Within the limits of experimental certainty the frequency separations of the members of the non-harmonic series originating in the 7μ band are constant. If they are constant and if an attempt is made to formulate equations to relate the long and short wave components independently then it must be recognized that one or both of the resulting polynomial equations must include a constant term. A short paper dealing with the appearance of constant terms in the equations relating the vibrational doublets in another instance has been submitted elsewhere for publication.

[1] Schaefer, Bormuth and Matossi, *Zeits. Physik*, **39**, 648 (1926).

[2] E. K. Plyler, *Phys. Rev.*, **33**, 948 (1929).

[3] Schaefer, Matossi and Aderhold, *Phys. Zeits.*, **30**, 581 (1929).

THE EFFECT OF THE ANNIHILATION OF MATTER ON THE WAVE-LENGTH OF LIGHT FROM THE NEBULAE

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§ 1. *Introduction.*—On the basis of the general theory of relativity, Einstein[1] and de Sitter[2] have each proposed a line element which might correspond to the large-scale metrical properties of the universe as a whole, neglecting local deviations due to the neighborhood of stars or stellar systems. Neither of these line elements, however, has proved entirely satisfactory.

The Einstein line element, which can be written in the form

$$ds^2 = -\frac{dr^2}{1 - \dfrac{r^2}{R^2}} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2, \qquad (1)$$

has a reasonably satisfactory derivation, since it can be obtained from the equations of general relativity, if we assume that the universe can be regarded on a large scale as though permanently filled with a homogeneous fluid, and assume that the cosmological constant Λ, which occurs in the relativity equations connecting metric and the distribution of matter, can be regarded as a parameter determined by the density and pressure of this fluid.[3] This line element, however, would not lead to any red shift in the light coming from distant objects, as is actually found in light from the extra-galactic nebulae.[4]

The de Sitter line element, which can be written in the form

$$ds^2 = -\frac{dr^2}{1-\frac{r^2}{R^2}} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + \left(1-\frac{r^2}{R^2}\right)dt^2, \quad (2)$$

can be derived from the equations of general relativity if we assume a completely empty universe, and assume that the cosmological constant Λ is a new fundamental constant connected with a natural curvature of free space. The theoretical basis for this line element is hence not entirely pleasing, since the actual presence of matter in the universe must be regarded as producing a disturbance away from the proposed form of line element, instead of actually determining that form as in the case of the Einstein line element. Furthermore, the actual red-shift in the light from the nebulae can be accounted for on the basis of the above equation only if we ascribe to these nebulae just the right velocities of recession to account for the observed facts. This seems an *ad hoc* method of accounting for the observations, but its arbitrary character appears somewhat diminished by the considerations of Robertson,[5] who has shown by a very clever transformation of coördinates that particles having the right velocities of recession to account for the linear increase in shift with distance would all be equivalent to each other, in the sense that observers located on different particles would obtain the same results for the relation between red shift and distance.

Since neither of these previously proposed line elements is completely satisfactory, it is next natural to inquire if some other form of static line element could account for the red shift in a more natural manner. I have already investigated this problem,[6] however, and found that the line elements of Einstein and de Sitter, including the special relativity case obtained with R = ∞, are the only static ones which would agree with those restrictions as to spherical symmetry, symmetry with respect to past and future time, uniform density of material throughout the universe, etc., which it seems natural to impose; and this conclusion has since been confirmed by Robertson.[7]

Thus it appears probable that no static line element will successfully account for the red shift from distant objects, and if this phenomenon is actually due to the metric for the universe instead of to some more immediate cause such as that suggested by Zwicky,[8] we must turn to the consideration of *non-static* line elements. Such a change in attack, however, is not an unnatural one to make since, as I pointed out in the article already mentioned,[6] static line elements take no cognizance of any universal evolutionary process which may be going on. Hence the actual line element may well be non-static if the matter in the universe is not in a steady state.

As the basic hypothesis for the present article we shall assume, in agreement with the contemporary opinion of astrophysicists, that the matter in the universe is not in a steady state but rather that a general transformation of matter into radiant energy is taking place throughout the universe at the rate necessary to account for the radiation from stellar objects. If such a process is going on, the line element for the universe cannot be static, but necessarily must be non-static, since matter and the radiation produced from it would not have the same effect on the gravitational field.

In the following sections, we shall attempt to deduce the form of line element which would correspond to the transformation of matter into radiation postulated above, and by introducing reasonable simplifications and assumptions, shall actually be able to obtain as a first approximation for the form of the line element the expression

$$ds^2 = -\frac{e^{2kt}}{\left(1 + \frac{4r^2}{R^2}\right)^2}(dx^2 + dy^2 + dz^2) + dt^2, \quad (3)$$

where R is a constant having approximately the same significance as in the Einstein line element and k a constant directly related, on the one hand, to the average rate of transformation of matter into radiation and, on the other hand, to the known shift in wave-length with distance. Furthermore by comparison with observational data, we shall show that the value of k necessary to account for the empirical relation between red shift and distance for the extra galactic nebulae found by Hubble and Humason,[4] falls close to the range of values which would correspond to the rates of transformation of matter into radiation for actual stars of different types.

