ON THE UNIFIED FIELD THEORY. I

BY TRACY YERKES THOMAS

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated September 30, 1930

In a number of notes in the Berlin *Sitzungsberichte*[1] followed by a revised account in the *Mathematische Annalen*,[2] Einstein has attempted to develop a unified theory of the gravitational and electromagnetic field by introducing into the scheme of Riemann geometry the possibility of distant parallelism. According to this view, there exists in each point of the underlying continuum of the world of space and time, a local cartesian coördinate system in which the Pythagorican theorem is satisfied. These local coördinate systems are determined by four independent vector fields with components $h^{\alpha}_{i}(x)$ depending on the coördinates x^i of the continuum. There is, therefore, associated with each point, a configuration consisting of four independent vectors; it is assumed that these vector configurations are in parallel orientation in such a way that the arbitrary orientation of the configuration at one point determines uniquely the orientation of the configurations at all points of the continuum. This affords the possibility of determining whether or not two vectors at different points are parallel, namely, by comparing their components in the local systems: Two vectors are parallel if the corresponding components are equal when referred to a local system of coördinates. It is in the theory of the space of distant parallelism that Einstein has hoped to find his long-sought unification of electricity and gravitation.

It is important to determine an exact relation between the coördinates z^i of the local system and the coördinates x^i of the space of distance parallelism. This problem is solved in the present communication in such a way that certain requirements specified precisely in Sect. 2 are satisfied. As so defined there is a certain analogy between the local coördinates and the normal coördinates introduced into the theory of relativity by Birkhoff[3] and later discussed by the writer.[4] The relation between the z^i and x^i coördinates enables us to construct a set of *absolute invariants* with respect to transformations of the x^i coördinates sufficient for the complete characterization of the space of distant parallelism. Equations in the local system, in which the coördinates z^i are interpretable as coördinates of time and space, can be transformed directly into equations of general invariantive character. In view of this property we are led to the construction of a *system of wave equations* as the equations of the combined gravitational and electromagnetic field. This system is composed of 16 equations for the determination of the 16 quantities h^{α}_{i} and is closely analogous to the system of 10 equations for the determination of the 10 components

$g_{\alpha\beta}$ in the original theory of gravitation. It is an interesting fact that the covariant components h^i_α of the fundamental vectors, when considered as electromagnetic potential vectors, satisfy in the local coördinate system the universally recognized laws of Maxwell for the electromagnetic field in free space, as a consequence of the field equations.

It is my intention to supplement the results of the present paper by a series of papers devoted to an existence-theoretic treatment of the field equations and related problems.

1. Let us denote by $h^\alpha_i(x)$ for $i = 1,2,3,4$, the components of four contravariant vector fields; we observe that the Latin letter i is thus used to denote the vector, and the Greek letter α to denote the component of the vector. In general, we shall adopt this convention of Einstein, i.e., we shall employ a Latin letter for an index which is of invariantive character with respect to the arbitrary transformations of the x^i coördinates, and a Greek letter in all contrary cases. Departures from this rule, as well as departures from the rule that an index which appears twice in a term is to be summed over the values of its range, will be such as are easily recognized on observation. By means of the quantities $h^\alpha_i(x)$ we impose on the underlying continuum its structure as a space of distant parallelism in accordance with the following

Postulates of Space Structure

A. *In any coördinate system (x) there exists a unique set of components $\Delta^\alpha_{\beta\gamma}$ of affine connection, for the determination of the affine properties of the continuum.*

B. *In any coördinate system (x) there exists a unique quadratic differential form $g_{\alpha\beta}dx^\alpha dx^\beta$ of signature -2, for the determination of the metric properties of the continuum.*

C. *At each point P of the continuum there is determined a configuration consisting of four orthogonal unit vectors issuing from P.*

D. *Corresponding vectors in the configurations, determined at two points P and Q of the continuum, are parallel.*

E. *The components h^α_i of the vectors determining the configuration at any point P of the continuum, are analytic functions of the x^i coördinates.*

The quadratic differential form in Postulate B will be referred to as the *fundamental form* and its coefficients $g_{\alpha\beta}$ as the components of the *fundamental metric tensor*. The four unit vectors with components $h^\alpha_i(x)$, which enter in Postulates C, D and E, will be referred to as the *fundamental vectors*. Since these vectors are orthogonal by Postulate C, the expression $g_{\alpha\beta}h^\alpha_i h^\beta_k$ is equal to zero for $i \neq k$; the condition that the fundamental vectors are unit vectors, likewise specified by Postulate C, means that for $i = k$, the expression $g_{\alpha\beta}h^\alpha_i h^\beta_k$ has the value ± 1. Hence we can write

$$g_{\alpha\beta}h_i^\alpha h_k^\beta = e_i\delta_k^i \tag{1.1}$$

where, following Eisenhart,[5] the convenient notation e_i for $+1$ or -1, is introduced. The fact that the fundamental form has signature -2 by Postulate B leads us to take

