Selected Chapters of Geometry

ETH Zürich, summer semester 1940

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(This is the text of a course that Heinz Hopf gave in the summer of 1940 at the ETH in Zürich. It has been reconstructed and translated by Hans Samelson from the notes that he took as student during the course and that resurfaced in 2002.)

Contents

Chapter I. Euler's Formula

I.1. The Euclidean and the Spherical Triangle

For the Euclidean triangle we have the well known fact "Sum of the angles $= \pi$" or $\sum \alpha - \pi = 0$.

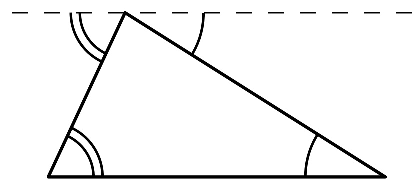


FIGURE 1

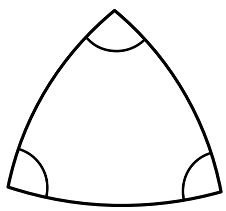


FIGURE 2

For a spherical triangle with angles α and area F on a sphere of radius r there is the "formula of the spherical excess" (excess of the sum of the angles over π):

$$\sum \alpha - \pi = F/r^2 \qquad \text{(Thomas Hariot, 1603.)}$$

This is proved with the help of the three great circles that form the triangle. These great circles decompose the sphere into six spherical 2-gons, three covering the original triangle and the other three its antipodal image. Thus these 2-gons cover the two triangles three times and the rest of the sphere exactly once.

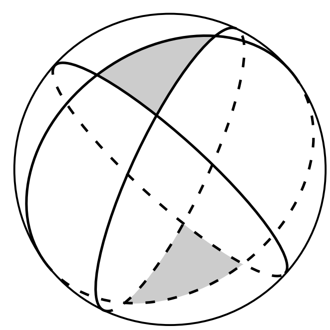


FIGURE 3

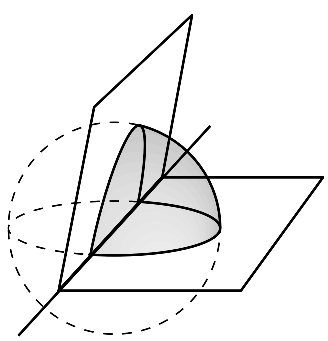


FIGURE 4

The surface area of the sphere is $4\pi r^2$. The area of a 2-gon of opening angle α is $4\pi r^2 \cdot \frac{\alpha}{2\pi} = 2\alpha r^2$. This gives us $2 \cdot \sum 2\alpha r^2 = 4\pi r^2 + 4F$ (left hand: the six 2-gons, right hand: the whole sphere plus twice the original triangle plus twice the antipodal one), which proves the formula. $\surd$

We apply this to get a relation for the vertex angles and edge angles of a tetrahedron. For each vertex of the tetrahedron we have $F/r^2 = \sum \epsilon - \pi$, where F is the area of the spherical triangle cut out by the corner from a sphere of radius r, and the ϵ are the edge angles of the tetrahedron at the edges going out from the vertex. Adding over the vertices this gives the formula of J.-P. Gua de Malves (around 1740? 1783?)

$$\sum F/r^2 = 2\sum \epsilon \; - 4\pi,$$

where the ϵ are the angles at the six edges. √

We measure all angles by the fraction they cut out from a sphere of (small) radius r with center at a vertex or an interior point of an edge and denote the angles $F/4\pi r^2$ at the vertices by γ (the new ϵ is the old one divided by 2π). The formula then becomes

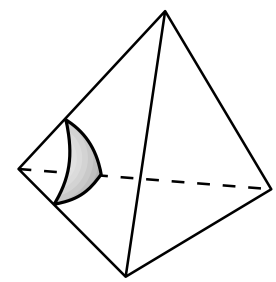


FIGURE 5

$$\sum \gamma - \sum \epsilon + 1 = 0.$$

This becomes more attractive if we write it as

$$\sum \gamma - \sum \epsilon + 4 \cdot \tfrac{1}{2} - 1 = 0.$$

Here we interpret each of the four $\frac{1}{2}$s as the "angle" at one of the faces, i.e. the fraction of the surface of a sphere with center at an interior point of the face cut out by the tetrahedron, and interpret the 1 similarly with a sphere at an interior point of the tetrahedron. Thus we have four sums in the formula going respectively over the vertices, the edges, the faces, and the last with only one term for the whole tetrahedron, all added together with alternating signs. This has a generalization to n dimensions.

I.2. The n-Dimensional Simplex, Euclidean and Spherical

A **Euclidean n-simplex** in k-space $\mathbb{R}^k$ (with $k \geq n$)is the convex hull of $n+1$ affine-independent points. (Affine-independent means: not contained in any linear variety of dimension $< n$.) Often one thinks of just the set of these points as the simplex. The points are called the vertices of the simplex. Any $p+1$ of the vertices (with $p \leq n$) determine a **boundary simplex**, or **face**, more precisely boundary p-simplex or p-face. There are $\binom{n+1}{p+1}$ p-faces of an n-simplex. Similar to what we did with the tetrahedron we assign to each face an angle, namely we consider a (small) sphere, of radius r say, with center at an interior point of the face, and take the fraction of the surface area of the sphere cut out by the n-simplex. (As before this works also for

$p = n-1$ or n: for each $(n-1)$-face we get $1/2$, and for the n-simplex itself we get 1.) Let w_p be the sum of these angles over all the p-faces, for $p = 0, 1, \dots, n$. (Note $w_{n-1} = (n+1)/2$ and $w_n = 1$.)

We claim:

$$\sum (-1)^p w_p = 0,$$

the alternating sum of the angle sums is 0; this generalizes what we found for the tetrahedron; it also means that we should rewrite our first equation $\sum \alpha - \pi = 0$ as $\sum \alpha - 3\pi + 2\pi = 0$.

The proof will involve a detour over the spherical simplex. So let S^n be a sphere of radius r in $\mathbb{R}^{n+1}$, centered at the origin O, and let E_i be $n+1$ hyperplanes (codimension 1) through O, in general position (meaning that the intersection of any r of them has dimension $n-r$, or also that the lines orthogonal to the E_i are independent). Each E_i is oriented by choosing one of the two (closed) half-spaces determined by it as "positive". The intersection of the positive half-spaces is a convex cone Γ; the intersection $\Gamma \cap S^n$ is a **spherical n-simplex**, say Σ, (of dimension n). The **faces** of Σ are the intersections of Σ and some subset of the E_i. (Each such face is a spherical simplex in the intersection of the E_i defining it; its dimension is one less than that of that intersection.)

The **(multihedral) angle** at a face is the angle of the E_i defining it, i.e. the fraction of the total volume of a small sphere around an interior point of the face, that is contained in Γ. (The angle at Σ itself is 1, the angle at a codimension-1 face is $1/2$.) As before we write w_p for the sum of all the angles at the faces of dimension p. We define $n+1$ functions f_i on $\mathbb{R}^{n+1}$ by $f_i = 1$ on the positive half space of E_i and 0 otherwise (the characteristic or indicator function of this half space). For any (integrable) function g on S^n we write $\int g$ for the integral of g over S^n, the latter provided with the usual area element. Clearly for any subset $(p_1, p_2, \dots, p_r)$ of $(1, 2, \dots, n+1)$ the value of $\int f_{p_1} f_{p_2} \cdots f_{p_r}$ is the angle of Σ at the corresponding face.

We now form the function $\phi = (1-f_1)\cdot(1-f_2)\cdot\ldots\cdot(1-f_{n+1})$. Clearly ϕ, restricted to S^n, is 1 on (the interior of) the spherical simplex antipodal to Σ and 0 otherwise; thus $\int \phi = A_\Sigma =$ area of Σ. On the other hand, by multiplying out, we have

$$\int \phi = \int 1 - \sum \int f_i + \sum \int f_{i_1} f_{i_2} - \ldots + (-1)^n \sum \int f_{i_1} f_{i_2} \cdots f_{i_n} + (-1)^{n+1} \sum \int f_1 f_2 \cdots f_{n+1}.$$

The individual terms here are easily seen to be the angle sums w_p introduced above. Writing A_n for the total surface area of S^n, we get

$$A_\Sigma = [w_n - w_{n-1} + w_{n-2} - \ldots + (-1)^n w_0]A_n + (-1)^{n+1}A_\Sigma.$$

(This is the analogon of the decomposition of the 2-sphere considered in the beginning.) √

We rewrite this as **Poincaré's formula**:

$$\sum_0^n (-1)^p\, w_p = \begin{cases} 2A_\Sigma/A_n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

As earlier, let Δ be a Euclidean simplex in $\mathbb{R}^n$. Viewing $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$, move the center of S^n to the point $(0,0,\ldots,0,r)$. Projecting Δ stereographically (and conformally) to a spherical simplex Σ on this S^n from $(0,0,\ldots,0,2r)$, and finally letting r go to ∞, we get the formula for the Euclidean simplex promised above.

I.3. Euler's Theorem on Polyhedra, other Proofs and Consequences

Let there be given a triangulation of the sphere, with v vertices, e edges, and f triangles. (The sphere is covered with a finite number of triangles; any two triangles either are disjoint or have a vertex in common, or have an edge (and its two vertices) in common.) For each triangle we have the relation $\sum \alpha - \pi = F/r^2$ of Section I, between its angles and its area. We add these f equations and get

$$v \cdot 2\pi - f\pi = 4\pi r^2/r^2 = 4\pi.$$

Thus we have $2v - f = 4$. But we also have $3f = 2e$. (Explode the sphere, or cut the triangles apart. Since there are f triangles, one sees $3f$ edges. Also, each edge belongs to two triangles; so one sees $2e$ edges.) Substituting $f = 2e - 2f$ into the first relation, we get

$$v - e + f = 2,$$

the famous Euler formula (1752) for a triangulation of the sphere. It extends easily to a more general situation:

We consider a decomposition of the sphere into (spherically) convex polyhedral cells. The boundary of each cell is a simple closed curve, consisting of a finite number, ≥ 3, of great circle arcs (the "edges"), which meet at the "vertices"; any two cells are either disjoint or have exactly one vertex in common or have an edge (with its two endpoints-vertices in common). More generally we allow cells that are star-shaped from one of their points; such a cell is in the obvious way the union of the spherical triangles formed by the central point and the edges of the cell. (Note that a convex cell is star-shaped from any interior point.)

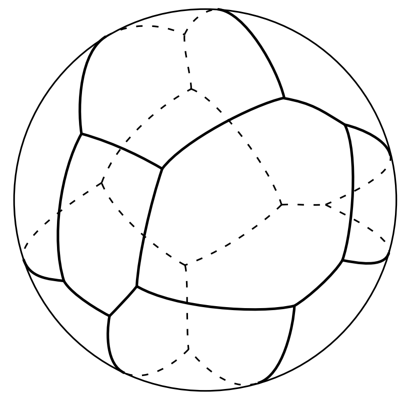


FIGURE 6

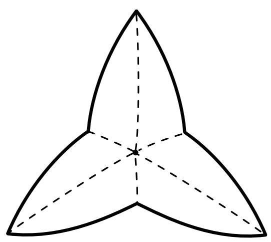


FIGURE 7

Again we let v, e and f be the number of vertices, edges and cells. What is the value of $v - e + f$?

We choose any cell that is not a triangle already, with $n > 3$ edges, and divide it, as just described, into triangles from the chosen center of the cell, arriving at a new cell decomposition of the sphere. Clearly the number v has increased by 1, e has increased by n, and f by $n-1$. The change in the combination $v-e+f$ is 0. We continue until we have only triangular cells left. Using the Euler formula for triangulations, proved above, we conclude that

$$v - e + f = 2$$

holds for any decomposition into star-shaped polyhedral cells.

