NORMALIZED GEOMETRIC SYSTEMS

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Communicated by E. H. Moore, January 18, 1921

The notion of norm or numerical value of a complex quantity, $c = a + b\sqrt{-1}$, namely, $|c| = \sqrt{a^2 + b^2}$, as it arises in algebra, has a more or less immediate generalization to more extensive matric systems. The three important properties: (1) $|c_1 + c_2| \leq |c_1| + |c_2|$; (2) $|c_1 . c_2| = |c_1| . |c_2|$; (3) $c|$ is the positive square root of a positive definite quadratic form, are carried over at the expense only of replacing (2) by (2′) $|c_1 . c_2| \leq |c_1| . |c_2|$ and allowing in place of (3), (3′) $|c|$ is the positive square root of a positive definite form, Hermitian or quadratic. By $c_1 . c_2$ in these geometrical examples is meant the inner product[1] or a generalization of it. Two other generalizations of norm have been of great importance. The first of these is that of the theory of algebraic numbers,[2] where (1) is dropped, (2) is retained, and in place of (3) one has, Norm of c is a certain function of the nth degree, n being the order of the algebraic field. The second is that of a general theory of sets as treated for example by Fréchet,[3] where (1) is retained, (2) and (3) are dropped. The theory of integral equations as usually developed is geometrical in an infinity of dimensions and retains (1), (2′), (3′). It is noted that instances in which (1) and (2′) are retained usually keep (3′) also. Now the importance of (3′) is chiefly that it implies (1) and (2′) with the conventions as to linearity and so forth usually assumed. The converse that (1) and (2′) imply (3′) is false. It is of interest to show that most of the familiar properties of the norm may be retained, in particular (1) and (2′), when the norm is positive definite but otherwise largely arbitrary.

Three discussions bearing on this topic may be referred to. First, a geometrical study involving points but not their duals, by Minkowski, in his *Geometrie der Zahlen*.[4] The great generality of the idea of norm is there beautifully developed although it is not carried so far as it is here; but since the concept of the point dual is not brought in by Minkowski, most of the ideas here discussed are not found there. A second discussion, involving inner products, but treating only a very special case of the non-quadratic norm for an infinite number of variables is given by F. Riesz[5] in examining the convergence of bilinear forms. The third discussion involving only a scalar system, and hence without inner products, between elements of different systems is given by Kürschàk.[6] It is perhaps the most suggestive system of scalars in the literature in which $||n||$ may be less than n, for n a natural number.

The following treatment relates under one head the notions of convex region,[4] the triangle property,[7] the linearly homogeneous property of distance[8] or norm, conjugate norms,[5] the inner product,[1] convergence of a bilinear form,[5] Minkowski's gauge form[4] and Schwarz's inequality[9] (so-called), as these occur in geometry, hypercomplex number theory, integral equations, and more generally, general linear analysis.[10]

The following theorem may be proved: *Every geometric system admitting a gauge set may be normalized by means of that set.* In order that there may be no ambiguity an extended sequence of definitions will be given to cover all terms used.

If P is a proposition concerning a system, S, it may be that another system, T, is such that the proposition P has a meaning for the system T. The content of the proposition may be seriously altered while its form is not affected except in the sense that S is replaced by T. We may employ the functional notations $P(S)$ and $P(T)$ to suggest that the form of the proposition is carried over unaltered from one system to the other. To avoid repetition certain propositional functions will be here listed for reference. These serve as definitions, whenever all of the terms appearing in a proposition have been themselves defined, for example, "*Reg* (L)," below, acquires a meaning only in a system in which 0_L and 1_L are defined.

