x^2 + y^2 + z^2 + p^2 = 3R^2.$$

5. If through D the middle point of the hypotenuse BC of a right-angled triangle DE be drawn at right angles to BC, meeting AC in E, prove that the rectangle $EC \,.\, CA$ is equal to half the square on BC.

6. Find the area of the maximum triangle which can be inscribed in a circle.

7. Shew that the polar equation to the normal of a conic, the focus being the pole, and semi-latus rectum $= c$, is

$$e \sin \theta + \sin (\theta - a) = \frac{c}{r} \cdot \frac{e \sin a}{1 + e \cos a}.$$

Paper XXXVII.

1. A man looks at a clock between the hours of 4 and 5, and again between the hours of 7 and 8, and he observes that in the interval the hour-hand and minute-hand have precisely exchanged their positions. Shew that at each observation the hands were equally inclined to the vertical.

2. Solve the equations

(1) $\sqrt{\frac{a}{b}(bx - a^2)} - \sqrt{\frac{b}{a}(ax - b^2)} = a - b,$

(2) $\left.\begin{aligned} x + y + \sqrt{x^2 - y^2} &= a \\ y\sqrt{x^2 - y^2} &= 2b^2 \end{aligned}\right\} .$

3. If $A + B + C$ be a multiple of $180°$, and

$$\sin 2A : \sin 2B : \sin 2C :: 5 : 4 : 3$$

prove that

$$\tan A = \pm 1, \tan B = \pm 2, \tan C = \pm 3.$$

4. Prove that

$$\frac{\cos\alpha + \cos(\alpha + 2\beta) + \ldots\ldots \text{ to } n \text{ terms}}{2\cos\{\alpha + (n-1)\beta\}}$$
$$- \Big[\cos(n-1)\beta + \cos(n-3)\beta \ldots\ldots$$
$$+ \cos\left\{\frac{3}{2} + \frac{(-1)^{n-1}}{2}\right\}\beta\Big] = \frac{1 + (-1)^{n-1}}{4}.$$

5. From a given point A without a circle any two straight lines APQ, ARS are drawn making equal angles with the diameter which passes through A, and cutting the circle in PQ, RS. Shew that PS and QR intersect in a fixed point.

6. Find the area of the greatest triangle which can be inscribed in an ellipse, and shew that the centre of the ellipse is the point of intersection of the bisectors of the sides of the triangle.

7. The vertex of one of the branches of an hyperbola is A : S is the nearer and S' the more remote focus; the focal distance PS' of a point P in this branch is bisected at U. Prove that the locus of the intersection of the directions of the lines PS, UA, is a similar hyperbola, the transverse axis of which is equal to AS.

PAPER XXXVIII.

1. A, B, C, D, and E play at cards. A deals first, and loses to each of the others as many counters as he has. B does the same, and similarly for C, D, and E. They then find that each player has 32 counters. How many had each at the beginning of the game?

2. Determine which is the greatest of the numbers

$$\sqrt{2},\ \sqrt[3]{3},\ \sqrt[4]{4},\ \sqrt[5]{5},\ \ldots\ldots$$

3. In a triangle the least angle is 45°, and the tangents of the angles are in A.P., and the area is 3 square yards. Shew how to solve the triangle.

4. If $a'b'c'$ be the sides of the triangle formed by joining the points of contact of the inscribed circle with the sides of a triangle, shew that

$$\frac{a'b'c'}{abc} = \frac{r^2}{2R^2}.$$

5. $ABCD$ is a parallelogram whose sides BC, CD, DA, AB are bisected in A', B', C', D'. Shew that AA', BB', CC', DD' include a parallelogram whose area $= \frac{1}{5}ABCD$.

6. On a given base AB any isosceles triangle PAB is described; and on AP as base another triangle QAP, similar to PAB, is described. Shew that the locus of Q is a parabola.

7. From a given point C are drawn a tangent to a given conic section, touching it at O, and any straight line whatever, intersecting it at P and P'. Prove that the sum of the cotangents of the angles POC, $P'OC$ is constant.

Paper XXXIX.

1. There are 11 routes from London to Cambridge, including routes viâ Oxford, and there are 13 from London to Oxford, including those viâ Cambridge. Find the number of direct routes between the several towns.

2. If p, q, r are all unequal positive integers, and x is positive and not equal to unity, prove that

$$p \cdot x^{q-r} + q \cdot x^{r-p} + r \cdot x^{p-q} > p + q + r.$$

3. Given a fixed point O without a fixed circle whose centre is C, if any straight line be drawn through O cutting the circle in P and P', shew that the circle described round PCP' will pass through a fixed point D in OC, and that if OC meet the fixed circle in A, AP bisects the angle OPD.

4. Let O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles of ABC. Shew that

$$OA \,.\, OO_1 = OB \,.\, OO_2 = OC \,.\, OO_3 = 4Rr.$$

5. Find a point such that the sum of the squares of the perpendiculars drawn from it to the sides of a given triangle shall be a minimum, and shew that the minimum value of the sum is

$$\frac{4\Delta^2}{a^2 + b^2 + c^2}.$$

6. Find the locus represented by the equations

$$x = A \sin\theta + B \cos\theta$$
$$y = A \cos\theta + B \sin\theta$$

where θ is a variable parameter. Employ it in the following example.

7. An equilateral triangle moves in a plane so that two of its angular points slide one on each of two rectangular axes. Prove that the third angular point lies on one of the conics $x^2 + y^2 \pm \sqrt{3} \,.\, xy = a^2$, $2a$ being a side of the triangle.

Paper XL.

1. Give a rule for determining by inspection the cube root of every perfect cube less than a million.

2. A farmer sold 10 sheep at a certain price, and 5 others at 10*s.* less per head. The sum he received for each lot was expressed in £'s by the same two digits. Find the price of each sheep.

3. In any triangle shew that

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = 4 \cdot \frac{1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A + \sin B + \sin C}.$$

4. If S denote the area, a, b, c the sides of a triangle ABC inscribed in a circle, shew that the perimeter of the triangle formed by drawing tangents to the circle at A, B, C will be equal to

$$\frac{abc \tan A \tan B \tan C}{2S}.$$

5. ABC is a triangle, and D the middle point of BC. Any straight line through C meets AD in E, and AB in F. Shew that $AE \cdot FB = 2AF \cdot ED$.

6. A variable ellipse always touches a fixed ellipse, and has a common focus with it. Find the locus of the second focus

(1) When the major axis is constant,

(2) When the minor axis is constant.

7. The locus of a point P such that the sum of the squares of the three normals drawn from P to the parabola $y^2 = 2px$ may equal a given quantity k^2 is the ellipse

$$x^2 + 3y^2 + 4px - 2p^2 = k^2.$$

Also find the position of a point P on a given line $y = mx + n$, such that the sum of the squares of the three normals may be a minimum, and shew that the point P is a vertex of that diameter of the ellipse whose equation is $3my + x + 2p = 0$.

PAPER XLI.

1. 5 men do $\cdot 600\dot{6}$ of a piece of work in 2·12 hours How long will 6 boys take to finish it, it being known that 3 men and 7 boys have done a similar piece of work in 3 hours?

2. Prove that, if $x < 1$

$$\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \ldots = \frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \ldots$$

3. If

$$\frac{\sin(a+\theta)}{\sin(a+\phi)} = \frac{\sin(\beta+\theta)}{\sin(\beta+\phi)},$$

shew that either a and β, or θ and ϕ differ by a multiple of π.

4. From the angular points of an equilateral triangle ABC, lines are drawn at a constant inclination $\frac{2m\pi}{6m+1}$ to the sides taken in order, so as to form another equilateral triangle $A_1B_1C_1$ within ABC; another equilateral triangle $A_2B_2C_2$ is formed in a similar manner within $A_1B_1C_1$; and so on. If S denote the area of the triangle ABC, S_1 the area of $A_1B_1C_1$, &c., X the area of the first of the triangles which is similarly situated to ABC, prove that

$$S_p \,.\, S_q = S \,.\, X$$

where p and q are integers, such that $p + q = 6m + 1$.

5. Any point E is taken outside a given circle; and on the chord of contact of tangents from E is taken a point F. Prove that the circles whose centres are E and F which cut the given circle orthogonally, cut one another orthogonally.

6. The tangent at any point P of an ellipse meets any pair of parallel tangents in M and N. Shew that the circle described on MN as diameter will meet the normal at P in points whose distance apart is equal to the diameter

conjugate to CP, and whose distances from the centre of the ellipse are respectively equal to the sum and difference of the semi-axes.

7. Given an ellipse and any triangle in its plane, through each angle draw a line to the opposite side so that side and line are parallel to a pair of conjugate diameters of the ellipse. Shew that the three lines are concurrent.

Paper XLII.

1. Prove that

$$(1 - x^2)^n = (1 + x)^{2n} - 2nx(1 + x)^{2n-1} + \frac{2n(2n-2)}{1 . 2} . x^2 . (1 + x)^{2n-2} - \dots .$$

2. Shew that

$$3m(3m + 1)^2 > 4(\lfloor 3m)^{\frac{1}{m}}.$$

3. ABC is a triangle such that if each of its angles be taken in succession as the unit of measurement, and the measures formed of the sums of the other two, these measures are in A.P. Shew that the angles of the triangle are in H.P. Also shew that only one of these angles can be greater than $\frac{2}{3}$ of a right angle.

4. Prove that if the tangents at B and C to the circle ABC meet in O, the chord of the circle drawn through O parallel to AB will be bisected by AC.

5. If P be any point on a parabola whose vertex is A, and if PR be perpendicular to AP, meeting the axis in R, shew that a circle whose centre is R and radius RP will pass through the ends of the ordinate to the parabola through R.

6. If from any fixed point on the axis of a parabola perpendiculars be drawn on tangents, the locus of their intersections with the focal distances of the points of contact is a circle.

7. From an external point O (h, k) two tangents OP, OQ are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Shew that the area of the triangle CPQ is

$$\frac{a^2b^2(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{3}{2}}}{b^2h^2 + a^2k^2}$$

and the area of the quadrilateral $OPCQ$ is

$$(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{1}{2}}.$$

Paper XLIII.

1. If

$$a(by + cz - ax) = b(cz + ax - by) = c(ax + by - cz),$$

and if $a + b + c = 0$, then will $x + y + z = 0$.

2. Shew that

$$1 + \frac{2^3}{\underline{|1}} + \frac{3^3}{\underline{|2}} + \frac{4^3}{\underline{|3}} + \ldots \equiv 15e.$$

3. Prove that

$$\frac{\sin 5\theta - \cos 5\theta}{\sin 5\theta + \cos 5\theta} = \tan\left(\theta - \frac{\pi}{4}\right)\frac{1 - 2\sin 2\theta - 4\sin^2 2\theta}{1 + 2\sin 2\theta - 4\sin^2 2\theta}.$$

4. $ACBP$ is a quadrilateral figure such that the angle APB (2β) is bisected by the diagonal CP. If $CA = a$, $CB = b$, and the angle $ACB = \alpha$, prove that

$$CP = \frac{ab}{\sin\beta} \cdot \frac{\sin(\alpha + 2\beta)}{\sqrt{a^2 + b^2 + 2ab\cos(\alpha + 2\beta)}}.$$

5. ABC is an isosceles triangle having each of the angles at the base double the third angle. Shew that the nine point circle of this triangle will intercept portions of the equal sides such that a regular pentagon can be inscribed in the circle having these portions as two of its sides.

Also, in *Euclid* IV. 10, find what portion of the circumference of the small circle is intercepted by the large circle.

6. If through a fixed point A a straight line be drawn meeting two fixed lines OD, OE in B and C respectively, and on it a point P be taken such that $AC \cdot AP = AB^2$, prove that the locus of P is a parabola which passes through A and O, and has its axis parallel to OD, and the tangent at A parallel to OE.

7. If ϕ be the excentric angle of any point P of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and if a parabola be described passing through the foci of the ellipse, and having P for its focus, shew that the latus rectum of this parabola is $2(a \mp b)\sin^2\phi$, and that the equation of its directrix is $x\cos\phi \pm y\sin\phi = a$. Also write down the equation of the parabola.

Paper XLIV.

1. Find the condition that the roots of the equation $ax^2 + 2bx + c = 0$ may be formed from those of $a'x^2 + 2b'x + c' = 0$ by adding the same quantity to each root.

2. If a, b, c are positive and x real, prove that the quantity $ax + b + \frac{c}{x}$ can never lie in value between the quantities $b \pm 2\sqrt{ac}$.

Hence prove that the fraction $\frac{Ax^2 + Bx + C}{A'x^2 + B'x + C'}$ has its limiting values when x is a root of the equation

$$(AB' - A'B)x^2 - 2(CA' - C'A)x + BC' - B'C = 0.$$

3. If $\sin A$ be the Arithmetic and $\sin B$ the Geometric Mean between $\sin C$ and $\cos C$, prove that

$$\cos 2A = \tfrac{1}{2}\cos 2B = \cos^2\left(\frac{\pi}{4} + C\right).$$

4. If ABC, ADE be two triangles of equal area, and having one angle in each equal, and placed so that BA

AE are in a straight line, as also CA, AD, and if BC, DE be produced to meet in F, shew that FA bisects CE and BD.

5. If PP' be a chord of a conic parallel to the transverse axis, and the two circles be drawn through a focus S touching the conic at P and P' respectively, prove that F, the second point of intersection of the circles will be at the intersection of PP' and ST, where T is the point of intersection of the tangents at P and P'.

Prove also that the locus of F for different positions of PP' will be a parabola with its vertex at S.

6. Shew that the locus of the intersection of normals to a parabola which make complementary angles with the axis is a parabola.

7. When the portion of the tangent to an ellipse intercepted between the axes is a minimum, shew that its length $= a + b$.

Paper XLV.

1. Find the sum of n terms of the series

$$1 + 2x + 3x^2 + \ldots . + (n - 1)x^{n-2} + nx^{n-1}.$$

2. Shew that the least value of

$$ae^{kx} + be^{-kx} \text{ is } 2\sqrt{ab}.$$

3. Find the sum to infinity of the series

(1) $\cos\theta \sin\theta + \cos^2\theta \sin 2\theta + \cos^3\theta \sin 3\theta + \ldots .$

(2) $(1+2)\log_e 2 + \dfrac{1+2^2}{\lfloor 2}(\log_e 2)^2 + \dfrac{1+2^3}{\lfloor 3}(\log_e 2)^3 + \ldots .$

4. Prove that

$$\cos x \sin\left(x + \frac{n\pi}{2}\right) + n\cos\left(x + \frac{\pi}{2}\right)\sin\left\{x + \frac{(n-1)\pi}{2}\right\} + \ldots .$$

$$+ \frac{n(n-1)\ldots.(n-r+1)}{\lfloor r}\cos\left(x + \frac{r\pi}{2}\right)\sin\left\{x + \frac{(n-r)\pi}{2}\right\}$$

$$+ \ldots . \text{ to } (n+1) \text{ terms} = 2^{n-1}\sin\left(2x + \frac{n\pi}{2}\right).$$

5. Prove that the area of any triangle is a mean proportional between the areas of two triangles formed, the one by joining the points of contact of any one of the four circles touching the sides of the triangle, and the other by joining the centres of the other three.

6. ABC is a triangle, and any straight line CF is drawn meeting AB in F. The angles BFC, AFC are bisected by FD, FE meeting the opposite sides in D, E. Prove that AD, BE, CF are concurrent.

7. If two straight lines, inclined to each other at a given angle α, be drawn from the focus of an ellipse, and tangents to the ellipse be drawn at their extremities, prove that the locus of the point of intersection of these tangents is an ellipse, a parabola, or an hyperbola, according as the eccentricity of the ellipse is respectively less than, equal to, or greater than $\cos\frac{\alpha}{2}$.

Paper XLVI.

1. Shew that three of the roots of the equation

$$x^6-1-(a+b+c)x(x^4+1)+(ab+bc+ca)x^2(x^2-1)$$
$$+(a+b+c-abc)x^3$$
$$=x^4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-x^3\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)-x^2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

are

$$a+\frac{1}{b+}\,\frac{1}{a+}\ldots,\quad b+\frac{1}{c+}\,\frac{1}{b+}\ldots,\quad c+\frac{1}{a+}\,\frac{1}{c+}\ldots.$$

Find the other roots.

2. Prove that

$$\log_e 2 = 2\left\{\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\ldots\right\}$$
$$+\left\{\frac{1}{3^2}+\frac{1}{2}\cdot\frac{1}{3^4}+\frac{1}{3}\cdot\frac{1}{3^6}+\ldots\right\}$$

3. Shew that

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}$$
$$+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=1.$$

4. The angles subtended by a hill at the base and summit of a tower of height a are respectively α and β. Prove that the height of the hill is given by the equation

$$x^2\cos\alpha\sin(\beta-\alpha)-ax\sin^2\alpha\sin\beta+a^2\sin^2\alpha\sin\beta=0.$$

5. Four circles are described, each passing through two adjacent angular points of a square, and also through a point P on one of the diagonals. A quadrilateral is described such that each angular point lies on the circumference of one of the circles, and each side passes through one of the angular points of the square. Shew that the quadrilateral may have a circle described about it with its centre at P, and that its diagonals are equal and at right angles.

6. Given the focus and auxiliary circle of an hyperbola, shew how to find any number of points on the curve.

7. Shew that chords of a rectangular hyperbola which subtend a right angle at one of the foci, touch a confocal and coaxial parabola.

Paper XLVII.

1. If the equation

$$\sqrt{x^2+x(b-c)+a^2}+\sqrt{x^2+x(c-a)+b^2}$$
$$+\sqrt{x^2+x(a-b)+c^2}=0$$

be rationalized it will take the form

$$x^4+2px^2+qx+r=0$$

where r is the same function of a^2, b^2, c^2 that p is of a, b, c, and

$$q=\frac{4}{3}(b-c)(c-a)(a-b).$$

2. If n things be arranged in any fixed order, and divided into two groups of r and $n - r$ things respectively, and the things in each group be then permutated in any way amongst themselves, $f(r)$ being the whole number of permutations that can thus be formed, prove that

$$\frac{1}{f(0)} + \frac{1}{f(1)} + \frac{1}{f(2)} + \dots + \frac{1}{f(n)} = \frac{2^n}{\lfloor n}.$$

3. Prove that

(1) $\tan 36° = \sqrt{5} \tan 18°$.

(2) $\tan 9° = \dfrac{\sqrt{5}+1}{4}(4 - \sqrt{10 + 2\sqrt{5}})$.

4. Given the base, area, and difference of the squares of the sides, shew how to construct the triangle.

5. From an external point O two tangents are drawn to a parabola, and from the points where they meet the directrix two other tangents are drawn meeting the tangents from O at A and B. Prove that AB passes through the focus S, and that OS is at right angles to AB.

6. If G be the centre of gravity of n equal particles arranged at equal intervals along the arc of a circle which subtends an angle $2a$ at the centre O, shew that, if r be the radius of the circle,

$$OG = \frac{r}{n} \cdot \frac{\sin \dfrac{na}{n-1}}{\sin \dfrac{a}{n-1}}.$$

Deduce the position of the centre of gravity of a circular arc.

7. From a given point (h, k) perpendiculars are drawn to the axes, inclination ω, and their feet are joined. Prove that the length of the perpendicular from (h, k) on this line is

$$\frac{hk \sin^2\omega}{\sqrt{h^2 + k^2 + 2hk \cos \omega}},$$

and that its equation is $hx - ky = h^2 - k^2$.