Here is another version (going back to Legendre): For any cell with n vertices we have

$$\sum \alpha - (n-2)\pi = F/r^2$$

as follows by division into triangles from the center and our earlier formula $\sum \alpha - \pi = F/r^2$ for a spherical triangle. Adding these equations over the cells and using the obvious relation $\sum n = 2e$ we get

$$v \cdot 2\pi - 2k \cdot \pi + 2f\pi = 4\pi,$$

i.e.

$$v - e + f = 2,$$

the Euler formula. $\surd$

I.4. Convex Polyhedra

A **convex polyhedron** in $\mathbb{R}^3$ is by definition a non-empty bounded set that can be written as the intersection of a finite number of closed half spaces (The complement of a plane in $\mathbb{R}^3$ is the union of two connected open sets, called open half spaces. The union of either one with the plane is the corresponding closed half space.) It is intuitively obvious and can be proved in detail that the (point set) boundary of such an object is the union of a finite number of plane convex polyhedra, the **faces**, and that for any two faces the intersection is as above on the sphere: it is either empty or a single vertex or an edge (with its endpoints) of each of the two faces. The polyhedron is the convex hull of the set of its vertices (the union of the sets of vertices of the various faces). As before we write v, e, and f for the number of vertices, edges, and faces.

Projecting from an interior point onto a sphere around the point we get a cell division of the sphere with the same numbers v, e, f. Thus we have the same Euler formula

$$v - e + f = 2$$

for convex polyhedra.

We think of all these convex polyhedra as cell divisions of the 2-sphere. Thus it is a property of the 2-sphere that $v - e + f$ for all these cell divisions always has the value 2. One expresses this by saying that the **Euler** (or sometimes **Euler-Poincaré**) **characteristic** $\chi(S^2)$ is 2.

We make an application of this: regular polyhedra, strictly speaking combinatorially regular polyhedra. A convex polyhedron is called (combinatorially) regular if

(1) The number of edges going out from a vertex is the same for all vertices, say m, and

(2) The number of edges of a face is the same for all the faces, say n. (For obvious reasons we shall assume $m, n \geq 3$.)

For such a polyhedron we ask for the possible values of v, e. f, m, n. I.e., what do regular polyhedra look like?

Our starting point is the Euler formula $v - e + f = 2$. Conditions (1) and (2) yield the relations

$$m \cdot v = 2e, \quad n \cdot f = 2e,$$

a Diophantine system of equations, i.e., we are looking for solutions in (positive) integers. We have $v = 2e/m$ and $f = 2e/n$ and so

$$v + f = e + 2 = 2(m + n)e/mn$$

or

$$1 = e \cdot (1/m + 1/n - 1/2).$$

This implies $1/m + 1/n > 1/2$. Clearly then not both m and n can be ≥ 4, and so one of them must be 3. Checking the possibilities for the other we find the table

m	3	3	4	3	5
n	3	4	3	5	3
v	4	8	6	20	12
e	6	12	12	30	30
f	4	6	8	12	20

Note the relation between the third and fourth and between the fifth and sixth column: The entries for m and n are interchanged, and so are those for v and f.

These are the the values of v, e, f that the Euler relation allows for regular (convex) polyhedra. Such polyhedra actually exist: they are the well known five Platonic polyhedra, Tetrahedron, Cube, Octahedron, Icasohedron, Dodecahedron. These polyhedra are not only combinatorially regular, but even metrically regular: All the vertices are congruent (i.e. the cones spanned by the faces at the vertices are congruent), and all the faces are congruent.

The cases $m = 2$ and $n = 2$ make sense in a way, as degenerate cell divisions of the sphere: For $m = 2$ and then $f = 2$ and $v = e = n$, any n, one takes the division of the sphere into upper and lower hemispheres, with n vertices on the equator. For $n = 2$ and $v = 2$, $f = e = m$, any m, there is dually the division of the sphere into strips (2-gons) by m meridians.

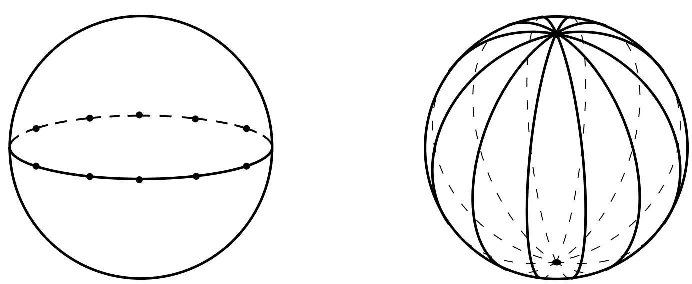


FIGURE 8

I.5. More Applications

We consider consequences of the Euler formula for arbitrary (not necessarily regular) convex polyhedra. Suppose every face has at least m edges and from every vertex emanate at least n edges; also $m \geq 3, n \geq 3$. We denote by f_3 the number of triangles among the faces, by f_4 the number of quadrangles, etc, and similarly by $v_3, v_4, \ldots$ the number of vertices of "order" $3, 4, \ldots$. We clearly have

$$f = f_3 + f_4 + \ldots \qquad v = v_3 + v_4 + \ldots$$
$$2e = 3f_3 + 4f_4 + \ldots \qquad 2e = 3v_3 + 4v_4 + \ldots.$$

Because of our restrictions on the polyhedron this refines to

$$2e = n \cdot f_n + (n+1) \cdot f_{n+1} + \ldots$$
$$2e = m \cdot v_m + (m+1) \cdot v_{m+1} + \ldots.$$

This implies

$$2e \geq n(f_n + f_{n+1} + \ldots) = n \cdot f$$
$$2e \geq m(v_m + v_{m+1} + \ldots) = m \cdot v,$$

or

$$1/n \geq f/2e; \quad \text{and} \quad 1/m \geq v/2k.$$

This in turn, using the Euler formula, means

$$1/n + 1/m \geq (2+k)/2k = 1/2 + 1/k$$

and so

$$1/n + 1/m > 1/2.$$

This is the same inequality that we had before, and we conclude:

a) At least one of m and n is 3, and

b) Both numbers are ≤ 5.

In words: On every convex polyhedron there is always either a triangle or a quadrangle or a pentagon, and similarly for vertices. And: there is always either a triangle or a vertex of order 3 (or both).

I.6. Extension of Legendre's Proof to n Dimensions

We start from Poincaré's formula for the spherical simplex:

$$\sum_0^n (-1)^r w_r = \begin{cases} 2A_\Sigma/A_n, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Let the sphere be triangulated:

a_0 vertices

a_1 edges

$\vdots \quad \vdots$

a_n n-simplices.

Now we add the Poincaré formula for the n-simplices and get, by summation of the angles at each vertex, edge, . . . :

$$\sum_0^n a_r = \left\{ \begin{matrix} 2, & n \text{ even} \\ 0, & n \text{ odd} \end{matrix} \right\} = 1 - (-1)^{n+1}$$

If a division of the sphere into convex cells is given, one proceeds by dividing each cell into simplices. Let $A_1, A_2, \dots, A_n$ be the numbers of cells of dimension $0, 1, \dots, n$. We are interested in $\sum_0^n (-1)^r A_r$. Dividing the cells into simplices is done by increasing dimension. The boundary of a cell x_r is a (polyhedral) sphere, divided into spherical simplices, and one projects this division from any inner point of the cell. The numbers A_r change at each step, but one shows that the alternating sum $\sum_0^n (-1)^r A_r$ doesn't change. At the end one has a simplicial division, for which the formula holds, and so the same formula holds for the original A_r.

To show that the sum doesn't change: the boundary of the cell x_r has α_i simplices of dimension $i = 0, 1, \dots, r-1$, and we know $\sum_0^{r-1} (-1)^t \alpha_t = 2$ or 0, for $r-1$ even or odd. In the subdivision of x_r we get

1	new vertex
α_0	new edges
α_1	new triangles
$\vdots$	
α_{r-2}	new $(r-1)$-simplices
α_{r-1}	new r-simplices.

However we lose the cell x_r itself, and so the change is

$$1 - \alpha_0 + \alpha_1 - \ldots + (-1)^r(\alpha_{r-1} - 1)$$

which is 0 as just noted.

I.7. Steiner's Proof of Euler's Theorem

For a triangle we have $\sum \alpha_i - \pi = 0$. For an n-gon we have $\sum \alpha_i - (n-2)\pi = 0$ (divide the n-gon into triangles from some interior point; "artificial" vertices on the edges, making the original n-gon into an m-gon with $m > n$, are allowed). These expressions can be written as

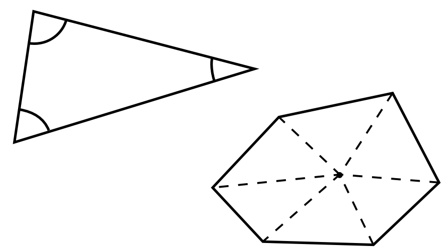


Figure 9

$$w_0 - w_1 + w_2 = 0$$

where w_0, w_1, w_2 mean the sum of the angles at the vertices, the edges, and "at the (one and only) face". I.e. $w_0 = \sum \alpha_i$, $w_1 = n\pi$, and $w_2 = 2\pi$.

Let now z be a convex cell, divided in some manner into convex cells. Let v, e, f be as usual the numbers of vertices, edges, and faces of this division. For each cell z_i of the subdivision we have the corresponding equation

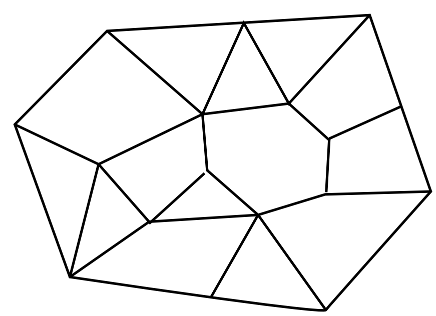


Figure 10

$$w_0^i - w_1^i + w_2^i = 0.$$

Also let W_0, W_1, W_2 be the angle sums for z itself (here we regard z as an n-gon with all the vertices of the subdivision on the boundary of z as vertices). We have $W_0 - W_1 + W_2 = 0$ and $W_2 = 1$. Finally let η and κ be the number of vertices and edges of the subdivision that lie in the interior of z. We clearly have

$\sum w_0^i = \eta = W_0$ (since every vertex in the interior contributes 1 to the sum)

$\sum w_1^i = \kappa + W_1$ (similarly)

$\sum w_2^i = f$ (clearly).

Taking the alternate sum we get

$$0 = \eta - \kappa + f + W_0 + W_1.$$

With $W_0 - W_1 + W_2 = 0$ and $W_2 = 1$ this gives

$$0 = \eta - \kappa + f - 1.$$

Finally let v' and e' be the numbers of vertices and edges on the boundary of z. Clearly we have $v' = e'$ and also $\eta + v' = v, \kappa + e' = e$. putting it all together we find

$$v - e + f = 1.$$

In words, the (Euler) characteristic of a subdivided convex cell is always equal to 1; it is in this sense that we say that the Euler characteristic of a cell (with no subdivision mentioned) is 1. That result is the basis for Steiner's proof of Euler's theorem:

So consider a (bounded) convex polyhedron in space. Choose a direction that is not parallel to any edge or face (no problem: one has to avoid only a finite number of planes and lines) and project the polyhedron in that direction into a plane. The image is a convex polygon. Its boundary is one-to-one image of a closed edge path on the polyhedron; it divides the polyhedron into two parts, each of which is projected one-to-one onto the image polygon. Thus for each one we have $v - e + f = 1$. By addition, noticing that for the common boundary curve the relation $v - e = 0$ holds, we get

$$v - e + f = 2.$$

I.8. The Euler Characteristic of a (Bounded Convex) 3-Cell

The cell is subdivided into smaller convex cells. We have as usual v vertices, e edges, f two-dimensional faces, and now also t three-dimensional cells. The **Euler characteristic** is the alternating sum $v - e + f - t$, and the main result will be that (again)

$$v - e + f - t = 1.$$

A convex cell has natural vertices (the extreme points), natural edges (joining some pairs of vertices), and natural faces which are convex 2-cells (maximal plane sets in the boundary of the cell). As before we have the notions of angle at a vertex, at an edge, at a 2-face, and now also “at a 3-face”; we take them normalized (full angle $= 1$); and the angle at the 3-face is (by definition: fraction within the cell of a small sphere at an interior point) 1. Adding up gives the angle sums w_0, w_1, w_2, w_3 at the vertices, edges, ... Similar to, but more complex than, our convention in the two-dimensional case of an n-gon, we now allow the cell to have a polyhedral subdivision of its boundary with its vertices, edges, faces, together with one 3-dimensional element, the cell itself, and the resulting angle sums w_i. The first result to be proved is

$$w_0 - w_1 + w_2 - w_3 = 0,$$

the alternating sum of the angle sums is 0.

