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The Point-Source Model with Coefficient of Opacity $k = k_1 \varrho T^{-3.5}$.

By **Bengt Strömgren.**

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The point-source model with $k = k_1 \varrho T^{-3.5}$ is studied according to the ideas of E. A. MILNE. The consequences of MILNES postulate of a stellar nucleus are discussed on the point-source model with the aid of numerical solutions of the equations of the problem. It is shown that on this model the existence of a nucleus of extreme density is connected with overcompressibility in the nucleus, as on the standard model. With the aid of the computed solutions the discordance between „astronomical" and „theoretical" coefficient of opacity on MILNE-models is discussed.

1. The problem of stellar structure has recently been considered by E. A. MILNE* on lines that differ from those followed by A. S. EDDINGTON. The present paper is concerned with questions that have arisen naturally out of MILNES investigations.

Quite general considerations on energy-generation in the stars and stability of the stars lead to the idea of a stellar nucleus of extreme density and temperature. The question of the existence of stellar configurations with a nucleus of this character was then examined. To treat this question certain models were investigated, primarily the so called standard model, with constant opacity and constant energy-generation per unit mass. The possibilities for configurations differing so widely from EDDINGTONS configuration on the same model arose from the singular nature of EDDINGTONS solution: considering mass M and radius R to be fixed and L to be a freely varying parameter, neighbouring solutions of EDDINGTONS solution are obtained which differ infinitely from EDDINGTONS solution.

It is therefore natural, for fixed M and R, to vary L freely, including all solutions corresponding hereto in the discussion. It is true, that these solutions perhaps will not all be physically possible: they have to satisfy a certain condition (which in particular EDDINGTONS solution is known to satisfy), viz. $\int_{\text{centre}}^{\text{boundary}} dM_r = M$; but it will be difficult a priori to survey the possibilities to satisfy the condition mentioned even for neighbouring solutions of EDDINGTONS solution, from just the reason that EDDINGTONS solution is singular.

* M. N. **91**, 4, 1930.

The procedure mentioned, for fixed M and R to construct solutions corresponding to all values of L and then to examine whether or not they satisfy the condition $\int_{\text{centre}}^{\text{boundary}} dM_r = M$, was adopted by MILNE and applied to the standard model. MILNES discussion was not completed, part of it being postponed for further quantitative investigations. It was shown however. independently by T. G. COWLING* and by myself**, that the discussion could be carried further without the quantitative investigation mentioned. The outcome was, that with an equation of state, that was the equation of state of a perfect gas at low densities, deviating for high densities in the sense of incompressibility, no solution with a nucleus was possible. MILNE***, considering the evidence for a stellar nucleus to be binding, concluded from this result, that under the extreme conditions near the centre, the equation of state would (at some critical density, cfr. p. 366) deviate in the sense of overcompressibility.

This result was reached by a discussion of the standard model. It is the object of the present paper to extend the discussion, using the point-source model with coefficient of opacity $k = k_1 \varrho T^{-3.5}$, this model being the most plausible model from physical reasons. (On this model the nature of the particular solution corresponding to the EMDEN-EDDINGTON solution on the standard model is known from a numerical investigation by EDDINGTON****).

Discussing the model mentioned we shall follow MILNE: we *postulate* a nucleus and from the conditions of the problem we *deduce* something of the physical properties of the nucleus. It will be difficult to discuss any result reached from the physical point of view, on account of the extreme conditions of the nucleus. As far as can be judged however, the result is quite plausible.

Section 2 contains a survey of the problem, in sections 3 and 4 the method used in the numerical integration of the equations of the model is developed; section 5 contains the results obtained and a discussion of the nature of the solutions, section 6 the interpretation of the results; section 7 is concerned with the mass-luminosity relation section 8 contains a summary of the discussion.

* M. N. **91**, 472, 1931.
** M. N. **91**, 466, 1931.
*** M. N. to appear.
**** The internal constitution of the stars, p. 124.

2. In the present investigation the variables and the differential equations that govern their variations are the classical ones of EDDINGTON. To fix the ideas the equations are given below.

$$\left.\begin{aligned} d(T^4) &= -\frac{3}{ac} k \varrho \frac{L_r}{4\pi r^2} dr \\ d(p+p') &= -\frac{GM_r}{r^2} \varrho\, dr \\ dM_r &= 4\pi r^2 \varrho\, dr \\ p+p' &= \varphi(\varrho, T) + \frac{a}{3} T^4 \left(= \frac{\overline{R}}{\mu} \varrho T + \frac{a}{3} T^4\right) \\ k &= k(\varrho, T)\left(= k_1 \frac{\varrho}{T^{3.5}}\right) \end{aligned}\right\} \quad (1)$$

r Distance from the centre,
T Temperature,
ϱ Density,
p Gas-pressure,
p' Radiation-pressure,
M_r Mass inside sphere of radius r,
L_r Net-flux through sphere of radius r,
k Coefficient of opacity,
μ Molecular weight,
a Stefans constant,
c Velocity of light,
$\overline{R}$ Gas constant.

The distance from the centre, r, is the independent variable, the dependent variables are T, ϱ, M_r and L_r. When for a certain r the variables T, ϱ, M_r and L_r are known, then dT, $d\varrho$ and dM_r can be calculated, provided the functions $\varphi(\varrho, T)$ and $k(\varrho, T)$ are known functions. Again, the boundary values of the variables are: $T = T_0$ (a certain boundary temperature), $\varrho = 0$, $M_r = M$ (the total mass), $L_r = L$ (the luminosity).

It is thus evident that, given the function $L_r(r)$, the integration from the boundary inwards can be carried out. The net-flux L_r is known to vary according to the equation

$$dL_r(r) = 4\pi r^2 \varrho\, \varepsilon(r)\, dr$$

where $\varepsilon(r)$ is the energy-generation per unit mass at the distance r from the centre. As long as the nature of the energy-generation in the star remains obscure, not much can be inferred of the function L_r. If however the idea of a stellar nucleus as the seat of energy-generation is adopted, it is natural to assume L_r = constant = L until extreme conditions are reached, assuming normal stellar matter to be inert. On this assumption the integration can be carried on from the boundary inwards, until conditions become extreme.

We thus choose the following procedure: we integrate from the boundary inwards with L_r = constant = L, until extreme conditions are reached; then, with the aid of the condition $\int_{\text{centre}}^{\text{boundary}} dM_r = M$, we can gain insight into the nature of the nucleus.

Consider now the integration inwards from the surface on the assumption L_r = constant = L. We shall first decide what equation of state is to be used. Until degeneracy sets in we shall use the equation of state of a perfect gas with constant molecular weight; when degeneracy sets in we shall not carry the actual integration further, qualitative arguments of the same nature as used in the discussion of the standard model (based on R. H. FOWLERS theorems) being sufficient.

The solution will depend on 5 parameters: The mass M, the radius R, the luminosity L, the intrinsic opacity k_1 and the molecular weight μ. (To know the perfect gas-part of the solution for any set of values of the 5 parameters it is necessary to carry out a double infinity of numerical integrations only, all solutions being derivable from these by three homology-transformations; this means a saving of labour but is not essential in the general discussion).

A way of choosing parameters naturally suggests itself: take for M, R and L the observed values for the sun, or any star for which they are known from observation, and take for k_1 and μ the values suggested by the theory of the physics of normal high-temperature matter (viz. $k_1 = 10^{26}$, $\mu = 2.2$). If this is done we obtain a solution for which $M_r = 0$ at $r > 0$, or the mass is used up before the centre is reached. This expresses the well-known fact of the discordance between „theoretical" and „astronomical" coefficient of opacity. With the values quoted for k_1 and μ nothing more can be done on the model used; the same is true of any model, in which sinks of energy are not allowed. So, to reach any results on the adopted model, we must choose greater k_1 or smaller μ (or both of course).