$$e_1 = 1,\ e_2 = e_3 = e_4 = -1;$$

more precisely, the signature -2 requires that one of the e_i have the value $+1$ and that the remaining e_i have values -1, the selection of the particular e_i to which the value $+1$ is assigned being an unessential matter of notation. From (1.1) we see that the product of the determinants $|g_{\alpha\beta}|$ and $|h_i^\alpha|^2$ is equal to -1. This shows that the determinant $|g_{\alpha\beta}|$ is *negative* and leads to the conclusion of the *independence* of the fundamental vectors. It is, therefore, possible to deduce from the fundamental vectors a system of four covariant vectors with components $h_\alpha^i(x)$ uniquely defined by the relations

$$h_i^\alpha h_\beta^i = \delta_\beta^\alpha,\ h_k^\alpha h_\alpha^i = \delta_k^i;$$

the components h_i^α will be called the covariant components of the fundamental vectors to distinguish them from the contravariant components h_i^α of these vectors. In consequence of the above relations, equations (1.1) can be solved so as to obtain

$$\boxed{g_{\alpha\beta} = \sum_{i=1}^{4} e_i h_\alpha^i h_\beta^i} \tag{1.2}$$

as the equations defining the components of the fundamental metric tensor.

The condition that the fundamental vectors be parallel as demanded by Postulate D has its analytical expression in the equations

$$\frac{\partial h_i^\alpha}{\partial x^\gamma} + h_i^\beta\ \Delta_{\beta\gamma}^\alpha = 0.$$

This gives

$$\boxed{\Delta_{\beta\gamma}^\alpha = h_i^\alpha \frac{\partial h_\beta^i}{\partial x^\gamma}} \tag{1.3}$$

as the equations which define the components $\Delta_{\beta\gamma}^\alpha$ of affine connection.

Let us now suppose the existence of another system of fundamental vectors in the same space of distant parallelism. Let us denote the contravariant components of these vectors by $*h_i^\alpha(x)$ in the (x) coördinate system, and let us put

$$*h_i^\alpha = a_i^k h_k^\alpha; \tag{1.4}$$

these equations define the quantities a_i^k as analytic functions of the x^i coördinates in consequence of Postulate E. Multiplying both members of (1.4) by $*h_\beta^i h_\alpha^j$ we obtain the equations

$$h_\beta^j = a_k^j {*h_\beta^k} \tag{1.5}$$

which represent the transformation induced by (1.4) on the covariant components h_α^i. On account of the uniqueness of determination of the components $\Delta_{\beta\gamma}^\alpha(x)$ demanded by Postulate A, we must have

$$h_i^\alpha \frac{\partial h_\beta^i}{\partial x^\gamma} = *h_i^\alpha \frac{\partial *h_\beta^i}{\partial x^\gamma}.$$

From (1.4) and these latter equations, it readily follows that *the quantities a_i^k are constants.* The quantities a_i^k are, however, not arbitrary constants, since they must satisfy a condition of orthogonality, namely,

$$\boxed{\sum_{i=1}^{4} e_i\, a_k^i\, a_l^i = e_k\, \delta_l^k} \tag{1.6}$$

which is obtained from (1.5) as the direct result of the uniqueness of determination of the fundamental form specified by Postulate B. Taking the determinant of both members of (1.6) we find that the determinant $|a_i^k|$ has the value ± 1. We can therefore define uniquely a set of quantities b_k^i by the equations

$$a_j^i\, b_k^j = \delta_k^i,\ a_k^j\, b_j^i = \delta_k^i.$$

Another form of the conditions (1.6) which is sometimes useful, can be derived in the following manner. Multiply both members of (1.6) by b_m^k so as to obtain

$$e_m a_l^m = e_l b_m^l,$$

or

$$a_l^m = e_l e_m b_m^l.$$

When we multiply both members of these latter equations through by $e_l a_l^k$ and sum of the index l, we find

$$\boxed{\sum_{l=1}^{4} e_l\, a_l^k\, a_l^m = e_k\, \delta_m^k}\,. \tag{1.7}$$

The transformation (1.4) in which the coefficients a_i^k are constants satisfying (1.6) or (1.7) will be called an *orthogonal transformation* of the components of the fundamental vectors.

2. The idea of the local coördinate system (z) is inherent in the idea of the configuration formed by the four vectors whose components h_i^α are subject to the orthogonal transformations (1.4), and, in fact, the construction of the local system was not carried beyond this stage by Einstein. Nevertheless a configuration of four vectors is not a coördinate system and we must face the problem of showing exactly how a system of local coördinates z^i can be defined. We require of the local system that it satisfy certain conditions which are stated precisely in the following

POSTULATES OF THE LOCAL SYSTEM

A. *With each point P of the space of distant parallelism there is associated a local coördinate system (z) having its origin at the point P.*

B. *The coördinate axes of the local system (z) at the point P are tangent to the fundamental vectors at P, in such a way that the z^i axis is tangent to the vector with components h_i^α and has its positive direction along the direction of this vector.*

C. *The interval ds is given by*

$$\boxed{ds^2 = (dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2} \tag{2.1}$$

at the origin of the local system.

D. *The paths[6] of the space of distant parallelism which pass through the origin of the local system (z) have the form*

$$\boxed{z^i = \xi^i v}, \tag{2.2}$$

where the ξ^i are constants and v is a parameter.

