PRINCIPLES

OF THE

ALGEBRA OF LOGIC

WITH EXAMPLES

BY

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PRINCIPLES

OF THE

ALGEBRA OF LOGIC.

'A generation will arise in which the leaders of education will know the value of logic, the value of mathematics, the value of logic in mathematics, and the value of mathematics in logic.'—DE MORGAN, *Syllabus*, p. 44.

'Shall we then err in regarding that as the true science of Logic, which, laying down certain elementary laws, confirmed by the very testimony of the mind, permits us thence to deduce, by uniform processes, the entire chain of its secondary consequences, and furnishes for its practical application methods of perfect generality. Let it be considered whether in any science, viewed either as a system of truth or as the foundation of a practical art, there can properly be any other test of the completeness and fundamental characters of its laws, than the completeness of its system of derived truths, and the generality of the methods which it serves to establish.'—BOOLE, *Laws of Thought*, p. 5.

'It is curious to compare the properties of these quaternion symbols with those of the Elective Symbols of Logic, as given in Boole's wonderful treatise on the *Laws of Thought;* and to think that the same grand science of mathematical analysis, by processes remarkably similar to each other, reveals to us truths in the science of *position* far beyond the powers of the geometer, and truths of deductive reasoning to which unaided thought could never have led the logician.'—PROFESSOR TAIT, *Quaternions*, p. 50.

PREFACE.

THESE 'Principles' were originally contributed to the Royal Society of Edinburgh in a Memoir received by the Secretary 9th October 1878, and in a supplementary paper received 5th November. I had the honour of reading an Abstract before the Society at the meetings of 16th December and 20th January. In the interval between the 5th November and the present time I have improved several of the demonstrations, introduced illustrative matter, and prepared the collection of examples. The work, in its present state, forms an elementary treatise on the science of Formal Reasoning.

I consider it proper to state that the theory of the operation of the mind in reasoning about Quality, which is advanced in this work, occurred to me five years ago; and that I have directed towards its development the whole of my subsequent study of the Mathematical, Physical, and Natural Sciences, which are embraced in the curriculum for the degree of Doctor of Science (Mathematics) at the University of Edinburgh.

ALEXANDER MACFARLANE.

EDINBURGH, *23d January* 1879.

TO THE

REV. PHILIP KELLAND, M.A., F.R.S.

PROFESSOR OF MATHEMATICS IN EDINBURGH UNIVERSITY

President of the Royal Society of Edinburgh

THIS WORK IS DEDICATED

AS A MARK OF RESPECT

BY

A FORMER PUPIL.

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ALGEBRA OF LOGIC.

I. THE SCIENCE OF FORMAL LOGIC AN ALGEBRA.

1. THOUGH it is evident *a priori* to one who reflects on the matter, that the theory of Necessity and the theory of Probability are the complementary parts of one whole, it is nevertheless true that the foundations of the general science, of which they form the parts, were not laid until quite recent times. The merit of conceiving and undertaking this important unification is due in some measure to De Morgan, but principally to Boole.

2. That the science of inference is capable of being treated analytically, may be inferred from the fact that the ordinary rules about Conversion and Syllogism are established by a comparison of circles, taken to represent the terms of the propositions considered. In one of the best modern manuals of Logic, it is stated that the testing whether a given combination of premises leads to a valid inference, and the proof of the validity or invalidity, must depend on the comparison of the spheres, within which, according to the premises, the notions under consideration find application; and that these spheres are made apparent to the senses by geometrical figures (especially by circles) whose reciprocal relations agree with the relations of the spheres of the notions to each other in all relations essential for demonstration. (*Ueberweg's Logic*, translated by Professor Lindsay, p. 379.) The introduction of these diagrams is commonly attributed to Euler.

3. Corresponding to this graphical method, which consists in the use of diagrams, there is an analytical method, which consists in the use of symbols. The relative advantages and disadvantages of the two, when applied to Quality, are precisely the same as when applied to Quantity. The diagram exhibits an individual case of the given data with all the clearness of the concrete; on the other hand, the analytical expression separates the essential relations from the accidental, with which they must be mixed up in any individual example.

4. The reason why the operations of Boole's calculus appear mysterious and its employment difficult, is, that the calculus is not founded upon a sufficient theory of the operation of the mind in reasoning about Quality. That it is not all that a Logical organon ought to be, is evident from what Venn says in *Mind*, vol. i. p. 484:—'The distinctive characteristic of Boole's system is the boldness, not to say audacity, with which he carries on his processes through stages which have no logical or other signification whatever, that is, which admit of no possible interpretation —provided only they terminate in an interpretable result.' Boole himself claims nothing higher for his calculus. He would, however, have objected to the statement which Professor Jevons makes (*Principles of Science*, p. 71), that 'Boole imported the conditions of number into the science of Logic, and produced a system which, though wonderful in its results, was not a system of logic at all.'

5. It is the object of this little work to investigate the foundations of the analytical method of reasoning about Quality, with special reference to the principles laid down by Boole as the basis of his calculus, and to the observations which have been published by various philosophers concerning these principles. I bring forward a new theory of the operation of the mind in reasoning about Quality, which enables me to correct Boole's principles, and place them on a clear rational basis. I endeavour to show that the

analytical method of reasoning about Quality is an Algebra, which coincides with the Algebra of Quantity when the symbols are integral, but is a generalised form of the latter when the symbols are fractional. The rest of the work is taken up with the investigation of problems by means of this algebraic organon,—especially such problems as are suggested by the ordinary Logic.

6. Logic, as the Algebra of Quality, is a *formal science.* It investigates the general properties of the symbol of Quality, and by means of these properties deduces equations which are true generally, or combines such equations with data of given forms. It is not its province to consider how a particular form of datum can in any case be asserted to be true—that subject of investigation being left to the Transcendental Logic; it is sufficient that examples of such a form occur, or may occur, in the practical or theoretical activities of mankind.

7. The properties of the symbol of Quality are not *laws of thought* in the common acceptation of that term. For the properties of the symbol of Quantity, on which the ordinary algebra is founded, are held not to be laws of thought, but to refer to the actual constitution of things; and there is no difference in the two methods, when developed, which indicates the existence of such a distinction. If the basis of the science of Quality is subjective, it is so only in the same sense in which the basis of the science of Quantity is subjective. There is ground for believing that the true reason why the former science has remained so stationary is, that there has been too much introspection into the individual mind in the hope of finding laws of thought there, and too little contemplation of the form and nature of the truths of Science. The logician assumes that all men reason equally well about Quality, fallacies being possible only by a momentary lapse of attention; but the mathematician never assumes that all men reason equally well about Quantity.

8. Boole entitled his great work on reasoning 'An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities,' and in several places he says that the Laws in question are subjective in a sense in which the Laws of Quantity are not. He considers

$$x^2=x$$

in particular to be a subjective law; but I have endeavoured to show (Art. 118) that it is a special condition, which the symbol of this Algebra must satisfy in order to be of a particular kind.

9. No one, I suppose, contends that the properties of the Chemical Symbol, or of the Quaternionic Symbols, are laws of thought. Since the corresponding properties of the different symbols differ greatly among one another, it is surely better in every case to consider the actual constitution of things as suggesting rules for thought to the mind, rather than the mind imposing laws of thought upon itself.

10. Logic, as the Algebra of Quality, is a true *organon.* It can determine whether a conclusion of a required form can be deduced from data of given forms; and if so, what that conclusion is. It can manipulate complex data, as is shown in the examples appended. Bacon's judgment—'*Syllogismus ad principia scientiarum non adhibetur, ad media axiomata frustra adhibetur, quum sit subtilitati naturae longe impar*'—however true of the scholastic exposition of the syllogism, does not apply to the Algebra of Quality; for the latter can be made to discover principles, and to imitate to some extent the subtlety of Nature. It may be said (to adapt a remark of De Moivre) that innumerable questions in the theory of necessary and probable reasoning can be solved without any manner of trouble to the imagination, by the mere force of the notation supplied by this Algebra.

The Algebra of Quantity is acknowledged to be the weapon for the philosopher who attacks the Experimental

Sciences; the Algebra of Quality is the weapon for the philosopher who attacks the Sciences of Observation.

11. Thus viewed, Formal Logic is not the *short and dry science* which even Kant held it to be. Any one who has studied Boole's Calculus, may well imagine that the theory of reasoning was not completed by Aristotle; and that so far from any 'System' of Logic having ever been written, there is still need to consider the foundations.

II. UNIVERSE AND CHARACTER.

12. Boole in his analysis of language draws no *distinction between Substantive and Adjective;* he considers their function in reasoning to be the same. He says (*Laws of Thought*, p. 27), 'The substantive proper and the adjective may indeed be regarded as differing only in this respect, that the former expresses the substantive existence of the individual thing or things to which it refers; the latter implies that existence. If we attach to the adjective the universally understood subject "being" or "thing," it becomes virtually a substantive, and may for all the essential purposes of reasoning be replaced by the substantive.' Accordingly, he uses the symbol x to denote 'men' or 'good things' or 'white things' or 'horned things,' as the case may be. For instance: he says, if x alone stands for 'white things' and y for 'sheep,' let xy stand for 'white sheep;' and in like manner if z stands for 'horned things' and x and y retain their previous interpretations, let zxy represent 'horned white sheep.'

13. Again; when investigating the operations of the mind, which are implied in the use of language as an instrument of reasoning, he finds no difference in the operation expressed by a substantive from that expressed by an adjective. He says that there is a universe of discourse; but this universe is not one described by a substantive. 'In every discourse,' he says, 'whether of the mind conversing with its own thoughts, or of the individual in his intercourse with others, there is an assumed or expressed limit within which the subjects of its operations are confined. The most unfettered discourse is that in which the words we use are understood in the widest possible application, and for them the limits of discourse are co-extensive with those of the universe itself. But more usually we confine ourselves to a less spacious field. Sometimes, in discoursing of men, we imply (without expressing the limitation) that it is of men only under certain circumstances and conditions that we speak, as of civilised men, or of men in the vigour of life, or of men under some other condition or relation. Now, whatever may be the extent of the field within which all the objects of our discourse are found, that field may properly be termed the universe of discourse.'—*Laws of Thought*, p. 42.

14. From the passage just quoted, as well as from many others, it appears that what Boole means by the universe of discourse is not the objects denoted by a Universal Substantive, but a definite part of the whole realm of things—a limited portion of the physical universe, with all the entities which are or can be imagined to be in it, whether mental or physical, ponderable or imponderable, atomic or complex.

15. The substantive 'men' expresses an operation of election from the universe of all the beings to which the term 'men' is applicable; the adjective 'good' in combination, as 'good men,' directs us still further to elect mentally from the class of men all those who possess the further quality of good; and if another adjective were prefixed to the combination, it would direct a similar operation upon 'good men.' In short, he supposes that the mind always proceeds along *the predicamental line;* whereas that is only one mode of its procedure.

16. In consequence of this analysis, the subjects of thought in Logic and in Arithmetic are said to be perfectly distinct; and it is not of any importance to compare the symbols of logic with the symbols of quantity generally. Attention is directed so exclusively to an Algebra in which the symbols x, y, z, etc., admit indifferently of the values 0 and 1, and of these alone, that some logicians have supposed that the symbols can have no other value.

17. Another consequence of this analysis is, that Boole is obliged to make a new and independent investigation of 'Secondary' Propositions. In the case of Secondary Propositions, the proper interpretation of the symbol 1 is held to be 'eternity,' or a part of eternity. The question is suggested, whether in the case of Primary Propositions 1 does not really represent space. He thinks not; because the sign of identity = connecting the members of the corresponding equation, implies that the things which they represent are identical, not simply that they are found in the same portion of space. The reason why the symbol 1 in Secondary Propositions represents not the universe of events, but the eternity in whose successive moments and periods they are evolved, is, that the same sign of identity connecting the logical members of the corresponding equations implies, not that the events which those members represent are identical, but that the times of their occurrence are the same.—*Laws of Thought*, p. 176.

18. The principles of the Calculus of Identity become much clearer, and their application greatly facilitated, by taking into consideration the *difference of the functions of the Substantive, and of the Adjective used in an attributive sense.* The objects expressed by the common noun, or rather universal term of a proposition, constitute the universe of the proposition—the actual whole considered by the mind in forming the judgment. The attributive adjectives, whether one or more, which appear in the proposition, refer to that subject, and not to things in general. In thinking of 'sheep that are white and horned,' I do not consider 'white things' or 'horned things.' It is even questionable whether the mind can consider some adjectives as denoting classes of things. Can we consider 'small things' or 'wise things' or 'primary things'? Boole remarks, with reference to this very attribute 'wise,' that, before denoting it by a symbol, we must consider whether it is to be used in its absolute sense or only relatively. But 'small' has no absolute sense. 'Nothing by itself,' Aristotle lays down in *The Categories*, 'is described as great or small. A mountain, for instance, may be said to be "little," and a millet seed "large," from the fact of the one being greater, and the other less, in respect of things of the same nature.' It is this reference to things of the same nature that I wish to draw attention to.

19. As quantities have a certain abstract meaning in themselves, but no definite meaning unless with reference to a given unit; so qualities have a certain abstract meaning in themselves, but no definite meaning unless when referred to a given universe.

20. Let the particular kind or collection of objects considered in any judgment or series of judgments be denoted by a capital letter U—a symbol used in an analogous sense by De Morgan. When the same kind or collection of objects is the subject of all the judgments considered, U need not be expressed, but is to be understood. Let an attribute, character, or quality, be denoted by a small letter, as x.

21. This modification of Boole's notation brings out the contrast between the Substantive and the Adjective; which is indeed only one form of the general contrast between that which is the subject of the operations of thought and the operations themselves. Another common form of the contrast is that, made prominent in the *Theory of Prob-*

ability, between the 'event' and 'the way in which it can happen.'

If U denote the 'Members of the House of Commons at the present time,' x may denote 'Liberal' or 'Conservative.' Or if U denote 'triangles,' x may denote 'isosceles' or 'equilateral.'

22. Boole, in his investigation, generally considers a combination of a Substantive with an Adjective prefixed. Some languages, however, show, by a difference of position—before or after the Substantive—that the Adjective may be used in two senses, viz., as forming part of the Substantive merely, or as equivalent to a relative phrase. In the English language, where the adjective is commonly placed before the substantive, the distinction referred to is brought out by emphasis. The prefixed adjective, when emphasised, is equivalent to a relative phrase; when not emphasised, it is part of the subject of thought.

23. For example: I find on looking up a Polyglot Bible that the proposition of Proverbs xv. 20 is expressed as follows:—

A wise son maketh a glad father.
Υἱὸς σοφὸς εὐφραίνει πατέρα.
Filius sapiens laetificat patrem.
Ein weiser Sohn erfreuet den Vater.
L'enfant sage réjouit son père.
Il figliuol savio rallegra il padre.
El hijo sabio alegra al padre.

Here the subject of discourse is 'Sons'; and it is to be observed that while the two Teutonic languages place the conditioning attribute before the subject, the others put it after.

24. Boole holds that Primary Propositions refer to things, and Secondary to facts; and that the idea of *time* is involved in the Secondary. Now there are propositions relating to facts which do not involve time, or a collection of portions of time, as the underlying subject; for example,

those which refer to *place* or a collection of places. We have not only the relative 'when,' but the relatives 'where' and 'who.' Hence if fact-propositions, which relate to the identity of portions of time, required a special investigation, those which relate to the identity of portions of space would also require a special investigation. If, however, we draw a contrast between the subject and its characters, one investigation suffices for all the different kinds of subject. Instead of two 1's, of which the one means the actually existent universe, and the other eternity, there is an infinite number of *U*'s, any one of which may be the subject of discursive thought.

25. This view of an essential difference in the functions of the Common Noun and Adjective is supported by *the results of philological research.* According to Max Müller (*Lectures on the Science of Language*, vol. i. p. 291), the component elements of language, which remain at the end of a complete grammatical analysis, are of two kinds, namely, roots *predicative* and roots *demonstrative.* In such a language as the Chinese, where the predicative root may by itself be used as a noun or a verb or an adjective, the noun is still distinguished from the verb by its collocation in the sentence. In the Aryan languages no predicative root can by itself form a word; in order to have a substantive it is necessary to add a demonstrative root, this forming the general subject of which the meaning contained in the root is to be predicated. If Boole's view of the operation of the mind were correct, we should have only predicative roots.

26. Aristotle, in his discussion of the Categories, draws a strong contrast between Substance (*οὐσία*) and Quality (*ποιόν*); and between Primary and Secondary Substances. By Substance is meant a particular thing (*τόδε τι*); by Quality that which is in a subject. In the case of the Primary Substance, the thing signified is individual and one in number; in the case of the Secondary, the thing signified

involves a quality, but so as to denote a particular kind of substance. It is the characteristic of Substance that being one and the same in number it can receive contraries; while it is the characteristic of Quality to be that with respect to which things are said to be like or unlike.

27. The Primitive judgment is so called because it does not refer to an exact subject, but to the whole external universe as one substance having all the physical changes which occur for accidents. For example, the judgment

It rains

refers to the states of a portion of the Physical Universe, and of these equates the 'present' with 'raining.'

28. The ordinary Eulerian Diagrams do not represent the whole of thought, but leave it indefinite; unless we suppose it to be represented by the finite sheet, on which the attributes are represented by circles.

29. *U is in general made up of a type and certain finite limitations.* The type corresponds to the predicative part of the Noun, of which Professor Max Müller speaks, and the limitations to the demonstrative part. The most common limitations are those of Space and Time; which, in this aspect, may be looked upon as logical variables. We may suppose the Time to be constant, and consider all the *U*'s throughout a given region; or we may suppose the individual to be constant, and consider its successive states within a given portion of time. The zoologist, when he compares the members of a genus, takes them in the adult state; when he considers the life-history of a particular form, he follows an individual through its cycle of states.

30. It is with the notion of the type that questions about Essence and Abstract Ideas are more properly concerned. Berkeley draws a distinction between two kinds of abstract ideas. 'As the mind frames to itself abstract ideas of qualities or modes, so does it by the same prescission or mental separation attain abstract ideas of the more compounded beings which include several co-existent qualities.'

—Professor Fraser's *Selections from Berkeley*, p. 16. He says also that there are two kinds of abstraction to correspond.

The distinction considered is correlative to that between Universe and Character.

31. *Arithmetical value of U.*—Since *U* signifies a definite collection of individuals of a given type, its arithmetical value must be an integer. The integer is in general *plural*, but may be *singular* or *infinite*. It is infinite when the individual parts are not discrete but continuous. Grammar recognises two of these cases.

It is interesting to consider how the subject of thought has naturally an integral value, while the operation of thought has naturally a fractional value,—how the relation between the symbols is mirrored in the relation between their kinds of quantity.

32. *U* also may be either *real* or *imaginary*. For instance, the judgment

τραγέλαφος ἐστι λευκός

The goat-stag is white

refers to an imaginary universe of goat-stags. When *U* is imaginary, it appears proper to indicate that fact by saying that the arithmetical value of *U* is 0; which is therefore, in this aspect, an integer.

33. The Universe holds the same position in the Algebra of Quality that the Unit does in the Algebra of Quantity. It may be said to be a *generalised unit*.

34. When *U* is used to denote the subject of thought, and x, y, etc., to denote operations on it, the symbols x, y, etc., have *a definite arithmetical value;* and as their meaning is supposed to be fixed throughout a discourse, their arithmetical value must also be supposed fixed. If x denotes a single positive attribute, its value is a fraction lying between 0 and 1; but if it is negative, its value lies between 0 and -1. Suppose that we have a complex character as xy; being compounded of two characters x and y, which are in their statement independent of one

another. It is then necessary to suppose that the arithmetical values of x and y are preserved independently of the combination; for these symbols depend on U only. But if x refers to U, and y not to U but to Ux, then the meaning of y is not independent of x; and y may have several arithmetical values according to the several orders it has in combination.

35. Thus 1 denotes 'all' or 'the whole'; while 0 denotes 'none.' 1 and 0 are to be considered as operating symbols of the same kind as x. *Some* is an indefinite operating symbol; but it generally carries the additional meaning of having an arithmetical value which is greater than nought.

36. It is very frequently necessary to express the arithmetical value of x. A convenient notation is $\bar{x}$. Boole uses for this purpose the circumlocution 'Prob. x'.

37. The mind, when reasoning on matters such as are discussed in the Theory of Probability, considers a particular class or kind of things; as has been well shown by Venn in his *Logic of Chance.* The p's and q's are in their first signification selective symbols, with arithmetical values lying between 0 and 1. A dependent event involves another event as a presupposition; and its arithmetical value depends on that connection—a circumstance which also shows us that attributes which are independent in their statement must be conceived as operating upon the universe directly.

38. The Algebra of Quality is the more general method. It discusses the relations of the characters of a Universe, whether that universe comprise one, several, or an infinite number of parts, and whether the characters change or are independent of time; whereas the Theory of Probability as commonly stated (see Venn's *Logic of Chance*, p. 5), supposes the universe to comprise a very large or infinite number of individuals, and the proper arithmetical value of p to be a certain limiting ratio, to which the actual value of p is continually approaching the greater the number of individuals in the universe.

39. There is an important *distinction among adjectives according as they involve being or having.* The contrary of the former is formed by prefixing *not*, that of the latter by prefixing *without.* Let us consider a collection of balls. The attributes which, in the Theory of Probability, such balls are commonly supposed to have are such as 'white,' 'red,' 'black.' These are attributes of *being*, since they apply to the whole of the ball, or at least to the whole of its surface. But we may consider a ball, which has stripes of different colours upon its surface; such attributes as 'having a white stripe' or 'having a red stripe' are adjectives of *having.* The adjective 'with a white stripe' applies to the ball as a whole, but 'white' applies only to the stripe. The former denote *qualities* proper, the latter what are called *characters* in the Natural Sciences. As the term character is the more general of the two, I prefer to use it instead of quality.

40. The ordinary doctrine of Subject and Predicate is often departed from in Manuals of Logic without the writer, apparently, being aware of it. For instance, Ueberweg, by contraposition, converts

Every regular figure may be inscribed in a circle,

into—

Every figure which cannot be inscribed in a circle is not regular,

thus leaving 'figure' the subject of thought unchanged, and manipulating only the characters 'regular' and 'inscribable in a circle.'

41. De Morgan brought the idea of the Universe into prominence. In his *Syllabus* (section 122) he defines it as meaning the whole extent of matter of thought under consideration. I do not consider this definition as sufficiently exact—as indicating precisely that which in the Algebra of Quality corresponds to the Unit of the Algebra of Quantity. With him the term does not denote a definite subject of operations, but rather limits within which elective, not selective, operations may be made. Hence his symbols are all integral. Instead of separating between U and x, as in Ux, he denotes the result by X.

III. THE SIGN =

42. By $$U\{x=y\}$$
is meant, that those U's which have the character x are identical with those which have the character y; or, in another aspect, that within the universe U the character x is equivalent to the character y.

It follows that *the equation is true arithmetically.* If
$$\dot{x}=y,$$
then $$\bar{x}=\bar{y};$$
the equivalence of two characters involves the equality of their arithmetical values.

43. Though the U's, which have the respective characters, are asserted to be identical with one another, the characters may not be identical. Their relation appears to be best expressed by the term *Equivalence.* If the proposition is real, the characters derive their equivalence from nature; but if it is verbal, they derive their equivalence from convention. Two characters may be equivalent *in themselves*, or *by definition*, or *by reality*. For example: let U denote triangles; x, equilateral; y, equiangular; then
$$U\{x=y\}$$
does not assert the identity, that is, the undistinguishable sameness of the characters 'equilateral' and 'equiangular,' but only their logical equivalence.

44. We may also have an equation of the form—
$$U\{x=\bar{x}\}$$
where $\bar{x}$ denotes the arithmetical value of x, and is supposed known. This equation also expresses the logical

equivalence of the members; but the second member is indefinite as regards identity. For example: let U denote 'an infinite series of throws with a die,' and x denote 'ace,' then $$U\{x = \tfrac{1}{6}\}.$$

This is an example of a form of equation which bulks largely in the Theory of Probability.

45. Since x and y, when regarded as operating on a definite universe, acquire an unambiguous and exact arithmetical value, no valid objection can be raised against expressing a judgment of the kind considered as *an equation.* It has, indeed, been rightly maintained by the opponents of the doctrine of the Quantification of the Predicate, that it is by their attributes that we think of things; but I hope to show that the supposed conclusion does not follow—that, therefore, the proposition cannot be expressed by means of an equation. The theory of reasoning about quality, advanced in these Principles is, *that the basis of the judgment alone is denotative; and that the members of the judgment, viz., the antecedent and the consequent, are both attributive.* The contrast comes out in the fact that the arithmetical value of the subject is integral, while that of the antecedent and of the consequent is generally fractional.

46. It appears to be of the essence of an Equation, that *its members be equal arithmetically.* This holds of the Chemical Equation. For example, in

$$C_2H_6O \quad = \quad H_2O + C_2H_4$$

Alcohol becomes Water and Olefiant Gas.

the arithmetical value of the right-hand member is equal to the arithmetical value of the left-hand member. Also in the Quaternion Equation—

$$\alpha + \beta = \gamma,$$

the arithmetical value of $\alpha + \beta$ is equal to that of γ.

IV. THE SIGNS + AND –

47. By $$U\{x+y\}$$
is meant

U's which are x together with U's which are y;

that is, the result of the addition of the selective operation x upon U to the selective operation y upon U. Read with respect to a single U, the expression means—

A U which is either x or y.

48. Suppose that we have a collection of cubes, and that some of them have a dot on their upper face, and some have a dash. Let the operation x denote the separation of those having a dot; and the operation y of those having a dash. Suppose x first performed; the collection is broken up into two parts, I. and II., viz., those which have a dot, and those which are without a dot. The subsequent operation y consists of two parts, viz., a selection of those having a dash, first among those having a dot, and secondly, among those without a dot; and therefore gives as a result four parts, 1, 2, 3, 4, of the Figure. $x+y$ denotes the addition of those having a dot to those having a dash, whether the first division, which consists of those having a dot and having a dash, vanish or does not vanish.

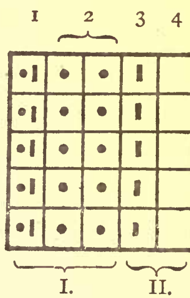


Fig. 1.

49. Since the sign – is by definition the sign of the operation which is the reverse of that denoted by +; and since the sign + has its ordinary algebraic meaning, the only difference consisting in the fact that the symbols joined involve identity; the sign – must have its ordinary algebraic meaning, but modified by the above-mentioned circumstance.

$$U\{x-y\}$$

means

U's which are x minus U's which are y.

x and y destroy one another, so far as they coincide; and the result in general consists of a positive and a negative part.

50. Consider the collection of cubes represented by fig. 1. $x-y$ denotes that those 'having a dot' are to be separated positively, and those 'having a dash' negatively. Those cubes which are both x and y (division 1) will be manipulated, but left out finally; the result of the operation being that division 2 of the figure is taken positively, and division 3 negatively.

51. If the operations x and y are formally independent, as in the above example, the arithmetical value of $x+y$ cannot be affected by any real relation existing between x and y. It is simply the sum of the arithmetical values of x and y. The operation y (fig. 1), though subsequent to x, applies to the whole of the universe; hence

$$\overline{x+y}=\bar{x}+\bar{y}.$$

$$=\tfrac{3}{5}+\tfrac{2}{5} \text{ in the case represented.}$$

The above is not the view commonly taken. De Morgan says (*Syllabus*, section 131), 'The more classes aggregated, so long as each class has something not contained in any of the others, the greater the extension of the aggregate term.'

52. If it were necessary for x and y in $x+y$ to be mutually exclusive, a restriction would be placed upon + which would prove fatal to the development of an Algebra of Quality. But the imposing of the restriction would be a mistake, for the basis of our operations is not a Unit but a Universe. $x+y$ does not necessarily denote a single character; but what it in general denotes is equally intelligible,—a summation of two characters.

Similarly, it is not necessary for x in $x-y$ to be inclusive of y, or for y to be inclusive of x. In the Algebra of Quantity these conditions hold; they do not hold in the Algebra of Quality, because the Universe is a generalised Unit. The Universe becomes a Unit when it contains only one individual.

53. *Definition of Single.*—A symbol x is said to be single when it does not select any member of the universe more than once, and always selects with the same sign.

Observation.—This embraces exclusion, and both positive and negative inclusion.

54. If we have $z=x+y$,
and z is a single character, then x and y must be mutually exclusive. It is in this way that the Algebra imposes the restriction when it is requisite. Similarly, if

$$z=x-y,$$

and z be single, then the x must be inclusive of the y.

55. *To express not x in terms of x.*

By not x is meant the whole excepting x; hence it is expressed by $1-x$.

The arithmetical value of not x is $\bar{1}-\bar{x}$. In the example of Art. 48, $1-x$ denotes 'not having a dot,' and forms part II. of the figure; while $1-y$ denotes 'not having a dash,' and forms parts 2 and 4.

56. *If x is positive and single; then $1-x$ is positive and single.*

For x being included in 1, and 1 being single and positive, the result of $1-x$ is single and positive. For an analytical proof, see Art. 120.