The evidence for and against low μ (great hydrogen-abundance) has been discussed by EDDINGTON. We shall not consider solutions with low μ in the present paper, reserving this case for a following investigation. Consequently we have to choose greater values of k_1; to investigate one single star of observed M, R and L we have to construct solutions for a whole range of k_1-values greater than the theoretical value.

The next step is to choose the star [i. e. to decide which set of (M, R, L)-values to use]. We shall in the present paper confine ourselves to discussion of stars of solar mass, the discussion of stars of other masses being reserved for continued investigation. Results reached for stars of solar mass will probably be typical for an extended range of masses.

Now it is known from observation that stars of solar mass are either stars on the main series or white dwarfs. Interpreting observational results schematically, i. e. ignoring the scattering of observed L- and R-values, we are led to consider (for stars of solar mass) two sets of values of (M, R, L): $(M_\odot, R_\odot, L_\odot)$ and one further set (more vaguely defined) corresponding to white dwarfs. The schematical interpretation is suggested, quite apart from observational evidence, by general theoretical considerations (cfr. p. 368).

We shall consider white dwarfs first. When for a white dwarf the integration is carried on from the boundary inwards, degeneracy very soon sets in, in fact when only a small fraction of the mass has been used up. Now, when degeneracy has set in, and when the ratio of radiation-pressure and gas-pressure is so small as it turns out to be in a white dwarf, the influence of change of source-distribution (e. g. from that of the standard model to that here adopted) is small, and cannot well be discussed at all till a theory of the coefficient of opacity of a degenerate gas has been developed. Moreover arguments regarding the nucleus will be different for ordinary stars and for white dwarfs. We shall therefore leave out the white dwarfs from the general discussion, referring to MILNES treatment. It may be noted however that the methods developed in the following sections are well suited for a discussion of the perfect gas fringes of white dwarfs.

We are thus led to consider solutions corresponding to the following values of the 5 parameters: $M = M_\odot$, $R = R_\odot$, $L = L_\odot$, $\mu = 2.2$ and k_1 varying through a certain range above its theoretical value. Studying these solutions we shall gain insight into the structure of stars of solar mass and "neighbouring" masses.

In the following sections we shall consider the problem of constructing the perfect gas-part of a solution corresponding to specified values of the parameters M, R, L, μ and k_1. We shall first treat the outer part of the star, defined as containing a negligible fraction of the mass, taking $M_r = M$, and then pass over to the integration through the interior.

3. Consider a solution corresponding to (M, R, L, μ, k_1). We want to know the values of T and ϱ at some specified point r below the surface, chosen so that M_r/M is greater than say 0.999 (or 0.9998); with the values of the parameters with which we are concerned a point can be chosen, satisfying the above requirement, so far below the surface that the temperature has risen to about one million degrees, so that at this point matter will be normal high-temperature matter.

Introducing instead of r the new variable

$$x = 1/r$$

we have from the equations on p. 347.

$$d\,p' = \frac{1}{4\,\pi\,c}\,k\,\varrho\,L_r\,d\,x, \tag{2}$$

$$d\,p + d\,p' = G M_r\,\varrho\,d\,x. \tag{3}$$

Now $L_r = L$ on the adopted model, and $M_r = M$ throughout the region considered. Integrating from the boundary (which need not be definite) up to the point considered, we get:

$$p' - p'_{\text{boundary}} = \frac{L}{4\,\pi\,c}\int k\,\varrho\,d\,x, \tag{4}$$

$$p + p' - p'_{\text{boundary}} = G M \int \varrho\,d\,x. \tag{5}$$

Proceeding with these equations we should encounter the well-known phenomenon of a non-terminating chromosphere; as we are not interested in the chromosphere in this connection, we shall consider only that part of the star which is inside the photosphere (cfr. T. G. COWLING*); consequently we shall ignore T_0^4 in comparison with T^4, or we shall put p'_{boundary} equal to zero:

$$p' = \frac{L}{4\,\pi\,c}\int k\,\varrho\,d\,x, \tag{6}$$

$$p + p' = G M \int \varrho\,d\,x. \tag{7}$$

Dividing these equations we get:

$$\frac{p}{p'} + 1 = \frac{4\,\pi\,G\,c\,M}{L}\,\frac{\int \varrho\,d\,x}{\int k\,\varrho\,d\,x} \tag{8}$$

or

$$\lambda + 1 = \frac{4\,\pi\,G\,c\,M}{\bar{k}\,L}, \tag{9}$$

* M. N. **91**, 92, 1930.

where we have put

$$\lambda = \frac{p}{p'}, \tag{10}$$

$$\frac{1}{\bar{k}} = \frac{\int \varrho\, d x}{\int k \varrho\, d x}. \tag{11}$$

If in this relation we introduce dp' instead of dx with the aid of (2), we get

$$\frac{1}{\bar{k}} = \frac{\int \frac{1}{k} d p'}{\int d p'} = \frac{\int \frac{1}{k} d p'}{p'} = \frac{\int \frac{1}{k} d (T^4)}{T^4}. \tag{12}$$

This shows that $1/\bar{k}$ is the mean value of $1/k$ taken with equal steps of T^4: hence $1/\bar{k}$ will not be far from $1/k$. We put:

$$\frac{1}{\bar{k}} = \frac{1}{k} \cdot f \tag{13}$$

With $f = 1$ and putting $k = \lambda k'$ ($k' \sim T^{-1/2}$) we should get a formula used by Eddington*. We insert in (9):

$$\lambda + 1 = \frac{4 \pi G c M}{k L} f = \frac{4 \pi G c M}{k_1 L} \frac{T^{3.5}}{\varrho} f$$

$$= \frac{4 \pi G c M}{k_1 L} \frac{3 \bar{R}}{a \mu} f \frac{1}{\lambda} T^{1/2} \tag{14}$$

or

$$\lambda (\lambda + 1) = \frac{4 \pi G c M}{k_1 L} \frac{3 \bar{R}}{a \mu} f T^{1/2}, \tag{15}$$

where

$$\lambda = \frac{p}{p'} = \frac{3 \bar{R}}{a \mu} \frac{\varrho}{T^3}. \tag{16}$$

Equation (15) determines λ and so ϱ as a function of T. Introducing $\varrho = \varrho(T)$ in (12) we can calculate f. It is convenient for $\lambda > 1$ to solve (15) as follows:

With $\lambda (\lambda + 1) = a$ put either

$$\lambda = \sqrt{a}\, e^{-\frac{1}{2\sqrt{a}}} \tag{17}$$

with

$$\log \lambda = \frac{1}{2} \log a - \frac{\log e}{2} \frac{1}{\sqrt{a}} \tag{17a}$$

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* M. N. **91**, 109, 1930.

or

$$\frac{1}{\lambda} = \frac{1}{a^{1/2}} + \frac{1}{2a} + \frac{1}{8a^{3/2}}. \tag{18}$$

The first form is very convenient for numerical computing, the second form will be used in analytical investigations.

The degree of approximation thus obtained is shown in Table 1. The logarithmic expression should only be used for $\lambda > 3$.

Table 1.

