PRACTICAL ASTRONOMY FOR ENGINEERS

SEARES

PRACTICAL ASTRONOMY
FOR ENGINEERS

PRACTICAL ASTRONOMY FOR ENGINEERS

BY

FREDERICK HANLEY SEARES

Professor of Astronomy in the University of Missouri
and Director of the Laws Observatory

COLUMBIA, MISSOURI
THE E. W. STEPHENS PUBLISHING COMPANY
1909

PREFACE

The following pages represent the result of several years' experience in presenting to students of engineering the elements of Practical Astronomy. Although the method and the extent of the discussion have been designed to meet the specialized requirements of such students, it is intended that the work shall also serve as an introduction for those who desire a broader knowledge of the subject.

The order of treatment and the methods proposed for the solution of the various problems have been tested sufficiently to establish their usefulness; and yet the results are to be regarded as tentative, for they possess neither the completeness nor the consistency which, it is hoped, will characterize a later edition. The volume is incomplete in that it includes no discussion of the principles and methods of the art of numerical calculation—a question fundamental for an appreciation of the spirit of the treatment. Difficulties inherent in this defect may be avoided by a careful examination of an article on numerical calculation which appeared in *Popular Astronomy*, 1908, pp. 349-367, and in the *Engineering Quarterly of the University of Missouri*, v. 2, pp. 171-192. The final edition will contain this paper, in a revised form, as a preliminary chapter. The inconsistencies of the work are due largely to the fact that the earlier pages were in print before the later ones were written, and to the further fact that the manuscript was prepared with a haste that permitted no careful interadjustment and balancing of the parts.

The main purpose of the volume is an exposition of the principal methods of determining latitude, azimuth, and time. Generally speaking, the limit of precision is that corresponding to the engineer's transit or the sextant. Though the discussion has thus been somewhat narrowly restricted, an attempt has been made to place before the student the means of acquiring correct and complete notions of the fundamental conceptions of the subject. But these can scarcely be attained without some knowledge of the salient facts of Descriptive Astronomy. For those who possess this information, the first chapter will serve as a review; for others, it will afford an orientation sufficient for the purpose in question. Chapter II blocks out in broad lines the solutions of the problems of latitude, azimuth, and time. The observational details of these solutions, with a few exceptions, are presented in Chapter IV, while Chapters V–VII consider in succession the special adaptations of the fundamental formulæ employed for the reductions. In each instance the method used in deriving the final equations originates in the principles underlying the subject of numerical calculation. Chapter III is devoted to a theoretical consideration of the subject of time.

It is not customary to introduce historical data into texts designed for the use of professional students; but the author has found so much that is helpful and stimulating in a consideration of the development of astronomical instruments, methods, and theories that he is disposed to offer an apology for the brevity of the historical sections rather than to attempt a justification of their introduction into a work mainly technical in character. To exclude historical material from scientific instruction is to disregard the most effective means of giving the student a full appreciation of the significance and bearing of scientific results. Brief though they are, it is hoped that these sections may incline the reader toward wider excursions into this most fascinating field.

The numerical solutions for most of the examples have been printed in detail in order better to illustrate both the application of the formulæ involved and the operations to be performed by the computer. Care has been taken to secure accuracy in the text as well as in the examples, but a considerable number of errors have already been noted. For these the reader is referred to the list of errata on page 132.

The use of the text should be supplemented by a study of the prominent constellations. For this purpose the "Constellation Charts" published by the editor of *Popular Astronomy*, Northfield, Minnesota, are as serviceable as any, and far less expensive than the average.

My acknowledgments are due to Mr. E. S. Haynes and Mr. Harlow Shapley, of the Department of Astronomy of the University of Missouri, for much valuable assistance in preparing the manuscript, in checking the calculations, and in reading the proofs.

F. H. SEARES.

LAWS OBSERVATORY,
UNIVERSITY OF MISSOURI,
June, 1909.

CONTENTS

CHAPTER I

INTRODUCTION—CELESTIAL SPHERE—COÖRDINATES

CHAPTER II

FORMULAE OF SPHERICAL TRIGONOMETRY—TRANSFORMATION OF COÖRDINATES—GENERAL DISCUSSION OF PROBLEMS

CHAPTER III

TIME AND TIME TRANSFORMATION

CHAPTER IV

INSTRUMENTS AND THEIR USE

CHAPTER V

THE DETERMINATION OF LATITUDE

PRACTICAL ASTRONOMY FOR ENGINEERS

CHAPTER I

INTRODUCTION—CELESTIAL SPHERE—COÖRDINATES.

1. The results of astronomical investigations.—The investigations of the astronomer have shown that the universe consists of the sun, its attendant planets, satellites, and planetoids; of comets, meteors, the stars, and the nebulae. The sun, planets, satellites, and planetoids form the solar system, and with these we must perhaps include comets and meteors. The stars and nebulae, considered collectively, constitute the stellar system.

The sun is the central and dominating body of the solar system. It is an intensely heated luminous mass, largely if not wholly gaseous in constitution. The planets and planetoids, which are relatively cool, revolve about the sun. The satellites revolve about the planets. The paths traced out in the motion of revolution are ellipses, nearly circular in form, which vary slowly in size, form, and position. One focus of each elliptical orbit coincides with the center of the body about which the revolution takes place. Thus, in the case of the planets and planetoids, one of the foci of each orbit coincides with the sun, while for the satellites, the coincidence is with the planet to which they belong. In all cases the form of the path is such as would be produced by attractive forces exerted mutually by all members of the solar system and varying in accordance with the Newtonian law of gravitation. In addition to the motion of revolution, the sun, planets, and some of the satellites at least, rotate on their axes with respect to the stars.

The planets are eight in number. In order from the sun they are: Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune. Their distances from the sun range from thirty-six million to nearly three thousand million miles. Their diameters vary from about three thousand to nearly ninety thousand miles. Nevertheless, comparatively speaking, they are small, for their collective mass is but little more than one one-thousandth that of the sun.

The planetoids, also known as small planets or asteroids, number six hundred or more, and relatively to the planets, are extremely small bodies—so small that they are all telescopic objects and many of them can be seen only with large and powerful instruments. Most of them are of comparatively recent discovery, and a considerable addition to the number already known is made each year as the result of new discoveries. With but few exceptions their paths lie between the orbits of Mars and Jupiter.

The only satellite requiring our attention is the moon. This revolves about the earth with a period of about one month, and rotates on its axis once during each revolution. Although one of the smaller bodies of the solar system it is, on account of its nearness, one of the most striking.

The solar and stellar systems are by no means coördinate parts of the universe. On the contrary, the former, vast as it is, is but an insignificant portion of the latter, for the sun is but a star, not very different on the average from the other stars whose total number is to be counted by hundreds of millions; and the space containing the entire solar system, including sun, planets, satellites, and planetoids, is incredibly small as compared with that occupied by the stellar system. To obtain a more definite notion of the relative size of the two systems consider the following illustration: Let the various bodies be represented by small spheres whose diameters and mutual distances exhibit the relative dimensions and distribution through space of the sun, planets, and stars. We shall thus have a rough model of the universe, and to make its dimensions more readily comprehensible let the scale be fixed by assuming that the sphere representing the sun is two feet in diameter. The corresponding diameters of the remaining spheres and their distances from the central body are shown by the following table.

OBJECT	DIAMETER	DISTANCE
Sun	2 feet	———
Mercury	0.08 inch	83 feet
Venus	0.21 inch	155 feet
Earth	0.22 inch	215 feet
Mars	0.12 inch	327 feet
Jupiter	2.42 inch	1116 feet
Saturn	2.02 inch	2048 feet
Uranus	0.97 inch	4118 feet
Neptune	0.91 inch	6450 feet
Nearest Star	Unknown	11000 miles

It will be seen that the distance of the outermost planet from the sun is represented in the model by about a mile and a quarter. On the same scale, the distance of the nearest star, the only one included in the table, is approximately equal to one-half the circumference of the earth. When it is remembered that this object is but one of perhaps two hundred million stars, the vast majority of which are probably at least one hundred times more distant, and further that each of these stars is a sun as our own sun, the very subordinate position of the solar system becomes strikingly apparent.

The fact that the sun is similar in size and chemical composition to millions of other stars at once raises the question as to whether they too are not provided with attendant systems of planets and satellites. A definite answer is wanting, although analogy suggests that such may well be the case. Bodies no larger than the planets and shining only by reflected light would be quite invisible, even in the most powerful telescopes, when situated at distances comparable with those separating us from the stars.

We do know, however, that in many instances two or more stars situated relatively near each other revolve about their common center of gravity thus forming binary or multiple systems. The discovery and study of these systems constitutes one of the most interesting and important lines of modern astronomical investigation.

The distances separating the various members of the solar system are such that the motions of the planets and planetoids with respect to the sun, and of the satellites relative to their primaries, produce rapid changes in their positions as seen from the earth. The stars are also in motion and the velocities involved are very large, amounting occasionally to a hundred miles or more per second of time, but to the observer on the earth, their relative positions remain sensibly unchanged. The distances of these objects are so great that it is only when the utmost refinement of observation is employed and the measures are continued for months and years, that any shift in position can be detected even for those which move most rapidly. With minor exceptions, the configuration of the constellations is the same as it was two thousand years ago when the observations upon which are based the earliest known record of star positions were made.

To the casual observer there is not a great deal of difference in the appearance of the stars and the planets. The greater size and luminosity of the former is offset by their greater distance. In ancient times the fundamental difference between them was not known, and they were distinguished only by the fact that the planets change their positions, while relatively to each other the stars are apparently fixed. In fact the word *planet* means literally, a moving or wandering star, while what appeared to the early observers as the distinguishing characteristic of the stars is shown by the frequent use of the expression *fixed stars.*

The nebulae are to be counted by the hundreds of thousands. They consist of widely extended masses of luminous gas, apparently of simple chemical composition. They are irregularly distributed throughout the heavens, and present the greatest imaginable diversity of form, structure, and brightness. Minute disc like objects, rings, double branched spirals, and voluminous masses of extraordinarily complex structure, some of which resemble closely the delicate high-lying clouds of our own atmosphere, are to be found among them. The brightest are barely visible to the unaided eye, while the faintest tax the powers of the largest modern telescopes. Their distances are of the same order of magnitude as those of the stars, and, indeed, there appears to be an intimate relation connecting these two classes of objects, for there is evidence indicating that the stars have been formed from the nebulae through some evolutionary process the details of which are as yet not fully understood.

The preceding paragraphs give the barest outline of the interpretation which astronomers have been led to place upon the phenomena of the heavens. The development of this conception of the structure of the universe forms the major part of the history of astronomy during the last four centuries. Many have contributed toward the elaboration of its details, but its more significant features are due to Copernicus, Kepler, and Newton.

Although the scheme outlined above is the only theory thus far formulated which satisfactorily accounts for the celestial phenomena in their more intricate relations, there is another conception of the universe, one far earlier in its historical origin, which also accounts for the more striking phenomena. This theory bears the name of the Alexandrian astronomer Ptolemy, and, as its central idea is immediately suggested by the most casual examination of the motions of the celestial bodies, we shall now turn to a consideration of these motions and the simple, elementary devices which can be used for their description.

2. **The apparent phenomena of the heavens.**—The observer who goes forth under the star-lit sky finds himself enclosed by a hemispherical vault of blue which meets in the distant horizon the seemingly flat earth upon which he stands. The surface of the vault is strewn with points of light of different brightness, whose number depends upon the transparency of the atmosphere and the brightness of the moon, but is never more than two or three thousand. A few hours observation shows that the positions of the points are slowly shifting in a peculiar and definite manner. Those in the east are rising from the horizon while those in the west are setting. Those in the northern heavens describe arcs of circles in a counter-clockwise direction about a common central point some distance above the horizon. Their distances from each other remain unchanged. The system moves as a whole.

The phenomenon can be described by assuming that each individual point is fixed to a spherical surface which rotates uniformly from east to west about an axis passing through the eye of the observer and the central point mentioned above. The surface to which the light-points seem attached is called the **Celestial Sphere.** Its radius is indefinitely great. Its period of rotation is one day, and the resulting motion of the celestial bodies is called the **Diurnal Motion** or **Diurnal Rotation.**

The daylight appearance of the heavens is not unlike that of the night except that the sun, moon, and occasionally Venus, are the only bodies to be seen in the celestial vault. They too seem to be carried along with the celestial sphere in its rotation, rising in the east, descending toward the west, and disappearing beneath the horizon only to rise again in the east; but if careful observations be made it will be seen that these bodies can not be thought of as attached to the surface of the sphere, a fact most easily verified in the case of the moon. Observations upon successive nights show that the position of this object changes with respect to the stars. A continuation of the observations will show that it apparently moves eastward over the surface of the sphere along a great circle at such a rate that an entire circuit is completed in about one month. A similar phenomenon in the case of the sun manifests itself by the fact that the time at which any given star rises does not remain the same, but occurs some four minutes earlier for each successive night. A star rising two hours after sunset on a given night will rise approximately $1^h\ 56^m$ after sunset on the following night. The average intervals for succeeding nights will be $1^h\ 52^m$, $1^h\ 48^m$, $1^h\ 44^m$, etc., respectively. That the stars rise *earlier* on successive nights shows that the motion of the sun over the sphere is toward the east. Its path is a great circle called the **Ecliptic.** Its motion in one day is approximately one degree, which corresponds to the daily change of four minutes in the time of rising of the stars. This amount varies somewhat, being greatest in January and least in July, but its average is such that a circuit of the sphere is completed in one year. This motion is called the **Annual Motion of the Sun.**

With careful attention it will be found that a few of the star-like points of light, half a dozen more or less, are exceptions to the general rule which rigidly fixes these objects to the surface of the celestial sphere. These are the planets, the wandering stars of the ancients. Their motions with respect to the stars are complex. They have a general progressive motion toward the east, but their paths are looped so that there are frequent changes in direction and temporary reversals of motion. Two of them, Mercury and Venus, never depart far from the sun, oscillating from one side to the other in paths which deviate but little from the ecliptic. The paths of the others also lie near the ecliptic, but the planets themselves are not confined to the neighborhood of the sun.

The sun, moon, and the planets therefore appear to move over the surface of the celestial sphere with respect to the stars, in paths which lie in or near the ecliptic. The direction of motion is opposite, in general, to that of the diurnal rotation. The various motions proceed quite independently. While the sun, moon, and planets move over the surface of the sphere, the sphere itself rotates on its axis with a uniform angular velocity.

These elementary facts are the basis upon which the theory of Ptolemy was developed. It assumes the earth, fixed in position, to be the central body of the universe. It supposes the sun, moon, and planets to revolve about the earth in paths which are either circular or the result of a combination of uniform circular motions; and regards the stars as attached to the surface of a sphere, which, concentric with the earth and enclosing the remaining members of the system, rotates from east to west, completing a revolution in one day.

3. Relation of the apparent phenomena to their interpretation.—The relation of the apparent phenomena to the conception of Ptolemy is obvious, and their connection with the scheme outlined in Section 1 is not difficult to trace. The celestial sphere is purely an optical phenomenon and has no real existence. The celestial bodies though differing greatly in distance are all so far from the observer that the eye fails to distinguish any difference in their distances. The blue background upon which they seem projected is due partly to reflection, and party to selective absorption of the light rays by the atmosphere surrounding the earth. As already explained, the stars are so distant that, barring a few exceptional cases, their individual motions produce no sensible variation in their relative positions, and, even for the exceptions, the changes are almost vanishingly small. On the other hand, the sun, planets, and satellites are relatively near, and their motions produce marked changes in their mutual distances and in their positions with respect to the stars. The annual motion of the sun in the ecliptic is but a reflection of the motion of the earth in its elliptical orbit about the sun. The monthly motion of the moon is a consequence of its revolution about the earth, and the complex motions of the planets are due, partly to their own revolutions about the sun, and partly to the rapidly shifting position of the observer. Finally, the diurnal rotation of the celestial sphere, which at first glance seems to carry with it all the celestial bodies, is but the result of the axial rotation of the earth.

In so far as the more obvious phenomena of the heavens are concerned there is no contradiction involved in either of the conceptions which have been devised for the description of their relations. That such is the case arises from the fact that we are dealing with a question concerning changes of relative distance and direction. Given two points, A and B, we can describe the fact that their distance apart, and the direction of the line joining them, are changing, in either of two ways. We may think of A as fixed and B moving, or we may think of B as fixed and A in motion. Both methods are correct, and each is capable of giving an accurate description of the change in relative distance and direction. So, in the case of the celestial bodies, we may describe the variation in their distances and directions, either by assuming the earth to be fixed with the remaining bodies in motion, or by choosing another body, the sun, as the fixed member of the system and describing the phenomena in terms of motions referred to it. The former method of procedure is the starting point for the system of Ptolemy, the latter, for that of Copernicus. Both methods are correct, and hence neither can give rise to contradiction so long as the problem remains one of motion.

Though two ways lie open before us, both leading to the same goal, the choice of route is by no means a matter of indifference, for one is much more direct than the other. For the discussion of many questions the conception of a fixed earth and rotating heavens affords a simpler method of treatment; but, when a detailed description of the motions of the planets and satellites is required, the Copernican system is the more useful by far, although the geocentric theory presents no formal contradiction unless we pass beyond the consideration of the phenomena as a case of relative motion, and attempt their explanation as the result of the action of forces and accelerations. If this be done, the conception which makes the earth the central body of the universe comes into open conflict with the fundamental principles of mechanics. With the heliocentric theory there is no such conflict, and herein lies the essence of the various so-called proofs of the correctness of the Copernican system.

The problems of practical astronomy are among those which can be more simply treated on the basis of the geocentric theory, and we might have proceeded to an immediate consideration of our subject from this primitive stand-point but for the importance of emphasizing the character of what we are about to do. For the sake of simplicity, we shall make use of ideas which are not universally applicable throughout the science of astronomy. We shall speak of a fixed earth and rotating heavens because it is convenient, and for our present purpose, precise; but, in so doing, it is important always to bear in mind the more elaborate scheme outlined above, and be ever ready to shift our view-point from the relatively simple, elementary conceptions which form a part of our daily experience, to the more majestic structure whose proportions and dimensions must ever be the delight and wonder of the human mind.

4. Relation of the problems of practical astronomy to the phenomena of the heavens.—The problems of practical astronomy with which we are concerned are the determination of latitude, azimuth, time, and longitude.