Paper XLVIII.

1. Prove that

$$\begin{vmatrix} (b-c)^2, & c^2, & b^2 \\ c^2, & (c-a)^2, & a^2 \\ b^2, & a^2, & (a-b)^2 \end{vmatrix} \equiv 16(u-bc)\,(u-ca)\,(u-ab)$$

where $2u = bc + ca + ab$.

2. Shew that if

$$\frac{\alpha^2(b-c)}{x-a} + \frac{\beta^2(c-a)}{x-b} + \frac{\gamma^2(a-b)}{x-c} = 0$$

has equal roots, then

$$\pm\,\alpha(b-c) \pm \beta(c-a) \pm \gamma(a-b) = 0.$$

3. Shew that the sum to infinity of the series

$a \sin\theta + a^2 \sin 2\theta + a^3 \sin 3\theta + \ldots .$ is $\dfrac{a\sin\theta}{1 - 2a\cos\theta + a^2}$.

4. If

$$y = \pi \cdot \frac{5}{4} \cdot \frac{17}{16} \cdot \frac{37}{36} \cdot \ldots . \text{ to infin.}$$

$$x = \cdot \frac{2}{1} \cdot \frac{10}{9} \cdot \frac{26}{25} \cdot \frac{50}{49} \cdot \ldots . \text{ to infin.}$$

prove that $4x^2 - y^2 = 4$.

5. Prove that the intersection of the diagonals of a square described on the hypotenuse of a right-angled triangle is equidistant from the two sides containing the right angle.

6. A straight line moves in such a manner that the sum of the squares on its distances from two given points is constant. Prove that it always touches an ellipse or hyperbola, the square on whose transverse axis is equal to twice the sum of the squares on the distances of the moving line from the given points, and the given points are on the conjugate axis at distances from the centre $= CS$.

7. If PM, PN be perpendiculars from any point of an ellipse on the axes, and the tangent at P meet the equi-conjugate diameters in Q and R, shew that the tangents from Q and R to the ellipse will be parallel to MN.

PAPER XLIX.

1. If

$$1 \cdot \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} + n \cdot \frac{\underline{|n}}{\underline{|r+1}\,\underline{|n-r-1}}$$

$$+ \frac{n\,.\,\overline{n-1}}{\underline{|2}} \cdot \frac{\underline{|n}}{\underline{|r+2}\,\underline{|n-r-2}} + \ldots \qquad = f_1(r)$$

$$f_1(0) \cdot \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} + f_1(1) \cdot \frac{\underline{|n}}{\underline{|r+1}\,\underline{|n-r-1}}$$

$$+ f_1(2) \cdot \frac{\underline{|n}}{\underline{|r+2}\,\underline{|n-r-2}} + \ldots \qquad = f_2(r)$$

$$f_m(0) \cdot \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} + f_m(1) \cdot \frac{\underline{|n}}{\underline{|r+1}\,\underline{|n-r-1}}$$

$$+ f_m(2) \cdot \frac{\underline{|n}}{\underline{|r+2}\,\underline{|n-r-2}} + \ldots = f_{m+1}(r)$$

Find the value of $f_m(n)$, and shew that it is equal to

$$\frac{\underline{|(m+1)n}}{\underline{\left|\frac{(m+2)n}{2}\right.}\,\underline{\left|\frac{mn}{2}\right.}} \text{ or } \frac{\underline{|(m+1)n}}{\underline{\left|\frac{(m+3)n}{2}\right.}\,\underline{\left|\frac{(m-1)n}{2}\right.}}.$$

according as m is even or odd.

2. If a, b, c be positive integers, shew that

$$a^{\frac{a}{a+b+c}} \,.\, b^{\frac{b}{a+b+c}} \,.\, c^{\frac{c}{a+b+c}} > \tfrac{1}{3}(a+b+c)$$

3. If the measures of the angles of a triangle referred to 1°, $100'$, $10{,}000''$ as units be in the proportion $2:1:3$, find the angles.

4. If $\tan^2\theta + \cot^2\theta - 1$ has its least value, shew that θ must be one of the angles $(n \pm \frac{1}{4})\pi$, where n is integral or zero.

5. Draw through a given point a straight line such that its two intercepts by three given straight lines meeting in a point may be in a given ratio.

6. $ABCD$ is a parallelogram, and E is a fixed point in BC. Divide AB, AD into any the same number of equal parts, and join E and C with corresponding points of AB and AD. Shew that the locus of intersection of these lines is an hyperbola or ellipse according as E lies on CB or CB produced through B.

7. Find the relation between the coefficients of the two equations

$$ax^2 + by^2 + 2cxy + 2a'x = 0$$
$$\alpha x^2 + \beta y^2 + 2\gamma xy + 2\alpha' x = 0$$

in order that the two curves may touch each other at two points.

PAPER L.

1. If $\xi = lx + my + nz$; $\eta = nx + ly + mz$; $\zeta = mx + ny + lz$; and if the same equations be true for all values of x, y, z, when ξ, η, ζ are interchanged with x, y, z respectively, shew that the real values of l, m, n are given by $-2l = m = n = \pm\frac{2}{3}$.

2. Find the coefficient of x^n in the expansion in ascending powers of x

$$\frac{x^2 + px + q}{(x-a)(x-b)(x-c)}.$$

3. If A, B, C are the angles of a triangle, prove that

$$\cos A \operatorname{cosec} B \operatorname{cosec} C + \cos B \operatorname{cosec} C \operatorname{cosec} A + \cos C \operatorname{cosec} A \operatorname{cosec} B = 2.$$

4. If a straight line join the points where an escribed circle touches the produced sides, and corresponding lines be drawn for the other escribed circles so as to form a triangle, prove that the lines joining corresponding vertices of the triangles are perpendicular to the sides of the original triangle, and are equal to the radii of the corresponding escribed circles.

5. The straight line which bisects the exterior angle A of a triangle cuts the base BC produced in D, and the circumscribed circle in E. Shew that the rectangle $AB \,.\, AC$ together with the square on AD = rectangle $DB \,.\, DC$; and that the rectangle $AB \,.\, AC$ = rectangle $AE \,.\, AD$.

6. In an ellipse, from the extremities of the diameter which is perpendicular to one of the equi-conjugate diameters chords are drawn parallel to the other. Prove that these chords are normal to the ellipse.

7. If P be any point on an ellipse, and L, M, N the points of contact of the ellipse with the three circles of curvature through P, shew that the area of the triangle LMN is constant, and equal to that of the maximum triangle which can be inscribed in the ellipse.

Paper LI.

1. Solve the equations

$$(1)\ \begin{cases} (b+c)x + (c+a)y + (a+b)z = 0 \\ (b-c)x + (c-a)y + (a-b)z = 0 \\ \dfrac{x}{\frac{c}{a}-\frac{a}{b}} + \dfrac{y}{\frac{a}{b}-\frac{b}{c}} + \dfrac{z}{\frac{b}{c}-\frac{c}{a}} = n(bc+ca+ab) \end{cases}$$

(2) $\sqrt{(2b - a + x)(b - 2a + x)} + \sqrt{(x - a)(x + b)}$
$= \sqrt{(x + b)(x + 2b - a) - 4ab}$.

(3) $\begin{cases} x^2y^2(x^4 - y^4) = a^2 \\ xy(xy^3 - 1)(x^2 + y^2) = a \end{cases}$

2. Shew that no number of the form $8x + 7$ can be the sum of 3 square numbers.

3. Prove that, if $\alpha, \beta, \gamma, \delta$ be all different

$$\frac{\cos 2\alpha}{\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}} + \frac{\cos 2\beta}{\sin\frac{\beta-\alpha}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\beta-\delta}{2}}$$
$$+ \frac{\cos 2\gamma}{\sin\frac{\gamma-\alpha}{2}\sin\frac{\gamma-\beta}{2}\sin\frac{\gamma-\delta}{2}} + \frac{\cos 2\delta}{\sin\frac{\delta-\alpha}{2}\sin\frac{\delta-\beta}{2}\sin\frac{\delta-\gamma}{2}}$$
$$= 8\sin\frac{\alpha+\beta+\gamma+\delta}{2}.$$

4. A straight line cuts three concentric circles in A, B, C, and passes at a distance p from their centre. Shew that the area of the triangle formed by the tangents at A, B, C is

$$\frac{BC \,.\, CA \,.\, AB}{2p}.$$

5. If any rectangle circumscribe an ellipse, prove that the perimeter of the parallelogram formed by joining the points of contact is equal to twice the diameter of the director circle.

6. The angular points of one triangle lie on the sides of another. If the latter triangle be thus divided into four equal parts, prove that the lines joining its angular points with the corresponding angular points of the former triangle will be bisected by the sides of the former triangle.

7. If $xy = c^2$ be the equation to a rectangular hyperbola, and x_1, x_2, x_3, x_4 be the abscissæ of the angular points and orthocentre of a triangle inscribed in it, shew that

$$x_1 x_2 x_3 x_4 = -c^4.$$

PAPER LII.

1. Prove that if n be a positive integer > 1,

$15^{2n} + 5^{4n} - 7 \cdot 5^{2n+1} - 3^{2n} + 34$ is divisible by 2304.

2. If D, E, F be the feet of the perpendiculars from A, B, C upon the opposite sides, and O the orthocentre of the triangle ABC, and p_1, p_2, p_3, p_4 be the lengths of the perpendiculars from A, B, C, O upon the sides of the triangle DEF, shew that p_1, p_2, p_3, p_4 are the roots of the equation

$$x^4 - 2Rx^3 + \left(\frac{\Delta^2}{R^2} - 2Rr' - r'^2\right)x^2 - \frac{\Delta^2}{R^2} \cdot r'^2 = 0$$

where R is the radius of the circle circumscribing ABC,

„ r' „ „ „ „ inscribed in DEF,

and Δ the area of the triangle ABC.

3. Shew that there can always be found n consecutive integers, no one of which is a prime number, however great n may be.

4. If from a point P in one of two circles, a straight line PQq be drawn cutting the other, prove that the rectangle $PQ \cdot Pq$ = twice the rectangle between PH the perpendicular from P on the chord of intersection, and AB the distance between the centres.

5. Through one of the points of intersection of two circles shew how to draw a straight line such that the product of the segments cut off from it by the circles shall be a maximum.

6. P, Q, R, S are any four points on a parabola. RS meets the diameter through Q in L, and PQ meets the diameter through S in K. Shew that KL is parallel to PR.

7. The points in which the tangents at the extremities of the transverse axis of an ellipse are cut by the tangent at any point of the curve are joined one with each focus. Prove that the point of intersection of the joining lines lies in the normal at the point.

Paper LIII.

1. Sum to n terms the series

(1) $1 - 2 - 3 - 2 + 1 + 6 + \ldots.$

(2) $2 + 5 + 12 + 31 + 86 + 249 + \ldots.$

(3) $\dfrac{2}{1 \,.\, 4 \,.\, 7} + \dfrac{5}{4 \,.\, 7 \,.\, 10} + \dfrac{8}{7 \,.\, 10 \,.\, 13} + \ldots.$

2. Prove that any even square $(2n)^2$ is equal to the sum of n terms of one series of integers in A.P., and that any odd square $(2n + 1)^2$ is equal to the sum of n terms of another increased by unity.

Also shew that the common difference in the two series is the same.

3. From the top of a hill the depression of a point on the plane below is 30°, and from a spot three-quarters of the way down the depression is 15°. If θ be the inclination of the hill, shew that

$$\tan\theta = \frac{3}{3\sqrt{3} - 2}.$$

4. The point in which the external bisector of one angle of a triangle again cuts the circumscribed circle is equidistant from the other two angular points of the triangle, and from the centres of two escribed circles.

5. Let AC, CB be diameters of two circles touching each other in C, and let AB be bisected in D. Shew that if a circle be described with centre D cutting the circles in E and F, the straight line EF will pass through C.

6. Find an expression for the product of the perpendiculars from the centre of the circumscribing circle and the orthocentre on any side of a triangle, and shew that it is the same for each of the sides.

7. A triangle is inscribed in a parabola, and another triangle similar and similarly situated circumscribes it. Prove that the sides of the former triangle are respectively four times the corresponding sides of the latter.

PAPER LIV.

1. Sum to n terms the series

(1) $4 + 18 + 48 + 100 + 180 + 294 + \ldots.$

(2) $5 + 11 + 22 + 41 + 74 + 133 + \ldots.$

(3) $$\frac{5}{1.2.3} + \frac{8}{2.3.4}\left(\frac{1}{3}\right) + \frac{15}{3.4.5}\left(\frac{1}{3}\right)^2 + \frac{34}{4.5.6}\left(\frac{1}{3}\right)^3 + \frac{89}{5.6.7}\left(\frac{1}{3}\right)^4 + \frac{252}{6.7.8}\left(\frac{1}{3}\right)^5 + \ldots.$$

2. If m and n be two prime numbers, shew that

$$m^{n-1} + n^{m-1}$$

when divided by mn leaves a remainder 1.

3. Two ships are sailing uniformly in parallel directions, and a person in one of them observes the bearing of the other to be $\alpha°$ from the North; p hours afterwards its bearing was $\beta°$; and q hours after that it was $\gamma°$. Prove that the course of the vessels is $\theta°$ from the North, where θ is given by

$$\tan\theta = \frac{p\sin\alpha\sin(\beta-\gamma) - q\sin\gamma\sin(\alpha-\beta)}{p\cos\alpha\sin(\beta-\gamma) - q\cos\gamma\sin(\alpha-\beta)}.$$

4. ABC is a straight line divided at any point B into two portions. ADB, CDB are similar segments of circles on AB and BC. Shew that if CD, AD be produced to meet the circles in F and E respectively, ABF and CBE are similar isosceles triangles.

5. The normal to an ellipse meets the axes in G and g. Find where the normal must be drawn in order that the triangle CGg may be a maximum.

6. If a point situated at the orthocentre of a triangle be acted on by three forces represented in magnitude and direction by its distances from the angular points of the triangle, shew that the resultant force will pass through the centre of the circumscribing circle, and will be represented in magnitude by twice the distance from the point to the centre.

7. From any point on the curve

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 p^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{b^2x^2}{a^2} + \frac{a^2y^2}{b^2}\right)$$

tangents are drawn to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Shew that the length of the chord of contact is equal to $2p$.

PAPER LV.

1. Shew that there cannot be found in any scale three different digits such that the three numbers formed from them by placing each digit differently in each number shall be in A.P., unless the radix of the scale exceed by unity a multiple of 3. If this condition be satisfied, and the radix be $3p + 1$, there are then $2(p - 1)$ such sets of digits; and the common difference of the progressions is in all of them the same.

2. Given the relations

$$a_1x^2 - a_2xy = (\mu_2 - \nu_2)\rho$$
$$b_1x^2 - b_2xy = (\nu_2 - \lambda_2)\rho$$
$$c_1x^2 - c_2xy = (\lambda_2 - \mu_2)\rho$$

in which the suffixes 1 and 2 may be interchanged throughout; given also

$$a_1\lambda_1 + b_1\mu_1 + c_1\nu_1 = a_2\lambda_2 + b_2\mu_2 + c_2\nu_2 = \sigma$$
$$\text{and } x^2 - y^2 = \rho^2,$$

shew that

$$\mu_1\nu_2 - \mu_2\nu_1 + \nu_1\lambda_2 - \nu_2\lambda_1 + \lambda_1\mu_2 - \lambda_2\mu_1 = \rho\sigma.$$

3. If a triangle be solved from the observed parts $C = 75°$, $b = 2$, $a = \sqrt{6}$, shew that an error of $10''$ in the value of C would cause an error of about $3''\cdot 66$ in the calculated value of B.

4. $ABCDE$ is a regular pentagon in a circle, and P the middle point of the arc AB. Prove that AP together with the radius of the circle is equal to PC.

5. Given the circumscribed and inscribed circles of a triangle, prove that the centres of the escribed circles lie on a fixed circle.

6. Any tangent to an ellipse meets the director circle in P and Q. Shew that CP and CQ are the directions of conjugate diameters.

7. Find the position of a point such that the sum of its distances from the vertices of a triangle may be a minimum.

PAPER LVI.

1. Solve the equations

$$(1)\quad \left.\begin{aligned} x^4y^3 + x^3y^4 &= 27(2x^2y^2 - x - y) \\ x^2y + xy^2 &= 3(4xy - x - y) \end{aligned}\right\}$$

$$(2)\quad \left.\begin{aligned} b^2y^{-1} + c^2z^{-1} &= x \\ c^2z^{-1} + a^2x^{-1} &= y \\ a^2x^{-1} + b^2y^{-1} &= z \end{aligned}\right\}.$$

$$(3)\quad (x^2 - a^2)\sqrt{4a^2 - x^2} = 2a^3.$$

2. Shew that when n is integral the value of the expression

$$4(1 + \sqrt{-3})^{6n-1} - (1 - \sqrt{-3})^{6n+1} \text{ is zero.}$$

3. If

$$\tan^2 x + \sec 2x = \frac{7\sqrt{3} - 10}{\sqrt{3}},$$

prove that

$$\cos 2x = -\frac{5 + 4\sqrt{3}}{23}.$$

4. If $A_1, A_2, \ldots . A_{2n+1}$ be the angular points of a regular polygon inscribed in a circle, and O any point on the circumference between A_1 and A_{2n+1}; prove that the sum of the lengths $OA_1 + OA_3 + OA_5 + \ldots .$ will be equal to the sum of $OA_2 + OA_4 + OA_6 + \ldots .$

5. Prove that if two chords be drawn through a fixed point in the interior of a circle at right angles to one another, the angular points of the quadrilateral formed by the four tangents at the ends of the chords will lie on a circle, the diagonals will pass through the fixed point, and the angles between them will be bisected by the chords.

6. If Q be a point on the major axis of an ellipse, O the centre, and P a point on the ellipse such that $OP = BQ$, shew that $AQ = SP$, and $A'Q = S'P$.

7. If from any point P of a parabola perpendiculars PN, PM be drawn on the axis and the tangent at the vertex, shew that the line MN always touches another parabola.

Paper LVII.

1. Find the sum to n terms of the series

(1) $6 + 7 + 18 + 45 + 94 + 171 + \ldots .$

(2) $7 + 13 - 7 - 181 - 1149 - 6111 - \ldots .$

(3) $$-\frac{8}{1 \cdot 2 \cdot 3} - \frac{2}{2 \cdot 3 \cdot 4} \cdot \frac{1}{2} + \frac{9}{3 \cdot 4 \cdot 5} \cdot \frac{1}{2^2} + \frac{28}{4 \cdot 5 \cdot 6} \cdot \frac{1}{2^3} + \frac{61}{5 \cdot 6 \cdot 7} \cdot \frac{1}{2^4} + \frac{120}{6 \cdot 7 \cdot 8} \cdot \frac{1}{2^5} + \ldots .$$

2. Assuming that

$$\sqrt{N} = A + \frac{1}{q_1 +} \frac{1}{q_2 +} \ldots . + \frac{1}{q_2 +} \frac{1}{q_1 +} \frac{1}{2A +} \ldots .$$

and that n is the number of the recurring quotients q_1, $q_2, \ldots . 2A$, if $\frac{P_n}{Q_n}, \frac{P_{2n}}{Q_{2n}}$ be the n^{th} and $2n^{\text{th}}$ convergents to $\sqrt{N}$, prove that

$$Q_{2n} = 2P_n Q_n; \text{ and } P_{2n} = 2P_n^2 + (-1)^{n+1}.$$

3. If
$\sin 2\phi - \sin\phi = 1 - \sqrt{2}\sin\theta$, and $\cos 2\phi + \cos^2\theta = 0$; find θ and ϕ.

4. Prove that in any triangle
$$(\sin A+\sin B)(\cos B+\cos C)(\cos C+\cos A)+(\sin B+\sin C)(\cos C+\cos A)(\cos A+\cos B)+(\sin C+\sin A)(\cos A+\cos B)(\cos B+\cos C)\equiv(\sin A+\sin B)(\sin B+\sin C)(\sin C+\sin A).$$

5. A hexagon, two of whose sides are of length a, two of length b, and two of length c is inscribed in a circle of diameter d. Prove that
$$d^3 = (a^2 + b^2 + c^2)d + 2abc.$$
and that the difference between the square of the area of the hexagon and the square of the area of a triangle whose sides are $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$ is $abcd + \frac{1}{4}d^4$.