The first instance and the basis for the proof is the formula of de Gua for a tetrahedron which we derived in the beginning ($\sum \gamma - \sum \epsilon + 1 = 0$ or $\sum \gamma - \sum \epsilon + 4 \cdot 1/2 - 1 = 0$); in the new notation this says

$$w_0 - w_1 + w_2 - w_3 = 0.$$

Next, let a convex 3-cell (possibly with subdivided boundary) be given. In the interior of each face introduce a new vertex, and join it by edges to the vertices on the boundary of the face. This divides the face into triangles. One sees easily that $w_0 - w_1 + w_2$ does not change under this process: for each face with r vertices there is now one more vertex with angles $1/2$, r new edges with angle $1/2$, and r new faces with angle $1/2$, and the original face, with angle $1/2$, has disappeared. For the moment the numbers v, e, f will refer to the cells of the (thus subdivided) boundary. We now introduce a new vertex in the interior of the 3-cell, and divide the cell into the tetrahedra obtained by connecting the simplices on the boundary with the new vertex. We write the formula of de Gua for each tetrahedron and add all the equations

$$\sum w_0^i - \sum w_1^i + \sum w_2^i - \sum w_3^i = 0.$$

Summing first over the contributions of the individual vertices, edges, faces, and tetrahedra The expression on the left can be rewritten as

$$1 + w_0 - (v + w_1) + (e + w_2) - f = 0.$$

the numbers $1, v, e, f$ here come from the full angles around the interior vertex and the edges, faces, tetrahedra issuing from it to the vertices, edges, and faces on the boundary. With $w_3 = 1$ this means

$$w_0 - w_1 + w_2 - w_3 + 2 - v + e - f = 0,$$

and with the Euler theorem for the boundary polyhedron ($v - e + f = 2$) there follows the promised relation

$$w_0 - w_1 + w_2 - w_3 = 0$$

for a 3-cell with subdivided boundary.

Now let us take a 3-cell z that is subdivided into smaller 3-cells (all convex of course), with v, e, f, t vertices, edges, faces, and 3-cells. (We should clarify the notion of subdivision: It consists of a finite family of convex 3-cells whose union is the given cell z, with the property that the intersection of any two is either empty or a common vertex or edge or face.) We let v° etc. be the numbers of those in the interior, and v' etc. the numbers of those on the boundary of z (note $t' = 0$). For z with its division of the boundary we have the equation $W_0 - W_1 + W_2 - W_3 = 0$, with $W_3 = 1$. For each of the smaller 3-cells we have $w_0 - w_1 + w_2 - w_3 = 0$. We add all these equations. The sum of the w_0 gives the angle sum at the vertices; similarly for edges, faces, 3-cells. for a vertex in the interior the sum of the angles around it is 1. Thus we get $\sum w_0 = v^\circ + W_0$, $\sum w_1 = e^\circ + W_1$, $\sum w_2 = f^\circ + W_2$, $\sum w_3 = t^\circ$, and so

$$v^\circ - e^\circ + f^\circ - t^\circ + W_0 - W_1 + W_2 = 0$$

or also, with $W_0 - W_1 + W_2 - W_3 = 0$ and $W_3 = 1$, $v^\circ - e^\circ + f^\circ - t^\circ = -1$. On the other hand, by Euler we have $v' - e' + f' = 2$. Altogether this gives

$$v - e + f - t = 1$$

for the characteristic of a 3-cell.

We state, but do not prove, the result for a convex n-cell (any $n = 0, 1, 2, 3 \ldots$): With a_i the number of i-cells, the formula is

$$a_0 - a_1 + a_2 - a_3 + \ldots (-1)^n a_n = 1.$$

Let us assume $a_n = 1$ so that the cells of dimension $< n$ form a subdivision of the boundary of the cell (the polyhedral version of the $(n-1)$-sphere S^{n-1}). Replacing $n-1$ by n and writing χ for the alternating sum of the numbers of cells of the various dimensions we arrive at

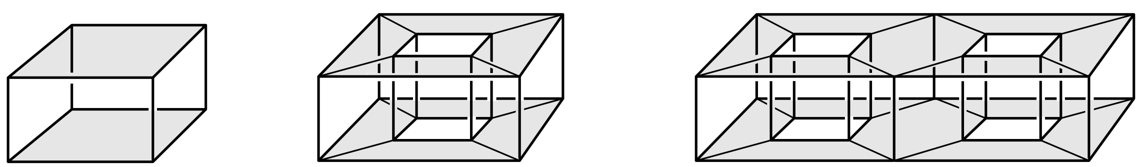
$$\chi(S^n) = 1 + (-1)^n.$$

The even-dimensional spheres have characteristic 2, the odd-dimensional ones have characteristic 0.

Here is a third proof for the Euler theorem, using the result that $v - e + f = 1$ for a (subdivided) 2-cell. Let a subdividion of the sphere into convex cells be given. The value of $v - e + f$ doesn't change if one subdivides an edge or a face (by introducing new vertices and edges within the face): This is clear for an edge (a new vertex replaces the edge with two edges), and, as just noted, a face contributes 1 to $v - e + f$ before and after subdivision. Now consider two subdivisions. By superimposing them one gets a third division which is a subdivision of each of the two. Since subdivision doesn't change $v - e + f$, it follows that all these values are equal. To find what the common value is one takes any simple division (e.g. the tetrahedron). $\surd$

As a matter of fact one can even allow more general cell divisions. E.g., divide the sphere into two hemispheres, with two (say antipodal, for beauty's sake) vertices on the common equator.

More generally in another direction, one can develop all this for the other closed surfaces (and even beyond that): one has division into cells as before, and it turns out that $v - e + f$ for a given surface always gives the same value, which then naturally is called the **(Euler) characteristic** of the surface. As an example, the characteristic of a torus is 0. Here are pictures for the "orientable" surfaces (Fig. 11)



$g = 0$ $g = 1$ $g = 2$

FIGURE 11

Here g is a number, called the **genus** of the surface, related to the characteristic by $\chi = 2 - 2g$.

Chapter II. Graphs

II.1. The Euler Formula for the Characteristic

A **graph** in space or the plane consists of a finite number of (closed) segments, such that any two either are disjoint or have exactly an endpoint in common. Abstractly a graph consists of two pieces of data: A finite set $\mathcal{V}$ of objects, called **vertices**, and a set $\mathcal{E}$ of unordered pairs of distinct vertices, called **edges**; usually every vertex belongs to at least one edge. (Sometimes one allows pairs (a,a), "loops", concretely an edge [not straight] from a vertex back to itself; sometimes one uses ordered pairs, interpreted as directed edges.) Every abstract graph can be realized in $\mathbb{R}^3$ (clear), but not necessarily in the plane (see below).

The number $v - e$ (number of vertices minus number of edges) is called the **characteristic** of the graph; subdivision (introducing a new vertex in an edge) doesn't change it.

A closed curve (often called a **circuit**) in a graph must of course consist of a number of edges in the usual cyclic arrangement; if it is simple (no "self-intersection"), it is called a **simple circuit**.

A graph is called a **tree** if it is connected and contains no simple circuit. Clearly a (nonempty) tree has at least two "free points" or "end points", meaning vertices that belong to only one edge, and has $v - e = 1$. The latter is proved by induction on e, by removing an edge with an endpoint, leaving a tree with smaller e.

Next, let $\mathcal{K}$ be a connected graph, but not a tree. By definition it contains a simple circuit. Remove one of the edges, say s, of this curve. The remaining graph $\mathcal{K}' = \mathcal{K} - s$ is clearly still connected. The number v is still the same, but e decreases by 1. We iterate this, with $\mathcal{K}'', \mathcal{K}''', \dots$, until we get to a $\mathcal{K}^p$ that is a tree. (This must happen!) Then $v - e$ has decreased by p, and for a tree the characteristic is 1; so we have $v - e = 1 - p$. As a consequence we see that it always takes the same number of such steps (removing edges from simple circuits), p, to reduce $\mathcal{K}$ to a tree. The number is called the connectivity of the graph, or also the first Betti number of $\mathcal{K}$. (The name refers to the Italian geometer Enrico Betti, 1823-1892, who introduced characteristic numbers of this kind for geometric figures, also in higher dimensions.) We see that a connected gaph is a tree if and only if its characteristic is 1. For an arbitrary connected graph the characteristic is ≤ 1. For a general $\mathcal{K}$ let $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_r$ be the components of $\mathcal{K}$ (maximal connected subsets). Each component $\mathcal{K}_i$ has v_i vertices, e_i edges, and connectivity p_i, with $v_i - e_i = 1 - p_i$. Put $p = \sum p_i$. then we have

$$v - e = r - p.$$

The left hand side involves numbers that are easily read off from the graph, but are not invariants under subdivision. The right hand side involves numbers that express geometric properties of the graph. One might call it the Euler formula for graphs.

II.2. Graphs in the Plane (and on the Sphere)

A graph $\mathcal{K}$ in the plane (or on the sphere) divides the plane (or sphere) into a certain number of connected domains or "faces" (components of the complement of the graph); let f be that number. With p as before we shall prove the main relation

$$f = p + 1.$$

We first state a lemma, which is actually the main part of the Jordan curve theorem for graphs (and which we prove below):

Lemma *Given a simple circuit in the plane, any two points close to and on opposite sides of an edge of the graph cannot be connected without meeting the graph.*

Here to "connect" two points P and Q means to give a set (a "chain" from P to Q) consisting of P, Q, some other points $P_1, P_2, \dots, P_m$ and the segments $PP_1, P_1P_2, \dots, P_mQ$.

Now to the proof of our relation:

1) The case $p = 0$ ($\mathcal{K}$ is a "forest"). We proceed by induction on the number e of edges. For $e = 0$ or 1 the matter is obvious. So suppose the statement holds for $e = n$, and take a forest with $n + 1$ edges. As we know, there exists an edge with a free endpoint (to which no other edge is attached); take such a one. Removing it leaves a forest $\mathcal{K}'$ with n edges (for which the statement holds). Any two points a and b not on $\mathcal{K}$ can then be connected by a finite chain of edges that doesn't meet $\mathcal{K}'$. Clearly one can then connect a and b without meeting $\mathcal{K}$: If the chain meets the removed edge one can modify it by going by going around the free end.

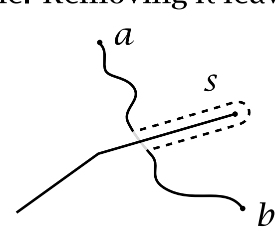


Figure 12

2) p arbitrary, > 0. We remove an edge k so that $\mathcal{K}' = \mathcal{K} - k$ has connectivity $p - 1$. We claim: The two sides of k belong to two different faces, this follows easily from

the above Lemma, applied to a circuit in $\mathcal{K}$ that contains k. Therefore the number of faces for $\mathcal{K}'$ is one less than that for $\mathcal{K}$. We iterate this. After p steps we arrive at a graph $\mathcal{K}^{(p)}$ with connectivity 0, and therefore, by point 1), with number of faces 1. We conclude $f = p + 1$. $\surd$

From $f = p + 1$ and $v - e = s - p$ follows the relation

$$v - e + f = s + 1.$$

This is a generalization of the Euler formula for convex polyhedra: Suppose we have a division of the 2-sphere into convex faces. The edges of the division form a graph, which is obviously connected (a path in the 2-sphere connecting two points on the graph can be modified by pushing the part of the path in any given face into the boundary of the face). Thus $s = 1$, and the above formula becomes $v - e + f = 2$.