(a) The latitude of a point on the earth may be defined roughly as its angular distance from the equator. It can be shown that this is equal to the complement of the inclination of the rotation axis of the celestial sphere to the direction of the plumb line at the point considered. If the inclination of the axis to the plumb line can be determined, the latitude at once becomes known.

(b) The azimuth of a point is the angle included between the vertical plane containing the rotation axis of the celestial sphere and the vertical plane through the object. If the orientation of the vertical plane through the axis of the sphere can be found, the determination of the azimuth of the point becomes but a matter of instrumental manipulation.

(c) Time measurement is based upon the diurnal rotation of the earth, which appears to us in reflection as the diurnal rotation of the celestial sphere. The rotation of the celestial sphere can therefore be made the basis of time measurement. To determine the time at any instant, we have only to find the angle through which the sphere has rotated since some specified initial epoch.

(d) As will be seen later, the determination of the difference in longitude of two points is equivalent to finding the difference of their local times. The solution of the longitude problem therefore involves the application of the methods used for the derivation of time, together with some means of comparing the local times of the two places. The latter can be accomplished by purely mechanical means, quite independently of any astronomical phenomena, although such phenomena are occasionally used for the purpose.

In brief, therefore, the solution of these four fundamental problems can be connected directly with certain fundamental celestial phenomena. Both latitude and azimuth depend upon the position of the rotation axis of the celestial sphere, the former, upon its inclination to the direction of the plumb line, the latter, upon the orientation of the vertical plane passing through it; while the determination of time and longitude involve the position of the sphere as affected by diurnal rotation.

A word more, and we are immediately led to the detailed consideration of our subject: The solution of our problems requires a knowledge of the position of the axis of the celestial sphere and of the orientation of the sphere about that axis. We meet at the outset a difficulty in that the sphere and its axis have no objective existence. Since our observations and measurements must be upon things which have visible existence, the stars for example, we are forced to an indirect method of procedure. We must make our measurements upon the various celestial bodies and then, from the known location of these objects on the sphere, derive the position of the sphere and its axis. This raises at once the general question of coördinates and coördinate systems to which we now give our attention.

5. Coördinates and Coördinate Systems.—Position is a relative term. We cannot specify the position of any object without referring it, either explicitly or implicitly, to some other object whose location is assumed to be known. The designation of the position of a point on the surface of a sphere is most conveniently accomplished by a reference to two great circles that intersect at right angles. For example, the position of a point on the earth is fixed by referring it to the equator and some meridian as that of Greenwich or Washington. The angular distance of the point from the circles of reference are its coördinates—in this case, longitude and latitude.

Our first step, therefore, in the establishment of coördinate systems for the celestial sphere, is the definition of the points and circles of reference which will form the foundation for the various systems.

The **Direction of the Plumb Line**, or the **Direction due to Gravity,** produced indefinitely in both directions, pierces the celestial sphere above in the **Zenith**, and below, in the **Nadir**. The plane through the point of observation, perpendicular to the direction of the plumb line, is called the **Horizon Plane.** Produced indefinitely in all directions, it cuts the celestial sphere in a great circle called the **Horizon**. Since the radius of the celestial sphere is indefinitely great as compared with the radius of the earth, a plane through the center of the earth perpendicular to the direction of gravity will also cut the celestial sphere in the horizon. For many purposes it is more convenient to consider this plane as the horizon plane.

The celestial sphere is pierced by its axis of rotation in two points called the **North Celestial Pole** and the **South Celestial Pole**, or more briefly, the **North Pole** and the **South Pole,** respectively. It is evident from the relations between the phenomena and their interpretation traced in Section 3 that the axis of the celestial sphere must coincide with the earth's axis of rotation.

Great circles through the zenith and nadir are called **Vertical Circles.** Their planes are perpendicular to the horizon plane. The vertical circle passing through the celestial poles is called the **Celestial Meridian**, or simply, the **Meridian.** Its plane coincides with the plane of the terrestrial meridian through the point of observation.

The vertical circle intersecting the meridian at an angle of ninety degrees is called the **Prime Vertical.** The intersections of the meridian and prime vertical with the horizon are the cardinal points, **North, East, South,** and **West.**

Small circles parallel to the horizon are called **Circles of Altitude** or **Almucanters.**

Great circles through the poles of the celestial sphere are called **Hour Circles.**

The great circle equatorial to the poles of the celestial sphere is called the **Celestial Equator.** The plane of the celestial equator coincides with the plane of the terrestrial equator.

Small circles parallel to the celestial equator are called **Circles of Declination.**

The ecliptic, already defined as the great circle of the celestial sphere followed by the sun in its annual motion among the stars, is inclined to the celestial equator at an angle of about 23½ degrees. The points of intersection of the ecliptic and the celestial equator are the **Equinoxes, Vernal** and **Autumnal,** respectively. The **Vernal Equinox** is that point at which the sun in its annual motion passes from the south to the north side of the equator; the **Autumnal Equinox,** that at which it passes from the north to the south.

The points on the ecliptic midway between the equinoxes are called the **Solstices, Summer** and **Winter,** respectively. The **Summer Solstice** lies to the north of the celestial equator, the **Winter Solstice,** to the south.

The coördinate systems most frequently used in astronomy present certain features in common, and a clear understanding of the underlying principles will greatly aid in acquiring a knowledge of the various systems. At the basis of each system is a **Fundamental Great Circle.** Great circles perpendicular to this are called **Secondary Circles.** One of these, called the **Principal Secondary,** and the fundamental great circle, form the reference circles of the system.

The **Primary Coördinate** is measured along the fundamental great circle from the principal secondary to the secondary passing through the object to which the coördinates refer. The **Secondary Coördinate** is measured along the secondary passing through the object from the fundamental great circle to the object itself. The fundamental great circle and the principal secondary intersect in two points. The intersection from which the primary coördinate is measured, and the direction of measurement of both coördinates, must be specified.

In practical astronomy three systems of coördinates are required. The details are shown by the following table. The symbol used to designate each coördinate is written after its name in the table.

It is sometimes more convenient to use as secondary coördinate the distance of the object from one of the poles of the fundamental great circle. Thus in System I we shall frequently use the distance of an object from

COÖRDINATE SYSTEMS.

System	Fundamental Great Circle	Secondary Circles	Principal Secondary	Coördinates	
				Primary	Secondary
I	Horizon	Vertical Circles	Meridian	Azimuth = A + from South toward West	Altitude = h + from Horizon upward
II	Celestial Equator	Hour Circles	Hour Circle coinciding with Meridian	Hour Angle = t + from Meridian toward West	Declination = δ + from Equator toward North; — toward South
III	"	"	Hour Circle through the Vernal Equinox	Right Ascension = α + from Vernal Equinox toward East	" "

the zenith, its **Zenith Distance** = z, instead of the altitude. Similarly, in Systems II and III we shall occasionally find that an object's distance from the north celestial pole, its **North Polar Distance** = π, is more convenient than declination. Between these alternative coördinates we have the relations:

$$z = 90° - h \qquad (1)$$

$$\pi = 90° - \delta \qquad (2)$$

The details of the various systems are also shown graphically in Fig. 1, which represents an orthogonal projection of the celestial sphere upon the horizon plane. In this projection all vertical circles become straight lines. All circles inclined to the horizon at an angle other than 90° become ellipses. The horizon, and all circles parallel to the horizon plane, remain circles.

6. Characteristics of the Three Systems. Changes in the Coördinates.— Coördinates are used both for the location of objects on the sphere by actual observation, and as a means of stating positions predicted on the basis of the laws which describe the motions of the various celestial bodies. The practical astronomer and the engineer have occasion to use them in both ways. It is essential that there be a clear understanding of the relative advantages of the various systems, of the changes which may occur in the different coördinates, and of the relations of the systems to each other. We now proceed to a discussion of the first two of these points. The relations between the systems will be discussed in Chapter II.

None of the coördinates defined above is absolutely constant for any of the celestial bodies. The changes which occur arise as the result of:

(a) a change in the position of the object,
(b) a change in the position of the reference circles,
(c) a change in the position of the observer,
(d) a bending of the light rays by the atmosphere surrounding the earth.

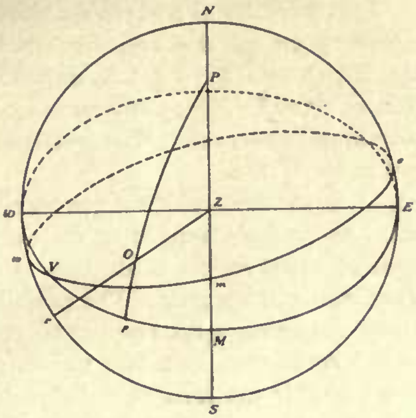

Fig. 1.

Point	Z	= Zenith		
Circle	$NESW$	= Horizon		
Point	P	= North Celestial Pole		
Line	NZS	= Celestial Meridian		
Line	WZE	= Prime Vertical		
Points	$N, E, S, W,$	= Cardinal Points		
Ellipse	WME	= Celestial Equator		
Ellipse	wme	= Ecliptic		
Point	V	= Vernal Equinox		
Point	O	= Any Celestial Object		
Line	ZOr	= Vertical Circle through O		
Arc	POp	= Hour Circle through O		
Arc Sr = Angle	SZO	= Azimuth of O = A	Coördinates System I	
Line	rO	= Altitude of O = h		
Line	ZO	= Zenith Distance of O = z		
Arc Mp = Angle	ZPO	= Hour Angle of O = t	Coördinates System II	
Arc	pO	= Declination of O = δ		
Arc	PO	= North Polar Distance of O = π		
Arc	Vp	= Right Ascension of O = α	Coördinates System III	
		δ and π same as in System II		

Any or all of these causes may enter to affect the position of an object, with the result that the number of possible variations with which we have to deal is considerable. In some instances, however, the variations are small quantities—so small that they can be disregarded in all but the most precise investigations. The small changes which cannot be neglected entirely are regarded as corrections, which, applied to the coördinates corresponding to a given position of the object, reference circles, and observer, give their values for some other position.

The bending of the light rays by the earth's atmosphere, a phenomenon known as **Refraction**, affects all of the coördinates but azimuth.[1] The amount of the refraction, which is always small, depends upon the conditions under which the object is observed. The allowance for its influence is therefore made by each individual observer. The method of determining its amount will be discussed in Section 8.

In the first system, the reference circles are fixed for any given point of observation. The azimuth and altitude of terrestrial objects are therefore constant, unless the point of observation is shifted. For celestial bodies, on the contrary, they are continuously varying. The positions of all such objects are rapidly and constantly changing with respect to the circles of reference, as a result of the diurnal rotation. For the nearer bodies, an additional complexity is introduced by their motions over the sphere and the changing position of the earth in its orbit. It appears, therefore, that azimuth and altitude are of special service in surveying and in geodetic operations, but that their range of advantageous application in connection with celestial bodies is limited, for not only are the azimuth and altitude of a celestial object constantly changing, but, for any given instant, their values are different for all points on the earth. But in spite of this disadvantage, altitude, at least, is of great importance. Its determination in the case of a celestial body affords convenient methods of solving two of the fundamental problems with which we are concerned, viz., latitude and time. Since the fundamental circle in the first system depends only upon the direction of the plumb line, the instrument required for the measurment of altitude is extremely simple, both in construction and use. In consequence, altitude is the most readily determined of all the various coördinates. The observational part of the determination of latitude and time is therefore frequently based upon measures of altitude, the final results being derived from the observed data by a process of coördinate transformation to be developed in Chapter II.

In the third system, the reference circles share in the diurnal rotation. Although not absolutely fixed on the sphere, their motions are so slow that the coördinates of objects, which, like the stars, are sensibly fixed, remain practically constant for considerable intervals of time. Right ascension and declination are therefore convenient for listing or cataloguing the positions of the stars. Catalogues of this sort are not only serviceable for long periods of time, but can also be used at all points on the earth. The latter circumstance renders right ascension and declination an advantageous means of expressing the positions of bodies not fixed on the sphere. For such objects we have only to replace the single pair of coördinates which suffices for a star, by a series giving the right ascension and declination for equi-distant intervals of time. Such a list of positions is called an **Ephemeris**. If the time intervals separating the successive epochs for which the coördinates are given be properly chosen, the position can be found for any intermediate

[1]The azimuth of objects near the horizon is also affected by refraction. The magnitude of the change in the coördinate is very small, however.

instant by a process of interpolation. The interval selected for the tabulation is determined by the rapidity and regularity with which the coördinates change. In the case of the sun, one day intervals are sufficient, but for the moon the positions must be given for each hour. For the more distant planets, whose motions are relatively slow, the interval can be increased to several days.

Collections of ephemerides of the sun, moon, and the planets, together with the right ascensions and declinations of the brighter stars, are published annually by the governments of the more important nations. That issued by our own is prepared in the Nautical Almanac Office at Washington, and bears the title "American Ephemeris and Nautical Almanac."

It is necessary to examine the character of the variations produced in the coördinates by the slow motion of the reference circles mentioned above. The mutual attractions of the sun, moon, and the planets produce small changes in the positions of the equator and ecliptic. The motion of the ecliptic is relatively unimportant. That of the equator is best understood by tracing the changes in position of the earth's axis of rotation. As the earth moves in its orbit, the axis does not remain absolutely parallel to a given initial position, but describes a conical surface. The change in the direction of the axis takes place very slowly, about 26000 years being required for it to return to its original position. During this interval the inclination of the equator to the ecliptic never deviates greatly from its mean value of about $23\frac{1}{2}°$. Consequently, the celestial pole appears to move over the sphere in a path closely approximating a circle with the pole of the ecliptic as center. The direction of the motion is counter-clockwise, and the radius of the circle equal to the inclination of the equator to the ecliptic. The actual motion of the pole is very complex; but its characteristic features are the progressive circular component already mentioned, and a transverse component which causes it to oscillate or nod back and forth with respect to the pole of the ecliptic. The result is a vibratory motion of the equator about a mean position called the **Mean Equator**, the mean equator itself slowly revolving about a line perpendicular to the plane of the ecliptic. The motion of the equator combined with that of the ecliptic produces an oscillation of the equinox about a mean position called the **Mean Vernal Equinox**, which, in turn, has a slow progressive motion toward the west. The resulting changes in the right ascension and declination are divided into two classes, called precession and nutation, respectively. **Precession** is that part of the change in the coördinates arising from the progressive westward motion of the mean vernal equinox, while **Nutation** is the result of the oscillatory or periodic motion of the true vernal equinox with respect to the mean equinox.

The amount of the precession and nutation depends upon the position of the star. For an object on the equator the maximum value of the precession in right ascension for one year is about forty-five seconds of arc or three seconds of time. For stars near the pole it is much larger, amounting in the case of Polaris, for example, to about 27^s. The annual precession in declination is relatively small, and does not exceed 20″ for any of the stars.

There remains to be considered the effect of the object's own motion and that of the observer. We have already seen how the changes arising from the motion of such objects as the sun, moon, and the planets can be expressed by means of an ephemeris giving the right ascension and declination for equi-distant intervals of time. For the stars the matter is much simpler. Their motions over the sphere are so slight as to be entirely inappreciable in the vast majority of cases, and for those in which the change cannot be disregarded, it is possible to assume that the motion is uniform and along the arc of a great circle. The change in one year is called the star's **Proper Motion.** If the right ascension and declination are given for any instant, t, and it is desired to find their values as affected by proper motion for any other instant t', it is only necessary to add to the given coördinates the products of the proper motion in right ascension and declination into the interval $t - t'$ expressed in years. The position of a star for a given initial epoch and its proper motion are therefore all that is required for the determination of its position at any other epoch, in so far as the position is dependent upon the star's own motion.

The motion of the observer may affect the position of a celestial object in two ways: First, the actual change in his position due to the diurnal and annual motions of the earth causes a change in the coördinates called **Parallactic Displacement.** Second, the fact that the observer is in motion *at the instant of observation* may produce an apparent change in the direction in which the object is seen, in the same way that the direction of the wind, as noted from a moving boat or train, appears different from that when the observer is at rest. The change thus produced is called **Aberration,** and is carefully to be distinguished from the parallactic displacement. Aberration depends only upon the observer's velocity, and not at all upon his position, except as position may determine the direction and magnitude of the motion. Parallactic displacement, on the contrary, depends on the distance over which the observer actually moves.

For the nearer bodies the parallactic displacement due to the earth's annual motion is large, and is included with the effect of the object's own motion in the ephemeris which expresses its positions. The variation arising from the rotation of the earth on its axis is far smaller, and can always be treated as a correction. In the case of the stars, the distances are so great that the maximum known parallactic displacement due to the earth's annual motion amounts to only three-quarters of a second of arc. For all but a few, a shift in the position of the earth from one side of its orbit to the other, a distance of more than 180,000,000 miles, reveals no measurable change in the coördinates. The displacement due to the earth's rotation is of course altogether inappreciable.

Parallactic displacement is usually called **Parallax,** and, when so spoken of, signifies specifically, the correction which must be applied to the observed coördinates of an object in order to reduce them to what they would be were the object seen from a standard position. For the stars, the standard position is the center of the sun; for all other bodies, the center of the earth.

Aberration is due to the fact that the velocity of the observer is a quantity of appreciable magnitude as compared with the velocity of light. For all stars not lying in the direction of the earth's orbital motion, the telescope must be inclined slightly in advance of the star's real position in order that its rays may pass centrally through both objective and eye-piece of the instrument. The star thus appears displaced in the direction of the observer's motion. The amount of the displacement is a maximum when the direction of the motion is at right angles to the direction of the star, and equal to zero when the two directions coincide. The rotation of the earth on its axis produces a similar displacement. The **Diurnal Aberration** is so minute, however, that it requires consideration only in the most refined observations.

The coördinates of the second system possess, to a certain degree, the properties of those of both Systems I and III. Hour angle, like azimuth and altitude, is a coördinate which varies continuously and rapidly, and is dependent on the position of the observer on the earth. The secondary coördinate, declination, is the same as in System III, and the remarks concerning it made above, apply with equal force here. The second system is of prime importance in the solution of the problems of practical astronomy, for it serves as an intermediate step in passing from System I to System III, or vice versa. It is also the basis for the construction of the equatorial mounting for telescopes, the form most commonly used in astronomical investigations.