§ 2. *The General Form of the Line Element.*—To determine the form of the line element, we shall begin by ascribing spatial spherical symmetry to it, in order to agree with the idea that matter and radiation are uniformly distributed on the average throughout the universe. And we shall also ascribe to it symmetry with respect to past and future time in order to agree with the principle of dynamical reversibility. We can then write the line element in the general form

$$ds^2 = -e^{\mu}(dx^2 + dy^2 + dz^2) + e^{\nu}dt^2, \quad (4)$$

where μ and ν are as yet undetermined functions of the distance from the origin $r = \sqrt{x^2 + y^2 + z^2}$ and of the time t.

As our next restriction on the form of the line element, we shall require that particles which are stationary in the chosen coördinates x, y, z shall remain permanently stationary and not be subject to accelerations. Without the possibility of imposing this restriction, it would be difficult to construct a stable model of the universe. For the acceleration of a particle, however, we have in accordance with the theory of relativity the well-known equation for a geodesic

$$\frac{d^2x_\alpha}{ds^2} + \{\rho\sigma,\alpha\} \frac{dx_\rho}{ds}\frac{dx_\sigma}{ds} = 0. \tag{5}$$

And considering for example accelerations in the x-direction, for particles having zero spatial velocities

$$\frac{dx}{ds} = \frac{dy}{ds} = \frac{dz}{ds} = 0. \tag{6}$$

equation (5) reduces to

$$\frac{d^2x}{ds^2} + \{44,1\}\left(\frac{dt}{ds}\right)^2 = 0. \tag{7}$$

Thus, if the acceleration d^2x/ds^2 is to be equal to zero, we are led to the conclusion that the line element must be such as to make the Christoffel 3-index symbol $\{44,1\}$ itself equal to zero. For the line element (4), however, we can easily calculate the value of this symbol (see equation 18 below), and obtain

$$\{44,1\} = -\frac{1}{2} g^{11} \frac{\partial g_{44}}{\partial x} = \frac{1}{2} e^{\mu+\nu}\frac{\partial \nu}{\partial x} = 0. \tag{8}$$

The result shows that ν must be independent of x and hence by symmetry independent of y and z as well.

Hence ν is a quantity which must be a function of the time t alone, and by introducing a new time variable, t', defined by the equation

$$e^{\nu(t)}dt^2 = dt'^2, \tag{9}$$

and then dropping the primes we can necessarily reduce the line element (4) to the form

$$ds^2 = -e^{\mu}(dx^2 + dy^2 + dz^2) + dt^2, \tag{10}$$

where μ still remains as an undetermined function of r and t.

Nevertheless, a restriction on the form of μ can now easily be obtained by the following consideration. For the proper spatial volume dV_0, contained between non-accelerated particles located on the boundary of the volume element determined by dx, dy, dz, we can evidently write

$$dV_0 = e^{3\mu/2}\, dx\, dy\, dz, \tag{11}$$

and the logarithmic differential of this with respect to the time will evidently be

$$\frac{\partial \log dV_0}{\partial t} = \frac{3}{2}\frac{\partial \mu}{\partial t}. \tag{12}$$

Since, however, in accordance with the line element (10) the proper time is the same for all stationary observers throughout the universe, we can maintain our original assumption of uniform conditions throughout the universe, only if the percentage change with the time in the proper volume enclosed by "stationary" particles is independent of position. We must hence have

$$\frac{\partial}{\partial r}\frac{\partial \log dV_0}{\partial t} = \frac{3}{2}\frac{\partial^2 \mu}{\partial r \partial t} = 0, \tag{13}$$

and can write

$$\frac{\partial^2 \mu}{\partial r\, \partial t} = 0 \qquad \text{or} \qquad \mu = f(r) + g(t), \tag{14}$$

where f and g are undetermined functions of r and t, respectively.

Summarizing the present status of our information as to the general form of the line element, we may now write

$$ds^2 = -\,e^{\mu}(dx^2 + dy^2 + dz^2) + dt^2$$

with

$$\begin{aligned} &g_{11} = g_{22} = g_{33} = -\,e^{\mu} && g_{44} = 1 \\ &g^{11} = g^{22} = g^{33} = -\,e^{-\mu} && g^{44} = 1 \\ &g_{\rho\sigma} = g^{\rho\sigma} = 0 \qquad (\rho \neq \sigma) && \sqrt{-g} = e^{3\mu/2} \end{aligned} \tag{15}$$

and

$$\mu = f(r) + g(t)$$
$$r = \sqrt{x^2 + y^2 + z^2}.$$

§3. *The Christoffel Symbols.*—To proceed further in our determination of the form of the line element, we must next calculate the values of the Christoffel 3-index symbols corresponding to the line element given. These are defined in accordance with the equation

$$\{\mu\nu,\sigma\} = \frac{1}{2}g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right), \tag{16}$$

which for our line element evidently reduces to

$$\{\mu\nu,\sigma\} = \frac{1}{2}g^{\sigma\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) \text{ (not summed)}, \tag{17}$$

and taking μ, ν and σ as *different* indices we obtain the four following cases.