57. When x and y are in their statement independent of one another, their combination by + and − is subject to the formal laws

$$x+y=y+x \qquad (1)$$
$$x+y+z=(x+y)+z \qquad (2)$$
$$x-y=-y+x. \qquad (3)$$

58. The truth of the formal law (1) may be seen by a consideration of the particular instance of the cubes with

different marks. If we first separate those having a dot, then separate those having a dash, and add the two results together; we shall get the same cubes in the final result as when we commence by separating those having a dash, then separate those having a dot, and add the two results together.

The formal law (2) shows that independent characters, which are not necessarily single, are subject to the same law of combination as independent characters, which are single.

The formal law (3) shows that – as a symbol of operation of the mind is quite as independent as +

V. THE SIGNS × AND ÷

59. By

$$Uxy$$

is meant

U's which are both x and y.

In common language 'both' is frequently dropped. According to Boole + expresses the signification of 'and'; while × properly expresses sequence. It is, however, an additional argument in favour of a difference of notation for the Universe and its characters, and generally between operations which are subordinate and operations which are co-ordinate; that, when it is done, + expresses 'or' and × expresses 'and.'

60. Consider, as before, a collection of cubes, Art. 48; but we shall now suppose that we have a great number. As before, let x denote 'having a dot,' and y 'having a dash.' Let x separate to the left hand (fig. 2); and then let y separate to the bottom. The operation xy is a consequence, being the separation of the cubes, which both have a dot and also have a dash.

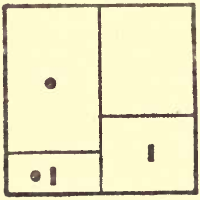


FIG. 2.

61. When the symbols x and y are independent—that is, when each refers to U simply—the compound symbol xy has for any given U *a definite arithmetical value.* This value, however, is not determinable from those of x and y; but they give limits to the value. Consider the example of the cubes (fig. 2). It is evident that xy cannot be greater (include more) than x, or greater than y; and that it cannot be less than (must include) $x+y-1$, or 0. Hence its arithmetical value cannot be greater than $\bar{x}$ or $\bar{y}$, and cannot be less than $\bar{x}+\bar{y}-1$, or 0.

62. It is important to draw a distinction between the independence of the operations x and y, and the relation of these to one another as characters. *The operation y is independent of x, when it is not confined by its expression to Ux, but applies to U.* In the Theory of Probability it is assumed that, when x and y are independent,

$$\overline{xy}=\bar{x}\times\bar{y}.$$

But this is an assumption resting on the supposition that the ratio of the Uy's in the x part to the x part is the same as the ratio of the Uy's in the not x part to the not x part. It is assumed that the two straight lines, which together separate the y in fig. 2, form one straight line. It is, no doubt, the best assumption to make, when we do not know how the two portions of the line lie; and is of the same nature as the doctrine that the values of two characters not being known, they are to be assumed to be equal. We have supposed that the Universe contains a great number of cubes, in order that each portion of the y line may be supposed to be straight.

63. Venn shows the true nature of the assumption in his *Logic of Chance*, p. 157:—'For the establishment of the rule under discussion, it is both necessary and sufficient that the division into classes caused by each of the above distinctions, should sub-divide each of the classes created by the other distinction in the same ratio in which it subdivides the whole. If the independence be granted and so defined as to mean this, the rule of course will stand, but, without especial attention being drawn to the point, it does not seem that the word would naturally be so understood.' The distinction may be called that between *formal* and *real independence.*

64. But one character may in its statement involve another character, so as to be formally dependent on the latter. Let U denote the universe of objects, a any character, and x a character which is formally dependent on a. Then Ua_x denotes

U's which are a and of these such as are x;

or U's which have a which have x.

The symbol x operates upon Ua, not upon U.

The arithmetical value of x is measured with reference to Ua, not to U. Hence

$$\bar{a}_x = a \times \bar{x}.$$

65. In this case, and this case only, does the mind proceed in the *predicamental line.* Boole supposes that it always proceeds in that manner. In Uxy he considers y to operate on Ux,—in fact, to mean Ux_y, and supposes that it can preserve the same signification in Uy. But in fact its meaning and arithmetical value may both be changed.

66. In the ordinary Algebra $\frac{1}{2}\ \frac{1}{3}$ means one-half of a third or one-third of a half. The one operation is formally dependent on the other. But we have shown, Art. **61**, that in the Algebra of Quality

$$\frac{1}{2}\ \frac{1}{3}$$

does not mean one-half of a third or a third of a half, and can be considered to be equivalent to such an expression only when the characters are really independent of one another. Thus the Algebra of Quality is more general than the Algebra of Quantity.

67. Suppose that we consider the 'Mammalia.'
Then an example of a_x is

having red blood corpuscles

which are oval;

and an example of a_{xy} is

having red blood corpuscles

which are circular and discoid.

It is the function of the *Relative Pronoun* to denote depend ence.

68. When x and y are formally independent of one another

$$xy = yx.$$

Also, when x, y and z are formally independent of one another,

$$x(y+z) = xy + xz.$$

69. Since a_x is equivalent to an independent character of arithmetical value $\bar{a}\,\bar{x}$, the laws that are true of independent characters apply to a_x as a whole.

70. Also, the same laws apply to characters which are co-ordinately dependent upon a common character. For example,

$$U\{a_x + a_y = a_{x+y}\}.$$

and $U\{a_{xy} = a_{yx}\}$.

71. Special laws.

$$U\{a_x = x_a\}.$$

Also, if $\quad a+b+c+\ldots+k=1,$

then $\quad a_x+b_x+c_x+\ldots+k_x=x.$

Observation.—It is supposed that x is such that it can be freed of its dependence on a; which is not possible when x refers to a noun contained in a.

72. *Deduction of the meaning of* ÷ *from that of* ×.

Let $xy = xz$,

then ÷ being the reverse of × is to have such a signification that

$$y = \frac{xz}{x}.$$

It is evident from fig. 3, which represents a case of

$$xy = xz$$

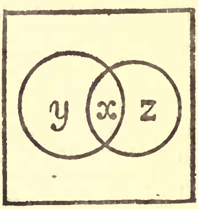


FIG. 3.

that $\frac{xz}{x}$ is indefinite. Hence the proper reading of

$$y=\frac{xz}{x}$$

is

$$y \text{ is equivalent to an } \frac{xz}{x}.$$

It is true in the case of fig. 3 that

$$yz=x,$$

hence $$z=\frac{x}{y},$$

which means

$$z \text{ is equivalent to an } \frac{x}{y}.$$

For there are evidently several z's which can satisfy the condition

$$yz=x,$$

although x and y be kept constant.

Cor. $\frac{x}{x}$ is not in general = 1.

For if $\frac{x}{x}$ were necessarily = 1, the truth of the equation

$$xy=xz$$

would necessarily involve that of

$$y=z.$$

But an inspection of the figure shows that this is not the case.

73. The condition $y=z$ is always satisfied by the fractions of ordinary Algebra; but it is not satisfied by the fractions of the Algebra of Quality, unless as a special case. This peculiarity $\left(\frac{x}{x} \text{ not necessarily equal to } 1\right)$ was clearly pointed out by Boole. It is commonly supposed, and I think that Boole himself supposed, that it is the only peculiarity of the symbols of this Algebra. For example, Professor Robertson Smith says:—'There is one limitation only to our right to

manipulate logical and mathematical identities by the same rules. From $xy=xz$ we cannot infer $y=z$.' *In Translation of Ueberweg*, p. 569. Now the fact is that each sign has a generalised meaning.

74. Boole interprets $\frac{0}{0}$ to mean an indefinite class symbol; but this interpretation, he says, cannot except upon the ground of analogy be deduced from its arithmetical properties, but must be established experimentally. *Laws of Thought*, p. 92. In Art. **166** I show that under a certain condition $\frac{0}{0}$ must be single within a certain part of the universe. When no condition is imposed upon it, its meaning is quite indeterminate.

75. *Definition of the Index sign.*

x^m, where m is an integral symbol, is defined to mean the selective operation x repeated m times. For instance,

$$x^2 = xx.$$

It is to be noted that x may be single or complex, and positive or negative or both.

76. *Law of Indices.*

$$x^m x^n = x^{m+n}.$$

Also $$\frac{1}{x^m x^n} = \frac{1}{x^{m+n}}.$$

But $\frac{x^m}{x^n}$ is not in general $=x^{m-n}$.

For if it were, $\frac{x^2}{x}$ would be $= x$; which is not necessarily true.

VI. RULE OF SIGNS.

77. If $+^2$ be defined to be equivalent to $+$, it follows by the usual proof, that

$$+- = -;$$
$$-+ = -;$$

and $$-- = +.$$

Observation.—It is, of course, immaterial in what connection $+-$ occurs; whether in $+(-x)$, or $+x(-y)$, or $\frac{+x}{-y}$. The superposition of the two mental operations can be considered apart from the accidents with which the operations are mixed up in use.

VII. INTEGRAL SYMBOLS.

78. *To find the meaning of* $\frac{m}{n}$, *where m and n are each integral.*

Let $$w = \frac{m}{n}.$$

Then $nw = m$. (Art. 72); for every definition applies conversely.

Since m and n are each integral (that is, *the whole* repeated a number of times), it is evident that the latter equation cannot be satisfied unless n is a divisor of m. Then $w =$ quotient. Thus $\frac{m}{n}$ is *impossible* unless n divides m.

COR. 1. $\frac{1}{1} = 1$. $\frac{4}{2} = 2$.

COR. 2. $\frac{1}{2}$ is impossible.

But $\frac{1}{2}x$ will be possible, when $x = 2y$. For then we have $\frac{2}{2}y$; which is $= y$.

Observation.—$\frac{1}{2}$ does not mean the same as $\frac{\text{I}}{2}$. For $\frac{1}{2}$ means *the whole upon two wholes;* whereas $\frac{\text{I}}{2}$ simply means one-half.

79. *To find the meaning of* $\frac{0}{n}$, *and* $\frac{n}{0}$; *where n is any integral symbol.*

First; for $$\frac{0}{n}.$$

Let $w = \frac{0}{n}$.

Then $nw = 0$ (by definition).

From this latter equation it is evident that w must be $= 0$.

COR. $\frac{0}{1} = 0$.

Second; for $\frac{n}{0}$.

Let $w = \frac{n}{0}$.

Then $0w = n$ (by definition).

Suppose w to be integral; the latter equation is then evidently impossible.

Suppose it to be fractional and single; the equation is then also impossible. So also, when w is supposed fractional and complex.

Hence $\frac{n}{0}$ is an *impossible* selective operation.

COR. *If* $x = \frac{1}{0} y$, *and* x *is possible; then* y *must be* $= 0$.

For $0x = y$,

but x being a possible selective symbol, $0x = 0$;

$$\therefore 0 = y.$$

80. *To find the meaning of* $\frac{0}{0}$.

Let $w = \frac{0}{0}$.

Then $0w = 0$.

The latter equation will be satisfied when w is an integral symbol; also when w is any fractional symbol, whether single or complex. Hence $\frac{0}{0}$ is quite indeterminate.

81. Since the symbols and the signs of the Algebra of Quality have a meaning, which is a generalised form of their meaning in the Algebra of Quantity, every theorem in the latter is true in the former, provided that any special conditions which have been introduced are removed. For example:—

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} \text{ is not } = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix};$$

$$\text{but it is } = \frac{abc}{abc} \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

The former equation can be deduced from the latter in the Algebra of Quantity, because in it

$$\frac{abc}{abc} = 1 \text{ always.}$$

82. The propositions contained in the *Theory of Determinants*, when so generalised, are of great interest and importance in the analysis of Quality. The ordinary demonstration of these propositions does not necessarily depend upon the peculiarities of the symbol of Quantity. It, however, proceeds on the assumption that the symbols are independent of one another.

83. In the preceding investigation of the fundamental definitions of this Algebra and their immediate consequences, I have endeavoured to imitate the clear logical manner in which the fundamental definitions of the Calculus of Quaternions are laid down by Professor Tait in his *Treatise on Quaternions.* The definitions given are perfectly exact, and in this respect stand in marked contrast to the vague indefinite statements about $+$ and $\times$ which are to be found in even the best manuals of Logic.

VIII. ON THE EQUATION AS EXPRESSING A GENERAL PROPOSITION.

84. Any relation between the characters of a definite universe can be expressed by means of an equation between the symbols which represent the characters. The logical equation is an analytical statement of a truth, which may require for its expression in language one or more general propositions—a statement which, from its brevity, clearness, and exactness, is of the greatest aid to the mind when reasoning from complex data. Whether other logical relations besides those of Quality can be stated more fully and concisely by means of symbols and the equation than by means of words and the proposition, is a different question.

85. *Every general proposition refers to a definite universe;* which is the subject of the judgment, and, it may be, of a series of judgments. For example;

All men are mortal,

refers to the universe 'men.'

No men are perfect,

refers to the same universe.

86. It is often said, for instance by Boole (*Laws of Thought*, chap. IV.), that

All men are mortal beings,

is a more complete statement of the judgment

All men are mortal,

or, at all events, expresses the same judgment of the mind. Now, in the one case, the subject of discursive thought is the class 'beings;' in the other, the class 'men.' Can judgments, which apply to different universes of objects, be the same? In considering whether, in the universe of 'men,' the attribute 'mortal' is equivalent to 'the whole,' I do not require to consider *beings in general.*

87. Let U denote men, and x mortal; then

All men are mortal

is expressed by

$$U\{1=x\}.$$

But if the judgment is

Beings which are human are mortal;

let U denote beings, x mortal, y human; then

$$U\{y=x-w\}$$

(where w denotes an unknown character, the complement of y to x) expresses the judgment considered.

Another form of the equation is

$$U\{y=yx\}.$$

Again; to express

No men are perfect;

$$U\{0=p\}.$$

And to express

Some men are wise;

$$U\{v=w\},$$

where v is a proper fraction, and greater than 0.

88. We may write these equations, dispensing with abbreviations, which is indeed a very good test of the soundness of a notation.

Men {all = mortal}.
Men {none = perfect}.
Men {some = wise}.

These are precise and undistorted representations of the judgments actually expressed by

All men are mortal.
No men are perfect.
Some men are wise.

89. We may consider the analytical expression of those examples of general judgments, which Aristotle discusses in his book on *Interpretation*.

πᾶς ἐστιν ἄνθρωπος δίκαιος.
οὐ πᾶς ἐστιν ἄνθρωπος δίκαιος.
πᾶς ἐστιν ἄνθρωπος οὐ δίκαιος.
οὐ πᾶς ἐστιν ἄνθρωπος οὐ δίκαιος.

The common universe of these propositions is ἄνθρωπος, and the quality considered is δίκαιος. The equations are respectively

$$1=\delta. \qquad (1)$$

$$1-v=\delta. \qquad (2)$$

$$1=1-\delta. \qquad (3)$$

$$1-v=1-\delta. \qquad (4)$$

$1-v$ denotes *all minus some, some being greater than nought;* which is equivalent to *not all.*

90. It is evident that *the grammatical subject of the proposition does not form one member of the equation.* The members of the equation may be the quantity and the predicate; as in the above examples, and in judgments such as

$$\text{Men}\{\tfrac{1}{12}=\text{shortsighted}\}.$$

Or, they may be the antecedent and consequent; as in

Triangles which are equilateral are equiangular,

$$T\{x=y-w\}.$$

91. Since a universe may contain one, several, or an infinite number of individual parts, and since a judgment always refers to a universe, it is evident that the judgment will be modified in an important manner according to the number of individual parts in its universe. The strictly *singular* judgment refers to a universe, which has only one individual part; as, for example,

A given man at a given time is sleeping.

It is to be observed that, in order to produce a singular universe, it is not enough that the substance be fixed; the time must also be fixed.

92. *When the judgment refers to a singular universe, the only values which any character can assume are* 1 *and* 0. The former is required to make an affirmation, and the latter to make a negation.

The given man at the given time is sleeping.
" " is not sleeping.
" " is not-sleeping.
" " is not not-sleeping.

are expressed respectively thus—

$$1=s. \quad (1)$$

$$0=s. \quad (2)$$

$$1=1-s. \quad (3)$$

$$0=1-s. \quad (4)$$

93. Aristotle remarks that here equation (3) follows from equation (2); but that in the more general case (Art. **89**) when U is plural, equation (3) does not follow from equation (2). It is only judgments of the kind considered in the preceding article that are *affirmative* or *negative* in the strict sense of these words.

94. There are singular judgments of another kind. For example;

Of all the planets Jupiter is the greatest;

the analytical expression of which is

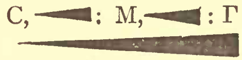
Planets {Jupiter = the greatest}.

'Jupiter' is a *singular character*, so is 'the greatest;' that is, they can apply to only one of the individual parts of the universe. Hence the singularity of this kind of judgments does not consist in the universe of the judgment being an individual, but in the characters being singular.

95. It is also important to consider the special nature of the judgment which relates to a universe having an *infinite* number of parts. For, questions arise as to how it is possible in such a case to assert relations among the characters.

96. As the relations of the characters of the universe may be given in an infinite variety of ways; the logical equation, in order to be adequate, must be capable of assuming an infinite variety of forms. The forms treated in manuals of Logic are but a few of the most common. Not only ought we to be able to state explicitly what is thought implicitly, but we ought to be able to state in one equation any one condition, which a number of characters satisfy; even though it take a great number of sentences to express that condition.

97. The reason why Formal Logic has so long been unable to cope with the subtlety of nature, is that too much attention has been given to *pictorial notations*. Arithmetic could never be developed by means of the Roman system of notation; and Formal Logic cannot be developed so long as Barbara is represented by



or even by the simpler spicular notation of De Morgan. We cannot manipulate data so crudely expressed; because the nature of the symbols has not been investigated, and laws of manipulation derived from their general properties.

98. A consideration of the nature of *propositions about abstract numbers* is of importance here, as throwing light upon the nature of the logical equation. Kant appears to be the only logician who has discussed them at all thoroughly. He says:—'The self-evident propositions as to the relation of numbers are certainly synthetical, but not universal, like those of geometry, and for this reason cannot be called axioms, but numerical formulæ. That

$$7+5=12$$

is not an analytical proposition. For neither in the representation of seven, nor of five, nor of the composition of the two numbers do I cogitate the number twelve. But although the proposition is synthetical, it is nevertheless only a singular proposition. In so far as regard is here had merely to the synthesis of the homogeneous (the units), it cannot take place except in one manner, although our *use* of these numbers is afterwards general.'—*Critique of Pure Reason.*

99. The full expression of the example discussed is

$$U\{7+5=12\};$$

where U may denote *any unit* or *any universe.* For it is a logical as well as an arithmetical proposition.

Similarly $$U\{7\times 5=35\},$$

whatever U be.

It is evident that the source of the nexus is not convention. The peculiarity is, that the equivalence is true independently of the nature of U.

100. It is also of importance to consider *the nature of the* *chemical symbols and of the chemical equation.* The chemist considers a universe of material molecules; and such symbols as H and Cl denote a distinctive character or set of characters possessed by some of the molecules. It is a property of these characters, that they are exclusive of each other. HCl does not denote matter which has the property H and also the property Cl; but matter which has an entirely distinct property HCl. The meaning of HCl is 'formed by the combination of H and Cl.' The symbols have fixed fractional values—that is, fixed proportions; in this respect differing from the symbols of Quality.

101. The chemical equation expresses how the distinctive characters of a particular set of molecules are changed into other distinctive characters, and what the latter are. For example;

$$K_2SO_4+BaCl_2=BaSO_4+2KCl,$$

Sulphate of potassium and chloride of barium become sulphate of barium and chloride of potassium,

expresses that in a particular quantity of matter the sum of distinctive characters

$$K_2SO_4+BaCl_2$$

becomes changed into the sum of distinctive characters

$$BaSO_4+2KCl.$$

The symbol = involves *becoming.* (Professor Crum Brown's *Manual of Chemistry*, p. 46.) Hence the chemical equation cannot be reversed without reversing the meaning of the copula =. In De Morgan's language, the copula here is inconvertible.

102. The above equation expresses completely the change to which it refers. Its right-hand member is as definite as its left-hand member, and the two are equal arithmetically. In particular cases we may require only the incomplete equation

$$K_2SO_4+BaCl_2=BaSO_4+\ ;$$

just as instead of the complete logical equation

$$x=y+z,$$

we may in particular circumstances consider only

$$x = y +$$

There is an important difference between what is requisite to give a complete scientific expression of a change or of a relation, and what is requisite to give an expression of such a part of that change or relation as may be necessary in a particular process of reasoning.

103. *The equations of Natural History are examples of logical equations.* The dental equation of Man is written

$$i\frac{2-2}{2-2};\ c\frac{1-1}{1-1};\ pm\frac{2-2}{2-2};\ m\frac{3-3}{3-3}=32.$$

But this truth may be written as a logical equation; thus—

$$\text{Man}\left\{1 = {}^{i}\frac{2-2}{2-2}\,{}^{c}\frac{1-1}{1-1}\,{}^{pm}\frac{2-2}{2-2}\,{}^{m}\frac{3-3}{3-3}\right\};$$

that is, considering the universe of men in the adult state, the whole is identical with those having incisors which are in number and arrangement $\frac{2-2}{2-2}$, having canines which are in number and arrangement $\frac{1-1}{1-1}$, having premolars which are in number and arrangement $\frac{2-2}{2-2}$, having molars which are in number and arrangement $\frac{3-3}{3-3}$. Each of the characters ${}^{i}\frac{2-2}{2-2}$, ${}^{c}\frac{1-1}{1-1}$, etc., must be equal to 1. In ${}^{i}\frac{2-2}{2-2}$, the character $\frac{2-2}{2-2}$ is dependent with respect to i, and is itself composed of four co-ordinate characters.

104. If we consider the universe of Mammals in the adult state; to what is the character

$${}^{i}\frac{2-2}{2-2}\,{}^{c}\frac{1-1}{1-1}\,{}^{pm}\frac{2-2}{2-2}\,{}^{m}\frac{3-3}{3-3}$$

equivalent? We have, I think,

$$\text{Mammalia}\left\{\text{bimana} + \text{quadrumana}_{\text{catarhina}} = i\,\frac{2-2}{2-2}\;c\,\frac{1-1}{1-1}\;pm\,\frac{2-2}{2-2}\;m\,\frac{3-3}{3-3}\right\};$$

and, if so, we have a statement which is as complete and exact as any chemical equation.

105. Professor Huxley, in his argument against the assertion that Biology differs from the Physico-chemical and Mathematical sciences in being 'inexact,' holds that the statements

A vertebrated animal has jaws which never open sideways, but always up and down,

and

An annulose animal has jaws which always open sideways, and never up and down,

are as *exact* as any of the propositions in Euclid. (*On the Value of the Natural History Sciences. Lay Sermons, Essays and Reviews*, p. 79.)

The above statements are exact, in the sense of being capable of expression by exact equations; thus

$$\text{Animals}\left\{\text{vertebrated} = \text{having jaws}_{\text{never opening sideways} \times \text{opening up and down.}}\right\}$$

$$\text{Animals}\left\{\text{annulose} = \text{having jaws}_{\text{opening sideways} \times \text{never opening up and down.}}\right\}$$

IX. THE PRINCIPLE OF IDENTITY AND THE AXIOMS OF IMMEDIATE INFERENCE.

106. Let U denote any universe; and x any character, positive or negative, integral or fractional; then

$$U\{x = x\}.$$

The Principle of Identity, when put into the above form, does not appear to merit the abuse which has been heaped upon

A is A

and

Ens est Ens.

Another form of this principle is

$$x-x=0,$$

which involves the meaning of the sign $-$.

107. An inference is said to be *immediate*, when it is deduced from an equivalent equation and an identical equation (or the former repeated) by means of the axioms of the science.

It has been a matter of perplexity to logicians, how a conclusion can be drawn from one premise. Ueberweg says:—'The immediateness in the so-called "immediate" inference is relative. It implies that this kind of inference does not require, as mediate inference does, the addition of a second datum to the first, but at once and of itself yields the derived judgment, which is nevertheless another judgment, and not merely another verbal expression.'—(*Lindsay's translation of Ueberweg*, p. 226.)

It is not very evident, how one equation can at once and of itself yield another, which has any claim to be called different. But if we consider that an identical equation can be formed, and that the axioms of the science, which depend upon what is essential in the symbols, allow us to combine these equations; a great part, if not the whole, of the mystery disappears.

108. *The general axioms of immediate inference are:—*

I. If an equation and an identical equation be added together; the resulting equation is true.

Thus if $\quad x=y;$

since $\quad z=z,$

$$x+z=y+z.$$

II. If an equation and an identical equation be subtracted the one from the other; the resulting equation is true.

Thus if $x=y$;
since $z=z$,

$$x-z=y-z.$$

III. If an equation and an identical equation be multiplied together; the resulting equation is true.

Thus if $x=y$;
since $z=z$,

$$xz=yz.$$

IV. If an equation and an identical equation be divided the one by the other; the resulting equation is true.

Thus if $x=y$;
since $z=z$,

$$\frac{x}{z}=\frac{y}{z}.$$

109. A common form of application of Axiom II. consists in subtracting the given equation from the identical equation

$$1=1.$$

For example; the universe being Triangles,

equilateral = equiangular;

∴ 1 − equilateral = 1 − equiangular.

This is one form of the process of *Contraposition.*

110. An equation, when multiplied by a fractional symbol, is still true (Axiom III.); but it is not so general as before. The multiplication of an equation by such a symbol means the introduction of an hypothesis. A proposition, which is true independently of an hypothesis, is true under the hypothesis; but not conversely.

111. The axiom that

When the signs of the terms of an equation are changed into their respective converse signs, the resulting equation is true,

reduces Axioms II. and IV. to cases of Axioms I. and III. respectively.

Thus if $x = y$;
then $-x = -y$,
where $+$ is changed into its converse $-$; and also

$$\frac{1}{x} = \frac{1}{y},$$

where $\times$ is changed into its converse $\div$.

112. *Given the principle of Identity*

x is equivalent to x,

to prove the principle

x is equivalent to not not x.

Since $x = x$,
and $0 = 1 - 1$,

$$x = 1 - 1 + x. \quad \text{II. Axiom of Immediate Inference.}$$
$$= 1 - (1 - x) \quad \text{by Rule of Signs.}$$
$$= \text{not not } x. \quad \text{(Art. 55.)}$$

Cor. $1 - x = 1 - \{1 - (1 - x)\}$.

Generally.—In the case of x, we have an even number of 1's alternately positive and negative; and in the case of $1 - x$, an odd number of 1's alternately positive and negative. It is customary with grammarians to point out, that an adjective, with an even number of *nots* before it, is equivalent to the simple adjective.

Observation. $x = 1 - (1 - x)$,
is true whatever U be; and also without x being restricted, as above, to be single.

When U is singular, x can only be 1 or 0 (Art. 92); and

$$x = 1 - (1 - x)$$

then becomes what is called by some *the Principle of Contradiction.*

113. *Given the principle of Identity*

x is equivalent to x,

to prove the principle of Excluded Middle

x + not x is equivalent to the whole.

$$x = x,$$
$$\therefore x - x = 0;$$
$$\text{but } 1 = 1,$$
$$\therefore x + 1 - x = 1. \quad \text{I. Axiom of Immediate Inference.}$$

That is, $x +$ not x is equivalent to the whole, whatever U be. U may be singular, plural or infinite.

The principle of Excluded Third, as enunciated by many logicians, presupposes that U is singular. For example; Ueberweg states it thus (*Translation*, p. 260)—

'Judgments opposed as contradictories (such as A is B, and A is not B) can neither both be false nor can admit the truth of a third or middle judgment, but the one or the other must be true.'

Let U be singular, then

$$U\{x + 1 - x = 1\}.$$

Now when U is singular; x can only be 1 or 0; and in consequence $1 - x$ can only be 0 or 1. Hence either

$$U\{x = 1\}$$

or else $$U\{x = 0\}$$

must be true.

When U is plural, it is of course not the case that either

$$U\{x = 1\}$$

or else $$U\{x = 0\}$$

must be true.

Observation. $$x + 1 - x = 1$$

does not necessarily presuppose any restriction upon the signification of x, such as to be single.

X. AXIOMS OF MEDIATE INFERENCE.

114. A *mediate* inference may be defined as, an inference from two or more independent equations by means of the axioms of the science.

115. The general axioms of mediate inference are :—

I. If the equations be added, the resulting equation is true. For example; if

$$x = y, \text{ and } a = b;$$

then $$x + a = y + b.$$

II. If two equations be subtracted the one from the other; the resulting equation is true. For example; if

$$x=y, \text{ and } a=b;$$

then $$x-a=y-b.$$

III. If two equations be multiplied together; the resulting equation is true.

$$x=y, \text{ and } a=b;$$

then $$ax=by.$$

IV. If two equations be divided the one by the other; the resulting equation is true.

$$x=y, \text{ and } a=b;$$

then $$\frac{x}{a}=\frac{y}{b}.$$

116. It is not necessary, that the two equations should contain a common character; but it is necessary that both should refer to the same universe: just as the ordinary algebraic equations in a given investigation must be supposed to refer to the same unit. In the latter sense only is it true, that no reasoning is possible without a *middle term.*

117. Kant would not consider the above to be true axioms. He says (*Critique of Pure Reason*, Meiklejohn's translation, p. 124):—'The propositions

If equals be added to equals, the wholes are equal,

and

If equals be taken from equals, the remainders are equal,

are analytical, because, I am immediately conscious of the identity of the production of the one quantity with the production of the other; whereas axioms must be *a priori* synthetical propositions.' But these propositions are not analytical, in the sense of depending upon convention for their truth; they are true of the symbols of Quantity, as symbols of Quantity; and only in that sense may be said to be analytical. An axiom appears to be precisely such a proposition.