7. Summary. Method of treating the corrections in practice.—It is to be remembered, therefore, that the azimuth and altitude of terrestrial objects are constant for a given point of observation, but change as the observer moves over the surface of the earth. For celestial objects they are not only different for each successive instant, but also, for the same instant, they are different for different points of observation. Right ascension and declination are sensibly the same for all points on the earth, and, in consequence, are used in the construction of catalogues and ephemerides. One pair of values serves to fix the position of a star for a long period of time. but for the sun, the moon, and the planets an ephemeris is required.

The corrections to which the coördinates are subject are proper motion, precession, nutation, annual aberration, diurnal aberration, parallax, stellar or planetary as the case may be, and refraction. Right ascension and declination are affected by all, but only planetary parallax, refraction, and diurnal aberration arise in practice in connection with azimuth and altitude, and of these three the last is usually negligible. In all cases these three are dependent upon local conditions, and consequently, their calculation and application are left to the observer. Since it is impracticable to include them in catalogue and ephemeris positions of right ascension and declination, there remains to be considered, as affecting such positions, proper motion, precession, nutation, annual aberration, and stellar parallax. The last is so rarely of significance in practical astronomy that it can be disregarded. As for the others, it is sometimes necessary to know their collective effect, and sometimes, the influence of the individual variations. It thus happens that we have different kinds of positions or places, known as mean place, true place, and apparent place.

The **mean place** of an object at any instant is its position referred to the mean equator and mean equinox of that instant. The mean place is affected by proper motion and precession.

The **true place** of an object at any instant is its position referred to the true equator and true equinox of that instant, that is, to the instantaneous positions of the actual equator and equinox. The true place is equal to the mean place plus the variation due to the nutation.

The **apparent place** of an object at any instant is equal to the true place at that instant plus the effect of annual aberration. It expresses the location of the object as it would appear to an observer situated at the center of the earth.

The positions to be found in star catalogues are mean places, and are referred to the mean equator and equinox for the beginning of some year, for example, 1855.0 or 1900.0. Such catalogues usually contain the data necessary for the determination of the precession corrections which must be applied to the coördinates in deriving the mean place for any other epoch. Modern catalogues also contain the value of the proper motion when appreciable. The nutation and annual aberration corrections are found from data given by the various annual ephemerides. The ephemerides themselves contain mean places for several hundred of the brighter stars; but the engineer is rarely concerned with these, or with the catalogue positions mentioned above, for apparent places are also given for the ephemeris stars, and these are all that he needs. The apparent right ascension and declination are given for each star for every ten days throughout the year. Apparent positions are also given by the ephemeris for the sun, the moon, and the planets, for suitably chosen intervals. Positions for all of these bodies for dates intermediate to the ephemeris epochs can be found by interpolation. With this arrangement, the special calculation of the various corrections necessary for the formation of apparent places is avoided entirely in the discussion of all ordinary observations. The observer must understand the origin and significance of all of the changes which occur in the coördinates, in order to use the ephemeris intelligently; but he has occasion to calculate specially only those which depend upon the local conditions affecting the observations, viz., diurnal aberration, parallax, and refraction. The first we disregard on account of its minuteness. There remains for the consideration of the engineer only refraction and parallax. The following is a brief statement of the methods by which their numerical values can be derived.

8. Refraction.—The velocity of light depends upon the density of the medium which it traverses. When a luminous disturbance passes from a medium of one density into that of another, the resulting change in velocity shifts the direction of the wave front, unless the direction of propagation is perpendicular to the surface separating the two media. Stated otherwise, a light ray passing from one medium into another of different density undergoes a change in direction, unless the direction of incidence is normal to the bounding surface. This change in direction is called **Refraction.** The incident ray, the refracted ray, and the normal to the bounding surface at the point of incidence lie in a plane. When the density of the second medium is greater than that of the first, the ray is bent toward the normal. When the conditions of density are reversed, the direction of bending is away from the normal.

The light rays from a celestial object which reach the eye of the observer must penetrate the atmosphere surrounding the earth. They pass from a region of zero density into one whose density gradually increases from the smallest conceivable amount to a maximum which occurs at the surface of the earth. The rays undergo a change in direction as indicated above. The effect is to increase the altitude of all celestial bodies, without sensibly changing their azimuth unless they are very near the horizon. For the case of two media of homogeneous density, the phenomenon of refraction is simple; but here, it is extremely complex and its amount difficult of determination. The course of the ray which reaches the observer is affected not only by its initial direction, but also by the refraction which it suffers at each successive point in its path through the atmosphere. The latter is determined by the density of the different strata, which, in turn, is a function of the altitude. This brings us to the most serious difficulty in the problem, for our knowledge of the constitution of the atmosphere, especially in its upper regions, is imperfect. To proceed, an assumption must be made concerning the nature of the relation connecting density and altitude. This, combined with the fundamental principles enunciated above, forms the basis of an elaborate mathematical discussion which results in an expression giving the refraction as a function of the zenith distance of the object, and the temperature of the air and the barometric pressure at the point of observation. This expression is complicated and cumbersome, disadvantages overcome, in a measure, by the reduction of its various parts to tabular form in accordance with a method devised by Bessel. With this arrangement, the determination of the refraction involves the interpolation and combination of a half dozen logarithms, more or less.

Various hypotheses concerning the relation between density and altitude have been made, each of which gives rise to a distinct theory of refraction, although the differences between the corresponding numerical results are slight. That generally used is due to Gylden. The tables based upon this theory are known as the Pulkova Refraction Tables, and can be found in the more comprehensive works on spherical and practical astronomy.

When the highest precision is desired these tables or their equivalent must be used, but for many purposes a simpler procedure will suffice. For example, the approximate expression,

$$r = \frac{983\,b}{460 + t} \tan z', \tag{3}$$

derived empirically from the results given by the theoretical development,[1] can be used for the calculation of the refraction, r, when the altitude is not less than 15°. In this expression, b is the barometer reading in inches; t, the temperature in degrees Fahrenheit; z, the observed or apparent zenith distance. The refraction is given in seconds of arc. The error of the result will rarely exceed one second.

For rough work the matter can be still further simplified by using mean values for b and t. For $b = 29.5$ inches, and $t = 50°$ Fahr. the coefficient of (3) is 57″, whence

$$r = 57'' \tan z'. \tag{4}$$

The values of r given by (4) can be derived from columns three and eight of Table I with either the apparent altitude or the apparent zenith distance as argument. For altitudes greater than 20° and normal atmospheric conditions, the error will seldom exceed a tenth of a minute of arc.

9. Parallax.—The parallax of an object is equal to the angle at the object subtended by the line joining the center of the earth and the point

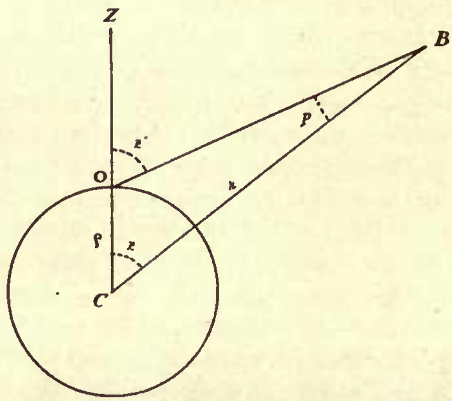

Fig. 2

of observation. Thus, in Fig. 2, the circle represents a section of the earth coinciding with the vertical plane through the object. C is the center of the earth, O the point of observation, Z the zenith, and B the object. The angles z' and z are the apparent and geocentric zenith distances, respectively. Their difference, which is equal to the angle p, is the parallax of B.

[1]This form was derived by Comstock, *Bulletin of the University of Wisconsin*, Science Series, v. 1, p. 60.

We have the relations

$$z = z' - p, \tag{5}$$

$$h = h' + p, \tag{6}$$

where h' and h are the apparent and geocentric altitudes, respectively. The effect of parallax, therefore, is to increase zenith distances and decrease altitudes,—just the opposite of that produced by refraction.

The parallax depends upon ρ the radius of the earth, r the distance of the object from the earth's center, and the zenith distance z' or z. From the triangle *OCB*

$$r \sin p = \rho \sin z'.$$

The angle p does not exceed a few seconds of arc for any celestial body excepting the moon. For this its maximum value is about 1°. We therefore write

$$p = \frac{\rho}{r} \sin z'. \tag{7}$$

The coefficient ρ / r, the value of the parallax when the body is the horizon, is called the **Horizontal Parallax.** Denoting its value by p_0 we have

$$p = p_0 \sin z'. \tag{8}$$

The value of p_0 varies with the distance of the object. It is tabulated in the *American Ephemeris* for the sun (p. 285), the moon (page IV of each month), and the planets (pp. 218-249). For the sun, however, the change in p_0 is so slight that we may use its mean value of 8″.8, whence

$$p = 8''.8 \sin z'. \tag{9}$$

The error of this expression never exceeds 0″.3. The values of p corresponding to (9) can be interpolated from columns four and nine of Table I.

For approximate work the solar parallax is conveniently combined with the mean refraction given by (4). The difference of the two corrections can be derived from the fifth and tenth columns of Table I with the apparent altitude or the apparent zenith distance as argument.

The preceding discussion assumes that the earth is a sphere. On this basis the parallax in azimuth is zero. Actually, the earth is spheroidal in form, whence it results that the radius, ρ, and consequently the angle *OBC*, do not, in general, coincide with the vertical plane through *B*; for the plumb line does not point toward the center of the earth, except at the poles and at points on the equator. The actual parallax in zenith distance is therefore slightly different from that given by (9), and in addition, there is a minute component affecting the azimuth. The influence of the spheroidal form of the earth is so slight, however, that it requires consideration only in the most precise investigations.

Finally, it should be remarked that the apparent zenith distance used for the calculation of the parallax is the observed zenith distance freed from refraction; that is, of the two corrections, refraction is to be applied first. The zenith distance thus corrected serves for the calculation of the parallax.

For the first system of coördinates, therefore, and the limits of precision here considered, the influence of both refraction and parallax is confined to the coördinate altitude, or its alternative, zenith distance. Hour angle, right ascension, and declination are all affected by both refraction and parallax, but, as these coördinates do not appear as observed quantities in the problems with which we are concerned, the development of the expressions which give the corresponding corrections is omitted.

TABLE I. MEAN REFRACTION AND SOLAR PARALLAX

Barometer, 29.5 in.; Thermometer, 50° Fahr.

h'	z'	r	p	$r-p$	h'	z'	r	p	$r-p$
15°	75°	3'.5	8".5	3'.4	40°	50°	1'.1	6".7	1'.0
20	70	2.6	8.3	2.5	50	40	0.8	5.7	0.7
25	65	2.0	8.0	1.9	60	30	0.6	4.4	0.5
30	60	1.6	7.6	1.5	70	20	0.4	3.0	0.3
35	55	1.3	7.2	1.2	80	10	0.2	1.5	0.1
40	50	1.1	6.7	1.0	90	0	0.0	0.0	0.0

The Refraction, r, and the Refraction—Solar Parallax, $r-p$, are to be subtracted from h', or added to z'.

The Solar Parallax, p, is to be added to h', or subtracted from z'.

CHAPTER II

FORMULÆ OF SPHERICAL TRIGONOMETRY—TRANSFORMATION OF COÖRDINATES—GENERAL DISCUSSION OF PROBLEMS.

10. The fundamental formulæ of spherical trigonometry.—Transformations of coördinates are of fundamental importance for the solution of most of the problems of spherical and practical astronomy. The relations between the different systems should therefore receive careful attention. The more complicated transformations require the solution of a spherical triangle, and, because of this fact, a brief exposition of the fundamental formulæ of spherical trigonometry is introduced at this point.

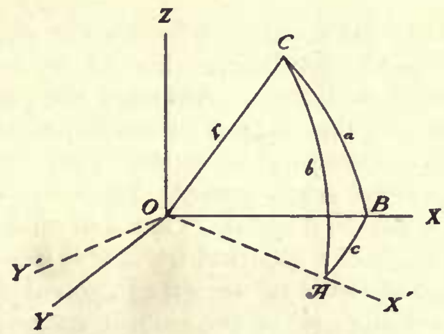

Fig. 3.

Let ABC, Fig. 3, be any spherical triangle. Denote its angles by A, B, and C; and its sides by a, b, and c. With the center of the sphere, O, as origin, construct a set of rectangular coördinate axes, XYZ, such that the XY plane contains the side c, and the X axis passes through the vertex B. Let the rectangular coördinates of the vertex C be x, y, and z. Their values in terms of the parts of the triangle and the radius of the sphere are

$$\begin{aligned} x &= r\cos a, \\ y &= r\sin a\,\cos B, \\ z &= r\sin a\,\sin B. \end{aligned} \qquad (10)$$

Construct a second set of axes, $X'Y'Z$, with the origin at O, the $X'Y'$ plane coinciding with the side c, and the X' axis passing through the vertex A. Let the coördinates of C referred to this system be x', y', and z'. We then have

$$\begin{aligned} x' &= \quad r\cos b, \\ y' &= -r\sin b\,\cos A, \\ z' &= \quad r\sin b\,\sin A. \end{aligned} \qquad (11)$$

The second set of rectangular axes can be derived from the first by rotating the first about the Z axis through the angle c. The coördinates of the first system can therefore be expressed in terms of those of the second by means of the relations

$$\begin{aligned} x &= x' \cos c - y' \sin c, \\ y &= x' \sin c + y' \cos c, \qquad\qquad (12) \\ z &= z'. \end{aligned}$$

Substituting into equations (12) the values of x, y, z, x', y', and z from (10) and (11), and dropping the common factor r, we obtain the desired relations

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \qquad (13)$$
$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \qquad (14)$$
$$\sin a \sin B = \sin b \sin A. \qquad (15)$$

These equations express relations between five of the six parts of the spherical triangle ABC, and are independent of the rectangular coördinate axes introduced for their derivation. Although the parts of the triangle in Fig. 3 are all less than 90°, the method of development and the results are general, and apply to all spherical triangles. These relations are the fundamental formulæ of spherical trigonometry. From them all other spherical trigonometry formulæ can be derived. They determine without ambiguity a side and an adjacent angle of a spherical triangle in terms of the two remaining sides and the angle included between them, provided the algebraic sign of the sine of the required side, or of the sine or cosine of the required angle, be known. Otherwise there will be two solutions.

Equations (13)–(15) are conveniently arranged as they stand if addition-subtraction logarithms are to be employed for their calculation. For use with the ordinary logarithmic tables, they should be transformed so as to reduce the addition and subtraction terms in the right members of (13) and (14) to single terms (*Num. Cal.* pp. 13 and 14).

Aside from the case covered by equations (13)–(15), two others occur in connection with the problems of practical astronomy, viz., that in which the given parts are two sides of a spherical triangle, and an angle opposite one of them, to find the third side; and that in which the three sides are given, to find one or more of the angles. The first of these can be solved for those cases which arise in astronomical practice by a simple transformation of (13), the details of which will be considered in connection with the determination of latitude. A solution for the third case can also be found by a rearrangement of the terms of (13). Thus,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \qquad (16)$$

Similar expressions for the angles B and C can be derived by a simple permutation of the letters in (16). Equation (16) affords a theoretically accurate solution of the problem; but, practically, the application of expressions of this form is limited on account of the necessity of determining the angles from their cosines. For numerical calculation it is important to have formulæ such that the angles A, B, and C can be interpolated from their tangents (*Num. Cal.* pp. 3 and 14). The desired relations can be derived by a transformation of (16), (*Chauvenet*, *Spherical Trigonometry*, §§ 12 and 16–18), giving

$$\tan^2 \tfrac{1}{2} A = \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}, \tag{17}$$

in which $s = \tfrac{1}{2}(a+b+c)$. Similar expressions for B and C can be derived by a permutation of the letters of (17). When the three angles of a spherical triangle are to be determined simultaneously, it is advantageous to introduce the auxiliary K, defined by the relation

$$K^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}. \tag{18}$$

Substituting (18) into (17), we find

$$\tan \tfrac{1}{2} A = \frac{K}{\sin(s-a)}. \tag{19}$$

The expressions $\tan \tfrac{1}{2}B$ and $\tan \tfrac{1}{2}C$ are similar in form.

Collecting results, the complete formulæ for the calculation of the three angles of a spherical triangle from the three sides are

$$s = \tfrac{1}{2}(a+b+c).$$

Form $s-a$, $s-b$, and $s-c$, and check by

$$(s-a)+(s-b)+(s-c) = s.$$

$$K^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s} \tag{20}$$

$$\tan \tfrac{1}{2} A = \frac{K}{\sin(s-a)}, \quad \tan \tfrac{1}{2} B = \frac{K}{\sin(s-b)}, \quad \tan \tfrac{1}{2} C = \frac{K}{\sin(s-c)},$$

$$\text{Check:} \quad \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C = \frac{K}{\sin s}.$$

Two solutions are possible. The ambiguity is removed if the quadrant of one of the half-angles of the triangle is known.

11. Relative positions of the reference circles of the three coördinate systems.—The transformation of the coördinates of one system into those of another requires a knowledge of the relative positions of the reference circles of the various systems.

In the case of Systems I and II the principal secondary circles coincide by definition. The fundamental great circles are inclined to each other at an angle which is constant and equal to the complement of the latitude of the place of observation. The proof of this statement can be derived from Fig. 4, which represents a section through the earth and the celestial sphere in the plane of the meridian of the point of observation, O. The outer circle represents the celestial meridian, and the inner, the terrestrial meridian of O, the latter being greatly exaggerated with respect to the former. Z and N are the zenith and the nadir; P and P', the poles of the celestial sphere; p and p', the poles of the earth; HH' and EE', the lines of intersection of the planes of horizon and equator, respectively, with the meridian plane. The plane of the celestial equator coincides with that of the terrestrial equator, which cuts the terrestrial meridian in ee'.

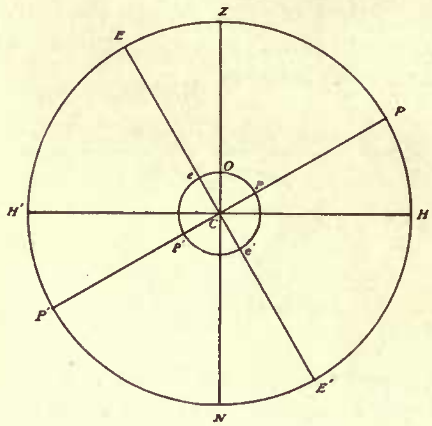

Fig. 4.