The above postulates give a complete geometrical characterization of the coördinates z^i of the local system. The *paths* which enter in Postulate D are, in general, defined as those curves which are given as solutions of the invariant system of equations

$$\frac{d^2x^\alpha}{dv^2} + \Lambda^\alpha_{\beta\gamma}\frac{dx^\beta}{dv}\frac{dx^\gamma}{dv} = 0, \tag{2.3}$$

where

$$\Lambda^\alpha_{\beta\gamma} = \frac{1}{2}(\Delta^\alpha_{\beta\gamma} + \Delta^\alpha_{\gamma\beta}); \tag{2.4}$$

such curves are generated by continuously displacing a vector parallel to itself along its own direction and are analogous to the straight lines of affine Euclidean geometry. If we denote the components of affine connection $\Lambda^{\alpha}_{\beta\gamma}$ by $\lambda^{i}_{jk}\ (z)$, when referred to the local coördinate system, we have in consequence of Postulate D that

$$\lambda^{i}_{jk}\, \xi^{j}\, \xi^{k} = 0$$

along a path through the origin of the local system. It follows immediately from these latter equations that the equations

$$\lambda^{i}_{jk} z^{j} z^{k} = 0 \qquad (2.5)$$

are satisfied identically in the local system. When we replace the components λ^{i}_{jk} in (2.5) by their values in terms of the components $\Lambda^{\alpha}_{\beta\gamma}$ we obtain a system of partial differential equations for the determination of the coördinates x^{i} in terms of the coördinates z^{i}, namely,

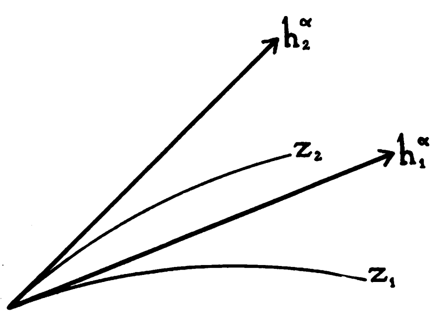
$$\left(\frac{\partial^2 x^{\alpha}}{\partial z^{j}\partial z^{k}} + \Lambda^{\alpha}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial z^{j}}\frac{\partial x^{\gamma}}{\partial z^{k}}\right) z^{j} z^{k} = 0. \qquad (2.6)$$

This system of equations possesses a unique solution $x^{\alpha} = \varphi^{\alpha}(z)$ satisfying a set of initial conditions

$$x^{\alpha} = p^{\alpha}\ (z^{i} = 0), \qquad (2.7)$$

$$\frac{\partial x^{\alpha}}{\partial z^{i}} = p^{\alpha}_{i}\ (z^{i} = 0), \qquad (2.8)$$

where p^{α} and p^{α}_{i} are arbitrary constants.[7] By Postulate A condition (2.7) is satisfied, provided that the constants p^{α} denote the coördinates of the point P; it remains to determine the values of the constants p^{α}_{i} in (2.8). For this purpose we consider the relation, imposed by Postulate B, between the local coördinates and the fundamental vectors at the point P; this is illustrated in the accompanying figure in which we have indicated only the positive z^1 and z^2 axes and the corresponding tangent vectors with components h^{α}_{1} and h^{α}_{2}, respectively. In consequence, we see that



$$\left(\frac{\partial x^{\alpha}}{\partial z^{i}}\right)_{z=0} = \sigma_{i} h^{\alpha}_{i}(p),$$

where the σ_i are positive constants. On account of this set of equations and the condition imposed by Postulate C, it readily follows that the σ_i are equal to ± 1. Hence $\sigma_i = +1$, since these constants are positive, and the above set of equations becomes

$$\left(\frac{\partial x^\alpha}{\partial z^i}\right)_{z=0} = h_i^\alpha(p). \tag{2.9}$$

The values $h_i^\alpha(p)$ are therefore to be ascribed to the above constants p_i^α. By repeated differentiation of (2.6) and use of the initial conditions (2.7) and (2.9), we determine the successive coefficients $H(p)$ of the power series expansions

$$x^\alpha = p^\alpha + h_i^\alpha\,(p)z^i - \frac{1}{2!}H_{ij}^\alpha\,(p)z^iz^j - \frac{1}{3!}H_{ijk}^\alpha(p)z^iz^jz^k - \ldots. \tag{2.10}$$

Thus we have that

$$H_{ij}^\alpha = \Lambda_{\beta\gamma}^\alpha h_i^\beta h_j^\gamma,$$

where

$$\Lambda_{\beta\gamma}^\alpha = \frac{1}{2}h_i^\alpha\left(\frac{\partial h_\beta^i}{\partial x^\gamma} + \frac{\partial h_\gamma^i}{\partial x^\beta}\right), \tag{2.11}$$

and

$$H_{ijk}^\alpha = \Lambda_{\beta\gamma\delta}^\alpha h_i^\beta h_j^\gamma h_k^\delta,$$

where

$$\Lambda_{\beta\gamma\delta}^\alpha = \frac{1}{3}\left[\left(\frac{\partial\Lambda_{\beta\gamma}^\alpha}{\partial x^\delta} + \frac{\partial\Lambda_{\gamma\delta}^\alpha}{\partial x^\beta} + \frac{\partial\Lambda_{\delta\beta}^\alpha}{\partial x^\gamma}\right) - 2(\Lambda_{\sigma\beta}^\alpha\,\Lambda_{\gamma\delta}^\sigma + \Lambda_{\sigma\gamma}^\alpha\,\Lambda_{\delta\beta}^\sigma + \Lambda_{\sigma\delta}^\alpha\,\Lambda_{\beta\gamma}^\sigma)\right], \tag{2.12}$$

etc. The jacobian of (2.10) does not vanish since it is equal to the determinant $|h_i^\alpha(p)|$ at the origin of the local system; hence (2.10) possesses a unique inverse. Either the transformation (2.10), or its inverse, gives a unique defination of the local coördinate system.