The "Addition Proposition,"[11] *Add.* (R): (*i*) R is a system comprising elements, r, and a rule of binary combination, $+$. (*ii*) For any r_1, r_2, of R, $r_1 + r_2$ is a uniquely defined element of R. (*iii*) For any r_1, r_2, of R, $r_1 + r_2 = r_2 + r_1$. (*iv*) For any r_1, r_2, r_3, of R, $r_1 + (r_2 + r_3) = (r_1 + r_2) + r_3$. (*v*) There is an element O_R of R (also denoted by merely O), such that for every r of R, $r + O_R = r$. (*vi*) For a given r_1 of R, there is not more than one element r of R, such that $r_1 + r = r_1$. (*vii*) For a given r_1 of R, there is one and only element, r, of R such that $r_1 + r = O_R$—this element, r, is denoted by $-r_1$.

The "Multiplication Proposition," *Mult.* $(R, S, T. P)$: (*i*) There are systems R, S, T, P, comprising elements, respectively, r, s, t, p, and two rules of binary combination $+$ and $.$, and such that with respect to $+$, the addition proposition is valid for each of the systems. (*ii*) For any r of R, s of S, t of T, $r.s$ and $s.t$ are uniquely defined, and $s.t$ is an element of P. (*iii*) For any r_1, r_2 of R, and s_1, s_2 of S, it is true that $r_1.(s_1 + s_2) = r_1.s_1 + r_1.s_2$ and $(r_1 + r_2).s_1 = r_1.s_1 + r_2.s_1$. (*iv*) For any r of R, s of S, t of T, it is true that $r.(s.t) = (r.s).t$. (*v*) For any r of R, and s of S, it is true that $O_R.s = O_T$, and $r.O_S = O_T$.

The "Commutative Proposition," *Com.* $(C.M.L.)$: (*i*) C is a subsystem of M. (*ii*) For each c of C and l of L, $c.l = l.c$. (*iii*) There exists an element 1_C, of C (also denoted merely by 1) such that for every l, $1_C.l = l$.

The "Regular Proposition," *Reg.* (L): (*i*) The elements of L fall into two mutually exclusive subsets L_{0*} and L_∞. (*ii*) The elements of L_{0*} fall into two mutually exclusive subsets, L_0 and L_*. (*iii*) O_L is an element of L_0. (*iv*) 1_L if it has been defined and exists, is an element of L_*. (*v*) For every l of L, $-l$ is in the same set as l. (*vi*) There exists at least one l_* other than 1_L .

The "Regular Multiplication Proposition," *R. Mult.* (R, S, T, P): (*i*) Reg. (R), Reg. (S), Reg. (T), Reg. (P). (*ii*) Mult. (R, S, T, P). (*iii*) Mult. $(R_{o*}, S_{o*}, T_{o*}, P_{o*})$. (*iv*) Mult. (R_o, S_o, T_o, P_o).

The "Regular Commutative Proposition," *R. Com.* (C, M, L): (*i*) Com. (C, M, L). (*ii*) For every c of C and l of L, $c_o.l_o$, $c_o.l_*$ and $c_*.l_o$ are of L_o; $c_*.l_*$ is of L_*; $c_\infty.l_*$, $c_*.l_\infty$, and $c_\infty.l_\infty$ are of L_∞. (*iii*) 1_C is of L_*. (*iv*) For every l_1 and l_2 of L, if neither l_1 nor l_2 is in L_∞, $l_1 + l_2$ is not in L_∞, and if both l_1 and l_2 are in L_o, $l_1 + l_2$ is in L_o.

The "Regular Conjugate Proposition," *R. Conj.* (C, R, S): For each r of R, there is a c of C and an s of S, such that (*i*) $r.s = s.r = c.c$; (*ii*) if r is of R_o, c is of C_o, s is of S_o; if r is of R_*, c is of C_*, s is of S_*; if r is of R_∞, c is of C_∞, s is of S_∞.

The "Normal Proposition," *Nrm.* (L): (*i*) Each l of L has associated with it a unique value, $\|l\|$. (*ii*) Reg. (L). (*iii*) For each l_o, $\|l_o\| = O$; for each l_*, $\|l_*\|$ is a positive real finite number; for each l_∞, $\|l_\infty\|$ is positively infinite. (*iv*) For each l of L, $\|-l\| = \|l\|$. (*v*) If 1_L exists, $\|l_L\| = 1$. (*vi*) There exists at least one l_* for which $\|l_*\|$ is different from unity.