XI. CONDITIONS FOR A CHARACTER BEING SINGLE.

118. *If x is single and positive, then*

$$x^2 = x;$$

and if x is single and negative, then

$$x^2 = -x:$$

and conversely.

For the definition of 'single,' see Art. **53**.

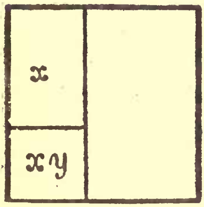


FIG. 4.

First; when x is positive.

Let y be another single and positive symbol, and let y be wholly included in x (fig. 4). Then

$$xy = y.$$

Let y expand until $y = x$;

then $\quad xx = x,$

that is, $\quad x^2 = x.$

Second; when x is negative.

Let $x = -1\, x'$; x' being positive.

Then $x^2 = -1^2\, x'^2.$

$= -1^2\, x'$ by first part.

$= x'$ by Rule of Signs.

$= -x.$

Hence, if x is single;

$$x^2 = \pm x.$$

By a consideration of the figure it is also evident that if

$$x^2 = x,$$

x must be single and positive;

and that if $\quad x^2 = -x,$

x must be single and negative.

Hence $\quad x^2 = \pm x$

is the necessary and sufficient condition for x representing a single character.

119. More generally :—If x is single and positive,

$$x^m = x;$$

and if x is single and negative,

$$x^m = (-1)^{m-1}\, x.$$

120. If x is single positively, then $1-x$ is single positively.

For $(1-x)^2 = 1-2x+x^2$. (Art. 68.)

$= 1-x + (x^2-x)$,

$= 1-x$ by given condition ;

$\therefore$ $1-x$ is single positively. (Art. 118.)

It is when $1-x$ is single positively, that it means the same as *not* x.

121. *Given that* $U\{x^2=x\}$;

to prove that $U\{x(1-x)=0\}$.

$x^2=x$,

and $x^2=x^2$, Principle of Identity.

therefore $x^2-x^2=x-x^2$; II. Axiom of Immediate Inference ;

that is, $0=x-x^2$,

therefore $0=x(1-x)$. Distributive Law, Art. 68.

Another Proof.

$x+1-x=1$, Principle of Excluded Middle.

$x=x$, Principle of Identity.

therefore $x(x+1-x)=x$; III. Axiom of Immediate Inference.

therefore $x^2+x(1-x)=x$; Distributive Law.

but $x^2=x$, given.

therefore $x(1-x)=0$. II. Axiom of Immediate Inference.

122. *Meaning of these equations.*

$$U\{x^2=x\}$$

means

U's which are x and x are identical with those which are x.

It is true only when x is positively single with reference to U.

Since x is single positively, $1-x$ must be single positively (Art. 120). Hence $1-x$ means 'not x.' Hence

$$U\{x(1-x)=0\}$$

means

U's which are x and not x are none.

Since $$U\{x^2 = \pm x\}$$
is the condition for x being single, relatively to the universe U;
$$U\{x\,(1 \pm x) = 0\}$$
is the general form of the other mode of stating the condition.

123. When U is singular, any fractional character x must be single (Art. **92**). Hence, when that condition is satisfied,
$$x^2 = \pm x$$
$$\text{and } x(1 \pm x) = 0$$
are true always. The latter is then the exact analytical expression of the Principle—

It is impossible for an object at a given time to have a quality and the contrary quality.

This is what is commonly called the *Principle of Contradiction;* but the most general form of the Principle is that enuntiated in Art. **122**, where U may be singular, plural, or infinite.

124. As the deduction and interpretation of
$$x\,(1 - x) = 0$$
is of great importance in the Algebra of Quality, I shall quote the original proof given by Boole. (*Laws of Thought*, p. 49.)

'*That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is* $x^2 = x$.'

Let us write this equation in the form
$$x - x^2 = 0$$
whence we have
$$x\,(1 - x) = 0,$$
both these transformations being justified by the axiomatic laws of combination and transposition. Let us, for simplicity of conception, give to the symbol x the particular

interpretation of *men*, then $1-x$ will represent the class of 'not men.' Now the formal product of the expressions of two classes represents that class of individuals which is common to them both. Hence $x(1-x)$ will represent the class whose members are at once 'men' and 'not men.' Now the formal product of the expressions of two classes represents that class of individuals which is common to them both. Hence $x(1-x)$ will represent the class whose members are at once 'men' and 'not men,' and the latter equation thus express the principle, *that a class whose members are at the same time men and not men does not exist.* In other words, that *it is impossible for the same individual to be at the same time a man and not a man.*

125. With reference to the above deduction, the Rev. Robert Harley observes as follows (*British Association Report for* 1866):—'From the logical equation $x^2=x$ the equation $x-x^2=0$ is derived by subtracting x^2 from both members, and the result is put under the form $x(1-x)=0$ by the law of distribution. It is to be observed, however, that at every step of the process the principle of identity $x=x$ is assumed, and in Boole's interpretation of the final result the same principle is used, for it is implied that the x without the brackets is identical with the x within. Further, in the final interpretation not only is the principle of contradiction employed, but the principle of excluded middle is also employed. For in interpreting $1-x$ to mean not x, it is tacitly assumed that every one of the things of which the universe represented by unity is made up is either x or not x. It would thus appear that these three principles of identity, contradiction, and excluded middle are incapable of being reduced to more elementary truths. They are axiomatic, and Boole made use of them unconsciously in framing his laws of logical interpretation.'

126. Boole's investigation can be defended from the above criticism; but the defence requires that the definitions and axioms of the science be laid down in an exact manner;

as I have endeavoured to do in these 'Principles.' Hence I shall consider how the above remarks affect my own investigation, and shall assume that x denotes a character referred to a definite universe.

127. It is true that the Principle of Identity

$$x=x$$

is assumed, in the sense assigned to it in Art. **106**. Mr. Harley considers it to mean that the one x as a symbol is identical with the other x as a symbol; whereas it must mean that the result of the operation denoted by the one symbol is identical with the result of the operation denoted by the other symbol.

In the proof that, under the given condition of

$$x^2=x,$$

$1-x$ means 'not x' (Art. **120**), the Principle of Excluded Middle brought in is

$$x+1-x=1;$$

which is only another form of

$$x=x. \qquad \text{(Art. 113.)}$$

It is to be observed that the Principle of Excluded Middle quoted in the second proof of Art. **121**, is that general form of the Principle which is mentioned at the end of Art. **113.**

The Principle of Contradiction is not brought in. The proof follows from the Distributive Law, and the given condition put into the form $x^2-x=0$ by means of the Principle of Identity and the Second Axiom of Immediate Inference.

128. What Mr. Harley seems to mean by the Principle of Contradiction is the truth expressed by

$$x=1-(1-x).$$

That principle has been deduced from

$$x=x \qquad \text{(Art. 112)};$$

and is distinct from

$$x(1-x)=0,$$

the one being an identity, and the other a condition.

129. With reference to the same deduction, Venn observes (*Mind*, vol. i. p. 491):—'Boole says that the axiom which is termed the Principle of Contradiction, and which "has been commonly regarded as the fundamental axiom of metaphysics, is but a consequence of a law of thought mathematical in its form," viz. the law whose expression is $x^2=x$. This is doubtless a very elementary truth, but to regard it as the *source* of the Law of Contradiction surely argues a strange inversion of order. However that law be regarded, nothing can well be considered more ultimate. We could not distinguish one thing from another without it; we could not, even to go no further than these symbols, distinguish x from what is not x without making use of it. And yet Boole gives a demonstration of their dependence, a demonstration every step of which demands the law several times over.'

130. However well this criticism may avail against Boole's doctrine, where no distinction is drawn between the subject and the operations of thought; I think that it is avoided by the investigation in these 'Principles.' But is that, by which we distinguish one thing from another, the Principle of Contradiction? It does not appear to be the principle considered in Art. **122**.

131. Boole, as above said, considers $x^2=x$ to be a *law of thought.* He also directs attention to the circumstance, that the equation which expresses that fundamental law of thought is *of the second degree.* (*Laws of Thought*, p. 50.) He ventures to assert, that, if that circumstance had not existed, the whole procedure of the understanding would have been different from what it is. In reply to a possible objection, that the existence of the equation $x^2=x$ necessitates the existence of the equation $x^3=x$, which is of the third degree, he answers :—

'The equation $x^3=x$ is not interpretable in the system of logic. For writing it in either of the forms

$$x(1-x)(1+x)=0$$
$$x(1-x)(-1-x)=0$$

we see that its interpretation, if possible at all, must involve that of the factor $1+x$, or of the factor $-1-x$. The former is not interpretable, because we cannot conceive of the addition of any class x to the universe 1; the latter is not interpretable, because the symbol -1 is not subject to the law $x(1-x)=0$, to which all class symbols are subject.'

132. Our position is, that $x^2=x$ does not express a law of thought; as the mind can comprehend with equal ease a selective symbol x, which does not satisfy that condition. It is simply the condition for the character being single and positive. If x is single negatively; then the equation

$$x^3=x,$$

is actually true. The factor $1+x$, considered as a compound character, is as interpretable as x; and though -1 does not satisfy the condition

$$x^2=x,$$

it satisfies the condition

$$x^2=-x.$$

The source, whence it arises that $x^2=\pm x$ is the necessary and sufficient condition for x being single, is not the mind, but the actual constitution of things; so that, being a law of things, it ought to be made a rule for the intellect.

133. Such, it appears to me, are the relations existing between the properties of the symbol of Quality, the Axioms of Operation, the forms of the Principle of Identity, and the forms of the condition which the symbol of Quality must satisfy in order to be single. A consideration of these relations is of great speculative importance; for they clearly show that the language of Mathematical Analysis is that scientific language whose existence has been dreamt of by some philosophers.

134. Leslie Ellis remarked that $x^2=x$ is not true, when x denotes a *relative* term, such as 'father.' The arguments, which accompany this statement, appear to show the existence of a supposition, that the organon for reasoning about

the characters of a definite universe can be applied to all departments of reasoning. It is thus of the greatest importance clearly to define the nature of that operation, which the Algebra of Quality denotes by x; as it is to such, and such only, that its method applies.

135. When a logical equation is given which involves x, we can eliminate x from the equation, if we know that $x^2=x$, or $x^2=-x$. (Art. **286.**) But this must be looked upon as another equation—as expressing another relation which the symbols must satisfy. Hence, it is no doubt more accurate to say that the elimination of n symbols requires $n+1$ independent equations to be given, than to say (with Boole) that no relation whatever can be proved to obtain between the number of terms to be eliminated and the number of propositions from which the elimination is to be effected.

136. *Given that x and y are each positive and single, to find the condition that $x+y$ may be single.*

$$(x+y)^2=x+y, \qquad \text{(Art. 118.)}$$
$$\therefore\ x^2+2xy+y^2=x+y,$$
that is, $x+2xy+y=x+y$, by given conditions;
$$\therefore\ 2xy=0,$$
that is, $xy=0$. (Art. **79.**)

See fig. 5, Art. **148.**

137. *Given that x and y are each positive and single; to find the condition* (1) *that $x-y$ may be single and positive; and* (2) *that $x-y$ may be single and negative.*

(1.) $(x-y)^2=x-y$, (Art. **118.**)
$$\therefore\ x^2+y^2-2xy=x-y.$$
that is, $x+y-2xy=x-y$, by given conditions;
that is, $2\,(y-xy)=0$.
that is, $y\,(1-x)=0$.

(2.) $(x-y)^2=-(x-y)$, (Art. **118.**)
$$\therefore\ x+y-2xy=-x+y.$$
$$\therefore\ 2\,(x-xy)=0.$$
$$\therefore\ x\,(1-y)=0.$$

The truth of these results can be seen from fig. 5.

138. *If each of the factors of a product is single, the product must be single.*

Consider the term xy; where $x^2 = \pm x$, and $y^2 = \pm y$.

Then $(xy)^2 = x^2 y^2$.

$= \pm x \pm y$, by given data.

$= \pm xy$, by Rule of Signs.

Hence xy must be single.

COR. If x and y are each single and positive, $x\,(1-y)$ must be single.

For $1-y$ is single (Art. **120**).

139. *If xy is single, and x single; y is not necessarily single.*

$(xy)^2 = xy$; (1) and $x^2 = x$. (2)

From (1) $x^2 y^2 = xy$,

∴ from (2) $xy^2 = xy$,

$$\therefore y^2 = \frac{x}{x} y;$$

but $\frac{x}{x}$ is not necessarily $=1$. (Art. **72**.)

Hence y^2 is not necessarily equal to y.

COR. The same is true, when $y = \frac{0}{0}$. Boole assumes that $\frac{0}{0}$ in such a case can be single only. It must be single within x.

140. *If x and y are each single and positive, then*

$$\frac{x^m}{y^n} = \frac{x}{y}.$$

For $x^m = x$, and $y^n = y$ (Art. **119**);

$$\text{therefore } \frac{x^m}{y^n} = \frac{x}{y}.$$

COR. 1. Let $y = x$.

$$\text{Then } \frac{x^m}{x^n} = \frac{x}{x}.$$

Hence in the Algebra of Quality

$$\frac{x^m}{x^n} \text{ not in general } = x^{m-n}.$$

COR. 2. Since 1 is single and positive, and 0 single and positive,

$$\frac{1^m}{0^n}=\frac{1}{0}.$$

$$\text{Also } \frac{0^m}{1^n}=\frac{0}{1}.$$

$$\text{Also } \frac{0^m}{0^n}=\frac{0}{0}.$$

Observation. $\frac{x^2}{y^2}=\left(\frac{x}{y}\right)^2$. See Art. 167.

141. *If x, y, z are each positive and single; to find* (1) *the condition for $(x+y)\,z$ being single;* (2) *the condition for $(x-y)\,z$ being single and positive;* (3) *the condition for $(x-y)z$ being single and negative.*

(1.) $$(x+y)^2\,z^2=(x+y)\,z, \qquad \text{(Art. 118.)}$$

that is, $x^2z^2+2xyz^2+y^2z^2=xz+yz$,

that is, $xz+2xyz+yz=xz+yz$.

$$\therefore\ xyz=0.$$

Observation.—The condition $xyz=0$ does not necessarily involve $xy=0$.

(2.) $$(x-y)^2\,z^2=(x-y)\,z. \qquad \text{(Art. 118.)}$$

$$\therefore\ xz-2xyz+yz=xz-yz.$$

$$\therefore\ yz\,(1-x)=0.$$

(3.) $$(x-y)^2\,z^2=-(x-y)\,z. \qquad \text{(Art. 118.)}$$

$$\therefore\ xz-2xyz+yz=-xz+yz.$$

$$\therefore\ xz\,(1-y)=0.$$

142. *If x, y, z, u are each positive and single; to find the conditions for $(x+y)\,(z+u)$ being single.*

$$(x+y)^2\,(z+u)^2=(x+y)\,(z+u). \qquad \text{(Art. 118.)}$$

that is, $xz+xu+yz+yu+2\{xzu+yzu+zxy+uxy\}$

$$=xz+xu+yz+yu,$$

that is, $xzu+yzu+zxy+uxy=0$.

Hence $\quad xzu=0 \quad yzu=0 \quad zxy=0 \quad uxy=0$;
since no term can be negative.

143. *If x_1, x_2, x_3 are each single and positive; to find the condition for $x_1+x_2+x_3$ being single and positive.*

$$(x_1+x_2+x_3)^2=x_1+x_2+x_3 \qquad \text{(Art. 118.)}$$

$$\therefore\ x_1+x_2+x_3+2(x_1x_2+x_1x_3+x_2x_3)=x_1+x_2+x_3.$$

$$\therefore\ x_1x_2+x_1x_3+x_2x_3=0.$$

144. *If x is positive and not negative; then*

$$x^2-x=w,$$

where w denotes a quantity which is positive and not negative.

Suppose that x consists of two positive single parts

$$x_1 \text{ and } x_2.$$

$$\text{Then } (x_1+x_2)^2-(x_1+x_2)=2x_1x_2;$$

and $2x_1x_2$ is positive and not negative. (Art. **138.**)

Similarly it may be shown that when x consists of n positive single parts, the expression x^2-x is positive and not negative.

145. *If $x=x_1+x_2+x_3+x_4$, where x is single, and x_1, x_2, x_3, x_4 each positive and not negative; then x_1, x_2, x_3, x_4 are each single; and every sum of two, and every sum of three, is also single*

x is single, and since it is equivalent to the sum of a number of positive terms, it must be positive; therefore

$$(x_1+x_2+x_3+x_4)^2=x_1+x_2+x_3+x_4. \quad \text{(Art. 118.)}$$

$$\textit{i.e.},\ x_1^2+x_2^2+x_3^2+x_4^2+2\{x_1x_2+\ldots+x_3x_4\}$$
$$=x_1+x_2+x_3+x_4.$$

Now $x_1^2=x_1+w_1$, where w_1 is positive and not negative (Art. **144**),

and
$$x_2^2=x_2+w_2,$$
$$x_3^2=x_3+w_3,$$
$$x_4^2=x_4+w_4,$$

$$\therefore\ x_1+x_2+x_3+x_4+w_1+w_2+w_3+w_4+2\{x_1x_2+\ldots+x_3x_4\}$$
$$=x_1+x_2+x_3+x_4;$$

$$\therefore\ w_1+w_2+w_3+w_4+2\{x_1x_2+\ldots+x_3x_4\}=0.$$

But each of these terms is positive only; hence each must be $=0$.

Hence $w_1=0$, and therefore $x_1^2=x_1$; that is, x_1 is single. So are x_2, x_3, x_4.

Also $x_1x_2=0$; therefore x_1+x_2 is single. (Art. **136.**)

Also $x_1x_2+x_1x_3+x_2x_3=0$;

therefore $x_1+x_2+x_3$ is single. (Art. **143.**)

The proposition of this Article enables us to draw deductions from that form of equation which Boole discusses very frequently—

$$w=A+0B+\frac{0}{0}C+\frac{1}{0}D.$$

XII. THE SIGNS OF INEQUIVALENCE > AND <.

146. Definition of *greater than* and *less than*.

$$x>y$$

denotes that $x-y$ is positive and positive only. Hence $y<x$ denotes that $y-x$ is negative and negative only.

COR. If x and y are each single and positive,

then $x>y$

means that the x includes the y;

and $y<x$

means that the y is included in the x.

147. *Axioms.*

I. If $x>y$, and a denote any character; then

$$x+a>y+a.$$

For $$x+a-(y+a)=x-y,$$

which is given to be positive only.

Similarly; it follows that

$$x-a>y-a.$$

II. If $x>y$, and a denote any symbol which is positive and positive only; then

$$ax>ay.$$

For $$ax-ay=a(x-y);$$

but $x-y$ is positive only, and a is positive only, therefore

$a(x-y)$ is positive only. (Art. **77.**)

If a is negative and negative only; then

$$ax<ay.$$

For $$ax - ay = a(x - y);$$
but $x-y$ is positive only, and a negative only; hence
$$a(x-y) \text{ is negative only.} \qquad \text{(Art. 77.)}$$

Note.—In what follows, unless the contrary is stated, the fundamental symbols employed are supposed to be single and positive.

XIII. DIVISION.

148. *To find the primary parts into which a universe is divided by any number of independent selective operations x, y, z, etc.*

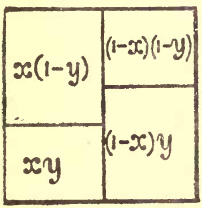


Fig. 5.

Suppose we have two symbols x and y.

Now $$x + (1 - x) = 1,$$
and $$y + (1 - y) = 1,$$
multiply these two equations together,
$$xy + x(1-y) + (1-x)y + (1-x)(1-y) = 1$$

Since x and y are each positive, the parts
$$xy,\ x(1-y),\ (1-x)y,\ (1-x)(1-y)$$
are each positive (Art. **120**). Also their sum is equivalent to the whole.

Since each of the terms is positive, and their sum single, they must be exclusive of one another (Art. **145**); that is, each term, each sum of two, each sum of three is single.

These then are the primary parts of 1, when we have two independent operations x and y. Fig. 5.

149. There are other sets of positive parts, the sum of which is equivalent to the whole; but they each consist of fewer terms than the primary set. For example: since
$$xy + x(1-y) = x;$$
$$x + (1-x)y + (1-x)(1-y) = 1,$$
where we have three positive parts.

150. Suppose that we have three symbols, x, y, z.

Multiply $xy+x(1-y)+(1-x)y+(1-x)(1-y)=1$

by $z+(1-z)=1$;

then $xyz+xy(1-z)+x(1-y)z+x(1-y)(1-z)+(1-x)yz$
$+(1-x)y(1-z)+(1-x)(1-y)z+(1-x)(1-y)(1-z)=1.$

It may be shown, as before, that xyz, $xy\,(1-z)$, etc., are each positive; and that they are exclusive of one another. Hence they are the primary parts, into which the universe is divided by the three independent operations x, y, z.

And so for any number of independent symbols.

151. The mode, in which three independent symbols divide a universe, can be well illustrated by means of a cube. The cube is supposed to contain a large number of individuals. The selection of the units according to the character x breaks up the cube into two parts, which are represented in fig. 6, as separated from one another by a single plane. The selection according to y again breaks up each of these two portions into two portions; the separating partition consists of two parts, which are not necessarily in one plane. Though the value of y is fixed (that of x being supposed to be fixed already) the two parts of the y partition are still free to vary, provided they cut off a sum equal to y. When in addition the value of one of the four parts, as xy, is given, the values of the other three follow necessarily. The selection according to the third character z breaks up the whole into eight compartments. The partition separating the z from the not z consists of four parts. When the value of z is fixed and of xz, yz, and xyz, then the eight parts are fully determined. The value of U is the remaining condition required to make up the eight.

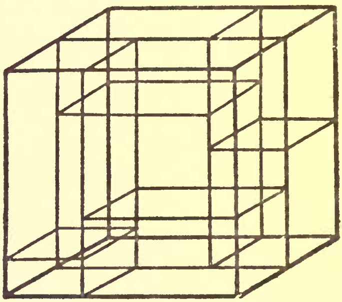


FIG. 6.

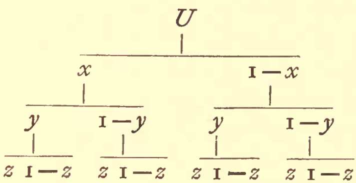
152. *The number of parts, into which a universe is primarily divided by n independent operations, is* 2^n.

For the parts, as above shown, are obtained by multipliers, each of which consists of two terms; and there is one multiplier for each symbol; hence when there are n symbols, there are 2^n parts.

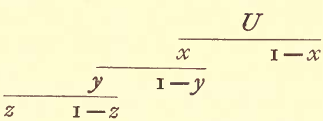
153. *Any single positive symbol, a, may be divided in the same manner as* 1.

For $$a(x+1-x)=a,$$
and $$a(y+1-y)=a,$$
therefore $$a^2\{xy+x(1-y)+(1-x)y+(1-x)(1-y)\}=a^2;$$
that is, $$a\{xy+x(1-y)+(1-x)y+(1-x)(1-y)\}=a.$$

154. The tree of Independent Division may be represented thus.



and so on. This form is exact and much more general than the tree of Porphyry. Porphyry's tree is a division of the particular universe of Substances, and is a mixture of the formal and the material. Only the first two parallel rows are given; which correspond to



the remaining part of the tree being left undeveloped.

155. Formal Division, when thus treated, is not so unimportant as it is commonly supposed to be. Were it kept in mind by men of science when classifying a given group of objects, according to a number of independent characters

ıld gain in clearness and exactness. ›ught first to be made ; then such parts, ı be left out of the *material* division. ınt to remember what can, and what the Algebra of Quality. By it we can

$+(1-x)y+(1-x)(1-y)=1$;

can tell us what, in any given case, or enable us to state any relation But there is nothing to prevent the supposing certain relations to be given proceeding to show what necessarily

succession of throws with two coins. the first head up, h_2 having the second the first tail up, and t_2 having the

$-t_1=1$, from the nature of the throw,
$-t_2=1$, " "

$+h_1t_2+t_1h_2+t_1t_2=1.$

ɛrs h_1h_2, h_1t_2, t_1h_2, t_1t_2 are thus subject : their sum is equivalent to the whole. completely, three independent data

ɛd that 'having the head of the first ; the head of the second coin up' are cters, and therefore must be denoted . The reason why they are denoted . *h* in the Theory of Probability, is, e arithmetical values, when *U* is sufficial.

conditions, sufficient to determine ȳ, are not given, we may consider ıssumptions to make—what are the ʼill lead us least astray. In making these assumptions we are guided by our knowledge of the nature of U and its characters h_1, h_2, etc.

The arithmetical values of the parts may be capable of determination *a priori*, when U is indefinitely large. In the above example, each of the parts will then be $=\frac{1}{4}$.

159. In the Linnæan Classification of Plants we have an example of a formal division by means of two characters and their subordinate characters. Linnæus considered the universe of

Plants having flowers

having stamens of equal length × having pistils.

Then

$$\text{Monandria} + \text{Diandria} + \text{Triandria} + \ldots$$
$$\ldots + \text{Dodecandria} + \text{Polyandria} = 1 \quad (1);$$

and

$$\text{Monogynia} + \text{Digynia} + \text{Trigynia} + \ldots$$
$$\ldots + \text{Dodecagynia} + \text{Polygynia} = 1 \quad (2).$$

By multiplying these two equations together we get the several formal classes.

If we consider the universe of

Plants having flowers;

then 'having stamens' and 'having pistils' are two independent characters; and 'one in number,' 'two in number,' etc., are dependent characters. The terms of (1) are necessarily exclusive of one another. They would not be so, if they meant

$$\text{having stamens}_{\text{one at least}}, \text{ having stamens}_{\text{two at least}}, \text{ etc.}$$

160. *Definition of Contrary.*—Let x denote a single positive selective operation; then $1-x$ is its formal contrary.

Observation.—If y is such that with respect to the given universe

$$y = 1 - x$$

in reality; then y is the *material*, but not the formal contrary of x. For example; in the case of the succession of throws with a couple of coins, considered in Art. **157**,

$$h_1 + t_1 = 1;$$
$$\therefore t_1 = 1 - h_1.$$

That is, 'having the tail of the first coin up' is identical with 'not having the head of the first coin up;' and it is therefore the material contrary of 'having the head of the first coin up.'

161. Consider the parts of the division

$$xy + x(1-y) + (1-x)y + (1-x)(1-y) = 1.$$

$x(1-y)$ agrees with xy in the factor x, but has the contrary factor as regards y. Hence $x(1-y)$ may be said to be a contradictory of the first degree of xy; and xy, a contradictory of the first degree of $x(1-y)$. Again; $(1-x)(1-y)$ is contrary to xy, both with respect to x, and with respect to y: hence it is a contradictory of the second degree of xy.

Definition.—A term, which is a product of factors each of the form x or $1-x$, is said to be *of the n^{th} order*, when it contains n such factors.

Definition.—Two terms of the n^{th} order are said to involve with respect to one another *a contradiction of the r^{th} degree*, when r factors of the one are contrary to r factors of the other.

162. *Definition.*—Two terms of the n^{th} order are said to be *opposites*, when they involve a contradiction of the n^{th} degree.

For example; xy and $(1-x)(1-y)$ in the one diagonal of the square (fig. 5) are opposites: so also $x(1-y)$ and $(1-x)y$ in the other diagonal. The two terms, which lie in any one of the four diagonals of the cube (fig. 6), are opposite to one another.

Since two opposite terms have $r=n$, we can speak of an opposition of the n^{th} degree.

Cor. An opposition of the first degree is a contrariety.

163. If it is objected that the above definitions are not in strict accordance with usage, I answer that they are at least exact and equal to the complexity of nature. It is impossible to use words in strict accordance with a usage which is contradictory. What some have denoted by contrary, others have denoted by contradictory.

The importance of an exact definition of these terms will be seen from a consideration of the use De Morgan makes of contrary. He says what is equivalent to the assertion that $(1-x)(1-y)$ is the contrary of $x+y$. Now

$$(1-x)(1-y)=1-x-y+xy,$$
$$=1-(x+y),$$

provided $$xy=0.$$

But xy is not necessarily $=0$. Hence if $1-x$ is defined to be the contrary of x; $(1-x)(1-y)$ is not the contrary of $x+y$.

164. *In a division by n independent selective symbols, each primary term has*

$$\frac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}$$

contradictories of the r^{th} degree.

The primary terms are formed by multiplying together n expressions, each of which is the sum of a symbol taken directly and the same symbol taken contrarily (Art. **148**). Let a denote directly and b contrarily; then the expansion of

$$(a+b)^n,$$

by the Binomial Theorem, groups together the contradictories of the different degrees.

a^n	denotes the particular term considered;			
$a^{n-1}b$	,,	a contradictory of the first degree;		
$a^{n-2}b^2$	,,	,,	second	,,
$a^{n-r}b^r$	,,	,,	r^{th}	,,
b^n	,,	,,	n^{th}	,,

Now

$$(a+b)^n=a^n+na^{n-1}b+\ldots+\frac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}a^{n-r}b^r+\ldots+b^n.$$

Hence the number of contradictories of the r^{th} degree with respect to the given term is

$$\frac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}.$$

COR. 1. A term has but one opposite.