3. Let us denote by $\bar{z}^i$ the coördinates of the local system determined by the point P and the components of affine connection $\bar{\Lambda}_{\beta\gamma}^\alpha(\bar{x})$ which result from the components $\Lambda_{\beta\gamma}^\alpha(x)$ by an analytic transformation T of the x^α coördinates. The relation between the z^i and $\bar{z}^i$ coördinates must then be such that

$$z^i = 0\ (\bar{z}^i = 0),\quad \frac{\partial z^i}{\partial \bar{z}^k} = \delta_k^i\ (\bar{z}^i = 0). \tag{3.1}$$

Also this relation must be such as to satisfy the system of equations

$$\left(\frac{\partial^2 z^i}{\partial\bar{z}^j\partial\bar{z}^k} + \lambda_{pq}^i\frac{\partial z^p}{\partial\bar{z}^j}\frac{\partial z^q}{\partial\bar{z}^k}\right)\bar{z}^j\bar{z}^k = 0. \tag{3.2}$$

Hence

$$\boxed{z^i = \bar{z}^i}. \tag{3.3}$$

This follows from the fact that (3.3) satisfies (3.2) and the initial conditions (3.1), and that (3.2) possesses a unique solution which satisfies the conditions (3.1). We can therefore say that the *local coördinates z^i remain unchanged when the underlying coördinates x^i undergo an arbitrary analytic transformation T.* In a similar manner we can show that *when the fundamental vectors undergo an orthogonal transformation* (1.4), *the local coördinates z^i associated with any point P likewise undergo an orthogonal transformation, i.e., a linear homogeneous transformation*

$$\boxed{z^i = a^i_k z^k_*}, \tag{3.4}$$

which leaves the form

$$\sum_{i=1}^{4} e_i z^i z^i$$

invariant. The behavior of the coördinates z^i of the local system (1) under arbitrary analytic transformations of the x^i coördinates, and (2) under orthogonal transformations of the fundamental vectors as described by the above italicized statements, is of great importance for the development of the theory of relativity.

4. If we transform the components of a tensor to a system of local coördinates z^i and evaluate at the origin of this system, we obtain a set of quantities which are of the nature of *absolute invariants* with respect to transformations of the x^i coördinates. To prove this formally we shall find it convenient to denote a set of tensor components $T^{\alpha\ldots\beta}_{\mu\ldots\nu}(x)$ with respect to the (x) system, by $t^{|i\ldots j|}_{|k\ldots m|}$ in the (z) coördinate system, i.e., we shall adopt the convention that in the (z) coördinate system, Latin letters when enclosed by $|\quad|$, correspond to indices of covariant or contravariant character. If we put

$$T^{i\ldots j}_{k\ldots m} = (t^{|i\ldots j|}_{|k\ldots m|})_{z=0},$$

then

$$T^{i\ldots j}_{k\ldots m}(x) = \overline{T}^{i\ldots j}_{k\ldots m}(\bar{x}),$$

in consequence of (3.3). The explicit expressions for these invariants are obtained by evaluating at $z^i = 0$ both members of the set of equations

$$t^{|i\ldots j|}_{|k\ldots m|} = T^{\alpha\ldots\beta}_{\mu\ldots\nu} \frac{\partial x^\mu}{\partial z^k} \cdots \frac{\partial x^\nu}{\partial z^m} \frac{\partial z^i}{\partial x^\alpha} \cdots \frac{\partial z^j}{\partial x^\beta}. \tag{4.1}$$

This gives

$$T^{i\ldots j}_{k\ldots m} = T^{\alpha\ldots\beta}_{\mu\ldots\nu} h^\mu_k \ldots h^\nu_m h^i_\alpha \ldots h^j_\beta.$$

Similarly, if we differentiate the components t any number of times and evaluate at the origin of the (z) system, we obtain a set of quantities, namely,

$$T^{i\ldots j}_{k\ldots m,\, p\ldots q} = \left(\frac{\partial^r t^{|i\ldots j|}_{|k\ldots m|}}{\partial z^p \ldots \partial z^q}\right)_{z\,=\,0},$$

each of which is an absolute invariant with respect to analytic transformations of the x^i coördinates. On account of this invariant property, the name *absolute derivatives* will be used to refer to these quantities. To derive the explicit formulae for any absolute derivative $T^{i\ldots j}_{k\ldots m,\, p\ldots q}$ we have merely to differentiate (4.1) with respect to $z^p \ldots z^q$ and evaluate at the origin of the (z) system. For example, the first absolute derivative of the covariant vector with components T_α is given by the formula

$$T_{k,p} = \left(\frac{\partial T_\beta}{\partial x^\gamma} - T_\alpha \Lambda^\alpha_{\beta\gamma}\right) h^\beta_k\, h^\gamma_p.$$