$$\begin{aligned}
\{\mu\mu,\mu\} &= \frac{1}{2} g^{\mu\mu} \frac{\partial g^{\mu\mu}}{\partial x_\mu} = \frac{1}{2} \frac{\partial \log g_{\mu\mu}}{\partial x_\mu} \\
\{\mu\mu,\nu\} &= -\frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x_\nu} \\
\{\mu\nu,\nu\} &= \{\nu\mu,\nu\} = \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\nu\nu}}{\partial x_\mu} = \frac{1}{2} \frac{\partial \log g_{\nu\nu}}{\partial x_\mu} \\
\{\mu\nu,\sigma\} &= 0.
\end{aligned} \tag{18}$$

Applying these equations, we can then easily calculate as the only non-vanishing Christoffel symbols, corresponding to the line element defined by (15)

$$\begin{array}{llll}
\{11,1\} = \frac{1}{2}\frac{\partial\mu}{\partial x} & \{22,2\} = \frac{1}{2}\frac{\partial\mu}{\partial y} & \{33,3\} = \frac{1}{2}\frac{\partial\mu}{\partial z} & \{41,1\} = \frac{1}{2}\frac{\partial\mu}{\partial t} \\
\{11,2\} = -\frac{1}{2}\frac{\partial\mu}{\partial y} & \{22,1\} = -\frac{1}{2}\frac{\partial\mu}{\partial x} & \{33,1\} = -\frac{1}{2}\frac{\partial\mu}{\partial x} & \{42,2\} = \frac{1}{2}\frac{\partial\mu}{\partial t} \\
\{11,3\} = -\frac{1}{2}\frac{\partial\mu}{\partial z} & \{22,3\} = -\frac{1}{2}\frac{\partial\mu}{\partial z} & \{33,2\} = -\frac{1}{2}\frac{\partial\mu}{\partial y} & \{43,3\} = \frac{1}{2}\frac{\partial\mu}{\partial t} \\
\{11,4\} = \frac{1}{2}e^\mu\frac{\partial\mu}{\partial t} & \{22,4\} = \frac{1}{2}e^\mu\frac{\partial\mu}{\partial t} & \{33,4\} = \frac{1}{2}e^\mu\frac{\partial\mu}{\partial t} & \\
\{12,2\} = \frac{1}{2}\frac{\partial\mu}{\partial x} & \{21,1\} = \frac{1}{2}\frac{\partial\mu}{\partial y} & \{31,1\} = \frac{1}{2}\frac{\partial\mu}{\partial z} & \\
\{13,3\} = \frac{1}{2}\frac{\partial\mu}{\partial x} & \{23,3\} = \frac{1}{2}\frac{\partial\mu}{\partial y} & \{32,2\} = \frac{1}{2}\frac{\partial\mu}{\partial z} & \\
\{12,1\} = \frac{1}{2}\frac{\partial\mu}{\partial y} & \{21,2\} = \frac{1}{2}\frac{\partial\mu}{\partial x} & \{31,3\} = \frac{1}{2}\frac{\partial\mu}{\partial x} & \\
\{13,1\} = \frac{1}{2}\frac{\partial\mu}{\partial z} & \{23,2\} = \frac{1}{2}\frac{\partial\mu}{\partial z} & \{32,3\} = \frac{1}{2}\frac{\partial\mu}{\partial y} & \\
\{14,1\} = \frac{1}{2}\frac{\partial\mu}{\partial t} & \{24,2\} = \frac{1}{2}\frac{\partial\mu}{\partial t} & \{34,3\} = \frac{1}{2}\frac{\partial\mu}{\partial t} &
\end{array} \tag{19}$$

§ 4. *The Contracted Riemann-Christoffel Tensor.*—Using these values of the Christoffel symbols, we can now calculate the components of the contracted Riemann-Christoffel tensor, which are given by the well-known equation

$$G_{\mu\nu} = -\frac{\partial}{\partial x_\alpha}\{\mu\nu,\alpha\} + \{\mu\alpha,\beta\}\{\nu\beta,\alpha\} + \frac{\partial^2}{\partial x_\mu \partial x_\nu}\log\sqrt{-g} - \{\mu\nu,\alpha\}\frac{\partial}{\partial x_\alpha}\log\sqrt{-g}. \tag{20}$$

As a result of the calculation, noting the restriction on the form of μ given by equation (14), we obtain as the only non-vanishing components of this tensor