6. Prove that if a circle be described with its centre on a fixed circle and passing through a fixed point, the perpendicular from the fixed point on the common chord is of constant length.

7. From the foci of an ellipse perpendiculars are let fall on the tangent at any point. With the feet of these perpendiculars as foci, an ellipse is described touching the major axis of the given ellipse. Prove that the point at which it touches the axis major will be the foot of the ordinate of the given point, and that the ellipse described will be similar to the given ellipse.

Paper LVIII.

1. Solve the equations

(1) $\dfrac{x^2}{a} + \dfrac{y^2}{b} = \dfrac{a^2}{x} + \dfrac{b^2}{y} = a + b.$

(2) $a(1-x^2+y^2)+b(xy-x^2)=b(1-x^2+y^2)+a(y^2-xy)=c.$

2. Shew that if $n > 3$

$$n^3 + \frac{n(n-1)}{\lfloor 2}(n-2)^3 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}(n-4)^3 + \dots$$
$$= n^2(n+3)2^{n-4}.$$

3. If

$$\frac{\sin(\alpha+\theta)}{\sin(\alpha+\phi)} + \frac{\sin(\beta+\theta)}{\sin(\beta+\phi)} = \frac{\cos(\alpha+\theta)}{\cos(\alpha+\phi)} + \frac{\cos(\beta+\theta)}{\cos(\beta+\phi)} = 2,$$

prove that either

$$\alpha \sim \beta = (2n+1)\frac{\pi}{2} \text{ or } \theta \sim \phi = 2n\pi.$$

4. Prove that if $x > \frac{1}{2}$

$$\tan\frac{1}{1+x^2} > \frac{1}{1+x+x^2} \text{ and } < \frac{1}{1-x+x^2}.$$

5. ABC is a triangle, and O the centre of its inscribed circle. Shew that AO passes through the centre of the circle which circumscribes BOC.

6. Find the equation to the straight line joining two given points on a parabola in terms of the ordinates of those points, and shew that if the difference of the ordinates be constant, the locus of the middle point of the chord is a parabola, which also envelopes the chord.

7. If P, p be corresponding points on an ellipse and the auxiliary circle, centre C, and if CP be produced to meet the auxiliary circle in q, and if Q be the point on the ellipse corresponding to q, prove that the tangent at Q is perpendicular to Cp, and that it cuts off from Cp a length equal to CP.

Paper LIX.

1. Convert $\frac{1}{17}$ into a circulating decimal, and explain why the period is such that its first sixteen multiples consist of the same digits in the same cyclical order.

2. Solve the equations

(1) $$\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b},$$

(2 $$\sqrt[4]{x}+\sqrt[4]{x-1}=\sqrt[4]{x+1}.$$

3. If $A+B+C=2\pi$, and if

$$\cos A=\frac{(d-a)(b-c)}{(d+a)(b+c)};\ \cos B=\frac{(d-b)(c-a)}{(d+b)(c+a)};$$

$$\cos C=\frac{(d-c)(a-b)}{(d+c)(a+b)},$$

prove that $\tan\frac{1}{2}A+\tan\frac{1}{2}B+\tan\frac{1}{2}C=\pm 1$.

4. ABC is a triangle inscribed in a circle, R is any point on the arc AB, a hexagon $ARBPCQ$ is completed, having its opposite sides parallel; triangles are formed by producing AR, BP, CQ and AQ, CP, BR respectively. Prove that these triangles are similar to ABC, and have their homologous sides parallel, and that the sum of the homologous sides is to the homologous side of ABC as

$$(\sin^2 A+\sin^2 B+\sin^2 C)\sin\theta : \sin A\sin B\sin C,$$

where θ is the angle between the homologous sides of the triangles and that of ABC.

5. The sides BC, CA, AB of a triangle cut a straight line in D, E, F. Through D, E, F three straight lines $OLDG$, $MEOH$, $KNOF$, having the common point O are drawn, cutting the sides CA, AB in L, G; AB, BC, in M, H; BC, CA in N, K. Prove that

$$\frac{AK.BG.CH}{AM.BN.CL}=\frac{AG.BH\ CK}{LA.MB.NC}=\frac{GD.HE\ KF}{LD.ME.NF}=\frac{HD.KE.GF}{ND.LE.MF}$$

6. A circle and a parabola touch one another at both ends of a double ordinate to the parabola. Prove geometrically that the latus rectum is a third proportional to the parts into which the abscissa of the points of contact is divided by the circle either internally or externally.

7. Prove that the equation of the locus of the points of intersection of pairs of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ inclined to one another at an angle α is

$$(x^2 + y^2 - a^2 - b^2)^2 = 4 \cot^2\alpha\, (a^2y^2 + b^2x^2 - a^2b^2).$$

Paper LX.

1. At the recent general election the whole number of Liberals returned was 15 more than the number of English Conservatives, and the whole number of Conservatives was 5 more than twice the number of English Liberals. The number of Scotch Conservatives was the same as the number of Welsh Liberals, and the Scotch Liberal majority was equal to twice the number of Welsh Conservatives, and was to the Irish Liberal majority as 2 : 3. The English Conservative majority was 10 more than the whole number of Irish members. The whole number of members is 652, of whom 60 are returned by Scotch constituencies. Find the number of each party returned by England, Scotland, Ireland, and Wales, respectively.

2. If $a < 1$, and $(1 + ax)(1 + a^3x)(1 + a^5x) \ldots\ldots$ be expanded in ascending powers of x, prove that the series is

$$1 + \frac{ax}{1-a^2} + \frac{a^4x^2}{(1-a^2)(1-a^4)} + \frac{a^9x^3}{(1-a^2)(1-a^4)(1-a^6)} + \ldots\ldots$$

3. If ABC be a triangle, shew that

$$\tan^2\frac{B}{2}\tan^2\frac{C}{2} + \tan^2\frac{C}{2}\tan^2\frac{A}{2} + \tan^2\frac{A}{2}\tan^2\frac{B}{2}$$

is always < 1;

Also shew that if one angle approach indefinitely near to two right angles, the least value of the expression is $\frac{1}{2}$.

4. If a straight line AB be bisected in C and produced to D so that sq. on AD = 3 sq. on CD, and if CB be bisected in E, shew that sq. on ED = 3 . sq. on EB.

5. The bisectors of the angles A, B, C of a triangle meet in O. Prove that the triangles OBC, OCA, OAB are proportional to $\sin A$, $\sin B$, $\sin C$.

6. Two tangents to an ellipse intersect at right angles. Prove that the sum of the squares on the chords intercepted on them by the auxiliary circle is constant.

7. Through a fixed point O a chord POQ of a hyperbola is drawn, and lines PL, QL are drawn parallel to the asymptotes. Shew that the locus of L is a similar and similarly situated hyperbola.

Paper LXI.

1. Prove that if x be less than 1,

$$\frac{x}{(1-x)(1-x^{2n+1})} + \frac{x^3}{(1-x^3)(1-x^{2n+3})} + \frac{x^5}{(1-x^5)(1-x^{2n+5})} + \dots \text{ to inf.}$$

$$= \frac{1}{1-x^{2n}}\left\{\frac{1}{1-x} + \frac{1}{1-x^3} + \frac{1}{1-x^5} + \dots + \frac{1}{1-x^{2n-1}}\right\}.$$

2. If A, B, C be the angles of a plane triangle, shew that the equation

$$\sqrt{y^2+z^2+2yz\cos A} + \sqrt{z^2+x^2+2zx\cos B} + \sqrt{x^2+y^2+2xy\cos C} = 0,$$

is identical with

$$(yz\sin A + zx\sin B + xy\sin C)^2 = 0.$$

3. Prove that if a quadrilateral be inscribed in a circle the (length)^2 of the line joining the points of intersection of opposite sides is

$$\frac{(ad+bc)(ab+cd)\{bd(c^2-a^2)^2 + ac(b^2-d^2)^2\}}{(b^2-d^2)^2(c^2-a^2)^2}.$$

4. The side BC of a triangle ABC is produced to D so that the triangles ABD, ACD are similar. Prove that AD touches the circle round ABC.

5. A series of confocal ellipses is cut by a confocal hyperbola. Prove that either focal distance of any point of intersection is cut by its conjugate diameter with respect to that particular ellipse in a point which lies on a circle.

6. P and D are any two points on an ellipse, and P', D' the corresponding points on the auxiliary circle. Shew that the tangents of inclination to the major axis of the two central radii, which bisect, the one PD, the other $P'D'$ are proportional to the lengths of the principal axes of the ellipse.

7. Shew that the envelope of the radical axis of a fixed circle and a variable point which lies on a fixed straight line is a parabola.

PAPER LXII.

1. Find the sum to n terms of the series

(1) $\dfrac{3}{1.4} + \dfrac{3.6}{1.4.7} + \dfrac{3.6.9}{1.4.7.10} + \dots.$

(2) $\dfrac{1}{1.3.5} + \dfrac{2}{3.5.7} + \dfrac{3}{5.7.9} + \dots.$

(3) $-3 + 2 + 13 + 28 + 39 + 26 - 55 - 296 \dots.$

2. If a, b, c are all real and positive, and if $a + b > c$, prove that

$$a^3 + b^3 + c^3 + 3abc > 2(a + b)c^2.$$

3. A circle is inscribed in a triangle, and a second triangle is formed whose sides are equal to the distances of the points of contact from the adjacent angular points of the triangle. If r be the radius of the circle inscribed in the first triangle, and ρ, ρ' the radii of the inscribed and circumscribed circles of the second triangle, shew that $r^2 = 2\rho\rho'$.

4. Given $\sin x = n\cos(x + a)$, expand x in ascending powers of n.

Prove that

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{2\sqrt{2}}.$$

5. AB is a chord of a conic. The tangents at A and B meet in T. Through B a straight line is drawn meeting the conic in C and AT in P. The tangent to the conic at C meets AT in Q. Shew that $TPQA$ is a harmonic range.

6. AB, BC, CD are three equal rods freely jointed at B and C. The rods AB, CD rest on two pegs in the same horizontal line so that BC is horizontal. If a be the inclination of AB, and β the inclination of the reaction at B to the horizon, prove that

$$3\tan a\tan\beta = 1.$$

7. Shew that, if θ denotes a variable angle, the envelope of the parabola

$$\frac{x^2}{a^2}\sin^2\theta - 2\frac{xy}{ab}\sin\theta\cos\theta + \frac{y^2}{b^2}\cos^2\theta + \frac{2x}{a}\cos\theta$$
$$+ \frac{2y}{b}\sin\theta - 2 = 0$$

is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Paper LXIII.

1. Find the cube root of $37 + 30\sqrt{3}$.

If the cube root of $a + \sqrt{b} + \sqrt{c} + \sqrt{d}$ can be expressed in the form $x + \sqrt{y} + \sqrt{z}$, prove that $\sqrt{\frac{bc}{d}}$,

$\sqrt{\dfrac{cd}{b}}$, $\sqrt{\dfrac{db}{c}}$ must all be rational. Also if $\sqrt{\dfrac{bc}{d}} = k$, then x^3 is a root of the equation

$$48t^2 + 6(5a - 9k)t + 3a^2 + d = 0.$$

Employ this method to shew that the cube root of the expression

$$16 + 14\sqrt{2} + 12\sqrt{3} + 6\sqrt{6} \text{ is } 1 + \sqrt{2} + \sqrt{3}.$$

2. Prove that $\dfrac{\cos x - \cos y \cos z}{\sin y \sin z}$ is approximately unaltered by simultaneous small increments $x'y'z'$ of x, y, and z, if

$$\frac{x'}{\tan\frac{x}{2}(\cos y + \cos z)} = \frac{y'}{\sin y} = \frac{z'}{\sin z}.$$

3. Shew that the difference between the sum of the reciprocals of all the even numbers and the sum of the reciprocals of all the odd numbers is ·69314718 nearly.

4. *AB* is any fixed straight line, *CD* a chord of a circle parallel to *AB*. *AC* being joined cuts the circle in *E*, and *BE* cuts the circle in *F*. Prove that *DF* will cut *AB* in a fixed point *G* which is the same for all chords.

5. *S* is the focus of an ellipse of eccentricity e. *F* is a fixed point on the major axis, and *P* any point on the curve. Shew that when *PF* is a minimum, $SP = \dfrac{1}{e} SF$.

6. A triangular lamina is supported at its three angular points, and a weight equal to that of the triangle is placed on it. Find the position of the weight if the pressures on the points of support are proportional to

$$4a + b + c,\ a + 4b + c,\ a + b + 4c,$$

a, b, c being the lengths of the sides of the triangle.

7. The diameter parallel to any focal chord of an ellipse is equal to the chord joining the points on the auxiliary circle which correspond to the extremities of the focal chord.

Paper LXIV.

1. Eliminate x, y, z from the equations

$$a^2 - \frac{1}{x} = b^2 - \frac{1}{y} = c^2 - \frac{1}{z},$$

$$a^2x^2 + b^2y^2 + c^2z^2 = 0\,;\; a^2x^3 + b^2y^3 + c^2z^3 = 0,$$

and shew that

$$a^{\frac{2}{3}}(b^2 - c^2)^{\frac{2}{3}} + b^{\frac{2}{3}}(c^2 - a^2)^{\frac{2}{3}} + c^{\frac{2}{3}}(a^2 - b^2)^{\frac{2}{3}} = 0.$$

2. Take all the integers from 2 to infinity, and raise each to all negative powers from 2 to infinity, and shew that the sum of all these terms is unity.

3. A workman is told to make a triangular enclosure of sides 50, 41, 21 yards respectively: but having made the first side 1 yard too long, what length must he make the other two sides in order that they may enclose the prescribed area with the prescribed length of fencing?

4. In any triangle ABC, prove that

$$\frac{b^{\frac{3}{2}} + c^{\frac{3}{2}}}{b^{\frac{1}{2}}c^{\frac{1}{2}}} \,.\, \cos A + \frac{c^{\frac{3}{2}} + a^{\frac{3}{2}}}{c^{\frac{1}{2}}a^{\frac{1}{2}}} \cos B + \frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}} \cos C$$
$$= a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}.$$

5. Given a triangle ABC, shew how to inscribe in it a triangle LMN so that the perimeter of LMN may be a minimum. Shew that L, M, N are then the feet of the perpendiculars from the angular points on the sides.

6. A boy stands on a sheet of ice balancing himself by means of a chair, but not leaning any of his weight on it. Shew that if the chair be heavier than the boy, he may incline his body to the vertical at an angle $\tan^{-1} 2\mu$; but

if the boy be heavier than the chair, he can only incline it to an angle $\tan^{-1}\dfrac{2\mu W'}{W}$, μ being the coefficient of friction between the boy and ice, and also between the chair and ice, W and W' being the weights of boy and chair respectively.

7. A line is drawn from the focus of a hyperbola parallel to an asymptote to meet the directrix. Prove that it is equal in length to half the latus rectum, and is bisected by the curve.

Paper LXV.

1. Find the sum to n terms of each of the following series

(1) $-1-3+3+23+63+129+\ldots.$

(2) $\dfrac{4}{3.8.13}+\dfrac{11}{8.13.18}+\dfrac{18}{13.18.23}+\ldots.$

(3) $\dfrac{1(2^3+3)}{\lfloor 3}+\dfrac{2(2^4+4)}{\lfloor 4}+\dfrac{3(2^5+5)}{\lfloor 5}+\ldots.$

2. Prove that

$$1^n-n.2^n+\frac{n(n-1)}{\lfloor 2}3^n-\ldots.(-1)^n(n+1)^n=(-1)^n\lfloor n.$$

3. Prove that when n is a positive integer

$$\sin^n\phi\cos n\theta+n\sin^{n-1}\phi\cos(n-1)\theta\sin(\theta-\phi)$$
$$+\frac{n(n-1)}{2}\sin^{n-2}\phi\cos(n-2)\theta\sin^2(\theta-\phi)+\ldots.$$
$$+\ldots.+\sin^n(\theta-\phi)=\sin^n\theta\cos n\phi.$$

4. A is the centre of a circle, and CAB a diameter. On CB produced take a point D so that $DB\,.\,DC^2=AD\,.\,AB^2$. With centre D and radius $=AB$ describe a circle cutting the given circle in E. Shew that the arc EB = [illegible] the circumference.

5. The six middle points of the sides and diagonals of a quadrilateral, the two points in which the opposite sides intersect, and the point in which the two diagonals intersect lie on a conic. Also shew that if the quadrilateral can be inscribed in a circle, the conic will be a rectangular hyperbola passing through the centre of the circle.

6. A bullet is fired in the direction towards a second equal bullet which is let fall at the same instant. Prove that the two will meet, and that if they coalesce, the latus rectum of their joint path will be $\frac{1}{4}$ the latus rectum of the original path of the first bullet.

7. The equation to a conic referred to the centre being $ax^2 + 2hxy + by^2 = c'$, and that referred to the axes being $a'x^2 + b'y^2 = c'$, prove that the sign of $a' - b'$ will be the same as that of h, provided the axis of x' make with the axis of x the least positive angle given by the equation

$$\tan 2\theta = \frac{2h}{a - b}.$$

Trace the curve

$$13x^2 + 2xy + 13y^2 - 22x + 50y - 23 = 0.$$

PAPER LXVI.

1. The equations

$$(1 + lx)\ (1 + ay) = 1 + lz$$
$$(1 + mx)\ (1 + by) = 1 + mz$$
$$(1 + nx)\ (1 + cy) = 1 + nz$$

cannot be true together unless

$$(b - c)\frac{a}{l} + (c - a)\frac{b}{m} + (a - b)\frac{c}{n} = 0.$$

If this condition holds, shew that

$$x = \frac{\frac{c}{n} - \frac{b}{m}}{b - c},$$

and that particular solutions for y and z will be

$$y = -\frac{1}{a},\ z = -\frac{1}{l},$$

with two similar sets. Also $y = 0$, and $x = z$ is another solution.