Now to the proof of the lemma above.

So let $\mathcal{P}$ be a circuit on the plane, let k be an edge of $\mathcal{P}$, and let a and b be two points on opposite sides and "very close" to k. We must prove that a and b belong to different faces (components of the complement of $\mathcal{P}$).

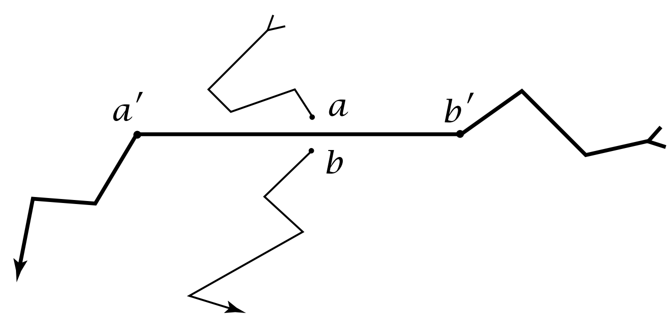


FIGURE 13

We need the concept of arc (more explicitly, PL arc; here PL means "piecewise linear"); this is a tree whose vertices have order at most two; in other words, a sequence $\{k_1, k_2, \ldots, k_n\}$ of edges such that the only intersections are between k_i and k_{i+1} for $i = 1, \ldots, n-1$ and there they consist of one point, an endpoint of each edge. The following is fairly easy to establish: If two points are connected by a continuous path in an open set, one can also connect them by a PL arc in that open set. (Approximate the path by a polygon; then eliminate unnecessary loops.)

So suppose a and b belong to the same face; as just noted, they can then be connected by a PL arc in that face, i.e., without meeting $\mathcal{P}$. Clearly one can arrange it so that this arc together with the segment ab is a circuit $\mathcal{Q}$. We note that $\mathcal{P}$ and $\mathcal{Q}$ have exactly one point in common (on ab), and that they are in general position. (Two graphs are in general position if no vertex of either is contained in an edge of the other, a condition easy to arrange by a slight shift of the vertices; and stable: if one has it, then sufficiently small shifts of the vertices will not destroy it.) Our claim will then follow from the following:

PROPOSITION. *Two circuits $\mathcal{D}$ and $\mathcal{E}$ in general position have an even number of points in common.*

For the proof we take a point c in the plane in general position to $\mathcal{D}$ and $\mathcal{E}$, i.e., meaning here that c does not lie on any line that contains two vertices of $\mathcal{D}$ and $\mathcal{E}$. The point c and any edge k of $\mathcal{E}$ span a triangle Δ_k, and this triangle is in general position to $\mathcal{D}$.

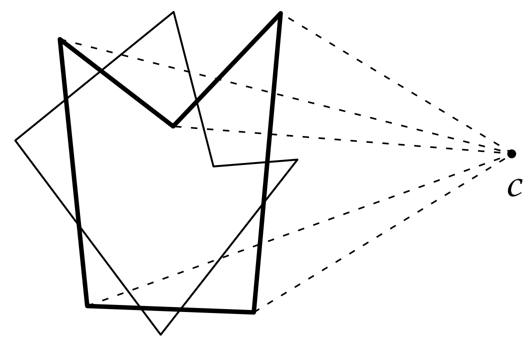


FIGURE 14

It is elementary that the complement of the triangle has two components, the interior and the exterior of the triangle. It follows that the intersection of $\mathcal{D}$ and Δ_k consists of an even number of points (going around $\mathcal{D}$ one changes from interior to exterior of Δ_k or conversely at every point of intersection). Thus the sum S over the edges of $\mathcal{E}$ of these numbers of intersection is also even. This sum consists of the contributions of the segments from c to the various vertices of $\mathcal{E}$ and the edges k of $\mathcal{E}$. On the other hand segment from c to any vertex of $\mathcal{E}$ belongs to exactly two of the Δ_k, and so the contribution of this segment to S is also even, and so is the sum over the vertices of $\mathcal{E}$.

It follows that the sum of the contributions of the edges k of $\mathcal{E}$ is also even, and that is our proposition. And so our Lemma is established, and so is our equation $f = p + 1$.

In particular for a circuit we have $v = e$ and $s = 1$ (s is the number of components of the graph). Consequently we have from $v - e = s - p$ that p is 1, and so $f = 2$, i.e., a circuit divides the plane into exactly two components (the interior, which is bounded, and the exterior, which is unbounded). This is the Jordan Curve Theorem for PL curves. $\surd$

All these arguments also hold on the sphere (with straight segments replaced by great circle segments, preferably short enough not to contain any two antipodal points); for general position of c and $\mathcal{E}$ one requires that no vertex of $\mathcal{E}$ is antipodal to c.

II.3. COMMENTS AND APPLICATIONS

We have the formula $v - e + f = 2$ for connected graphs in the plane. That leads to the question: Can every abstract graph be realized in the plane? We will allow here to subdivide the edges by introducing new vertices in the interior of the edges (this doesn't change $v - e$). In the following v and e will mean the numbers of original vertices and edges, before any subdivision.

Suppose we have a graph, embedded in the plane (with possible subdivision of the edges). Let f_n be the number of faces whose boundary is formed by n (original) edges. Clearly f_1 and f_2 are zero except for the trivial cases when $e \le 2$. Ruling out these cases, we will have

$$f = f_3 + f_4 + \dots$$

and

$$3f \le 3f_3 + 4f_4 + 5f_5 + \dots .$$

We also have

$$3f_3 + 4f_4 + \dots \le 2e.$$

To see this, cut the plane along all those edges that belong to two different faces. (Equality holds here if every edge belongs to two different faces. Consider the graph in the plane consisting of a triangle and an extra edge from a vertex to some point within the triangle.) Altogether we find

$$2e \ge 3f,$$

and so, because of $v - e + f = 2$,

$$2e \ge 6 - 3v + 3e,$$

i.e.

$$3(v - 2) \ge e \quad \text{for every connected graph in the plane.}$$

In particular for $v = 5$ we get $e \le 9$. Thus the graph A consisting of 5 vertices and all possible edges between them, the "complete graph" on 5 vertices, which has 10 edges, cannot be embedded in the plane. (Note that the edges would be allowed to be subdivided.)

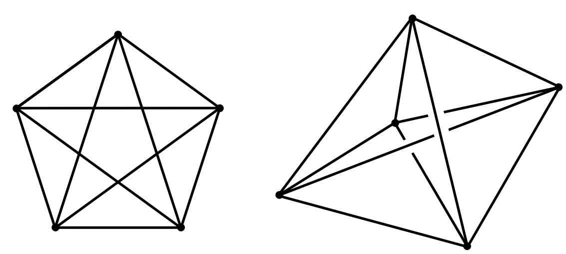


FIGURE 15

There is the dual statement: A map on the sphere, consisting of 5 (simply connected) countries such that every two have an edge in common, does not exist.

A second example: Suppose f_3 is 0. Then we have

$$2e \ge 4f_4 + 5f_5 + \dots \ge 4f.$$

Using $v - e + f = 2$ again we find

$$2(v-2) \geq e, \quad \text{if } f_3 = 0.$$

An example for this, B: We have six points, arranged in two rows of three each. Each point of row I is connected to each point of row II by an edge; no other edges. This cannot be realized in the plane. If it were, there wouldn't be any triangles (a triangle would have to contain at least two points from one row; but such points are not connected by an edge); thus $f_3 = 0$. Also $v = 6$. By the last result we would have $e \leq 8$; but e is 9.

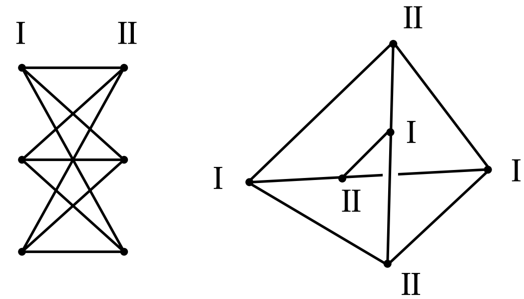


FIGURE 16

Thus a graph that contains A or B as subgraph cannot be embedded in the plane. The converse also holds: If a graph cannot be embedded in the plane, than it contains A or B as subgraph (Kuratowski, Whitney).

II.4. A RESULT OF CAUCHY'S

Another application of $v - e + f = 2$, which will be used later: Let K be a graph in the plane, and let it (i.e., its edges) be colored with two colors. The edges going out from a vertex divide a small circle around the vertex into a number of sectors. We are interested in the sectors whose bounding edges are of different colors; let z be their number. One sees, by going around the circle, that z is even.

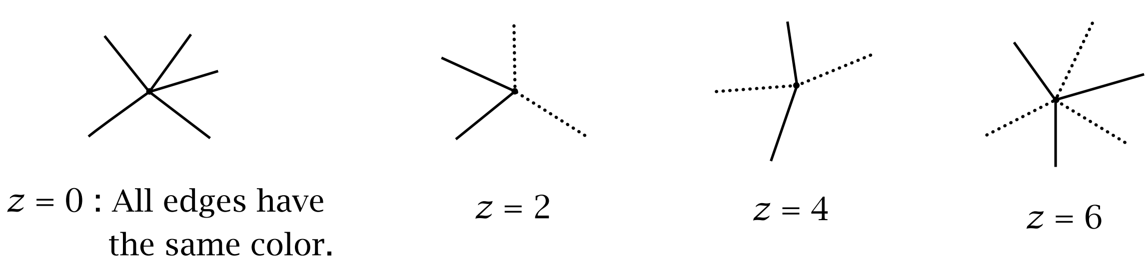


FIGURE 17

Vertices with $z \geq 4$ are called crossing points.

Here is Cauchy's result:

Let $\mathcal{K}$ be a (finite, with at least one edge) graph in the plane that is colored with two colors. Then $\mathcal{K}$ has at least one vertex which is not a crossing point; in fact it has at least two such.

Proof: We can assume $\mathcal{K}$ connected, and also $f_1 = f_2 = 0$. Then we have

$$f_3 + f_4 + \ldots = f$$
$$3f_3 + 4f_4 + \ldots = 2e.$$

Thus

$$\begin{aligned} 2e &\geq 3f_3 + 4(f_4 + f_5 + \ldots) \\ &\geq 3f_3 + 4(f - f_3) \\ &\geq 4f - f_3. \end{aligned}$$

And so

$$f_3 \geq 2(2f - e).$$

We now use the coloring. Let v_i be the number of vertices with $z = i$. So then

$$v_0 + v_2 + v_4 + \ldots = v.$$

For each vertex p let w_p be the number of edges that go out from p. So

$$w = \sum w_p = 2e.$$

Also let w^* be the number of sectors whose two edges carry different colors. Then we have

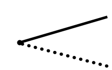


FIGURE 18

$$2v_2 + 4v_4 + \ldots = w^*.$$

Now we consider $w - w^*$. In every triangle in our graph there is at least one sector whose edges have equal colors and which is thus not counted in w^*. Thus

$$w - w^* \geq f_3.$$

Because of $w = 2e$, $f_3 \geq 4f - 2e$ and $v - e + f = 2$ we have

$$w^* \leq 4(e - f) = 4v - 8.$$

This means

$$2v_2 + 4v_4 + \ldots \leq 4v_0 + 4v_2 + \ldots - 8.$$

We rewrite this:

$$8 + 2v_2 + 4v_4 + \ldots \leq 4v_0 + 4v_2 + 4v_4 + \ldots$$
$$8 + 6v_6 + 8v_8 + \ldots \leq 4v_0 + 2v_2 + 4v_6 + 4v_8 + \ldots$$

This gives $8 \leq 4v_0 + 2v_2$, and so finally

$$v_0 + v_2 \geq 2. \qquad \surd$$

Chapter III. The Four Vertex Theorem and Related Matters

III.1. The Theorem of Friedrich Schur (Erhard Schmidt's Proof)

Let C_0 be a curve in the plane (differentiability class C^2, say) that together with the chord connecting the endpoints a_0 and b_0 forms the boundary of a convex set in the plane.