$\sqrt{a}$	λ	$\sqrt{a}\, e^{-\frac{1}{2\sqrt{a}}}$	$\frac{1}{\lambda}$	$\frac{1}{a^{1/2}} + \frac{1}{2a} + \frac{1}{8a^{3/2}}$
10	9.5125	9.5123	0.105 125	0.105 125
8	7.5156	7.5153	0.133 057	0.133 057
5	4.5249	4.5242	0.221 00	0.221 00
4	3.5311	3.5300	0.283 20	0.283 20
3	2.5414	2.5394	0.393 48	0.393 52
2	1.5616	1.5576	0.640 4	0.640 6
$\sqrt{2}$	1	0.9930	1	1.001 3
1	0.6180	0.6065	1.618 1	1.625 0
0.8	0.4434	0.4282	2.255	2.275
0.5	0.2071	0.1839	4.829	5.000

From (18) we get:

$$\frac{1}{\lambda} = \frac{1}{(\alpha f T^{1/2})^{1/2}} + \frac{1}{2(\alpha f T^{1/2})} + \frac{1}{8(\alpha f T^{1/2})^{3/2}} \tag{19}$$

with

$$\alpha = \frac{4\pi G c M}{k_1 L} \frac{3\overline{R}}{a\mu}. \tag{20}$$

We can now find an expression for f; we have defined f by the equations:

$$\left.\begin{aligned} \frac{1}{k} f &= \frac{\int \frac{1}{k} d(T^4)}{T^4} = \frac{4\int \frac{1}{k_1} T^{3.5} \frac{T^3}{\varrho} d\,T}{T^4} = \frac{1}{k_1} \frac{3\overline{R}}{a\mu} \frac{4\int T^{3.5} \frac{1}{\lambda} d\,T}{T^4} \\ \frac{1}{k} &= \frac{1}{k_1} \frac{3\overline{R}}{a\mu} \frac{1}{\lambda} T^{1/2} \end{aligned}\right\} \tag{21}$$

from which we get:

$$f \frac{1}{\lambda} = \frac{4\int T^{3.5} \frac{1}{\lambda} d\,T}{T^{4.5}} = \frac{16 \int \nu^{17} \frac{1}{\lambda} d\,\nu}{\nu^{18}} \tag{22}$$

with

$$\nu = \alpha^{1/2} T^{1/4}. \tag{23}$$

Introducing in (22) an expression:

$$\frac{1}{f} = a_1\left(1 + a_2 \frac{1}{\nu} + a_3 \frac{1}{\nu^2}\right) \tag{24}$$

and using (19) we can determine a_1, a_2 and a_3. It is easily found that

$$a_1 = \frac{17}{16},$$

$$a_2 = -\frac{1}{33}\left(\frac{17}{16}\right)^{1/2},$$

$$a_3 = \frac{1}{33^2}\frac{17}{16},$$

so that

$$\frac{1}{f} = \frac{17}{16}\left(1 - \frac{1}{33}\left(\frac{17}{16}\right)^{1/2}\frac{1}{\nu} + \frac{1}{33^2}\frac{17}{16}\frac{1}{\nu^2}\right) \tag{25}$$

or

$$f = \frac{16}{17}\left(1 + \frac{1}{33}\left(\frac{17}{16}\right)^{1/2}\frac{1}{\nu} + 0\cdot\frac{1}{\nu^2}\right). \tag{25a}$$

Putting

$$\Theta = \left(\frac{16}{17}\right)^{1/2}\nu = \left(\frac{16}{17}\right)^{1/2}\alpha^{1/2}\,T^{1/4} = \left[\frac{16}{17}\frac{4\pi G c M}{k_1 L}\frac{3\overline{R}}{a\mu}\,T^{1/2}\right]^{1/2} \tag{26}$$

we finally have

$$\frac{1}{f} = \frac{17}{16}\left(1 - \frac{1}{33}\frac{1}{\Theta} + \frac{1}{33^2}\frac{1}{\Theta^2}\right) \tag{27}$$

$$f = \frac{16}{17}\left(1 + \frac{1}{33}\frac{1}{\Theta} + 0\cdot\frac{1}{\Theta^2}\right). \tag{27a}$$

Introducing (27) in (19) we get:

$$\frac{1}{\lambda} = \frac{1}{\Theta} + \frac{16}{33}\frac{1}{\Theta^2} + \frac{40}{363}\frac{1}{\Theta^3}. \tag{28}$$

The formulae now developed determine λ as a function of T, and so ϱ as a function of T; we now turn to the determination of T as a function of r or rather of $x = 1/r$. The differential equation for T is [cfr. (2)]:

$$dT = \frac{3}{4a}\frac{1}{4\pi c}\frac{k\varrho}{T^3}L\,dx = \frac{3}{4a}\frac{1}{4\pi c}\frac{a\mu}{3\overline{R}}\lambda k L\,dx. \tag{29}$$

We now substitute for λk from (14), which we write:

$$\lambda\left(1 + \frac{1}{\lambda}\right) = \frac{4\pi G c M}{kL}f \tag{30}$$

getting

$$\left(1 + \frac{1}{\lambda}\right)dT = \frac{3}{4a}\frac{1}{4\pi c}\frac{a\mu}{3\overline{R}}\frac{4\pi G c M}{L}f L\,dx$$

or

$$\left(1+\frac{1}{\lambda}\right)dT = \frac{\mu G M}{4\bar{R}} f\,d\,x. \tag{31}$$

Integrating from the boundary up to the point considered we have:

$$\int_{T_0}^{T} \frac{1}{f}\left(1+\frac{1}{\lambda}\right)dT = \frac{\mu G M}{4\bar{R}}(x - x_0) \tag{32}$$

where

$$x_0 = \frac{1}{R}. \tag{33}$$

We now have to evaluate the integral introducing expressions (27) and (28) for $1/f$ and $1/\lambda$. The result may be written

$$x - x_0 = \frac{4\bar{R}}{\mu G M}(\sigma T - \sigma_0 T_0) \tag{34}$$

$$\sigma = 1 + \frac{4}{3}\frac{32}{33}\frac{1}{\Theta} + \frac{992}{1089}\frac{1}{\Theta^2} + \frac{420}{1089}\frac{1}{\Theta^3}. \tag{35}$$

In forming the expression for σ all three terms were retained in the expression for $1/\lambda$, thus making $1 + 1/\lambda$ a four-term expression, while for $1/f$ a three-term expression was used; the two expressions are about equally accurate for the values of λ with which we shall be concerned.

To facilitate computation according to the formulae developed tables have been computed giving λ and σ with argument $\Theta^2 = A\,T^{1/2}$ and $\sigma_0 T_0$ with argument $\log k_1$, the latter table being applicable only for the sun and with $\mu = 2.2$.

Table 2.

$\log A\sqrt{T} - 0.262$	$\log \lambda$	$\log \sigma$	$\log A\sqrt{T} - 0.262$	$\log \lambda$	$\log \sigma$
0.8	0.469	0.1672	2.1	1.167	0.0371
0.9	0.526	0.1489	2.2	1.218	0.0331
1.0	0.581	0.1326	2.3	1.270	0.0295
1.1	0.637	0.1181	2.4	1.321	0.0262
1.2	0.692	0.1052	2.5	1.372	0.0234
1.3	0.746	0.0937	2.6	1.423	0.0209
1.4	0.800	0.0834	2.7	1.474	0.0186
1.5	0.853	0.0744	2.8	1.525	0.0166
1.6	0.907	0.0662	2.9	1.575	0.0147
1.7	0.959	0.0590	3.0	1.626	0.0131
1.8	1.012	0.0525	3.1	1.676	0.0117
1.9	1.064	0.0468	3.2	1.727	0.0104
2.0	1.115	0.0417			

Table 3.