The "Normal Multiplication Proposition," *N. Mult.* (R, S, T, P): (*i*) Nrm. (R), Nrm. (S), Nrm. (T), Nrm. (P). (*ii*) Mult. (R, S, T, P). (*iii*) $\|s.t\| \leq \|s\|.\|t\|$.

The "Normal Commutative Proposition," *N. Com.* (C, M, L): (*i*) Com. (C, M, L). (*ii*) For every c of C and l of L, for which $\|c\|.\|l\|$ exists, $\|c.l\| = \|c\|.\|l\|$. (*iii*) $\|1_C\| = 1$. (*iv*) For every l_1 and l_2 of L, $\|l_1 + l_2\| \leq \|l_1\| + \|l_2\|$.

The "Normal Conjugate Proposition," *N. Conj.* (C, R, S): For each r of R, there is a c of C and an s of S, such that (*i*) $r.s = s.r = c.c$, (*ii*) $\|r\| = \|c\| = \|s\|$.

The "Linear Proposition," *Lin.* (C, M, L): (*i*) Mult. (M, M, L, L). (*ii*) Mult. (M, L, M, L). (*iii*) Com. (C, M, L).

In the above, L is called a *linear system*, with M as *system of multipliers*, and C as *commutative subsystem* of M.

The "Hypernumber Proposition," *Hyp.* (C, H): (*i*) Lin. (C, C, C). (*ii*) Lin. (C, H, H).

In the above, H is called a *system of hypernumbers* with C as commutative subsystem.

The "Vector Proposition," *Vect.* (C, H, V): (*i*) Hyp. (C, H). (*ii*) Lin. (C, H, V).

In the above, V is called a *system of vectors*, with H as *associated system of hypernumbers*.

The "Regular Linear Proposition," *R. Lin.* (C, M, L): (*i*) R. Mult. (M, M, L, L). (*ii*) R. Mult. (M, L, M, L). (*iii*) R. Com. (C, M, L).

The "Regular Hypernumber Proposition," *R. Hyp.* (C, H): (*i*) R. Lin. (C, C, C). (*ii*) R. Lin. (C, H, H).

The "Regular Vector Proposition," *R. Vect.* (C, H, V): (*i*) R. Hyp. (C, H). (*ii*) R. Lin. (C, H, V).

The "Normal Linear Proposition," *N. Lin.* (C, M, L): (*i*) N. Mult. (M, M, L, L). (*ii*) N. Mult. (M, L, M, L). (*iii*) N. Com. (C, M, L).

The "Normal Hypernumber Proposition," *N. Hyp.* (C, H): (*i*) N. Lin. (C, C, C). (*ii*) N. Lin. (C, H, H).

The "Normal Vector Proposition," *N. Vect.* (C, H, V): (*i*) N. Hyp. (C, H). (*ii*) N. Lin. (C, H, V).

The "Geometric Proposition," *Geom.* (C, H, X, U): (*i*) Vect. (C, H, X). (*ii*) Vect. (C, H, U). (*iii*) Mult. (H, X, U, H). (*iv*) Mult. (H, U, X, H).

The "Regular Geometric Proposition," *R. Geom.* (C, H, X, U): (*i*) R. Vect. (C, H, X). (*ii*) R. Vect. (C, H, U). (*iii*) R. Mult. (H, X, U, H). (*iv*) R. Mult. (H, U, X, H). (*v*) R. Conj. (C, X, U). (*vi*) R. Conj. (C, U, X).

The "Normal Geometric Proposition," *N. Geom.* (C, H, X, U): (*i*) N. Vect. (C, H, X). (*ii*) N. Vect. (C, H, U). (*iii*) N. Mult. (H, X, U, H). (*iv*) N. Mult. (H, U, X, H). (*v*) N. Conj. (C, X, U). (*vi*) N. Conj. (C, U, X).