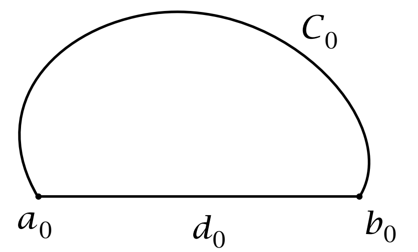


Figure 19

Let κ_0 be the curvature of C_0. (We take curvature in the space sense, i.e., non-negative by definition, even for plane curves. The signed curvature of a curve in an oriented plane will only occur incidentally, if at all.) Let C be another curve in space, of the same length L as C_0, endpoints a and b, and suppose that we have $\kappa_0(s) \geq \kappa(s)$ for all s (here s refers to arclength on C_0 and C).

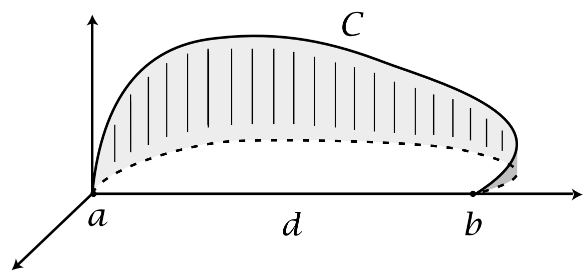


Figure 20

F. Schur's Theorem. *The inequality $d_0 = |a_0 b_0| \leq |ab| = d$ holds for the lengths of the chords. Equality holds iff $\kappa_0(s) = \kappa(s)$ for all s, and C lies in a plane.*

Proof: We project C to the x_1-axis in space. The projection of the chord is

$$d_1 = x_1(L) - x_1(0) = \int_0^L \dot{x}_1 \, ds = \int_0^L \cos\tau \, ds$$

where τ is the angle between the tangent vector of C and the (positive) x_1-axis. Similarly we have $d_0 = \int_0^L \cos\tau_0 \, ds$ where τ_0 is the angle between the tangent vector and the chord of C_0. To complete the proof, it will be enough to show that for a suitable choice of the x_1-direction the inequalities

$$\pi \geq \tau_0(s) \geq \tau(s)$$

hold for all s. Namely then we have $\cos\tau \geq \cos\tau_0$ for all s, and so $d_1 \geq d_0$ and then also $d \geq d_0$.

Now for the choice of the x_1-axis: Let c_0 be the (or a) point on C_0 at maximal distance from the chord ab, and let c be the corresponding point (same s-value) on C. Take the x_1-axis parallel to the tangent vector to C at c.

Next we consider curvature from the point of view of the spherical map ("Gauss map") which assigns to each s-value the tangent vector of the curve (C_0 or C), or rather the point on the unit 2-sphere defined by the tangent vector: the length of the spherical image of any arc on the curve is equal to the integral of the curvature over the arc. For C_0 convexity implies that the spherical image of the part from c_0 to b_0 is (part of) a great semicircle and that for any subarc beginning at c_0 the integral $\int \kappa_0(s)\,ds$ over the arc equals the angle τ_0 at the other endpoint. For C on the other hand the corresponding τ is less than or equal to the integral $\int \kappa(s)\,ds$ (since great circle arcs minimize the distance between the endpoints). Putting this together we find

$$\tau_0 = \int \kappa_0(s)\,ds \geq \int \kappa(s)\,ds \geq \tau,$$

as we wanted to show. Similarly for the part of C_0 from a_0 to c_0. This finishes the proof.

For the case of equality of the chord lengths we must have $\tau_0(s) = \tau(s)$ for all s; in particular $\tau_0(L) = \tau(L)$. So the spherical image of the part of C from c to b connects two points at (spherical) distance $\tau_0(L) = \int \kappa_0\,ds$, which is $\int \kappa(s)\,ds$. The latter being the length of that spherical image, that curve must be the great circle arc between its endpoints. That in turn makes the curve C plane and convex, and so congruent to C_0. $\surd$

[Hopf illustrated F. Schur's theorem by saying: "Look at your index finger. When you straighten it, from a crooked position, the distance between its base and its tip increases".]

This theorem applies also to curves with corners. Here a corner contributes to the integral of the curvature the value of the angle between the incoming and the outgoing tangent at the corner (taken with a sign forthe integral of the signed curvature for a plane curve), and one adds to the curvature inequality hypothesis the requirements: the angles of C are less than or equal to the corresponding ones for C_0. The proof then goes through as before. If the points c_0 and c above are corners of C_0 and C, one has to be careful with the choice the line through c that is to be the x_1-direction: Namely one should take that line in the plane of the incoming and the outgoing tangent to C at c such that the angles of these two tangents with the line are both greater than or equalto the corresponding angles for C_0 at c_0; that is possible.

Parts of either curve in the theorem may be straight line segments. (If C_0 has a straight part, so is the corresponding part of C.) In particular both curves may be polygons, with corners at corresponding points; the second polygon doesn't have to be plane. The hypothesis is then that corresponding edges have the same length, and that the angles of the second polygon are greater than or equal to those of the first polygon. Note that the angles at the ends of the base line play no role.

F. Schur's Theorem for the case of polygons in the plane goes back to Cauchy and is usually called Cauchy's Lemma. Here are the hypotheses for it: Let $\mathcal{C}$ and $\mathcal{D}$ be two convex polygons in the plane, with vertices $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$, and interior angles α_i and β_i (understood to be strictly between 0 and π). Suppose

(a) $A_iA_{i+1} = B_iB_{i+1}$ for $i = 1, 2, \dots, n-1$ (the edges A_iA_{i+1} and B_iB_{i+1} are equal in length), and

(b) $\alpha_i \le \beta_i$ for $i = 2, 3, \dots, n-1$ (the interior angles of $\mathcal{C}$ at those A_i are less than or equal to the corresponding angles of $\mathcal{D}$ at B_i). (Nothing is said about the angles at A_1, A_n, B_1, B_n; the sides A_1A_n and B_1B_n are sometimes called "the base line".)

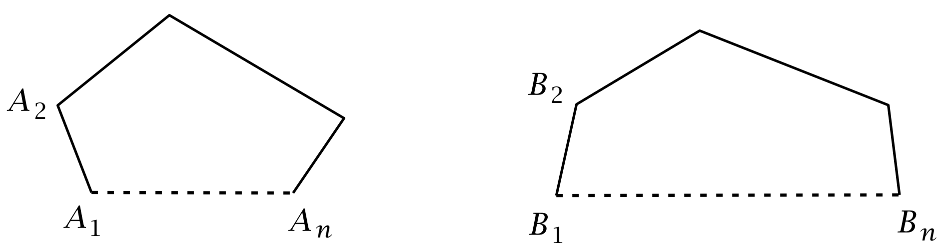


FIGURE 21

The result is then:

CAUCHY'S LEMMA. *The side A_1A_n is less than or equal to B_1B_n in length, with equality if and only if each α_i is equal to the corresponding β_i, for $i = 2, 3, \dots, n-1$.*

Cauchy tried to prove the lemma as a simple generalization of the standard fact for triangles that increasing an angle, without changing the lengths of its legs, increases the base. He started with the triangle $A_1A_2A_n$, increasing the angle at A_2 (and taking the rest of the polygon along) until α_2 has increased to β_2. This produces a new polygon and makes A_1A_n longer. And now he says: iterate — operate on the angle at the new vertex A_3', etc. However, the new polygon may no longer be convex, and then the construction fails — the next step may decrease the base (there are easy examples). E. Steinitz noticed the mistake and gave a considerably more complicated argument to prove the lemma; we shall not go into details.

As noted, Cauchy's Lemma in the plane is part of the Theorem of F. Schur. In fact the Lemma holds also *for convex polygons on a sphere* (edges are great circle arcs, and the polygons should each lie in some hemisphere). The case of spherical polygons can be derived from the plane case as follows: With the notations above for the spherical polygons, project C from the center of the sphere to a suitable plane, obtaining a plane convex polygon C'. Now modify $\mathcal{D}$ by moving each vertex B_i along the ray from the center of the sphere through B_i to the same distance from the center that A_i has moved to under the projection. The new polygon $\mathcal{D}'$ has straight edges, but is likely to not be plane anymore. One verifies easily that corresponding edges of C' and $\mathcal{D}'$ are equal and that the angles at corresponding vertices are $\geq$ or $=$ just as for the original polygons. Schur's Theorem now tells us that the base of $\mathcal{D}'$ is at least as long as that of C', with equality only in the case of congruence of the two polygons. Finally one verifies that the same alternative holds for the bases of the original spherical polygons. $\surd$

III.2. The Theorem of W. Vogt

Let C be a curve in the plane that together with the chord between its endpoints a and b forms the boundary of a convex set, and suppose that the curvature decreases monotonically along C from a to b. (Reminder: this is the curvature in the non-negative sense.)

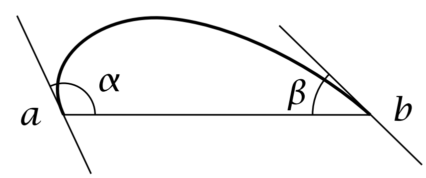


FIGURE 22

THEOREM OF W. VOGT *The angle between C and the chord at a is greater than or equal to that between C and the chord at b; equality holds if and only if the curvature is constant on C.*

PROOF: For once we now use the usual signed curvature in the plane; we denote it by k (so $\kappa = |k|$). With enough differentiability our curvature assumption amounts to $k(s) < 0$ and $\dot{k}(s) \geq 0$ for all s. We may assume that $\dot{k}(s)$ is not identically 0. We also have $y(s) > 0$ except at the endpoints. It follows that

$$0 < \int y\dot{k}\,ds = yk\,\big|_a^b - \int \dot{y}k\,ds.$$

Here $yk\,|_a^b$ is 0, because $y = 0$ at the endpoints. Further, with the usual angle τ we have $\dot{y} = \sin\tau$ and $k = \dot{\tau}$, and so $-\dot{y}k\,ds = d\,\cos\tau$. Integrating we get $0 < \cos\tau\,|_a^b$, i.e., $\cos\beta > \cos\alpha$ or $\alpha > \beta$. $\surd$

Here is a second proof (the theorem isn't that important; but the proofs are amusing). Let p be the midpoint of ab, and let q be the "midpoint" of the curve C (point with $s = 1/2 \cdot$ length of the curve). The chord pq divides the figure into two figures as envisioned for F. Schur's theorem: convex over the base pq (each with a corner). The curvature inequality for the two curves is satisfied (by the monotonicity of κ on C), except possibly for the condition on the angles at the corner. If the latter held also, the two curves would have to have different bases (unless κ is constant on C, trivial case). But they don't and so the condition on the angles at the corners cannot hold, and that is just what Vogt's theorem says. $\surd$

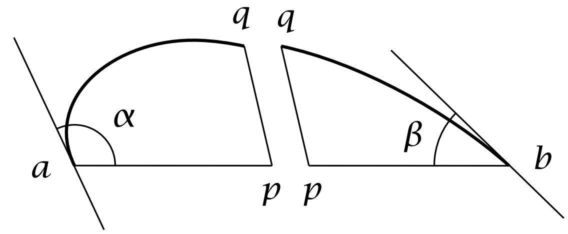


FIGURE 23

III.3. MUKHOPADAYA'S FOUR VERTEX THEOREM

Let C be a closed convex curve of class C^2; write k for its curvature function, the Four Vertex Theorem says:

THEOREM. *There are at least two points on C where k has a local maximum and at least two where k has a local minimum, such that the maxima separate the minima on C.*

Points where k has a local maximum or minimum are sometimes called vertices of the curve (not too good a term). The "such that..." phrase is added here to rule out the case where the maxima and minima each form a nondegenerate arc on the curve.