For when $r=n$,

$$\frac{n(n-1)\dots(n-r+1)}{1.\ 2\ \dots\ r}=1.$$

COR. 2. When there are n symbols, there are 2^{n-1} pairs of opposites.

For each term has one opposite; and there are 2^n terms (Art. 152). But only one-half of these 2^n pairs will be different; hence the number of pairs of opposites is

$$2^{n-1}.$$

XIV. EXPANSION OF A FUNCTION OF A NUMBER OF INDEPENDENT SINGLE AND POSITIVE OPERATIONS IN TERMS OF THE PRIMARY PARTS INTO WHICH THE UNIVERSE IS DIVIDED BY THE OPERATIONS.

165. *To expand* $\frac{1}{x}$ *in terms of* x *and* $1-x$, *which are the parts into which the universe is divided by the selective operation* x.

$$\text{Let } \frac{1}{x}=ax+b\,(1-x);$$

a and b being independent of x.

The symbol x, being single and positive, can assume the value 1 and the value 0. Hence the above equation, being an identity, must be true when either of these values is substituted for x.

Let $x=1$. Then $1-x=0$; and $b0$ also $=0$, for b is an absolute constant. Hence $a=1$.

Let $x=0$. Then $a0=0$, because a is also an absolute constant. Hence $b=\frac{1}{0}$.

$$\text{Thus } \frac{1}{x}=x+\frac{1}{0}\,(1-x).$$

Observation 1.—The above is the expansion for $\frac{1}{x}$, whether $\frac{1}{x}$ occur by itself or as a factor in an expression. If it occur by itself, and is limited to being a possible operation; then $1-x$, being the factor of $\frac{1}{0}$, must be $=0$ (Art. 79). Hence $x=1$.

Observation 2.—The general expansion for x is

$$x=x+0\,(1-x),$$

which is true whether x occur by itself or as a factor.

166. *To expand* $\frac{x}{x}$.

$$x=x+0(1-x),$$

$$\text{and } \frac{1}{x}=x+\frac{1}{0}(1-x);$$

$$\therefore\ x\frac{1}{x}=x^2+0\ x(1-x)+\frac{1}{0}x(1-x)+\frac{1}{0}0(1-x)^2.$$

$$\text{Now } x^2=x, \qquad \text{given condition.}$$

$$\therefore\ (1-x)^2=1-x\,;$$

$$\text{and } x(1-x)=0.$$

Hence the third term consists of an absolute constant $\frac{1}{0}$ multiplied by 0; it therefore vanishes. The fourth term has the constant $\frac{1}{0}$ multiplied by a constant 0; its co-efficient is therefore $\frac{0}{0}$.

$$\text{Thus } \frac{x}{x}=x+\frac{0}{0}(1-x).$$

This expansion, of course, can be obtained directly.

Cor. 1. $\frac{x}{x}$ is always possible. For no impossible co-efficients occur in its expansion.

Cor. 2. $\frac{x}{x}$ is not always single.

For example; suppose that we consider

$$x = x(y+z),$$

where y and z denote the circles of fig. 7, and x the shaded parts of these circles.

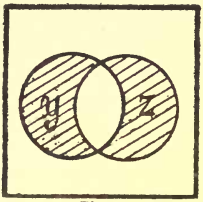


Fig. 7.

Then $y+z = \frac{x}{x}$,

$$= x + \frac{0}{0}(1-x),$$

for x is single and positive.

Now $y+z$ recurs; hence $\frac{x}{x}$ is not necessarily single. In the case of $\frac{x}{x}$ represented by the figure,

$$\frac{0}{0}(1-x) = 2yz.$$

COR. 3. When $\frac{x}{x}$ is single; $\frac{0}{0}$ is restricted to being single within $1-x$.

For each of the terms of the expansion is positive only; hence each must be single (Art. **145**). Therefore $\frac{0}{0}$ must be single within $1-x$ (Art. **139**).

167. *When* $\frac{x}{x}$ *is taken indefinitely,* $\left(\frac{x}{x}\right)^2 = \frac{x^2}{x^2}$.

For $\left(\frac{x}{x}\right)^2 = \left\{ x + \frac{0}{0}(1-x) \right\}^2$,

$$= x^2 + 2\frac{0}{0}x(1-x) + \left(\frac{0}{0}\right)^2(1-x)^2,$$

$$= x + \left(\frac{0}{0}\right)^2(1-x); \qquad \text{for } x^2 \text{ is given } = x.$$

Now $\left(\frac{0}{0}\right)^2$ cannot be more indeterminate than $\frac{0}{0}$.

(Art. **80**.)

$\therefore \left(\frac{x}{x}\right)^2$ taken generally $= x + \frac{0}{0}(1-x)$.

$$\text{But } \frac{x^2}{x^2} = \frac{x}{x}, \text{ for } x^2 = x;$$

$$= x + \frac{0}{0}(1 - x).$$

$$\text{Hence } \left(\frac{x}{x}\right)^2 \text{ taken generally } = \frac{x^2}{x^2}.$$

168. *To expand* $x+y$.

Let $x+y = axy + bx(1-y) + c(1-x)y + d(1-x)(1-y)$.

Since x and y are each positive, they can each assume the values 1 and 0. If they are not restricted by any condition, there are four sets of singular values, viz. :—

$x=1$	$y=1$	(1)
$x=1$	$y=0$	(2)
$x=0$	$y=1$	(3)
$x=0$	$y=0$	(4)

The above equation, being an identity, is true when any one of these sets of values is substituted for x and y. Put in the first set, then

$$2 = a + b0 + c0 + d0.$$

Since b, c, d, do not involve any variable;

$$b0 = 0, \; c0 = 0, \; d0 = 0.$$

Hence $$a = 2.$$

Similarly by means of the other sets of values, we get

$$b = 1; \; c = 1; \; d = 0.$$

Therefore $x+y = 2xy + x(1-y) + (1-x)y + 0(1-x)(1-y)$.

Observation 1.—The above is the general expansion for $x+y$, that is, the expansion which is true, whether $x+y$ occur by itself or as a factor. In the former case the equation can be reduced to

$$x+y = 2xy + x(1-y) + (1-x)y.$$

Observation 2.—If $x+y$ be restricted to be single, then each of the terms must be single. (Art. **145**.) Hence $2xy$ must be single; which necessitates

$$xy = 0.$$

ı that $x+y$ is single; we can expand t datum.

rt. 136); hence

$$y)+c(1-x)y+d(1-x)(1-y).$$

sets of singular values are possible,

$=1 \qquad y=0$

$=0 \qquad y=1$

$=0 \qquad y=0.$

e to determine the three constants $b, c, d.$

$-y.$

independent symbols, there are four can form four equations to determine as in the case of $x+y$. We get

$, b=1, c=-1, d=0.$

$$-x(1-y)-(1-x)y+0(1-x)(1-y).$$

by itself, the expansion can be re-

$$=x(1-y)-(1-x)y.$$

$$=x+0(1-x),$$

$$=y+\frac{1}{0}(1-y), \qquad \text{(Art. 165.)}$$

ı

$$y)+0(1-x)y+\frac{1}{0}\,0\,(1-x)(1-y).$$

$$y)+0(1-x)y+\frac{0}{0}(1-x)(1-y).$$

he factor 0 is not a part but a co-

s an expression by itself, the third

.

COR. 2. When in addition $\frac{x}{y}$ is possible, the second term may be cancelled.

For the expansion must be possible, and it contains only positive terms; hence each term must be possible. Therefore $x(1-y)=0$ (Art. 79).

Now $\frac{1}{0}$ is an absolute constant, since we are supposing $\frac{x}{y}$ to occur by itself; hence the $\frac{1}{0}\times 0$ formed by the second term is $=0$.

The truth of this last statement may be seen better by means of the following considerations.

When $\frac{x}{y}$ is restricted to be possible, y cannot take the value 0 without x taking the value 0 simultaneously. Hence we have only three sets of singular values, viz. :—

$$x=1 \qquad y=1$$
$$x=0 \qquad y=1$$
$$x=0 \qquad y=0.$$

These ought to be sufficient to determine the expansion; but they will be so, only if we write

$$\frac{x}{y}=axy+c(1-x)y+d(1-x)(1-y);$$

where the second term of the general expression is omitted.

$$\text{Hence } \frac{x}{y}=xy+\frac{0}{0}(1-x)(1-y).$$

COR. 3. If in addition $\frac{x}{y}$ be restricted to be single, then the fourth term of the expansion is single (Art. 145).

171. *To expand $\frac{x}{y}$ with respect to y alone.*

$$\text{Since } \frac{1}{y}=y+\frac{1}{0}(1-y).$$

$$\frac{x}{y}=x\left\{y+\frac{1}{0}(1-y)\right\}.$$

If $\frac{x}{y}$ is possible, then the factor multiplying $\frac{1}{0}$ must be $=0$. Hence $x(1-y)=0$.

$$\therefore \frac{x}{y}=xy+\frac{0}{0}.$$

We are not entitled to say that the $\frac{0}{0}$ which occurs here is $=0$; because the 0 of the numerator depends upon the co-efficient $\frac{x}{0}$ as well as upon $1-y$.

If in addition $\frac{x}{y}$ is single; then $\frac{0}{0}=\frac{0}{0}(1-xy)$.

But $xy=x$, for $x(1-y)=0$;

$$\therefore \frac{x}{y}=xy+\frac{0}{0}(1-x).$$

This expansion differs from that obtained by expanding with respect to both x and y in not having the second term multiplied by $1-y$. It is therefore less definite.

172. *To expand* $\frac{1}{x+y}$.

By applying the method of Art. **168** we obtain

$$\frac{1}{x+y}=\frac{1}{2}xy+x(1-y)+(1-x)y+\frac{1}{0}(1-x)(1-y).$$

This is the general expansion for $\frac{1}{x+y}$.

When we consider $\frac{1}{x+y}$ by itself, and restrict it to be possible; then the constituent multiplying the co-efficient $\frac{1}{0}$ must be $=0$ (Art. **79**). Also the constituent multiplying the co-efficient $\frac{1}{2}$ must be $=0$; for xy does not contain 2 as a factor (Art. **78**).

173. *To expand* $\frac{x+y}{x+y}$.

Since $x+y=2xy+x(1-y)+(1-x)y+0(1-x)(1-y)$,

(Art. **168**.)

and $\frac{1}{x+y}=\frac{1}{2}xy+x(1-y)+(1-x)y+\frac{1}{0}(1-x)(1-y)$;
(Art. 172.)

multiply together, then

$$\frac{x+y}{x+y}=\frac{2}{2}xy+x(1-y)+(1-x)y+\frac{0}{0}(1-x)(1-y),$$

for all the other terms vanish, on account of their constituents being $=0$;

but $\frac{2}{2}=1$ (Art. 78),

$$\therefore \frac{x+y}{x+y}=xy+x(1-y)+(1-x)y+\frac{0}{0}(1-x)(1+y).$$

This result may be easily verified by expanding directly. I have deduced it by means of the results of Arts. 168 and 172, for the purpose of showing that when the function expanded is a factor of an expression, its true expansion consists of all the terms, even though some of these have impossible or zero co-efficients.

174. *To expand* $\frac{x}{y+z}$.

Since $x=x+0(1-x)$, (Art. 165.)

and $\frac{1}{y+z}=\frac{1}{2}yz+y(1-z)+(1-y)z+\frac{1}{0}(1-y)(1-z)$
(Art. 172.)

$$\frac{x}{y+z}=\frac{1}{2}xyz+xy(1-z)+x(1-y)z+\frac{1}{0}x(1-y)(1-z)$$
$$+\frac{0}{2}(1-x)yz+0(1-x)y(1-z)+0(1-x)(1-y)z$$
$$+\frac{0}{0}(1-x)(1-y)(1-z).$$

When $\frac{x}{y+z}$ occurs by itself or multiplied by an absolute constant, the fifth, sixth, and seventh terms vanish. (Art. 79.)

When in addition $\frac{x}{y+z}$ is possible, the constituents of the first and fourth terms must each be equal to 0; and therefore the terms themselves equal to 0. Then

$x(1-y)z+\frac{0}{0}(1-x)(1-y)(1-z);$

itions $xyz=0$, and $x(1-y)(1-z)=0$.

s single, then $\frac{0}{0}(1-x)(1-y)(1-z)$

DEFINITION.

on is the establishment by conven-
between a complex expression and a

t may be said—

Let $1-x=x'$.

is established between the significa-
racter $1-x$ and the simple character
, if adopted in a particular course of
ered to throughout; but it is some-
or one course of reasoning only, but

established for the sake of gaining
d also for the purpose of expressing
e) the result of the complex expres-

y

Let $x_y=x'$.

e symbol the result of the product of
r y upon the character x, on which

tional equation, usually considered
ead of 'Definition,' is related to the
pe is

Let $Ux=U'$.

ual symbol denoting the universe $\dot{U}x$.

Conventional equations of this kind are usually of a permanent nature.

178 As an example of

$$U' = Ux,$$

I take the definition of 'Vertebrata' given by Nicholson (*Manual of Zoology*, p. 383). When expressed analytically, it is as follows:—

Vertebrata
=Animals {having a body composed of a number of definite segments arranged along a longitudinal axis}
× having a nervous system
in its main masses dorsal × completely shut off by a partition from the hæmal region of the body
× having limbs
not more than four in number × turned away from the neural aspect of the body
× {having a bony axis known as the vertebral column + without a vertebral column × having a notochord}.

Here x is of the form $a_x\, b_{yz}\, c_{(1-u)\,v}\{d+(1-d)e\}$; and affords a good example of the import of dependent characters.

XVI. INFERENCE FROM ONE OR MORE EQUATIONS OF THE FORM $x=m$; THAT IS, FROM CATEGORICAL JUDGMENTS.

179. *Form of the equation.* An unknown single character of the first order and with unity for its co-efficient is equated to a known character, or else to a known arithmetical value. These are written

$$x=m,$$

$$\text{and } x=\overline{m},$$

respectively. The former is read

U's which are x are identical with *U*'s which are m;

and the latter

U's which are x have the arithmetical value of *U*'s which are m.

It is generally sufficient, when an investigation has to be made, to make it for the former kind of equation only; as the result obtained is generally true for the latter kind of equation, provided m be changed into $\bar{m}$.

180. The primary categorical forms comprise four special cases of the equation

$$x=m.$$

When

$m=1$; the equation becomes the universal affirmative.
$=0$; ,, universal negative.
$=v$ (a fraction greater than 0) ,, particular affirmative.
$=1-v$; ,, particular negative.

Hence $x=m$ may be called the general form of the categorical judgment. $\bar{m}$ is the definite *some* considered in the Theory of Probability.

181. The term *categorical* is frequently ascribed to judgments which differ in an important respect from those to which the term is primarily defined to apply. The forms

$$xy=1,\ xy=0,\ xy=v,\ xy=1-v$$

are not distinguished from the forms

$$x=1,\ x=0,\ x=v,\ x=1-v.$$

For instance; one of the examples of a universal negative categorical judgment given by Ueberweg is (*System of Logic*, p. 308)

No innocent person is unhappy.

But this is converted into

No unhappy person is innocent.

In the new form of the statement 'person' remains where it was before; hence 'persons' and not 'innocent persons' is the true subject of thought. The analytical expressions for these judgments are

Persons {innocent unhappy$=0$},

and Persons {unhappy innocent$=0$};

both of which are of the form

$$xy=0.$$

The above equation contains a term of the second order, and is therefore very different from one which contains a term of the first order only. The equation

$$xy = m,$$

where the term on the left-hand side is of the second order, may be called the general form of the hypothetical equation of the first degree.

182. *Immediate inference by Contraposition.*

$$\text{If } x = m,$$
$$\text{then } 1 - x = 1 - m. \qquad \text{(Art. 108.)}$$

In the same manner,

$$\text{if } x = \overline{m},$$
$$\text{then } 1 - x = 1 - \overline{m}.$$

The latter is a well-known proposition in the Theory of Probability, and is one of the logical theorems which are constantly assumed in stating arithmetical problems.

COR. 1. Let $x = 1$. Then $1 - x = 0$.

COR. 2. Let $x = v$, where v is greater than 0. Then $1 - x = 1 - v$; where $1 - v$ must be less than 1 and may be 0.

It is said that no inference can be drawn by contraposition from

$$x = v.$$

But this is not correct. It follows necessarily that

$$1 - x = 1 - v;$$

what does not follow, is, that $1 - v$ must be greater than 0 when v is greater than 0.

183. *Conversion of $x = m$.*

The only way in which the equation

$$x = m$$

admits of conversion, consists in exchanging the position of its members thus—

$$m = x.$$

It is only to such equations, as involve two unknowns x and

y, that a process of conversion in the proper sense of the term applies. For instance

$$xy=m$$

may be changed into $yx=m$;

and $x=y-w$

into $y=x+w$.

184. An attempt, however, is frequently made to convert the categorical form after the manner of an hypothetical of the first degree. It is, for instance, said that

All men are fallible;

can be converted into

Some fallibles are men.

But several logicians have observed that such a procedure confounds Accident with Substance. The universe of the judgment is confounded with the characters of the judgment; selective symbols are changed into elective symbols. The result in the above or any other instance does not recommend the process. The given judgment really means

Men {1=fallible};

which can be written

Men {fallible=1}.

The inference by contraposition is

Men {0=not fallible}.

185. It is important to consider the differences in the signification of

$$x=1,$$

according to the nature of U as regards variables. If *Personality* (which is the logical Space) is constant, and *State* (which is the logical Time) varies; then

$$x=1$$

expresses that in every state of the given individual the character x is present.

Similarly, when the State is fixed, and the Personality varies;

$$x=1$$

expresses that in every individual in the given state, the character x is present.

When both Personality and State vary;

$$x=1$$

expresses that in every individual in every state, the character x is present.

Examples of these respective cases are—

This crow is always black.
All crows at a given age are black.
All crows are always black.

186. The formal treatment of the equation

$$x=1$$

is the same, whether the equation express an essential property, a general property, or an inseparable accident (for the given universe). The Algebra of Logic makes no difference in this respect between the *necessary truth*

Triangles {equilateral = equiangular},

and the *observed fact*

Animals {having condyles on the occipital bone two in number × possessing red blood corpuscles non-nucleated = suckle their young.}

187. *Given* $x=m$ *and* $y=n$; *to find what can be concluded about* xy, $x(1-y)$, $(1-x)y$, *and* $(1-x)(1-y)$.

First, for xy.

$$\text{Since } (1-x)(1-y)=1-x-y+xy,$$

$$xy=x+y-1+(1-x)(1-y);$$

but $(1-x)(1-y)$ is positive only; (Arts. **120** and **77**.)

therefore xy is greater than $x+y-1$;

that is, $xy>x+y-1$. (Art. **146**.)

Again; since $x(1-y)=x-xy$,

$$xy=x-x(1-y);$$

but $x(1-y)$ is positive only; (Art. **120**.)

therefore xy is less than x;

that is, $xy<x$. (Art. **146**.)

Similarly it may be shown that $xy<y$.

Second, for $x(1-y)$.

$$\text{Since } xy=x+y-1+(1-x)(1-y),$$
$$\text{let } y \text{ be changed into } 1-y;$$
$$\text{then } x(1-y)=x-y+(1-x)y;$$
$$\text{but } (1-x)y \text{ is positive only;}$$
$$\text{therefore } x(1-y)>x-y.$$

And in a similar manner it may be shown that

$$x(1-y)<x;$$
$$\text{and } <1-y.$$

Third, for $(1-x)y$.

$(1-x)y$ is the same as $x(1-y)$, excepting that x and y have replaced one another. Hence

$$(1-x)y>-x+y;$$
$$<1-x;$$
$$<y.$$

Fourth, for $(1-x)(1-y)$.

Change x into $1-x$, and y into $1-y$, in the limits for xy. Then

$$(1-x)(1-y)>-x-y+1;$$
$$<1-x;$$
$$<1-y.$$

188. Hence by putting in the given data

$$x=m, \text{ and } y=n,$$

we obtain

$$xy>m+n-1;$$
$$<m;$$
$$<n.$$
$$x(1-y)>m-n;$$
$$<m;$$
$$<1-n.$$
$$(1-x)y>-m+n;$$
$$<1-m;$$
$$<n.$$
$$(1-x)(1-y)>-m-n+1;$$
$$<1-m;$$
$$<1-n.$$

Observation.—When the data are of the form

$$x=\overline{m}, \qquad y=\overline{n},$$

the above inequalities are still true, provided the right-hand members be stroked with a vinculum.

189. From the above we infer; that any two terms, which involve two degrees of contradiction, have the sum of their minor limits $=0$; and the sum of their corresponding major limits $=1$. Also that any two terms, which involve but one degree of contradiction, have the sum of their minor limits $=\pm\left\{2\binom{m}{n}-1\right\}$ and the sum of their corresponding major limits $=2\left\{\begin{matrix} m \\ n \\ 1-m \\ 1-n \end{matrix}\right\}$, or 1.

190. Any one of the equations

$$xy=x+y-1+(1-x)(1-y),$$
$$xy=x-x(1-y),$$
$$xy=y-y(1-x),$$

suffices to give xy exactly, provided each one of the terms of the right-hand member is given as a datum.

191. *Example.*—Suppose that of the persons on board a ship which was wrecked, the passengers formed two-thirds; and those that were saved in the wreck three-fourths. How many passengers must have been saved, how many lost; how many of the crew must have been saved, how many lost?

Let p denote the passengers; c the crew; s saved; l lost.

Then $p+c=1$; $\therefore c=1-p$.

Also $s+l=1$; $\therefore l=1-s$.

And $p=\overline{\tfrac{2}{3}}$; $s=\overline{\tfrac{3}{4}}$.

Now $ps>p+s-1$,

$$\therefore >\overline{\tfrac{2}{3}}+\overline{\tfrac{3}{4}}-\overline{1},$$
$$\therefore >\overline{\tfrac{5}{12}}.$$

The passengers saved formed at least five-twelfths.

Since the limit of $(1-p)(1-s)$ is the same as that for ps but with a contrary sign;

$$(1-p)(1-s) > -\tfrac{5}{12}.$$

The crew lost may have been none.

$$\text{Again.}\quad (1-p)s > -p+s,$$
$$\therefore\ > -\tfrac{2}{3}+\tfrac{3}{4},$$
$$\therefore\ > \tfrac{1}{12}.$$

The crew saved formed at least one-twelfth.

Since the limit for $p(1-s)$ is the same as that for $(1-p)s$ but taken with the contrary sign,

$$p(1-s) > -\tfrac{1}{12}.$$

The passengers lost may have been none.

192. Suppose the additional information given:—that the crew lost formed one-sixth.

$$\text{Then } (1-p)(1-s)=\tfrac{1}{6}.$$
$$\text{Now } ps=p+s-1+(1-p)(1-s), \quad \text{(Art. 190.)}$$
$$=\tfrac{2}{3}+\tfrac{3}{4}-1+\tfrac{1}{6},$$
$$=\tfrac{7}{12}.$$
$$\text{Also } p(1-s)=1-s-(1-s)(1-p), \quad \text{(Art. 190.)}$$
$$=\tfrac{1}{4}-\tfrac{1}{6},$$
$$=\tfrac{1}{12}.$$
$$\text{Also } (1-p)s=1-p-(1-p)(1-s), \quad \text{(Art. 190.)}$$
$$=\tfrac{1}{3}-\tfrac{1}{6},$$
$$=\tfrac{2}{12}.$$

Suppose in addition that the persons on board were 120 in number.

$$\text{Then}\quad U=120.$$
$$\text{And}\quad Ups=70. \qquad (1)$$
$$Up(1-s)=10. \qquad (2)$$
$$U(1-p)s=20. \qquad (3)$$
$$U(1-p)(1-s)=20. \qquad (4)$$

Also, when the data of Art. 191 only are given;

$$Ups > 50. \qquad (1)$$
$$U(1-p)s > 10. \qquad (2)$$

193. *Given*

$$x=m\,;\ y=n\,;\ z=p\,;$$

to find what can be concluded about xyz.

Since $$xy=x+y-1+(1-x)(1-y)\,;$$
multiply by z,

then $$xyz=xz+yz-z+(1-x)(1-y)z. \quad \text{(Art. 108.)}$$
But $$xz=x+z-1+(1-x)(1-z),$$
and $$yz=y+z-1+(1-y)(1-z),$$
$$\therefore\ xyz=x+y+z-2+(1-x)(1-z)+(1-y)(1-z)$$
$$+(1-x)(1-y)z. \qquad (1)$$

Now $(1-x)(1-z)+(1-y)(1-z)+(1-x)(1-y)z$, being a sum of terms each of which cannot be negative, cannot itself be negative;

$$\therefore\ xyz>x+y+z-2. \qquad \text{(Art. 146.)}$$

Again, $$xy=x-x(1-y)\,;$$
multiply by z,

then $$xyz=xz-x(1-y)z\,; \qquad (2)$$
$$=x-x(1-z)-x(1-y)z$$
$$<x.$$

For the remainder is minus the sum of two terms each of which can be positive only. (Art. **146.**)

Similarly it can be proved that

$$xyz<y, \text{ and } <z.$$

Hence, by putting in the given values,

$$xyz>m+n+p-2.$$
$$<m.$$
$$<n.$$
$$<p.$$

194. Other two equations may be derived from (1) by cyclical change of the symbols; but, as the terms of the first order enter similarly into (1), the derived equations will give the same limit as (1) gives.

Also, two other equations may be derived from (2) by cyclical change of the symbols. Each of these yields two forms; but only three limits of the kind considered can be derived.

195. *The limits for any of the other terms of the third order can be deduced from those for xyz.*

For example; to find those for $x(1-y)(1-z)$.

$$1-y=1-n,$$

$$\text{and } 1-z=1-p.$$

$$\therefore x(1-y)(1-z) > m+1-n+1-p-2;$$

$$> m-n-p.$$

Also. $x(1-y)(1-z) < m.$

$$< 1-n.$$

$$< 1-p.$$

196. *Generally*

$$x_1 x_2 x_3 \ldots x_n > x_1 + x_2 + x_3 + \ldots + x_n - n + 1.$$

Suppose it true for n; and multiply by x_{n+1}.

Then

$$x_1 x_2 \ldots x_n x_{n+1} > \{x_1 + x_2 + x_3 + \ldots + x_n - n + 1\} x_{n+1},$$

$$> x_1 + x_2 + x_3 + \ldots + x_n + (n - n + 1) x_{n+1} - n,$$

(Art. 187.)

$$> x_1 + x_2 + x_3 + \ldots + x_n + x_{n+1} - (n+1) + 1.$$

Hence if the inequality is true for n, it is also true for $n+1$.

But it is true for 2; hence it is true generally.

Cor. If r of the factors are direct, and $n-r$ contrary; the minor limit becomes

$$x_1 + x_2 + \ldots + x_r - x_{r+1} - x_{r+2} - \ldots - x_n - r + 1.$$

For the minor limit is

$$x_1 + x_2 + \ldots + x_r + 1 - x_{r+1} + \ldots + 1 - x_n - n + 1,$$

$$= x_1 + x_2 + \ldots + x_r - x_{r+1} - \ldots - x_n + n - r - n + 1,$$

$$= \quad ,, \quad ,, \quad -r+1.$$

197. De Morgan makes some investigations upon the subject of the preceding ten articles under the name of Numerically Definite Syllogisms. The fundamental data with which he deals are inequalities, and he always supposes m to be arithmetical. But there is nothing to prevent us from supposing that the fundamental data are equations, and investigating what follows. A slight reflection upon the nature of statistical data will serve to show the importance of such an investigation.

198. *Given* $a_x = m$, $a_y = n$, $a = p$; *to find limits to* a_{xy}.

$$a_{xy} = a_{x+y-1+(1-x)(1-y)},$$
$$= a_x + a_y - a + a_{(1-x)(1-y)}. \qquad \text{(Art. 70.)}$$
$$\therefore \quad a_{xy} > a_x + a_y - a.$$
$$\text{Also } a_{xy} = a_{x-x(1-y)},$$
$$= a_x - a_{x(1-y)}. \qquad \text{(Art. 70.)}$$
$$\therefore \quad a_{xy} < a_x.$$
$$\text{Similarly } a_{xy} < a_y.$$
$$\text{Hence } a_{xy} > m + n - p.$$
$$\text{but} \quad < m.$$
$$\text{and} \quad < n.$$

COR. Let $a = 1$.
Then we get the results of Art. 188.

199. *To find limits to* $\frac{x}{y}$, *when it is single.*

Since $\frac{x}{y}$ is given to be single.

$$\frac{x}{y} = xy + \frac{0}{0}(1-x)(1-y),$$

where $\frac{0}{0}(1-x)(1-y)$ is single. (Art. 170, COR. 3.)

Hence $\frac{0}{0}(1-x)(1-y) < 1-x$ and $< 1-y$;

also $xy < x$ and $< y$;

$$\therefore \frac{x}{y} < x + 1 - x, \textit{ i.e. } < 1. \qquad (1)$$
$$< x + 1 - y. \qquad (2)$$
$$< 1 - x + y. \qquad (3)$$
$$< y + 1 - y, \textit{ i.e. } < 1. \qquad (4)$$

But as $xy = x$ (Art. 170), the expansion can be reduced to

$$\frac{x}{y} = x + \frac{0}{0}(1-y); \qquad (5)$$

therefore, the important limits are—

$$\frac{x}{y} < x + 1 - y,$$

$$\text{and } \frac{x}{y} > x.$$

COR. 1. Let $y=1$.