Now, by definition the arc eO measures the latitude, φ, of the point O But,

$$\text{Arc } EZ = \text{Arc } eO = \varphi, \qquad (21)$$

whence

$$H'E = 90^\circ - \varphi, \qquad (22)$$

which was to be proved. It thus appears that the second system can be derived from the first by rotating the first about an axis passing through the east and west points, through an angle equal to the co-latitude of the place.

It is to be noted, further, that

$$\text{Arc } ZP = 90^\circ - \varphi = \text{Co-latitude of } O, \qquad (23)$$

and

$$\text{Arc } HP = \varphi. \qquad (24)$$

From (21) and (24) it follows that **the latitude of any point on the earth is equal to the declination of the zenith of that point. It is also equal to the altitude of the pole as seen from the given point.**

Systems II and III have the same fundamental great circle, viz., the celestial equator. The principal secondary of the third system does not maintain a fixed position with respect to that of the first, but rotates uniformly in a clockwise direction as seen from the north side of the equator.

Let Fig. 5 represent an orthogonal projection of the celestial sphere upon the plane of the equator as seen from the North. P is the north celestial pole; M, the point where the meridian of O intersects the celestial equator; and V, the vernal equinox. The arc MBV therefore measures the instantaneous position of the principal secondary of the third system with respect to that of the first. This arc is equal to the hour angle of the vernal equinox, or the right ascension of the observer's meridian. It is called the Sidereal Time $= \theta$. We thus have the following important definition:

The sidereal time at any instant is equal to the hour angle of the vernal equinox at that instant. It is also equal to the right ascension of the observer's meridian at the instant considered.

It follows, therefore, that the third system can be derived from the second by rotating the second system about the axis of the celestial sphere through an angle equal to the sidereal time.

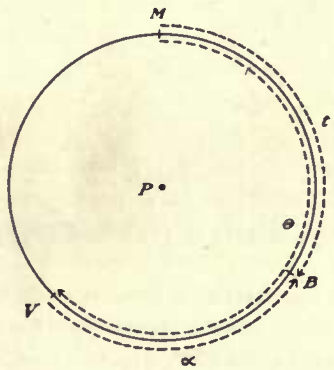

Fig. 5.

Finally, the third system can be derived from the first by rotating the first into the position of the second, and thence into the position of the third.

Briefly stated, the transformation of coördinates involves the determination of the changes arising in the coördinates as a result of a rotation of the various systems in the manner specified above. It is at once evident that the transformation of azimuth and altitude into hour angle and declination requires a knowledge of the latitude; of hour angle and declination into right ascension and declination, a knowledge of the sidereal time; while, to pass from azimuth and altitude to right ascension and declination, both latitude and sidereal time are required. It is scarcely necessary to add that the reverse transformations demand the same knowledge.

12. Transformation of azimuth and zenith distance into hour angle and declination.—The transformation requires the solution of the spherical triangle ZPO, Fig. 1, p. 11. The essential part of Fig. 1 is reproduced in Fig. 6 upon an enlarged scale. An inspection of the notation of p. 11 shows that the parts of the triangle ZPO can be designated as shown in Fig. 6.

Assuming the latitude, φ, to be known, it is seen that the transformation in question involves the determination of the side $\pi = 90° - \delta$ and the adjacent angle t in terms of the other two sides, $90° - \varphi$ and $z = 90° - h$, and the angle $180° - A$ included between them. Equations (13)–(15) are directly applicable, and it is only necessary to make the following assignment of parts:

$$\begin{aligned} a &= 90° - \delta, & A &= 180° - A, \\ b &= z, & B &= t, \\ c &= 90° - \varphi. \end{aligned} \qquad (25)$$

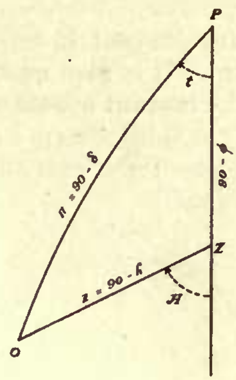

Fig. 6.

The substitution of (25) into (13), (14), and (15) gives

$$\sin\delta = \cos z \sin\varphi - \sin z \cos\varphi \cos A, \qquad (26)$$
$$\cos\delta \cos t = \cos z \cos\varphi + \sin z \sin\varphi \cos A, \qquad (27)$$
$$\cos\delta \sin t = \sin z \sin A. \qquad (28)$$

To adapt these formulæ for use with the ordinary logarithmic tables, the auxiliary quantities m and M, defined by

$$\begin{aligned} m \sin M &= \sin z \cos A, \\ m \cos M &= \cos z, \end{aligned}$$

are introduced (*Num. Cal.* p. 14).

Substituting these relations into (26) and (27) and collecting results, we have for the calculation

$$\begin{aligned} m \sin M &= \sin z \cos A, \\ m \cos M &= \cos z, \\ \cos\delta \sin t &= \sin z \sin A, \\ \cos\delta \cos t &= m \cos(\varphi - M), \\ \sin\delta &= m \sin(\varphi - M). \end{aligned} \qquad (29)$$

In formulæ (26)–(28) we have three equations for the determination of two unknown quantities, t and δ; in (29), five equations are given for the determination of the four unknowns, M, m, t, and δ. In both cases one more condition is available than is required for the theoretical solution of the problem, a point of great practical importance, as it affords a means of testing the accuracy of the numerical solution.

The order of solution is as follows: First, determine M and m from the first two equations. Whatever the values of z and A, there will always be two pairs of values of M and m satisfying these equations. For one, m will be positive; for the other, negative. It is immaterial, so far as the final values of t and δ are concerned, which of the two solutions we adopt. For simplicity, however, we assume that m is always positive. This makes the algebraic signs of $\sin M$ and $\cos M$ the same as those of the right-hand members of the first and second equations, respectively, of (29). The difference of the logarithms of the right-hand members of these equations equals $\log \tan M$, from which the angle M is determined, the quadrant being fixed by a consideration of the algebraic signs of any two of the three functions, $\sin M$, $\cos M$, and $\tan M$. M, $\log \sin M$, and $\log \cos M$ are interpolated with a single opening of the table. The difference of the last two must equal $\log \tan M$, which affords a partial check. The subtraction of $\log \sin M$ from $\log m \sin M$ gives $\log m$. The addition of this result to $\log \cos M$ must agree with the value of $\log m \cos M$ from the second of (29), which gives a second partial check. The values of M and m thus derived are to be substituted, along with z and A, into the right-hand members of the last three of (29) for the completion of the calculation. The left members of the third and fourth of (29) are of the same form as the first two, which makes it possible to determine t and $\cos \delta$ by an application of the process employed for finding m and M, care being taken to apply the checks at the points indicated above. The algebraic sign of $\cos \delta$ is necessarily positive, since δ must always lie between $+90°$ and $-90°$, which fixes the quadrant of t. It is to be noted that this limitation upon the sign of $\cos \delta$ removes the ambiguity existing in the solution of the general spherical triangle which was mentioned on p. 22. The hour angle, t, and $\log \cos \delta$ having been found, the next step is the determination of $\log \sin \delta$ from the last of (29). The values of $\log \cos \delta$ and $\log \sin \delta$ must correspond to the same angle. This affords a third partial check. The determination of δ and the application of the check can be accomplished in either of two ways: We may interpolate δ from the *smaller* of the two functions $\log \cos \delta$ and $\log \sin \delta$, and check by comparing the other function with the value interpolated from the tables with the calculated δ as argument; or we may interpolate δ from $\log \tan \delta$, which is found by subtracting $\log \cos \delta$ from $\log \sin \delta$. With the value of δ thus derived, $\log \sin \delta$ and $\log \cos \delta$ are interpolated from the table. The interpolated values must agree with those resulting from the last three equations of (29). The former method is shorter; the latter, more precise in the long run, although not necessarily so in any specific case. In practice, the first method is usually sufficient.

In applying the checks it is to be noted that the accumulated error of calculation (*Num. Cal.* pp. 4 and 12) may produce a disagreement of one, and in rare instances, of two units in the last place of decimals. Great care must be exercised with the algebraic signs of the trigonometric functions and in assigning the quadrants of the angles. Otherwise, an erroneous computation may apparently check. The check quantities must agree both in absolute magnitude and algebraic sign.

The calculation of t and δ from equations (26)–(28) with the aid of addition-subtraction logarithms is accomplished by an application of the method used for the solution of the last three of (29). The only differences which occur are to be found in the details of the combination of the quantities which enter into the right members of the two groups of equations.

Example 1. For a place of observation whose latitude is 38° 56′ 51″, the azimuth of an object is 97° 14′ 12″ and its zenith distance 62° 37′ 49″. Find the corresponding hour angle and declination.

The calculation for equations (26)–(28), using addition-subtraction logarithms, appears in the first column; that for equations (29), made with the ordinary tables, is in the second column. For the first, δ is derived from $\log \sin \delta$, which, in this case, is smaller than $\log \cos \delta$. In the calculation of (29), δ is determined from $\log \tan \delta$. The arguments for the check quantities, $\sin \delta$ and $\cos \delta$, need not ordinarily be written down. They are inserted here in order to illustrate the application of the control. The abbreviation *log* is not prefixed to the arguments, although the majority of the numbers appearing in the computation are logarithms. Its omission saves time and produces no confusion.

A	97° 14′ 12″
z	62 37 49
φ	38 56 51

$\sin A$	9.99653	$\sin A$	9.99653
$\sin z$	9.94844	$\sin z$	9.94844
$\cos A$	9.10026_n	$\cos A$	9.10026_n
$\sin \varphi$	9.79838	$m \sin M$	9.04870_n
$\cos \varphi$	9.89082	$m \cos M$	9.66250
$\cos z$	9.66250	$\tan M$	9.38620_n
$\cos z \sin \varphi$	9.46088	M	− 13°40′34″
$\sin z \cos \varphi \cos A$	8.93952_n	φ	38 56 51
A	9.47864	$\varphi - M$	52 37 25
B	0.11430	$\sin M$	9.37371_n
$\cos z$ co φ	9.55332	$\cos M$	9.98751
$\sin z \sin \varphi \cos A$	8.84708_n	$\log m$	9.67499
B	0.70624	$\sin (\varphi - M)$	9.90018
A	0.61112	$\cos (\varphi - M)$	9.78322
$\cos \delta \sin t$	9.94497	$\cos \delta \sin t$	9.94497
$\cos \delta \cos t$	9.45820	$\cos \delta \cos t$	9.45821
$\tan t$	0.48677	$\tan t$	0.48676
t	71°56′36″	t	71°56′35″
t	$4^h 47^m 46^s.4$	t	$4^h 47^m 46^s.3$
$\sin t$	9.97806	$\sin t$	9.97806
$\cos t$	9.49130	$\cos t$	9.49131
$\cos \delta$	9.96691	$\cos \delta$	9.96691
$\sin \delta$	9.57518	$\sin \delta$	9.57517
δ (from $\sin \delta$)	+ 22° 5′ 8″	$\tan \delta$	9.60826
$\cos \delta$	9.96660 Ck.	δ	+ 22° 5′ 5″
		$\sin \delta$	9.57517 Ck.
		$\cos \delta$	9.96691 Ck.

13. Transformation of hour angle and declination into azimuth and zenith distance.—The transformation can be effected by solving (29) in the reverse order to that followed in Section 12. It is better, however, to use equations of the same form as those appearing in the preceding section, thus reducing the two problems to the same type. As before, two sides and the included angle are given, to find the remaining side. With the following assignment of parts

$$\begin{aligned} a &= z, & A &= t, \\ b &= 90^\circ - \delta, & B &= 180^\circ - A, \\ c &= 90^\circ - \varphi, \end{aligned} \tag{30}$$

we find by substituting into (13), (14), and (15),

$$\cos z = \sin\delta \sin\varphi + \cos\delta \cos\varphi \cos t, \tag{31}$$
$$\sin z \cos A = -\sin\delta \cos\varphi + \cos\delta \sin\varphi \cos t, \tag{32}$$
$$\sin z \sin A = \cos\delta \sin t. \tag{33}$$

These are of the same general form as (26), (27), and (28). Applying the same principle as before, we derive

$$\begin{aligned} n \sin N &= \sin\delta, \\ n \cos N &= \cos\delta \cos t, \\ \sin z \sin A &= \cos\delta \sin t, \\ \sin z \cos A &= n \sin(\varphi - N), \\ \cos z &= n \cos(\varphi - N). \end{aligned} \tag{34}$$

The two groups (31)–(33) and (34) give the required transformation. The former can be used with addition-subtraction logarithms; the latter, with the ordinary tables. A comparison of these equations with groups (26)–(28) and (29) shows that the same arrangement of calculation can be used for both transformations. The unknowns are involved in the same manner in both cases, with the exception that the sine and cosine of z are interchanged in the left members of (31)–(34) as compared with the corresponding functions of δ in (26)–(29).

In the solution of (31)–(33) and (34), the quadrant of A is fixed by the fact that $\sin z$ is necessarily positive, since z is always included between 0° and $+180^\circ$. This eliminates the ambiguity attached to the solution of the general spherical triangle.

14. Transformation of hour angle into right ascension, and vice versa.—Since the coördinate declination is common to Systems II and III, the transformation of the coördinates of one of these systems into those of the other requires only a knowledge of the relation between hour angle and right ascension.

In Fig. 5, p. 25, let B be the intersection of the hour circle through any celestial body with the celestial equator. We then have by definition

Arc $MB = t =$ Hour angle of object,
Arc $VM = \alpha =$ Right ascension of object,
Arc $MVB = \theta =$ Sidereal time,

whence

$$\alpha = \theta - t, \tag{35}$$

$$t = \theta - \alpha. \tag{36}$$

Equations (35) and (36) express the required transformations. The same result can be derived from Fig. 1, p. 11, the point p, in this figure, corresponding to B in Fig. 5.

Example 2. In a place of observation whose latitude is 38° 58′ 53″, the hour angle of an object is $20^h 19^m 41^s.8$, and its declination − 8° 31′ 47″. Find the corresponding azimuth and zenith distance.

The calculation by equations (31)–(33) is in the first column; that by equations (34), in the second.

$$t = 20^h\ 19^m\ 41^s.8 = 304°\ 55'\ 27''$$
$$\delta = -8°\ 31'\ 47''$$
$$\varphi = 38\ \ 58\ \ 53$$

$\sin t$	9.91377_n	$\sin t$	9.91377_n
$\cos\delta$	9.99517	$\cos\delta$	9.99517
$\cos t$	9.75777	$\cos t$	9.75777
$\sin\varphi$	9.79870	$n \sin N$	9.17121_n
$\cos\varphi$	9.89061	$n \cos N$	9.75294
$\sin\delta$	9.17121_n	$\tan N$	9.41827_n
$\sin\delta \sin\varphi$	8.96991_n	N	− 14°40′ 50″
$\cos\delta\cos\varphi\cos t$	9.64355	φ	38 58 53
B	0.67364	$\varphi - N$	53 39 43
A	0.57016	$\sin N$	9.40386_n
$\sin\delta\cos\varphi$	9.06182_n	$\cos N$	9.98559
$\cos\delta\sin\varphi\cos t$	9.55164	$\log n$	9.76735
A	9.51018	$\sin(\varphi - N)$	9.90608
B	0.12180	$\cos(\varphi - N)$	9.77273
$\sin z \sin A$	9.90894_n	$\sin z \sin A$	9.90894_n
$\sin z \cos A$	9.67344	$\sin z \cos A$	9.67343
$\tan A$	0.23550_n	$\tan A$	0.23551_n
A	300° 10′ 31″	A	300° 10′ 29″
$\sin A$	9.93676_n	$\sin A$	9.93677_n
$\cos A$	9.70126	$\cos A$	9.70126
$\sin z$	9.97218	$\sin z$	9.97217
$\cos z$	9.54007	$\cos z$	9.54008
z (from $\cos z$)	69° 42′ 32″	z (from $\cos z$)	69° 42′ 30″
$\sin z$	9.97218 Ck.	$\sin z$	9.97217 Ck.

Example 3. What is the right ascension of an object whose hour angle is $17^h 21^m 34^s.6$, when the sidereal time is $21^h 14^m 52^s.8$?

By equation (35)

$$\begin{aligned} \theta &= 21^h 14^m 52^s.8 \\ t &= 17 \quad 21 \quad 34.6 \\ \alpha &= \;\;3 \quad 53 \quad 18.2, \text{ Ans.} \end{aligned}$$

Example 4. What is the hour angle of an object whose right ascension is $8^h 12^m 34^s.8$, when the sidereal time is $6^h 6^m 28^s.7$?

By equation (36)

$$\begin{aligned} \theta &= \;\;6^h \;\; 6^m 28^s.7 \\ \alpha &= \;\;8 \quad 12 \quad 34.8 \\ t &= 21 \quad 53 \quad 53.9, \text{ Ans.} \end{aligned}$$

15. Transformation of azimuth and altitude into right ascension and declination, or vice versa.—These transformations are effected by a combination of the results of Sections 12–14. For the direct transformation, determine t and δ by (26)–(28) or (29), and then α by (35). For the reverse calculate t by (36), and then A and z by (31)–(33) or (34).

Example 5. What is the right ascension of the object whose coördinates, at the sidereal time $17^h 21^m 16^s.4$, are those given in Example 1?

The hour angle found in the solution of Example 1 by equations (26)–(28) is $4^h 47^m 46^s.4$. This, combined with $\theta = 17^h 21^m 16^s.4$ in accordance with equation (35), gives for the required right ascension $12^h 33^m 30^s.0$.

Example 6. At a place whose latitude is $38° 38' 53''$, what are the azimuth and zenith distances of an object whose right ascension and declination are $9^h 27^m 14^s.2$ and $-8° 31' 47''$, respectively, the sidereal time being $5^h 46^m 56^s.0$?

By equation (36), $t = 20^h 19^m 41^s.8$. We have, further, $\delta = -8° 31' 47''$ and $\varphi = 38° 38' 53''$. These quantities are the same as those appearing in Example 2. The solution by equations (39) gave $A = 300° 10' 29''$, $z = 69° 42' 30''$.

16. Given the latitude of the place, and the declination and zenith distance of an object, to find its hour angle, azimuth, and parallactic angle.—We have given three sides of the spherical triangle *ZPO*, Fig. 6, p. 26, to find the three angles, the parallactic angle being the angle at the object. The parallactic angle is not used in engineering astronomy, although its value is frequently required in practical astronomy proper.