$\log k_1 - 26$	$\sigma_0 \cdot T_0 \cdot 10^{-6}$	$\log k_1 - 26$	$\sigma_0 \cdot T_0 \cdot 10^{-6}$	$\log k_1 - 26$	$\sigma_0 \cdot T_0 \cdot 10^{-6}$
2.0	0.0155	1.3	0.0084	0.6	0.0063
1.9	0.0138	1.2	0.0079	0.5	0.0061
1.8	0.0124	1.1	0.0075	0.4	0.0060
1.7	0.0112	1.0	0.0072	0.3	0.0058
1.6	0.0104	0.9	0.0070	0.2	0.0058
1.5	0.0096	0.8	0.0066	0.1	0.0057
1.4	0.0089	0.7	0.0064	0.0	0.0056

Below the formulae necessary for computing ϱ and $x - x_0$ for given T are compiled:

$$A = \frac{16}{17} 4 \pi G c \frac{3 \overline{R}}{a} \cdot \frac{M}{k_1 L \mu} = [0.262] \cdot \frac{M}{M_\odot} \cdot \frac{10^{26} L_\odot}{k_1 L} \cdot \frac{2.2}{\mu}$$

$$\lambda = \lambda (A \sqrt{T}) \text{ from Table 2,}$$

$$\frac{\varrho}{T^3} = \frac{a \mu}{3 \overline{R}} \lambda = [7.832 - 30] \frac{\mu}{2.2} \lambda$$

$$\sigma = \sigma (A \sqrt{T}) \text{ from Table 2.}$$

$\sigma_0 T_0$ from Table 3 in the case of the sun and $\mu = 2.2$

$$12 + \log (x - x_0) = \log \{\sigma T - \sigma_0 T_0\} - 5.9185 + \log \frac{2.2}{\mu} \cdot \frac{M_\odot}{M} \cdot$$

Example: $M = M_\odot$, $R = R_\odot$, $L = L_\odot$, $\mu = 2.2$, $k_1 = 27.4$

$\log T = \mathbf{6.0}$

$\text{Arg} = 1.6$	$\log \lambda = 0.907$
$\log \sigma = 0.0662$	$\log \text{const}_2 = 7.832 - 30$
$\log \sigma \frac{T}{10^6} = 0.0662$	$\log \frac{\varrho}{T^3} = 8.739 - 30$
$\sigma \frac{T}{10^6} = 1.1647$	$\log T^3 = 18.000$
$\sigma_0 \frac{T_0}{10^6} = 0.0089$	$\log \varrho = \mathbf{6.739 - 10}$

$$\sigma \frac{T}{10^6} - \sigma_0 \frac{T_0}{10^6} = 1.1558$$

$$\log \left(\sigma \frac{T}{10^6} - \sigma_0 \frac{T_0}{10^6} \right) = 0.0629$$

$$\log \text{const}_1 = 0.0815$$

$$12 + \log (x - x_0) = \mathbf{0.1444}.$$

By the procedure developed, for the values of the parameters with which we are concerned in the present paper, three-figure values of T^4 and ϱ can be obtained for a point well below the surface. From this point numerical integration is started inwards.

Thus far we have been considering the construction of solutions on our model. With one exception — the procedure to evade the chromosphere — we have not in this section gone beyond the differential equations of the model: all we have done in this section (with the exception mentioned) could be substituted by numerical integration according to the differential equations of the model. The above treatment however lends itself very easily to a discussion of the influence of deviation of conditions in the outer layers of the model from conditions in a real star.

Such deviations are due to differences in μ and k in the low-temperature region. With the aid of the relations developed in this section we can discuss the influence of change of μ and k in the low-temperature region on the values of T and ϱ at some specified point so far below the surface, that T is about 10^6. We shall first discuss the dependence of ϱ on T; this is controlled by the mean value of $1/k$ defined in (12). It is seen at once that the influence of the values of k in the low-temperature region is quite small, the number of steps of T^4 that fall in the low-temperature region being small compared with the number of steps in the high-temperature region. In determining ϱ from T the value of the molecular weight used is that at the point considered, i. e. a μ-value in the high-temperature region. So the ϱ-value corresponding to a specified T-value is changed very little when conditions in the low-temperature region change. We next turn to the relation between depth under the surface and temperature. Here the influence of the low-temperature region is somewhat greater, corresponding to the fact, that the controlling factor is a mean for equal steps of T; still the number of steps through the high-temperature region is greater than the number of steps through the low-temperature region. Increased radiation-pressure means greater influence of this region. We see that changes in the low-temperature region are equivalent to changes in two parameters, e. g. R and k_1, so that the equivalent change of say k_1 is quite small and the equivalent change of R is small, less than say 10%. (That solutions near singular solutions are very sensitive to small changes in the parameters does not interest us so much in *this* connection). Also it is seen that the particular value of the boundary temperature T_0 is of no great importance. The statements thus obtained are in accord with generally accepted views on the importance of conditions in the outer layers of the stars in the problem of stellar structure.

With the aid of the relations in this section we shall discuss one more question, the question of the upper limit of L. On the standard model there is an upper limit for L, while this is not the case on the model here adopted. On the standard model, with k = constant = $\varkappa$ (14) becomes:

$$\lambda + 1 = \frac{4\pi G c M}{\varkappa L} \tag{36}$$

which implies

$$\frac{4\pi G c M}{\varkappa L} > 1 \tag{37}$$

or

$$L < \frac{4\pi G c M}{\varkappa}, \tag{38}$$

while on our model with $k = k_1 \frac{\varrho}{T^{3.5}}$ there is no corresponding condition due to the greater elasticity of this model, according to which k decreases for increasing L. This difference between the models should not be stressed.

In this section we have aimed at three-figure accuracy in T^4 and ϱ. This accuracy has only significance on the model adopted. The same accuracy will be aimed at in the numerical integrations of the following section.

4. We shall now consider a transformation of the differential equations which makes them more suited for numerical integration.

In the part of the star where we can put $M_r = M$ we have, when radiation-pressure is negligible, $T \sim (x - x_0)$ and $\varrho \sim (x - x_0)^{3.25}$, so that in this case $\log T$ and $\log \varrho$ are linear functions of $\log (x - x_0)$. For the solutions which we have to consider deviations from linearity will be small. Introducing $y = \log T$, $z = \log \varrho$ and $t = \log (x - x_0)$ we shall get equations that are well suited for the integration through the outer half (say) of the star. Again, near the centre we have $\log (x - x_0) \rightarrow \log x$, and, as we shall find, we have in this region $T \sim x^{1 - 1/11}$ and $\varrho \sim x^{3 - 1/11}$ so that for this region also the variables are well suited. We shall use these variables throughout the star. Introducing further $u = M_r/M$ and arranging the equations for practical computation we get the following system of equations:

$$\left.\begin{aligned} dy &= \gamma \frac{\varrho^2}{T^{6.5}} \frac{x - x_0}{T} dt \\ dz &= \alpha u \frac{x - x_0}{T} dt - dy \left\{1 + \beta \frac{T^3}{\varrho}\right\} \\ du &= \zeta \frac{x - x_0}{x^4} dt \end{aligned}\right\} \tag{39}$$

with

$$y = \log T \qquad \alpha = \frac{\mu G}{R} M$$

$$z = \log \varrho \qquad \beta = \frac{\mu}{R} \frac{4a}{3}$$

$$u = \frac{M_r}{M} \qquad \gamma = \frac{3}{16 \pi a c} k_1 L$$

$$t = \log(x - x_0) \qquad \zeta = \frac{4\pi}{\log e}.$$

$$x = \frac{1}{r}, \; x_0 = \frac{1}{R}$$

The function $\nu = \zeta \frac{x - x_0}{x^4}$ was calculated in advance for all values of t for which it would be needed. The numerical integration was performed according to the usual scheme of GAUSS-ENCKE; the interval was taken as 0.1 up to about $r = {}^1/_2\, R$, then it was taken as 0.05. Thus $r = {}^1/_2\, R$ was reached after about 10 intervals, $r = {}^1/_{10}\, R$ after about 30 intervals, $r = {}^1/_{100}$ after 50 and $r = {}^1/_{400}$ after about 60 intervals. Thus for a solution approaching the centre most intervals fall in the central region.