$$
\begin{aligned}
G_{11} &= \frac{\partial^2\mu}{\partial x^2} + \frac{1}{2}\frac{\partial^2\mu}{\partial y^2} + \frac{1}{2}\frac{\partial^2\mu}{\partial z^2} + \frac{1}{4}\left(\frac{\partial\mu}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial\mu}{\partial z}\right)^2 - \frac{1}{2}e^{\mu}\frac{\partial^2\mu}{\partial t^2} - \frac{3}{4}e^{\mu}\left(\frac{\partial\mu}{\partial t}\right)^2 \\
G_{22} &= \frac{\partial^2\mu}{\partial y^2} + \frac{1}{2}\frac{\partial^2\mu}{\partial x^2} + \frac{1}{2}\frac{\partial^2\mu}{\partial z^2} + \frac{1}{4}\left(\frac{\partial\mu}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial\mu}{\partial z}\right)^2 - \frac{1}{2}e^{\mu}\frac{\partial^2\mu}{\partial t^2} - \frac{3}{4}e^{\mu}\left(\frac{\partial\mu}{\partial t}\right)^2 \\
G_{33} &= \frac{\partial^2\mu}{\partial z^2} + \frac{1}{2}\frac{\partial^2\mu}{\partial x^2} + \frac{1}{2}\frac{\partial^2\mu}{\partial y^2} + \frac{1}{4}\left(\frac{\partial\mu}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial\mu}{\partial y}\right)^2 - \frac{1}{2}e^{\mu}\frac{\partial^2\mu}{\partial t^2} - \frac{3}{4}e^{\mu}\left(\frac{\partial\mu}{\partial t}\right)^2 \\
G_{44} &= \frac{3}{2}\frac{\partial^2\mu}{\partial t^2} + \frac{3}{4}\left(\frac{\partial\mu}{\partial t}\right)^2 \qquad (21) \\
G_{12} &= G_{21} = \frac{1}{2}\frac{\partial^2\mu}{\partial x\partial y} - \frac{1}{4}\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial y} \\
G_{13} &= G_{31} = \frac{1}{2}\frac{\partial^2\mu}{\partial x\partial z} - \frac{1}{4}\frac{\partial\mu}{\partial x}\frac{\partial\mu}{\partial z} \\
G_{23} &= G_{32} = \frac{1}{2}\frac{\partial^2\mu}{\partial y\partial z} - \frac{1}{4}\frac{\partial\mu}{\partial y}\frac{\partial\mu}{\partial z}
\end{aligned}
$$

In addition we can calculate from the above results for the invariant spur of this tensor

$$
\begin{aligned}
G = g^{\mu\nu}G_{\mu\nu} = e^{-\mu}\Bigg[2\left\{\frac{\partial^2\mu}{\partial x^2} + \frac{\partial^2\mu}{\partial y^2} + \frac{\partial^2\mu}{\partial z^2}\right\} \\
+ \frac{1}{2}\left\{\left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial y}\right)^2 + \left(\frac{\partial\mu}{\partial z}\right)^2\right\}\Bigg] + 3\frac{\partial^2\mu}{\partial t^2} + 3\left(\frac{\partial\mu}{\partial t}\right)^2 \qquad (22)
\end{aligned}
$$

and introducing the variable $r = \sqrt{x^2 + y^2 + z^2}$, this can be rewritten as

$$
G = -e^{-\mu}\left[2\frac{\partial^2\mu}{\partial r^2} + \frac{4}{r}\frac{\partial\mu}{\partial r} + \frac{1}{2}\left(\frac{\partial\mu}{\partial r}\right)^2\right] + 3\frac{\partial^2\mu}{\partial t^2} + 3\left(\frac{\partial\mu}{\partial t}\right)^2. \qquad (23)
$$

§ 5. *The Dependence of the Line Element on the Radius.*—We have now obtained sufficient material so that we can return to the problem of determining the form of the line element. In accordance with the fundamental equation of general relativity connecting the metric with the distribution of matter and energy, the invariant G is connected with the proper density of matter ρ_0 by the equation

$$
8\pi\rho_0 = G - 4\Lambda, \qquad (24)
$$

where Λ is the cosmological constant. We have assumed from the start, however, that the average properties of the universe are independent of position, and on account of the large scale of our interests are neglecting the local differences due to the neighborhood of individual stars or stellar systems. Hence we must take ρ_0 as independent of r, and since Λ is in any case a constant we must have G also independent of r. Returning then to the expression for G given by equation (23), and remembering in accordance with equation (14) that $\partial\mu/\partial t$ is independent of r, we can write

$$e^{-\mu}\left[2\frac{\partial^2\mu}{\partial r^2}+\frac{4}{r}\frac{\partial\mu}{\partial r}+\frac{1}{2}\left(\frac{\partial\mu}{\partial r}\right)^2\right]+A=0, \tag{25}$$

where A is a quantity which is independent of r.

We have thus obtained a second-order differential equation to be solved for μ as a function of r. A first integral of this equation can be found in the following form,[9] as can be verified by resubstitution into equation (24),

$$\frac{\partial\mu}{\partial r}=-\frac{2}{r}\left(1\pm\sqrt{1-\frac{Ae^{\mu}r^2}{6}+\frac{C_1e^{-\mu/2}}{A^{1/2}r}}\right)$$

where C_1 is an arbitrary constant of integration. For our case, nevertheless, this constant must evidently have the value zero and the minus sign must be chosen on the right-hand side of the equation, since otherwise the quantity $\partial\mu/\partial r$ would become infinite at the origin. Making this choice, however, we find that $\partial\mu/\partial r$ approaches zero instead of infinity at the origin, in satisfactory agreement with our experimental knowledge that the special theory of relativity is approximately valid for a limited region in space and time.

Giving C_1 then the value zero, and using the minus sign, the equation reduces to

$$\frac{\partial\mu}{\partial r}=-\frac{2}{r}+\frac{2}{r}\sqrt{1-\frac{Ae^{\mu}r^2}{6}}, \tag{26}$$

and this can be integrated to give us

$$e^{\mu}=\frac{24C_2}{A(1+C_2r^2)^2}, \tag{27}$$

where C_2 is the arbitrary constant of integration. This final result can easily be verified by substitution back into equation (26) or better yet by substitution into the original equation (25) itself.