Equations (20) are directly applicable for the solution of the problem. Assigning the parts of the triangle as in (30), and, further, writing the angle $C = q =$ parallactic angle, we have for the calculation.

$$a = z, \quad b = 90° - \delta, \quad c = 90° - \varphi,$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$\text{Check:} \quad (s-a) + (s-b) + (s-c) = s,$$
$$K^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}, \tag{37}$$
$$\tan \tfrac{1}{2} t = \frac{K}{\sin(s-a)}, \quad \cot \tfrac{1}{2} A = \frac{K}{\sin(s-b)}, \quad \tan \tfrac{1}{2} q = \frac{K}{\sin(s-c)}.$$
$$\text{Check:} \quad \tan \tfrac{1}{2} t \cot \tfrac{1}{2} A \tan \tfrac{1}{2} q = \frac{K}{\sin s}$$

Object $\left\{ \begin{matrix} \text{west} \\ \text{east} \end{matrix} \right\}$ of meridian, $\tfrac{1}{2} t$, $\tfrac{1}{2} A$, $\tfrac{1}{2} q$ in $\left\{ \begin{matrix} \text{first} \\ \text{second} \end{matrix} \right\}$ quadrant.

In engineering astronomy the determination of the hour angle, t, is usually all that is required. For this case it is simpler to use equation (17). The formulæ are

$$a = z, \quad b = 90° - \delta, \quad c = 90° - \varphi,$$
$$s = \tfrac{1}{2}(a + b + c).$$
$$\text{Check:} \quad (s - a) + (s - b) + (s - c) = s, \tag{38}$$
$$\tan^2 \tfrac{1}{2} t = \frac{\sin(s - b)\sin(s - c)}{\sin s \sin(s - a)}$$

where $\tfrac{1}{2} t$ is to be taken in the first or second quadrant according as the object is west or east of the meridian at the time of observation.

For those cases in which the object is more than two and one-half or three hours from the meridian, equation (16) written in the form

$$\cos t = \frac{\cos z - \sin \delta \sin \varphi}{\cos \delta \cos \varphi}, \tag{39}$$

will usually give satisfactory results. In any case, (39) affords a valuable control upon the value of t given by (38). The numerator of the right member of (39) is readily calculated by means of addition-subtraction logarithms.

17. Application of transformation formulæ to the determination of latitude, azimuth, and time.—It was shown in Section 4 that the solution of the fundamental problems of practical astronomy requires the determination of the position of the axis of the celestial sphere and the orientation of the sphere as affected by the diurnal rotation. In practice this is accomplished indirectly by observing the positions of various celestial bodies with respect to the horizon, the observed data being combined with the known position of the bodies on the sphere for the determination of the position of the sphere itself. The means for effecting the coördinate transformation hereby implied are to be found in the formulæ of Sections 12–16.

Although the most advantageous determination of latitude, azimuth, and time requires a modification of these formulæ, it is, nevertheless, easy to see that the solution of the various problems is within our grasp, and that the

Example 7. For a place whose latitude is 38°56′51″, find the hour angle, azimuth, and parallactic angle of an object east of the meridian whose declination and zenith distance are — 8° 16′ 14″ and 54° 16′ 12″, respectively.

Equations (37) are used for the solution, which is given below in the column on the left. If only the hour angle were required, equations (38) or (39) would be used. As an illustration of the application of these formulæ, the problem is also solved on this assumption. The first ten lines of the computation for (38), being the same as that for (37), are omitted. The remainder of the calculation for (38) occupies the upper part of the right-hand column. The solution by (39) is in the lower part of this column. The object is rather too near the meridian for the satisfactory use of equation (39), although it happens that the resulting value of the hour angle agrees well with that from (37) and (38).

δ	—8° 16′ 14″	sin $(s-b)$	8.78890
φ	38 56 51	sin $(s-c)$	9.88892
$z=a$	54 16 12	cosec $(s-a)$	0.13218
b	98 16 14	cosec s	0.00927
c	51 3 9	$\tan^2 \frac{1}{2} t$	8.81927
$2s$	203 35 35	$\tan \frac{1}{2} t$	9.40964_n
s	101 47 48 Ck.	$\frac{1}{2} t$	165° 35′ 46″
$s-a$	47 31 36	t	331 11 32
$s-b$	3 31 34		
$s-c$	50 44 39		
sin $(s-a)$	9.86782		
sin $(s-b)$	8.78890		
sin $(s-c)$	9.88892		
cosec s	0.00927		
log K^2	8.55491		
log K	9.27746_n	sin δ	9.15790_n
$\tan \frac{1}{2} t$	9.40964_n	sin φ	9.79838
$\cot \frac{1}{2} A$	0.48856_n	cos δ	9.99546
$\tan \frac{1}{2} q$	9.38854_n	cos φ	9.89082
$\tan \frac{1}{2} t \cot \frac{1}{2} A \tan \frac{1}{2} q$	9.28674_n	cos z	9.76639
K cosec s	9.28673_n Ck.	sin δ sin φ	8.95628_n
$\frac{1}{2} t$	165° 35′ 46″	A	9.18989
$\frac{1}{2} A$	162 0 48	B	0.06252
$\frac{1}{2} q$	166 15 10	cos z — sin δ sin φ	9.82891
t	331 11 32	cos δ cos φ	9.88628
A	324 1 36	cos t	9.94263
q	332 30 20	t	331° 11′ 34″

adaptation of the equations to any special case is only a matter of detail. Consider for a moment either equations (26)–(28) or (31)–(33). Both groups involve the five quantities A, z, t, δ, and φ; but, since $t = \theta - \alpha$, we may regard them as functions of the six quantities A, z, α, δ, φ, and θ. If, therefore, the zenith distance of a star of known right ascension and declination be measured, either group will enable us, theoretically at least, to determine φ, A, and θ—the latitude, the azimuth of the star, and the sidereal time. The azimuth of the star being known, the azimuth of any other object, a distant terrestrial mark, for example, can be found by applying to the calculated position of the star the difference in azimuth of the star and the mark. The latter can be observed directly with any instrument adapted for the measurement of horizontal angles. Further, the sidereal time, as will be shown in Chapter III, bears an intimate relation to all of the other kinds of time, so that, if the sidereal time has been found, the determination of the others becomes but a matter of calculation.

Practically, such a solution would be complicated. It is simpler to determine φ, A, and θ separately, assuming for the calculation of each that one or both of the others are known.

For example, equation (31) is a function of z, δ, φ, and $t = \theta - a$. Let it be assumed that the zenith distance of a star of known right ascension and declination has been measured and that the time of observation has been noted. The substitution of the resulting data into (31) leads to the determination of the only remaining unknown, namely, the latitude, φ.

Again, the elimination of z from (32) and (33) gives an expression for A as a function for φ, δ, and $t = \theta - a$. Let it be assumed that φ and θ are known. The azimuth of a star of known right ascension and declination can therefore be calculated. The calculated azimuth applied to the observed difference in azimuth of star and mark gives the azimuth of the mark.

Finally, equations (38) and (39) express the hour angle, t, as a function of z, φ, and δ. If the zenith distance of a star of known right ascension and declination be measured in a place of known latitude, the hour angle can be calculated. Equation (35), in the form $\theta = t + a$, then gives the sidereal time of observation.

The solutions thus outlined require, for the determination of latitude, a knowledge of the time; for the determination of time, a knowledge of the latitude; and, for azimuth, both time and latitude. For the first two, time and latitude, it might appear that the methods proposed are fallacious. If each is required for the determination of the other, how can either ever be determined? The explanation is to be found in the fact that the formulæ can be arranged in such a way that an approximate value for either of these quantities suffices for the determination of a relatively precise value of the other. Thus, a mere guess as to the time will lead to a relatively accurate value of the latitude, which, in turn, can be used for the determination of a more precise value of the time. The process can be repeated as many times as may be necessary to secure the desired degree of precision. The principle involved in the procedure thus outlined is called the **Method of Successive Approximations.** In numerical investigations it is of great importance. The method amounts, practically, to replacing a single complex process by a series, consisting of repetitions of some relatively simple operation. Ordinarily, the success of the method depends upon the number of repetitions or approximations which must be made in order to arrive at the desired result. If the convergence is rapid, so that one or two approximations suffice, the saving in time and labor as compared with the direct solution is frequently very great. Indeed, in some instances, the method of successive approximations is the only method of procedure, the direct solution being impossible as a result of the complexity of the relation connecting the various quantities involved.

The general method of procedure for the solution of the problems of latitude, time, and azimuth has been outlined. There remains the formulation of the details. But, before proceeding to a detailed development, we must consider the subject of time in its theoretical aspects—the different kinds of time, their definition and their relations. Chapter III will be devoted to this question. We must also consider the various astronomical instruments that find application in engineering astronomy—their characteristics and the conditions under which they are employed, since the nature of the data obtained through their use will influence the arrangement of the solutions. Chapter IV is therefore devoted to a discussion of various astronomical instruments.

In arranging the details of the methods for the determination of latitude, time, and azimuth, it is to be remembered that the various problems are not merely to be solved, but they are to be solved with a definite degree of precision, and with a minimum expenditure of labor. This requirement renders the question one of some complexity, for the precision required may vary within wide limits. For many purposes approximate results will suffice, and it is then desirable to sacrifice accuracy and thus reduce the labor involved. On the other hand, in astronomical work of the highest precision, no means should be overlooked which can in any way contribute toward an elimination or reduction of the errors of observation and calculation.

The problems with which we have to deal therefore present themselves under the most diverse conditions, and, if an intelligent arrangement of the methods is to be accomplished, one must constantly bear in mind the results which will be established in the two following chapters, as well as those already obtained in the discussion of the principles of numerical calculation.

CHAPTER III

TIME AND TIME TRANSFORMATION

18. **The basis of time measurement.**—The rotation of the earth is the basis for the measurement of time. Since motion is relative, we must specify the object to which the rotation is referred. By referring to different objects, it is obvious that we may have several different kinds of time. Actually, the rotation of the earth is referred to three different things: the apparent, or true, sun, a fictitious object called the mean sun, and the vernal equinox. In practice, however, we turn the matter about and take the apparent diurnal rotations of these objects with reference to the meridian of the observer, considered to be fixed, as the basis of time measurement. We have, accordingly, three kinds of time: **Apparent, or True, Solar Time, Mean Solar Time,** and **Sidereal Time.**

19. **Apparent, or True, Solar Time**=A.S.T.—The apparent, or true, solar time at any instant is equal to the hour angle of the apparent, or true, sun at that instant.

The interval between two successive transits of the apparent, or true, sun across the same meridian is called an **Apparent, or True, Solar Day**=A. S. D.

The instant of transit of the apparent sun is called **Apparent Noon**=A. N.

In astronomical practice the apparent solar day begins at apparent noon. It is subdivided into 24 hours, which are counted continuously from 0 to 24. The earth revolves about the sun in an elliptical orbit, the sun itself occupying one of the foci of the ellipse. The earth's motion is such that the radius vector connecting it with the sun sweeps over equal areas in equal times. Since the distance of the earth from the sun varies, it follows that the angular velocity of the earth in its orbit is variable. Hence, the angular motion of the sun along the ecliptic, which is but a reflection of the earth's orbital motion, is also variable. The projection into the equator of the motion along the ecliptic is likewise variable, not only because the ecliptical motion is variable, but also on account of the fact that the angle of projection changes, being 0 degrees at the solstices, and about 23½ degrees at the equinoxes. Apparent solar time is not, therefore, a uniformly varying quantity, nor are apparent solar days of the same length.

The adoption of such a time system for the regulation of the affairs of everyday life would bring with it many inconveniences, the first of which would be the impossibility of constructing a timepiece capable of following accurately the irregular variations of apparent solar time. On this account there has been devised a uniformly varying time, based upon the motion of a fictitious body called the mean sun.

20. **Mean Solar Time**=M. S. T.—The mean sun is an imaginary body supposed to move with a constant angular velocity eastward along the equator, such that it completes a circuit of the sphere in the same time as the apparent, or true, sun. Further, the mean sun is so chosen that its right ascension differs as little as possible, on the average, from that of the true sun.

The **Mean Solar Time** at any instant is equal to the hour angle of the mean sun at that instant.

The interval between two successive transits of the mean sun across the same meridian is called a **Mean Solar Day**=M. S. D.

The instant of transit of the mean sun is called **Mean Noon**=M. N.

Mean solar time is a uniformly varying quantity and all mean solar days are of the same length. Mean solar time is the time indicated by watches and clocks, generally, throughout the civilized world, and the mean solar day is the standard unit for the measurement of time.

In astronomical practice the mean solar day begins at mean noon. It is subdivided into 24 hours which are numbered continuously from 0 to 24. The astronomical date therefore changes at noon. But since a change of date during the daylight hours would be inconvenient and confusing for the affairs of every-day life, the **Calendar Date,** or **Civil Date,** is supposed to change 12 hours before the transit of the mean sun, *i.e.* at the midnight preceding the astronomical change of date. Further, in most countries, the hours of the civil mean solar day are not numbered continuously from 0 to 24, but from 0 to 12, and then again from 0 to 12, the letters A. M. or P. M. being affixed to the time in order to avoid ambiguity. For example the civil date 1907, Oct. 8, 10^h A. M., is equivalent to the astronomical date, 1907, Oct. 7, 22^h. The astronomical day Oct. 8 did not begin until the mean sun was on the meridian on Oct. 8 of the calendar.

From the manner of definition, it is evident that at any instant the mean solar time for different places not on the same meridian is different. If each place were to attempt to regulate its affairs in accordance with its own local mean solar time, confusion would arise, especially in connection with railway traffic. To avoid this difficulty all points within certain limits of longitude use the time of the same meridian. The meridians selected for this purpose are all an exact multiple of 15 degrees from the meridian of Greenwich, with the result that all timepieces referred to them indicate at any instant the same number of minutes and seconds, and differ among themselves, and from the local mean solar time of the meridian of Greenwich, by an exact number of hours. The system thus defined is called **Standard Time.**

Although, theoretically, all points within 7½ degrees of longitude of a standard meridian use the local mean solar time of that meridian, actually, the boundaries separating adjacent regions whose standard times differ by one hour are quite irregular.

The standard meridians for the United States are 75, 90, 105, and 120 degrees west of Greenwich. The corresponding standard times are **Eastern, Central, Mountain,** and **Pacific.** These are slow as compared with Greenwich mean solar time by 5, 6, 7, and 8 hours, respectively.

21. Sidereal Time.—The sidereal time at any instant is equal to the hour angle of the true vernal equinox at that instant. (See p. 25.)

The interval between two successive transits of the true vernal equinox across the same meridian is called a **Sidereal Day**=S. D.

The instant of transit of the true vernal equinox is called **Sidereal Noon**=S. N.

Since the precessional and nutational motions of the true equinox are not uniform, sidereal time is not, strictly speaking, a uniformly varying quantity, but practically it may be considered as such, for the variations in the motion of the equinox take place so slowly that, for the purposes of observational astronomy, all sidereal days are of the same length.

The importance of sidereal time in the transformation of the coördinates of the second system into those of the third, and vice versa, has already been shown in Sections 11 and 14. It also plays an important role in the determination of time generally, for sidereal time is more easily determined than either apparent or mean solar time.

The usual order of procedure in time determination is as follows: Every observatory possesses at least one sidereal timepiece whose error is determined by observations on stars. The true sidereal time thus obtained is transformed into mean solar time by calculation, and used for the correction of the mean solar timepieces of the observatory. Certain observatories, in particular the United States Naval Observatory at Washington, and the Lick Observatory at Mt. Hamilton in California, send out daily over the wires of the various telegraph companies, series of time signals which indicate accurately the instant of mean noon. These signals reach every part of the country, and serve for the regulation of watches and clocks generally.

22. The Tropical Year.—Several different kinds of years are employed in astronomy. The most important are the tropical and the Julian. The **Tropical Year** is the interval between two successive passages of the mean sun through the mean vernal equinox. Its length is 365.2422 M. S. D. During this interval the mean sun makes one circuit of the celestial sphere from equinox to equinox again, in a direction opposite to that of the rotation of the sphere itself, whence it follows that during a tropical year the equinox must complete 366.2422 revolutions with respect to the observer's meridian. We therefore have the important relation:

$$\text{One Tropical Year} = 365.2422 \text{ M. S. D.} = 366.2422 \text{ S. D.} \qquad (40)$$

In accordance with a suggestion due to Bessel, the tropical year begins at the instant when the mean right ascension of the mean sun plus the constant part of the annual aberration is equal to 280° or $18^h\ 40^m$. The symbol for this instant is formed by affixing a decimal point and a zero to the corresponding year number; thus for 1909, the beginning of the tropical year is indicated by 1909.0. This epoch is independent of the position of the observer on the earth and does not, in general, coincide with the beginning of the calendar year, although the difference between the two never exceeds a fractional part of a day.

23. The Calendar.—For chronological purposes the use of a year involving fractional parts of a day would be inconvenient. That actually used has its origin in a decree promulgated by Julius Caesar in 45 B. C. which ordered that the calendar year should consist of 365 days for three years in succession, these to be followed by a fourth of 366 days. The extra day of the fourth year was introduced by counting twice the sixth day before the calends of March in the Roman system. In consequence such years were long distinguished by the designation bissextile, although they are now called **Leap Years.** The years of 365 days are **Common Years.** With this arrangement the average length of the calendar year was 365¼ days. This period is called a **Julian Year,** and the calendar based upon it, the **Julian Calendar.**

The difference between the Julian and the tropical years is about 11^m. In order to avoid the gradual displacement of the calendar dates with respect to the seasons resulting from the accumulation of this difference, a slight modification in the method of counting leap years was introduced in 1582 by Pope Gregory XIII. The accumulated difference amounts approximately to three days in 400 years, and, as the Julian year is longer than the tropical, the Julian calendar falls behind the seasons by this amount. Gregory therefore ordered that the century years, all of which are leap years under the Julian rule, should not be counted as such unless the year numbers are exactly divisible by 400. At the same time it was ordered that 10 days should be dropped from the calendar in order to bring the date of the passage of the sun through the vernal equinox back to the 21st of March, where it was at the time of the Council of Nice in 325 A. D. The Julian system thus modified is called the **Gregorian Calendar.** The revised rule for the determination of leap years is as follows: *All years whose numbers are exactly divisible by four are leap years, excepting the century years. These are leap years only when exactly divisible by four hundred. All other years are common years.* The average length of the Gregorian calendar year differs from that of the tropical year by only 0.0003 day or 26^s. In the modern system the extra day in leap years appears as the 29th of February.