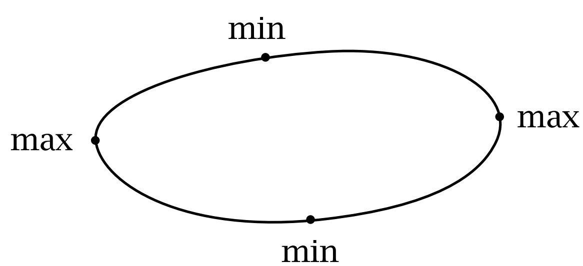


FIGURE 24

First proof, with F. Schur's Theorem: Suppose there is only one local maximum, at P, and only one local minimum, at Q. Clearly one can find two points R_1 and R_2, one on each of the two arcs of C from P to Q, that divide C into two parts of equal length and at which k has the same value. Since k is monotone decreasing on the two arcs from P to Q, the value of k at any point of the "upper" arc R_1PR_2 is greater than the value of k at any point of the "lower" arc

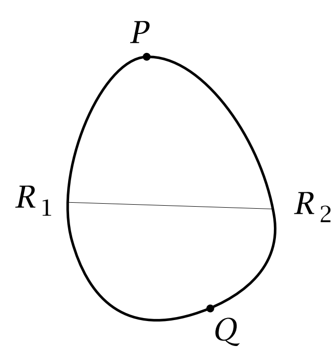


FIGURE 25

R_1QR_2. By Schur's Theorem, applied to the two curves R_1PR_2 and R_1QR_2 and their common chord R_1R_2, this chord, viewed as chord of the "upper" arc, would have to be longer than if viewed as chord of the "lower" arc. Contradiction.

Second proof, with Vogt's theorem: With P and Q as before, consider the two arcs from P to Q on C and their common chord PQ. By Vogt's Theorem, for each arc the angle between arc and chord at P is greater than that at Q. But the two angles at P add up to the same value as those at Q, namely π. Contradiction. $\surd$

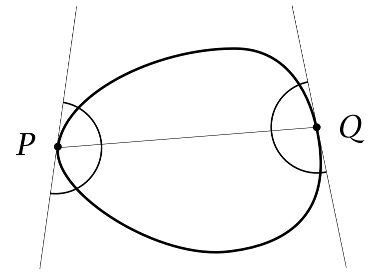


FIGURE 26

III.4. Cauchy's Congruence Theorem for Convex Polyhedra

Our subjects are bounded polyhedra (bounded sets with non-empty interior in 3-space that can be written as convex hull of a finite number of points or also the intersection of a finite number of half-spaces). By vertices, edges, and faces we mean the actual geometric ones (although one might want to consider subdivisions).

The **dihedral angle** or interior angle of two faces of a convex polyhedron with a common edge is that region determined by the planes carrying the two faces that contains the polyhedron, or also the real number measuring it cf. beginning of Ch. I. Just as for interior angles of convex polygons, we require the dihedral angles of a convex polyhedron to lie in the open interval $(0, \pi)$.

An **isomorphism** of two polyhedra $\mathcal{P}$ and $\mathcal{Q}$ is a one-one correspondence between their sets of vertices that "preserves edges and faces", meaning: If two vertices of $\mathcal{P}$ form an edge, then the corresponding vertices of $\mathcal{Q}$ form an edge there, and conversely; and if some vertices of $\mathcal{P}$ are the vertices of a face of $\mathcal{P}$, then the corresponding vertices of $\mathcal{Q}'$ form a face there, and conversely. Such an isomorphism is called an **isometry** if it preserves the "metric data", meaning: The length of any edge of $\mathcal{P}$ equals that of the corresponding edge of $\mathcal{Q}'$, and the interior angle of a face of $\mathcal{P}$ at a vertex equals the corresponding angle on $\mathcal{Q}$ (or maybe more intuitive: any face of $\mathcal{P}$ is congruent to the corresponding face of $\mathcal{Q}$).

Here is Cauchy's result:

THEOREM *If the two (compact) convex polyhedra $\mathcal{P}$ and $\mathcal{Q}$ are isometric, then they are congruent (i.e., there is a rigid motion of 3-space, possibly orientation-reversing, that sends $\mathcal{P}$ to $\mathcal{Q}$).*

For the proof we consider the dihedral face angles. Clearly Cauchy's Theorem amounts to the statement that the dihedral face angles along two corresponding edges of the two polyhedra are equal.

So let $\mathcal{K}$ be the (abstract) graph consisting of those edges of $\mathcal{P}$ (or the isomorphic $\mathcal{Q}$), for which the corresponding dihedral angles of $\mathcal{P}$ and $\mathcal{Q}$ are not equal, and their vertices. We two-color $\mathcal{K}$ by giving an edge the first color if the dihedral angle at the edge in $\mathcal{P}$ is greater than that at the corresponding edge in $\mathcal{Q}$, and the second if it is less than it. $\mathcal{P}$ and $\mathcal{Q}$ being congruent is then equivalent to $\mathcal{K}$ being empty. Assuming $\mathcal{K}$ not empty, we are in position to apply the last result of the section on Graphs in the Plane: There is the function z that to each vertex assigns the number of sectors with different colors on the two sides. And there is the result that there are at least two vertices in the graph at which z takes the value 0 or 2. Cauchy's Lemma will show that that is impossible, and that will prove the Congruence Theorem.

So let P be a vertex with $z = 0$. One takes a small sphere of radius r with P as center and intersects the polyhedron $\mathcal{P}$ with it, obtaining a spherical convex polygon C with vertices $A_1, A_2, \ldots, A_n$. We get another such polygon $\mathcal{D}$ with vertices $B_1, B_2, \ldots, B_n$ on a sphere (of the same radius) on $\mathcal{Q}$ at the corresponding vertex Q.

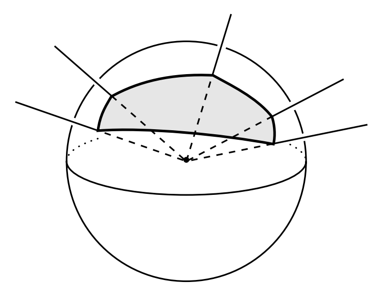


FIGURE 27

By the isometry assumption corresponding edges of C and $\mathcal{D}$ have equal length (the edges are, up to a factor r, the interior angles of faces of $\mathcal{P}$ and $\mathcal{Q}$ at P and Q). The present hypothesis $z = 0$ means that the angle of C at any A_i is greater than or equal to the angle of $\mathcal{D}$ at B_i. Since also A_1A_n equals B_1B_n, Cauchy's Lemma implies that the two polygons are congruent. This means that the dihedral face angles at the edges issuing from P on C equal the corresponding angles on $\mathcal{D}$.

Now the other case, where z at P is 2. There are then two indices, which one may take to be 1 and r, such that for i from 1 to r the angle at A_i is greater than that at B_i, and for i from $r+1$ to n the angle at A_i is less than or equal to that at B_i. On C let R and S be the midpoints of the edges A_rA_{r+1} and A_1A_n and introduce the new edge RS, creating two "halfpolygons" with the common edge RS; similarly we have T and U and the edge TU for $\mathcal{D}$.

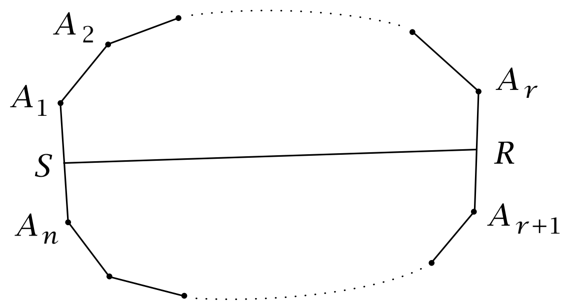


FIGURE 28

Applying Cauchy's Lemma to the half-polygons $A_1 \ldots A_r RS$ and $B_1 \ldots B_r TU$ one finds

$$RS \geq TU.$$

Arguing the same way with the other two half-polygons one finds

$$RS \leq TU.$$

The result is $RS = TU$, and then Cauchy's Lemma says that the angles at A_i and B_i are equal for all i, and that the two polygons $\mathcal{C}$ and $\mathcal{D}$ are congruent. As explained above, this implies that the two polyhedra $\mathcal{P}$ and $\mathcal{Q}$ are congruent. This finishes the proof of Cauchy's Theorem. $\surd$

Chapter IV. The Isoperimetric Inequality

Occasionally this inequality will be referred to as "the I-I".

An old version is "Dido's Problem": What route should a man take to walk from sunup to sundown and to enclose as much land as possible? The answer is: he should walk on a circular path. — We look at simple closed curves in the plane, of a given length L (some differentiability assumptions) and ask: Which curve encloses the largest area? The answer is that it is the circle, of radius $L/2\pi$; it encloses an area $F = L^2/4\pi$. Thus the "Isoperimetric Inequality" for the plane says: The area enclosed by any simple closed curve of length L is less than or equal to $L^2/4\pi$ and equality holds only for the circle. We shall look at proofs of this, and at generalizations.

Jakob Steiner set up a procedure that to every simple closed curve in the plane, not a circle, constructs another curve, of the same length as the given one, that encloses a larger area. (Note: This doesn't prove by itself that the circle encloses the largest area. There could be other curves that enclose a larger area with no one having the largest area. For the existence proof see, e.g., W. Blaschke, *Kreis und Kugel.*)

Here is the process:

I. Let C be not convex. Take a double support tangent and reflect the part of the curve between the two points of contact.

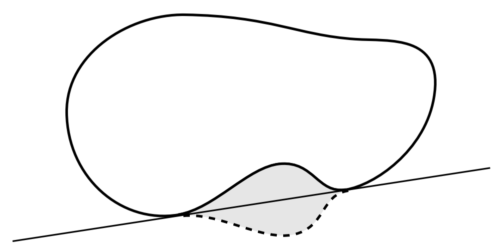


FIGURE 29

II. Now let C be convex. Take two points a and b on it that halve the perimeter. The chord ab divides curve and area into two parts F_1 and F_2. Suppose F_1 is larger. Then replace the part F_2 by the mirror image of F_1 across ab.

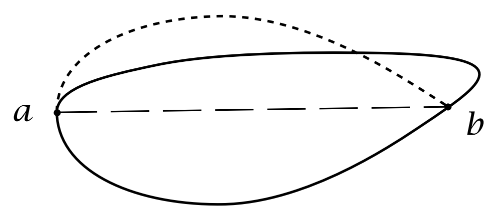


FIGURE 30

III. Suppose both areas are equal. If both arcs C_1, C_2 were semicircles, then C would be a circle. So suppose C_1 is not a semicircle. Then (by Thales) there is some point c on it such that the angle $acb = \gamma$ is not a right angle.

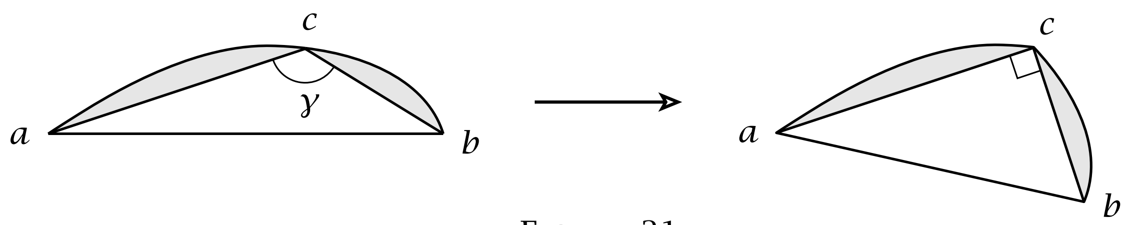


FIGURE 31

With a and c fixed and the length of bc fixed, turn bc around c to change γ to $\pi/2$, taking along the lune over bc. The resulting figure has area larger than that of F_1. Add to it its mirror image across (the new) ab. Now we have a curve of the same length as the old C, but with larger area. √

IV.1. PROOFS. H.A. SCHWARZ (1884), A. HURWITZ, ERHARD SCHMIDT

First some generalities on area. The beginning is the definition of the area A of a rectangle of sides l and w by $A = l \cdot w$. Next is the definition of the area under a curve $y = f(x)$ in the x-y plane as $\int_a^b f(x)\,dx$.