We have thus obtained the desired information as to the dependence of the line element on the radius r. And noting in accordance with equation (14) that e^{μ} can in any case be taken as the product of a function of r and t, we can now write

$$e^{\mu}=\frac{e^{g(t)}}{\left(1+\frac{r^2}{4R^2}\right)^2} \tag{28}$$

or for the line element as a whole

$$ds^2 = -\frac{e^{g(t)}}{\left(1 + \frac{r^2}{4R^2}\right)^2}(dx^2 + dy^2 + dz^2) + dt^2, \tag{29}$$

where we have substituted $1/4R^2$ for the constant C_2, in order to facilitate the comparison of this expression with that for the Einstein line element.

§ 6. *The Distribution of Matter and Radiation in the Universe.*—We may now investigate the distribution of matter and radiation in the universe which would correspond to this line element. Substituting the value for e^{μ} given by equation (28) into equations (21) for the Riemann-Christoffel tensor and raising suffixes for later convenience, we can now calculate as the *only* non-vanishing components of this tensor

$$G_1^1 = G_2^2 = G_3^3 = \frac{2}{R^2}e^{-g} + \frac{1}{2}\frac{d^2g}{dt^2} + \frac{3}{4}\left(\frac{dg}{dt}\right)^2$$

$$G_4^4 = \frac{3}{2}\frac{d^2g}{dt^2} + \frac{3}{4}\left(\frac{dg}{dt}\right)^2 \tag{30}$$

with

$$G = \frac{6}{R^2}e^{-g} + 3\frac{d^2g}{dt^2} + 3\left(\frac{dg}{dt}\right)^2.$$

These quantities, however, are related to the energy-momentum tensor T_μ^ν in accordance with the fundamental equations

$$-8\pi T_\mu^\nu = G_\mu^\nu - \frac{1}{2}Gg_\mu^\nu + \Lambda g_\mu^\nu. \tag{31}$$

Moreover, in applying this equation to cosmological considerations, our scale of interest is so large, that we shall regard our simplified model of the universe as though filled with a perfect fluid, and can then use for the energy-momentum tensor the well-known expression for such a fluid

$$T^{\mu\nu} = (\rho_{00} + p_0)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} - g^{\mu\nu}p_0, \tag{32}$$

where ρ_{00} is the proper macroscopic density of the fluid, p_0 its proper pressure, and the quantities dx_μ/ds are macroscopic "velocities." Furthermore, in order to construct a stable model of the universe we have already imposed, in § 2, the condition that particles which are stationary in our coördinates shall not be subject to acceleration, and hence can take all the "velocities" dx_μ/ds as zero, except for the case $\mu = 4$. Under these circumstances, using our line element, and lowering suffixes we easily find as the only non-vanishing components for the energy-momentum tensor

$$T_1^1 = T_2^2 = T_3^3 = -p_0 \qquad T_4^4 = \rho_{00}. \tag{33}$$

Combining equations (30), (31) and (33), we now readily obtain for the proper pressure p_0, proper macroscopic density ρ_{00}, and proper density of matter ρ_0 (see equation 24)

$$\begin{aligned} 8\pi p_0 &= -\frac{1}{R^2}e^{-g} - \frac{d^2g}{dt^2} - \frac{3}{4}\left(\frac{dg}{dt}\right)^2 + \Lambda \\ 8\pi\rho_{00} &= \frac{3}{R^2}e^{-g} + \frac{3}{4}\left(\frac{dg}{dt}\right)^2 - \Lambda \\ 8\pi\rho_0 &= \frac{6}{R^2}e^{-g} + 3\frac{d^2g}{dt^2} + 3\left(\frac{dg}{dt}\right)^2 - 4\Lambda. \end{aligned} \tag{34}$$

§ 7. *The Dependence of the Line Element on Time.*—We must now return to the still unsolved problem of the dependence of the line element on the time, i.e., the dependence of g on t. With an exact knowledge of the changes taking place in pressure and density, we could theoretically use equations (34) to determine the exact dependence of g on t. In the absence of exact knowledge, however, we can start by assuming some simple empirical formula, as expressing the dependence of g on t, and then use what meager information we do have to evaluate as many constants as possible in this empirical formula.

As a simple empirical formula for the dependence of g on t, the following naturally suggests itself

$$g = a + bt + ct^2 + \ldots \tag{35}$$

as giving the value of g in the neighborhood of the time $t = 0$.

From the form of the line element (29), it is evident that the first constant in this formula a merely depends on the choice of units for laying off the system of spatial coördinates, and hence would naturally be taken as zero in order that the line element should reduce to the special relativity form at $t = 0$ and in the neighborhood of the origin $r = 0$. For the second constant b, it will be better to use the symbol $2k$ in order to avoid fractions in the later calculations. As for the higher terms, we do not as yet have sufficient information for their evaluation and we shall have to omit them, introducing at this point the assumption that, in the neighborhood of the time $t = 0$, the quantity g can be given with sufficient approximation by a formula which is linear in t.

We can then write as our chosen empirical expression for the dependence of g on t,

$$g = 2kt \qquad \frac{dg}{dt} = 2k \qquad \frac{d^2g}{dt^2} = 0 \tag{36}$$

and for the line element as a whole

$$ds^2 = -\frac{e^{2kt}}{\left(1 + \frac{r^2}{4R^2}\right)^2}(dx^2 + dy^2 + dz^2) + dt^2. \qquad (37)$$

§ 8. *Relation of the Quantity k to the Rate of Transformation of Matter into Radiation.*—With the help of our explicit empirical expression for g, and its differential coefficients we can now obtain explicit expressions for the pressure and density in our model of the universe.