We next consider the absolute derivatives of the fundamental vectors, using the covariant components h^i_α of these vectors as the basis of discussions. Let us denote these components by $A^i_{|j|}$ when referred to the (z) coördinate system. Then

$$A^i_{|j|} = h^i_\alpha \frac{\partial x^\alpha}{\partial z^j}. \tag{4.2}$$

In general, the absolute derivatives are defined by the equations

$$h^i_{j,k\ldots m} = \left(\frac{\partial^s A^i_{|j|}}{\partial z^k \ldots \partial z^m}\right)_{z\,=\,0}. \tag{4.3}$$

We note that

$$A^i_{|j|} = h^i_\alpha h^\alpha_j = \delta^i_j \quad (z^i = 0); \tag{4.4}$$

also that

$$\boxed{h^i_{j,k} = \frac{1}{2}\left[\frac{\partial h^i_\alpha}{\partial x^\beta} - \frac{\partial h^i_\beta}{\partial x^\alpha}\right] h^\alpha_j h^\beta_k} \tag{4.5}$$

and

$$\boxed{\begin{aligned} h^i_{j,kl} = \Bigg[\frac{\partial^2 h^i_\alpha}{\partial x^\beta \partial x^\gamma} &- \frac{\partial h^i_\alpha}{\partial x^\sigma}\Lambda^\sigma_{\beta\gamma} - \frac{\partial h^i_\sigma}{\partial x^\beta}\Lambda^\sigma_{\alpha\gamma} \\ &- \frac{\partial h^i_\sigma}{\partial x^\gamma}\Lambda^\sigma_{\alpha\beta} - h^i_\sigma \Lambda^\sigma_{\alpha\beta\gamma}\Bigg] h^\alpha_j h^\beta_k h^\gamma_l \end{aligned}}. \tag{4.6}$$

The special formulae (4.5) and (4.6) will be important in our later work.

By (4.3) the absolute derivative $h^i_{j,k\ldots m}$ is symmetric in the indices $k \ldots m$. Hence, these quantities satisfy the identities

$$\boxed{h^i_{j,k\ldots m} = h^i_{j,p\ldots q}}, \tag{4.7}$$

where $p\ldots q$ denotes any permutation of the indices $k\ldots m$. To derive other identities satisfied by the $h^i_{j,k\ldots m}$ we observe that the equations

$$\left(\frac{\partial A^i_{|j|}}{\partial z^k}\right) z^j z^k = 0 \tag{4.8}$$

which are equivalent to (2.5), are satisfied identically in the (z) coördinate system. By repeated differentiation of (4.8) and evaluation at the origin of the (z) system, we obtain

$$\boxed{S\ (h^i_{j,k\ldots m}) = 0}, \tag{4.9}$$

where the symbol S is used to stand for the summation of all the terms obtainable from the one in parenthesis by permutating the indices $jk\ldots m$ cyclically. As special cases of the identities (4.9) let us observe that

$$h^i_{j,k} + h^i_{k,j} = 0 \tag{4.10}$$

and

$$h^i_{j,kl} + h^i_{k,lj} + h^i_{l,jk} = 0. \tag{4.11}$$

The identities (4.7) *and* (4.9) *constitute a complete set of identities of the absolute derivatives* $h^i_{j,k\ldots m}$, in the sense that any other identity satisfied by these quantities is derivable from these identities.[8]

When we make an orthogonal transformation (1.4) of the components of the fundamental vectors, the components $A^i_{|j|}$ go over into a set of components $^*A^i_{|j|}$ referred to the (z_*) system, which are related to the $A^i_{|j|}$ by

$$^*A^i_{|j|}\, a^p_i = A^p_{|q|} a^q_j; \tag{4.12}$$

this follows from (1.5), (3.4), and (4.2). Differentiating both members of (4.12) with respect to $z^k_*\ldots z^m_*$ and evaluating at the origin of the local system, we obtain

$$^*h^i_{j,k\ldots m}\, a^p_i = h^p_{q,r\ldots t}\, a^q_j \ldots a^t_m. \tag{4.13}$$

We express this result by saying that the *absolute derivatives* $h^i_{j,k\ldots m}$ *constitute the components of a tensor with respect to orthogonal transformations of the fundamental vectors.* A similar discussion can, of course, be made on the basis of the contravariants components h^α_i of the fundamental vectors; also equations similar to (4.13) give the transformation of the components $T^{i\ldots j}_{k\ldots m}$ or more generally of the components $T^{i\ldots j}_{k\ldots m,p\ldots q}$, induced by the transformation (1.4) of the fundamental vectors.[9]

5. It is the sense of the local coördinate system (z) that the coördinate

z^1 is interpretable as the time coördinate and z^2, z^3, z^4, as rectangular cartesian coördinates of space. Let us, therefore, put

$$z^1 = t,\ z^2 = x,\ z^3 = y,\ z^4 = z,$$

in accordance with the unusual designations. More precisely, we should say that z^1 has the significance of a time coördinate and z^2, z^3, z^4, the significance of rectangular cartesian coördinates x, y, z *at the origin of coördinates*, i.e., in the infinitesimal neighborhood of the origin, since the interval ds has the exact form (2.1) only at the origin of the local system. In conformity with this interpretation of the coördinates of the local system, we now impose on the space of distant parallelism, the following

POSTULATE OF THE UNIFIED FIELD

The equations

$$\boxed{\frac{\partial^2 A^i_{|j|}}{\partial x^2} + \frac{\partial^2 A^i_{|j|}}{\partial y^2} + \frac{\partial^2 A^i_{|j|}}{\partial z^2} - \frac{\partial^2 A^i_{|j|}}{\partial t^2} = 0} \qquad (5.1)$$

are satisfied at the origin of the local system.