2. Eliminate ϕ, ϕ' from the equations

$$r = \frac{ab\cos(\theta - \phi)}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} = \frac{ab\cos(\theta - \phi')}{\sqrt{a^2\sin^2\phi' + b^2\cos^2\phi'}},$$

$$\text{and } \tan\phi\tan\phi' = -\frac{b^2}{a^2},$$

and shew that $2r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

3. Eliminate ϕ from the equations

$$a^3y\sin\phi + b^3x\cos\phi + ab(a^2\sin^2\phi + b^2\cos^2\phi) = 0$$
$$ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2.$$

4. The triangle ABC is right-angled at A, and on the sides AB, AC are described squares $ADEB$, $AFGC$. Prove that BG and CE meet on the perpendicular from A on BC.

5. TP and TQ are tangents to a parabola, and O is the centre of the circle round TPQ. Shew that the angle TSO is a right angle.

6. Forces P and Q act at a point O, and their resultant is R. If any transversal cut their directions in the points L, M, N respectively, shew that

$$\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}.$$

7. Prove that in order to produce the greatest deviation in the direction of a smooth billiard ball of diameter a by impact on another equal ball at rest, the former must be projected in a direction making an angle

$$\sin^{-1}\frac{a}{c}\sqrt{\frac{1-e}{3-e}}$$

with the line (of length c) joining the two centres, e being the coefficient of elasticity.

Paper LXVII.

1. If $x^2 + ax + b$ and $x^2 + a'x + b'$ have a common measure of the first degree, prove that their L.C.M. is

$$x^3 + \frac{ab - a'b'}{b - b'} \cdot x^2 + \frac{a'b^2 - ab'^2}{a'b - ab'} x + \frac{bb'(a - a')}{b - b'}$$

2. Two persons A and B set out together to walk from P to Q, each completing the distance in 4 hours. A increases his rate at the end of each quarter of an hour by the same quantity, and B increases his by that same quantity at the end of each half hour. After one hour they are $\frac{3}{16}$ of a mile apart, and after 2 hours they are $\frac{1}{4}$ of a mile apart. Find the distance from P to Q.

3. Shew that if a quadrilateral whose sides taken in order are a, b, c, d be such that a circle can be inscribed in it, the circle is the greatest when the quadrilateral can be inscribed in a circle. When this is the case, shew that the square on the radius of the inscribed circle is

$$\frac{abcd}{(a + c)(b + d)}.$$

4. A tree blown over by the wind falls on another, partially uprooting it. Shew that it is not possible for both their tops to be in the same straight line as before.

5. Describe a parabola passing through three given points, and having its axis parallel to a given straight line.

Hence shew how to inscribe in a given parabola a triangle having its sides parallel to three given straight lines.

6. A quadrilateral $ABCD$ is composed of four unequal beams joined at the extremities, and is compressed by a force P along the diagonal AC. Prove that the force Q along the other diagonal BD requisite to resist compression is given by $\dfrac{P \,.\, AC}{Q \,.\, BD} = \dfrac{AO \,.\, OC}{BO \,.\, OD}$, O being the intersection of the diagonals.

7. A particle is projected from a platform with velocity V, and elevation β. On the platform is a telescope fixed at elevation α. The platform moves horizontally in the plane of the particle's motion, so as to keep the particle always in the centre of the field of view of the telescope. Shew that the original velocity of the platform must be $V \,.\, \dfrac{\sin(\alpha - \beta)}{\sin \alpha}$, and its acceleration $g \cot \alpha$.

Paper LXVIII.

1. Resolve into its component factors the expression

$$(a^3 + b^3 + c^3)xyz + (b^2c + c^2a + a^2b)(y^2z + z^2x + x^2y)$$
$$+ (bc^2 + ca^2 + ab^2)(yz^2 + zx^2 + xy^2) + (x^3 + y^3 + z^3)abc + 3abcxyz.$$

2. If m, n, p be prime numbers, shew that the expression

$$(np)^{m-1} + (pm)^{n-1} + (mn)^{p-1}$$

leaves a remainder 1 when divided by mnp.

3. Circles are described on the sides a, b, c of a triangle as diameters. Prove that the diameter D of a circle which touches the three externally is such that

$$\sqrt{\frac{D}{s-a} - 1} + \sqrt{\frac{D}{s-b} - 1} + \sqrt{\frac{D}{s-c} - 1} = \sqrt{\frac{s}{D-s}}$$

where $2s = a + b + c$.

4. ABC is a triangle, and AD, BE, CF are the perpendiculars on the opposite sides, intersecting in P. D', E', F' are the middle points of these sides. Shew that the

three straight lines which join D', E', F' with the middle points of AP, BP, CP respectively are equal and concurrent; and the three straight lines which join D', E', F' with the middle points of AD, BE, CF respectively meet in a point (the Symmedian point).

5. From the vertex A of a parabola AY is drawn perpendicular to the tangent at P, and YA produced to meet the curve in Q. Shew that if PQ meet the axis in R, $AR = 2AS$.

6. ABC is a triangle, and P any point in the plane of ABC. P is acted on by forces represented by PA, PB, PC. If the magnitude of their resultant is constant, shew that the locus of P is a circle.

Hence find the position of P when there is equilibrium.

7. AB is the range of a projectile on a horizontal plane. Shew that if t be the time from A to any point P of the trajectory, and t' the time from P to B, the vertical height of P above AB is $\frac{1}{2}gtt'$.

Paper LXIX.

1. A certain number is divided into two parts in the ratio $x : y$, the former part being 120. When divided into two parts in the ratio $x : z$ the former part is 140; and when divided into two parts in the ratio of $y : z$ the former part is 126. Find the number.

2. Solve the equations

(1) $(27x + 4)^2 + (17x + 253)^2 = (32x + 45)^2$.

(2) $$x^2 + a(2x + y + z) = y^2 + b(x + 2y + z)$$
$$= z^2 + c(x + y + 2z) = (x + y + z)^2.$$

(3) $yz - x^2 = a$; $zx - y^2 = b$; $xy - z^2 = c$.

3. O is a fixed point in the plane of a circle, and P, Q two variable points on its circumference. Shew that if the sum of the squares on OP, OQ be constant, the middle point of PQ will lie on a fixed straight line, and the line PQ will envelop a parabola.

4. If $\log \sin (\theta + \phi\sqrt{-1}) = a + \beta\sqrt{-1}$, where θ, ϕ, a, β are real, prove that

$$2 \cos 2\theta = \epsilon^{2\phi} + \epsilon^{-2\phi} - 4\epsilon^{2a}, \ \cos(\theta - \beta) = \epsilon^{2\phi} \cos(\theta + \beta).$$

5. If A be the vertex, and BC the base of the isosceles triangle in *Euclid* IV. 10, and if two circles be drawn passing through A and touching BC at its extremities, then if these two circles cut the sides in D, E, and one another in F, FD and FE will be perpendicular respectively to the tangents to the circles at A.

6. A ball whose coefficient of restitution is e, is projected at right angles to a plane (angle a) from a point on the plane with velocity v. Shew that before ceasing to bound it will have described along the plane a distance

$$\frac{2v^2 \sin a}{g \cos^2 a} \cdot \frac{1}{(1 - e)^2}.$$

7. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation to an ellipse, shew that the locus of a point whose polar is a normal chord of the ellipse is

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = a^4 c^4.$$

PAPER LXX.

1. Solve the equations

(1) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0,$

(2) $x^4 + \frac{1}{4} = x\sqrt{2}\sqrt{x^4 - \frac{1}{4}}.$

(3) $(a - 1)(x + 1)(x + a^3) = x(a^4 - 1).$

2. Out of m persons who are sitting in a circle three are selected at random. Prove that the chance that no two of those selected are sitting next one another is

$$\frac{(m - 4)(m - 5)}{(m - 1)(m - 2)}$$

3. Shew how to draw a pair of equal circles on two parallel sides of a parallelogram as chords so as to touch each other; and shew that the circles so drawn on the two pairs of parallel sides intersect at angles equal to those of the parallelogram.

4. If D, E, F be the points of contact of the inscribed circle with BC, CA, AB respectively, shew that if the squares on AD, BE, CF are in Arithmetic Progression, the sides of the triangle are in Harmonic Progression.

5. Given in position four tangents to a parabola, find geometrically the focus, vertex, latus rectum, and points of contact.

6. Two equal particles on two inclined planes are connected by a string which lies wholly in a vertical plane perpendicular to the line of intersection of the planes, and passes over a smooth peg vertically above this line. If when the particles are on the point of motion the portions of the string make equal angles with the vertical, shew that the difference between the inclinations of the planes must be twice the angle of friction.

7. If a point be supposed to begin moving with an acceleration equal and opposite to that of gravity from the point of projection at the instant of projection, prove that at any subsequent time the particle will be moving directly away from the point and with a velocity which in the time elapsed would have carried it over the distance between them.

Paper LXXI.

1. Solve the equations

$$(1)\quad a + (b + x)\sqrt{\frac{x^2 + a^2}{x^2 + b^2}} = b + (x + a)\sqrt{\frac{x^2 + b^2}{x^2 + a^2}}.$$

$$(2)\quad \begin{cases} x^2 + a^2 + y^2 + b^2 = \sqrt{2}\{x(a + y) - b(a - y)\} \\ x^2 - a^2 - y^2 + b^2 = \sqrt{2}\{x(a - y) + b(a + y)\} \end{cases}$$

2. A candidate is examined in 3 papers to each of which is assigned m marks as a maximum. His total on the 3 papers is $2m$. Shew that there are

$$\frac{(m+1)(m+2)}{2}$$

ways in which this can occur.

3. Shew that if A, B, C, D be any plane angles

$$\cos B \sin\frac{A+B}{2}\sin\frac{C-D}{2}+\cos C\sin\frac{A+C}{2}\sin\frac{D-B}{2}$$
$$+\cos D\sin\frac{A+D}{2}\sin\frac{B-C}{2}$$
$$=2\sin\frac{C-D}{2}\sin\frac{D-B}{2}\sin\frac{B-C}{2}\sin\frac{A+B+C+D}{2}.$$

4. Shew that the value of θ which is $< 90°$ which satisfies the equation

$$7\tan^2\theta+8\sqrt{3}\tan\theta=1$$

is $3° \, 59' \, 16''\cdot 2$, having given $\log_{10} 2 = \cdot 3010300$;

$$L\sin 33° \, 59' = 9\cdot 7473743\,;\; L\sin 34° = 9\cdot 7475617.$$

5. A regular tetrahedron and a cube have the same volume, and the middle points of the faces of each are joined, thus forming another tetrahedron and an octohedron; and the centres of the faces of the octohedron are joined, thus forming a cube. Prove that the volume of this cube is equal to that of the smaller tetrahedron.

6. O is a fixed point in the plane of an ellipse; OE, OD are two straight lines parallel to any two conjugate diameters of the ellipse, and cutting in E and D a fixed straight line DE, which is parallel to the diameter conjugate to CO, where C is the centre of the ellipse. Shew that the circle round EOD passes through a fixed point.

7. Particles are projected from the same point with equal velocities. Prove that the vertices of their paths lie on an ellipse. If they be all equally elastic, and impinge on a vertical wall, prove that the vertices of their paths after impact lie on an ellipse.

PAPER LXXII

1. Establish the identities

(1) $(x^2 + 2yx)^3 + (y^2 + 2zx)^3 + (z^2 + 2xy)^3$
$- 3(x^2+2yz)(y^2+2zx)(z^2+2xy) \equiv (x^3+y^3+z^3-3xyz)^2.$

(2) $(ca-b^2)(ab-c^2)+(ab-c^2)(bc-a^2)+(bc-a^2)(ca-b^2)$
$\equiv (bc+ca+ab)(bc+ca+ab-a^2-b^2-c^2).$

2. Solve the equation

$$\cos x + \sin 3x + \cos 5x + \sin 7x + \ldots . + \sin (4n-1)x = \tfrac{1}{4}(\sec x + \operatorname{cosec} x).$$

3. Eliminate ϕ between

$$x\cos(\phi + a) + y\sin(\phi + a) = a\sin 2\phi$$
$$y\cos(\phi + a) - x\sin(\phi + a) = 2a\cos 2\phi$$

and shew that

$$(x\cos a + y\sin a)^{\frac{2}{3}} + (x\sin a - y\cos a)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

4. If from any point A in the plane of an ellipse perpendiculars AM, AN be drawn to the equi-conjugate diameters, shew that the diagonal of the parallelogram constructed with AM, AN for adjacent sides is perpendicular to the polar of A.

5. Let the circle inscribed in the triangle ABC touch the sides in D, E, F. Take $CD' = BD$, and let AD' cut the circle in P, Q, P being nearest A. Prove geometrically that $AP \cdot BC = AE \cdot PD'$.

6. Two weights P and Q, whose coefficients of friction are μ_1, μ_2, each less than $\tan a$, on a rough inclined plane (angle a), are connected by a string which passes through a fixed pulley A in the plane. Prove that if the angle PAQ be the greatest possible, the squares of the weights of P and Q are in the ratio $1 - \mu_2{}^2 \cot^2 a : 1 - \mu_1{}^2 \cot^2 a$.

7. A particle whose elasticity is ϵ is projected from a given point O in a horizontal plane with a velocity v in a

direction inclined at an angle a to the horizon. Shew that the distance of the point of n^{th} impact from O is

$$\frac{v^2 \sin 2a}{g} \cdot \frac{1 - \epsilon^n}{1 - \epsilon},$$

and that the time which elapses before the n^{th} impact is

$$\frac{2v \sin a}{g} \cdot \frac{1 - \epsilon^n}{1 - \epsilon}.$$

Paper LXXIII.

1. If

$$P = \cfrac{a}{a + \cfrac{b}{b + \cfrac{c}{c + \dots}}} \quad \text{and} \quad Q = \cfrac{a}{b + \cfrac{b}{c + \cfrac{c}{d + \dots}}}.$$

shew that $P(a + 1 + Q) = a + Q$.

2. If a, b, c denote the sides of a triangle, shew that

$$\tfrac{2}{3}(a + b + c)(a^2 + b^2 + c^2) > a^3 + b^3 + c^3 + 3abc.$$

3. If straight lines be drawn from the angles of a triangle ABC to the centre of the inscribed circle, cutting the circumference in D, E, F, shew that the angles D, E, F of the triangle DEF are respectively equal to

$$\tfrac{1}{4}(\pi + A),\ \tfrac{1}{4}(\pi + B),\ \tfrac{1}{4}(\pi + C).$$

4. Three circles have two common tangents. Shew that the square of the tangent drawn to any one of them S from either point of intersection of the other two S_1, S_2 is equal to the rectangle contained by the parts of either common tangent intercepted between S, S_1 and S, S_2.

5. PSQ, a focal chord of a conic, is produced to meet the directrix in K, and KM, KN are drawn through the feet of the ordinates PM, QN of P and Q. If KN and PM intersect in R, shew that $PR = PM$, and if KM and QN intersect in R', shew that $QR' = QN$.

6. Shew that the envelope of all equilateral hyperbolas which are concentric and which cut orthogonally the same straight line is

$$x^2 - y^2 - a^2 = 3(axy)^{\frac{2}{3}}$$

the centre being the origin, and the axis of x parallel to the given straight line

7. Two particles, each of mass m, are at rest side by side, when one is struck by an impulse B in a given direction, while a constant force F begins at the same instant to act upon the other in the same direction. If after traversing a space s in time t, they are again side by side, prove that $2B = Ft$; and $2B^2 = m \,.\, F \,.\, s$.

Paper LXXIV.

1. Eliminate x, y, z from the equations

$$\left.\begin{aligned} ax + yz &= bc \\ by + zx &= ca \\ cz + xy &= ab \\ xyz &= abc \end{aligned}\right\}$$

and shew that $b^3c^3 + c^3a^3 + a^3b^3 = 5a^2b^2c^2$.

2. A number of three digits in the scale of 7 also consists of three digits when expressed in the scale of 10, and the digits in the former scale are respectively double those in the latter scale. Find the number.

3. Find θ and ϕ from

$$\left.\begin{aligned} p\sin^4\theta - q\sin^4\phi &= p \\ p\cos^4\theta - q\cos^4\phi &= q \end{aligned}\right\}.$$

4 Two circles intersect in A and B, and through A a chord PAQ is drawn cutting off from each circle segments containing equal angles. Prove that the tangents at P and Q intersect on AB.

5. MT, MT' are two tangents drawn from any point M to an ellipse whose foci are S, H. Along these tangents take lengths MO, MO' equal respectively to MS and MH. Shew that $OO' = AA'$.

6. Two parallel smooth vertical walls are at a distance a from each other, and an elastic particle is projected from a point in one wall so as to impinge on the other wall in the vertical plane perpendicular to the two walls. Prove that if, on its return, the particle strike the wall horizontally, the direction of projection is given by the equation

$$\sin 2\,\theta = \frac{2ga}{v^2} \cdot \frac{1+e}{e},$$

v being the velocity of projection, and e the coefficient of elasticity.

7. Prove that the centre of gravity of a wedge bounded by two similar equal and parallel triangular faces and three rectangular faces coincides with that of six equal particles placed at its angular points.

Paper LXXV.

1. Eliminate x, y, z from the equations

$$x^3 - a^3 = y^3 - b^3 = z^3 - c^3 = xyz$$

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} = \frac{d^3}{x+y+z}.$$

2. If $6n$ tickets numbered $0, 1, 2, \ldots . 6n - 1$, be placed in a bag, and three drawn out, shew that the chance that the sum of the numbers on them is $6n$ is

$$\frac{3n}{(6n-1)(6n-2)}.$$

3. The perpendiculars from the angular points of an acute-angled triangle ABC on the opposite sides meet in P, and PA, PB, PC are taken for the sides of a new

triangle. Find the condition that this should be possible; and if it is, and α, β, γ be the angles of the new triangle, shew that

$$1 + \frac{\cos\alpha}{\cos A} + \frac{\cos\beta}{\cos B} + \frac{\cos\gamma}{\cos C} = \tfrac{1}{2}\sec A \sec B \sec C.$$

4. In a convex polygon of an odd number of sides the middle points of all are fixed except one, which describes a curve. Prove that the angular points of the polygon describe equal curves.

5. The major axis of an ellipse is harmonically divided externally in E, and internally in F, and EPQ is any chord through E. If PF, QF produced cut the curve in P', Q', prove that

$$\left(\frac{EP}{EQ}\right)^2 = \frac{FP \,.\, FQ'}{FQ \,.\, FP'}.$$

6. OA, OB, OC, ... are any number of fixed straight lines drawn from a point O, and spheres are described on them as diameters. Any straight line OX is drawn through O, and a point P taken on it so that OP is equal to the algebraic sum of the lengths intercepted on OX by the spheres. Find the locus of P.

7. A series of n elastic spheres whose masses are 1, e, e^2, ... are at rest, separated by intervals with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Prove that the final kinetic energy of the system is $\frac{1}{2}(1 - e + e^n)u^2$.

Paper LXXVI.

1. A person has £1,583 17*s.* 11*d.* stock in the 3 per cents. and £982 12*s.* 6*d.* stock in the $3\frac{1}{2}$ per cents. He transfers a certain sum from the former to the latter when the stocks are at 91 and 98 respectively, and thus makes the income derived from each the same. How much stock has he finally in the 3 per cents?

2. If x be small compared with N^2, prove that

$$\sqrt{N^2 + x} = N + \frac{x}{4N} + \frac{Nx}{2(2N^2 + x)} \text{ approximately,}$$

and shew that the error is of the order $\frac{x^4}{N^7}$.

Ex. Shew that $\sqrt{101} = 10\frac{401}{8040}$ to 8 places of decimals.