Substituting equations (36) into the expression for pressure given in (34) we obtain

$$8\pi p_0 = -\frac{1}{R^2}e^{-2kt} - 3k^2 + \Lambda. \qquad (38)$$

Owing to the low temperature of intergalactic space, however, it is evident that the pressure that should be ascribed to our simplified model of the universe must be nearly zero over that small time range in the neighborhood of $t = 0$ where we assume our formula valid. This can be obtained from equation (38) only if we have the approximate relation between constants

$$\Lambda = \frac{1}{R^2} + 3k^2 \qquad (39)$$

and we shall introduce this equation below, where it will be seen that its use is equivalent to neglecting the pressure in comparison with the density of the universe at time $t = 0$.

Turning now to the expression for the proper density of matter given by equations (34), and substituting equations (36) we obtain

$$8\pi\rho_0 = \frac{6}{R^2}e^{-2kt} + 12k^2 - 4\Lambda$$

and making use of (39) this can be written in the form

$$8\pi\rho_0 = \frac{6}{R^2}e^{-2kt} - \frac{4}{R^2}. \qquad (40)$$

With the help of this expression connecting density and time, we can now obtain an expression for the rate at which matter is changing over into radiation. For the proper volume dVo associated with the coordinate range $dx\ dy\ dz$ we can evidently write, in accordance with the line element (37)

$$dV_0 = \frac{e^{3kt}}{\left(1 + \frac{r^2}{4R^2}\right)^3}dx\,dy\,dz. \qquad (41)$$

Hence using equation (40) for the proper density of matter, we can write for the mass of matter M included within a given range of coördinates x, y, z

$$M = A \left\{ \frac{6}{R^2} e^{kt} - \frac{4}{R^2} e^{3kt} \right\} \tag{42}$$

where the quantity A, whose magnitude depends on the range of coördinates taken, is a constant independent of the time. This mass, however, which is included within a given coördinate range, can only change with the time through the transformation of matter into radiation, since our model of the universe has been so constructed that material particles are stationary in the coördinates used and thus do not cross any coördinate boundary. Hence differentiating equation (42) with respect to the time, and dividing by M itself, we easily obtain for the percentage rate of the transformation of matter into radiation at the time $t = 0$ the very simple expression

$$-\left[\frac{1}{M}\frac{dM}{dt}\right]_{t=0} = 3k. \tag{43}$$

Equation (43) gives us an intermediate interpretation for the quantity k in terms of the average rate at which matter is disappearing throughout the universe.[10]

§ 9. *Relation of the Quantity k to the Red Shift in the Light from Distant Objects.*—It is also possible to obtain an interpretation for the quantity k in terms of the red shift in the light from the nebulae.

In accordance with the general theory of relativity, we can secure an expression for the velocity of light by setting ds equal to zero in our formula for the line element. Doing so and restricting ourselves to distances where r is small compared with R, we obtain for the velocity of light

$$\frac{dl}{dt} = e^{-kt}. \tag{44}$$

The restriction to small distance r introduces considerable simplification, since it limits our considerations to a region in which space is approximately Euclidean so that the coördinate distance l can easily be correlated with the distance as measured by usual methods. Some restriction is in any case necessary, however, since our formula for the line element can certainly claim validity only for times in the neighborhood of $t = 0$, and hence could not be applied with confidence if the light required too long a time to reach the observer.

Consider now a light impulse which leaves a nebula at the time t_1 and reaches the observer at time t_2, after traveling the distance Δl as measured in our system of coördinates. By integrating equation (44) we then obtain

$$\Delta l = \int_{t_1}^{t_2} e^{-kt}dt = \frac{1}{k}\left(e^{-kt_1} - e^{-kt_2}\right). \tag{45}$$

Since the nebula will be stationary in our system of coördinates, we obtain on differentiating this expression with respect to the time of departure t_1

$$0 = \frac{1}{k}\left(- e^{-kt_1}\, k + e^{-kt_2} k \frac{dt_2}{dt_1}\right)$$

or solving for dt_2/dt_1, we can write

$$\frac{dt_2}{dt_1} = e^{k(t_2 - t_1)}. \tag{46}$$

We have thus obtained an expression connecting the short time interval dt_1 between the departure of two light impulses from the nebula with the time interval dt_2 between their reception. And considering dt_1 as the period of the light on emission and dt_2 its period on reception we can at once write the relation

$$\frac{\lambda + \delta\lambda}{\lambda} = e^{k(t_2 - t_1)}, \tag{47}$$

where $\delta\lambda$ is the shift in wave-length of the light from the nebula as compared with that from a similar terrestrial source.

Moreover, considering times t_1 and t_2 in the neighborhood of $t = 0$, which in any case is necessary in order to accord with our assumptions, this formula for the change in wave-length can be rewritten in a more familiar form, since by expanding (45) and (47) we obtain the approximate relations

$$\Delta l \approx t_2 - t_1 \quad \text{and} \quad \frac{\delta\lambda}{\lambda} \approx k(t_2 - t_1)$$

and by combining can write

$$\frac{\delta\lambda}{\lambda} = k\ \Delta l. \tag{48}$$

We have thus secured another interpretation of the quantity k, as the constant which occurs in the empirically verified linear formula connecting red shift with distance.