According to the above postulate, the field equations of the combined gravitational and electromagnetic field are of the nature of a system of wave equations—a type of equation which has already shown itself to be of fundamental importance for the study of gravitational or electromagnetic phenomena. The relationship of the above system of field equations to the field equations of the earlier theory of gravitation and to the Maxwell equations of the electromagnetic field is discussed in the following section.

When (5.1) is expressed in general invariative form, we have

$$\boxed{\sum_{k=1}^{4} e_k h^i_{j,kk} = 0} \qquad (5.2)$$

as the proposed system of field equations for the unified field theory. The system (5.2) is obviously invariant with respect to transformations of the x^i coördinates, since the factors which compose the system are themselves directly invariant; it is also invariant with respect to orthogonal transformations (1.4) of the fundamental vectors. To prove this we observe that

$$*h^i_{j,kl} a^p_i = h^p_{q,rs} a^q_j a^r_k a^s_l$$

from (4.13). Putting l equal to k in these equations and multiplying through by e_k, we obtain

$$\left(\sum_{k=1}^{4} e_k *h^i_{j,kk}\right) a^p_i = \left(\sum_{r=1}^{4} e_r h^p_{q,rr}\right) a^q$$

on making use of (1.7). This proves the assertion of invariance of (5.2) with respect to orthogonal transformations of the fundamental vectors.

6. If we denote the components of electric force by X, Y, Z and the components of magnetic force by α, β, γ then Maxwell's equations for the electromagnetic field in free space, are

$$\left\{\begin{aligned} &\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = -\frac{\partial \alpha}{\partial t} \\ &\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} = -\frac{\partial \beta}{\partial t} \\ &\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = -\frac{\partial \gamma}{\partial t} \\ &\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0 \end{aligned}\right. \tag{6.1}$$

and

$$\left\{\begin{aligned} &\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} = \frac{\partial X}{\partial t} \\ &\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} = \frac{\partial Y}{\partial t} \\ &\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} = \frac{\partial Z}{\partial t} \\ &\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0, \end{aligned}\right. \tag{6.2}$$

where x, y, z are rectangular cartesian coördinates and t is the time. It is possible to write the above equations in a more contracted form. For this purpose we define a set of skew-symmetric quantities F_{jk} as the elements of the matrix

$$\left\| \begin{matrix} 0 & X & Y & Z \\ -X & 0 & -\gamma & \beta \\ -Y & \alpha & 0 & -\alpha \\ -Z & -\beta & \alpha & 0 \end{matrix} \right\|$$

and then construct the following two sets of equations

$$F_{jk,l} + F_{kl,j} + F_{lj,k} = 0 \tag{6.3}$$

and

$$\sum_{k=1}^{4} e_k F_{jk,k} = 0, \tag{6.4}$$

where $F_{jk,l}$ is now used to denote the ordinary partial derivative of F_{jk} with respect to z^l. The expanded forms of (6.3) and (6.4) give equations (6.1) and (6.2), respectively. Functions φ_j of the coördinates x, y, z and the time t are called electromagnetic potentials if they are such that

$$F_{jk} = \frac{\partial \varphi_j}{\partial z_k} - \frac{\partial \varphi_k}{\partial z^j}. \tag{6.5}$$

If the F_{jk} have the form (6.5), equations (6.3) are satisfied identically.

Now consider the covariant components h^i_α referred to the local system, i.e., the components $A^i_{|j|}$, and put

$$\varphi_j = A^i_{|j|} \tag{6.6}$$

for a fixed value of the index i; also define a set of quantities F_{jk} by (6.5), using the φ_j given by (6.6) for that purpose. Then

$$F_{jk} = \frac{\partial A^i_{|j|}}{\partial z^k} - \frac{\partial A^i_{|k|}}{\partial z^j}. \tag{6.7}$$

Differentiating and evaluating at the origin of the local system, we have

$$F_{jk,l} = h^i_{j,kl} - h^i_{k,jl};$$

the absolute derivatives $F_{jk,l}$ given by these equations satisfy the first set of Maxwell's equations (6.3) in consequence of the identities (4.11). Moreover,

$$\sum_{k=1}^{4} e_k F_{jk,k} = \sum_{k=1}^{4} (e_k h^i_{j,kk} - e_k h^i_{k,jk}) = \frac{3}{2} \sum_{k=1}^{4} e_k\, h^i_{j,kk}.$$

Hence, the second set of Maxwell's equations (6.4) is satisfied in consequence of the field equations (5.2). In other words, *the covariant components of the fundamental vectors when considered as electromagnetic potential vectors, satisfy, in the local coördinate system, the universally recognized laws of Maxwell for the electromagnetic field in free space, as a consequence of the field equations* (5.2). This fact strongly suggests that the components h^i_α will play the rôle of electromagnetic potentials in the present theory.