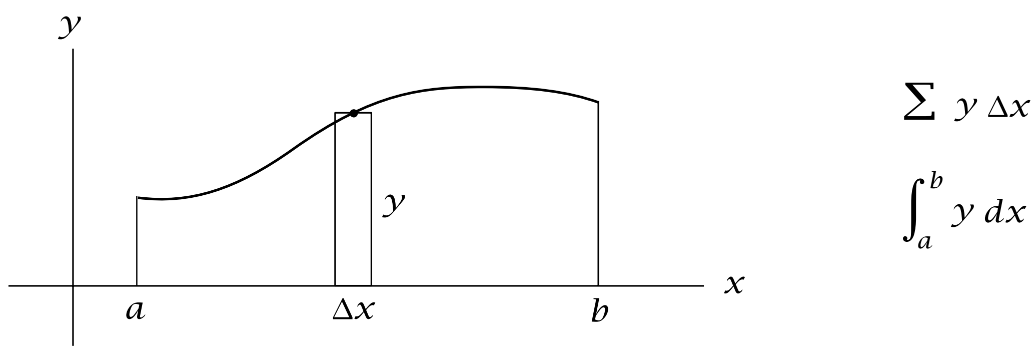


FIGURE 32

Here $(f(x), x)$ can be thought of as a parametric representation of a curve, and it makes sense to change parameter, say by $x = \phi(t), \alpha \le t \le \beta$ (pieceway class C^1, say); arclength s is a possibility. In detail we have $\int_a^b y\,dx = \pm\int_\alpha^\beta y\dot{x}\,dt$, with the $+$-sign for $a = \phi(\alpha)$, $b = \phi(\beta)$ (orientation preserving) and the $-$-sign for $a = \phi(\beta)$, $b = \phi(\alpha)$ (orientation reversing). (By the way, ϕ doesn't have to be monotone, and some of its values may fall outside $[a, b]$, as long as f is defined there.)

With this said, instead of the function f we consider a closed curve C (simple closed, say) in the plane, in some parametrization, and form $\int_C y\,dx$. This gives us the area enclosed by the curve, up to a $\pm$-sign; the integral over the "upper" part of C gives the area under that part, and the integral over the "lower" part subtracts the area under that part. The value of the integral will be negative if we traverse C in the usual positive sense (interior to the left); and so the area enclosed by C is given by $\int_C -y\,dx$. By analogy, and taking orientation into account, the area is also given by $\int_C x\,dy$, or then also by $1/2 \int_C x\,dy - y\,dx$.

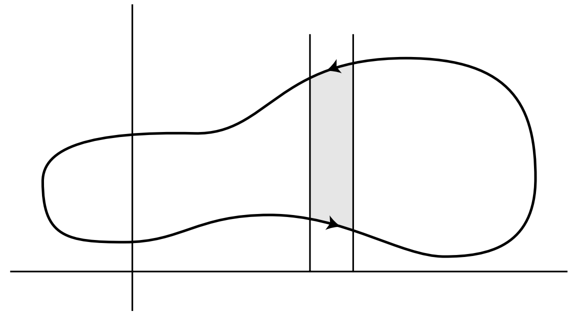


FIGURE 33

So now let C be such a simple closed curve, with length L and area A. Enclose it in a vertical strip, touching it at a and b, which divide the curve into an upper and a lower arc, C^+ and C^-. In the strip construct a circle $\mathcal{K}$ that touches the sides at α and β. Call the radius ρ and choose the coordinate origin at the center of $\mathcal{K}$.

Map C to $\mathcal{K}$ by projecting vertically, upper (lower) arc to upper (lower) arc. On $\mathcal{K}$ use the parameter inherited from C; this is not necessarily monotone, but runs around the circle once. C is described by $(x(t), y(t))$ and $\mathcal{K}$ by $(x(t), \eta(t))$, with the same $x(t)$ for both and with $\eta(t) = \pm\sqrt{\rho^2 - x^2}$. So we have

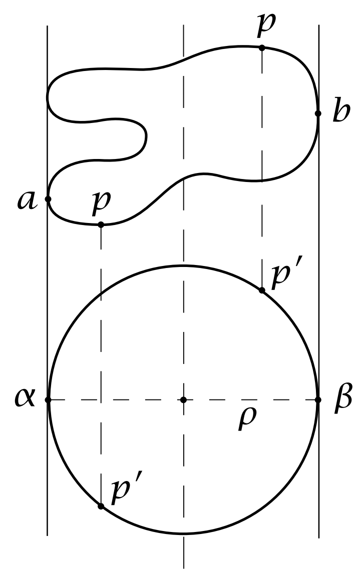


FIGURE 34

$$A = \int x\dot{y}\,dt, \quad A_{\mathcal{K}} = -\int \eta\dot{x}\,dt = \pi\rho^2$$

and therefore

$$A + \pi\rho^2 = \int (x\dot{y} - \eta\dot{x})\,dt.$$

Now we use the Schwarz inequality for the vectors (x, η) and $(\dot{y}, -\dot{x})$ to get

$$A + \pi\rho^2 \le \int \sqrt{x^2 + \eta^2} \cdot \sqrt{\dot{y}^2 + \dot{x}^2}\,dt = \rho \cdot L.$$

Furthermore there is the standard inequality $2\sqrt{uv} \le u + v$ for $u, v \ge 0$, with equality only if $u = v$. This gives $2\sqrt{A \cdot \pi\rho^2} \le \rho \cdot L$, that is

$$4\pi A \le L^2,$$

the isoperimetric inequality.

Now for the =-sign.

1) We must have $2\sqrt{A \cdot \pi\rho^2} = A + \pi\rho^2$, which requires $A = \pi\rho^2$. This means that ρ is determined by A and so is independent of the direction of the strip that encloses C. In other words, C is a curve of constant width 2ρ.

2) We must have equality in the Schwarz inequality. This means that the vectors (x, η) and $(\dot{y}, -\dot{x})$ are parallel. Assuming that t is arc length on C (and so $\dot{x}^2 + \dot{y}^2 = 1$) we have then

$$x = \pm\rho\dot{y}, \quad \eta = \mp\rho\dot{x}.$$

Now we use the constant width again to enclose the curve in a horizontal strip of width 2ρ, so that it is now enclosed in a square of side 2ρ. We take the center of the square as origin of the coordinate system and so as center of the circle $\mathcal{K}$, which thus serves as comparison circle also for the horizontal strip. The point x, y on C now projects to the point (ξ, y) on $\mathcal{K}$, with $\xi^2 + y^2 = \rho^2$.

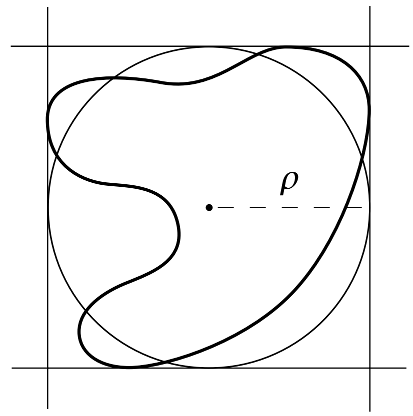


FIGURE 35

Repeating the previous in the horizontal direction we get the equations

$$\xi = \pm\rho\dot{y}, \quad y = \mp\rho\dot{x}.$$

From the last two sets of equations we deduce

$$x^2 + y^2 = \rho^2.$$

The curve C is identical with the circle $\mathcal{K}$, as we claimed. $\surd$

IV.2. The Isoperimetric Inequality in $\mathbb{R}^n$

Briefly, the inequality says: *Among all closed $(n-1)$-dimensional surfaces in $\mathbb{R}^n$ with given surface area A the sphere alone encloses the largest volume.*

This of course requires definitions of “closed surface of dimension $n-1$”, “surface area”, “volume”, and of “enclosed” (Jordan theorem in $\mathbb{R}^n$); we shall not enter into that, and take all for granted. (By the way, the surface doesn’t have to be connected; under surface area and volume one must understand the sum of those for the components. One only has to check the elementary fact that two spheres enclose a smaller volume than the one sphere with surface area equal to the sum of the other two. This in turn amounts to the inequality $a^p + b^p \le (a+b)^p$ for $a, b \ge 0$, $p \ge 1$.)

We write A_r and V_r for surface area and volume of the sphere of radius r in $\mathbb{R}^n$. (These are known quantities, involving π; we shall not need them.) Consider a closed surface $\mathcal{F}$ of surface area A and enclosed volume V. With this A take the sphere $\mathcal{K}$ of radius r with $A_r = A$. Then our inequality says

$$V_r \ge V,$$

with equality only in case $\mathcal{F}$ is equal to (congruent with) the sphere $\mathcal{K}$. Naturally we have $V_r = r^n \cdot V_1$ and $A_r = r^{n-1} \cdot A_1$ and also $V_r = \int_0^r A_t \, dt$ and $A_r = dV_r/dr$; in particular $A_1 = n \cdot V_1$. Now $A_r = A$ gives $r^{n-1} = A/A_1$. With $V_r \ge V$ we get $V_r^{n-1} = r^{(n-1)n} V_1^{n-1} \ge V^{n-1}$, or

$$(A/A_1)^n \ge (V/V_1)^{n-1}.$$

With $A_1 = n \cdot V_1$ we rewrite this as

$$A^n \ge C_n \cdot V^{n-1},$$

where $C_n = n^n V_1$ is a dimension constant. For the plane this reads $L^2 \ge 4\pi A$; for 3-space: $A^3 \ge 36\pi V^2$. One can also write the inequality in the form $\sqrt[n-1]{A/A_1} \ge \sqrt[n]{V/V_1}$.

We begin, as for the plane, by representing volume as an $(n-1)$-dimensional surface integral. A (piece of) surface in n-space can be described, in terms of an orthonormal coordinate system, by giving one of the coordinates, x_i, as function (class C^2 say) of the others. The volume "under" the surface $\mathcal{F}$ is then given by

$$\int (-1)^{(i-1)} x_i \, dx_1 \, dx_2 \ldots \widehat{dx_i} \ldots dx_n$$

where as usual the ^ means that that term is to be skipped. The minus-sign is there for orientation reasons, similar to what happened to $\int y\,dx$ in the plane for a curve. For a closed bounded surface the orientation is so chosen that the outer normal together with that of the surface gives the basic orientation of $\mathbb{R}^n$. The integral is computed locally, in terms of local parameters $u_1, \ldots, u_{n-1}$ with, e.g.,

$$dx_2 \, dx_3 \cdots dx_n = \frac{\partial(x_2, \ldots, x_n)}{\partial(u_1, \ldots, u_{n-1})} \cdot du_1 \cdots du_{n-1}.$$

The coordinate patches cover the surface. They don't all have to agree with the orientation of the surface, but the surface should be covered once more positively than negatively (degree 1). The same cancelation argument as in the plane case will show that the integral (which can also be written $\int 1/n \cdot \sum(-1)^{(i-1)} x_i\, dx_1\, dx_2 \ldots \widehat{dx_i} \ldots dx_n$) gives the volume enclosed by the surface. (This is a special case of Stokes's theorem.)

The proof goes in two steps, I. for surfaces of revolution, II. for general ones.

I. Let $\mathcal{F}$ be an $(n-1)$-dimensional surface of revolution, with the x_n-axis as axis of rotation, and $(n-2)$-dimensionl spheres as orbits of the rotation. We form a coaxial spherical cylinder that contains and touches the surface along one or several $(n-2)$-spheres, of radius ρ. Choose one; it divides $\mathcal{F}$ into an upper and a lower part. We choose a sphere $\mathcal{K}$ inscribed into the cylinder, with the origin of the coordinate system at the center of the sphere.