Then equation (5) becomes $\frac{x}{y}=x$.

COR. 2. Let $x=1$ and $y=1$.

Then equation (5) becomes $\frac{x}{y}=1$.

XVII. INFERENCE FROM ONE OR MORE EQUATIONS OF THE FORM $xy=m$; THAT IS, FROM HYPOTHETICAL JUDGMENTS.

200. *Form of the equation.*—We have an unknown single character of the second order equated to a known symbol, which may either be a symbol of identity or express arithmetical value.

$$xy=m$$

means that

U's which are x and y are identical with U's which are m.

$$xy=\overline{m}$$

means that

U's which are x and y have an arithmetical value equal to that of Um; or, simply, that xy has an arithmetical value $\underline{m}$.

201. A particular case is

$$xy=x;$$

which is an equation *universal* with respect to x, and means that

U's which are x are y.

$$xy=y$$

is the other universal form.

$$xy=0$$

means that

U's which are x and y are none.

202. *Varieties of the equation $xy=m$.*

There are four varieties—

$$xy=m. \qquad (1)$$
$$x(1-y)=m. \qquad (2)$$
$$(1-x)y=m. \qquad (3)$$
$$(1-x)(1-y)=m. \qquad (4)$$

Each of these has two universal forms, viz.

(1) $xy=x$; $xy=y$.
(2) $x(1-y)=x$; $x(1-y)=1-y$.
(3) $(1-x)y=1-x$; $(1-x)y=y$.
(4) $(1-x)(1-y)=1-x$; $(1-x)(1-y)=1-y$.

203. *The right-hand member, m, may assume a great variety of forms.*

For example—

$$xy=1-x.$$
$$xy=1-y.$$
$$xy=x+y-1.$$

Or we may have

$$xy=\bar{1}-\bar{x}.$$
$$xy=\bar{1}-\bar{y}.$$
$$xy=\bar{x}+\bar{y}-\bar{1}.$$

The latter examples differ from the former in that the equivalence asserted does not involve identity.

Observation.—The equation

$$xy=x+y-1$$

is an equation of Condition; whereas the equation

$$xy=x+y-1+(1-x)(1-y)$$

is an Identity.

204. *If $xy=m$, then $xy=\bar{m}$; but not conversely.*

For $xy=m$ expresses the identity of the U's which are x and y with those which are m. Hence the arithmetical value of xy must be equal to that of m; that is,

$$xy=\bar{m}.$$

But suppose it is given that

$$xy=\bar{m}.$$

Since xy may in general have the quantitative value $\bar{m}$ in a plurality of ways, and

$$xy=m$$

is only one of these ways; it does not follow that

$$xy=m.$$

COR. *If $xy=\bar{x}$, then $xy=x$.*

For xy can have the quantitative value $\bar{x}$ in only one way; and $$xy=x$$ is one way.

205. *There are certain forms of the equation*

$$xy=m,$$

which can be satisfied only by one or both of the symbols having a singular value.

First example; $xy=1-x$.

Then $xy+x=1$,

$\therefore xy=0$, (Art. **136**) for 1 is single.

$\therefore 1-x=0$, by given equation.

$\therefore x=1$;

and $\therefore y=0$.

Second example; $xy=x+y$.

Then $xy=0$, (Art. **136**.)

$\therefore x+y=0$,

$\therefore x=0$; and $y=0$.

For both terms are positive only.

Third example; $xy=x(1-y)$.

Then $2xy=x$,

$\therefore xy=0$; for x is single.

$\therefore x=0$.

y is left undetermined.

206. The equation may express a condition, which can be reduced to a simpler condition.

For example; let $xy=(1-x)(1-y)$.

Then $xy=1-x-y+xy$;

$\therefore 0=1-x-y$;

$\therefore x+y=1$.

207. *Given* $xy=m$;
to find $x(1-y)$, $(1-x)y$, $(1-x)(1-y)$ *in terms of* x, y, *and* m.

First, $$x(1-y)=x-xy,$$ $$=x-m.$$

Second, $$(1-x)y=y-m.$$

Third, $$(1-x)(1-y)=1-x-y+xy,$$ $$=1-x-y+m.$$

The last equation, which gives the expression for the opposite of the given term, is commonly called the inference obtained by Contraposition. The other two equations give expressions for those terms, which involve only one degree of contradiction.

COR. 1. Let $xy=x$.

Then $x(1-y)=0$. (1)

$(1-x)y=y-x$. (2)

$(1-x)(1-y)=1-y$. (3)

COR. 2. Let $xy=x+y-1$.

Then $x(1-y)=1-y$. (1)

$(1-x)y=1-x$. (2)

$(1-x)(1-y)=0$. (3)

COR. 3. Let $xy=0$.

Then $x(1-y)=x$. (1)

$(1-x)y=y$. (2)

$(1-x)(1-y)=1-x-y$. (3)

208. Let $xy=v$, where v is greater than 0.

Then $x(1-y)=x-v$;

but v, though restricted to being greater than 0, may be equivalent to x. Hence $x(1-y)$ is not necessarily greater than 0.

It may be shown in a similar manner that $(1-x)y$ is not necessarily greater than 0.

Again. $$(1-x)(1-y)=1-x-y+v;$$
but v may be equivalent to $x+y-1$,

$\therefore$ $(1-x)(1-y)$ is not necessarily greater than 0.

The last statement contains what is meant by the proposition, that the particular affirmative (hypothetical) judgment does not admit of Contraposition.

209. *The Conversion of an Equation consists in making two symbols replace one another as far as possible.*

As applied to equations of the form

$$xy=m$$

the Conversion may be *Simple*, or *per Accidens.*

210. The Conversion is *Simple*, when x and y enter similarly into the equation.

For example: if $xy=0$;

then $yx=0$.

211. The Conversion is *per Accidens*, when x and y do not enter similarly into the equation.

For example: let $xy=x$.

Then $yx=y-(y-x)$,

but $y>x$,

$\therefore y-x$ is in general greater than 0.

Hence the equation $yx=y$

does not in general follow; the additional condition required being

$$y=x.$$

212. *Given* $xy=m$, *and* $yz=n$;
to find minor limits to xz, $x(1-z)$, $(1-x)z$, *and* $(1-x)(1-z)$ *expressed in terms of* m, n, x, y, z.

First for xz.

Since $xz=x+z-1+(1-x)(1-z)$;

multiply by y,

then $xyz=xy+yz-y+(1-x)y(1-z)$,

$\therefore xyz>xy+yz-y$,

for $(1-x)y(1-z)$ is positive only.

But $xz>xyz$,

$\therefore xz>xy+yz-y$.

Hence, by putting in the given data,

$$xz>m+n-y.$$

Similarly; since

$$x(1-y)z > x(1-y)+(1-y)z-(1-y);$$

and $xz > x(1-y)z$;

$$\therefore xz > x(1-y)+(1-y)z-(1-y),$$

that is, $> -xy-yz-1+x+y+z$,

that is, $> -m-n-1+x+y+z$.

Second for $x(1-z)$.

Since $zy=n$,

$$(1-z)y=y-n.$$

Thus z and $1-z$ replace one another in one of the given conditions, and so also n and $y-n$. The results, when similarly transformed, will be true.

Hence $x(1-z) > m+y-n-y$,

that is, $> m-n$.

Also $x(1-z) > -m-(y-n)-1+x+y+(1-z)$,

that is $> -m+n+x-z$.

In the same manner it may be shown that

$$(1-x)z > -m+n;$$

and $> m-n-x+z$.

Also that

$$(1-x)(1-z) > -m-n+y;$$

and $> m+n+1-x-y-z$.

Cor. Let $y=1$.

Then $xz > m+n-1$, and > 0.

$x(1-z) > m-n$, and > 0.

$(1-x)z > -m+n$, and > 0.

$(1-x)(1-z) > -m-n+1$, and > 0.

These are the results of Art. **188**.

The second limit ought in each case to become 0; for when $y=1$, $1-y=0$.

213. *The sum of the two limits for any one of the terms is independent of* m *and* n.

Since $xyz > m+n-y$,

and $x(1-y)z > -m-n-1+x+y+z$,

add, then $xz > -1+x+z$;

which is independent of m and n.

By adding together the two limits of any other term, it will be found that the result is independent of m and n.

Hence; when each of the two limits is >0, each limit must be $<-1+x+z$.

When each of the two limits is <0, each must be $>-1+x+z$.

When one is >0, and the other <0; the former is $>-1+x+z$, and the latter is $<-1+x+z$.

214. *Given $xy=m$, and $(1-y)z=n$; to find minor limits to xz.*

Here the common symbol occurs contrarily in the two data; whereas in the former case (Art. **212**) it occurred similarly.

$$\text{Since } (1-y)z=n,$$
$$yz=z-n.$$

Hence, if we make n and $z-n$ replace one another in the results of the former case, these results become true for the present case.

$$\text{Thus } xz>m+(z-n)-y,$$
$$\text{that is, } >m-n-y+z.$$
$$\text{Also } xz>-m-(z-n)-1+x+y+z,$$
$$\text{that is, } >-m+n-1+x+y.$$

Similarly for the other terms.

215. *The sum of the corresponding limits of any two terms, which are opposite to each other, is equal to 0.*

Consider for example those limits of xz and of $(1-x)(1-z)$ which involve y.

They are obtained from the equation

$$y\{xz-(1-x)(1-z)=x+z-1\},$$

by supposing $(1-x)y(1-z)=0$ and $xyz=0$ respectively.

Hence they are equivalent but of opposite sign; and therefore their sum is $=0$.

216. *A limit, when it assumes either of the forms*

$$-m+x,\ -m+y,\ -n+y,\ -n+z,$$

cannot be negative.

Consider the first form.

$m = xy,$ by given equation.

and $xy < x,$ (Art. **187**.)

$\therefore m < x,$

that is, $x - m$ cannot be negative.

The same proof applies to the other forms.

Example of application.

Given $xy = m$ (1); $z(1-y) = 1-y$ (2).

From (1) $x(1-y) = x - m;$

$\therefore xz > x - m + 1 - y - (1-y),$ (Art. **212**.)

that is, $xz > x - m;$

but $x - m$ cannot be negative,

$\therefore$ the limit for xz cannot be negative.

217. *A limit, which is of either of the forms*

$$m - x - y + 1, \qquad n - y - z + 1,$$

cannot be negative.

Consider the first form.

$m = xy,$ by given equation.

and $xy > x + y - 1,$ (Art. **187**.)

$\therefore m > x + y - 1;$

that is, $m - x - y + 1$ cannot be negative.

It can be shown in a similar manner that $n - y - z + 1$ cannot be negative.

Example of application.

Given $xy = m$ (1); $yz = z$ (2).

From (1) $(1-x)(1-y) = 1 - x - y + m;$

from (2) $(1-y)(1-z) = 1 - y;$

$\therefore (1-x)(1-z) > 1 - x - y + m.$ (Art. **212**.)

$\therefore$ the limit for $(1-x)(1-z)$ cannot be negative.

218. *A limit, which is of either of the forms*

$$m, \qquad n,$$

cannot be negative.

For $m = xy$, and xy cannot be negative.

Example of application.

Given $xy = m$ (1); $yz = y$ (2).

therefore $xz > m + y - y,$

$> m.$

Hence the limit for xz cannot be negative.

219. *When the minor limit is equal to a factor of the term of which it is a limit, the inequivalence becomes an equivalence.*

For any term, which can form the left-hand member of the inequivalences considered, cannot be greater than any of its factors.

Example of application.

$$\text{Let } xy=x \quad (1); \qquad z(1-y)=z \quad (2).$$

$$\text{Then from (2)} \qquad (1-z)y=y;$$

$$\text{therefore } x(1-z)>x+y-y,$$

$$>x;$$

$$\text{but } x \text{ is a factor of } x(1-z),$$

$$\text{therefore } x(1-z)=x.$$

This corresponds to what is called a *universal* conclusion. Nothing can be more elegant than the manner in which the general analytical conclusion becomes universal, when the data necessitate the conclusion to be universal.

220. *When the limit is equal to the common symbol, or its contrary, the conclusion does not necessarily become universal.*

For neither y nor $1-y$ is a factor of the left-hand members of the inequivalences considered, viz. xz, $x(1-z)$, etc.

Example of application.

$$\text{Let } xy=y \quad (1); \qquad yz=y \quad (2).$$

$$\text{then } xz>y+y-y,$$

$$>y;$$

but y is not a factor of xz,

$$\therefore\ xz=y \text{ is not necessarily true.}$$

Observation.—If we consider the limits of xyz, then

$$xyz>y$$

$$\text{becomes } xyz=y.$$

221. *To find the condition, to which the sum or difference (as the case may be) of* m *and* n *must be subject, in order that a given limit may be positive only.*

First; when m and n each enter positively into the limit.

Example: $m+n-y.$

This expression cannot be negative, if $m+n>y$; that is, if $m+n$ includes y.

Second; when m and n each enter negatively into the limit.

Example: $-m-n-1+x+y+z$.

This expression cannot be negative, if $m+n<x+y+z-1$; that is, if $m+n$ is included in $x+y+z-1$.

Third; when m and n enter, the one positively, and the other negatively.

Example: $m-n-y+z$.

This expression cannot be negative, if $m-n>y-z$; that is, if $m-n$ include $y-z$.

Another form of writing this condition is

$$n-m<z-y,$$

where the signs of the members have been changed, and in consequence the sign of inequivalence also changed.

222. Suppose, for example, that

$$xy=m;\ yz=n.$$

Then the limit for	xyz	is positive only; if	$m+n>y$.
,,	$x(1-y)z$	,,	,, $<x+y+z-1$.
,,	$xy(1-z)$	,,	$m-n>0$.
,,	$x(1-y)(1-z)$	,,	,, $<x-z$.
,,	$(1-x)yz$	,,	,, <0.
,,	$(1-x)(1-y)z$	,,	,, $>x-z$.
,,	$(1-x)y(1-z)$	,,	$m+n<y$.
,,	$(1-x)(1-y)(1-z)$	,,	,, $>x+y+z-1$.

Again; suppose that

$$xy=m;\ (1-y)z=n.$$

Then the limit for	xyz	is positive only; if	$m-n>y-z$.
,,	$x(1-y)z$	,,	,, $<x+y-1$.
,,	$xy(1-z)$	,,	$m+n>z$.
,,	$x(1-y)(1-z)$	,,	,, $<x$.
,,	$(1-x)yz$	,,	,, $<z$.
,,	$(1-x)(1-y)z$	,,	,, $>x$.
,,	$(1-x)y(1-z)$	,,	$m-n<y-z$.
,,	$(1-x)(1-y)(1-z)$	,,	,, $>x+y-1$.

223. The above investigation applies to equations of the form

$$xy = \bar{m}\,;\ yz = \bar{n}.$$

Let $x = \frac{2}{10}$; $y = \frac{4}{10}$; $z = \frac{7}{10}$.

Then	maximum	value of	$\bar{m}+\bar{n} = \frac{6}{10}$.
,,	minimum	,,	,, $= \frac{1}{10}$.
,,	maximum	,,	$\bar{m}-\bar{n} = \frac{1}{10}$.
,,	minimum	,,	,, $= -\frac{4}{10}$.

The limit for	xyz	is positive;	if $\bar{m}+\bar{n} > \frac{4}{10}$.
,,	$x(1-y)z$	,,	,, $< \frac{3}{10}$.
,,	$xy(1-z)$	,,	$\bar{m}-\bar{n} > \bar{0}$.
,,	$x(1-y)(1-z)$	,,	,, $< -\frac{5}{10}$.
,,	$(1-x)yz$	,,	,, $< \bar{0}$.
,,	$(1-x(1-y)z$	,,	,, $> -\frac{5}{10}$.
,,	$(1-x)y(1-z)$	,,	$\bar{m}+\bar{n} < \frac{4}{10}$.
,,	$(1-x)(1-y)(1-z)$	,,	,, $> \frac{3}{10}$.

Suppose that the given equations are of the form

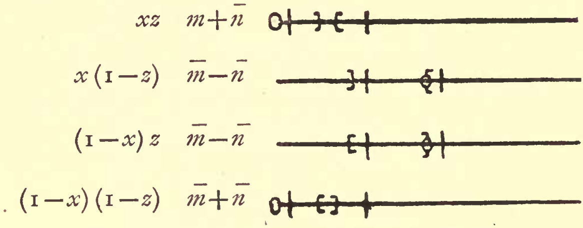
$$xy = \bar{m}\,;\ (1-y)\,z = \bar{n}\,;$$

and let x, y, z have the same arithmetical values as before.

Then	maximum	value of	$\bar{m}+\bar{n} = \frac{8}{10}$.
,,	minimum	,,	,, $= \frac{3}{10}$.
,,	maximum	,,	$\bar{m}-\bar{n} = -\frac{1}{10}$.
,,	minimum	,,	,, $= -\frac{6}{10}$.

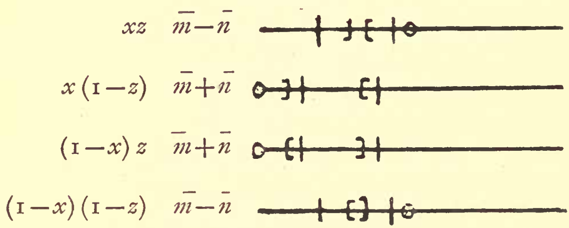
The limit for	xyz	is positive;	if $\bar{m}-\bar{n} > -\frac{3}{10}$.
,,	$x(1-y)z$	,,	,, $< -\frac{4}{10}$.
,,	$xy(1-z)$	,,	$\bar{m}+\bar{n} > \frac{7}{10}$.
,,	$x(1-y)\ 1-z)$	,,	,, $< \frac{2}{10}$.
,,	$(1-x)yz$	,,	,, $< \frac{7}{10}$.
,,	$(1-x)(1-y)z$	,,	,, $> \frac{2}{10}$.
,,	$(1-x)y(1-z)$	,,	$\bar{m}-\bar{n} < -\frac{3}{10}$
,,	$(1-x)(1-y)(1-z)$	,,	,, $> -\frac{4}{10}$.

224.



$$xy=\bar{m}\ ;\ yz=\bar{n}.$$
$$x=\tfrac{2}{10}\ ;\ y=\tfrac{4}{10}\ ;\ z=\tfrac{7}{10}.$$

FIG. 8.



$$xy=\bar{m}\ ;\ (1-y)\,z=\bar{n}.$$
$$x=\tfrac{2}{10}\ ;\ y=\tfrac{4}{10}\ ;\ z=\tfrac{7}{10}.$$

FIG. 9.

These relations are represented graphically in figs. 8 and 9. The values of $\bar{m}+\bar{n}$ are represented by distances from 0 along the top line of fig. 8. Its possible values, under the given conditions

$$x=\tfrac{2}{10},\ y=\tfrac{4}{10},\ z=\tfrac{7}{10},$$

lie between the straight strokes. Its minimum value, when the limit for xyz is positive, is represented by the distance from 0 to a stroke pointing in the positive direction ; and its maximum value, when $x(1-y)z$ is positive, is represented by the distance from 0 to the stroke pointing in the negative direction.

Similarly, the values of $\overline{m}-\overline{n}$ are represented by lengths from o along the second line; when positive by a length to the right, when negative by a length to the left. Its possible values, under the given conditions, lie between the tall strokes. Its values, when the limit for $xy(1-z)$ is positive, are represented by distances cut off by the stroke pointing to the right; and its values, when the limit for $x(1-y)(1-z)$ is positive, by distances cut off by the stroke pointing to the left.

And so on for the others.

225. When $\overline{m}+\overline{n}$ (or $\overline{m}-\overline{n}$) has a value inside [ ], then the two limits are simultaneously positive; when inside] [, the two limits are simultaneously negative; when outside [ ] or] [, the one limit is positive and the other negative.

226. *Given* $xy=m$, $yz=n$; *to find what relations necessarily exist between $m+n$ or $m-n$ and x, y, z.*

Write each of the equations in its four equivalent forms, as follows—

$$xy=m. \quad (1)$$
$$(1-x)y=y-m. \quad (2)$$
$$x(1-y)=x-m. \quad (3)$$
$$(1-x)(1-y)=1-x-y+m. \quad (4)$$
$$yz=n. \quad (5)$$
$$(1-y)z=z-n. \quad (6)$$
$$y(1-z)=y-n. \quad (7)$$
$$(1-y)(1-z)=1-y-z+n. \quad (8)$$

From (1) and (6) $xy(1-y)z>m+z-n-1$; (Art. 188)

$$\text{but } y(1-y)=0,$$
$$\therefore 0>m+z-n-1;$$
$$\therefore m-n<1-z.$$

From (1) and (8) we obtain by similar reasoning

$$0 > m + 1 - y - z + n - 1;$$
$$\therefore\ m + n < y + z.$$

By this method we obtain

From (1) and	(6),	$m - n < 1 - z$.
„	(8),	$m + n < y + z$.
(2)	(6),	„ $> y + z - 1$.
„	(8),	$m - n > -z$.
(3)	(5),	„ $> -(1 - x)$.
„	(7),	$m + n > x + y - 1$.
(4)	(5),	„ $< x + y$.
„	(7),	$m - n < x$.

227. *Given* $xy = m,\qquad (1 - y)\,z = n;$
to find what relations necessarily exist between $m + n$ *or* $m - n$ *and* x, y, z.

Write each of the equations in its four forms as follows—

$$xy = m. \qquad (1)$$
$$(1 - x)\,y = y - m. \qquad (2)$$
$$x\,(1 - y) = x - m. \qquad (3)$$
$$(1 - x)\,(1 - y) = 1 - x - y + m. \qquad (4)$$
$$yz = z - n. \qquad (5)$$
$$(1 - y)\,z = n. \qquad (6)$$
$$y\,(1 - z) = y - z + n. \qquad (7)$$
$$(1 - y)\,(1 - z) = 1 - y - n. \qquad (8)$$

Then by applying the method of the preceding Article we obtain—

From (1) and	(6),	$m + n < 1$.
„	(8),	$m - n < y$.
(2)	(6),	$m - n > -(1 - y)$.
„	(8),	$m + n > 0$.
(3)	(5),	$m + n > x + z - 1$.
„	(7),	$m - n > x + y - z - 1$.
(4)	(5),	$m - n < x + y - z$.
„	(7),	$m + n < x + z$.

Observation.—The merit of this investigation is its exhaustiveness. The truth of any of the individual results can be established directly. Consider, for example, the first of the above relations.

$$xy<y \quad \text{and} \quad (1-y)z<1-y,$$
$$\therefore xy+(1-y)z<y+1-y,$$
$$\text{that is, } m+n<1.$$

228. *To find expressions for the limits independently of one of the two data m, n.*

First; when the datum, which is to be eliminated, enters positively.

Example.—If $xy=m$, (1) and $(1-y)z=n$; (2) then it can be shown (Art. **214**) that

$$(1-x)y(1-z)>-m+n+y-z.$$

From (2) $\quad n>-y+z$ and >0, (Art. **217**.)

$$\therefore (1-x)y(1-z)>-m; \qquad (3)$$
$$\text{and } >-m+y-z. \qquad (4)$$

Second; when the datum, which is to be eliminated, enters negatively.

Example.—Under the same conditions as above,

$$(1-x)(1-y)(1-z)>m-n+1-x-y.$$

From (2) $\quad n<1-y$ and $<z$,

$$\therefore (1-x)(1-y)(1-z)>m-x; \qquad (5)$$
$$\text{and } >m+1-x-y-z. \qquad (6)$$

Observation 1.—In each case one of the pair of expressions obtained cannot be positive.

The expression (3) cannot be positive; for m is positive only.

The expression (5) cannot be positive; for m cannot include more that x.

Observation 2.—The truth of the other two results may be verified as follows:—

The inequality (4) asserts that

$$(1-x)y(1-z)>-xy+y-z;$$
$$\text{that is, } (1-x)y(1-z)>y(1-x)-z,$$
$$\text{that is, } z>y(1-x)z;$$

which is evidently true. (Art. **193**.)

The inequivalence (6) asserts that

$$(1-x)(1-y)(1-z) > xy + 1 - x - y - z;$$

that is, ,, $> (1-x)(1-y) - z$,

that is, $z > (1-x)(1-y)z$;

which is evidently true. (Art. **193.**)

229. *To find the condition which m or n must satisfy, in order that it may be impossible for the limit of a given term to be negative.*

When the condition for m is required, n must be eliminated from the general form of the limit in the manner shown in the preceding Article. One of the results obtained is necessarily negative; hence not more than one condition can be found. If m enters positively into the other result; then the required condition is, that m include the remainder of the expression with its sign changed. If it enters negatively; then it must be included in the remainder.

For example; $m > x + y + z - 1$,

is the condition which m must satisfy under the data of Article **226**, in order that it may be impossible for the limit of $(1-x)(1-y)(1-z)$ to be negative.

Similarly, if $m < y - z$,

the limit for $(1-x)y(1-z)$ cannot be negative.

230. Two Middle Terms.

Given $xy = m$, $yz = n$, $zu = p$;

to find those minor limits to the terms of the fourth order having x and u as factors, which can be expressed in terms of m, n, p, x, y, z or u.

$$xyz > m + n - y, \quad \text{(Art. 212.)}$$

and $zu = p$,

$$\therefore xyzu > m + n - y + p - z; \quad \text{(Art. 212.)}$$

that is, $> m + n + p - y - z$. (1)

Also, $x(1-y)z > -m - n - 1 + x + y + z$, (Art. **212.**)

and $zu = p$,

$$\therefore x(1-y)zu > -m - n - 1 + x + y + z + p - z;$$

that is, $> -m - n + p - 1 + x + y$. (2)

$$\text{Again } xy(1-z)>m-n,$$
$$\text{and } (1-z)u=u-p,$$
$$\therefore xy(1-z)u>m-n+u-p-(1-z);$$
$$\text{that is, } >m-n-p-1+z+u. \qquad (3)$$
$$\text{Also, } x(1-y)(1-z)>-m+n+x-z, \text{ (Art. } \mathbf{212}.)$$
$$\text{and } (1-z)u=u-p,$$
$$\therefore x(1-y)(1-z)u>-m+n-p-1+x+u. \quad (4)$$

Now there are four and only four terms of the second order which can be formed by means of the middle symbols y and z, viz.—

$$yz,\ (1-y)z,\ y(1-z),\ (1-y)(1-z).$$

Hence the above are all the terms of the fourth order which contain x and u as factors.

COR. *Each of the four limits obtained is a limit to xu.*

For xu is greater than each of the terms.

231. The limits for the terms of the fourth order, which have x and $1-u$, or $1-x$ and u, or $1-x$ and $1-u$ as factors, can be found in a similar manner, or deduced from the above results by making the appropriate transformations.

For example; to deduce the limit for

$$(1-x)(1-y)(1-z)(1-u).$$
$$xyzu>m+n+p-y-z;$$
$$\text{and } (1-x)(1-y)=1-x-y+m.$$
$$(1-y)(1-z)=1-y-z+n.$$
$$(1-z)(1-u)=1-z-u+p.$$
$$\therefore (1-x)(1-y)(1-z)(1-u)>1-x-y+m+1-y-z+n$$
$$+1-z-u+p-(1-y)-(1-z).$$
$$\therefore \qquad >m+n+p+1-x-y-z-u.$$

232. The *Sorites* is the simplest example possible of a conclusion derived from three equations of the kind considered. The premises are—

U's which are x are y; $xy=x$. (1)
U's which are y are z; $yz=y$. (2)
U's which are z are u; $zu=z$. (3)

Now $xyzu > m+n+p-y-z$, (Art. 224.)
$\therefore > x+y+z-y-z$, by premises,
that is, $> x$,
$= x$. (Art. 219.)

The conclusion is read—

U's which are x are u and y and z.

What is commonly called the conclusion is

U's which are x are u;

which is evidently only a part of the full conclusion.

We have shown in the preceding Article that

$$(1-x)(1-y)(1-z)(1-u) > m+n+p+1-x-y-z-u.$$

Hence by the premises $> x+y+z+1-x-y-z-u$,
$> 1-u$.

U's which are not u are not x and not y and not z.

This conclusion is complementary to the other.

Observation.—The investigation of the two preceding Articles shows how much truth there is in the statement that no conclusion can be drawn when more than one of the premises is not universal.

233. *Given the conclusions of Art.* 230; *to deduce minor limits to the terms of the third order having x and u as factors.*

It has been shown that

$$xyzu > m+n+p-y-z. \quad (1)$$

$$x(1-y)zu > -m-n+p-1+x+y. \quad (2)$$

$$xy(1-z)u > m-n-p-1+z+u. \quad (3)$$

$$x(1-y(1-z)u > -m+n-p-1+x+u. \quad (4)$$

By adding (1) to (3); $xyu > 2m-1-y+u$. (5)

" (2) " (4); $x(1-y)u > -2m-2+2x+y+u$. (6)

" (1) " (2); $xzu > 2p-1+x-z$. (7)

" (3) " (4); $x(1-z)u > -2p-2+x+z+2u$. (8)

Now $y,\ 1-y,\ z,\ 1-z$

comprise all the different expressions which can form the third factor of a term of the third order having x and u

constant factors. Hence the above comprise all the required limits.