The Gregorian calendar was soon adopted by all Roman Catholic countries and by England in 1752. Russia and Greece and other countries under the dominion of the Eastern or Greek Church, still use the Julian Calendar, which, at present, differs from the Gregorian by 13 days.

24. Given the local time at any point, to find the corresponding local time at any other point.—From the definitions of apparent solar, mean solar, and sidereal time, it follows that at any instant the difference between two local times is equal to the angular distance between the celestial meridians to which the times are referred. But this is equal to the angular distance between the geographical meridians of the two places, *i.e.* their difference of longitude.

Let T_e = the time of the eastern place,
T_w = the time of the western place,
L = longitude difference of the two places,

We then have the relations:

$$T_e = T_w + L$$
$$T_w = T_e - L, \qquad (41)$$

Equations (41) are true whether the times be apparent solar, mean solar, or sidereal.

Example 8. Given, Columbia mean solar time $12^h\ 14^m\ 16^s.41$, find the corresponding Greenwich mean solar time.

$$\begin{aligned} T_W &= 12^h\ 14^m\ 16^s.41 \\ L &= 6\quad 9\quad 18.33 \\ T_e &= 18\quad 23\quad 34.74\quad \textit{Ans.} \end{aligned}$$

Example 9. Given, Greenwich mean solar time 1907, Oct. 6 $3^h\ 14^m\ 21^s$, find the corresponding Washington mean solar time.

$$\begin{aligned} T_e &= 1907,\ \text{Oct. } 6\quad 3^h\ 14^m\ 21^s \\ L &= \qquad\qquad\quad 5\quad 8\quad 16 \\ T_W &= 1907,\ \text{Oct. } 5\quad 22\quad 6\quad 5\quad \textit{Ans.} \end{aligned}$$

Example 10. Given, central standard time 1907, Oct. 12 $6^h\ 18^m\ 0^s$ A.M., find the corresponding Greenwich mean solar time, astronomical and civil.

$$\begin{aligned} T_W &= 1907,\ \text{Oct. } 12\quad 6^h\ 18^m\ 0^s\ \text{A.M.} \\ &= \qquad\quad \text{Oct. } 11\quad 18\quad 18\quad 0\ \text{astronomical} \\ L &= \qquad\qquad\qquad 6\quad 0\quad 0 \\ T_e &= 1907,\ \text{Oct. } 12\quad 0\quad 18\quad 0\ \text{astronomical} \\ &= \qquad\quad \text{Oct. } 12\quad 0\quad 18\quad 0\ \text{P.M. civil} \end{aligned} \Bigg\}\ \textit{Ans.}$$

Example 11. Given, central standard time 1907, Oct. 11 $0^h\ 3^m\ 16^s.18$ P.M. find the corresponding Columbia mean solar lime, civil and astronomical.

$$\begin{aligned} T_e &= 1907,\ \text{Oct. } 11\quad 0^h\ 3^m\ 16^s.18\ \text{P.M.} \\ L &= \qquad\qquad\qquad\quad 9\quad 18.33 \\ T_W &= 1907,\ \text{Oct. } 11\quad 11\quad 53\quad 57.85\ \text{A.M. civil} \\ &= \qquad\quad \text{Oct. } 10\quad 23\quad 53\quad 57.85\ \text{astronomical} \end{aligned} \Bigg\}\ \textit{Ans.}$$

25. Given the apparent solar time at any place, to find the corresponding mean solar time, and vice versa.—From equation (36), $t = \theta - \alpha$, and the definitions of mean solar and apparent solar time, we find

$$\text{M. S. T.} = \theta - \text{R. A. of M. S.},$$
$$\text{A. S. T.} = \theta - \text{R. A. of A. S.}$$

whence

$$\text{M. S. T.} - \text{A. S. T.} = \text{R. A. of A. S.} - \text{R. A. of M. S.}$$

The difference

$$E = \text{M. S. T.} - \text{A. S. T.} \qquad (42)$$

is called the **Equation of Time.** The equation of time varies irregularly throughout the year, its maximum absolute value being about 16^m. It is sometimes positive, and sometimes negative, since the right ascension of the apparent sun is sometimes smaller and sometimes greater than that of the mean sun. The right ascension of the apparent sun is calculated from the known orbital motion of the earth. The right ascension of the mean sun is known from its manner of definition. This data suffices for the calculation of E, whose values are tabulated in the various astronomical ephemerides. In the *American Ephemeris* they are given for instants of Greenwich apparent noon on page I for each month, and for Greenwich mean noon, on page II. The former page is used when apparent time is converted into the corresponding mean solar time, and the latter when apparent solar time is to be found from a given mean solar time. The algebraic sign of E is not given in the *American Ephemeris,* but the column containing its values is headed by a precept which indicates whether it is to be added to or subtracted from the given time. Values of E for times other than Greenwich apparent noon and Greenwich mean noon must be obtained by interpolation. This operation is facilitated by the use of the hourly change in E printed in the columns headed "Difference for 1 Hour," which immediately follow those containing the equation of time. If the time to be converted refers to a meridian other than that of Greenwich, the corresponding Greenwich time must be calculated before the interpolation is made. Note that for each date the difference of the right ascension of the apparent, or true, sun in column two of page II, and the right ascension of the mean sun in the last column of the same page, is equal to the corresponding value of E, in accordance with the definition.

Example 12. Given, Greenwich apparent solar time 1907, Oct. 15 $2^h\ 6^m\ 12^s.06$, find the corresponding Greenwich mean solar time.

E for Gr. A. N. 1907, Oct. 15		13^m	$56^s.75$	(Eph. p. 164)
Change in E during $2^h\ 6^m\ 12^s$			+ 1.20	
E (to be subtracted from A. S. T.)		13	57.95	
Gr. A. S. T. 1907, Oct. 15	2	6	12.06	
Gr. M. S. T. 1907, Oct. 15	1	52	14.11	*Ans.*

Example 13. Given, Greenwich mean solar time 1907, Oct. 15 $1^h\ 52^m\ 14^s.11$, find the corresponding Greenwich apparant solar time.

E for Gr. M. N. 1907, Oct. 15		13^m	$56^s.88$	(Eph. p. 165)
Change in E during $1^h\ 52^m\ 14^s$			+ 1.07	
E (to be added to M. S. T.)		13	57.95	
Gr. M. S. T. 1907, Oct. 15	1	52	14.11	
Gr. A. S. T. 1907, Oct. 15	2	6	12.06	*Ans.*

Example 14. Given, central standard time 1907, Oct. 20 $11^h\ 18^m\ 12^s.2$ A.M., find the corresponding Columbia apparent solar time.

C. S. T.	1907, Oct. 20	11^h	18^m	$12^s.2$	A.M.
L			9	18.3	
Columbia M. S. T.	1907, Oct. 19	23	8	53.9	astronomical
Gr. M. S. T.	1907, Oct, 20	5	18	12.2	
E for Gr. M. N.	1907, Oct. 20		14	58.63	(Eph. p. 165)
Change in E during $5^h\ 18^m\ 12^s$				+ 2.39	
E (to be added to Columbia M. S. T.)			15	1.0	
Columbia A. S. T.	1907, Oct. 19	23	23	54.9	*Ans.*

Example 15. Given Columbia apparent solar time 1907, Oct. 19 $23^h 23^m 54^s.9$, find the corresponding central standard time.

Columbia A. S. T.	1907, Oct. 19	23^h	23^m	$54^s.9$		
L		6	9	18.3		
Gr. A. S. T.	1907, Oct. 20	5	33	13.2		
E for Gr. A. N. 1907, Oct. 20			14	58.52		
Change in E during $5^h 33^m 13^s$				+ 2.50		
E (to be sub. from Columbia A. S. T.)			15	1.0		
Columbia M. S. T.	1907, Oct. 19	23	8	53.9		
L			9	18.3		
C. S. T.	1907, Oct. 20	11	18	12.2	A.M.	*Ans.*

26. Relation between the values of a time interval expressed in mean solar and sidereal units. Equation (40) is the fundamental relation connecting the units of mean solar and sidereal time. If we let

I_s = the value of any interval I in mean solar units,
I_m = the value of I in sidereal units,

we find from (40)

$$I_s = I_m + \frac{I_m}{365.2422} \qquad (43)$$

$$I_m = I_s - \frac{I_s}{366.2422} \qquad (44)$$

Writing

$$\text{III} = \frac{1}{365.2422} \qquad \text{II} = \frac{1}{366.2422}$$

(43) and (44) become

$$I_s = I_m + \text{III} I_m \qquad (45)$$
$$I_m = I_s - \text{II} I_s \qquad (46)$$

Assuming $I_m = 24^h$ we find from (45)

$$24^h\ 0^m\ 0^s.000 \text{ M. S.} = 24^h\ 3^m\ 56^s.555 \text{ Sid.}$$

Similarly, by supposing $I_s = 24^h$ we obtain from (46)

$$24^h\ 0^m\ 0^s.000 \text{ Sid.} = 23^h\ 56^m\ 4^s.091 \text{ M. S.}$$

Hence

$$\left.\begin{array}{l}\text{Gain of } \theta \text{ on M. S. T. in 1 M. S. D.} = \text{III}24^h = 236^s.555 \\ \text{Gain of } \theta \text{ on M. S. T. in 1 S. D.} = \text{II}24 = 235.909\end{array}\right\} \qquad (47)$$

and further

$$\begin{aligned} &\text{Gain of } \theta \text{ on M. S. T. in 1 M. S. hour} = \text{III}1^h = 9^s.8565 \\ &\text{Gain of } \theta \text{ on M. S. T. in 1 \quad S. hour} = \text{II}1 \quad = 9.8296 \end{aligned} \qquad (48)$$

For many purposes these expressions may be replaced by the following approximate relations:

$$\begin{aligned} \text{III}24^h &= 4^m(1 - 1/70), & \text{Error} &= 0^s.016 \\ \text{II}24 &= 4\ (1 - 1/60), & \text{Error} &= 0.081 \\ \text{III}1 &= 10^s(1 - 1/70), & \text{Error} &= 0.0006 \\ \text{II}1 &= 10\ (1 - 1/60), & \text{Error} &= 0.0037 \end{aligned} \qquad (49)$$

Equations (45) and (46) may be used for the conversion of the value of a time interval expressed in mean solar units into its corresponding value in sidereal units, and vice versa. The calculations are most conveniently made by Tables II and III printed at the end of the *American Ephemeris.* Table II contains the numerical values of $\text{II}I_s$, while Table III gives those of $\text{III}I_m$, the arguments being the values of I_s and I_m, respectively. It will be observed that the first factors of $\text{II}I_s$ and $\text{III}I_m$ indicate the table, and the second the argument which is to be used for the interpolation.

In case tables are not available the conversion can be based upon equations (47) or (48), or more simply, upon (49), provided the highest precision is not required.

Example 16. Given the mean solar interval $16^h\ 18^m\ 21^s.20$, find the equivalent sidereal interval.

By Eq. (45)

$$\begin{aligned} I_m &= 16^h\ 18^m\ 21^s.20 \\ \text{III}I_m &= \ \ 2\ \ 40.72 \quad \text{(Eph. Table III)} \\ I_s &= 16\ \ 21\ \ \ 1.92 \quad \textit{Ans.} \end{aligned}$$

The calculation of $\text{III}I_m$ by the third of (49) is as follows:

$$\begin{aligned} I_m = 16^h.306 \qquad 10^s I_m &= 163^s.06 \\ 1/70 \times 10^s I_m &= 2.33 \\ \text{III}I_m &= 160.73 = 2^m\ 40^s.73 \end{aligned}$$

The value thus found differs only $0^s.01$ from that derived from Table III of the Ephemeris.

Example 17. Given the sidereal interval $20^h\ 28^m\ 42^s.17$, find the equivalent mean solar interval.

By Eq. (46)

$$\begin{aligned} I_s &= 20^h\ 28^m\ 42^s.17 \\ \text{II}I_s &= \ \ 3\ \ 21.29 \quad \text{(Eph. Table II)} \\ I_m &= 20\ \ 25\ \ 20.88 \quad \textit{Ans.} \end{aligned}$$

The calculation of $\text{II}I_s$ by the last of (49) is as follows:

$$\begin{aligned} I_s = 20^h.478 \qquad 10^s I_s &= 204^s.78 \\ 1/60 \times 10^s I_s &= 3.41 \\ \text{II}I_s &= 201.37 = 3^m\ 21^s.37 \end{aligned}$$

27. Relation between mean solar time and the corresponding sidereal time.—In Section 14 it was shown that the relation connecting the hour angle of an object with the sidereal time is

$$t = \theta - \alpha$$

where α represents the right ascension of the object. Applying this equation to the mean sun, we find

$$M = \theta - R \qquad (50)$$

in which R represents the right ascension of the mean sun, and M its hour angle. The latter, however, is equal by definition to the mean solar time. Equation (50) therefore expresses a relation between mean solar time and the corresponding sidereal time, which can be made the basis for the conversion of the one into the other. The transformation requires a knowledge of R, the right ascension of the mean sun, at the instant to which the given time refers. We now turn our attention to a consideration of the methods which are available for the determination of this quantity.

28. The right ascension of the mean sun and its determination.—It is shown in works on theoretical astronomy that the right ascension of the mean sun at any instant of Greenwich mean time is given by the expression

$$\begin{aligned} R_G = 18^h\ 38^m\ 45^s.836 &+ (236^s.555 \times 365.25)t \\ &+ 0^s.0000093t^2 \\ &+ \text{nutation in right ascension,} \end{aligned} \qquad (51)$$

in which t is reckoned in Julian years from the epoch 1900, Jan. $0^d\ 0^h$ Gr. M. T. It thus appears that the increase in the right ascension of the mean sun is not strictly proportional to the increase in the time. This, in connection with equation (50), shows that sidereal time is not a uniformly varying quantity, a fact already indicated in Section 21. The nutation in right ascension oscillates between limits which are approximately $+1^s$ and -1^s with a period of about 19 years. Its change in one day is therefore very small, and, as the same is true of the term involving t^2 in (51), it follows that the increase in the right ascension of the mean sun in one mean solar day is sensibly $236.^s555$. From equation (50) it is seen that the gain of sidereal on mean solar time during any interval is equal to the increase in R during that interval; and, indeed, we have exact numerical agreement between the change in the latter for one mean solar day, as given by equation (51), and the gain of the former during the same period as shown by the first of (47). From this it follows that the methods given in Section 26, including Tables II and III of the *Ephemeris* and the approximate relations (49), can equally well be applied to the determination of the increase in R, provided only that the interval for which the change is to be calculated is small enough to render the variations in the last two terms of (51) negligible.

To facilitate the solution of problems in which R is required, its precise numerical values are tabulated in the various astronomical ephemerides for every day in the year. In the *American Ephemeris* they are given for the instant of Greenwich mean noon, and are to be found in the last column of page II for each month. If these tabular values be represented by R_0, and if R_L represent the right ascension of the mean sun at the instant of mean noon for a point whose longitude west of Greenwich is L, it follows from the preceding paragraph that

$$R_L = R_0 + \text{III}L, \qquad (52)$$

for L is equal to the time interval separating mean noon of the place from the preceding Greenwich mean noon. Further, the value of R at any mean time, M, at a point whose longitude west of Greenwich is L is given by

$$R = R_L + \text{III}M, \qquad (53)$$

or

$$R = R_0 + \text{III}L + \text{III}M. \qquad (54)$$

Equations (52) and (53), or their equivalent, (54), suffice for the determination of R at any instant at any place when the value of R_0 for the preceding mean noon is known. For a given place the term IIIL is a constant. Its value can be calculated once for all, and can then be added mentally to the value of R_0 as the latter is taken from the *Ephemeris*. The quantity IIIM may be derived from Table III of the Ephemeris with M as argument.

If an *Ephemeris* is not available the values of R can still be found; approximately at least, by the use of Tables II-IV, page 46. The first of these contains the values of R_0 computed from (51) for the date Jan. 0 for each of the years 1907-1920. Denoting these by R_{00} and neglecting the variations in the last two terms of (51) we have for Greenwich mean noon of any other date

$$R_0 = R_{00} + \text{III}D \qquad (55)$$

where D indicates the number of mean solar days that have elapsed since the preceding Jan. 0. Substituting (55) into (54),

$$R = R_{00} + \text{III}L + \text{III}(D + M). \qquad (56)$$

The value of D may be obtained from Table III by adding the day of the month to the tabular number standing opposite the name of the month in question. M is conveniently expressed in decimals of a day by means of Table IV. The value thus found is to be combined with D. If the precise value of III, viz., $236.^s555$, be used, the uncertainty in R derived from (56) will be only that arising from the neglect of the variation in the last terms of (51). If care be taken to count D from the *nearest* Jan. 0 the error will never exceed $0.^s3$ or

TABLE II

RIGHT ASCENSION OF THE MEAN SUN FOR THE EPOCH JAN. 0^d 0^h GR. M. T.

Year	R_{oo}	Year	R_{oo}
1907	18^h 36^m $0^s.47$	1914	18^h 37^m $13^s.62$
1908	35 3.04	1915	36 16.64
1909	38 2.28	1916	35 19.63
1910	37 5.09	1917	38 19 07
1911	36 7.99	1918	37 21.85
1912	35 10.98	1919	36 24.51
1913	38 10.57	1920	35 28.05

TABLE III

NUMBER OF DAY IN YEAR

Date	D	
	Common Year	Leap Year
Jan. 0	0	0
Feb. 0	31	31
Mar. 0	59	60
Apr. 0	90	91
May 0	120	121
June 0	151	152
July 0	181 — 184	182 — 184
Aug. 0	212 — 153	213 — 153
Sept. 0	243 — 122	244 — 122
Oct. 0	273 — 92	274 — 92
Nov. 0	304 — 61	305 — 61
Dec. 0	334 — 31	335 — 31

PRECEPT: Add the day of the month to the tabular value corresponding to the given month. The use of the negative values gives the day number from the *following* Jan. 0.