3. Prove that whatever be the values of A, B, C,

(1) $\cos 2(A+B+C)+\cos(2A+B+C)+\cos(A+2B+C)$
$+\cos(A+B+2C)+\cos(B+C)+\cos(C+A)+\cos(A+B)$
$= 8\cos(A+B+C)\cos\frac{B+C}{2}\cos\frac{C+A}{2}\cos\frac{A+B}{2} - 1.$

(2) $\{\sin B+\sin C-\sin(B+C)\}\{\sin C+\sin A-\sin(C+A)\}$
$\{\sin A+\sin B-\sin(A+B)\}$
$= 16\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\{\sin A + \sin B + \sin C$
$- \sin(A+B+C)\}.$

4. If two equal circles intersect in A and B, and with A as centre any circle be described cutting the equal circles in D, D', E, E' respectively, prove that B, D, E or B, D', E' are collinear.

5. A circle is described passing through two points Q, Q' on a parabola, and T the intersection of the tangents at Q and Q'. Shew that the chord it intercepts on the diameter through T is equal to the sum of the focal distances of Q and Q'.

6. Any tangent to the hyperbola $4xy = ab$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points P, Q. Shew that the normals to the ellipse at P and Q meet on a fixed diameter of the ellipse.

7. A shot of m lbs. is fired from a gun of M lbs. placed on a smooth horizontal plane, and elevated at an angle a. Prove that if the muzzle velocity of the shot be v, the range will be

$$2 \cdot \frac{v^2}{g} \cdot \frac{\left(1 + \frac{m}{M}\right)\tan a}{1 + \left(1 + \frac{m}{M}\right)^2 \tan^2 a}$$

Paper LXXVII.

1. If

$$(a_2 - a_3)^2 + (b_2 - b_3)^2 = f^2$$
$$(a_3 - a_1)^2 + (b_3 - b_1)^2 = g^2$$
$$(a_1 - a_2)^2 + (b_1 - b_2)^2 = h^2$$

shew that

$$\frac{2g^2h^2 + 2h^2f^2 + 2f^2g^2 - f^4 - g^4 - h^4}{(a_2b_3 - a_3b_2 + a_3b_1 - a_1b_3 + a_1b_2 - a_2b_1)^2} = 4.$$

2. If ρ, l, m, n, be the reciprocals of the radius of the inscribed circle of a triangle and of the distances of its centre from the angular points, prove that

$$\rho^3 - \rho(l^2 + m^2 + n^2) - 2lmn = 0.$$

3. In every tetrahedron the sum of the squares of the six edges is equal to four times the sum of the squares on the lines which join the middle points of the opposite edges.

4. Prove geometrically that the perpendicular from the focus of a conic on any tangent and the central radius to the point of contact of the tangent will intersect on the directrix.

5. A beam AB lies horizontally upon two others at points A and C. Prove that the least horizontal force

applied at B in a direction perpendicular to AB which is able to move the beam is the less of the two forces

$$\mu W \cdot \frac{b - a}{2a - b} \text{ and } \frac{\mu W}{2},$$

where $AB = 2a$, $AC = b$, W is the weight of the beam, and μ is the coefficient of friction.

6. Prove that the time in which a projectile moves from one extremity to the other of a focal chord is equal to that in which it falls vertically from rest through a distance equal to the length of the chord.

7. A system of circles is described passing through the vertex of a given triangle, and cutting the base in a given point. Prove that the chord joining the points where the two sides of the triangle are cut by any circle of the system always touches a parabola.

Paper LXXVIII.

1. Prove that the numbers 220 and 284 are such that the sum of the aliquot parts of each is equal to the other.

2. If

$$ax + by = \sqrt[3]{cx + dy} = \sqrt[5]{ex + fy}$$

prove that

$$x = \frac{d - bk}{ad - bc}\sqrt{k},$$

where k is given by the equation

$$(ad - bc)k^2 - (fa - be)k + fc - de = 0.$$

Hence solve

$$7x - 11y = \sqrt[3]{x + y} = \sqrt[5]{x + 9y},$$

also solve

$$x \cdot y = \sqrt[n+1]{4x - 5y} = \sqrt[2n+1]{3x - 2y}.$$

3. A circle of radius R touches externally three circles which also touch each other externally, and whose radii are r_1, r_2, r_3.

Prove that

$$(1)\quad \sqrt{\frac{r_2+r_3+R}{r_1}}+\sqrt{\frac{r_3+r_1+R}{r_2}}+\sqrt{\frac{r_1+r_2+R}{r_3}} = \sqrt{\frac{r_1+r_2+r_3}{R}}$$

$$(2)\quad \frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\pm 2\sqrt{\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}}.$$

Also, if A, B, C be the centres of the circles r_1, r_2, r_3, and P the centre of the circle R, and if d, d_1 be the respective distances of P and A from BC, then $\frac{d}{R} \sim \frac{d_1}{r_1} = 2$.

4. If O be the centre of one of the escribed circles of a triangle, P one point of its intersection with the circumscribing circle, and if OP produced meet the latter circle again in Q, shew that OQ is equal to the diameter of the circumscribing circle.

5. In a given acute-angled triangle ABC inscribe a triangle whose sides shall be bisected by the lines joining the angular points of the triangle ABC to the centre of the circle circumscribing it.

6. A point V is taken on an ordinate PM, produced, of a parabola, and ME is taken on MP a mean proportional between PM and MV. If the diameters through E and V meet the curve in R and Q, prove that PQ meets the axis in the foot of the ordinate of R.

7. A bucket and a counterpoise, connected by a string passing over a pulley, just balance one another, and an elastic ball is dropped into the centre of the bucket from a distance h above it. Find the time that elapses before the

ball ceases to rebound, and prove that the whole descent of the bucket during this interval is

$$\frac{4mh}{2M + m} \cdot \frac{e}{(1 - e)^2}$$

where m and M are the masses of the ball and the bucket, and e is the coefficient of restitution.

Paper LXXIX.

1. For a house occupied by B, A pays a rent of £40 per annum by equal payments at the end of each quarter. B pays A by equal payments in advance at the beginning of each month. How much a month ought B to pay in order that at the end of the year, with simple interest reckoned at $3\frac{1}{3}$ per cent. per annum, A may have recovered the value of his own four payments with one-tenth additional?

2. If $a + b + c + d = 0$, prove that

$$(a^3 + b^3 + c^3 + d^3)^2 = 9(bcd + cda + dab + abc)^2$$
$$= 9(bc - ad)(ca - bd)(ab - cd).$$

3. Sum to infinity the series

$$m \cos\theta - \tfrac{1}{3}m^3 \cos 3\theta + \tfrac{1}{5}m^5 \cos 5\theta \ldots\ldots,$$ where m is less than unity;

and prove that the series has always the same sign as $m \cos\theta$.

4. If in BA, CA two sides of a triangle ABC two points D, E be taken respectively, such that $BA : AC :: EA : AD$, and G the middle point of DE be joined to A, and if BH, CK be constructed in the same way as AG, shew that AG, BH, CK intersect in a fixed point O.

Prove also that if from O perpendiculars be drawn to the sides of the triangle, the sum of their squares is a minimum.

5. If one of each of the 5 kinds of regular polyhedrons be inscribed in the same sphere, prove that their edges will be in the ratio of

$$2\sqrt{2} : 2 : \sqrt{6} : \sqrt{5} - 1 : \sqrt{\tfrac{6}{5}(5 - \sqrt{5})}.$$

6. If ϕ be the excentric angle of any point P of an ellipse, and CP be produced to Q, and tangents be drawn from Q to the ellipse, prove that the excentric angles of the points of contact will be

$$\phi \pm \cos^{-1}\frac{CP}{CQ}.$$

7. If a weight P balance a weight W in that system of pulleys in which each pulley hangs by a separate string, shew that if P be changed to P' and W to W', and all the pulleys be of equal weight, P' will descend with acceleration f such that

$$f[2^{2n}P' + W' + \tfrac{1}{3}(2^n + 1)(2^nP - W)]$$
$$= 2^n g\{2^n(P' - P) + W - W'\}.$$

Paper LXXX.

1. If

$$\frac{P}{pa^2+2qab+rb^2} = \frac{Q}{pac+q(bc-a^2)-rab} = \frac{R}{pc^2-2qca+ra^2},$$

prove that P, p; Q, q; and R, r may be interchanged without altering the equalities.

2. A man goes in for an examination in which there are four papers with a maximum of m marks for each paper. Shew that the number of ways of getting half marks on the whole is

$$\tfrac{1}{3}(m + 1)(2m^2 + 4m + 3).$$

3. Prove that

$$\cos 2\alpha \cot\tfrac{1}{2}(\gamma-\alpha)\cot\tfrac{1}{2}(\alpha-\beta)+\cos 2\beta \cot\tfrac{1}{2}(\alpha-\beta)\cot\tfrac{1}{2}(\beta-\gamma)$$
$$+\cos 2\gamma \cot\tfrac{1}{2}(\beta-\gamma)\cot\tfrac{1}{2}(\gamma-\alpha)$$
$$=\cos 2\alpha+\cos 2\beta+\cos 2\gamma+2\cos(\beta+\gamma)+2\cos(\gamma+\alpha)$$
$$+2\cos(\alpha+\beta).$$

4. The sides of a quadrilateral touch a circle in $ABCD$. If a circle can be described about the same quadrilateral, shew that the middle points of the chords AB, BC, CD, DA lie on another circle, and if R, r be the radii of the circum- and in-scribed circles, and d the distance between their centres, the product of the diagonals of the quadrilateral $= \dfrac{8R^2r^2}{R^2 - d^2}$.

Also if θ be the angle between the diagonals,

$$\tan\frac{\theta}{2} = \frac{\text{area of quadrilateral}}{\text{product of two opposite sides}}$$

Hence shew that when the diagonals are at right angles, the rectangles contained by the opposite sides are equal.

5. If a parallelogram be described about an ellipse having its sides parallel to a pair of conjugate diameters, and P be the point of contact of one of the sides QQ', and the normal at P meet the axes of the ellipse in G and G', then QG is perpendicular to $Q'G'$.

6. A parabola circumscribes a right-angled triangle. Taking the sides of the triangle as axes, shew that the locus of the foot of the perpendicular from the right angle on the directrix is the quintic

$$2xy(x^2+y^2)(hy+kx)+h^2y^4+k^2x^4=0,$$

and that the axis is one of the family of straight lines whose general equation is

$$y = mx - \frac{m^3h - k}{1+m^2},$$

m being an arbitrary parameter, and $2h$, $2k$ the sides of the triangle.

7. A brass figure $ABDC$ of uniform thickness, bounded by a circular arc BDC ($>$ a $\frac{1}{2}$ circle) and two tangents AB, AC inclined at an angle 2α, is used as a letter weigher. The centre of the circle O is a fixed point about which the machine turns freely, and a weight P is attached at A, and the weight of the machine is W. The letter to be weighed is suspended from a clasp (whose weight may be neglected) at D on the rim of the circle, and OD is perpendicular to OA. The circle is graduated and read by a pointer hanging vertically from O. When there is no letter attached, the point A is vertically below O, and the pointer indicates zero. Shew that if $P = \frac{1}{3}W \,.\, \sin^2\alpha$, the reading of the machine will be $\frac{1}{3}W$ when OA makes with the vertical an angle

$$\tan^{-1}\frac{(\pi + 2\alpha)\sin^2\alpha + 2\sin\alpha\cos\alpha}{(\pi + 2\alpha)\sin^3\alpha + 2\cos\alpha}.$$

Paper LXXXI.

1. If p be a prime number, and x not divisible by p, shew that

$$x^{p^r - p^{r-1}} = 1 + \text{multiple of } p.$$

2. A point O is taken within a circle of radius a, at a distance b from the centre, and points $P_1, P_2, \ldots. P_n$ are taken on the circumference so that $P_1P_2, P_2P_3, \ldots. P_nP_1$, subtend equal angles at O. Prove that

$$OP_1 + OP_2 + \ldots. + OP_n$$
$$= (a^2 - b^2)\left(\frac{1}{OP_1} + \frac{1}{OP_2} + \ldots. + \frac{1}{OP_n}\right).$$

3. If $OFAE$ be a parallelogram and BOC any straight line cutting the sides AF, AE in B and C, prove that

$$BA \,.\, AF + CA \,.\, AE = AO^2 + BO \,.\, OC.$$

4. A triangle is inscribed in an ellipse so that each side is parallel to the tangent at the opposite angle. Prove that the sum of the squares on the sides : sum of the squares on the axes of the ellipse :: 9 : 8.

5. If the diameter through a point P on a parabola meet the tangent at the vertex in Z, and the focal distance of Z meet the normal at P in R, P and R will be equidistant from the tangent at the vertex.

6. Two forces P and Q act at an angle a and have a resultant R. If each force be increased by R, and θ be the angle which the new resultant makes with R, prove that

$$\tan\theta = \frac{(P - Q)\sin a}{P + Q + R + (P + Q)\cos a}.$$

7. Chords are drawn joining any point of a vertical circle with its highest and lowest points. Prove that if a heavy particle slide down the latter chord, the parabola which it will describe after leaving the chord will be touched by the former chord.

Also shew that the locus of the points of contact will be a circle.

Paper LXXXII.

1. Prove that

$$2^n = 1 + \frac{(n+1)n}{\lfloor 2} + \frac{(n+1)n(n-1)(n-2)}{\lfloor 4} + \dots.$$

2. A, B, C, D, E . . . . being angular points of a regular polygon of n sides, join each pair of alternate points AC, BD, CE, ... and find the area of the star-like figure thus formed.

3. In any triangle ABC if E, F be the points where perpendiculars from B, C meet AC, AB, prove that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

4. P is the orthocentre of a triangle ABC, and O the centre of its circum-circle. A', B', C' are the centres of the circum-circles of the triangles BPC, CPA, APB respectively. Shew that O is the orthocentre of $A'B'C'$, P the centre of its circum-circle, A, B, C those of the circum-circles of $B'OC'$, $C'OA'$, $A'OB'$ respectively, and that all the eight triangles above mentioned have the same nine-point circle.

5. If an ellipse inscribed in a triangle has either one focus at the orthocentre, or one focus at the centre of the circumscribed circle, or its centre at the centre of the nine point circle of the triangle, then the other two properties are also true of it.

6. Two circular cylinders of unequal radii rest on a rough horizontal plane with their axes parallel, and on them rests a rough uniform beam of weight W, being supported by a force P parallel to its length. The axis of the beam is at right angles to the axes of the cylinders, and inclined at an angle α to the horizontal plane. Prove that

$$P = W \,.\, \tan\frac{\alpha}{2}.$$

7. PSp is a focal chord of a parabola, A its vertex. If PS make an angle of 60° with the axis, shew that the times in which a body moving in the parabola under a force to the focus would go from P to A and from A to p are as 27 to 5.

PAPER LXXXIII.

1. The University pays rates at $40d.$ in the £, the town at $15d.$ The former pays $\frac{1}{4}$ of the whole rates, and the latter $\frac{3}{4}$. Compare their rateable values.

Also, if the rates were equalised, what proportion of their present payment would the University save?

2. If n is integral, prove that so also are the expressions

$$(1)\quad \frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}.$$

$$(2)\quad \frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}.$$

3. D, E, F are the middle points of the sides of the triangle ABC. From the intersection of AD, BE, CF perpendiculars are let fall on the sides of the triangle. Shew that the radius of the circle through their feet is

$$\frac{4}{3}\,\frac{AD\,.\,BE\,.\,CF}{a^2+b^2+c^2}.$$

4. Two parabolas have a common focus and their axes in the same direction, and a line is drawn through the focus cutting the parabolas in four points. Shew that the tangents at these points form a rectangle, one of whose diagonals goes through the focus, and the other is perpendicular to the axis.

5. A circle is described about a triangle ABC, and PQ is a chord parallel to the tangent at A cutting AB, AC in R and S. If PM, QN be perpendiculars on BC, shew that

$$\frac{QR\,.\,QS}{PR\,.\,PS}=\frac{QN}{PM}.$$

6. Prove that if two heavy particles projected in the same vertical plane at the same instant from two given points with the same velocity meet, the sum of the inclinations of the directions of projection must be constant.

Also shew that if the particles be projected with a constant velocity, the locus of the point of meeting for different directions of projection is a parabola.

7. Three forces P, Q, R act respectively along the sides BC, CA, AB, of a triangle ABC. If the line of action of their resultant be the line joining the centre of the circum-circle with the orthocentre, prove that

$$\frac{P\cos B\cos C}{\cos^2 B-\cos^2 C}=\frac{Q\cos C\cos A}{\cos^2 C-\cos^2 A}=\frac{R\cos A\cos B}{\cos^2 A-\cos^2 B}.$$

Paper LXXXIV.

1. Prove that

$$\left\{b^2c^2(a+d)+a^2d^2(b+c)\right\}(b-c)(a-d)$$
$$\left\{c^2a^2(b+d)+b^2d^2(c+a)\right\}(c-a)(b-d)$$
$$\left\{a^2b^2(c+d)+c^2d^2(a+b)\right\}(a-b)(c-d)=0.$$

2. Prove that the square of the continued fraction

$$\frac{a}{b+}\,\frac{a}{b+}\cdots\cdot$$

is the continued fraction

$$\frac{a^2}{2a+b^2-}\,\frac{a^2}{2a+b^2-}\cdots\cdot$$

3. If p, q, r be the lengths of the bisectors of the angles of a triangle produced to meet the circumscribed circle, and u, v, w the lengths of the perpendiculars of the triangle produced to meet the same circle, then

$$p^2(w-v)+q^2(u-w)+r^2(v-u)=0.$$

4. A triangle ABC is inscribed in a circle. The points a, b are diametrically opposite to A, B; ad is drawn parallel to BC to meet the circle in d, and the straight line db meets AC and CB in e and f respectively. If O be the centre prove that Oe is parallel to BC, and that $be=ef=eC$.

5. A parabola P and hyperbola H have a common focus, and the asymptotes of H are tangents to P. Prove that the tangent at the vertex of P is a directrix of H, and that the tangent to P at the point of intersection passes through the further vertex of H.

6. TP and TQ are tangents to an ellipse. If forces proportional to PT, QT be applied at P and Q to the ellipse in the directions of these lines, shew that the ellipse will remain at rest if the centre is fixed.

7. If ϕ be the eccentric angle of a point P on an ellipse, shew that the eccentric angle of the point where the normal at P cuts the ellipse again is given by

$$b^2 \cot \tfrac{1}{2}(\theta + \phi) + a^2 \tan \phi = 0.$$

Hence shew that the locus of the middle points of normal chords of the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 \left(\frac{x^2}{a^6} + \frac{y^2}{b^6}\right) = \frac{(a^2 - b^2)^2}{a^6 b^6} x^2 y^2.$$

Paper LXXXV.

1. Two successive digits being put together to form a number give the product of two other successive digits; and if the first of these latter be inserted between the former, the result is the square of three times the remaining one. What are the digits?

2. If $ABCD$ be any quadrilateral, and if on its sides as hypotenuses right-angled isosceles triangles be described, all external, shew that twice the square on either straight line joining the vertices of opposite triangles will be equal to $a^2 + b^2 + c^2 + d^2 + da\,(\sin A - \cos A) + ab\,(\sin B - \cos B) + bc\,(\sin C - \cos C) + cd\,(\sin D - \cos D)$ where a, b, c, d are the sides AB, BC, CD, DA respectively.