COR. 1. Each of these four limits is a limit to xu.

COR. 2. Each of these limits is less than the one which can be obtained directly.

For example; consider (5).

$$xy=m, \text{ and } u=u,$$

$$\therefore xyu > m+u-1. \qquad \text{(Art. 188.)}$$

$$\text{Now } (2m-1-y+u)-(m+u-1)=m-y,$$

but $m-y$ is necessarily negative,

$\therefore$ the limit (5) is less than the one obtained directly.

Observation.—The integer 2 appears in each of these limits. Its presence indicates that the conclusion is *a fortiori*.

234. *To deduce a minor limit to the term of the second order having x and u as factors.*

Add (5) and (6);

$$\text{then } xu > -3+2x+2u. \qquad (9)$$

By adding (7) and (8) we get the same expression.

COR. 1. The total number of limits for xu which can be deduced by the method under consideration is 3^2.

For it is $\qquad =2^2+2\cdot 2+1,$

$$\therefore \ =3^2.$$

COR. 2. The sum of the primary limits for xu is independent of m, n, and p.

For the sum of (1), (2), (3), (4) is $-3+2x+2u$.

235. *To prove that*

$$xy > nx+ny-2n+1,$$

where n denotes any integral symbol.

Suppose it is true for any one n.

$$\text{Then } xy > nx+ny-2n+1,$$

$$\text{and } xy > x+y-1. \qquad \text{(Art. 187.)}$$

$$\therefore x^2y^2 > nx+ny-2n+1+x+y-1-1; \text{ (Art. 187.)}$$

$$\text{that is, } xy > (n+1)x+(n+1)y-2(n+1)+1.$$

Hence, if it is true for any n, it is true for $n+1$. But it is true for 1; hence for 2, and hence generally.

236. *Given* $xy=m$; $yz=n$; $zu=p$; *to find the relations which necessarily exist between* $m+n+p$, $m+n-p$, $m-n+p$, $m-n-p$, *and* x, y, z, u.

Write the equations as follows :—

$$xy=m. \quad (1)$$
$$x(1-y)=x-m. \quad (2)$$
$$(1-x)y=y-m. \quad (3)$$
$$(1-x)(1-y)=1-x-y+m. \quad (4)$$
$$yz=n. \quad (5)$$
$$y(1-z)=y-n. \quad (6)$$
$$(1-y)z=z-n. \quad (7)$$
$$(1-y)(1-z)=1-y-z+n. \quad (8)$$
$$zu=p. \quad (9)$$
$$z(1-u)=z-p. \quad (10)$$
$$(1-z)u=u-p. \quad (11)$$
$$(1-z)(1-u)=1-z-u+p. \quad (12)$$

From (1), (5), (11) $xyz(1-z)u>m+n-p+u-2$. (Art. **193**.)

but $z(1-z)=0$,

$\therefore 0>m+n-p+u-2$,

that is, $m+n-p<2-u$.

The truth of this result may be verified as follows :—

It asserts that $0>xy+yz-zu+u-2$,

that is, $2>xy+\{yz+u(1-z)\}$;

now $yz+u(1-z)<1$, and $xy<1$; hence their sum is <2.

By the above method it may be shown

from (1)	(5)	(11)	that $m+n-p<2-u$
,,	,,	(12)	$m+n+p<1+z+u$
,,	(6)	(9)	$m-n+p<2-y$
,,	,,	(10)	$m-n-p<2-y-z$
,,	(7)	(9)	$m-n+p<2-z$
,,	,,	(10)	$m-n-p<2-2z$
,,	,,	(11)	$m-n-p<2-z-u$
,,	,,	(12)	$m-n+p<1+u$
,,	(8)	(9)	$m+n+p<1+y+z$
,,	,,	(10)	$m+n-p<1+y$
,,	,,	(11)	$m+n-p<1+y+z-u$
,,	,,	(12)	$m+n+p<y+2z+u$

By taking (2) instead of (1) inequivalences involving $>$ will be obtained. And so on.

237. *To find the condition or conditions which the sum or difference (as the case may be) of any two of the three m, n, p, must satisfy in order that it may be impossible for a given limit to be negative.*

Suppose that m and n are the known data. If p occurs positively in the given limit, then either of its minor limits is to be substituted; if it occurs negatively, then either of its major limits is to be substituted.

Example of first case.

When $xy=m$, $yz=n$, $zu=p$,

$$xyzu>m+n+p-y-z.$$

Now $p>z+u-1$ and >0,

$$\therefore\ xyzu>m+n-y+u-1;$$

and also $>m+n-y-z$.

Hence the required conditions are

$$m+n>y+1-u;$$

and $m+n>y+z$.

The latter of these conditions is impossible.

Example of second case.

Under the same conditions as above,

$$xy(1-z)u>m-n-p-1+z+u.$$

Now $p<z$ and $<u$,

$$\therefore\ xy(1-z)u>m-n-1+u;$$

and also $>m-n-1+z$.

Hence the required conditions are

$$m-n>1-u,$$

and $m-n>1-z$.

Both of these are possible.

238. *To find the condition which any one of the three m, n, p, must satisfy, in order that it may be impossible for a given limit to be negative.*

Consider the condition for m.

Suppose *first* that n and p each enter positively into the given limit.

Example.—When $xy=m$, $yz=n$, $zu=p$,

$$xyzu>m+n+p-y-z.$$

Now $n>y+z-1$ and >0,

$p>z+u-1$ and >0;

hence $n+p>y+z-1+z+u-1.$ (1)

$>y+z-1.$ (2)

$>z+u-1.$ (3)

$>0.$ (4)

Therefore the required conditions are

$m>1-z+1-u$ (1)

$m>1$ (2)

$m>y+1-u$ (3)

$m>y+z$ (4)

The conditions (2), (3), (4) are evidently impossible.

Second; when n and p enter with opposite signs.

Example.—Under the same conditions as above

$$x(1-y)zu>-m-n+p-1+x+y.$$

Now $n<y$ and $<z$,

$p>z+u-1$ and >0;

$\therefore\ -n+p>-y+z+u-1.$ (1)

$>-y.$ (2)

$>u-1.$ (3)

$>-z.$ (4)

Therefore the required conditions are—

$m<x+z+u-2.$ (1)

$<x-1.$ (2)

$<x+y+u-2.$ (3)

$<x+y-z-1.$ (4)

The condition (2) is evidently impossible.

The condition (3) is impossible; because

$m>x+y-1,$

$\therefore\ m>x+y-1-(1-u).$

Similarly $m>x+y-1-z.$

Third; when n and p each enter negatively.

Example.—Under the same conditions as above,

$$xy(1-z)u > m-n-p-1+z+u.$$

Now $n < y$ and $< z$,

$p < z$ and $< u$;

$\therefore\ n+p < y+z$ (1)

$< y+u.$ (2)

$< z+z.$ (3)

$< z+u.$ (4)

Hence the required conditions are—

$m > y+1-u.$ (1)

$> y+1-z.$ (2)

$> z+1-u.$ (3)

$> 1.$ (4)

The conditions (1), (2), (4) are impossible.

239. *If m, n, p, be all eliminated from a limit in the above manner, only one of the resulting expressions is such as can be positive.*

Consider the third example of the preceding Article—

$$xy(1-z)u > m-n-p-1+z+u.$$

Now $m > x+y-1$ and > 0,

$n < y$ and $< z$,

$p < z$ and $< u$,

$\therefore\ m-n-p > x-1-z$ (1)

$> x-1-u$ (2)

$> x+y-1-2z$ (3)

$> x+y-1-z-u$ (4)

$> -y-z$ (5)

$> -y-u$ (6)

$> -2z$ (7)

$> -z-u.$ (8)

Hence $xy(1-z)u > x+u-2$ (1)

$> x+z-2$ (2)

$> x+y-z+u-2$ (3)

$> x+y-2$ (4)

$> -y+u-1$ (5)

$> -y+z-1$ (6)

$> -z+u-1$ (7)

$> -1.$ (8)

Only (3) can be positive. It is the limit obtained by means of the proposition of Art. **196**.

240. *Given a system of $n+1$ equations of the form*

$$x_0 x_1 = m_1, \left\{\frac{x_1}{1-x_1}\right\} x_2 = m_2, \ldots \left\{\frac{x_n}{1-x_n}\right\} x_{n+1} = m_{n+1},$$

where there are n middle terms; then there are 2^n limits of the first kind, $n2^{n-1}$ of the second kind, and generally $\frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1}2^{n-r+1}$ of the r^{th} kind.

The number of primary terms formed by n independent symbols is 2^n. (Art. **152**.) Hence there are 2^n limits of the *first* kind.

A limit of the *second* kind is formed by adding any pair of terms which involve only one degree of contradiction.

But a term of the n^{th} order has n contradictories of the first degree. (Art. **164**.) So for each of the 2^n terms; but only one-half of the sums so obtained are different. Hence there are $n2^{n-1}$ limits of the second kind.

A limit of the *third* kind is formed by adding any four terms, which differ with respect to two symbols only. For example;

$$\{x_1x_2 + x_1(1-x_2) + (1-x_1)x_2 + (1-x_1)(1-x_2)\}x_3x_4\ldots x_n.$$

Now each term has $\frac{n(n-1)}{1 \cdot 2}$ contradictories of the second degree (Art. **164**). Hence there are $\frac{n(n-1)}{1 \cdot 2}2^n$ sums; but of these only one-fourth are different. Therefore there are $\frac{n(n-1)}{1 \cdot 2}2^{n-2}$ limits of the third kind.

Generally for the limits of the r^{th} kind.

They are obtained by adding terms, which differ in $r-1$ factors. But there are to each term

$\frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1}$ contradictories of the $r-1^{th}$ degree.

(Art. **164**.)

But of the sums obtained only $\frac{1}{2^{r-1}}$ are different; hence there are

$$\frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1}2^{n-r+1} \text{ limits of the } r^{th} \text{ kind.}$$

COR. *The total sum of the limits is* 3^n.

For it is $=2^n+n2^{n-1}+\ldots+\frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1}2^{n-r+1}+\ldots+1$,

$$=(2+1)^n,$$
$$=3^n.$$

241. ONE MIDDLE TERM OF THE SECOND ORDER.

Given $xyz=m$, $yzu=n$;
to find limits to $xyzu$, *etc., in terms of* m, n, x, y, z.

$$x\cdot yz=m \qquad yz\cdot u=n,$$
$$\therefore xyzu>m+n-yz, \qquad \text{(Art. 188.)}$$
$$>m+n-y. \qquad (1)$$
$$\text{and } >m+n-z. \qquad (2)$$

Again. $x(1-yz)=x-m$ and $u(1-yz)=u-n$,

$$\therefore x(1-yz)u>x-m+u-n-(1-yz), \quad \text{(Art. 188.)}$$
$$\text{that is } >-m-n-1+x+u+yz.$$
$$\text{Now } yz>y+z-1 \text{ and } >0,$$
$$\therefore x(1-yz)u>-m-n-2+x+y+z+u. \qquad (3)$$
$$\text{and also } >-m-n-1+x+u. \qquad (4)$$

COR. xu is greater than any one of the above limits.

For it is greater than $xyzu$, or $x(1-yz)u$.

242. *Given* $xyz=m$ *and* $y(1-z)u=n$;
to find minor limits to $xyzu$, *etc., in terms of* m, n, x, y, z, u.

$$y(1-z)u=n,$$
$$\therefore yzu=yu-n;$$
$$\text{and } xyz=m,$$
$$\therefore xyzu>m-n+yu-yz. \qquad \text{(Art. 188.)}$$
$$\text{Now } yu-yz>y+u-1-y. \qquad (1)$$
$$>y+u-1-z. \qquad (2)$$
$$>0-y. \qquad (3)$$
$$>0-z. \qquad (4)$$

$$\text{Therefore } xyzu > m - n - 1 + u. \qquad (1)$$
$$> m - n - 1 + y - z + u. \qquad (2)$$
$$> m - n - y. \qquad (3)$$
$$> m - n - z. \qquad (4)$$

The limits for any of the other terms of the fourth order can be reduced in a similar manner.

XVIII. ON CERTAIN FORMS OF THE DISJUNCTIVE EQUATION.

243. One species of disjunctive equation consists of a single positive character equated to the sum of a number of positive characters. The general form of the equation is given in Art. **145**, and several properties are there deduced. Its chief function is to express the characters which a given character comprises.

The simplest case is when the disjunct member consists of two terms only, as

$$x = y + z\,;$$

the reading of which is

U's which are x are identical with those that are y together with those that are z.

244. Another species of disjunctive equation consists of a single positive character x equated to the sum of a number of terms multiplied by the character x. The principal forms are

$$x = x\{y(1-z) + z(1-y)\}. \qquad (1)$$
$$\text{and } x = x\{yz + y(1-z) + z(1-y)\}. \qquad (2)$$

We have already discussed

$$x = xyz\,;$$

and the remaining one of the four

$$x = x\{yz + y(1-z) + (1-y)z + (1-y)(1-z)\}$$

is an identity. (Art. **153**.)

Equation (1) is read

U's which are x are either y and not z or z and not y.

Equation (2) is read

U's which are x are either y and z or y and not z or z and not y.

COR. Since $yz+y(1-z)=y$,
equation (2) can be reduced to

$$x=x\{y+(1-y)z\};$$

that is, U's which are x are either y or not y and z.

Similarly $x=x\{z+(1-z)y\}$.

Observation.—Here x is a conditioning character. When it is put equal to 1, the equation ceases to be conditional.

245. The equation

$$x=y+z$$

may take the form

$$x=y+w,$$

where w is indefinite, but single and positive. From the nature of the equation, it follows that

$$wy=0.$$

The equation $x=y+w$ is read

Only U's which are x can be y;

and it is evidently the equational mode of stating the inequality,

$$x>y.$$

246. *Conversion of the disjunctive equation*

$$x=y+z.$$

By the Conversion of a disjunctive equation is meant its transformation into an equation having one of the terms of the disjunct member as the new left-hand member.

If $x=y+z$,
then $y=x-z$;

that is,

U's which are y are identical with those which are x excepting those which are z.

$$\text{If } x=y+w,$$
$$\text{then } y=x-w\,;$$

that is,

U's which are y must be x.

If the original equation is written

$$x>y,$$

then the form deduced by conversion is

$$y<x.$$

247. *Contraposition of the disjunctive equation*

$$x=y+z.$$

The contraposition of a disjunctive equation consists in deducing the expression for the contrary of the left-hand member.

$$\text{If } x=y+z,$$
$$\text{then } 1-x=1-y-z\,;$$

that is,

U's which are not x comprise all excepting those which are y and excepting those which are z.

$$\text{If } x=y+w,$$
$$\text{then } 1-x=1-y-w\,;$$

that is,

U's which are not x are not y.

Observation.—If $\quad x=y+w$

is expressed by $\quad x>y\,;$

then $\quad 1-x=1-y-w$

is expressed by $\quad 1-x<1-y.$

248. *If $x=y+z$; then $xy=y$, and $xz=z$: but not conversely.*

$$x=y+z,$$

multiply by y,

$$\text{then } xy=y^2+yz$$
$$=y. \qquad \text{(Art. 145.)}$$

Similarly $\quad xz=z.$

Given $\quad xy=y$ and $xz=z$;

$$\text{then } xy+xz=y+z, \qquad \text{(Art. 291.)}$$
$$\therefore x(y+z)=y+z,$$
$$\therefore x=\frac{y+z}{y+z}.$$

But $\frac{y+z}{y+z}$ is not necessarily $=y+z$; (Art. **173.**) hence it does not follow that $x=y+z$.

249. *If* $x=y+w$, *where* w *has the meaning assigned to it in* Art. **245**: *then the derived equation*

$$xy=y$$

is equivalent to the original.

$$xy=y,$$

$$\therefore x=\frac{y}{y},$$

$$=y+\frac{0}{0}(1-y).$$

Now $\frac{0}{0}(1-y)$ is single, since x is single (Art. **145**), and it is such that

$$\frac{0}{0}(1-y)y=0,$$

and it is not restricted in any other manner. Hence it is equivalent to w.

Observation. $x=y+w$, $x>y$, and $xy=y$, express the same truth. To express the truth considered by means of an equation, either three symbols must be used, or else a term of the second order introduced.

250. *Inference from two disjunctive equations having a common character.*

First; when the character occurs oppositely in the two equations.

$$x=y+z. \quad (1) \qquad x'=1-y+z'. \quad (2)$$

Add; $$x+x'=1+z+z',$$

that is, $x=1-x'+z+z'$.

Second; when the character occurs similarly in the two equations.

$$x=y+z. \quad (1) \qquad x'=y+z'. \quad (2)$$

Subtract; $$x-x'=z-z',$$

that is, $x=x'+z-z'$.

Observation.—When z and z' are each indefinite, the former conclusion amounts to

Only U's which are x are not x'.

The latter conclusion under the same conditions amounts to

Some U's which are x are x'.

For $x > z$, by equation (1).

251. *If* $x = x(y+z)$, *then* $x = x\{y(1-z)+z(1-y)\}$; *and conversely.*

$$x = x(y+z),$$
$$\therefore\ x\{1-y-z\}=0,$$
$$\therefore\ x^2\{1-y-z\}^2=0,$$

that is, $$x\{1+y+z-2y-2z+2yz\}=0,$$
that is, $$x\{1-y(1-z)-z(1-y)\}=0,$$
that is, $$x = x\{y(1-z)+z(1-y)\}.$$

As this proof can be reversed, the converse proposition must be true.

Observation.—The process here exemplified of squaring an expression which is equal to 0 is of great importance. The expression

$$x\{1-y(1-z)-z(1-y)\}$$

is of a form which cannot be negative; while the expression

$$x\{1-y-z\}$$

is of a form which may be negative. See Art. 289.

252. *To convert the equation*

$$x = x(y+z).$$

$$x = xy + xz,$$
$$\therefore\ xy = x(1-z),$$
$$\therefore\ y = \frac{x}{x}(1-z),$$

$$= 0xz + x(1-z) + \frac{0}{0}(1-x)z + \frac{0}{0}(1-x)(1-z),$$

$$= x(1-z) + \frac{0}{0}(1-x).$$

Cor. 1. Multiply by x,

then $xy = x(1-z)$:

which involves as a part of its truth

$$xy = xy(1-z),$$

i.e. U's which are x and y are not z.

COR. 2. Multiply by $1-y$,

$$\text{then } 0 = x(1-y)(1-z) + \frac{0}{0}(1-x)(1-y);$$

$\therefore x(1-y)(1-z) = 0$, for neither term can be negative;

$$\therefore x(1-y) = x(1-y)z;$$

i.e. U's which are x and not y are z.

Similarly it may be shown that

$$xz = x(1-y);$$

$$\text{and } x(1-z) = x(1-z)y.$$

253. *To convert the equation*

$$x = x\{y + (1-y)z\}.$$

$$\text{Now} \quad x = xy + x(1-y)z,$$

$$\therefore xy(1-z) = x(1-z),$$

$$\therefore y = \frac{x(1-z)}{x(1-z)};$$

$$= \frac{0}{0}xz + x(1-z) + \frac{0}{0}(1-x)z + \frac{0}{0}(1-x)(1-z),$$

$$= x(1-z) + \frac{0}{0}(1-x) + \frac{0}{0}xz.$$

This expression differs from the corresponding one obtained from

$$x = x(y+z)$$

in having $\frac{0}{0}xz$ in addition.

COR. 1. Multiply by x,

$$\text{then } xy = x(1-z) + \frac{0}{0}xz;$$

hence xy is not, as previously, identical with $x(1-z)$, but includes it.

COR. 2. Multiply by $1-y$,

$$\text{then } 0 = x(1-y)(1-z) + \frac{0}{0}(1-y)(1-x) + \frac{0}{0}x(1-y)z;$$

therefore, since none of the terms of the right-hand member can be negative,

$$x(1-y)(1-z)=0,$$

that is, $x(1-y)=x(1-y)z$.

Similarly $z=x(1-y)+\frac{0}{0}(1-x)+\frac{0}{0}xy$.

254. *If* $x=x(y+z)$; *then* $xy=xy(1-z)$, *and* $x(1-y)=x(1-y)z$; *and conversely.*

$$x=x(y+z),$$

$$\therefore x=x\{y(1-z)+z(1-y)\}, \qquad \text{(Art. 251.)}$$

Multiply by y; then

$$xy=xy(1-z). \qquad (1)$$

Multiply by $1-y$; then

$$x(1-y)=x(1-y)z. \qquad (2)$$

To prove the converse.

From (1) $xyz=0$.

From (2) $x(1-y)(1-z)=0$,

that is, $x\{1-y-z+yz\}=0$,

$\therefore$ by (1) $x=x(y+z)$.

Observation.—It is commonly said that the disjunctive proposition

U's are either *y* or *z*

is equivalent to the four hypothetical propositions

U's which are *y* are not *z*. (1)
U's which are not *y* are *z*. (2)
U's which are *z* are not *y*. (3)
U's which are not *z* are *y*. (4)

Now from $1=y+z$

there follows, by making $x=1$ in the above equations:—

$$y=y(1-z). \qquad (1)$$

$$1-y=(1-y)z. \qquad (2)$$

$$z=z(1-y). \qquad (3)$$

$$1-z=(1-z)y. \qquad (4)$$

But (1) and (2) without (3) and (4) are together equivalent to the given equation; and so are (3) and (4) without (1) and (2).

255. *If* $x=x\{y+(1-y)z\}$; *then* $x(1-y)=x(1-y)z$: *and conversely.*

$$x=x\{y+(1-y)z\},$$
$$=x\{yz+y(1-z)+(1-y)z\},$$
$$=x\{1-(1-y)(1-z)\}, \qquad \text{(Art. 148.)}$$
$$\therefore x(1-y)(1-z)=0;$$
$$\text{that is, } x(1-y)=x(1-y)z.$$

Since the proof can be reversed, the converse proposition must be true.

XIX. THE ARISTOTELIAN FORMS OF INFERENCE.

256. I shall assume that the Aristotelian moods do not consist of the mere application of a general proposition to an individual case, but of the combination of two general propositions. I shall further assume that the data are equations not of the first but of the second order; in other words, that the premises each involve one hypothesis.

They are mere cases of the general form of inference discussed in Article **212**, viz.—

$$\text{If } xy=m, \text{ and } yz=n,$$
$$\text{then } xyz>m+n-y;$$
$$\text{and *a fortiori* } xz>m+n-y.$$

The analytical reductions, which are sometimes required to fit the premises of a mood for being put into the above formula, correspond to the reductions indicated by the significant letters in the name of the mood.

Observation.—The fact that the number of Aristotelian moods is a prime number suffices to throw doubt upon the completeness of the scheme.

257. *The First Figure.*

Barbara. *U*'s which are y are z; (1)

U's which are x are y. (2)

$$y=yz. \quad (1) \qquad x=xy. \quad (2)$$

$$\therefore xz > x+y-y;$$

$$=x \qquad \text{(Art. 219)},$$

that is, *U*'s which are x are z.

Celarent. *U*'s which are y are not z; (1)

U's which are x are y. (2)

$$y=y(1-z). \quad (1) \qquad x=xy. \quad (2)$$

$$\therefore x(1-z) > x+y-y;$$

$$=x;$$

that is, *U*'s which are x are not z.

Darii. *U*'s which are y are z; (1)

U's which are x and y are some. (2)

$$y=yz. \quad (1) \qquad xy=v. \quad (2)$$

$$\therefore xz > v+y-y;$$

$$> v;$$

that is, *U*'s which are x and z are some.

Ferio. *U*'s which are y are not z; (1)

U's which are x and y are some. (2)

$$y=y(1-z). \quad (1) \qquad xy=v. \quad (2)$$

$$\therefore x(1-z) > v+y-y;$$

$$> v;$$

that is, *U*'s which are x and not z are some.

258. *The Second Figure.*

Cesare. The premises are

$$z=z(1-y). \quad (1) \qquad \text{and } x=xy. \quad (2)$$

Since the first premise contains $1-y$, and the second y, one of the two must be transformed. In this case it is the transformation of the first premise which leads to the ordinary conclusion.

From (1) $zy=0$,

$$\therefore y=y-yz,$$

$$=y(1-z).$$

Hence $$x(1-z)>x+y-y,$$
$$=x.$$

The transformation here made corresponds to that indicated by the letter *s* in the name of the mood.

Observation.—The conclusion in the above case is deduced more shortly by multiplying together the two data.

Camestres. $z=zy.$ (1) and $x=x(1-y).$ (2)

From (2) $$y=y(1-x);$$
$$\therefore (1-x)z>y+z-y;$$
$$=z;$$
$$\therefore x(1-z)=x.$$

Here both equation (2) and the immediate conclusion are transformed by a process which corresponds to simple conversion.

Festino. $z=z(1-y).$ (1) and $xy=v.$ (2)

From (1) $$(1-z)y=y,$$
$$\therefore x(1-z)>v+y-y,$$
$$>v.$$

Baroco. $z=zy.$ (1) and $x(1-y)=v.$ (2)

It is necessary to transform (1) so as to contain $1-y$.

Now $$(1-z)(1-y)=1-y-z+yz \quad \text{always,}$$
$$=1-y \quad \text{by (1)}$$

Hence $$x(1-z)>v+1-y-(1-y)$$
$$>v.$$

The method of reduction here employed corresponds to the contraposition of the major premise. The analytical investigation shows the reason why the mood gave so much trouble to the Scholastic logicians, and also shows how readily problems, which are difficult to the unaided mind, may be solved with the help of a true organon.

The *ductio per impossibile* is as follows.

If $x(1-z)$ is not greater than 0, it must be equal to 0; for it cannot be negative.

Let it be equal to 0.

Then $xz = x.$

Multiply (1) by x, then $xz = xzy.$

Hence $x = xy$

that is $x(1-y) = 0.$

But $x(1-y) = v$ by (2)

hence $x(1-z)$ not $= 0$

therefore $x(1-z)$ is greater than 0.

Observation.—Here $1-y$ is a factor of the left-hand member of the conclusion. In the rest of the moods y is the corresponding factor.

259. *The Third Figure.*

Darapti. $y = yz.$ (1) and $y = yx.$ (2)

Hence $xz > y + y - y,$

$> y.$

The conclusion is commonly said to be

U's which are x and z are some;

but it evidently amounts to more, viz.,

U's which are x and z include those which are y.

Felapton. $y = y(1-z).$ (1) and $y = yx.$ (2)

Hence $x(1-z) > y + y - y,$

$> y.$

Disamis. $yz = v.$ (1) and $y = yx.$ (2)

Hence $xz > v + y - y,$

$> v.$

Datisi. $y = yz.$ (1) and $xy = v.$ (2)

Hence $xz > v.$

Bocardo. $y(1-z) = v.$ (1) and $y = yx.$ (2)

Hence $x(1-z) > v.$

Ferison. $y = y(1-z).$ (1) and $xy = v.$ (2)

Hence $x(1-z) > v.$

260. *The Fourth Figure.*

Bamalip. $z = zy.$ (1) and $y = yx.$ (2)

Hence $xz > y + z - y,$

$= z.$

The conclusion is commonly said to be

U's which are x and z are some.

But it really amounts to

U's which are x include those that are z,

or, in other words,

Only U's which are x can be z.

The character x is a *conditio sine quâ non* with respect to the character z.

Calemes. $z=zy$. (1) and $y=y(1-x)$. (2)

$$\text{Hence } (1-x)z>y+z-y,$$
$$=z.$$

The conclusion in this form means

Only U's which are not x can be z;

but it can be converted into

$$x=x(1-z),$$

that is,

U's which are x are not z.

Dimatis. $zy=v$. (1) and $y=yx$. (2)

Hence $xz>v$.

Fesapo. $z=z(1-y)$. (1) and $y=yz$. (2)

From (1) $y=y(1-z)$,

$$\therefore x(1-z)>y+y-y,$$
$$>y.$$

This conclusion is not particular merely, but asserts that the character $x(1-z)$ is a *conditio sine quâ non* with respect to the character y.

Fresison. $z=z(1-y)$. (1) and $xy=v$. (2)

From (1) $y=y(1-z)$,

$$\therefore x(1-z)>v+y-y,$$
$$>v.$$

Here the conclusion is merely particular. For an investigation of De Morgan's Syllogistic forms see the Examples, page 135.

XX. ON PROBABILITY.

261. It is important to draw a sharp distinction between the combining of two general propositions, and the application of a general proposition to an individual case. The former of these processes is investigated by the Theory of Necessity; the latter by the Theory of Probability. Any one who clearly perceives this distinction, is in no danger of falling into the absurdity of maintaining that true syllogistic reasoning involves the fallacy of the Circle.

262. *The Probability that a given member of a universe has a character x is measured by the arithmetical value of x referred to that universe.*

The probability here spoken of, or at least its measure, is entirely objective. The value of this objective probability may not be fully known; in which case we consider its most probable value. The latter depends upon the state of our knowledge concerning the objective probability, and in consequence varies with that state.