§ 10. *Comparison of the Magnitudes of k as Determined from the Red Shift and from the Rate of Transformation of Matter into Radiation.*—We must now compare the magnitude of k, as determined from observations on the red shift in the light from the extra-galactic nebulae, with the values that would correspond to the rates of transformation of matter into radiation occurring in the case of stars of different types.

Expressing the distance Δl to the nebulae in light years, it is found that equation (48) fits the experimental data of Hubble and Humason[4] on the change of red shift with distance with an approximate value of the constant

$$k = 5.1 \times 10^{-10} \text{ (years)}^{-1}. \tag{49}$$

On the other hand for the rate of transformation of matter into radiation in the stars we obtain a wide range of values depending on the type of star considered. The rates for a series of typical stars are given in the table below, together with the corresponding values of k as calculated from equation (43). The first two columns of the table are taken from Jeans, *Astronomy and Cosmogony*, page 125.

GENERATION OF ENERGY BY TYPICAL STARS

STAR	ERGS PER GRAM PER SECOND	$-\frac{1}{M}\frac{dM}{dt}$ PER YEAR	k YEARS^{-1}
H. D. 1337 A	15,000	5.2×10^{-10}	1.7×10^{-10}
B. D. 6° 1309 A	(11,000)	3.9×10^{-10}	1.3×10^{-10}
V Puppis A	1,100	3.9×10^{-11}	1.3×10^{-11}
Betelgeux	(300)	1.1×10^{-11}	3.7×10^{-12}
Capella A	48	1.7×10^{-12}	6.3×10^{-13}
Sirius A	29	1.0×10^{-12}	3.3×10^{-13}
Sun	1.90	6.6×10^{-14}	2.2×10^{-14}
α Centauri B	0.90	3.2×10^{-14}	1.1×10^{-14}
60 Kruger B	0.02	7.0×10^{-16}	2.3×10^{-16}

It will thus be seen that the value, $k = 5.1 \times 10^{-10}$ years^{-1}, determined from the red shift falls close to the range of values that would be determined from known rates for the change of matter into radiation. Nevertheless, the value from the red shift certainly seems high when account is taken of the preponderance of stars of the type of the sun or later, and we must return to this again when we try to estimate the validity of the suggested explanation of the red shift.

§ 11. *Conclusion.*—In conclusion, I wish to make several comments concerning different features of the line element that we have obtained. I also desire to make some statement as to what degree of probability we could feel justified in ascribing to this tentative explanation of the red shift.

a. Relation to the Einstein Line Element.—A close relationship will be noted between the line element which we have obtained in this work—

$$ds^2 = -\frac{e^{2kt}}{\left(1 + \frac{r^2}{4R^2}\right)^2}(dx^2 + dy^2 + dz^2) + dt^2 \tag{50}$$

and that of Einstein, since by an easy transformation of coördinates we can change his line element from the usual form given by equation (1), § 1, to the form[11]

$$ds^2 = -\frac{1}{\left(1 + \frac{r^2}{4R^2}\right)^2}(dx^2 + dy^2 + dz^2) + dt^2, \tag{51}$$

which bears a close relation to our line element. And on the other hand, by the reverse transformation we could, if desired, re-express our line element in a form more similar to the usual Einstein one.

The quantity k, which occurs in the term that distinguishes our line element from the Einstein one, is proportional to the average rate at which matter is changing into radiation, and if we should take this to be zero, our line element would reduce to that of Einstein. Hence with considerable correctness we can describe the model of the universe which corresponds to our line element as a *non-static* Einstein universe—necessarily non-static since it takes into account a general process of transformation of matter into radiation neglected in the derivation of the older formula.

b. Relation to the de Sitter Line Element.—Some measure of relationship can also be noted between our line element and that of de Sitter, since Robertson[5] has shown by his transformation of coördinates that the de Sitter line element can be written in the form

$$ds^2 = -e^{2kt}(dx^2 + dy^2 + dz^2) + dt^2 \tag{52}$$

which appears to have considerable relation to ours.

For several reasons, however, I think that too much stress should not be laid on the similarity between expressions (50) and (52).

In the first place, our line element has been derived on the definite hypothesis of a universe containing matter which is gradually being transformed into radiation, while the de Sitter line element can only be derived by definitely neglecting the effect of any matter which actually may be present in the universe.

In the second place, the term involving the time t enters our equations because of our hypothesis of the transformation of matter into radiation, and the sign and magnitude which we obtain for the constant k from a consideration of known rates of transformation accounts correctly for the sign and approximately for the order of magnitude of the observed red shift in the light from distant objects. On the other hand, in the case of the de Sitter line element in the form given by equation (52), we should have no *a priori* reason which would guide us in choosing either the sign or magnitude of the constant k. Either sign would lead to agreement with the gravitational equations for an empty universe, and in the absence of knowledge as to the magnitude of the red shift, any reasonably small value of k would seem equally possible.