3. On each side of a triangle ABC is described a square externally. The vertices of these squares are joined in pairs by three straight lines such that no line cuts a square. If these three lines be taken as the sides of a triangle DEF, prove that the area of DEF is three times that of ABC.

If from the triangle DEF we form another triangle in the same manner in which DEF was obtained from ABC, shew that the sides of this new triangle are respectively equal to three times the sides of the triangle ABC.

4. If PQ be a focal chord of a parabola, and R any point on the diameter through Q, show that the focal chord parallel to $PR = \frac{PR^2}{PQ}$.

5. A hyperbola is placed in a vertical plane with its transverse axis horizontal. Prove that when the time of descent down a diameter is a minimum, the conjugate diameter is equal to the distance between the foci.

6. Prove that $r = a\,(1 - \frac{1}{2}\,e^2 \sin^2 \theta)$ is the polar equation to an ellipse of which the fourth power of the eccentricity may be neglected.

7. The Mississippi rises in lat. 45° and enters the sea in lat. 30°. Shew that owing to the spheroidal form of the earth, of which the greatest radius is 4,000 miles and eccentricity ·08, the mouth of the river is about $3\frac{1}{4}$ miles higher, *i.e.* further from the earth's centre, than the source.

Paper LXXXVI.

1. An event happens on an average once a year. Shew that the chance that it will not happen in any particular year is $\frac{1}{e}$.

2. If A, B, C be the angles of a triangle, shew that

(1) $\sin 2A + \sin 2B + \sin 2C = 2(\sin A + \sin B + \sin C)$
$(\cos A + \cos B + \cos C - 1)$.

(2) $(\sin A + \sin B + \sin C)^2 + (\cos A + \cos B + \cos C - 1)^2$
$+ 4\,(\cos A + \cos B + \cos C - 1) = 4\,(\sin B \sin C$
$+ \sin C \sin A + \sin A \sin B)$.

3. If the radii of the four circles which touch the sides of a triangle be in continued proportion, shew that the triangle must be right-angled.

4. Any point D is taken on the circumference of the circle round the triangle ABC of which the sides AB, AC are equal. Prove that the ratio of the difference or sum of DB and DC to DA is constant according as D and A lie on the same or opposite sides of BC respectively.

5. QVQ' is a chord of an ellipse parallel to one of the equiconjugate diameters. The normals at Q and Q' meet in O. Shew that a circle will go round $QCOQ'$.

6. Produce the normal at any point A of an ellipse outwards to a length AM equal to the radius of curvature. Shew that the circle described on AM as diameter cuts orthogonally the director circle.

7. An elastic sphere A impinges upon an elastic sphere B at rest. Determine the motions of the two spheres after collision supposing the original direction of A's velocity to be inclined at an angle α to the line joining the centres of the two spheres at the moment of collision.

Paper LXXXVII.

1. An A.P., G.P., and H.P. have each a and $a + b$ for their first two terms. If the third terms be x, y, z respectively, prove that

$$y (x - 3z)^2 + 4z (y^2 + xz) = 12yz^2.$$

and if l be the n^{th} term of the A.P., shew that

$$\frac{1}{n}\{a+(a+b)+....+l\}^2-\{al+(a+b)(l-b)+....+la\}$$
$$= \frac{1}{12}n(n^2 - 1)b^2.$$

2. Shew that

$$\sqrt{6} = \frac{5}{2}\left\{1 - \frac{1}{\lfloor 2}\cdot\frac{2^3.3}{5^4} - \frac{1.3.5}{\lfloor 4}\cdot\frac{2^6.3^2}{5^8} - \frac{1.3.5.7.9}{\lfloor 6}\cdot\frac{2^9.3^3}{5^{12}}\cdots\right\}$$

3. Shew that the expression

$$\sin\alpha\sin\beta\{\operatorname{cosec}\alpha\operatorname{cosec}(\alpha+\beta)+\operatorname{cosec}(\alpha+\beta)\operatorname{cosec}(\alpha+2\beta)$$
$$+\operatorname{cosec}(\alpha+2\beta)\operatorname{cosec}(\alpha+3\beta)\}=\sin 3\beta\operatorname{cosec}(\alpha+3\beta).$$

4. Shew that there are eleven pairs of regular polygons which satisfy the condition that the measure of an angle in one in degrees is equal to the measure of an angle of the other in grades, and find the number of sides in each.

5. Shew that the centres of the four circles circumscribing the triangles formed by four straight lines lie on a circle.

6. Find the locus of a point from which two tangents can be drawn at right angles, one to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ the other to } \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1;$$

and shew that the curve lies entirely in the space between two circles of radii

$$\sqrt{\alpha^2+\beta^2} \text{ and } \sqrt{a^2+b^2}$$

respectively, and that it touches both these circles.

7. Three particles are projected from the same point in different directions. Shew that after a time t they form a triangle whose area $\propto t^2$.

If the direction of projection of two of the particles are in the same vertical plane, shew that the plane of this triangle will pass through the point of projection after a time

$$\frac{2}{g}\cdot\frac{uv\sin(\beta-\alpha)}{u\cos\alpha-v\cos\beta}$$

where u, v are the initial velocities, and α, β the initial elevations of the two particles.

Paper LXXXVIII.

1. Solve the equations:—

(1) $\sqrt{x+a}+\sqrt{x+2a}+\sqrt{x+3a}=\sqrt{x+6a}.$

(2) $\sqrt{x^2+a^2}.\sqrt{x^2+b^2}+x\left\{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}\right\}=nb^2+x^2,$

(3) $$\begin{aligned} x^2+y^2+z^2 &= a^2+2x(y+z)-x^2 \\ &= b^2+2y(z+x)-y^2 \\ &= c^2+2z(x+y)-z^2. \end{aligned}$$

2. If $\dfrac{x\cos 3\theta+y\sin 3\theta}{\cos^3\theta}=\dfrac{y\cos 3\theta-x\sin 3\theta}{\sin^3\theta}$ $=x^2+y^2$, shew that $x^2+y^2+x=2$.

3. ABC is a triangle and $2s$ its perimeter. If α, β, γ be the angles of the triangle whose sides are equal to the radii of the escribed circles of ABC, prove that

$$a(s-a)\cos^2\frac{\alpha}{2}+b(s-b)\cos^2\frac{\beta}{2}+c(s-c)\cos^2\frac{\gamma}{2}$$
$$=\frac{3}{2}(bc+ca+ab-s^2).$$

4. A straight line AB is divided at C so that the rectangle $AB\,.\,BC$ is equal to the square on AC; and on BC as base is described an isosceles triangle BDC having its two sides equal to AC. Prove that the straight line joining A to the middle point of BD will divide CD at E so that $CE : ED :: BC : CA$.

5. Assuming only the focus and directrix definition of a parabola, prove that one arm of a right angle envelops the curve if the other arm always passes through a fixed point, and the angular point always lies on a fixed straight line.

6. From any point four normals are drawn to a rectangular hyperbola. Shew that the points where they meet the curve are such that each is the orthocentre of the triangle formed by joining the other three.

7. In the differential axle, if the ends of the chain, instead of being fastened to the axles, are joined together so as to form another loop in which another pulley and weight are suspended, find the least force which must be applied along the chain in order to raise the greater weight, the different parts of the chain being all vertical.

Paper LXXXIX.

1. Solve the equations

$$(1)\quad \sqrt[4]{\frac{a+x}{a-x}} + \sqrt[4]{\frac{a-x}{a+x}} = \sqrt[4]{b}.$$

$$(2)\quad \left.\begin{aligned}(x^2+1)(x+y)^2 + 2(x+y) &= 135\\ x(x+y)(x+y+1) &= 60\end{aligned}\right\}.$$

$$(3)\quad \left.\begin{aligned}y+z+yz &= 3\\ z+x+zx &= -1\\ x+y+xy &= -1\end{aligned}\right\}.$$

2. Prove that any equation holding between the sines and cosines of the angles A, B, C of any triangle will still be true when A, B, C are replaced by $2A$, $2B$, $2C$ respectively, provided the cosines have their signs changed. Also that it will be true when A, B, C are replaced by $5A$, $5B$, $5C$ respectively, provided that the sines have their signs changed.

Prove that in any triangle

$$\sin 10A + \sin 10B + \sin 10C = 4\sin 5A \sin 5B \sin 5C.$$

Also shew that the sum of the cotangents of

$$\frac{5\pi + A}{2^5},\ \frac{5\pi + B}{2^5},\ \frac{5\pi + C}{2^5},$$

is equal to their product.

3. An equilateral triangle is constructed with its angular points on three given parallel straight lines whose distances apart are a, b, c. Prove that its area is

$$\frac{1}{2\sqrt{3}}(a^2 + b^2 + c^2).$$

4. P, Q, R are three points on a circle whose centre is C. ACB is the diameter bisecting QR, and intersecting PQ, PR in M and N. Shew that the triangles QCM, RCN are similar.

5. A number of parabolas whose axes are parallel have a common tangent at a given point. Shew that if parallel tangents be drawn to all the parabolas, the points of contact will lie on a straight line passing through the given point.

6. Shew that the radius of the circle inscribed in the semi-ellipse cut off by the minor axis is

$$\frac{b}{a}\sqrt{a^2 - b^2}.$$

7. A railway train is running smoothly along a curve at the rate of 60 miles an hour, and in one of the cars a pendulum which would oscillate seconds ordinarily is observed to oscillate 121 times in 2 minutes. Shew that the radius of the curve is nearly a quarter of a mile.

Paper XC.

1. Sum to n terms the series

(1) $3 + 2 + 29 + 36 + 137 + 122 + 429 + 200 + \ldots.$

(2) $3 + \frac{8}{6} + \frac{20}{6^2} + \frac{50}{6^3} + \frac{128}{6^4} + \frac{338}{6^5} + \frac{920}{6^6} + \ldots.$

(3) $\frac{1}{1 \cdot 3} + \frac{2}{1 \cdot 3 \cdot 5} + \frac{3}{1 \cdot 3 \cdot 5 \cdot 7} + \ldots.$

(4) $1 \cdot 2^2 + n \cdot 2 \cdot 3^2 + \frac{n(n-1)}{\lfloor\underline{2}} \cdot 3 \cdot 4^2$

$$+ \frac{n(n-1)(n-2)}{\lfloor\underline{3}} \cdot 4 \cdot 5^2 + \ldots,$$

2. When x is indefinitely diminished, find the value of the expression

$$\frac{1-\cos 2x+\cos 4x-\cos 6x+\cos 8x-\cos 10x-\cos 14x+\cos 16x}{3-4\cos 2x+\cos 4x}$$

3. ABC is a triangle. Through A, B, C are drawn the straight lines A_1B_1, B_1C_1, C_1A_1 at right angles respectively to the sides AB, BC, CA, and forming the triangle $A_1B_1C_1$. The triangle $A_2B_2C_2$ is formed in a similar way from the triangle $A_1B_1C_1$, and so on. If $A_nB_nC_n$ be the n^{th} triangle so formed, prove that the radius of the circle circumscribing this triangle is

$$R\left(\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}\right)^n$$

where R is the radius of the circle circumscribing the triangle ABC.

4. Two parallelograms $ACBD$, $A'CB'D'$ have a common angle at C. Prove that DD' passes through the intersection of $A'B$ and AB'.

5. Two equal parabolas A and B have the same vertex and axis, but are turned in opposite directions. Shew that the locus of poles with respect to B of tangents to A is the parabola A.

6. PQ is a normal to an ellipse at P, and PCP' a diameter. CD is conjugate to CP. If PM be an ordinate and PQ, QP' meet the major axis in G and U respectively, prove that $GM \cdot CU = CD^2$.

7. A ball is projected against a smooth vertical wall. Find the direction in which it may be projected with the least velocity so that it shall return to the point of projection.

Paper XCI.

1. Between two quantities a Harmonic mean is inserted, and between each adjacent pair a Geometric mean is inserted. It is found that the three means thus inserted are in Arithmetic Progression. Prove that the quantities are in the ratio $7 - 4\sqrt{3} : 1$.

2. Prove that

$$(m + \tfrac{3}{2})(m + \tfrac{7}{2})(m + \tfrac{11}{2}) \dots \left(m + \frac{4n-1}{2}\right) > \left\{\frac{\lfloor m + 2n}{\lfloor m}\right\}^{\frac{1}{2}}$$

3. Shew how to find a series of triangles whose sides are in A.P. with a common difference 1, and whose areas are rational. Shew that the five least 'integral' values of the mean side are 4, 14, 52, 194, 724.

4. A triangle is inscribed in a circle so as to have its orthocentre at a given point. Prove that the middle points of its sides lie on a fixed circle.

5. Qq is a diameter of an ellipse. SP is drawn through the focus S parallel to the tangents at Q and q, and these tangents are intersected by the tangent at P in T and t respectively. Shew that the sum of TQ and tq is equal to the major axis of the ellipse.

6. A smooth parabolic arc is placed with its axis vertical and vertex upwards, and on it, at the extremities of a focal chord rest two weights $W_1 W_2$ which are connected by a fine string passing over the vertex. Shew that the length of the focal chord is

$$\tfrac{1}{4} \text{ lat. rect.} \times \left(\frac{W_1}{W_2} + \frac{W_2}{W_1}\right)^2.$$

7. Find the angle at which a body must be projected in order to strike at right angles a given plane, which is at right angles to the plane of projection, and passes through the point of projection, and shew that it is independent of the velocity of projection.

Paper XCII.

1. Prove that the expression $a^3 + b^3 + c^3 - 3abc$ is unaltered if we substitute for a, b, c the quantities $\sigma - a$, $\sigma - b$, $\sigma - c$ respectively, where $3\sigma = 2(a + b + c)$.

2. Prove that with a certain convention with regard to sign the roots of the equation

$$x^4 - \frac{abc}{S}x^3 + \frac{a^2 + b^2 + c^2}{2}x^2 - S^2 = 0,$$

are the radii of the four circles touching the sides of a triangle of area S.

3. If S_r denote the series

$$\frac{n^{r-1}}{\lfloor r-1} + \frac{(2n)^{r-2}}{\lfloor r-2} + \frac{(3n)^{r-3}}{\lfloor r-3} + \ldots.$$

prove that

$$S_{k+1} = S_k + nS_{k-1} + \ldots + \frac{n^{k-1}}{\lfloor k-1}S_1 + \frac{n^k}{\lfloor k}.$$

4. AB is the diameter of a semi-circle, $P, Q, R, \ldots K$ are any number of points on the circumference taken in order from A. Shew that the square on AB is not less than the sum of the squares on $AP, PQ, QR, \ldots KB$.

5. PQ is a chord of a parabola which is normal at P, and PK is another chord equally inclined to the axis of the parabola. Shew that PKQ is a right angle.

6. A uniform cylinder is supported in a horizontal position by a prop under the middle of the axis. Prove that if a be the length of the axis, and a cylindrical bore be made of length l, having the same axis as the former cylinder, by which $\frac{1}{n}^{\text{th}}$ of the substance of every transverse section is removed, the prop must be moved through a distance

$$\frac{l}{2} \cdot \frac{a - l}{na - l}.$$

Also shew that when the prop is furthest from the original position, it must be under the extremity of the bore.

7. Prove that at the equator a shot fired westward with velocity 8333 or eastward with velocity 7407 metres per second, will, if unresisted, move horizontally round the earth in one hour and twenty minutes, and one hour and a half respectively, given that a quadrant of the earth's equatorial circumference is 10^9 centimetres.

Paper XCIII.

1. Shew that if x, y, z are unequal, the equation

$$\frac{x}{1-x^2}\left\{\frac{z}{1+z^2}-\frac{y}{1+y^2}\right\}+\frac{y}{1-y^2}\left\{\frac{x}{1+x^2}-\frac{z}{1+z^2}\right\}$$
$$+\frac{z}{1-z^2}\left\{\frac{y}{1+y^2}-\frac{x}{1+x^2}\right\}=0.$$

may be reduced to

$$xyz(yz+zx+xy)-(x+y+z)=0.$$

2. Find x from the equation

$$\tan^{-1}(x+1)+\cot^{-1}(x-1)=\sin^{-1}\tfrac{4}{5}+\cos^{-1}\tfrac{3}{5}.$$

3. On the base BC of a triangle ABC two points Q, R are taken so that $BQ = QR = RC$. Shew that

$$\sin BAR \sin CAQ = 4 \sin BAQ \sin CAR.$$

4. If two equal circles be described intersecting in A and B, and through A a chord be drawn cutting them in C and D, shew that CD is bisected by the circle on AB as diameter.

5. In a given plane is drawn a series of confocal conics upon which stand cones with the vertical angles right angles. Shew that the locus of their vertices is given by the intersection of an hyperbola whose vertices are the foci of the conics and a circle concentric with the hyperbola and passing through its foci.

6. If the normals drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from any point on the normal at h, k meet the ellipse in P, Q, R, prove that the sides of the triangle PQR touch the parabola

$$\left(\frac{xh}{a^2} + \frac{yk}{b^2} + 1\right)^2 = \frac{4hkxy}{a^2b^2}.$$

7. Two equal circles touch each other, and from the point of contact two points move on the circles with equal velocities in opposite directions. Prove that one will appear to the other to move on a circle the radius of which is equal to the diameter of either of the fixed circles.

Paper XCIV.

1. Find the real roots of the equations

$$\begin{aligned} x^2 + z'^2 + y'^2 &= a^2 \quad ; \quad & y'z' + x'(y + z) &= bc \\ z'^2 + y^2 + x'^2 &= b^2 \quad ; \quad & z'x' + y'(z + x) &= ca \\ y'^2 + x'^2 + z^2 &= c^2 \quad ; \quad & x'y' + z'(x + y) &= ab. \end{aligned}$$

2. Find θ from the equations

(1) $$a^2 + \sqrt{1 - a^2\cot^2\theta}\,\sqrt{1 - a^2\tan^2\theta} = b^2 + \sqrt{1 - b^2\cot^2\theta}\,\sqrt{1 - b^2\tan^2\theta}.$$

(2) $$a\cos\theta + b\sin\theta = \frac{a + b}{\sqrt{2}}.$$

(3) $$4\sin^3\theta - \sin 3\theta = \frac{1}{\sqrt{2}}.$$

3. If A, B, C be the angles of a triangle, shew that

$$\sec^2 B + \sec^2 C + 2\sec B\sec C\cos A = \sec B\sec C\sin A(\tan B + \tan C).$$

4. If CA, CB be two fixed tangents to a circle, and D, E their middle points, shew that the perpendicular distance of any point P on the circumference of the circle from DE is proportional to the square on PC.

5. If A', B', C' be any points on the sides of the triangle ABC, prove that $AB' \,.\, BC' \,.\, CA' + B'C \,.\, C'A \,.\, A'B =$ area of triangle $A'B'C' \times$ twice the diameter of the circle circumscribing ABC.

6. Two tangents OA, OB are drawn to an ellipse whose foci are S, H and centre C. N is the middle point of AB. Shew that

$$OA \,.\, OB : OS \,.\, OH :: ON : OC.$$

7. A perfectly elastic ball is thrown into a smooth cylindrical well from a point in the circumference of the circular mouth. Shew that if the ball be reflected any number of times from the surface of the cylinder, the intervals between the reflections will be equal.