263. *If x is equivalent to a function of a number of characters, the probability of an individual U being x is equal to that function taken arithmetically.*

For every logical equation is true arithmetically. As an example,

$$\text{let } x=\frac{y}{z},$$

$$\text{then } x=y+\frac{0}{0}(1-z) \qquad \text{(Art. 199.)}$$

$$\therefore \bar{x}=\bar{y}+\frac{0}{0}\overline{(1-z)}.$$

Hence the results of the preceding Articles may be made to yield theorems in Probability by taking the right-hand members of the equations arithmetically, and supposing *U* to denote an individual member of the universe.

264. *The probability of a U which is x being y is measured by the arithmetical value of y in Ux_y.*

Let $Ux = U'$.

The probability of a U' being y is measured by the arithmetical value of y in $U'y$. (Art. 262.)

But Ux is identical with U'; therefore the probability of a U which is x being y is measured by the arithmetical value of y in Ux_y.

COR. *The probability of a U which is x being y is equal to* $\frac{\overline{xy}}{\bar{x}}$.

For it is equal to that of y in xy;

$$\therefore \text{ equal to } \frac{\bar{x}_y}{\bar{x}} \qquad \text{(Art. 64.)}$$

$$\therefore \text{ equal to } \frac{\overline{xy}}{\bar{x}}.$$

265. *If the probability of a U which is x being y is equal to the probability of a U which is not x being y, then either is equal to the probability of a U being y.*

$$\frac{\overline{xy}}{\bar{x}} = \frac{\overline{(1-x)y}}{\overline{1-x}} \text{ given.}$$

$$\therefore \frac{\overline{xy}}{\bar{x}} = \frac{\bar{y} - \overline{xy}}{1 - \bar{x}};$$

$$\therefore \frac{\overline{xy}}{\bar{x}} = \bar{y}.$$

Any one of the three equations

$$\frac{\overline{xy}}{\bar{x}} = \frac{\overline{(1-x)y}}{\overline{1-x}} = \bar{y}$$

expresses the condition for x and y being *really* independent of one another. (Art. 62.)

266. *Required the probability of a U which is x being y, having given that*

(1) $U\{x = y + z\}$;

or (2) $U\{x = y - z\}$.

First Case. Probability required $=\frac{\overline{xy}}{\bar{x}}$ (Art. 264. Cor.)

$$=\frac{\bar{y}}{\bar{y}+\bar{z}} \qquad \text{by (1).}$$

Second case. Probability required $=\frac{\overline{xy}}{\bar{x}}$

$\therefore$ by (2). $$=\frac{\bar{y}-\bar{z}}{\bar{y}-\bar{z}} \qquad =\bar{1}.$$

267. *Given* $x=x_1+x_2+x_3+x_4$; *required the probability of a U which is x being either* x_1 *or* x_2.

Probability required $=\frac{\overline{x(x_1+x_2)}}{\bar{x}}$ (Art. 264),

$$=\frac{\overline{(x_1+x_2+x_3+x_4)(x_1+x_2)}}{\bar{x}_1+\bar{x}_2+\bar{x}_3+\bar{x}_4},$$

$$=\frac{\bar{x}_1+\bar{x}_2}{\bar{x}_1+\bar{x}_2+\bar{x}_3+\bar{x}_4} \qquad \text{(Art. 145.)}$$

268. *Given* $x=y+z-u$, *required* (1) *the probability that a U which is x is y, and* (2) *the probability that it is not y.*

Since $$z-u=z(1-u)-u(1-z),$$
$$x=y+z(1-u)-u(1-z).$$

Now x cannot be negative, and $z(1-u)$ and $u(1-z)$ are exclusive of one another; therefore y must include $u(1-z)$.

Hence $$x=y\{1-u(1-z)\}+z(1-u).$$

(1.) Probability that a U which is x is y $=\frac{\overline{xy}}{\bar{x}}$

$$=\frac{\overline{y\{1-u(1-z)\}}}{\bar{y}+\bar{z}-\bar{u}}, \text{ from given equation}$$

$$=\frac{\bar{y}-\overline{u(1-z)}}{\bar{y}+\bar{z}-\bar{u}},$$

$$=\frac{\bar{y}-\bar{u}+\overline{uz}}{\bar{y}+\bar{z}-\bar{u}}.$$

Suppose that $\overline{uz}$ is not known. Then we affirm that

$$\text{prob.} > \frac{\bar{y}-\bar{u}}{\bar{y}+\bar{z}-\bar{u}}, \text{ and } < \frac{\bar{y}}{\bar{y}+\bar{z}-\bar{u}}.$$

(2.) Probability that a U which is x is not $y = \dfrac{\overline{x(1-y)}}{\bar{x}}$

$$= \frac{\overline{z(1-u)}}{\bar{y}+\bar{z}-\bar{u}},$$

$$= \frac{\bar{z}-\overline{zu}}{\bar{y}+\bar{z}-\bar{u}}.$$

Hence if $\overline{zu}$ is not known, we can affirm that

$$\text{prob.} < \frac{\bar{z}}{\bar{y}+\bar{z}-\bar{u}}.$$

269. *Given the probability that a U is x, and the probability that a U which is x is y; to find the probability that a U is y.*

The data are $x=\bar{p}$, and $x_y=\bar{p}\bar{q}$. It is required to find $\bar{y}$.

Now $y=\dfrac{xy}{x}$,

$$=\frac{x_y}{x},$$

$$=xx_y+\frac{1}{0}x_y(1-x)+0(1-x_y)x+\frac{0}{0}(1-x_y)(1-x),$$

$$=x_y+\frac{0}{0}(1-x).$$

$$\therefore \bar{y}=\bar{x}_y+\frac{0}{0}(\bar{1}-\bar{x}),$$

$$=\bar{p}\bar{q}+\frac{0}{0}(\bar{1}-\bar{p}).$$

Hence $\bar{y}>\bar{p}\bar{q}$ and $<\bar{p}\bar{q}+\bar{1}-\bar{p}$.

The problem can also be investigated as follows:—

$$(1-x)(1-y)=1-x-y+xy,$$

$$\therefore y=1-x+xy-(1-x)(1-y). \qquad (1)$$

$$=xy+(1-x)y. \qquad (2)$$

Hence $y<1-x+xy$ and $>xy$.

Hence $\bar{y}<\bar{1}-\bar{p}+\bar{p}\bar{q}$ and $>\bar{p}\bar{q}$.

XXI. FUNDAMENTAL RELATIONS BETWEEN THE SINGLE FUNCTIONS OF A NUMBER OF INDEPENDENT CHARACTERS.

270. By the fundamental relations between the single functions of the characters $x_1, x_2, \ldots x_n$ is meant the different forms of the relation connecting the part $x_1\ x_2 \ldots x_n$ with the other parts into which the universe is divided by the characters.

The relations in the case of two independent characters are given in Article **190.** They suffice to combine any number of premises, which are linked together by common characters after the manner shown in Article **240.** As, however, the data of a problem are not necessarily of that nature, it is important to consider all the fundamental relations which exist in a given case.

271. *To find the fundamental relations, when there are three characters* x, y, z.

Expand the term which is the opposite of xyz,

$$(1-x)(1-y)(1-z)=1-x-y-z+yz+zx+xy-xyz. \quad (1)$$

Expand a term which involves two degrees of contradiction,

$$(1-x)(1-y)z=z-xz-yz+xyz. \quad (2)$$

Expand a term which involves one degree of contradiction,

$$(1-x)yz=yz-xyz. \quad (3)$$

There are other two equations similar to (2), and other two similar to (3); hence, if

$$xyz=0+xyz$$

is also counted, there are 2^3 altogether.

The other relations which exist may be derived by transforming these fundamental relations. For example: change z into $1-z$, then (2) becomes

$$(1-x)(1-y)(1-z)=1-z-x(1-z)-y(1-z)+xy(1-z).$$

272. *When there are n characters, there are* 2^n *fundamental relations.*

For each primary part into which the universe is divided by the n characters yields a fundamental relation; and there are 2^n such parts. (Art. **152**.)

273. Consider the relation

$$(1-x)(1-y)(1-z)+x+y+z-yz-zx-xy+xyz=1.$$

If the arithmetical values of any seven of the eight terms are known, then that of the remaining term can be found. If the arithmetical values of any six of them are known, then that of the remainder can be found; and so on.

Suppose that the remainder can be reduced to two parts t_1 and t_2 having the same sign.

$$\text{Then } t_1+t_2=\bar{n};$$
$$\therefore t_1=\bar{n}-t_2.$$

Hence t_1 and t_2 must each be less than $\bar{n}$.

Suppose that the two parts t_1 and t_2, to which the remainder is reduced, occur with opposite signs.

$$\text{Then } t_1-t_2=\bar{n};$$
$$\therefore t_1>\bar{n}, \text{ and } t_2>-\bar{n}.$$

274. The following example illustrates the manner in which these relations can be applied to solve questions in Probability.

Given $x=\bar{a}$; $y=\bar{b}$; $xz=\bar{a}\bar{p}$; $yz=\bar{b}\bar{q}$;

and $(1-x)(1-y)z=0$;

required $\bar{z}$.

By (2) (Art. **271**) $(1-x)(1-y)z=z-xz-yz+xyz$.

$$\therefore z=xz+yz-xyz \quad \text{by last datum;}$$

$$\therefore z<xz+yz-x-yz+1. \quad (1) \qquad \text{(Art. 188.)}$$
$$<xz+yz-y-xz+1. \quad (2)$$
$$<xz+yz. \quad (3)$$
$$\therefore z<\bar{1}-\bar{a}+\bar{a}\bar{p}. \quad (1)$$
$$<\bar{1}-\bar{b}+\bar{b}\bar{q}. \quad (2)$$
$$<\bar{a}\bar{p}+\bar{b}\bar{q}. \quad (3)$$

By relation (3) (Art. 271) $(1-x)yz=yz-xyz.$

$$\therefore (1-x)yz=yz+z-xz-yz, \text{ by the last datum.}$$

$$\therefore z=xz+(1-x)yz.$$

$$\therefore z>\bar{a}\bar{p}.$$

It may be shown in a similar manner that

$$z>\bar{b}\bar{q}.$$

Boole deduces these results by a much longer process.

XXII. GENERAL METHOD OF DEDUCING A CONCLUSION OF A REQUIRED FORM FROM GIVEN DATA.

275. When we know a necessary relation existing between the forms of the given data and of the required conclusion, and between these forms alone, we can find the conclusion by simply substituting the data in the necessary relation. If the known relation involves other forms in addition, it may be made to yield limits to the required conclusion. If, however, all the forms of the given data do not exist in the relation, the limits deduced may not coincide with the actual limits necessitated by the data.

276. I have discovered a method, by means of which it is easy to find the necessary relation which exists between the forms of any given data and the form of the required conclusion. The forms are supposed to be functions of single (Art. 118) symbols.

Equate the form of the conclusion to the sum of the forms of the data each multiplied by a constant, and with a constant added; then substitute each of the sets of singular values which the characters can assume. The equations obtained express the relations which necessarily exist between the constants; they may suffice to determine each of the *constants exactly, or they may require one or more of the constants to have indefinite values.*

It is supposed that the functions do not contain any inverse factors.

277. *Given $x=m$, $x(1-y)=n$, $(1-x)y=p$; required xy.*

Let $xy=a+bx+cx(1-y)+d(1-x)y$.

Since x and y are given to be single and (we may suppose) positive, each can assume the values 1 and 0. Hence

$$\begin{array}{lll} \text{from } x=1, y=1, & 1=a+b. & (1) \\ \text{from } x=1, y=0, & 0=a+b+c. & (2) \\ \text{from } x=0, y=1, & 0=a \qquad\quad +d. & (3) \\ \text{from } x=0, y=0, & 0=a. & (4) \end{array}$$

The determinant of the right-hand members of these equations is

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

which is equal to 1. Hence a definite relation exists between the form of the conclusion and the forms of the given data. Definite values can be found for the constants, viz.—

$$a=0, \; b=1, \; c=-1, \; d=0.$$

Hence
$$xy=x-x(1-y),$$
$$=m-n.$$

The problem here discussed can of course be solved by much simpler methods. Its solution by the general method is introduced to illustrate that method.

278. *Given $y=m$, $xy=n$; required x.*

Let $x=a+by+cxy$.

$$\begin{array}{ll} \text{Then } 1=a+b+c. & (1) \\ 1=a. & (2) \\ 0=a+b & (3) \\ 0=a. & (4) \end{array}$$

The determinant of the right-hand members is

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

Since the elements of the fourth column vanish, the value of the determinant is 0. Hence there is no *definite* relation between the forms considered.

The fourth row is identical with the second; but the left-hand member of (4) is not identical with that of (2). The equations can be made identical by putting in $\frac{0}{0}$ instead of 1 and of 0, and restricting $\frac{0}{0}$ within the limits 1 and 0. Then we have

$$1 = a + b + c. \qquad (1)$$

$$\frac{0}{0} = a. \qquad (2)$$

$$0 = a + b. \qquad (3)$$

The determinant of these equations is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

the value of which is 1. And

$$a = \frac{0}{0}, \; b = -\frac{0}{0}, \; c = 1.$$

Hence
$$x = \frac{0}{0} - \frac{0}{0} y + xy,$$

$$= \frac{0}{0}(1 - y) + xy,$$

$$= \frac{0}{0}(1 - m) + n.$$

Since $\frac{0}{0}$ lies between 1 and 0,

$$x > n \text{ and } < 1 - m + n.$$

Observation.—The fundamental relations

$$(1-x)(1-y)=1-x-y+xy,$$
$$\text{and } (1-y)\,x=x-xy,$$

readily show that

$$x<1-y+xy \text{ and } >xy.$$

279. The following problem is an instance of the general problem investigated in the preceding Article :—

The probability that it thunders upon a given day is p, the probability that it both thunders and hails is q, but of the connection of the two phenomena of thunder and hail, nothing further is supposed to be known. Required the probability that it hails on the proposed day.—Boole, *Laws of Thought*, p. 276.

Let $U=$ a succession of states of the atmosphere at a given place, an individual state being of the length of a day.

$x=$containing a hailstorm; $y=$containing a thunderstorm.

$$\text{Then } y=\bar{p},\ \ xy=\bar{q}.$$

Hence, by the preceding Article,

$$x=\frac{0}{0}\,(\overline{1-\bar{p}})+\bar{q}.$$

Hence x cannot be greater than $\bar{q}+\overline{1-\bar{p}}$, and cannot be less than $\bar{q}$.

280. Let us take the three data

$$y=m,\ \ xy=n,\ \ 1-y=p,$$

which are equivalent to two independent data, and inquire how this is detected by the method, when we seek for the relation connecting x with the given functions.

$$\text{Let } x=a+by+cxy+d\,(1-y).$$

$$\begin{aligned} \text{Then } 1&=a+b+c. && (1)\\ 1&=a \qquad\quad +d. && (2)\\ 0&=a+b. && (3)\\ 0&=a \qquad\quad +d. && (4)\end{aligned}$$

The determinant of these equations is

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

Now this determinant has the sum of its second and fourth columns identical with its first column; hence its value is 0. Thus the analytical reason why the determinant vanishes indicates the real reason of the insufficiency of the data.

281. *Given* $xy+x(1-y)+(1-x)y=p$,
and $x(1-y)+(1-x)y+(1-x)(1-y)=q$;
to find $x(1-y)+y(1-x)$.

Let $x(1-y)+y(1-x)=a+b\{x+(1-x)y\}$
$+c\{x(1-y)+(1-x)y+(1-x)(1-y)\}$.

Then $0=a+b$. (1)

$1=a+b+c$. (2)

$1=a+b+c$. (3)

$0=a \quad +c$. (4)

The determinant is

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

in which the second and third rows are identical. But the equations (2) and (3) have the same left-hand member. Hence one of them may be struck out.

Then $$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1;$$

and $a=-1$, $b=1$, $c=1$.

Hence $\quad x(1-y)+y(1-x)=-1+p+q$.

The above is the general solution of the following problem:—

The probability that one or both of two events happen is $\bar{p}$, that one or both of them fail is $\bar{q}$. What is the probability that only one of these happens?

It is $-\bar{1}+\bar{p}+\bar{q}$.

282. *Given x, y, xz, yz, and $(1-x)(1-y)z$; required z.*

Let $z=ax+by+cxz+dyz+e(1-x)(1-y)z$.

Then by means of the eight sets of singular values, which the three characters can assume, we obtain the equations

$$1=a+b+c+d. \qquad (1)$$
$$0=a+b. \qquad (2)$$
$$1=a\quad+c. \qquad (3)$$
$$0=a. \qquad (4)$$
$$1=\quad b\quad+d. \qquad (5)$$
$$0=\quad b. \qquad (6)$$
$$1=\qquad e. \qquad (7)$$
$$0=0. \qquad (8)$$

The determinant expression is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Since the determinant contains five columns and seven rows, two of the equations are identical or must be made identical. Now the sum of the third and fifth rows is identical with the first, but the values of the corresponding equations are 2 and 1 respectively. Denote this value by $\frac{0}{0}$, subject to the restriction that it cannot be greater than 2 nor less than 1. Also the sum of the fourth and sixth rows is identical with the second, and the values of the corresponding equations are equal. Hence the system of equations reduces to

$$\frac{0}{0}=a+b+c+d. \qquad (1)$$
$$1=a \quad +c. \qquad (2)$$
$$0=a. \qquad (3)$$
$$0= \quad b. \qquad (4)$$
$$1= \quad e. \qquad (5)$$

From these equations we obtain

$$a=0, \quad b=0, \quad c=1, \quad d=\frac{0}{0}-1, \quad e=1.$$

Since the limits of $\frac{0}{0}$ are 2 and 1, the limits of $\frac{0}{0}-1$ are 1 and 0; hence $d=\frac{0}{0}$, where $\frac{0}{0}$ cannot be greater than 1 nor less than 0.

Thus $$z=xz+\frac{0}{0}yz+(1-x)(1-y)z.$$

Observation.—This is a more general form of the problem considered in Article 274.

283. It is evident from the manner in which the equations are derived, that when an additional datum is given, the only alteration made in the equations obtained for the fewer data is the introduction of the new constant into some of the equations.

284. Suppose that the forms considered involve three characters x, y, z, and that each has z as a factor. Then the term independent of the characters may be dispensed with; and four of the eight equations reduce to the form

$$0=0.$$

For, z being a factor of each of the terms, the four sets of singular values in which $z=0$ cause each side of the identity to become 0.

XXIII. ON BOOLE'S GENERAL METHOD.

285. The method of the preceding chapter, while admirably adapted for problems in which the data are what may be called explicit equations, is not adapted for problems in which the data are implicit equations. For the latter we require the special method invented by Boole. I propose to consider the elements of that method in the light of the principles of this work.

286. *To eliminate* x *from an equation* $f(x)=0$, *when it is given that* $x(1-x)=0$.

$$\text{Let } f(x)=ax+b(1-x).$$

Since $x(1-x)=0$, x can assume the value 1 and the value 0. By putting these singular values of x into the above identity, we get $f(1)=a$, and $f(0)=b$.

Hence $$f(1)x+f(0)(1-x)=0.$$

$$\therefore \{f(1)-f(0)\}x=-f(0). \qquad (1)$$

$$\text{and } \{f(1)-f(0)\}(1-x)=f(1). \qquad (2)$$

Multiply together (1) and (2),

then $$\{f(1)-f(0)\}^2 x(1-x)=-f(1)f(0);$$

but $$x(1-x)=0,$$

$$\therefore f(1)f(0)=0.$$

Observation 1.—The condition $f(1)f(0)=0$ will be satisfied independently of $x(1-x)=0$, if $f(1)-f(0)=0$; that is, if the given function does not contain x. It may also be satisfied independently of either of these conditions; for $f(1)-f(0)$ generally involves fractional symbols.

Observation 2.—The equation $f(1)f(0)=0$ is true, when $x=1$, and when $x=0$; therefore when $x=\frac{0}{0}$, where $\frac{0}{0}$ denotes any value lying between 1 and 0. The contradictory values of x are united into one in the same manner as those of a in Art. **278**. Hence

$$f(1)f(0)=0$$

expresses a relation among certain characters of the universe U, which holds true whatever the value of the character x.

Observation 3.—If the given equation is $f(x)=m$; it can be shown that the result of the elimination of x from the equation is

$$m^2-\{f(0)+f(1)\}m+f(1)f(0)=0.$$

287. *To eliminate x and y from an equation* $f(x,y)=0$; *x and y being such that* $x(1-x)=0$, *and* $y(1-y)=0$.

As x and y are independent, x can be eliminated first, and then y. The elimination of x by the preceding Article gives

$$f(1,y)f(0,y)=0.$$

By a double application of the same process with respect to y, we get

$$f(1,1)f(1,0)f(0,1)f(0,0)=0.$$

288. The method is illustrated by the following simple example.

Let the equation be $x=y+z$.

To eliminate x, write the equation in the form

$$f(x)=x-y-z=0.$$

Now $f(1)=1-(y+z)$, and $f(0)=-(y+z)$,

$$\therefore \{1-(y+z)\}(y+z)=0,$$

that is $$yz=0,$$

provided y and z as well as x are known to be single and positive.

To eliminate y, write the equation in the form

$$f(y)=y+z-x=0.$$

Then $$(1+z-x)(z-x)=0,$$

that is, $$z(1-x)=0.$$

289. *The second power of any expression which is a function of single symbols cannot be negative.*

For the expression can be expanded in terms of the primary parts into which the universe is divided by the symbols. The second power of the expansion is equal to the sum of the second powers of the terms of the expansion, as the products all vanish. The second power of each term is composed of the second power of the co-efficient multiplied by the second power of the part. The former cannot be negative (Art. 77) and the latter cannot be negative (Art. 138); hence their product cannot be negative. But the sum of a number of terms, each of which cannot be negative, cannot itself be negative.

290. *To prove that* $x+y+z-2xy-2xz+2yz$ *cannot be negative.*

For it is equal to $\{x-y-z\}^2$.

It is evident that the expression will be as negative as possible, when $x=1, y=1$. Its value is then z.

291. *If each equation of the system*

$$x_1=0, \quad x_2=0, \quad \ldots \quad x_n=0,$$

has its left-hand member of a form which cannot be negative; then the equation

$$x_1+x_2+\ldots+x_n=0$$

is equivalent to the system of equations.

When the equations of the system are true, the compound equation is true, for it is formed by adding the equations of the system. (Art. 115.)

When the compound equation is true, each equation of the system must be true; because the left-hand member of the compound equation is a sum of functions each of which cannot be negative.

292. *If any one equation of the above system,* $x_r=0$, *has a left-hand member* x_r *of a form which can be negative; then* $x_r^2=0$ *and not* $x_r=0$ *must be put into the compound equation.*

If x_r were put into the compound equation, then the truth of each of the original equations could not be inferred necessarily from the truth of the compound equation. But x_r^2 cannot be negative (Art 289). Hence when $x_r^2=0$ is put into the compound equation, its truth and the truth of the others can be inferred from the truth of the compound equation. Now from $x_r^2=0$ we can deduce $x_r=0$, whether x_r is or is not of a form which can be negative.

Hence the equations of the system can be inferred back from the truth of the compound equation, provided x_r be squared.

293. *Example.*

The equation $x=y$ gives $x-y=0$. Now $x-y$ is of a form which can be negative. Hence it is $(x-y)^2=0$ which must be put into an equation, which is to be equivalent to $x=y$ together with other equations.

It is curious to observe that $(x-y)^2=0$ gives

$$x-2xy+y=0,$$

that is, $x(1-y)=0$, and $y(1-x)=0$;

that is, $x=xy$, and $y=yx$.

Thus the single equation $x=y$ must for the purpose considered be resolved into the two equations $x=xy$ and $y=yx$.

EXAMPLES.

At page 56 of his *Syllabus*, De Morgan gives the following twenty-four forms of inference, under the name of Metaphysical Syllogisms :—

1. Dependent of dependent is dependent.
2. Essential of independent is independent.
3. Independent of essential is independent.
4. Essential of essential is essential.
5. Dependent of inessential is inessential.
6. Inessential of dependent is inessential.
7. Repugnant of alternative is dependent.
8. Repugnant of independent is inalternative.
9. Independent of alternative is irrepugnant.
10. Alternative of repugnant is essential.
11. Alternative of inessential is irrepugnant.
12. Inessential of repugnant is inalternative.
13. Dependent of repugnant is repugnant.
14. Essential of irrepugnant is irrepugnant.
15. Irrepugnant of repugnant is independent.
16. Essential of alternative is alternative.
17. Dependent of inalternative is inalternative.
18. Inalternative of alternative is inessential.
19. Repugnant of essential is repugnant.
20. Repugnant of irrepugnant is inessential.
21. Irrepugnant of dependent is irrepugnant.
22. Alternative of dependent is alternative.
23. Alternative of inalternative is independent.
24. Inalternative of essential is inalternative.

In each case three characters of a given universe are considered. Let the first-mentioned character be denoted by x, the second by y, the third by z. The first term in the proposition expresses the relation existing between x

and y; the second that existing between y and z; and the third the relation which exists in consequence between x and z.

By x being	dependent on y	is meant	$x(1-y)=0$.
,,	independent of	,,	$x(1-y)=v$.
,,	essential to	,,	$(1-x)y=0$.
,,	inessential to	,,	$(1-x)y=v$.
,,	repugnant to	,,	$xy=0$.
,,	irrepugnant to	,,	$xy=v$.
,,	alternative to	,,	$(1-x)(1-y)=0$.
,,	inalternative to	,,	$(1-x)(1-y)=v$.

Some of these syllogisms coincide with Aristotelian forms, if the latter have the meaning assigned to them in Articles **256-260**. The truth of the conclusions can be proved by means of the theorem enuntiated in Article **256**—excepting those of the sixteenth and twenty-second forms, which are not true as enuntiated.

Consider the sixteenth. The data are

$$xy=y \quad (1) \qquad \text{and} \quad y+(1-y)z=1. \quad (2).$$

Substitute xy for y in (2),

then $$xy+(1-xy)z=1,$$

that is, $$(1-xy)(1-z)=0.$$

Hence xy, but not y, is alternative to z with respect to 1. It is evident that x is alternative to z with respect to y. The conclusion of the twenty-second form must be corrected by introducing the same condition.

25. For every man in the house there is a person who is aged; some of the men are not aged. It follows that some persons in the house are not men. De Morgan, *Syllabus*, p. 29.

Let U=people in the house;

$$m=\text{man}, \qquad \text{and} \quad a=\text{aged}.$$

Then $m<\bar{a}$ (1). and $\bar{m}(1-a)=\bar{v}$. (2).

From (1) $$1-\bar{m}>\bar{1}-\bar{a};$$

but from (2) $\bar{1}-\bar{a}$ is greater than 0.

$$\therefore 1-m \text{ is greater than } 0.$$

26. Most men in a certain company have coats; most men in the same company have waistcoats; therefore some in the company have coats and waistcoats. De Morgan.

27. Of the inhabitants of a certain country those having black hair form one-tenth, and those who are short-sighted one-twelfth. What proportion must be (1) black-haired and short-sighted, (2) black-haired and not short-sighted, (3) not black-haired and short-sighted, (4) neither black-haired nor short-sighted?

28. Eighteen out of twenty-one *Y*'s are *X*; fifteen out of twenty-one *Y*'s are *Z*; therefore twelve *Z*'s are *X*. De Morgan.

Here the *Y*'s form the universe, and the arithmetical value of the universe is given.

29. For every *Z* there is an *X* which is *Y*; some *Z*'s are not *Y*'s. De Morgan, *Syllabus*, p. 29.

The two conclusions which De Morgan deduces can be proved in the following manner:—

The data are $xy > \bar{z}$ (1). and $z(1-y) = \bar{v}$. (2).

From (2) $(1-z)y = \bar{y} + \bar{v} - \bar{z}$,

$$\therefore x(1-z) > \bar{z} + \bar{y} + \bar{v} - \bar{z} - \bar{y}$$
$$> \bar{v}.$$

Again, from (1) $(1-x)(1-y) > \bar{1} - \bar{x} - \bar{y} + \bar{z}$,

$$\therefore (1-x)z > \bar{1} - \bar{x} - \bar{y} + \bar{z} + \bar{v} - \bar{1} + \bar{y}.$$
$$> \bar{z} + \bar{v} - \bar{x}.$$

30. Of a certain community, which comprises 12,000 persons, three-fourths are Presbyterian, two-thirds Liberal, and one-sixth on the electoral roll. What is the least and what the greatest possible strength of each of the twelve classes formed by taking two of the characters; as, for example, Presbyterian and Liberal, Presbyterian and not Liberal, etc.? Also what is the greatest and what the least possible strength of each of the eight classes formed by taking three characters; as, for example, not Presbyterian and not Liberal and not on the electoral roll?

31. In a series of twelve throws with two dice, ace and

two occurred together in one-half of the throws, ace and not-three occurred together in one-third, and ace occurred in two-thirds. How often, at least, must two and three have occurred together?

32. In a series of throws with two dice, ace and two occurred together in one-sixth; ace and four in one-twelfth. How often, at least, must two and not four have occurred together?

33. Of the matriculated students of a certain University, two-thirds are medical; three-fourths are under the age of 21; and five-sixths are natives of the British Islands. Also the medical under the age of 21 form one-half; and the colonial or foreign under the same age one-twelfth. The number of students is 2400. What is the least, and what the greatest possible strength of each of the remaining ten classes of the second order; also of each of the eight classes of the third order?