Finally, it should be noted that the term involving the time enters the equation (52) in the exact way demanded by the de Sitter hypothesis of a substantially empty universe, while it is presumable that the term involving the time in our equation (50) gives only a first approximation of a more correct form that would correspond to a more exact fitting of our equation to the changing rate at which matter is beng transformed into radiation. Indeed, it is almost certain that the form that we have taken is only a first approximation, since by carrying out a double differentiation of equation (42) it can be shown that our form leads, at time $t = 0$, to a rate of transformation of matter into radiation which is increasing with the time, and this result does not appear probable.

c. The Recession of the Nebulae.—The system of coördinates used in this article was chosen in such a way that particles having zero coördinate velocity would continue to have no motion in these coördinates. From the form of our line element (50) it is evident, however, that the distance from the origin to a nebula permanently located at $r = \sqrt{x^2 + y^2 + z^2}$ *when measured with rigid scales* would have the value

$$\int ds_0 = e^{kt} \int_0^r \frac{dr}{1 + \frac{r^2}{4R^2}} \tag{53}$$

and hence be increasing with the time. Thus, in this sense and in others that will be evident, we can speak of a motion of recession of the nebulae, and can think of the red shift as correlated with this motion. Such a recessional motion, however, is not to be regarded as an arbitrary device for accounting for the red shift, but as a necessary accompaniment of the change in gravitational field which is produced by the transformation of matter into radiation.

d. The Validity of the Explanation.—Finally, a few remarks which attempt to estimate the validity of this attempted explanation of the red shift will not be out of place.

First of all, it should be clearly recognized that the explanation is definitely founded on the hypothesis of a continuous and uncompensated process taking place throughout the universe; and the explanation would certainly fall to the ground, if in reality the universe should prove to be in a steady state, the mass of the stars being continuously replenished by some cyclical process whose steps are now unknown. There is, indeed, little evidence in favor of such a cycle and astrophysicists are not inclined to this view; nevertheless I myself and many others would be glad to give it credence if we could.

In the second place, account must be taken of the simplifying assumptions that have been made in the course of the argument. These have been of two kinds—the mathematical approximations that have been introduced in taking only two terms of our empirical formula (35) for the dependence of g on t, and the high degree of over-simplification introduced into the model of the universe that we have used for our calculations. I do not think that the effect of the mathematical approximations is necessarily negligible since the observational determination of the connection between red shift and distance has perhaps not yet been pushed far enough to discover deviations from a linear relation. On the other hand the over-simplification of the model may have considerable effect on the results. The model that we have used for our calculations is essentially that of a universe filled with a homogeneous distribution of matter and radiation with a continuous increase in the radiation at the expense of the matter. In the actual universe, however, at least a large portion of the matter is really concentrated in stellar systems, and in addition to the transformation of matter into radiation which seems to be taking place inside these systems, there appears over a long time range to be an excess in the outflow of radiation accompanied by a cooling and readjustment in the structure of the systems. At a later time I shall perhaps be able to carry out the computations for a less unduly simplified model and further investigate the effect of the mathematical approximations.

In the third place, we must not forget the fact that the value of k determined from the red shift, although falling close to the range that corresponds to the actual rates of transformation of matter into radiation found in the stars of different types, is nevertheless considerably higher than would be expected in view of the large preponderance of stars of the type of the sun or later. This is certainly a serious difficulty for the theory, although not necessarily a fatal objection until we have made calculations using less approximation and simplification and have obtained more certain information concerning the rates of dissipation of mass in the nuclei of the nebulae, in dust which may be present throughout the whole of intergalactic space, and possibly also in those as yet unknown processes which are the cause of the cosmic rays.

In conclusion, we can at least say that in order to get a satisfactory line element for the universe as a whole we must certainly take into account the possibility of general evolutionary processes taking place throughout the universe. And the non-static line element which we thus obtain may provide the explanation for the whole or a part of the red shift in the light from the extra-galactic nebulae.

[1] Einstein, *Berl. Ber.*, 1917, p. 142.

[2] de Sitter, *Monthly Notices, R. A. S.*, **76**, **77**, **78**, 1916–1917.

[3] Einstein's original derivation contained an equivalent to the assumption that the pressure of this fluid was zero. This is not necessary, however, see Tolman, *Proc. Nat. Acad. Sci.*, **14**, 350 (1928), and reference 6 below.

[4] See Hubble, *Proc. Nat. Acad. Sci.*, **15**, 168 (1929); and Humason, *Ibid*, **15**, 167 (1929).

[5] Robertson, *Phil. Mag.*, **5**, 835 (1928).

[6] Tolman, *Proc. Nat. Acad. Sci.*, **15**, 297 (1929).

[7] Robertson, *Ibid.*, **15**, 822 (1929).

[8] Zwicky, *Ibid.*, **15**, 773 (1929).

[9] For a solution giving the first integral of this equation I am indebted to my colleague, Professor Paul S. Epstein, whom I wish to thank also in this place.

[10] Two remarks can be made which may make it easier to understand the treatment given in this Section. (1) The symbol ρ_0 signifies the proper density as measured by an observer actually moving with the material at the point in question. It hence applies to matter alone, as the rest mass of a light quantum is zero; and since matter is stationary in our coördinates it gives the desired proper density of matter in those coördinates. (2) Noting that the macroscopic density of radiation is three times its pressure, it can be shown from the equations of this section that the rate of decrease of material energy in a given coördinate range is compensated for by the rate of increase of radiation in that range.

[11] Einstein, *Berl. Ber.*, 1918, p. 448.