Table 4.

$M = M_\odot$, $R = R_\odot$, $L = L_\odot$, $\mu = 2.2$.

$t = \log\left(\frac{1}{r} - \frac{1}{R}\right)$	$\log k_1 = 26.2$			$\log k_1 = 27.0$		
	$z = \log \varrho$	$y = \log T$	$u = \frac{M_r}{M}$	$z = \log \varrho$	$y = \log T$	$u = \frac{M_r}{M}$
0.0	7.04–10	5.902	1.000	6.52–10	5.871	1.000
0.1	7.37	6.003	1.000	6.86	5.974	1.000
0.2	7.70	6.104	1.000	7.20	6.077	1.000
0.3	8.02	6.205	0.999	7.53	6.179	1.000
0.4	8.35	6.305	0.999	7.87	6.282	1.000
0.5	8.68	6.406	0.997	8.20	6.384	0.999
0.6	9.00	6.506	0.994	8.53	6.486	0.998
0.7	9.33	6.606	0.986	8.86	6.588	0.995
0.8	9.65	6.706	0.970	9.19	6.689	0.989
0.9	9.96–10	6.804	0.939	9.52	6.790	0.979
1.0	0.26	6.899	0.881	9.84–10	6.890	0.957
1.1	0.54	6.990	0.785	0.15	6.988	0.919
1.2	0.78	7.074	0.641	0.45	7.083	0.857
1.3	0.97	7.147	0.459	0.72	7.173	0.763
1.4	1.09	7.207	0.267	0.96	7.256	0.638
1.5	1.14	7.252	0.101	1.16	7.330	0.493
1.6				1.31	7.394	0.348
1.7				1.42	7.446	0.223
1.8				1.48	7.489	0.130
1.9				1.51	7.524	0.068
2.0				1.51	7.551	0.030

$t = \log\left(\frac{1}{r} - \frac{1}{R}\right)$	$\log k_1 = 27.4$			$\log k_1 = 27.8$			$\log k_1 = 27.4$ (Revision)		
	$z = \log \varrho$	$y = \log T$	$u = \frac{M_r}{M}$	$z = \log \varrho$	$y = \log T$	$u = \frac{M_r}{M}$	$z = \log \varrho$	$y = \log T$	$u = \frac{M_r}{M}$
0.0	6.19–10	5.834	1.000	(5.89–10)	(5.806)	(1.000)			
0.1	6.54	5.940	1.000	(6.24)	(5.912)	(1.000)			
0.2	6.89	6.046	1.000	(6.58)	(6.018)	(1.000)			
0.3	7.23	6.152	1.000	(6.93)	(6.123)	(1.000)			
0.4	7.58	6.256	1.000	7.27	6.228	1.000	7.60–10	6.263	1.000
0.5	7.92	6.361	1.000	7.61	6.333	1.000	7.93	6.366	1.000
0.6	8.25	6.465	0.999	7.96	6.438	1.000	8.27	6.469	0.999
0.7	8.59	6.568	0.998	8.30	6.543	0.999	8.60	6.572	0.998
0.8	8.93	6.671	0.995	8.64	6.647	0.997	8.94	6.674	0.994
0.9	9.26	6.774	0.989	8.98	6.751	0.994	9.27	6.776	0.988
1.0	9.59	6.875	0.976	9.31	6.854	0.988	9.59	6.877	0.976
1.1	9.91–10	6.976	0.955	9.64	6.956	0.976	9.92–10	6.977	0.954
1.2	0.23	7.075	0.918	9.97–10	7.058	0.956	0.23	7.076	0.917
1.3	0.53	7.170	0.860	0.29	7.157	0.923	0.53	7.171	0.859
1.4	0.80	7.261	0.777	0.60	7.254	0.873	0.81	7.262	0.775
1.5	1.06	7.346	0.670	0.88	7.347	0.804	1.06	7.346	0.668
1.6	1.27	7.423	0.547	1.15	7.435	0.718			
1.7	1.45	7.491	0.424	1.38	7.516	0.618			
1.8	1.59	7.551	0.314	1.60	7.591	0.516			
1.9	1.70	7.602	0.227	1.78	7.658	0.420			
2.0	1.78	7.646	0.164	1.93	7.719	0.339			
2.1	1.84	7.685	0.122	2.07	7.774	0.276			
2.2	1.90	7.719	0.095	2.19	7.824	0.228			
2.3	1.96	7.750	0.079	2.30	7.872	0.195			
2.4	2.01	7.780	0.069	2.42	7.917	0.172			
2.5	2.07	7.809	0.063	2.53	7.961	0.156			
2.6	2.14	7.839	0.059	2.65	8.006	0.145			
2.7	2.22	7.870	0.057	2.78	8.052	0.137			
2.8	2.30	7.902	0.056	2.93	8.100	0.132			
2.9	2.41	7.938	0.055	3.09	8.152	0.128			
3.0	2.53	7.978	0.054	3.26	8.207	0.125			
3.1	2.67	8.022	0.054	3.45	8.266	0.123			
3.2	2.83	8.072	0.054	3.65	8.330	0.121			
3.3	3.01	8.126	0.054	3.87	8.398	0.120			
3.4	3.20	8.187	0.053	4.10	8.471	0.119			
3.5	3.42	8.253	0.053	4.35	8.547	0.118			
3.6	3.65	8.324	0.053	4.61	8.627	0.117			
3.7	3.90	8.399	0.053	4.88	8.710	0.116			

5. The solutions thus found by integration are given above (Table 4). The solutions tabulated correspond to the following values of the parameters: $M = M_\odot$, $R = R_\odot$, $L = L_\odot$ and $\mu = 2.2$ for all solutions, and $\log k_1 = 26.2$, 27.0, 27.4 and 27.8 respectively. The computations were made with one more figure than given in the tables.

The starting values have not all been computed by the method developed in section 3, an approximation not so exact having been used at an earlier stage. This is of no importance; the differences can be seen from the table for $\log k_1 = 27.4$.

The solutions corresponding to $\log k_1 = 27.4$ and $\log k_1 = 27.8$ have been computed so far into the star that the nature of the degenerate solution can be ascertained (cfr. p. 365). Degeneracy will set in a few intervals beyond the limit of the table. From the numbers it is seen that extreme conditions have not yet been reached, so it is justified that we have used $L_r = \text{const.} = L$.

We shall now discuss the asymptotic behaviour of the solutions as $r \to 0$ or $x \to \infty$. We shall first discuss the behaviour of perfect gas-solutions continuing them analytically to the centre disregarding change of equation of state.