The "Gauge Proposition," *Gge.* (C, H, R, S): (*i*) For each r of R_G there is an s of S_G, such that $r.s = s.r = 1_C$, while for this s and any other element r' of R_G, $\|r'.s\| = \|s.r'\| \leq 1$. (*ii*) For each r_* of R_* there is an r of R_G, and a c of C_*, such that $r_* = c.r = r.c$.

The "Gauge Geometric Proposition," *Gge. Geom.* (C, H, X, U): (*i*) R. Geom. (C, H, X, U). (*ii*) N. Hyp. (C, H). (*iii*) There exists a subset X_G of X_*, and a subset U_G of U_*. (*iv*) Gge. (C, H, X, U). (*v*) Gge. (C, H, U, X).

The "Division Proposition," *Div.* (C, S, T): For each s of S_*, there is at least one t of T_*, such that (*i*) $s.t = t.s. = 1_C$, (*ii*) $\|s\|.\|t\| = 1$.

The "Norm Product Proposition," *N. Prod.* (R): For every r_1, r_2 of R, for which $\|r_1\|.\|r_2\|$ is defined, $\|r_1.r_2\| = \|r_1\|.\|r_2\|$.

A system (C, H, X, U) is said to be a *geometric system* if Geom. (C, H, X, U) is valid with respect to it. It is further said to *admit a gauge set*, if Gge. Geom. (C, H, X, U) also is valid. To *normalize* a geometric system is to define $\|x\|$ and $\|u\|$ in such a manner that N. Geom. (C, H, X, U) is valid, while any geometric system for which N. Geom. (C, H, X, U) holds is said to be a *normal geometric system*.

A geometric system admitting a gauge set may be normalized but leads to a special type of normal geometric system, since for any geometric system admitting a guage set, the following additional propositions may be proved: Div. (C, C, C), Div. (C, H, H), N. Prod. (C), N. Prod. (H), and for the normalized system obtained the following also are provable: Div. (C, X, U), Div. (C, U, X). For the general normal geometric system none of these six propositions may be true.

To normalize a geometric system admitting a guage set, define for every x_G and u_G, $\|x_G\| = 1$, $\|u_G\| = 1$, and for every $c.x_G$ and $c.u_G$, $\|c.x_G\| = \|c\|$, $\|c.u_G\| = \|c\|$. The theorem then follows without difficulty.

For the general normal geometric system, the *relative inclination*, θ, of two elements, x of X_*, and u of U_*, may be defined by, $\|x.u\| = \|x\|.\|u\| \cos \theta$. Thus x of X_*, and u of U_* are mutually orthogonal if and only if $\|x.u\| = O$. Two elements, x of X_* and u of U_*, are mutually *conjugate* if and only if $x.u = u.x = c.c$ while also $\|x\| = \|u\| = \|c\|$. Two elements, x of X_*, and u of U_*, are mutually *reciprocal* if and only if $x.u = u.x = 1_C$ while also $\|x\|.\|u\| = 1$. The sets (x) and (u) for which $\|x\| = 1$, and $\|u\| = 1$, respectively, are Convex. The relation $\|r_1 + r_2\| \leq \|r_1\| + \|r_2\|$ is called the Triangle Inequality of the norm. The relation $\|c.l\| = \|c\|.\|l\|$ is called the Linearly Homogeneous Property of the norm. The expressions $\|x\|$ and $\|u\|$ are said to be Conjugate Norms and when continuous, each is a "Gauge Form." The expressions $x.u$ and $u.x$ are Inner Products. Each of the bilinear forms $x.u$ and $u.x$ "converges" if $\|x\|$ and $\|u\|$ are finite, since $\|x.u\| \leq \|x\|.\|u\|$, and $\|u.x\| \leq \|u\|.\|x\|$, which are statements of the general form of Schwarz's Inequality.