Taking account of the field equations (5.2) we easily deduce the system of equations

$$\sum_{k=1}^{4} e_k g_{ij,kk} = 2 \sum_{k,l=1}^{4} e_k\, e_l\, h^l_{i,k}\, h^l_{j,k}. \tag{6.8}$$

Or, denoting the components $g_{\alpha\beta}$ by $B_{|ij|}$ when referred to the local coordinate system, we have

$$\frac{\partial^2 B_{|ij|}}{\partial x^2} + \frac{\partial^2 B_{|ij|}}{\partial y^2} + \frac{\partial^2 B_{|ij|}}{\partial z^2} - \frac{\partial^2 B_{|ij|}}{\partial t^2} = -2 \sum_{k,l=1}^{4} e_k e_l h^l_{i,k} h^l_{j,k} \tag{6.9}$$

at the origin of the local system. Now in the earlier theory of gravitation, the field equations could be given a form analogous to (6.9) in which the right members were equal to zero;[3] since the nature of the solution of (6.8) is determined by the form of the left members of (6.9) it follows that the functions $g_{\alpha\beta}$ possess the same general character as in the earlier theory of gravitation. Moreover, the interpretation of the quantities h^i_α as electromagnetic potentials would lead us to expect that in a purely gravitational field, i.e., more precisely, in a comparatively inappreciable electromagnetic field, the square of the $h^i_{j,k}$ would be negligibly small quantities. In this case the right members of (6.9) vanish approximately and we are left with the system

$$\frac{\partial^2 B_{|ij|}}{\partial x^2} + \frac{\partial^2 B_{|ij|}}{\partial y^2} + \frac{\partial^2 B_{|ij|}}{\partial z^2} - \frac{\partial^2 B_{|ij|}}{\partial t^2} = 0$$

as a first approximation. It is, therefore, to be expected that the quantities $g_{\alpha\beta}$ will successfully assume the rôle of *gravitational potentials* as in the previous theory of gravitation.

[1] A. Einstein, *Berliner Berichte*, 1928–30.

[2] A. Einstein, "Auf die Riemann-Metrik und den Fern-Parallelismus gegründete einheitliche Feldtheorie," *Math. Ann.*, **102** (1930), pp. 685–697.

[3] G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press (1923), pp. 124 and 228.

[4] T. Y. Thomas, "The Principle of Equivalence in the Theory of Relativity," *Phil. Mag.*, **48** (1924), pp. 1056–1068.

[5] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press (1926), Chap. II.

[6] This has reference to the *path* in the sense in which that word is used in "The Geometry of Paths," *Trans. Am. Math. Soc.*, **25** (1923), pp. 551–608.

[7] It is obvious that the system of differential equations (2.6) possesses a unique formal solution

$$x^\alpha = p^\alpha + p^\alpha_i z^i + \frac{1}{2!} p^\alpha_{ij} z^i z^j + \dots \tag{a}$$

such that the conditions (2.7) and (2.8) are satisfied. A proof of convergence of the series (*a*) can be given in the following manner. Consider a system of equations

$$\frac{\partial^2 X^\alpha}{\partial z^j \partial z^k} - F^\alpha_{\beta\gamma} \frac{\partial X^\beta}{\partial z^j} \frac{\partial X^\gamma}{\partial z^k} = 0, \tag{b}$$

where the $F^\alpha_{\beta\gamma}$ are analytic functions of the variables X^α. The conditions of integrability of (*b*) are that

$$\frac{\partial F^\alpha_{\beta\gamma}}{\partial X^\delta} - \frac{\partial F^\alpha_{\beta\delta}}{\partial X^\gamma} + F^\alpha_{\mu\delta} F^\mu_{\beta\gamma} - F^\alpha_{\mu\gamma} F^\mu_{\beta\delta} = 0$$

identically. These conditions are satisfied by taking all the functions $F^\alpha_{\beta\gamma}$ equal to one another in accordance with the equations

$$F^{\alpha}_{\beta\gamma} = \frac{M}{1 - \frac{X^1 + \ldots + X^4}{\rho}},$$

ere M and ρ are constants; the system (b) then possesses a unique solution

$$X^{\alpha} = P^{\alpha} + P^{\alpha}_{i} z^{i} + \frac{1}{2!} P^{\alpha}_{ij} z^{i} z^{j} + \ldots, \tag{c}$$

isfying the conditions

$$X^{\alpha} = P^{\alpha} \quad (z^i = 0),$$

$$\frac{\partial X^{\alpha}}{\partial z^i} = P^{\alpha}_{i} \quad (z^i = 0).$$

w choose the above constants M and ρ so that each function $F^{\alpha}_{\beta\gamma}$ dominates the corre-
nding functions $\Lambda^{\alpha}_{\beta\gamma}$, and, furthermore, choose the P^{α} and P^{α}_{i} such that

$$P^{\alpha} \geqq | p^{\alpha} |, \quad P^{\alpha}_{i} \geqq | p^{\alpha}_{i} |.$$

view of the fact that (c) is likewise a solution of the system

$$\left(\frac{\partial^2 X^{\alpha}}{\partial z^j \partial z^k} - F^{\alpha}_{\beta\gamma} \frac{\partial X^{\beta}}{\partial z^j} \frac{\partial X^{\gamma}}{\partial z^k}\right) z^j z^k = 0,$$

s then easily seen that each expansion (c) dominates the corresponding expansion (a). e convergence of the expansions (a) within a sufficiently small neighborhood of the ues $z^j = 0$, is therefore established.