From EDDINGTONS work we know that there exists one solution for which $\varrho \to 0$ as $x \to \infty$. In our table this would fall between the solutions corresponding to $\log k_1 = 27.0$ and $\log k_1 = 27.4$. Call the corresponding k_1-value k_1^0; in the present case $\log k_1^0$ is about 27.05.

For $k < k_1^0$ the solutions are of the type with $M_r = 0$ at $r > 0$. Following MILNES notation for the standard model we can say that for $k < k_1^0$ the solutions are of the collapsed type. For $k = k_1^0$ we have EDDINGTONS solution. For $k_1 > k_1^0$ we have solutions which, following MILNE, we can call centrally-condensed solutions, with $\varrho \to \infty$ and $T \to \infty$ as $x \to \infty$.

Table 5.

t_1	t_2	u_1	u_2	$u_1 + u_2$
		$\log k_1 = 27.0.$ $\bar{t} = 1.500.$		
1.0	2.0	0.957	0.030	0.987
1.1	1.9	0.919	0.068	0.987
1.2	1.8	0.857	0.130	0.987
1.3	1.7	0.763	0.223	0.986
1.4	1.6	0.638	0.348	0.986
1.5	1.5	0.493	0.493	0.986
		$\log k_1 = 26.2.$ $\bar{t} = 1.328.$		
1.1	1.556	0.785	0.028	0.813
1.2	1.456	0.641	0.168	0.809
1.3	1.356	0.459	0.349	0.808
1.328	1.328	0.404	0.404	0.808
		$\log k_1 = 27.8.$ $\bar{t} = 1.73_2.$		
0.4	3.064	1.000	0.124	1.124
0.6	2.864	1.000	0.129	1.129
0.8	2.664	0.997	0.140	1.137
1.0	2.464	0.988	0.161	1.149
1.2	2.264	0.956	0.206	1.162
1.4	2.064	0.873	0.297	1.170
1.6	1.864	0.718	0.453	1.171
1.732	1.732	0.585	0.585	1.170

At this point we may note a symmetry-property of the solution corresponding to $k_1 = k_1^0$; $\log k_1$ is close to $\log k_1^0$ and the corresponding solution shows the following symmetry-property, as far as can be judged from the computed numbers: $u(t)$ is point-symmetric with regard to a certain point $t = \bar{t}$. This is shown by Table 5. For each solution in the table $\bar{t}$ was computed as the point for which $d^2u/dt^2 = 0$.

As far as can be judged the symmetry is perfect for the solution corresponding to $k_1 = k_1^0$, while solutions corresponding to $k < k_1^0$ and $k > k_1^0$ respectively deviate in opposite directions.

We are thus led to the following empirical rule: If for a solution corresponding to $k_1 = k'_1$:

$u < \frac{1}{2}$ when $\frac{d^2 u}{d t^2} = 0$, then $k'_1 > k_1^0$, and we shall have $u = 0$ at $r > 0$

$u = \frac{1}{2}$ when $\frac{d^2 u}{d t^2} = 0$, then $k'_1 = k_1^0$, and we have Eddingtons solution

$u > \frac{1}{2}$ when $\frac{d^2 u}{d t^2} = 0$, then $k'_1 > k_1^0$, and we shall have $\varrho \to \infty$ as $r \to 0$.

By this rule the nature of a solution may be inferred, when the integration has been carried up to the point $\bar{t}$ where $d^2u/dt^2 = 0$. Also it is seen from the numbers in the table, that $u(\bar{t})$ varies practically linearly with $\log k_1$; thus it is easy to interpolate a value of k_1^0, when two solutions have been computed up to the point $\bar{t}$.

It will be interesting to study the properties mentioned for other values of the mass. It may be noted that Eddingtons solution for mass $5.02\, M_\odot$ does not lend itself to an investigation of the symmetry-property as it does not satisfy the boundary conditions of the model accurately enough (though the accuracy was ample for Eddingtons special purpose).

Also it may be noted in passing that the excess of $\log k_1^0$ on the $\eta k =$ const.-model over $\log k_1^0$ on the point source model is about the same for solar mass and mass $5.02\, M_\odot$, as was to be expected.

We have seen that for $k_1 < k_1^0$, $M_r = 0$ at $r > 0$, and for $k_1 = k_1^0$, $\varrho \to 0$ as $r \to 0$. Further it is known from Eddingtons investigation that there is only one solution with $\varrho \to 0$ as $r \to 0$. We now have to investigate the behaviour of solutions corresponding to $k_1 > k_1^0$ as $r \to 0$ or $x \to \infty$.

As we cannot have $\varrho \to 0$ we see that $dp'/dx \to 0$ with $T \to$ const. is impossible, so $T \to \infty$ as $x \to \infty$. Also $\varrho \to$ const. is impossible, since with $\varrho \to$ const., $u \to$ const. > 0 we have $dp/dx \to$ const. > 0, and thus $dT/dx \to$ const. > 0 in contradiction with

$$\frac{dT}{dx} = \frac{\text{const.}}{T^{6.5}} \to 0;$$

$\varrho \to$ const., $u \to 0$ is seen to be in contradiction with the differential equations. Hence we have $\varrho \to \infty$ as $x \to \infty$. We shall now prove that $u \to$ const. as $x \to \infty$ is impossible. If $u \to$ const. $= u_0$ the relations developed for the outer part of the star become applicable. In fact, integrating inwards from some arbitrary fixed point a, to a variable point b, we have asymptotically for $(x_a, x_b) \to \infty$

$$p'_b - p'_a = \frac{L}{4\pi c}\int_a^b k\,\varrho\,dx \tag{40}$$

$$p_b + p'_b - p_a - p'_a = GM u_0 \int \varrho\,dx. \tag{41}$$

Now for fixed x_a, as $x_b \to \infty$, we have $\frac{p'_a}{p'_b} \to 0$ and $\frac{p_a}{p_b} \to 0$. Consequently

$$p'_b = \frac{L}{4\pi c}\int_a^b k\,\varrho\,dx \tag{42}$$

$$p_b + p'_b = GM u_0 \int_a^b \varrho\,dx \tag{43}$$

as in the outer part of the model except for the factor u_0. As before we derive:

$$\lambda + 1 = \frac{4\pi G c M}{kL} u_0 f \tag{44}$$

with f nearly equal to one, and:

$$\lambda(\lambda + 1) = \frac{4\pi G c M}{k_1 L} u_0 f \frac{3\overline{R}}{a\mu} T^{1/2}. \tag{45}$$

This shows that $\lambda \to \infty$ as $x \to \infty$ with $\lambda \sim T^{1/4}$; again [cfr. (31)] we have as $x \to \infty$:

$$dT = \frac{\mu G M}{4\overline{R}} u_0 f\,dx, \tag{46}$$

so $T \sim x$ as $x \to \infty$. Thus $\varrho \sim \lambda T^3 \sim x^{3.25}$. Integrating again from a to b we get from this:

$$M_b - M_a = \int_a^b \frac{4\pi\varrho}{x^4}\,dx \sim \int_a^b \frac{x^{3.25}}{x^4}\,dx \sim x_b^{1/4} - x_a^{1/4} \to \infty \tag{47}$$

in contradiction with the assumption $u \to u_0$. Thus it is proved that u cannot tend to a finite limit as $r \to 0$.