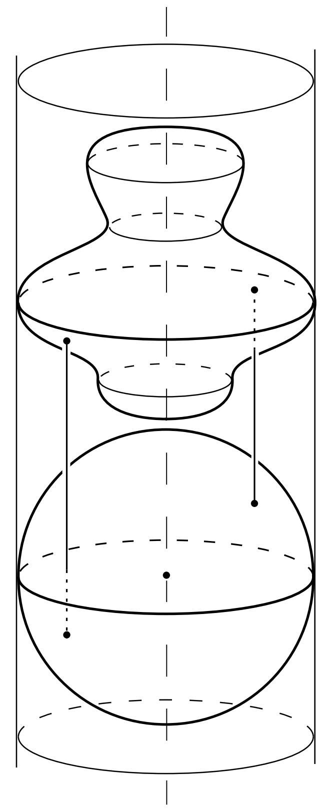


FIGURE 36

As in the plane case, $\mathcal{F}$ is projected to $\mathcal{K}$ along the x_n-axis, with x_n going to ξ, upper part to upper part and lower to lower. The projection is of degree 1 and so gives a parametric representation of $\mathcal{K}$. This yields $\int (-1)^{n-1}\, \xi\, dx_1\, dx_2 \ldots dx_{n-1} = V_\rho = \rho^n V_1$.

Now a small computation is needed. Let us write X for the vector $(x_1, x_2, \ldots, x_n)$. If the x_i are functions of parameters u_α, $\alpha = 1, 2, \ldots, n-1$, we write X_α for the vector of partial derivatives of the x_i with respect to u_α. Also write D_i for the signed functional determinant

$$(-1)^{i-1} \frac{\partial(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_{n-1})}$$

and write D for the vector $(D_1, \ldots, D_n)$. D is orthogonal to all X_α, as one sees by expanding the determinant $det\,[X_\alpha, X_1, X_2, \ldots, X_{n-1}]$ along the first column X_α; the D_i are just the minors that appear, and the determinant is 0 because it has two equal columns. Write ν for the corresponding unit vector $1/|D| \cdot D$, with $|D| = \left(\sum D_i^2\right)^{1/2}$; it is the unit normal to the surface $\mathcal{F}$.

$|D|^2$ can be written as $det\,[D, X_1, \ldots, X_{n-1}]$, as one sees again by expanding along the first column. For the matrix $M = [\nu, X_1, \ldots, X_{n-1}]$ one has then $det\, M = |D|$; this is then the $(n-1)$-dimensional area of the parallelepiped spanned by the X_α. The product $M^\top \cdot M$ (here $M^\top$ is the transpose) is the Gramian of the vectors involved, and the 1,1-minor is the Gramian $[X_\alpha \cdot X_\beta] = G$ of the X_α. $det\, M^\top \cdot M = det\, G$ is usually denoted by g, and thus finally $|D| = (det\, G)^{1/2} = \sqrt{g}$; with all this $\sqrt{g}\, du_1 du_2 \ldots du_{n-1}$ is usually called the area element dA of $\mathcal{F}$, and $\int \sqrt{g}\, du_1 \ldots du_{n-1}$ is the area A of the surface.

Now back to the volume integrals for $\mathcal{F}$ and $\mathcal{K}$. We add the first $n-1$ of the integrals for $\mathcal{F}$ and the last one (with ξ for x_n) for $\mathcal{K}$. With Ξ denoting the vector $(x_1 \ldots, x_{n-1}, \xi)$ this results in

$$(n-1)V + \rho^n V_1 = \int \Xi \cdot \nu \, dA.$$

By the Schwarz inequality, with $|\Xi| = \rho$ and $|\nu| = 1$, the right hand side is $\leq \int \rho \cdot dA = \rho \cdot A$. As in the plane case there is the arithmetic-geometric mean inequality, which gives

$$n\rho \sqrt[n]{V_1 \cdot V^{n-1}} \leq (n-1)V = \rho^n V_1 \leq \rho \cdot A$$

and so

$$A \geq n \sqrt[n]{V_1 \cdot V^{n-1}},$$

which is one form of the Isoperimetric Inequality.

Now the =-sign. First, in the Schwarz inequality there must be equality, so that Ξ and ν are parallel. i.e.,

$$\Xi = \pm\rho \cdot \nu.$$

$\mathcal{K}$ and $\mathcal{F}$ have parallel normals. Near any point on $\mathcal{F}$ in the interior of the cylinder the normal is not horizontal and so $\mathcal{F}$ can be described there by $x_n = f(x_1, \ldots, x_{n-1})$ with the normal $(f_{x_1}, \ldots, f_{x_{n-1}}, -1)$. For $\mathcal{K}$ this same normal appears as $(\xi_1, \ldots, \xi_{x_{n-1}}, -1)$. One concludes that $f = \xi + const$ and that the shape of $\mathcal{F}$ is as follows: An upper hemisphere touching the cylinder along its boundary, then a piece of the cylinder, of height h say, going down, and at the bottom the lower hemisphere.

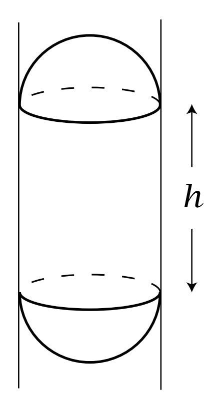


FIGURE 37

Second, there must be equality in the arithmetic-geometric mean inequality which implies that the n terms in the sum must be equal to each other, and so $\rho^n V_1 = V$, which requires $h = 0$ and $\mathcal{F}$ is a sphere. $\surd$

II. The I-I for a general surface. One employs a process called symmetrization, which transforms the surface into a surface of revolution with the same volume and smaller surface area. Take a "vertical" line L. For any value h consider the horizontal $(n-1)$-plane at level h and intersect it with $\mathcal{F}$, getting one or several surfaces in an $\mathbb{R}^{n-1}$. (One has to choose the line well, to get decent surfaces as sections. There will be a finite number of h-levels where the section has singularities, but very simple ones.) Write $B(h)$ for the total $(n-2)$-dimensional surface area of these surfaces, and write $Q(h)$ for the $(n-1)$-dimensional area of the closure of the intersection of the plane with the interior of $\mathcal{F}$. In that plane one forms the $(n-2)$-sphere of the same ($(n-1)$-dimensional) volume $Q(h)$ and places its center on the line L, obtaining a surface of revolution $\tilde{\mathcal{F}}$ of the same enclosed volume as $\mathcal{F}$ (by Cavalieri's principle); its surface area is denoted by $\tilde{A}$, and one has to prove $A \geq \tilde{A}$, with equality only for a surfaces of revolution. With $\tilde{B}(h)$ for the surface area of the sphere, one has of course $B(h) \geq \tilde{B}(h)$.

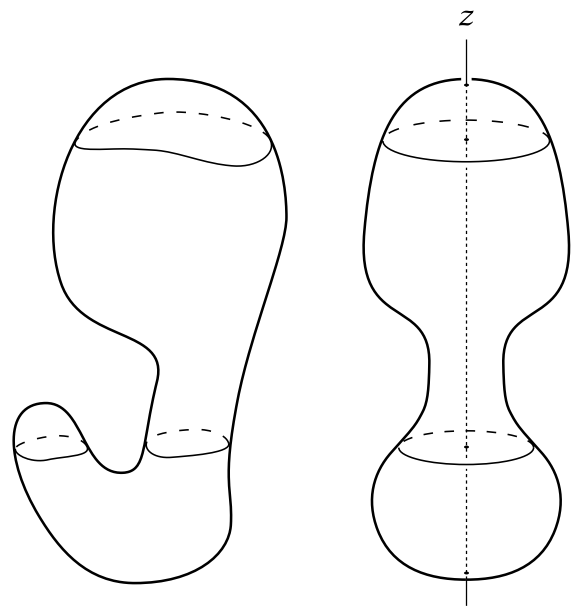


FIGURE 38

We will give the proof for $n = 3$, with x, y, z instead of x_1, x_2, x_3. First an integral inequality:

Let $p(t), q(t)$ be two (continuous) functions, defined on the interval $[0, T]$. Then

$$\left(\int_0^T p\,dt\right)^2 + \left(\int_0^T q\,dt\right)^2 \leq \left(\int_0^T \sqrt{p^2+q^2}\,dt\right)^2$$

with equality only if p/q is constant.

For the proof one defines two functions $x(t), y(t)$ by $x' = p$, $y' = q$ and $x(0) = y(0) = 0$. The inequality appears then as

$$\sqrt{x(T)^2 + y(T)^2} \leq \int_0^T \sqrt{(x')^2 + (y')^2}dt,$$

which is simply the well known fact that the straight segment is the shortest connection between its endpoints, and also makes clear when the =-sign holds. In particular, for $p = 1$ one has

$$\int_0^T \sqrt{1+q^2} \geq \sqrt{T^2 + \left(\int_0^T q\,dt\right)^2}.$$

with equality only if q is constant.

Consider a piece of $\mathcal{F}$ between two values z_1, z_2 of z. Parametrize it by z and s, arc length on each curve (or curves) $z = \text{const}$. The area element dA works out to be $\sqrt{1+D_z^2}\,ds\,dz$, where D_z is the functional determinant $x_s y_z - x_z y_s$ (D_x and D_y equal y_s and $-y_s$, respectively, with $x_s^2 + y_s^2 = 1$). Thus

$$\int \sqrt{1+D_z^2}\,ds\,dz$$

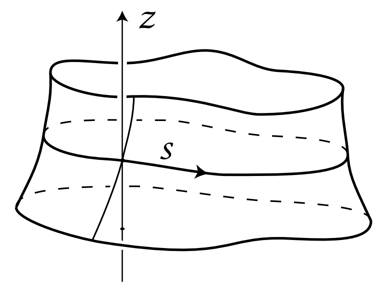


FIGURE 39

is the surface area of our piece of $\mathcal{F}$. Similarly, for the $(n-1)$-area difference of two sections we have

$$Q(z_1) - Q(z_2) = \int dx\,dy = \int D_z\,dx\,dy,$$

the integral extended over the piece of surface under consideration. Letting z_2 go to z_1 one gets

$$Q'(z) = \int D_z\,ds,$$

where now the integral goes over the boundary curve of the section at level z. Letting $L(z)$ be the length of that boundary curve, the inequality above gives

$$\int_0^{L(z)} \sqrt{1+D_z^2}\,ds \geq \sqrt{L(z)^2 + \left(\int D_z\,ds\right)^2}$$

and so, integrating over z,

$$A \geq \int \sqrt{L(z)^2 + Q'(z)^2}\,dz.$$

The analogous inequality holds for the surface of revolution $\tilde{\mathcal{F}}$; however by rotational symmetry $\tilde{D}_z$ is independent of s, and so equality holds. Combining this with the inequality $B(h) \geq \tilde{B}(h)$, one finally gets

$$A \geq \int \sqrt{L(z)^2 + Q'(z)^2}\,dz \geq \int \sqrt{\tilde{L}(z)^2 + \tilde{Q}'(z)^2}\,dz = \tilde{A},$$

which, as noted above, establishes the I-I.

Now once more for the $=$-sign. In the last formula there are two $\geq$-signs, which now become $=$-signs. The second one requires $L(z) = \tilde{L}(z)$ for all z, which by the I-I in the plane means that the horizontal sections of $\mathcal{F}$ are all circles. ($Q(z)$ equals $\tilde{Q}(z)$ by definition.) The first one requires D_z to be independent of s, from our integral inequality and the relation $Q'(z) = \int D_z\,ds$. This means that the third component $D_z/\sqrt{1 + D_z^2}$ is constant along each circle $z = \text{const}$, or that the angle of the normal to $\mathcal{F}$ and the z-axis is constant. One verifies that then the centers of the horizontal circles making up $\mathcal{F}$ must lie on a vertical line. Then $\mathcal{F}$ is a surface of revolution, and we are done. $\surd$

(What needs to be done to extend the result to any n? Certainly the integral ineqality has to put in a more general form.)