The proof is analogous to that given in my paper "The Identities of Affinely Con- ted Manifo'ds," *Math. Zeit.*, 25 (1926), pp. 714–722.

A method of covariant differentiation based on the non-symmetric components affine connection $\Delta^{\alpha}_{\beta\gamma}$ is used by Einstein.[2] The point of view adopted in the present estigation in which covariant differentiation, or more precisely absolute differentia- n, is brought into relationship with the local coördinate system, necessarily depends on ymmetric connection; for this reason I have rejected the method of Einstein. It uld be possible also to define covariant or absolute derivatives on the basis of the ristoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ derived from the components $g_{\alpha\beta}$ and, in fact, this has been ad- cated by T. Levi-Civita, "Vereinfachte Herstellung der Einsteinschen einheitlichen ldgleichungen," *Berliner Berichte*, **1929.** This method for the construction of invari- ts can be brought into relationship with the local coördinates; in fact, it is only neces- y to replace Postulate D of the *Postulates of the Local System* by a similar postulate on *geodesics* of the space of distant parallelism. My primary objection to this is that we a good deal of the simplification inherent in the above theory. For example, in ce of the quantities $h^{i}_{j,k}$ given by (4.5) we would have the more complicated set of ariants

$$\left(\frac{\partial h^{i}_{\alpha}}{\partial x^{\beta}} - \Gamma^{\sigma}_{\alpha\beta} h^{i}_{\sigma}\right) h^{\alpha}_{j} h^{\beta}_{k}$$

investigation. Now there is a certain psychological influence exerted by the method itself upon the investigator. What I mean is that one is naturally led to the construction of special invariants which, on the basis adopted for the formation of invariants, are represented by simple analytical expressions. So, for example, the field equations proposed by Einstein[2] have a very simple analytical form in terms of the covariant derivatives used by him, but these same equations would be considerably more complicated in form if expressed in terms of the covariant derivatives used by Levi-Civita; also the simple form (5.1) or (5.2) of the field equations assumed in the above investigation is peculiar to the method of absolute differentiation which I have adopted. Thus the different methods of construction of invariants will lead us, in practice, to the assumption of different systems of field equations; these systems of field equations must, roughly speaking, be of the same general character but will nevertheless, not be precisely equivalent, considered as systems of partial differential equations.

It is possible to develop a process by which the above methods are brought into relationship with one another and which will, moreover, permit the ready construction of invariants depending on combinations of these methods. This process has its geometrical foundation in the study of those surfaces $x^\alpha = f^\alpha(u, v)$ which are defined as solutions of the system of equations

$$\left\{\begin{aligned} &\frac{\partial^2 x^\alpha}{\partial u \partial v} + \Delta^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial u}\frac{\partial x^\gamma}{\partial v} = 0, \\ &\frac{\partial^2 x^\alpha}{\partial u^2} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial u}\frac{\partial x^\gamma}{\partial u} = 0 \quad (v = 0), \\ &\frac{\partial^2 x^\alpha}{\partial v^2} + \Lambda^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial v}\frac{\partial x^\gamma}{\partial v} = 0 \quad (u = 0), \end{aligned}\right. \qquad (a)$$

and so constitutes a generalization of the process of covariant differentiation or extension, as developed in *The Geometry of Paths*. In this way we are led to a set of relations $x^\alpha = g^\alpha(y, z)$ which, for z^i = const., denote a transformation to a system of coördinates y^i, and which for y^i = const. denote a transformation to a system of coördinates z^i. If $t(y, z)$ represents the components of a tensor either in the (y) or (z) coördinate system, then

$$\left(\frac{\partial^r t(y, z)}{\partial y^\alpha \ldots \partial y^\beta \partial z^\gamma \ldots \partial z^\delta}\right) y = 0, z = 0$$

defines, when considered as a function of the x^i coördinates, the components of a tensor in the (x) coördinate system. As an alternative method of procedure the system (a) can be replaced by the system

$$\begin{aligned} &\left(\frac{\partial^2 x^\alpha}{\partial y^j \partial z^k} + \Delta^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial z^k}\right) y^j z^k = 0, \\ &\left(\frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} + \Gamma^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\right) y^j y^k = 0 \quad (z = 0), \\ &\left(\frac{\partial^2 x^\alpha}{\partial z^j \partial z^k} + \Lambda^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial z^j}\frac{\partial x^\gamma}{\partial z^k}\right) z^j z^k = 0 \quad (y = 0). \end{aligned} \qquad (b)$$

The consideration of system (b) enables us, moreover, by imposing initial conditions corresponding to (2.9), to develop a theory of absolute invariants which is a generalization of that of the above investigation. The details of this process will not be developed here as it is not necessary for our work on the theory of relativity.