It is interesting to compare this with T. G. Cowlings * result concerning the point-source model with $k = \text{const.} = \varkappa$. For $k = \text{const.} = \varkappa$ (44) becomes:

$$\lambda + 1 = \frac{4\pi G c M}{\varkappa L} u_0. \tag{44a}$$

With

$$u_0 = \frac{\varkappa L}{4\pi G c M} \tag{48}$$

we find for $x \to \infty$:

as

$$\left.\begin{aligned} \lambda &\to 0 \\ u &\to u_0 \end{aligned}\right\}. \tag{49}$$

Introducing $\varrho \sim \lambda T^3$ in (47) and using this result, it is seen that $M_a - M_b \to 0$, and we have not the contradiction of our model. Thus with $k = \text{const.} = \varkappa$ a point-mass $\frac{\varkappa L}{4\pi G c}$ is possible, as was shown by Cowling. Again the difference between the models is due to the greater elasticity of the model with $k = k_1 \frac{\varrho}{T^{3.5}}$, according to which $k \to 0$ when $\lambda \to 0$ (cfr. p. 357).

We have shown that u cannot tend to a finite limit. Then, if radiation-pressure is negligible, a solution with

$$\left.\begin{aligned} T &\sim x^{1-\frac{1}{11}} \\ \varrho &\sim x^{3-\frac{1}{11}} \\ u &\sim x^{-\frac{1}{11}} \end{aligned}\right\} \tag{50}$$

as $x \to \infty$ is possible. This implies $\frac{k}{u} \sim x^{-\frac{2}{11}}$, so on that solution $\frac{p'}{p} \to 0$ as $x \to \infty$. According to the computed numbers in Table 4 the solutions are approaching a solution as (50). We can write (50) in the variables of the numerical integration:

$$\left.\begin{aligned} \frac{dy}{dt} &\to \frac{10}{11} \\ \frac{dz}{dt} &\to \frac{32}{11} \\ \frac{d\log u}{dt} &\to -\frac{1}{11} \end{aligned}\right\}. \tag{50a}$$

* M. N. **91**, 92, 1930.

We have thus far been considering the perfect gas-part of the solutions, or the analytical continuation to the centre disregarding change of equation of state. Summing up the results obtained, we see:

1. For $k_1 < k_1^0$ $M_r = 0$ at $r > 0$.
2. For $k_1 = k_1^0$ we have EDDINGTONS solution.
3. For $k_1 > k_1^0$ we have $T \to \infty$ and $\varrho \to \infty$ as $r \to 0$, and the condition $\int_{\text{centre}}^{\text{boundary}} dM_r = M$ is satisfied.

Conditions are thus strictly analogous to conditions on the standard model studied by MILNE.

We now proceed to discuss conditions taking account of change of equation of state.

On the standard model the perfect gas-part of the solution is an "$n = 3$"-polytrope. At a certain definite point on the "$n = 3$"-polytrope degeneracy sets in; at this point a new solution is started inwards, which on the standard model is an "$n = 3/2$"-polytrope. The "$n = 3/2$"-polytrope is definite being determined by the values of r, ϱ and M_r on the "$n = 3$"-polytrope at the point where degeneracy sets in.

What we have to do now is to consider the analagous problem on the point-source model. We shall consider first the form of the solution corresponding to degeneracy. When degeneracy has set in, the temperature does not enter in the equation of state, and so temperature influences the density-distribution only through light-pressure. The influence of light-pressure is probably small in the cases under consideration and cannot be discussed from lack of knowledge of the interaction of light with degenerate matter; we shall neglect light-pressure for the present. Then the density distribution will be independent of the source-distribution, and we shall have conditions identical with conditions on the standard model (cfr. p. 349). With an equation of state $p = K\varrho^{5/3}$ the solution running in from the interface will be an "$n = 3/2$"-polytrope. More generally the density-distribution of the degenerate part of the star will be ruled by the equations:

$$\left.\begin{aligned} dP &= GM_r\varrho\,dx \\ dM_r &= -\frac{4\pi\varrho}{x^4}dx \\ P &= P(\varrho) \end{aligned}\right\}. \tag{51}$$

$P = P(\varrho)$ expressing the equation of state.

The next step will be to ascertain the nature of the "$n = 3/2$"-polytrope running in from the interface. The method is the same as that employed in the discussion of the fitting of "$n = 3/2$"-polytropes on to "$n = 3$"-polytropes*. Let the values of the variables r, ϱ and M_r at the interface be r', ϱ' and M_r'. Further let:

$$\beta' = 1 - \frac{k' L}{4 \pi G c M_r'} . \tag{52}$$

We shall take for the equation of state inside the interface:

$$\left.\begin{aligned} \beta' P &= \frac{K}{\mu^{5/3}} \varrho^{5/3} \\ K &= 1.01 \times 10^{13} \end{aligned}\right\} . \tag{53}$$

Recalling the above discussion we see that much weight should not be attached to the factor β'; in fact we could equally well have omitted the factor β'. We shall however retain β' to show that the uncertainty with regard to this point is of no great importance. We can now write down relations according to which the nature of the "$n = 3/2$"-solution can be ascertained, with the aid of R. H. FOWLERS theorems. Compute:

$$(\eta\, \psi^{1/4})_E = \left(\frac{8 \pi G}{5 K}\right)^{1/2} \cdot \beta'^{1/2} r' \varrho'^{1/6} \tag{54}$$

then with EMDENS tables for the polytrope "$n = 3/2$" compute the corresponding two values:

$$s_1 = \left(\frac{3}{2} \frac{\eta}{\psi} \left|\frac{d\psi}{d\eta}\right|\right)_{E_1}, \quad s_2 = \left(\frac{3}{2} \frac{\eta}{\psi} \left|\frac{d\psi}{d\eta}\right|\right)_{E_2} \tag{55}$$

and compare these with:

$$s = \frac{r}{\varrho} \left|\frac{d\varrho}{dr}\right| = \left|\frac{dz}{dt}\right| \cdot \frac{x}{x - x_0} \simeq \left|\frac{dz}{dt}\right| . \tag{56}$$

We then have the following cases:

1. If $(\eta\, \psi^{1/4}) > 1.915$, then the "$n = 3/2$"-polytrope is of the collapsed type.
2. If $(\eta\, \psi^{1/4}) < 1.915$ and either $s < s_1$ or $s > s_2$, then the "$n = 3/2$"-polytrope is of the collapsed type.
3. If $(\eta\, \psi^{1/4}) < 1.915$ and $s_1 < s < s_2$, then the "$n = 3/2$"-polytrope is of the centrally-condensed type.
4. If $(\eta\, \psi^{1/4}) < 1.915$ and $s = s_1$ or $s = s_2$, then the "$n = 3/2$"-polytrope is EMDENS particular solution.
5. If $(\eta\, \psi^{1/4}) = 1.915$ we have $s_1 = s_2$; if $s \neq s_1 = s_2$, then the "$n = 3/2$"-polytrope is of the collapsed type; if $s = s_1 = s_2$, then the "$n = 3/2$"-polytrope is EMDENS particular solution.

* M. N. 91, 466, 1931.

Applying these rules to the numerical values of Table 4 it is found that the "$n = 3/2$"-polytropes are of the centrally-condensed type, the numbers leading in fact to the inequalities under 3.

For $\log k_1 = 27.4$ we have the following numbers:

$$t = 3.6 \qquad \eta\,\psi^{1/4} = 0.34$$
$$\log r = 8.40 \qquad s_1 = 0.06$$
$$\log \varrho = 3.65 \qquad s = 2.4$$
$$\beta = 0.88 \qquad s_2 = 1.5 \cdot 10^4.$$

Degeneracy sets in beyond $t = 3.6$, where $\eta\,\psi^{1/4}$ is smaller so that the lower limit is still lower and the higher limit still higher.