Shew also that if the ball be projected horizontally in a direction making an angle $\frac{\pi}{n}$ with the tangent to the circle at the point of projection, it will reach the surface of the water at the instant of the n^{th} reflection if the space due to the velocity of projection be

$$\frac{(\text{radius})^2}{\text{depth}}\left(n \sin \frac{\pi}{n}\right)^2$$

PAPER XCV.

1. Find the sum to n terms of ·

(1) $1 \,.\, 3 + 2 \,.\, 4 + 3 \,.\, 5 + \ldots\ldots$

(2) $1 \,.\, 2^2 + 2 \,.\, 3^2 + 3 \,.\, 4^2 + \ldots\ldots$

(3) $\frac{1}{1^2 \,.\, 3^2} + \frac{2}{3^2 \,.\, 5^2} + \frac{3}{5^2 \,.\, 7^2} + \frac{4}{7^2 \,.\, 9^2} + \ldots\ldots$

2. In each of two triangles the angles are in G.P. The least angle of one of them is three times the least angle of the other, and the sum of the greatest angles is 240°. Find the circular measures of the angles.

3. Prove that

$$\log_e 2 = \tfrac{3}{10}\left\{\log_e 10 + \frac{1}{2^7} + \frac{1}{2}\cdot\frac{3}{2^{14}} + \frac{1}{3}\cdot\frac{3^2}{2^{21}} + \dots\right\}$$

and shew that the coefficient of x^n in the expansion of $\{\log\ (1+x)\}^2$ is

$$\frac{2(-1)^n}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}\right).$$

4. D, E, F are the middle points of the sides BC, CA, AB of a triangle ABC. Through A a straight line is drawn cutting DF, DE in M and N respectively. Shew that BM and CN are parallel.

5. Two conjugate diameters of an ellipse are cut by the tangent at any point P in M and N. Shew that the area of the triangle CPM varies inversely as the area of the triangle CPN.

6. Given the centre of a conic and two tangents making angles α, β with the axis of x, shew that the locus of the foci is the rectangular hyperbola

$$y^2 - x^2 + 2xy\cot(\alpha+\beta) = k^2$$

7. Given that a quadrant of the earth's surface is 10^9 centimetres, and that the mean density of the earth is 5·67, prove that the unit of force will be the attraction of two spheres each of 3926 grammes, whose centres are a centimetre apart, the acceleration of gravity at the earth's surface being 981; a centimetre, second, and gramme being the units of length, time, and mass.

Paper XCVI.

1. If l_1, l_2, l_3 are the lowest common multiples of B and C, of C and A, of A and B respectively; if g_1, g_2, g_3 are the highest common divisors of the same pairs; and if L, G, are the lowest common multiple and highest common divisors of A, B, and C; prove that

$$\frac{L^2}{G^2} = \frac{l_1 l_2 l_3}{g_1 g_2 g_3}.$$

2. If

$$\begin{aligned} x &= 2\cos(\beta - \gamma) + \cos(\theta + \alpha) + \cos(\theta - \alpha) \\ &= 2\cos(\gamma - \alpha) + \cos(\theta + \beta) + \cos(\theta - \beta) \\ &= -2\cos(\alpha - \beta) - \cos(\theta + \gamma) - \cos(\theta - \gamma) \end{aligned}$$

prove that $x = \sin^2\theta$, provided that the difference between any two of the angles α, β, γ neither vanishes nor equals a multiple of π and $\alpha + \beta$ is not a multiple of π.

3. If the diameter of a circle be the fourth proportional to the three sides of an inscribed triangle, prove that the area of the triangle is equal to half the square described on its least side.

4. ABC is a triangle inscribed in a circle, radius R, and D is any point on the circumference. If d denote the length of the perpendicular from D on the pedal line of the triangle ABC with respect to D, shew that

$$4dR^2 = DA \,.\, DB \,.\, DC.$$

5. The tangent at a point P of a rectangular hyperbola meets a diameter QCQ' in T. Shew that CQ and TQ' subtend equal angles at P.

6. Two tangents TP, TQ are drawn to an ellipse at right angles. If S be a focus, shew that

$$\sin^2 SPT + \sin^2 SQT = \text{const.}$$

7. If in the second system of pullies there are n strings at the lower block, prove that the upward acceleration of W due to a power P will be

$$\frac{nP - W}{n^2P + W} \cdot g.$$

Paper XCVII.

1. A person has n sewing machines, each of which requires one worker, and will yield each day it is at work q times the worker's wages as nett profits. The machines are never all in working order at once, and at any time it is equally likely that 1, 2, 3, or any other number of them are out of repair. The worker's wages must be paid whether there is a machine for him to work or not. Prove that the most profitable number of workers to be permanently employed will be the integer nearest to

$$\frac{nq}{q+1} - \frac{1}{2}.$$

2. If $A + B + C = \pi$, and

$$\sin^3\theta = \sin(A - \theta)\sin(B - \theta)\sin(C - \theta),$$

then will

$$\cot\theta = \cot A + \cot B + \cot C.$$

3. TP and TQ are two tangents to a circle, and R is any point in the straight line which joins their middle points. Shew that TR is equal to the tangent from R to the circle.

4. If P be any point of an hyperbola whose foci are S and H, and if the tangent at P meet an asymptote in T, shew that the angle between that asymptote and HP is double the angle STP.

5. If normals be drawn to the parabola $y^2 = lx$ from any point of the curve

$$y^2 + \frac{2}{27l}(l - 2x)^3 + a^2 = 0$$

the area formed by the three points at which the normals meet the curve is constant.

Hence find the locus of the points from which only two separate normals can be drawn to the parabola.

6. The normals at three points of a parabola are concurrent. Shew that a rectangular hyperbola can be described through these three points having for one of its asymptotes the axis of the parabola, and that the centre of this hyperbola always lies on the axis of the parabola.

7. Pendulums which beat seconds correctly in London ($g = 32{\cdot}19$) and Edinburgh ($g = 32{\cdot}20$) respectively are changed in station. If started simultaneously from the vertical position towards the left, after how many seconds will they again be both vertical and moving leftwards?

Paper XCVIII.

1. The income-tax is levied on the average of three years' income. Shew that if a man's income increase either in A.P. or G.P. so will his income-tax, the percentage being supposed uniform.

2. If

$$u_n = n(1 + k)u_{n-1} - n(n - 1)ku_{n-2} \text{ and } u_2 = 2u_1k,$$

shew that

$$\frac{u_1}{\lfloor 0 \lfloor 1} + \frac{u_2}{\lfloor 1 \lfloor 2} + \frac{u_3}{\lfloor 2 \lfloor 3} + \dots = u_1 e^k.$$

3. Prove that

$$\sin\theta \,.\, \frac{\sin\theta}{1} - \sin 2\theta \,.\, \frac{\sin^2\theta}{2} + \sin 3\theta \,.\, \frac{\sin^3\theta}{3} - \dots$$
$$= \cot^{-1}(1 + \cot\theta + \cot^2\theta).$$

4. If O be the centre of the escribed circle which touches BC, and the other two sides produced, shew that

$$BC \cdot OA^2 - CA \cdot OB^2 - AB \cdot OC^2 = AB \cdot BC \cdot CA.$$

5. Shew that the locus of the vertex of a right circular cone which contains a given ellipse is a hyperbola.

6. Two rings, each of weight w, slide upon a vertical semi-circular wire with the diameter horizontal and convexity upwards. They are connected by a light string of length $2l$ (supposed less than $2a$, the diameter of the semicircle) on which is slipped a ring of weight W. Shew that when the two rings that slide on the semicircle are as far apart as possible, the angle 2α subtended by them at the centre is given by the equation

$$(W + 2w)^2 \tan^2(\alpha + \epsilon)\,(l^2 - a^2 \sin^2\alpha) = W^2 a^2 \sin^2\alpha$$

where $\tan\epsilon$ is the coefficient of friction between the rings and the wire.

7. BC is the horizontal base, AB the vertical axis of a cycloid, and PQ is a line unwrapped from the arc PA. Shew that the time of sliding down PQ from rest under the action of gravity is always the same however far the line is unwrapped.

Paper XCIX.

1. Solve the equations

(1) $(x^3 + 3x^2 + 34x + 37)^{\frac{1}{3}} - (x^3 - 3x^2 + 34x - 37)^{\frac{1}{3}} = 2.$

(2) $\left.\begin{aligned} x^4 - 8xy^3 + 48(y^2 - 1) &= 0 \\ x^3y - 8y^4 - 6(x^2 - 4) &= 0 \end{aligned}\right\}.$

(3) $\left.\begin{aligned} \sqrt{1 - 16y^2} - \sqrt{1 - 16x^2} &= 2(x + y) \\ x^2 + y^2 + 4xy &= \frac{1}{5} \end{aligned}\right\}.$

2. Prove that the distance from A to the centre of the circle inscribed in the triangle ABC is $b \sec \frac{B}{2} \sin \frac{C}{2}$.

If x, y, z denote these distances from A, B, C respectively, prove that

$$a^4x^4 + b^4y^4 + c^4z^4 + (a + b + c)^2x^2y^2z^2$$
$$= 2(b^2c^2y^2z^2 + c^2a^2z^2x^2 + a^2b^2x^2y^2).$$

3. Prove that in any quadrilateral the sum of the squares on the four lines drawn from the middle point of the line joining the middle points of the two opposite sides to the angular points of the quadrilateral is equal to the sum of the squares of the lines joining the middle points of the opposite sides, and of the line joining the middle points of the diagonals.

4. If OP, OQ be two tangents to an ellipse, and CP', CQ' the parallel semi-diameters, and S, H the foci, shew that

$$OP \cdot OQ + CP' \cdot CQ' = OS \cdot OH.$$

5. Two ellipses have one common focus and equal major axes. One ellipse revolves about this focus in its own plane. Prove that its chord of intersection with the other ellipse envelops a central conic confocal with the fixed ellipse.

6. $ABCD$ is a quadrilateral, and O the intersection of its diagonals. Prove that the centre of gravity of the quadrilateral is the same as that of five particles at A, B, C, D, O the mass of the particle at O being unity, and that at any vertex (as A) being the ratio of its distance (OA) from O to the diagonal through it (AC).

7. The series of quantities $v_1, v_2 \ldots . v_n$ obey the law

$$v_r = v_{r-1} + v_{r-2} \text{ and } v_2 = \lambda v_1.$$

Prove that $v_r^2 \sim v_{r+1}v_{r-1} = (\lambda^2 - \lambda - 1)v_1^2$.

Paper C.

1. If

$$yz + \frac{x^3}{y+z} = zx + \frac{y^3}{z+x} = xy + \frac{z^3}{x+y},$$

x, y and z being supposed unequal, prove that each of these quantities is equal to $xy + yz + zx$; and that

$$x + y + z = 0.$$

2. Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta + \tanh 2\phi}{\tan 2\theta - \tanh 2\phi}\right) + \tan^{-1}\left(\frac{\tan\theta - \tanh\phi}{\tan\theta + \tanh\phi}\right)$$
$$= \tan^{-1}(\cot\theta \coth\phi)$$

where tanh and coth are defined by the equations

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \text{ and } \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

3. O, A, B, C, D are points on a circle. Prove that the feet of the perpendiculars from O on the pedal lines with regard to O of the four triangles formed by joining the points A, B, C, D lie on one straight line.

4. Two given ellipses in the same plane have a common focus, and one revolves about the common focus while the other remains fixed. Prove that the locus of the point of intersection of their common tangents is a circle.

5. PSp, QSq, RSr are any three focal chords of a parabola; QR meets the diameter through p in A, RP meets the diameter through q in B, and QP meets the diameter through r in C. Shew that the points A, B, C lie on a straight line through the focus.

6. From n circular laminæ of equal thickness having radii in the ratios 1 : 3 : 5 &c., sectors are cut having the same vertical angle. These sectors are placed one on the other in their order of magnitude, so that their centres are in a vertical line, and their middle radii in one vertical

plane, and on the same side of the line of centres. Shew that the distance of the centre of gravity of the whole pile from this line is to the distance of the centre of gravity of the first as $3n(2n^2 - 1) : 4n^2 - 1$.

7. Two particles are connected by a string of given length which passes over a small smooth pulley fixed at the top of two inclined planes having a common height. Supposing that one particle moves on each plane so that the whole motion is in one vertical plane, find the locus of their centre of gravity, and shew that it describes a straight line with uniform acceleration

$$g \cdot \frac{w_1 \sin \alpha - w_2 \sin \beta}{(w_1 + w_2)^2} \cdot \sqrt{w_1^2 + w_2^2 + 2w_1w_2 \cos(\alpha + \beta)}$$

w_1, w_2 being the weights of the particles, and α, β the inclinations of their respective planes.

APPENDIX.

APPENDIX.

Paper VI.

7. An ellipse is described having for axes the tangent and normal at any point P of a fixed ellipse, and touching one of the axes of the fixed ellipse at its centre. Prove that the locus of the focus of the moving ellipse is two circles, of radii $a \pm b$.

Paper VII.

5. If $AA'BB'$, $BB'CC'$, $CC'AA'$ be three circles, and the straight lines AA', BB', CC' cut the circle $A'B'C'$ again in α, β, γ, respectively, the triangle $\alpha\beta\gamma$ will be similar to ABC.

Paper XI.

6. Prove that the asymptotes of the curve

$$11x^2 + 24xy + 4y^2 - 2x + 16y + 11 = 0,$$

are given by the equation

$$(11x + 2y + 9)(x + 2y - 1) = 0.$$

Trace the curve, find the lengths of its axes, and prove that the equation of its director circle is

$$x^2 + y^2 + 2x - 2y = 1.$$

PAPER XII.

7. AC, CB are chords at right angles in a circle, P is any point on the circumference. PA, PB, PC represent forces. Shew that the locus of the extremity of the straight line which represents their resultant is a circle.

PAPER XVII.

7. Along the sides of a regular hexagon taken in order act 6 forces represented by 1, 2, 3, 4, 5, 6 respectively. Prove that their resultant will be represented by 6, and that its direction will be parallel to one of the sides, and at a distance from the centre of the hexagon equal to $3\frac{1}{2}$ times the radius of the inscribed circle.

PAPER XVIII.

1. A road runs from A to meet another at right angles. Shew that there are two points on the second road which may be reached in the same time from A whether we travel by road or across country, the rates of travelling by road and across country being as 7 : 5. Also shew that for places between these the quickest route is across country, and the quickest for all other places is by road.

2. If 16 be added to the product of four consecutive odd or even numbers, the result is always a square number. For odd numbers its last digit in four cases out of five is 1, in the remaining case 5. For even numbers the last digit in four cases out of five is 6, in the remaining case 0.

3. If l, m, n be the distances of any point in the plane of a triangle ABC from its angular points, and d its distance from the circum-centre, shew that

$$l^2\sin 2A + m^2\sin 2B + n^2\sin 2C = 4(R^2 + d^2)\sin A\sin B\sin C,$$

R being the radius of the circum-circle.

7. A conic passes through the centres of the four circles which touch the sides of a triangle. Prove that the locus of its centre is the circumscribing circle.

PAPER XIX.

2. Solve the equations

(1) $x^4 + a^4 = 4ax(x^2 + a^2)$.

(2) $\frac{x}{a} + \frac{b}{y} + \frac{c}{z} = \frac{a}{x} + \frac{y}{b} + \frac{c}{z} = \frac{a}{x} + \frac{b}{y} + \frac{z}{c} = 1$.

4. If R be the radius of the circum-circle, shew that the area of the triangle

$$= \tfrac{2}{3} R^2 \left\{\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B)\right\}.$$

5. If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by two lines equally inclined to the bisector, one terminated by the base and the other by the circum-circle.

PAPER XX.

3. If the area of a quadrilateral be

$$\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

shew that it can be inscribed in a circle.

4. AB, CD are chords of a circle intersecting in O, and AC, DB meet at P. If circles be described about the triangles AOC, BOD, the angle between their tangents at O will be equal to APB, and their other common point will lie on OP.

5. A uniform rod AB rests with its ends on a rough circular wire in a vertical plane, and the equilibrium is limiting. Shew that the vertical through the centre of the rod meets the circle through A, B, and the centre of the wire in two points, in one of which the directions of the resultant actions at A and B meet.

6. $2a$ and $2b$ are the major and minor axes of an ellipse. With centre O as centre, and radii a, b, $a + b$, circles are

described, and a radius vector $OPQR$ is drawn meeting them respectively in P, Q, R. If a parallel to the minor axis drawn through P meet a parallel to the major axis drawn through Q in S, then S is a point on the ellipse, and SR is the normal at S.

7. Defining the angle at which two circles cut to be that in which no part of either circle lies, prove that if the circles

$$(x - b)(x - b') + y^2 = 0, \quad (x - a)(x - a') + y^2 = 0,$$

cut at an angle θ,

$$(a - a')^2 (b - b')^2 \sin^2\theta + 4(b' - a)(b - a)(b' - a')(b - a') = 0.$$

Paper XXI.

1. The population of a town at the end of any year can be found by subtracting eleven times the population at the end of the previous year from ten times the population at the end of the succeeding year. Nine years ago the population was 1210, eleven years ago it was 1000. Prove that it increases in G P.

2. Prove that $5 \sin^{-1} \dfrac{1}{\sqrt{50}} + 2 \sin^{-1} \dfrac{3}{25\sqrt{10}} = \dfrac{\pi}{4}$.

3. Sum to infinity the series

(1) $\sin\theta + \dfrac{1}{2}\sin 2\theta + \dfrac{1}{2^2}\sin 3\theta + \dfrac{1}{2^3}\sin 4\theta + \dots$

(2) $\sin\theta - \dfrac{1}{3}\sin 3\theta + \dfrac{1}{5}\sin 5\theta - \dfrac{1}{7}\sin 7\theta + \dots$

(3) $\sin\theta + \dfrac{1}{4}\sin 3\theta + \dfrac{1}{4^2}\sin 5\theta + \dfrac{1}{4^3}\sin 7\theta + \dots$

4. The three perpendiculars from the angles A, B, C of a triangle on the opposite sides meet the sides in D, E, F. If D, E, F be given, shew how to construct the triangle ABC.

5. P is the orthocentre of a triangle, Q any point on the circum-circle. Shew that PQ is bisected by the pedal line of the triangle with respect to the point Q.

6. If λ be a variable parameter, the locus of the vertices of the hyperbolas represented by

$$x^2 - y^2 + \lambda xy = a^2$$

is the curve

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

Paper XXII.

2. If the impossible root of $x^3 + qx + r = 0$ be $\alpha + \beta\sqrt{-1}$, shew that $\beta^2 = 3\alpha^2 + q$.

4. Through the angular points of a triangle ABC draw straight lines perpendicular to the lines bisecting the angles. If Δ, P be the area and perimeter of the original triangle, Δ', P' those of the new triangle, prove that

(1) $4\Delta\Delta' = Pabc$;

(2) $PP' = 4\Delta'\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)$.

6. If $ABCD$ be a quadrilateral inscribed in a circle, and the sides be produced to meet in F and G, prove that the bisectors of the angles at F and G meet at right angles.

7. Chords of a hyperbola are drawn through a fixed point. Shew that the locus of their middle points is a hyperbola similar to the original hyperbola or its conjugate.

Paper XXIII.