TABLE IV

HOURS AND MINUTES INTO DECIMALS OF A DAY

Hour	Decimals of a Day	Min.	Decimals of a Day	Min.	Decimals of a Day
1	0.042	1	0.001	10	0.007
2	0.083	2	1	20	0.014
3	0.125	3	2	30	0.021
4	0.167	4	3	40	0.028
5	0.208	5	3	50	0.035
6	0.250	6	4	60	0.042
7	0.292	7	5		
8	0.333	8	6		
9	0.375	9	6		
10	0.417	10	0.007		
11	0.458				
12	0.500				

PRECEPT: When the given hour is greater than 12, drop 12^h from the argument and add 0^d500 to the result given by the table. Thus, for $17^h\,28^m$ enter the table with the argument $5^h\,28^m$, giving 0^d228, whence $17^h\,28^m = 0^d228 + 0^d500 = 0^d728$.

$0^s.4$. This requires that for $D > 183^d$ the negative value of Table III be employed, together with the value of R_{00} for the *following* Jan. 0.

If a somewhat greater uncertainty is permissible, the result may be more expeditiously found by using $4^m(1-1/70)$ for III. If D be reckoned from the nearest Jan. 0 as above, the corresponding error will not exceed 3^s.

Example 18. Find the right ascension of the mean sun for the epoch 1907, June 16 $8^h\ 21^m\ 14^s.00$ Columbia M. S. T.

By Equation (54)

	$R_0 = 5^h\ 34^m\ 25^s.10$	(Eph. p. 93)
$L = 6^h\ 9^m\ 18^s.33$	$\mathrm{III}L = \quad 1\ \ 0.67$	(Eph. Table III)
$M = 8\ \ 21\ \ 14.00$	$\mathrm{III}M = \quad 1\ \ 22.34$	(Eph. Table III)
	$R = 5\ \ 36\ \ 48.11$	*Ans.*

By Equation (56)

$(D+M) = 167^d348$ (Tables III and IV)	$R_{00}(1907) = 18^h\ 36^m\ 0^s.5$	(Table II)
$4^m(D+M) = 669^m.392$	$\mathrm{III}L = \quad 1\ \ 0.7$	
$1/70 \times 4^m(D+M) = 9^m.563$	$\mathrm{III}(D+M) = 10\ \ 59\ \ 49.7$	
	$R = 5\ \ 36\ \ 51$	*Ans.*

Example 19. Find the right ascension of the mean sun for the epoch 1909, Sept. 21 $19^h\ 26^m\ 24^s$ Columbia M. S. T.

$D + M = -101^d + 0^d810$ (Tables III and IV)	$R_{00}(1910) = 18^h\ 37^m\ 5^s.1$	(Table II)
$= -100^d190$	$\mathrm{III}L = \quad 1\ \ 0.7$	
$4^m(D+M) = -400^m.760$	$\mathrm{III}(D+M) = -6\ \ 35\ \ 2.1$	
$1/70 \times 4^m(D+M) = -\ \ 5.725$	$R = 12\ \ 3\ \ 4$	*Ans.*

The precise value given by (54) is $12^h\ 3^m\ 5^s.21$.

29. Given the mean solar time at any instant to find the corresponding sidereal time.—From equation (50) we find

$$\theta = M + R \tag{57}$$

Introducing the value of R from (53) we have

$$\theta = M + R_L + \mathrm{III}M, \tag{58}$$

where

$$R_L = R_0 + \mathrm{III}L. \tag{59}$$

Equations (59) and (58) solve the problem.

Equation (58) may be interpreted as follows: R_L is the right ascension of the mean sun at the preceding mean noon for a place in longitude L west of Greenwich. It is therefore also equal to the hour angle of the vernal equinox at that instant, *i.e.* to the sidereal time of the preceding mean noon at the place considered. Now M is the mean time interval since preceding mean noon, and by (45) $M + \mathrm{III}M$ is the equivalent sidereal interval. The right member of (58) therefore expresses the sum of the sidereal time of the preceding mean noon and the number of sidereal hours, minutes, and seconds that have elapsed since noon. In other words it is the sidereal time corresponding to the mean time, M, as indicated by the equation.

In case the *Ephemeris* is not at hand, R may be obtained from (56) and substituted into (57) for the determination of θ. The uncertainty in the sidereal time thus found will be the same as that of R derived from (56).

Oftentimes a rough approximation for θ is all that is required. In such cases the following, designed for use at the meridian of Columbia, is useful:

$$\theta = 18^h\ 37^m.7 + M + 4^m(1 - 1/70)(D + M). \qquad (60)$$

The first term in the right member of this formula is the average value of R_{oo} plus the constant term IIIL, which for Columbia may be taken equal to 1^m. The expression can be adapted for use at any other meridian by introducing the appropriate value of IIIL. The maximum error in the value of θ derived from (60) is $1.^m7$.

Example 20. Given Columbia mean solar time $16^h\ 27^m\ 32^s.17$ on 1909, Nov. 16, find the corresponding sidereal time.

By equations (58) and (59)

$$\begin{aligned} M &= 16^h\ 27^m\ 32^s.17 \\ R_o &= 15\ \ 39\ \ 39.98 \\ \text{III}L &= \quad\ \ 1\ \ \ 0.67 \\ \text{III}M &= \quad\ \ 2\ \ 42.23 \\ \theta &= \ \ 8\ \ 10\ \ 55.05 \quad \textit{Ans.} \end{aligned}$$

By equations (56) and (57)

$$\begin{aligned} D + M &= -45^d + 0^d686 \\ &= -44^d314 \\ 4^m(D+M) &= -177^m.256 \\ 1/70 \times 4^m(D+M) &= -\quad 2.532 \end{aligned}$$

$$\begin{aligned} M &= 16^h\ 27^m\ 32^s.2 \\ R_{oo} &= 18\ \ 37\ \ \ 5.1 \\ \text{III}L &= \quad\ \ 1\ \ \ 0.7 \\ \text{III}(D+M) &= -2\ \ 54\ \ 43.4 \\ \theta &= \ \ 8\ \ 10\ \ 55 \quad \textit{Ans.} \end{aligned}$$

By equation (60)

$$\begin{aligned} & 18^h\ 37^m.7 \\ M &= 16\ \ 27.5 \\ 4^m(1 - 1/70)(D+M) &= -2\ \ 54.7 \\ \theta &= \ \ 8\ \ 10.5 \quad \textit{Ans.} \end{aligned}$$

30. Given the sidereal time at any instant to find the corresponding mean solar time.—We make use of equation (50), viz.

$$M = \theta - R$$

Substituting as in Section 29 we have

$$M = \theta - R_L - \text{III}M$$

or

$$(1 + \text{III})M = \theta - R_L \qquad (61)$$

Multiplying equations (45) and (46), member by member, and dropping the common factor I_m I_s we find

$$(1 + \text{III})(1 - \text{II}) = 1$$

Combining this with (61) we find

$$M = \theta - R_L - \text{II}(\theta - R_L), \tag{62}$$

where, as before,

$$R_L = R_0 + \text{III}L \tag{63}$$

Equations (63) and (62) solve the problem.

Equation (62) is susceptible of an interpretation similar to that given (58) in the preceding section. Since θ is the given sidereal time, and R_L the sidereal time of the preceding mean noon, $\theta—R_L$ is the sidereal interval that has elapsed since noon. To find the equivalent mean time interval we must, in accordance with equation (46), subtract from $\theta—R_L$ the quantity $\text{II}(\theta—R_L)$. The right member of (62) therefore expresses the number of mean solar hours, minutes, and seconds that have elapsed since the preceding mean noon, *i.e.* the mean solar time corresponding to the given θ.

Example 21. Given, 1908, May 12, Columbia sidereal time $1^h 7^m 19^s.27$, find the corresponding central standard time.

By equations (62) and (63)

$\theta =$	1^h	7^m	$19^s.28$	
$R_L =$	3	20	25.46	
$\theta - R_L =$	21	46	53.82	
$\text{II}(\theta - R_L) =$		3	34.10	(Eph. Table II)
$M =$	21	43	19.72	
$L =$		9	18.33	
C. S. T. =	9	52	38.05	A.M. May 13. *Ans.*

CHAPTER IV

INSTRUMENTS AND THEIR USE

31. Instruments used by the engineer.—The instruments employed by the engineer for the determination of latitude, time and azimuth are the watch or chronometer, the artificial horizon, and the engineer's transit or the sextant. The following pages give a brief account of the theory of these instruments and a statement of the methods to be followed in using them.

The use of both the engineer's transit and the sextant presupposes an understanding of the vernier. In consequence, the construction and theory of this attachment is treated separately before the discussion of the transit and sextant is undertaken.

TIMEPIECES

32. Historical.—Contrivances for the measurement of time have been used since the beginning of civilization, but it was not until the end of the sixteenth century that they reached the degree of perfection which made them of service in astronomical observations. The pendulum seems first to have been used as a means of governing the motion of a clock by Bürgi of the observatory of Landgrave William IV at Cassel about 1580, though it is not certain that the principle employed was that involved in the modern method of regulation. However this may be, the method now used was certainly suggested by Galileo about 1637; but Galileo was then near the end of his life, blind and enfeebled, and it was not until some years later that his idea found material realization in a clock constructed by his son Vincenzio. It remained for Huygens, however, the Dutch physicist and astronomer, to rediscover the principle, and in 1657 give it an application that attracted general attention. Some sixty years later Harrison and Graham devised methods of pendulum compensation for changes of temperature, which, with important modifications in the escapement mechanism introduced by Graham in 1713, made the clock an instrument of precision. Since then its development in design and construction has kept pace with that of other forms of astronomical apparatus.

The pendulum clock must be mounted in a fixed position. It can not be transported from place to place, and it does not, therefore, fulfill all the requirements that may be demanded of a timepiece. By the beginning of the eighteenth century the need of accurate *portable* timepieces had become pressing, not so much for the work of the astronomer as for that of the navigator. The most difficult thing in finding the position of a ship is the determination of longitude. At that time no method was known capable of giving this with anything more than the roughest approximation, although the question had been attacked by the most capable minds of the two centuries immediately preceding. The matter was of such importance that the governments of Spain, France, and the Netherlands established large money prizes for a successful solution, and in 1714 that of Great Britain offered a reward of £20,000 for a method which would give the longitude of a ship within half a degree. With an accurate portable timepiece, which could be set to indicate the time of some standard meridian before beginning a voyage, the solution would have been simple. Notwithstanding the stimulus of reward no solution was forthcoming for many years. In 1735 Harrison succeeded in constructing a chronometer which was compensated for changes of temperature; and about 1760 one of his instruments was sent on a trial voyage to Jamaica. Upon return its variation was found to be such as to bring the values of the longitudes based on its readings within the permissible limit of error.

The ideal timepiece, so far as uniformity is concerned, would be a body moving under the action of no forces, but in practice this can not be realized. The modern timepiece of precision is a close approximation to something equivalent, but falls short of the ideal. Thus far it has been impossible completely to nullify the effect of certain influences which affect the uniformity of motion. Changes in temperature, variations in barometric pressure, and the gradual thickening of the oil lubricating the mechanism produce irregularities, even when the skill of the designer and clockmaker is exercised to its utmost. No timepiece is perfect. We can say only that some are better than others. Further, it is impossible to set a timepiece with such exactness that it does not differ from the true time by a quantity greater than the uncertainty with which the latter can be determined. Thus it happens that a timepiece seldom if ever indicates the true time; and, in general, no attempt is made to remove the error. The timepiece is started under conditions as favorable as possible, and set to indicate approximately the true time. It is then left to run as it will, the astronomer, in the meantime, directing his attention to a precise determination of the amount and the rate of change of the error. These being known, the true time at any instant is easily found.

33. Error and rate.—The error, or correction, of a timepiece is the quantity which added algebraically to the indicated time gives the true time. The error of a timepiece which is slow is therefore positive. If the timepiece is fast the algebraic sign of its correction is negative.

The error of a mean solar timepiece is denoted by the symbol ΔT; of a sidereal timepiece, by $\Delta\theta$. To designate the timepiece to which the correction refers subscripts may be added. Thus the error of a Fauth sidereal clock may be indicated by $\Delta\theta_F$; of a Negus mean time chronometer, by ΔT_N. Sometimes it is convenient to use the number of the timepiece as subscript.

If θ' be the indicated sidereal time at a given instant, and $\Delta\theta$ the corresponding error of the timepiece, the true time of the instant will be

$$\theta = \theta' + \Delta\theta'. \tag{64}$$

The analogous formula for a mean solar timepiece is

$$T = T' + \Delta T'. \tag{65}$$

The **daily rate**, or simply the **rate**, of a timepiece is the change in the error during one day.

If the error of a timepiece increases algebraically, the rate is positive; if it decreases, the rate is negative. The symbols $\delta\theta$ and δT with appropriate subscripts are used for the designation of the rates of sidereal and mean solar timepieces, respectively. The hourly rate, *i.e.* the change during one hour, is sometimes more conveniently employed than the daily rate.

It is convenient, but in no wise important, that the rate of a timepiece should be small. On the other hand, it is of the utmost consequence that the rate should be constant; for the reliability of the instrument depends wholly upon the degree to which this condition is fulfilled.

Generally it is impossible to determine by observation the error at the instant for which the true time is required. We must therefore be able to calculate its value for the instant in question from values previously observed. If the rate is constant this can be done with precision; otherwise, the result will be affected by an uncertainty which will be the greater, the longer is the interval separating the epochs of the observed and the calculated errors.

If $\Delta\theta$ and $\Delta\theta'$ be values of the observed error for the epochs t and t', the daily rate will be given by

$$\delta\theta = \frac{\Delta\theta' - \Delta\theta}{t' - t} \qquad (66)$$

in which $t' - t$ must be expressed in days and fractions of a day. The rate having thus been found, the error for any other epoch, t'', may be calculated by the formula

$$\Delta\theta'' = \Delta\theta' + \delta\theta\,(t'' - t') \qquad (67)$$

Example 22. The error of a sidereal clock was $+5^m\ 27^s.61$ on 1909, Feb. 3, at $6^h.4$ sidereal time, and $+5^m\ 33^s.10$ on 1909, Feb. 11, at $5^h.2$; find the daily rate, and the correction on Feb. 14 at $7^h.6$ sidereal time.

We have $\Delta\theta = +5^m\ 27^s.61$, $\Delta\theta' = +5^m\ 33^s.10$, and

$$t' - t = 11^d\ 5^h.2 - 3^d\ 6^h.4 = 7^d\ 22^h.8 = 7^d.95.$$

Equation (66) then gives $\delta\theta = +5^s.49/7.95 = +0^s.69$, which is the required value of the rate.

To find the error for Feb. 14, $7^h.6$, we have

$$t'' - t' = 14^d\ 7^h.6 - 11^d\ 5^h.2 = 3^d\ 2^h.4 = 3^d.1,$$

whence by equation (67)

$$\Delta\theta'' = +5^m\ 33^s.10 + 3.1 \times 0^s.69 = +5^m\ 35^s.24. \quad \textit{Ans.}$$

34. Comparison of timepieces.—It is frequently necessary to know the time indicated by one timepiece corresponding to that shown by another. The determination of such a pair of corresponding readings involves a **comparison** of the two timepieces. To make such a comparison the observer must be able accurately to follow, or count, the seconds of a timepiece without looking at the instrument. It is desirable, moreover, that he should be able to do this while engaged with other matters, such as entering a record in the observing book, etc.

Pendulum clocks usually beat, or tick, every second; and chronometers, every half second. The beats of the ordinary watch are separated by an interval of a fifth of a second. With each beat the second hand of the timepiece moves forward by an amount corresponding to the interval separating the beats—a whole second space for the clock, a half second space for the chronometer, and a fifth of a second for the watch.

If the beats of two timepieces coincide, a comparison is easily made. The observer has only to pick up the beat from one, then, following mentally, look at the other and note the hour, minute, and second corresponding to a definite time on the first. After noting the reading of the second, the observer should look again at the first before dropping the count, to make sure that the indicated number of seconds and the count were in agreement at the instant of comparison If the beats of the timepieces do not coincide, and it is desired to obtain a comparison with an uncertainty less than the beat interval, the observer must estimate from the sound the magnitude of the interval separating the ticks. He will then note the hour, minute, second, and tenth of a second on the second timepiece corresponding to the beginning of a second on the first.

When a watch is to be compared with a clock or a chronometer, the count should be taken from the latter. The tenths of a second on a watch corresponding to the beginning of a second on the clock or chronometer may be estimated by noting the position of the watch second hand with respect to the two adjacent second marks at the instant the beat of the clock or chronometer occurs. The comparison will then give the hour, minute, second, and zero tenths on the clock or chronometer corresponding to a certain hour, minute, second, and tenth of a second on the watch.

If a sidereal and a mean solar timepiece are to be compared, a very precise result may be obtained by the method of **coincident beats**. It was shown in Section 26 that the gain of sidereal on mean solar time is about ten seconds per hour, or one second in six minutes. The ticks of a solar and a sidereal timepiece, each beating seconds, must therefore coincide once every six minutes. If one of the timepieces beats half seconds, the coincidences will occur at intervals of three minutes. A comparison is made by noting the times indicated by the two instruments at the instant the beats coincide. If carefully made, the uncertainty of the comparison will not exceed one or two hundredths of a second.

Example 23. On 1907, Oct. 29, five comparisons of a watch were made with the Fauth sidereal clock of the Laws Observatory. The means of the comparisons are $\theta_F = 18^h\ 23^m\ 0^s.00$, and $T_W = 4^h\ 3^m\ 16^s.12$ P.M. The error of the Fauth clock was $-29^s.72$, and the longitude west of Greenwich is $6^h\ 9^m\ 18^s.33$. Find the error of the watch referred to central standard time.

From θ_F and $\Delta\theta_F$ find θ by (64). The sidereal time is then to be transformed into C. S. T. by (62) and the first of (41). The resulting C. S. T. compared with T_W gives the error of the watch.

θ_F	18^h	23^m	$0^s.00$	
$\Delta\theta_F$			-29.72	
θ	18	22	30.28	
R_L	14	27	40.63	
$\theta - R_L$	3	54	49.65	
$\text{II}(\theta - R_L)$			38.47	
Col. M. S. T.	3	54	11.18	
L		9	18.33	
C. S. T.	4	3	29.51	P. M.
T_W	4	3	16.12	P. M.
ΔT_W			$+13.39$	*Ans.*

Example 24. On 1907, Oct. 30, civil date, the Fauth sidereal clock of the Laws Observatory read $14^h\ 28^m\ 9^s.75$, when the Riggs clock, a central standard timepiece, indicated $0^h\ 5^m\ 17^s.00$ P.M. The error of Riggs was $+4^s.82$; find the correction to the Fauth clock.