The following possibilities for a normal geometrical system may be emphasized: (*i*) The product among hypernumbers need not be commutative. (*ii*) The product involving two x's or two u's need not have a meaning. (*iii*) The system C may be an integral system in which division is not in general possible. (*iv*) $\|n_C\|$ may be less than n, where by n_C is meant $1_C + 1^C + \ldots + 1_C$ to n terms, for $n > 1$. (*v*) The conjugate of a given element need not be unique. (*vi*) The set of elements (x) for which $\|x\| = 1$, may be dense but not continuous, as for example the rational points on a circle. In particular, the above theory is applicable to the Geometry of Numbers of Minkowski,[4] to a system where C is a Kürschak valuated field,[6] and to a system where H is a quaternion field.[12]

While in every normal geometric system each x of X_* has a conjugate, u of U_*, which has again the given x as its conjugate, the conjugate relation is not in general a simple one. The geometric systems for which (3′) is satisfied are identified by the semi-linear property that if x_1 and u_1 be conjugates, and x_2, u_2 be conjugates, then $c_1.x_1 + c_2x_2$ and $c'_1.u_1 + c'_2.u_2$ will be conjugates where c_1, c_2, c'_1, c'_2 are of C, while c and c' are themselves conjugates in a more elementary sense.

[1] Due to Grassmann, *Geometrische Analyse*, 1847, paragraph 7, p. 16; cf. Gibbs, *Vector Analysis* (E. B. Wilson).

[2] Suggested by Gauss, Dirichlet, *Werke*, I, p. 539. Extended use by Kronecker, "De unitatibus complexis," *Werke* I, p. 14.

[3] Fréchet, M., Sur quelques points..., *Rend. Circ. M. Palermo*, **22**, 1906 (1–74).

Consult A. D. Pitcher and E. W. Chittenden, "On the Foundations of the Calcul Fonctionelle of Fréchet," *Trans. Amer. Math. Soc.*, **19**, 1918 (66–78).

[4] Minkowski, H., *Geometrie der Zahlen*, p. 9 (Ed. 1910), Leipzig.

[5] Riesz, F., *Les systèmes d'equations linéaires*, Paris (1913).

[6] Kürschàk, J., Ueber Limesbildung..., *Crelle*, **142**, 1913 (211).

[7] Proved in elementary geometry for triangles, namely (1) above.

[8] Identified since C. Wessel (1799), Gauss, and Hamilton.

[9] Moore, E. H., *Fifth Int. Cong. Math.* **1912**, I (253).

[10] Moore, E. H., *Bull. Amer. Math. Soc.*, (Ser. 2), **18**, 1912 (334–362), and later papers.

[11] Cf. Moebius, Der barycentrische Calcul, (1897), *Werke*, I.

[12] Cf. Hurwitz, A., *Zahlentheorie der Quaternionen*, Berlin (1919).

PROBLEMS OF POTENTIAL THEORY

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Communicated by E. H. Moore, January 18, 1921

1. *The Equations of Laplace and Poisson.*—As is well known, Bôcher considered the integral form of Laplace's equation:[1]

$$\int \frac{\partial u}{\partial n}\, ds = 0, \tag{1}$$

and showed that it was entirely equivalent to the differential form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

for functions u continuous with their *first* partial derivatives over any "Weierstrassian" region (as Borel would call it[2]); in fact he showed that any such solution of (1) possessed continuous derivatives of all orders and satisfied (2) at every point. One can go still further, and consider solutions of (1) which have merely summable derivatives, and of the first order, with practically the same result.

THEOREM I.—If the function u is what we shall call a "potential function for its gradient vector ∇u,"[3] the components of the latter being summable superficially in the Lebesgue sense, and if the equation

$$\int \nabla_n u\, ds = 0 \tag{1}$$

is satisfied for every curve of a certain class,[4] then the function u has merely unnecessary discontinuities, and when these are removed by changing the value of u in the points at most of a point set of superficial measure zero, the resulting function has continuous derivatives of all orders and satisfies (2) at every point.

Let us pass on to the equation

$$\int \frac{\partial u}{\partial n}\, ds = \int \rho\, d\sigma \tag{3}$$

in which a curvilinear integral on the left is equal to a superficial integral on the right, and this equality is a generalization of Poisson's equation