Quite generally we must have $\log(\beta'\, r'\, \varrho'^{1/6}) > 9.6$, if there shall be any possibility for a collapsed "$n = 3/2$"-polytrope, and on the computed solutions conditions are far from this.

We have thus reached the conclusion that, on the point-source model centrally-condensed "envelope" necessarily implies centrally-condensed "$n = 3/2$"-polytrope running in from the interface.

We have herewith arrived at the same conclusion that was reached on the standard model. In fact we see that discussing conditions of the nucleus we shall arrive at the same conclusions on the point-source model as on the standard model.

6. From investigations on the standard model it was concluded (MILNE), that the equation of state $P = P(\varrho)$ [cfr. (51)] at the extreme conditions near the centre was of the following type: at a certain ϱ, ϱ_{crit}, ϱ rises rapidly to a value $\varrho_{\max}$ far greater than for example the ϱ according to $\beta' P = K\varrho^{5/3}$: postulating a nucleus we arrive at a solution, which with some "normal" equation of state as (53) leads to a point-mass; hence the equation of state must be of the kind described, in order that all mass shall be got rid of, till the centre is reached. What we have shown in the previous section is that a discussion on the point-source model leads to the same result.

We can now ask whether we can obtain more detailed information of the equation of state of the nucleus by this way of reasoning. As far as the solution through the envelope is accurate, and as far as the fit-solution of (51) is accurate the following result holds: for each k_1 there corresponds a $\varrho_{\max}$ to every ϱ_{crit} given by the condition:

$$\varrho_{\max} = \frac{M_{r\,|\,\text{crit}}}{\frac{4}{3}\pi\, r^3_{\text{crit}}} \tag{57}$$

if $M_{r|\text{crit}}$ and r_{crit} denote the values of M_r and r at which, on that particular solution, ϱ becomes equal to ϱ_{crit}.

Again, if from physical consideration ϱ_{crit} and $\varrho_{\max}$ were known, that particular k_1 could be selected, which besides leading to a nucleus satisfied the condition $\int_{\text{centre}}^{\text{boundary}} dM_r = M.$

To approach the question in the manner described above it is essential that the computed solution be as near the truth as possible; some uncertainty is introduced by the discordance between "astronomical" and "theoretical" coefficient of opacity; further the equation of state $P = P(\varrho)$ for $\varrho < \varrho_{\text{crit}}$ should be studied basing on relativistic mechanics (relativistic mechanics introduces no change in the equation of a perfect gas).

From physical considerations* it can only be said that it is highly plausible, that at a certain density ϱ_{crit} the nuclei come so near one another that the nature of the interaction is abruptly changed, this then giving rise to an abrupt change in the equation of state in the sense of a very high $\varrho_{\max}$. So little is known however of the interaction of nuclei at small distances, that it is as yet difficult to make any prediction of the values of ϱ_{crit} and $\varrho_{\max}$.

7. We shall now discuss the bearings of the theory on the so called mass-luminosity relation. We know that, for given M and R, L must be chosen so that the condition $\int_{\text{centre}}^{\text{boundary}} dM_r = M$ is satisfied; the possible L-values will depend on the values of μ and k_1, on the source-distribution and on the form of the equation of state. If the equation of state of a perfect gas were valid unlimitedly we should have that L could have all values greater than a certain L on the model here considered, and all values between two limits on the standard model. In this case the condition $\int_{\text{centre}}^{\text{boundary}} dM_r = M$ would lead to no relation between M, R and L. For other types of equations of state, with a maximum density, the situation is different, the condition mentioned leading to a relation between M, R and L. It is however difficult at present to predict luminosities according to this relation for lack of knowledge of the equation of state throughout the star (cfr. for the discussion of the connection between predicted luminosity and equation of state the discussion of the connection between k_1, ϱ_{crit} and $\varrho_{\max}$ on p. 366).

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* I am indepted to Dr. L. Landau for discussion on this point.

Source-distribution has no great influence (when sinks of energy are excluded). The success of EDDINGTONS mass-luminosity relation in predicting the form of the curve is probably connected with the actual form of the equation of state: probably ϱ_{crit} and ϱ_{max} have such values that the corresponding L is not far from EDDINGTONS L (so that an increase in the difference of say 50% when passing from one mass to another will not be very important).

EDDINGTONS relation has the form:

$$L = \varphi(M)\, R^{-1/2}.$$

This form of the dependence on R corresponds to the existence of a homology-transformation with $R \rightarrow AR$ and $L \rightarrow A^{-1/2}L$. The homology-transformation mentioned is however only applicable as far as the laws of a perfect gas hold; there will be modifications in the dependence on R due to deviation from the laws of a perfect gas. (Strictly speaking the boundary condition $T = T_0$ is not invariant under the transformation mentioned. This is of no practical importance; cfr. the remark on the influence of T_0 on p. 356.)

Consider a star of mass M of (say) uniform composition. If we knew the equation of state and the correct values of k and μ for all values of T and ϱ, and further if we knew the activity of subatomic energy-sources for all values of T and ϱ, then we could predict the radius R and luminosity L of the star (this was first explicitly stated by H. VOGT). For there would be two conditions to satisfy: $\int_{\text{centre}}^{\text{boundary}} dM_r = M$ and $\int_{\text{centre}}^{\text{boundary}} dL_r = L$ leading to definite values of R and L. For solar mass probably two sets of values would satisfy the equations, corresponding to a main-series star and a white dwarf. Allowance being made for scatter due to different chemical composition this is in general accord with observation. Giants however fall outside the scheme; as giants are exceptions this is of no great importance [if for giant masses the intersection between the lines in an (R, L)-diagram corresponding to the two conditions were oblique, then small changes of chemical composition might lead to so different values of R for the same mass as are observed.]

As yet a discussion of the condition $\int_{\text{centre}}^{\text{boundary}} dL_r = L$ has not been possible; it may be hoped however, that such a discussion will be possible on the nucleus-theory.

8. MILNES views on stellar structure have not been generally accepted*. The main arguments of the critics seem to be:

I. The existence of solutions of the centrally-condensed type had not been proved.

II. Results might depend to a high degree on the nature of the model discussed, so results obtained on the standard model might not be typical.

III. The discordance between "theoretical" and "astronomical" coefficient of opacity is made worse, perhaps much worse.

We have discussed these points:

I. Following MILNE we postulate the existence of centrally-condensed configurations; then the objection should be formulated, that we have not discussed the consequences. This gap has however been filled after the objection was formulated and it was found (MILNE) that it could be concluded that overcompressibility must set in at a certain critical density.

II. Again the objection should be formulated, that in discussing the consequences of the postulate one should not restrict oneself to the standard model. It has been shown in the present paper that on the point-source model with $k = k_1 \varrho T^{-3.5}$ the results obtained are analogous to those obtained on the standard model.

III. It is true that k_1 of centrally-condensed configurations is greater than EDDINGTONS value (k_1^0); it need not however be much greater and the increase is partly balanced by the concentration of energy-sources towards the centre (cfr. Table 4); a discrepancy factor of 15 as against EDDINGTONS 10 is quite plausible. If the nucleus had to contain say 90% of the mass, then the discrepancy factor would be much greater (cfr. the numbers in Table 4 and Table 5), but there is no particular reason to assume this.

Accepting MILNES views we are thus led, I think, to a physically plausible stable model according to which the process of energy-liberation can be understood.

Copenhagen, Observatory, 1931 March 27th.

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* Confer Nature **127**, 130, 1931 and Observatory February 1931.