1. If $x\left(1 - \frac{mzy}{x^3}\right) = y\left(1 - \frac{mxz}{y^3}\right) = z\left(1 - \frac{myx}{z^3}\right)$,

and x, y, z be unequal, prove that each member of these equations

$$= x + y + z - m.$$

2. A besieged garrison is provisioned for a certain number of days; after 10 days one-sixteenth of the men are killed in a sortie, when it is calculated that by diminishing the daily rations by one-fifth it will be able to hold out for 30 days longer than was first supposed. Subsequently 150 men with a quantity of provisions equal to half what is still left come in; by which it will be enabled to increase the time it can still hold out by one-fourth. How many men were there originally? and for how long was it provisioned? *Ans.* 800 men. 100 days.

Paper XXV.

6. Out of a wooden cylinder is cut a cone of the same base, and the hole is filled up with lead. If lead be nine times as heavy as wood, and if the centre of gravity of the whole be at the vertex of the cone, shew that

the height of the cone : the height of the cylinder :: $\sin 18^\circ : 1$

Paper XXX.

1 Prove that (1) the coefficient of $x^{m+1}y^{n+1}$ in the expansion of

$$\frac{(1-x)(1-y)}{1-x-y} \text{ is } \frac{(m+n)\,!}{m\,!\,n\,!}.$$

(2) the coefficient of x^{n-1} in the expansion of

$$\left\{(1-x)(1-cx)(1-c^2x)(1-c^3x)\right\}^{-1},$$

in ascending powers of x is

$$\frac{(1-c^n)(1-c^{n+1})(1-c^{n+2})}{(1-c)(1-c^2)(1-c^3)}.$$

5. A circle is described about a triangle ABC, and from any point D lines DB, DC are drawn cutting the circle in two points P and Q whose pedal lines intersect in S. Prove that the angle S is equal to the difference between the angles A and D.

Paper XXXI.

1. In a bag there is a number of tickets marked with the natural numbers from 1 to n^2+1. Every number is marked on each of r tickets, and every square number m^2 confers a prize of m shillings. A person can draw one ticket from the bag. Shew that the value of his expectation is

$$\frac{n^2(n+1)^2}{2(n^2+1)(n^2+2)}.$$

2. If $(1+x+x^2)^n = P_0 + P_1x + P_2x^2 + \dots + P_rx^r + \dots$ prove (1) $P_n = P_0^2 - P_1^2 + P_2^2 - \dots$

$$(2)\ P_n = \frac{1}{(n-1)!}\left\{\frac{(2n-1)!}{n!} - n\,.\,\frac{(2n-4)!}{(n-3)!} + \frac{n\,.\,(n-1)}{2!}\,.\,\frac{(2n-7)!}{(n-6)!} - \dots\right\}.$$

5. A, B, C, D are four points not in one plane. If AB is perpendicular to CD, and AC is perpendicular to BD, then will AD be perpendicular to BC.

6. TP, TQ are tangents to a parabola whose focus is S. LM, a third tangent, cuts them in L and M. Prove that the triangles SPL, STM are similar.

Hence shew that $TL : LP :: QM : MT$.

Paper XXXII.

1. Of three events it is 2 to 1 against the first and second happening, 3 to 2 against the second and third, and 9 to 1 against the first and third. Shew that the odds against all three happening are $5\sqrt{3}-1$ to 1.

2. O is the centre of gravity of a triangle. AO, BO, CO are produced to points D, E, F such that $AD = l\,.\,AO$, $BE = m\,.\,BO$, $CF = n\,.\,CO$. Find the values of l, m, n so that the sides of the triangle DEF may pass through the points A, B, C.

5. A, B, C, D are four points in space. AB, AC are divided in E, F so that $AE : EB :: AF : FC$. DB, DC are divided in G, H so that $DG : GB :: DH : HC$. Shew that the lines GF and HE will intersect.

Paper XXXIII.

1. A river flows from P to Q, a distance of 12 miles, at a uniform rate. B starts at 12 o'clock from Q to row to P, and A starts at 5 minutes past 12 to row from Q to P and back again. A overtakes B a mile from Q; he rows on to P, and at once turning back meets B two miles below A. A reaches Q 35 minutes after B reaches P. Find the times at which A passed B, and the rate of the stream.

5. Any point P is taken on a given segment of a circle described on a line AB, and perpendiculars AG and BH are let fall on BP and AP respectively. Prove that GH touches a fixed circle.

Paper XXXIV.

1. A policeman walks round his beat uniformly during his hours of duty. Shew that the chance of my meeting him, if I walk *in the opposite direction* down a street, which is $\frac{1}{n}$th of his beat, at a rate m times his, is $\frac{1+m}{mn}$, where $m > 1$.

Also solve the problem when the condition in italics is removed.

2. If
$$b \cdot \frac{y}{z} + c \cdot \frac{z}{y} = a,$$
$$c \cdot \frac{z}{x} + a \cdot \frac{x}{z} = b,$$
$$a \cdot \frac{x}{y} + b \cdot \frac{y}{x} = c,$$

then will

$$x^{-3} + y^{-3} + z^{-3} + x^{-1}y^{-1}z^{-1} = 0,$$
$$a^3x^3 + b^3y^3 + c^3z^3 + abcxyz = 0,$$
$$a^3 + b^3 + c^3 = 5abc.$$

4. Prove that the distance between the centre of the inscribed circle and the orthocentre of a triangle is

$$2R\,(\text{vers}\,A\ \text{vers}\,B\ \text{vers}\,C - \cos A\cos B\cos C)^{\frac{1}{2}},$$

where R is the radius of the circum-circle.

Paper XXXV.

2. Solve the equations:—

(1) $$\frac{x - \sqrt{x^2 - 1}}{\sqrt{x + \sqrt{x^2 - 1}}} = \sqrt[4]{x^2 - 1}\left\{\sqrt{x^2 + x} - \sqrt{x^2 - x}\right\}.$$

(2) $x(y + z)^2 = 1 + a^3$; $x + y = \frac{3}{2} + x$; $yz = \frac{3}{16}$.

(3) $$\left.\begin{aligned} a(y - z) + b(z - x) + c(x - y) &= 0 \\ (x - y)(y - z)(z - x) &= d^3 \\ x + y + z &= e \end{aligned}\right\}.$$

3. Prove that in a triangle where $a < c$,

$$\frac{\cos nA}{b^n} = \frac{1}{c^n}\left\{1 + n\,.\,\frac{a}{c}\cos B + \frac{n(n+1)}{2\,!}\frac{a^2}{c^2}\cos 2B + \frac{n(n+1)(n+2)}{3\,!}\frac{a^3}{c^3}\cos 3B + \ldots\right\}.$$

Paper XLIII.

6. An ellipse and hyperbola are described so that the foci of each are at the extremities of the transverse axis of the other.

Prove that the tangents at their points of intersection meet the conjugate axis in points equidistant from the centre.

Paper XLIV.

1. If x, y, z are in G.P. when θ_1 is subtracted from each; and z, y, x are in G.P. when θ_2 is subtracted from each; and x, z, y are in G.P. when θ_3 is subtracted from each; prove that

$$\frac{1}{\theta_1 - x} + \frac{1}{\theta_2 - y} + \frac{1}{\theta_3 - z} = 0.$$

4. If TA, TB be tangents meeting a circle in A and B, and $TCQD$ be any chord meeting the circle in C and D, whilst Q is the middle point of the chord CD, shew that TQ bisects the angle AQB, and the length of TQ varies as the sum of the lengths of AQ and BQ.

5. T is any point on the tangent to a parabola at Q. Prove that the tangent at T to the circle round TQS touches the parabola.

Paper LI.

1. Prove that the value of the expression

$$\sqrt{a^2 \cos^2\phi + b^2 \sin^2\phi} + \sqrt{a^2 \sin^2\phi + b^2 \cos^2\phi}$$

lies between $a + b$ and $\sqrt{2a^2 + 2b^2}$.

Prove also that

$$\frac{1}{\sqrt{a^2 \cos^2\phi + b^2 \sin^2\phi}} + \frac{1}{\sqrt{a^2 \sin^2\phi + b^2 \cos^2\phi}}$$

lies between $\dfrac{1}{a} + \dfrac{1}{b}$ and $\dfrac{4}{\sqrt{2a^2 + 2b^2}}$.

5. ABC is a triangle inscribed in a conic whose centre is O, and Oa, Ob, Oc are drawn to the middle points of the chords. From any point P on the conic, Pa, $P\beta$, $P\gamma$ are drawn parallel to Oa, Ob, Oc to meet the sides in a, β, γ. Prove that the points a, β, γ are collinear.

Paper LV.

3. The sides of a triangle are in A.P., and its area is to that of an equilateral triangle of the same perimeter :: 3 : 5.

Shew that the greatest angle is 120.°

5. PAQ, PBC are two semi-circles which touch internally at P, PQC being the common diameter. Through P draw a secant PAB such that the area of the triangle ABC may be a max., and shew that for this position of the secant the area of the triangle QAB is also a max.

Paper LVII.

2. In the continued fraction

$$\frac{1}{(1-x)+}\,\frac{x}{(1-x^3)+}\,\frac{x^3}{(1-x^5)+}\,\frac{x^5}{(1-x^7)+}\cdots$$

shew that the n^{th} convergent is $\dfrac{\sigma_n}{\sigma_n - 1}$,

where $\sigma_n = x^{-1^2} - x^{-2^2} + x^{-3^2} - x^{-4^2} + \ldots$
$+ (-1)^{n-1} \,.\, x^{-n^2}.$

7. Prove geometrically that if a line be drawn through a focus of a central conic making a constant angle with a tangent, the locus of the point of intersection is a circle.

Paper LIX.

5. ABC is a triangle, O any point, in the same plane or not; P, Q, R points in OA, OB, OC. BR, CQ intersect in L; CP, AR in M; AQ, BP in N. OL, OM, ON cut BC, CA, AB in D, E, F. Prove that AD, BE, CF are concurrent.

6. A semicircular piece of paper is folded over so that a particular point P on the bounding diameter lies on the circular boundary. Shew that the crease-line always touches a fixed conic.

7. A straight line of given length moves so that its extremities always lie (1) on a fixed ellipse, (2) on a fixed parabola. Find the locus of its middle point in the two cases.

Paper LX.

2. Shew how to find n if the sum of n terms of the series

$$1 + 5 + 9 + 13 + \ldots$$

be a perfect square; and find the first two values of n greater than unity.

4. From a point A on the outer of two concentric circles tangents AP, AQ are drawn to the inner. AP, QP meet the outer again in T, R. Prove that

$$RP : RQ :: RT^2 : RA^2.$$

Paper LXI.

4. The sum of the reciprocals of the distances of a fixed point from tangents to a circle at the extremities of any chord through the point is constant.

Paper LXVII.

7. In the system of pulleys in which each string is attached to a bar supporting the weight, find at what point of the bar the weight must be attached if there are two movable pulleys.

Also shew that if the weight be then doubled, it will descend with acceleration $= \dfrac{g}{15}$.

PAPER LXX.

2. If

$$x_2x_3 + y_2y_3 = x_3x_1 + y_3y_1 = x_1x_2 + y_1y_2 = 1,$$

and

$$d_1 = x_2y_3 - x_3y_2, \quad d_2 = x_3y_1 - x_1y_3, \quad d_3 = x_1y_2 - x_2y_1,$$

shew that $\qquad d_1 + d_2 + d_3 = d_1d_2d_3.$

PAPER LXXV.

6. P is any point on a conic circumscribing the triangle ABC, and the diameters which bisect the chords parallel to PA, PB, PC meet the tangents at A, B, C in the points D, E, F respectively. Shew that D, E, F lie on the polar of P.

PAPER LXXVI.

7. A particle of elasticity e is projected from a point in the wall of a square room in a direction whose projection on the floor makes an angle θ with the wall. Shew that if the particle after striking each wall in succession returns to the point of projection, then

$$e(\mu + 1) \cot\theta = e\mu + 1,$$

$\mu : 1$ being the ratio in which a horizontal line in the side of the wall is divided by the point of projection.

PAPER LXXVII.

5. If in a rough inclined plane the ratio of the greatest force to the least force which, acting parallel to the plane, will just support a given weight on the plane be equal to the ratio of the weight to the pressure on the plane, prove that the coefficient of friction is $\tan\alpha \,.\, \tan^2\dfrac{\alpha}{2}$, where α is the inclination of the plane.

PAPER LXXVIII.

7. A projectile is discharged with velocity v at an elevation a, and n seconds afterwards a second one is discharged after it so as to strike it. If v', a' be its velocity and elevation, prove that

$$2\,vv' \sin(a - a') = (v \cos a + v' \cos a')\, gn.$$

PAPER LXXIX.

3. Prove that

$$(1)\ 1 = \tan\frac{\pi}{2^{n+1}}\left\{\tan\frac{\pi}{2^{n+1}} + 2\tan\frac{\pi}{2^n} + \ldots + 2^{n-2}\tan\frac{\pi}{2^3} + 2^{n-1}\right\}.$$

$$(2)\ 2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\,\frac{\cos\theta}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}} + \frac{1}{2^2}\,\frac{\cos\theta\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}\cos^2\frac{\theta}{2^3}} + \ldots$$

and sum to n terms the series

$$\sec^2\theta + 2^2\sec^2 2\theta + 2^4\sec^2 2^2\theta + \ldots + 2^{2n-2}\sec^2 2^{n-1}\theta.$$

4. On the sides of a triangle as bases are described externally three similar isosceles triangles. Prove geometrically that the lines joining the vertices of these triangles with the opposite vertices of the given triangle are concurrent.

5. Shew that the equation of the envelope of a circle described upon a chord of the circle $(x - a)^2 + y^2 = c^2$ passing through the origin as diameter is

$$(x^2 + y^2 + a^2 - c^2)(x^2 - 2ax + y^2 + a^2 - c^2) = a^2y^2.$$

Prove also that the maximum distance of a point on the envelope from the centre of the given circle is $c\sqrt{2}$.

Paper LXXX.

7. A tennis ball is served from a height of 8 feet. It just touches the net at a point where the net is 3 ft. 3 in. high, and hits the service line, 21 feet from the net. The horizontal distance of the server from the foot of the net is 39 feet. Prove that the angle which the direction of projection makes with the horizontal is $\tan^{-1}\frac{137}{1365}$; and that the horizontal velocity of the ball is about 171 feet per second, the plane of projection being perpendicular to the plane of the net.

Paper LXXXV.

4. S and H are the foci of a hyperbola, and PT, the tangent at P, cuts an asymptote in T. Prove that the angle $STP = PHT$.

Paper LXXXVIII.

7. One end of a string is fixed to a beam, from which it passes downwards and under a movable pulley of weight P, then over a fixed pulley, and then under a second movable pulley of the same weight, and then the other end is attached to the first movable pulley. A weight W is attached to the second movable pulley, and all the straight portions of the string are vertical.

Prove that there will be equilibrium if $W = P$.

Also, if $W > P$, the downward acceleration of W will be

$$\frac{W - P}{W + 5P}g.$$

Paper XCII.

7. If on a rectangular billiard table whose sides are a, b, a ball describe a rectangle whose sides are c, d, prove that the coefficient of elasticity between the ball and the sides of the table is

$$\left(\frac{ad - bc}{bd - ac}\right)^2 \text{ or } \left(\frac{bd - ac}{ad - bc}\right)^2.$$

Paper XCIII.

5. The envelope of a perpendicular drawn to a normal to a parabola at the point where the normal cuts the axis is a parabola. Prove also that the focal vector of the point of the parabola at which the normal is drawn meets the envelope at the point where the perpendicular touches it.

6. Shew that if

$$x_1^3 + y_1^3 = x_2^3 + y_2^3 = x_3^3 + y_3^3 = a^3,$$

and $\quad x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0,$

then $\quad x_1x_2x_3 + y_1y_2y_3 = a^3.$

A straight line cuts in 3 real points the curve $x^3 + y^3 = a^3$. Shew that their centroid, if it lie on either axis of coordinates, will be at the origin.

Paper XCIV.

1. Prove that

$$3 \cdot 81^{n+1} + (16n - 54)\,9^{n+1} - 320n^2 - 144n + 243$$

is a multiple of 2^{12}.

5. The diameter d of a circle is divided into $2n$ equal parts, and straight lines are drawn from any point in the circumference to each point of division. If $a_1, a_2 \ldots a_{2n-1}$ be the lengths of the lines so drawn, prove that in the limit, when the number of parts is increased indefinitely,

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a^2_{2n-1} = \frac{d^2}{2}.$$

Paper XCV.

7. From a point at a distance d from a plane whose inclination is β, two particles are projected simultaneously with velocities u and v in two different directions parallel to the plane and at right angles to each other. Prove that

they will strike the plane simultaneously at points A and B such that

$$AB^2 = \frac{2d}{g}(u^2 + v^2) \sec \beta.$$

Paper XCVI.

1. If $p = a + \frac{x^2}{a}$, $q = b + \frac{x^2}{b}$, $r = c + \frac{x^2}{c}$, prove that

$$\frac{1}{a}\left\{1 - \frac{q-r}{b-c}\right\} = \frac{1}{b}\left\{1 - \frac{r-p}{c-a}\right\} = \frac{1}{c}\left\{1 - \frac{p-q}{a-b}\right\};$$

and eliminate x, y, z from the equations

$$p = x - \frac{yz}{x},\ q = y - \frac{zx}{y},\ r = z - \frac{xy}{z},\ \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$$

2. If a, b, c, d be the sides of a quadrilateral taken in order, and ϕ the angle between the diagonals, shew that the area of the quadrilateral is

$$\tfrac{1}{4}(a^2 - b^2 + c^2 - d^2) \tan \phi.$$

Paper XCVII.

1. Prove that the max. and min. values of

$$x^3 + 3px^2 + 3qx + r$$

are

$$2p^3 - 3pq + r \pm 2(p^2 - q)^{\frac{3}{2}}.$$

4. If tangents be drawn to a fixed circle from any point on another circle, the envelope of the chord of contact is a conic.

7. If a particle of mass m fall down a cycloid under the action of gravity starting from the cusp, prove that the pressure of the particle upon the cycloid at any point is $2mg \cos \psi$, where ψ is the inclination to the horizon of the tangent to the cycloid at the point; also shew that the resultant acceleration $= g$.

Paper XCVIII.

6. Two equal uniform ladders, each of length l and weight w, are freely jointed at A and are connected by a rope PQ. A man whose weight is W goes b feet up one of the ladders. If the ground be smooth, prove that the tension of the rope

$$= \frac{Wb + wl}{2a} \cdot \frac{c}{\sqrt{a^2 - c^2}},$$

where $2c$ is the length of the rope in feet, and

$$a = AP = AQ.$$

Paper XCIX.

4. Two triangles BAC, $BA'C$ are inscribed in a circle on the common base BC, and the pedal lines of the triangles BAC, $BA'C$ are formed with regard to the points A' and A respectively. Shew that these two lines and the nine points' circles of the two triangles intersect in the same point.

Paper C.

2. Eliminate θ, having given

$$x \cos(\theta - \alpha) + y \cos\theta = 2a \sin(\theta + \gamma) \cos\theta \cos(\theta - \alpha),$$

$$x \sin(\theta - \alpha) + y \sin\theta = 2a \left\{\sin(\theta + \gamma) \sin\theta \cos(\theta - \alpha) - \cos\beta \cos\theta\right\},$$

$$\alpha + \beta + \gamma = \pi.$$

THE END.

RICHARD CLAY AND SONS, LIMITED, LONDON AND BUNGAY.