34. From P follows Q, and from R follows S; but Q and S cannot both be true; therefore P and R cannot both be true.

Let U = the collection of times considered. p = when P is true, q = when Q is true, etc.

Then the data are

$$p=pq \quad (1). \qquad r=rs \quad (2). \qquad qs=0. \quad (3).$$

Multiply together (1) and (2),

$$\text{then } pr=pqrs,$$
$$=0. \qquad \text{by (3).}$$

35. Every X is one only of the two, P or Q; every Y is both P and Q, except when P is M, and then it is neither; therefore no X is Y. De Morgan, *Formal Logic*, p. 124.

Let U denote the subject of which x, y, p, q are characters. Then m is not a character of U, but of Up. Hence the data are

$$x=x\{p(1-q)+q(1-p)\} \qquad (1)$$
$$y=y\{qp_{1-m}+(1-p)(1-q)\} \qquad (2)$$

Multiply together the two equations; then each of the four terms of the right-hand member will vanish, because

$$q(1-q)=0, \quad p(1-p)=0, \quad \text{and } (1-p)p_{1-m}=0.$$

Hence $xy=0$.

36. Every X is either P, Q, or R; but every P is M, every Q is M, every R is M; therefore every X is M. De Morgan.

37. *Dilemma.* What is A is either B or C; what is D is neither B nor C. Therefore what is D is not A.

38. *Constructive Dilemma.* U's which are x are y; U's which are z are y; and U's are either x or not x and z. Therefore U's are y.

The data are $x=xy$ (1) $z=zy$ (2)

and $x+(1-x)z=1$ (3)

Substitute for x and z in (3) by means of (1) and (2).

Then $xy+zy-xz=1$,

$$\therefore y=\frac{1+xz}{x+z},$$

$$=xz+x(1-z)+(1-x)z+\frac{1}{0}(1-x)(1-z),$$

$$=1.$$

Hence U's are y.

39. The Method of Difference, which bulks so largely in some *Systems* of Logic, can be disposed of in a few lines.

In the states of a certain substance when the property A is present, the property B is also present; and when the property A is absent, the property B is also absent. What follows?

The data are—

$a=ab$ (1) $(1-a)=(1-a)(1-b)$. (2)

From (2) $1-a=1-a-b+ab$,

$\therefore b=ab$,

$\therefore b=a$ from (1).

Hence the states in which the property A is present are identical with the states in which the property B is present.

40. The members of a board were all of them either bondholders or shareholders, but not both; and the bondholders, as it happened, were all on the board. What conclusion can be drawn? Venn, *Mind*, vol. i. p. 487, discussed by Jevons, *Principles of Science*, p. 90.

Let U=members of the company.

d=director b=bondholder s=shareholder.

Then $$d=d\{b(1-s)+s(1-b)\}. \quad (1)$$
and $$b=bd. \quad (2)$$

Put (1) into the form
$$d\{1-b(1-s)-s(1-b)\}=0,$$
and (2) into the form
$$b(1-d)=0;$$
then their left-hand members consist of functions which cannot be negative. Hence these two equations are together equivalent to
$$d\{1-b(1-s)-s(1-b)\}+b(1-d)=0. \quad \text{(Art. 291.)}$$

By means of this equation any one of the three d, b, s, can be expressed in terms of the remaining two. For example
$$d=\frac{b}{b-\{1-b(1-s)-s(1-b)\}},$$
$$=\frac{1}{0}bs+b(1-s)+\frac{0}{0}(1-b)s+0(1-b)(1-s).$$
$$=b(1-s)+\frac{0}{0}(1-b)s, \text{ from the nature of } d,$$
with the condition $bs=0$,
$$=b+\frac{0}{0}s \qquad \text{by taking in } bs=0.$$

For the sake of illustrating its application I have here employed Boole's Method of dealing with implicit equations. The conclusions can be deduced by a simpler method. Thus—

$$d=d(b+s) \quad (1). \qquad \text{(Art. 251.)}$$
and $$db=b. \quad (2).$$

Hence $d = b + ds$;

which is equivalent to $d = b + \frac{0}{0}s$.

Multiply by s,

then $ds = bs + ds$,

$\therefore\ bs = 0$.

41. It has been observed that—In a certain class of substances, where the properties A and B are present, the property C is present; and where B and C are present, A is present. Does it follow that B is present where C and A are present?

Let U = the class of substances.

a = having the property A, etc.

The data are $ab = abc$ and $bc = bca$.

Hence $ab(1-c) + bc(1-a) = 0$. (Art. **291.**)

$$\therefore\ ca = \frac{b(a+c)}{2b},$$

$$= \frac{2}{2}abc + \frac{1}{2}ab(1-c) + \frac{0}{0}a(1-b)c + \frac{0}{0}a(1-b)(1-c)$$

$$+ \frac{1}{2}(1-a)bc + \frac{0}{2}(1-a)b(1-c) + \frac{0}{0}(1-a)(1-b)c$$

$$+ \frac{0}{0}(1-a)(1-b)(1-c).$$

Now $\frac{2}{2} = 1$. (Art. **78.**)

And $\frac{1}{2}$ is impossible. (Art. **78.**) Therefore $ab(1-c) = 0$ and $bc(1-a) = 0$; which are the given equations taken each by itself. The sixth term is cancelled. (Art. **79.**)

Add together the terms whose co-efficient is $\frac{0}{0}$,

then $ca = abc + \frac{0}{0}(1-b)$.

Hence it does not follow that B is present where C and A are present.

42. Suppose that an analysis of the properties of a particular class of substances has led to the following general conclusions, viz.:—

1st, That wherever the properties *A* and *B* are combined, either the property *C*, or the property *D*, is present also; but they are not jointly present.

2d, That wherever the properties *B* and *C* are combined, the properties *A* and *D* are either both present with them, or both absent.

3d, That wherever the properties *A* and *B* are both absent, the properties *C* and *D* are both absent also; and *vice versa*, where the properties *C* and *D* are both absent, *A* and *B* are both absent also.

Let it then be required from the above to determine what may be concluded in any particular instance from the presence of the property *A* with respect to the presence or absence of the properties *B* and *C*, paying no regard to the property *D*. Boole, *Laws of Thought*, p. 118.

Here the universe under consideration is a particular class of Substances. Let $a=$ having the property *A*;
$a'=$ without the property *A*; and so on.

Then
$$ab=ab\,(c+d). \qquad (1)$$
$$bc=bc\,(ad+a'd'). \qquad (2)$$
$$a'b'=c'd'. \qquad (3)$$

Since the first equation, when put into the form
$$ab\{1-c-d\}=0,$$
contains a function which can be negative, it must be squared before being put into the equivalent equation. (Art. **292**.) This squaring amounts to writing the equation in the form
$$ab=ab\{cd'+dc'\}.$$

Similarly (3) must be squared. Hence the equivalent equation is
$$ab\{1-cd'-dc'\}+bc\{1-ad-a'd'\}+a'b'(1-c'd')$$
$$+c'd'(1-a'b')=0.$$

As d enters into this equation, and is not to enter into the required conclusion, it must be eliminated by Art. 286. Now

$$f(1)=abc+bca'+a'b',$$

and $$f(0)=abc'+bca+a'b'c+c'(1-a'b'),$$

∴ $$abc+a'b'c=0$$

is true whatever value d may have, consistent with being positive and single.

Hence $$a=\frac{b'c}{c},$$

$$=b'c+\frac{0}{0}bc'+\frac{0}{0}b'c',$$

$$=b'c+\frac{0}{0}c'.$$

Hence the substances, which have the property A, are identical with those which have the property C and are without the property B, together with an indefinite part of those which are without the property C.

The conclusion can be deduced very simply by the following method—

$$ab=ab(c+d). \qquad (1)$$

$$bc=bc\{ad+(1-a)(1-d)\}. \qquad (2)$$

$$(1-a)(1-b)=(1-c)(1-d). \qquad (3)$$

Multiply together (1) and (2),

then $$abc=abc(acd+ad),$$

$$=2\ abcd;$$

but abc is single,

$$\therefore\ abc=0. \qquad (4)$$

Again, multiply (3) by c,

then $c(1-a)(1-b)=0$,

$\therefore\ c(1-a-b)=0$, by (4)

$$\therefore\ a=\frac{c(1-b)}{c}.$$

This expression for a coincides with that obtained above.

43. Find also what may be concluded from the presence of the property C with reference to the properties A and B.

44. In a certain class of substances, those which have the property A and the property X, together with those which have the property B and are without the property X, are identical with those which have the property C. Also those which have the property D and the property X, together with those which have the property E and are without the property X, are identical with those which have the property F. What relation necessarily exists between the properties A, B, C, D, E, F?

The data are—

$$ax+b(1-x)=c. \qquad (1)$$
$$dx+e(1-x)=f. \qquad (2)$$

From (1) $$x=\frac{c-b}{a-b}.$$

From (2) $$x=\frac{f-e}{d-e}.$$

Hence, as the equations are true simultaneously,

$$\frac{c-b}{a-b}=\frac{f-e}{d-e};$$
$$\therefore (c-b)(d-e)=(a-b)(f-e);$$
$$\therefore af+bd+ce=ae+bf+cd.$$

45. In a certain succession of throws with six dice, those having a two and an ace, together with those having a three and a six, were identical with the whole; and those having a four and an ace, together with those having a five and a six, were identical with the whole. What relations existed between having a two, having a three, having a four, and having a five?

'Having a two' here means 'having one two at least.'

The equations are—

$$ax+by=1. \qquad (1)$$
$$cx+dy=1. \qquad (2)$$

Multiply (1) by d and (2) by b, and subtract;

then $(ad-bc)x=d-b$;

$$\therefore x=\frac{d-b}{ad-bc}.$$

Expand $\frac{d-b}{ad-bc}$ in terms of the primary parts into which the universe is divided by a, b, c, d. Then since x is possible, each of the co-efficients of $\frac{1}{0}$ must be 0. We thus obtain

$ab(1-c)(1-d)=0$ (1) $\quad (1-a)b(1-c)(1-d)=0$ (2)

$(1-a)(1-b)cd=0$ (3) $\quad (1-a)(1-b)(1-c)d=0$ (4)

The equations (1) and (2) can be combined into the equivalent equation

$$b(1-c)(1-d)=0. \quad (5)$$

Similarly (3) and (4) into $(1-a)(1-b)d=0$. (6)

Equation (5) can be put into the form

$$b=b\{c+(1-c)d\},$$

that is, the throws having a three had a four or, if not a four, a five.

Similarly $d=d\{a+(1-a)b\}$.

From the expansion for y other two relations can be obtained, viz.—

$$a=a\{c+(1-c)d\};$$

and $c=c\{a+(1-a)b\}$.

46. Boole solves the following highly complex problem :—

Let the observations of a class of natural productions be supposed to have led to the following general results :—

1st, That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.

2d, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.

3d, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.

Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D. *Laws of Thought*, p. 146.

The equations are

$$(1-a)(1-c)=(1-a)(1-c)e\{b(1-d)+d(1-b)\} \quad (1)$$
$$ad(1-e)=ad(1-e)\{bc+(1-b)(1-c)\} \quad (2)$$
$$ab+ae(1-b)=d(1-c)+c(1-d) \quad (3)$$

The function of (3) must be squared before being put into the single equation. (Art. 292.) Then e can be eliminated by means of Article 286. By expanding the function for a in terms of b, c, d, we get

$$a=c(1-d)+d(1-c)+(1-b)(1-c)(1-d). \quad (1)$$

By taking the function for b in terms of a, c, d and expanding it, we obtain

$$b=(1-a)(1-c)(1-d)+\frac{0}{0}\{ac(1-d)+a(1-c)d+(1-a)cd\}; \quad (2)$$

and $\quad acd=0 \quad (3), \quad (1-a)c(1-d)=0, \quad (4)$

$$(1-a)(1-c)d=0. \quad (5)$$

By means of (3), (4), and (5) equation (2) can be reduced to

$$b=\frac{0}{0}\{c+ad\}+(1-a)(1-c).$$

47. A certain philosopher has observed that his library satisfies the following conditions:—

(1.) Languages, which contain Dictionaries and Poetry, contain Mathematics.

(2.) Languages, which contain Novels and Mathematics, contain Natural Philosophy.

(3.) Languages, which contain Natural Philosophy, contain either Mathematics or Dictionaries, but not both.

(4.) Languages, which contain no Novels, contain either Poetry or no Natural Philosophy.

(5.) Languages, which contain Novels, Dictionaries, and no Mathematics, contain either Poetry and no Natural Philosophy, or Natural Philosophy and no Poetry.

(6.) Languages, which contain Poetry and Natural Philosophy, contain Dictionaries and Mathematics, or no Dictionaries.

Are these data all independent of one another? Is there any special relation between two only of the subjects? What are the relations among the subjects independently of Natural Philosophy?

48. To find $(1-x-y)^2$ in terms of $x+y$, $x-y$, and xy.

Let $(1-x-y)^2=a+b\,(x+y)+c\,(x-y)+dxy$.

Then
$$\begin{aligned} 1&=a+2b &&&+d. \\ 0&=a+b &&+c. \\ 0&=a+b &&-c. \\ 1&=a. \end{aligned}$$

The value of the determinant of the right-hand members is 2. When the equations are solved by the method of determinants, we get

$$a=\frac{2}{2},\quad b=-\frac{2}{2},\quad c=\frac{0}{2},\quad d=\frac{4}{2}.$$

Hence by Articles **78** and **79**

$$a=1,\qquad b=-1,\qquad c=0,\qquad d=2.$$

and
$$(1-x-y)^2=1-(x+y)+2xy.$$

I introduce this example to illustrate the truth and power of the method by indeterminate co-efficients.

49. There are three bags, one containing 2 white and 3 red balls, the second containing 5 white and 2 red balls, the third containing 4 white and 7 red balls; and all the bags are equally likely to be drawn from. A white ball has been drawn; what is the chance that it was drawn from the first bag? Gross, *Algebra*, p. 225.

Let $U=$ succession of drawings of a ball.

$a=$ from the first bag ; $b=$ from the second bag ;

$c=$ from the third bag.

$a_w=$from the first bag and white,

$a_r=$ „ and red.

The data are

$$a=\tfrac{1}{3} \qquad b=\tfrac{1}{3} \qquad c=\tfrac{1}{3}$$

$$a_w=\tfrac{1}{3}\,\tfrac{2}{5} \qquad a_r=\tfrac{1}{3}\,\tfrac{3}{5}$$

$$b_w=\tfrac{1}{3}\,\tfrac{5}{7} \qquad b_r=\tfrac{1}{3}\,\tfrac{2}{7}$$

$$c_w=\tfrac{1}{3}\,\tfrac{4}{11} \qquad c_r=\tfrac{1}{3}\,\tfrac{4}{11}$$

Now $a+b+c=1,$

$$\therefore\ w=a_w+b_w+c_w. \qquad \text{(Art. 71.)}$$

Probability required $=\dfrac{\overline{a_w}}{w}$

$$=\frac{\frac{1}{3}\,\frac{2}{5}}{\frac{1}{3}\left\{\frac{2}{5}+\frac{5}{7}+\frac{4}{11}\right\}}.$$

$$=\frac{\frac{2}{5}}{\frac{2}{5}+\frac{5}{7}+\frac{4}{11}}.$$

50. Suppose that in the drawing of balls from an urn attention had only been paid to those cases in which the balls drawn were either of a particular colour, 'white,' or of a particular composition, 'marble,' or were marked by both these characters, no record having been kept of those cases in which a ball that was neither white nor of marble had been drawn. Let it then have been found, that whenever the supposed condition was satisfied, there was a probability $\bar{p}$ that a white ball would be drawn, and a probability $\bar{q}$ that a marble ball would be drawn : and from these data alone let it be required to find the probability that in the next drawing, without reference at all to the condition above mentioned, a white ball will be drawn ; also the probability that a marble ball will be drawn. Boole, *Laws of Thought*, p. 262.

$U=$ long succession of drawings of a ball from the urn ;

$x=$ producing a white ball,

$y=$ producing a marble ball.

Let $$x+(1-x)y=a.$$

Then $$a_x=\bar{a}\bar{p}, \quad \text{and } a_y=\bar{a}\bar{q}.$$

It is required to find $\bar{x}$ and $\bar{y}$.

Now $$a_x=\{x+(1-x)y\}x,$$
$$=x.$$

Similarly $$a_y=y.$$

Hence $$\bar{p}=\frac{\bar{x}}{\overline{x+(1-x)y}} \qquad (1) \qquad \text{(Art. 64.)}$$

and $$\bar{q}=\frac{\bar{y}}{\overline{x+(1-x)y}} \qquad (2)$$

From (1) and (2) $$\bar{p}\bar{y}=\bar{q}\bar{x}. \qquad (3)$$

From (1) $$\bar{x}=\bar{p}\{\overline{x+(1-x)y}\},$$
$$=\bar{p}\bar{x}+\bar{p}\bar{y}-\bar{p}\overline{xy};$$

hence from (3) $$\frac{\bar{x}\{\bar{p}+\bar{q}-\bar{1}\}}{\bar{p}}=\overline{xy}.$$

Boole proceeds further on the assumption that $\overline{xy}=\bar{x}\bar{y}$. That is only the most probable assumption among an infinite number, and the value for $\bar{x}$ deduced by its means is only the most probable value of the probability of a given drawing producing a white ball. Let the assumption be introduced,

then $$\bar{y}=\frac{\bar{p}+\bar{q}-\bar{1}}{\bar{p}}.$$

And similarly $$\bar{x}=\frac{\bar{p}+\bar{q}-\bar{1}}{\bar{q}}.$$

51. The probability that a witness A speaks the truth is $\bar{p}_1$, the probability that another witness B speaks the truth is $\bar{p}_2$, and the probability that they disagree in a statement is $\bar{p}_3$. What is the probability that if they agree, their statement is true? Boole, *Laws of Thought*, p. 279.

Let $U=$ Testimonies by A and by B about a fact;
$x=$ in which A told the truth,
$y=$ in which B told the truth.

Then the data are—

$$x=\bar{p}_1 \quad (1) \qquad y=\bar{p}_2 \quad (2)$$

and

$$x(1-y)+y(1-x)=\bar{p}_3. \quad (3)$$

It is required to find the value of the dependent xy in

$$\{xy+(1-x)(1-y)\}_{xy}.$$

$$\text{Required value} = \frac{\overline{xy}}{\overline{xy+(1-x)(1-y)}}. \qquad \text{(Art. 64.)}$$

Now from (3) $2\overline{xy}=\bar{x}+\bar{y}-\bar{p}_3,$

$=\bar{p}_1+\bar{p}_2-\bar{p}_3.$ from (1) and (2)

Also from (3) $\overline{xy+(1-x)(1-y)}=\overline{1}-\bar{p}_3.$

$$\text{Hence required value} = \frac{1}{2}\frac{\bar{p}_1+\bar{p}_2-\bar{p}_3}{\overline{1}-\bar{p}_3}.$$

Cor. The value of the dependent $(1-x)(1-y)$ in

$$\{xy+(1-x)(1-y)\}_{(1-x)(1-y)}$$

$$\text{is} \quad \frac{\overline{2}-\bar{p}_1-\bar{p}_2-\bar{p}_3}{\overline{2}-(\overline{1}-\bar{p}_3)}.$$

To deduce these conclusions Boole employs a long and elaborate process. He does not see that $\overline{xy}$ can be deduced by very simple algebraic operations from the given data. At the end of his investigation he remarks that the number of the data exceeds that of the simple events which they involve. The considerations of Article **151** show that the statement is not correct—that the data are just sufficient to determine all the single functions of x and y necessarily.

52. *A* says that *B* says that a certain event took place; required the probability that the event did take place, p_1 and p_2 being *A*'s and *B*'s respective probabilities of speaking the truth. *Mathematics from the Educational Times*, vol. xxvii.

We must suppose that a succession of statements of *A* about *B*'s statements about an event taking place have been observed. It is required to deduce the probability that one of these statements refers to an event which actually took place.

Let $U=$ statements of A about B's statements about an event taking place.

$x=$ which truly reported a statement made by B;

$y=$ which truly reported the event.

The data are—

$$x=\bar{p}_1 \quad (1) \qquad x_y=\bar{p}_1\bar{p}_2. \quad (2)$$

and $\bar{y}$ is required.

Now $xy=x_y$,

$$\therefore \quad y=\frac{x_y}{x},$$

$$=x_y x+\frac{1}{0}x_y(1-x)+\frac{0}{1}(1-x_y)x+\frac{0}{0}(1-x_y)(1-x),$$

$$=x_y+\frac{0}{0}(1-x).$$

$$\therefore \quad \bar{y}=\bar{p}_1\bar{p}_2+\frac{0}{0}(\overline{1-\bar{p}_1}).$$

Hence $\qquad \bar{y}>\bar{p}_1\bar{p}_2$;

and $\qquad <\bar{p}_1\bar{p}_2+\overline{1-\bar{p}_1}$.

There has been much discussion about the true answer to the above question. No fewer than four different solutions are given in the Reprint, viz.—

Todhunter's Algebra $\qquad p_1p_2+(1-p_1)(1-p_2)$.

Artemas Martin $\qquad p_1\{p_1p_2+(1-p_1)(1-p_2)\}$.

American Mathematicians and Woolhouse $\qquad p_1p_2$.

Cayley $\qquad p_1p_2+\beta(1-p_1)(1-p_2)+\kappa(1-\beta)(1-p_1)$.

The meaning of β and κ in Cayley's solution is explained by the following paragraph:—

'B told A that the event happened, or he did not tell A this; the only evidence is A's statement that B told him that the event happened; and the chances are p_1 and $1-p_1$. But, in the latter case, either B told A that the event did not happen, or he did not tell him at all; the chances (on the supposition of the incorrectness of A's statement) are β and $1-\beta$; and the chances of the three cases are thus p_1, $\beta(1-p_1)$, and $(1-\beta)(1-p_1)$. On the suppositions of the first and second cases respectively, the chances for the event having happened are p_2 and $1-p_2$; on the supposition of the third case (viz., here there is no information as to the event) the chance is κ, the antecedent probability; and the whole chance in favour of the event is

$$p_1p_2+\beta(1-p_1)(1-p_2)+\kappa(1-\beta)(1-p_1).'$$

Todhunter assumes that $\frac{0}{0}$ is equal to $1-p_1$; and Woolhouse assumes that it is equal to 0. Cayley's solution expresses $\frac{0}{0}$ in terms of the unknown quantities β and κ.

The next example contains the solution of the general problem, when there are n persons.

53. A_1 says that A_2 says that A_3 says . . . that A_n says that a certain event took place. The probabilities of A_1, A_2, A_3, . . . A_n respectively speaking the truth are $p_1, p_2, p_3, \ldots p_n$. Required the probability that the event took place.

Let $U=$ succession of statements of A_1 about A_2 saying etc.

x_1 = which reported truly a statement of A_2
x_2 = " " A_3
x_3 = " " A_4
" " " "
x_{n-1} = " " A_n
x_n = which reported truly the event.

Let $x_1 \cdot x_2$ denote that x_2 is formally dependent on x_1; that is, $x_{1_{x_2}}$.

Then $x_1 \cdot x_2 \cdot x_3 \ldots x_{n-1} \cdot x_n = \bar{p}_1\bar{p}_2\bar{p}_3 \ldots \bar{p}_{n-1}\bar{p}_n$
and $x_1 \cdot x_2 \cdot x_3 \ldots x_{n-1} = \bar{p}_1\bar{p}_2\bar{p}_3 \ldots \bar{p}_{n-1}$.

Now $$x_n = \frac{x_1 \cdot x_2 \cdot x_3 \ldots x_{n-1} \cdot x_n}{x_1 \cdot x_2 \cdot x_3 \ldots x_{n-1}}$$

$$= x_1 \cdot x_2 \cdot x_3 \ldots x_{n-1} \cdot x_n + \frac{0}{0}(1 - x_1 \cdot x_2 \cdot x_3 \ldots x_{n-1}),$$

$$= \bar{p}_1\bar{p}_2 \ldots \bar{p}_{n-1}\bar{p}_n + \frac{0}{0}(1-\bar{p}_1\bar{p}_2 \ldots \bar{p}_{n-1}).$$

Cor. 1. Suppose that each of the persons always reports truly.

Then p_1, p_2, etc., each equal to 1;

and therefore $$x_n = 1 + \frac{0}{0} \times 0 = 1.$$

COR. 2. Suppose that each person excepting A_n always reports truly.

Then $$x_n = \bar{p}_n.$$

COR. 3. Suppose that A_n always reports falsely.

Then $$\bar{p}_n = 0,$$

and $$x_n = \frac{0}{0}(1 - \bar{p}_1 \bar{p}_2 \ldots \bar{p}_{n-1}).$$

COR. 4. Suppose that any other of the persons excepting A_n always reports falsely.

Then $$x_n = \frac{0}{0}.$$

COR. 5. Suppose that each reports falsely as often as truly.

Then each $\bar{p}$ is equal to $\frac{1}{2}$.

Hence $$x_n = \left(\frac{1}{2}\right)^n + \frac{0}{0}\left\{1 - \left(\frac{1}{2}\right)^{n-1}\right\}.$$

When n is infinitely great, the range between the limits is 1; that is, there is complete uncertainty.

54. A goes to hall p times in a consecutive days and sees B there q times. What is the most probable number of times that B was in hall in the a days? Whitworth, *Choice and Chance*, p. 239.

Let U = the consecutive days;
x = on which A went to hall,
y = on which B went to hall.

The data are—

$$U = \bar{a}, \qquad Ux = \bar{p}, \qquad Uxy = \bar{q};$$

and it is required to find Uy.

Now $$y = xy + \frac{0}{0}(1 - x),$$

$$\therefore Uy = \bar{q} + \frac{0}{0}(\bar{a} - \bar{p}).$$

$$\therefore Uy > \bar{q} \text{ and } < \bar{q} + \bar{a} - \bar{p}.$$

We cannot proceed further without assuming a condition which is not given.

Assume that B is as likely to go to hall when A does not go, as when A does go; then

$$Uy = \bar{q} + \frac{\bar{q}}{\bar{p}}(\bar{a} - \bar{p}),$$

$$= \frac{\bar{q}\bar{a}}{\bar{p}}.$$

55. The probabilities of two causes A_1 and A_2 are $\bar{a}_1$ and $\bar{a}_2$ respectively. The probability that if the cause A_1 present itself, an event E will accompany it (whether as a consequence of the cause A_1 or not) is $\bar{p}_1$, and the probability that if the cause A_2 present itself, that event E will accompany it, whether as a consequence of it or not, is $\bar{p}_2$. Moreover, the event E cannot appear in the absence of both the causes A_1 and A_2. Required the probability of the event E. Boole, *Laws of Thought*, p. 321.

The causes A_1 and A_2 and the event E are characters of a definite universe. Let them be denoted by x, y, z respectively.

The data are—

$$x = \bar{a}_1 \quad (1) \qquad y = \bar{a}_2 \quad (2)$$

$$xz = \bar{a}_1\bar{p}_1 \quad (3) \qquad yz = \bar{a}_2\bar{p}_2 \quad (4)$$

and $$(1-x)(1-y)z = 0. \quad (5)$$

Article **282** shows that the probability required cannot be determined exactly from the data. The limits, between which it must lie, are given in Article **274**.

Boole gives the relations which exist among the data. These relations can be determined very readily by means of the method of Article **226**. Thus—

$$1 - x = \bar{1} - \bar{a}_1 \quad \text{and} \quad xz = \bar{a}_1\bar{p}_1,$$

$$\therefore 0 > \bar{1} - \bar{a}_1 + \bar{a}_1\bar{p}_1 - \bar{1};$$

$$\text{that is, } \bar{a}_1 > \bar{a}_1\bar{p}_1.$$

Similarly $$\bar{a}_2 > \bar{a}_2\bar{p}_2.$$

Again $$x(1-z) = x - xz = \bar{a}_1 - \bar{a}_1\bar{p}_1,$$

$$\text{and } yz = \bar{a}_2\bar{p}_2,$$

$$\therefore\ 0 > \bar{a}_1 - \bar{a}_1\bar{p}_1 + \bar{a}_2\bar{p}_2 - \bar{1};$$

that is, $\bar{a}_2\bar{p}_2 < \bar{1} - \bar{a}_1 + \bar{a}_1\bar{p}_1.$

Similarly $\bar{a}_1\bar{p}_1 < \bar{1} - \bar{a}_2 + \bar{a}_2\bar{p}_2.$

And there is no other way in which the process can be applied to the given data; hence these are all the relations which exist among them. The solution of this problem was discussed by Cayley, Boole, and Wilbraham in the *Philosophical Magazine*, 4th series, vol. vi. p. 259; vol. vii. p. 29 and p. 465; vol. viii. p. 87 and p. 431. Wilbraham gives as the solution

$$z = a_1p_1 + a_2p_2 - \xi,$$

where ξ is necessarily less than either a_1p_1 or a_2p_2. He maintains that we can get no further in the solution without further assumptions or data, and remarks that the disadvantage of Boole's method in such cases is, that it does not show us whether the problem is really determinate.

The method by indeterminate co-efficients gives a solution which agrees with Wilbraham's (Art. 282), and in every case supplies the desiderated information as to whether the proposed problem is determinate.

What is given by Boole's solution is not the mathematical probability of the event E, but the most probable value of the probability which can be deduced from the given data.

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