The reading of Riggs combined with its error by (65) gives the true C. S. T. From this the Columbia M. S. T. is found by the second of (41). This converted into the corresponding θ by (58) and compared with the reading of Fauth gives $\Delta\theta_F$.

In problems in which the given time is near noon, great care must be exercised in determining the date for which R_0 is to be taken from the Ephemeris. In the present case, the astronomical date for the 90th meridian is Oct. 30, for the true C. S. T. shows that the instant of mean noon had passed; but at Columbia mean noon had not yet arrived. Since R_0 is always to be taken from the Ephemeris for the *preceding local* mean noon, the date to be used is Oct. 29.

T_R	0^h	5^m	$17^s.00$	P.M.
ΔT_R			$+4.82$	
C. S. T.	0	5	21.82	P.M., Oct. 30, civil date
L		9	18.33	
Col. M. S. T.	23	56	3.49	Oct. 29, astronomical
R_L	14	27	40.63	
$\text{III}M$		3	55.91	
θ	14	27	40.03	
θ_F	14	28	9.75	
$\Delta\theta_F$			-29.72	*Ans.*

Example 25. When the error of a timepiece, *a*, is given and it is required to find the corrections of two others *b* and *c*, the observations and reductions may be controlled by a circular comparison, *i. e.* by comparing *a* and *b*, *a* and *c*, and *b* and *c*. The first comparison leads to the error of *b*. The given error of *a*, and that calculated for *b*, may then be used to reduce the second and third comparisons. Each of these leads to a value of the error of *c* and the two results must agree within the uncertainty of the observations and calculations.

The Fauth sidereal clock, a Bond sidereal chronometer and the Gregg and Rupp central standard clock of the Laws Observatory were compared in this manner on 1902, April 18. The bracketed numbers are the results of the comparisons. $\Delta\theta_F = +1^m\ 14^s.3$, find $\Delta\theta_B$ and $\Delta T_{G\,\&\,R}$.

θ_B	10^h	8^m	$45^s.0$	}
θ_F	10	7	27.1	}
$\Delta\theta_F$	$+1$		14.3	
θ	10	8	41.4	
$\Delta\theta_B$			-3.6	

$T_{G\,\&\,R}$	8^h	26^m	$30^s.8$	P.M. }
θ_F	10	9	35.0	}
$\Delta\theta_F$		1	14.3	
θ	10	10	49.3	
C. S. T.	8	35	4.0	
$\Delta T_{G\,\&\,R}$		$+8$	33.2	

$T_{G\,\&\,R}$	8^h	28^m	$25^s.0$	P.M. }
θ_B	10	12	47.5	}
$\Delta\theta_B$			-3.6	
θ	10	12	43.9	
C. S. T.	8	36	58.3	
$\Delta T_{G\,\&\,R}$		$+8$	33.3	

The second and third comparisons are reduced by the method used for Ex. 23. The details of the conversion of θ into C. S. T. are omitted. The two values of $\Delta T_{G \& R}$ present a satisfactory agreement.

Example 26. Given thirty comparisons of a Waltham watch and a Bond sidereal chronometer made at intervals of one minute; to find the rate per minute of the watch referred to the chronometer, a precise value of the watch time corresponding to the first chronometer reading, and the average uncertainty of a single comparison.

The interval between any two chronometer readings minus the difference between the corresponding watch readings is the loss of the watch as compared with the chronometer during the interval. The quotient of the loss by the interval in minutes is a value of the relative rate per minute. Thus, if

I_C = interval between two chronometer times,
I_W = interval between two watch times,
R = relative rate of watch per minute,

then

$$R = \frac{I_C - I_W}{I_C} \qquad (a)$$

The solution of the first part of the problem may therefore be accomplished by grouping the comparisons in pairs and applying equation (a). The mean of the resulting values of R will then be the final result. The selection of the comparisons for the formation of the pairs requires careful attention if the maximum of precision is to be secured. To obtain a criterion for the most advantageous arrangement, consider the resultant error of observation in R when derived from equation (a). Denoting the influence of the errors in the observed watch times upon the interval I_W by e we find for the error of R

$$\mathrm{E}[R] = \frac{e}{I_C} \qquad (b)$$

Since e is independent of the length of the interval separating the comparisons, it follows from (b) that the precision of R increases with the length of this interval.

It is desirable for the sake of symmetry in the reduction that the separate values of R should be of the same degree of precision; and it is important to arrange the calculation so that any irregularity in the relative rate will be revealed. The reduction will then give not only the quantitative value of the final result, but at the same time will throw light upon the reliability of the instruments employed.

We are thus led to the following grouping of the comparisons: 1 and 16, 2 and 17, 3 and 18. ...... 15 and 30; or, in general, the nth comparison is paired with the $(15 + n)$th. The fourth column of the table gives the values of I_W corresponding to this choice. The first of these is derived by subtracting the first T_W from the sixteenth; the second, by subtracting the second T_W from the seventeenth, and so on. The 15 values of I_W substituted into equation (a), together with the constant value $I_C = 15^m$, would give 15 separate values for R. The first of these would depend upon data secured during the first 15 minutes of the observing period; the last, upon those obtained during the last 15 minutes; while the intermediate values of R would correspond to various intermediate 15-minute intervals. Any irregularity in the rate will therefore reveal itself in the form of a progressive change in the separate values of R. But, since I_C is assumed to be constant throughout, equation (a) shows that constancy of I_W will be quite as satisfactory a test of the reliability of the timepieces as constancy in R. It is not necessary, therefore, to calculate the separate values of the relative rate; and for the derivation of the final result we adopt the simpler procedure of forming the mean of the values of I_W, which we then substitute into (a) with $I_C = 15^m$. We thus find mean $I_W = 14^m\ 57^s.65$, whence the mean relative rate of the watch referred to the chronometer is $0^s.157$ per minute of chronometer time.

WATCH AND CHRONOMETER COMPARISON

No.	θ_B	T_W	I_W	$(n-1)R$	T_W	v
1	$0^h\ 45^m\ 0^s.0$	$10^h\ 25^m\ 25^s.9$	$14^m\ 57^s.6$	$0^s.00$	$25^s.90$	$-^s.11$
2	46 0.0	26 25.8	.6	0.16	25.96	−.17
3	47 0.0	27 25.7	.4	0.31	26.01	−.22
4	48 0.0	28 25.5	.5	0.47	25.97	−.18
5	49 0.0	29 25.1	.8	0.63	25.73	+.06
6	50 0.0	30 25.0	.7	0.78	25.78	+.01
7	51 0.0	31 24.8	.8	0.94	25.74	+.05
8	52 0.0	32 24.7	.7	1.10	25.80	−.01
9	53 0.0	33 24.4	.7	1.26	25.66	+.13
10	54 0.0	34 24.2	.8	1.41	25.61	+.18
11	55 0.0	35 24.1	.7	1.57	25.67	+.12
12	56 0.0	36 23.9	.7	1.73	25.63	+.16
13	57 0.0	37 23.8	.7	1.88	25.68	+.11
14	58 0.0	38 23.7	.7	2.04	25.74	+.05
15	59 0.0	39 23.7	.4	2.20	25.90	−.11
16	1 0 0.0	40 23.5	15) 9.8	2.36	25.86	−.07
17	1 0.0	41 23.4	14 57.65	2.51	25.91	−.12
18	2 0.0	42 23.1	15 0.00	2.67	25.77	+.02
19	3 0.0	43 23.0	15) 2.35	2.83	25.83	−.04
20	4 0.0	44 22.9	$R = 0^s.157$	2.98	25.88	−.09
21	5 0.0	45 22.7		3.14	25.84	−.05
22	6 0.0	46 22.6		3.30	25.90	−.11
23	7 0.0	47 22.4		3.45	25.85	−.06
24	8 0.0	48 22.1		3.61	25.71	+.08
25	9 0.0	49 22.0		3.77	25.77	+.02
26	10 0.0	50 21.8		3.92	25.72	+.07
27	11 0.0	51 21.6		4.08	25.68	+.11
28	12 0.0	52 21.5		4.24	25.74	+.05
29	13 0.0	53 21.4		4.40	25.80	−.01
30	14 0.0	54 21.1		4.55	25.65	+.14
Precise Comp.	0 45 0.00	10 25 25.79			30)23.69	+1.36
				$M_0 =$	25.79	−1.35
				Rem. =	− 0.01	+0.01
						30) 2.71

Average Residual = $\pm 0^s.09$

An examination of the individual values of I_W for the given problem affords no certain evidence of a variability of the relative rate.

As for the second requirement of the problem, it is evident that were the observations perfectly made, with a watch whose relative rate was zero, the seconds and tenths of seconds of all the watch readings would have been the same. Had they been made with the same errors of observation as actually occurred, but with a watch of zero relative rate, they would have differed among themselves only by the errors of observation. The mean of all the seconds readings would then have given a precise value of the watch time corresponding to the first chronometer reading. The given problem may be reduced to this case by correcting each watch reading by the effect of the rate during the interval separating it from the first observation. To accomplish this we have only to add to the readings, in order, the quantities $0R$, $1R$, $2R$, ... $29R$; or, in general, to the nth reading, $(n-1)R$. The values of these corrections are in column five of the table, and the watch times, corrected for rate, in column six. These results are given to two places of decimals in order to keep the errors of calculation small as compared with the errors of observation. The mean of the values of T_W, $10^h\ 25^m\ 25^s.79$, is the required precise watch reading corresponding to the first chronometer reading, $0^h\ 45^m\ 0^s.00$.

To obtain a notion of the uncertainty of a single comparison, consider the corrected watch readings, T_W. If the *true* value of R has been used in applying the corrections for rate, and if the *true* value of the first watch reading were known, the actual error of this and of each of the remaining readings could at once be found by forming the difference between the true value and each of the corrected watch times. The average of the errors would then indicate the precision of the comparisons. But the true values of R and of the first comparison are not known and cannot be found. We must therefore proceed as best we may; and, accordingly, we use for the true relative rate the value calculated above, and for the true value of the first watch reading, the mean of all the corrected readings. The differences between each corrected watch time and the mean of them all are called **residuals**. The residuals will differ but little from the corresponding errors, for the calculated value of R and the mean T_W will differ but little from the quantities they are taken to represent. Although the average of the residuals will not exactly equal the average of the errors, it may be accepted, nevertheless, as a measure of the precision of the observations; for, barring a constant systematic error, it is evident that the more accurate the observations, *i.e.* the smaller their variations among themselves, the less will be the average residual.

Denoting the residuals by v, and the mean of the corrected watch times by M_0, we have

$$v = M_0 - T'_W \qquad (c)$$

The values of v formed in accordance with (c) are in the last column of the table.

A valuable control may be applied at this point. It is easily shown that if the exact value of M_0 be used for the formation of the residuals, their algebraic sum must be zero. (Num. Comp. p. 17.) If, however, an approximation for M_0 is used, the algebraic sum of the residuals will equal the negative value of the remainder in the division which gives as quotient the value used as a mean.

In the present case the algebraic sum of the residuals is $+0.01$; the remainder is -0.01, which checks the formation of the mean and the residuals. The average residual, without regard to algebraic sign, is $\pm 0^s.09$. This we may accept as the average uncertainty of a single comparison.

The principles illustrated in the preceding reduction find frequent application in the treatment of the data of observation. The example is typical and the methods followed in the discussion should receive careful attention. In particular, the grouping of the observations for the determination of the mean value of R should be examined; and the student should investigate for himself the precision of the result when such combinations of the comparisons as 1 and 2, 2 and 3, ... 29 and 30; 1 and 2, 3 and 4, ... 29 and 30; 1 and 30, 2 and 28, ... 15 and 16; etc, are employed in place of that actually used.

Example 27. To determine the average uncertainty of a single comparison of two timepieces by the method of coincident beats.

Ten successive coincidences of the beats of a Bond sidereal chronometer with those of a Gregg & Rupp mean time clock are taken as the basis of the investigation. The method used for the reduction is similar to that employed in Ex. 26. The comparisons are in the second and third columns of the table. Since the chronometer beats half-seconds and the clock seconds, the interval between the successive coincidences is that required for the clock to lose

$0^s.5$ as compared with the chronometer. Denote the true value of this interval by I. To exhibit the influence of the errors of observation we find what the clock readings would have been had they all been made at the same instant as the first. This is done by subtracting from the readings, in order, $0I$, $1I$, $2I$, ... $9I$. The numerical values of the corrections are in column five, and the reduced clock readings themselves, in column six. The value to be used for I is one-fifth of the average of the intervals between the nth and the $(n+5)$th clock readings. The individual values of these intervals are in column four. Their mean is $14^m 55^s.2$, whence $I = 2^m 59^s.04$. The variations in the values of T' represent the influence of the errors of observation. The average residual for the reduced clock readings is $\pm 2^s.94$, which may be accepted as the average uncertainty of the time of a coincidence. Since the clock loses 1^s in 358^s, the corresponding average uncertainty of a comparison is $\pm 0^s.008$.

COMPARISON BY COINCIDENT BEATS.

No.	θ	T	I	$(n-1)I$	T'	v
1	$17^h\ 35^m\ 57^s.0$	$2^h\ 6^m\ 0^s.0$	$14^m\ 45^s.0$	$0^m\ 0^s.0$	$2^h\ 5^m\ 60^s.0$	$-\ 3^s.3$
2	38 52.5	8 55.0	62.0	2 59.0	56.0	+ 0.7
3	41 53.0	11 55.0	60.0	5 58.1	56.9	− 0.2
4	44 52.5	14 54.0	54.0	8 57.1	56.9	− 0.2
5	47 49.0	17 50.0	55.0	11 56.2	53.8	+ 2.9
6	50 44.5	20 45.0	5)276.0	14 55.2	49.8	+ 6.9
7	53 57.0	23 57.0	5)14 55.2	17 54.2	62.8	− 6.1
8	56 55.5	26 55.0	$I =$ 2 59.04	20 53.3	61.7	− 5.0
9	59 49.0	29 48.0		23 52.3	55.7	+ 1.0
10	18 2 46.5	32 45.0		26 51.4	53.6	+ 3.1
					10)567.2	+ 14.6
					$M_0 =$ 56.7	− 14.8
					Rem = + 0.2	− 0.2
						10) 29.4

Clock loses 1^s in 358^s.
Average uncertainty of a single comparison $= \pm 2^s.94/358 = \pm 0^s.008$.

Average Residual $= \pm 2^s.94$

35. The care of timepieces.—All timepieces should be wound at regular intervals. They should be protected from moisture, electrical and magnetic influences, and extremes of temperature, especially the direct rays of the sun. They yield the best results when at rest, absolutely untouched, except as winding may be necessary. Portable instruments must not be subjected to violent shocks, jolts, or oscillatory motions. Chronometers are particularly sensitive to such disturbances, especially oscillations. Timepieces of this sort are usually hung in gimbals, mounted in a substantial wooden case. When at rest, or when subjected to the long periodic motions of a ship, they should hang free in the gimbals in order that the mechanism may remain constantly horizontal in position. When transported from place to place on land, the gimbals should be locked. Otherwise the unavoidable jarring may produce oscillations sufficient to change appreciably the error and the rate. If the journey is such that shocks can not be avoided, it is safer to stop the instrument and insert thin wedges of cork between the balance wheel and the supporting frame, using just sufficient force to hold them in place. In this way the delicate pinions of the balance may be guarded from injury. The chronometer, so far as possible, should be kept in a fixed position with respect to the points of the compass.

THE ARTIFICIAL HORIZON

36. Description and use.—The artificial horizon consists of a shallow dish filled with mercury. The force of gravity brings the surface to a horizontal position, and the high reflective power of the metal makes it possible to see the various celestial bodies reflected in the surface. Any given object and its image will be situated on the same vertical circle, and the angular distance of the image below the surface will be equal to that of the object above. The angular distance between the object and its image is therefore twice its apparent altitude. Strictly speaking, this is true only when the eye of the observer is at the surface of the mercury, but for distant objects the error is insensible.

The measurement of the distance between the object and its image therefore affords a means of determining the altitude of a celestial body, and in this connection the artificial horizon is a valuable accessory to the sextant. It can also be used to advantage with the engineer's transit for the elimination of certain instrumental errors.

The artificial horizon is usually provided with a glass roof to protect the surface of the mercury from disturbances by air currents. It is important that the plates of glass should be carefully selected in order that the light rays traversing them may not be deflected from their course. The effect of any non-parallelism of the surfaces may be eliminated by making an equal number of settings with the roof in the direct and reversed position, reversal being accomplished by turning the roof end for end.

THE VERNIER

37. Description and theory.—The vernier is a short graduated plate attached to scales for the purpose of reducing the uncertainty of measurement. It takes its name from its inventor, Pierre Vernier, who in 1631 described its construction and use. In its usual form the graduations are such that the total number of vernier divisions, which we may denote by n, is equal to $n-1$ divisions of the scale, the graduation nearest the zero of the scale marking the zero of the vernier. The vernier slides along the scale, the arrangement being such that the angle, or length, to be measured corresponds to the distance between the zeros of the scale and of the vernier. When the zero of the vernier stands opposite a graduation of the scale, the desired reading is given directly by the scale. Usually this will not occur, and the vernier is then used to measure the fractional part of the scale division included between the last *preceding* scale graduation and the zero of the vernier.

The difference between the values of a scale and a vernier division is called the **least reading** $= l$ of the vernier. If

d = value of one division of the scale,
d' = value of one division of the vernier,

then, for the method of graduation described above,

$$(n-1)\,d = nd'$$

whence

$$l = d - d' = \frac{d}{n} \qquad (68)$$

The least reading of the vernier is therefore $1/n$th of the value of a scale division.

Now, for an arbitrary setting of the vernier, consider the intervals between the various vernier graduations and the nearest preceding graduations of the scale, beginning with the zero of the vernier and proceeding in order in the direction of increasing readings. The first interval is the one whose magnitude is to be determined by the vernier. Denote its value by v. Since a vernier division is less than a scale division by the least reading, l, it follows that the interval between the second pair of graduations will be $v - l$; that between the third $v - 2l$; and so on, each successive interval decreasing by l. By proceeding far enough we shall find a pair for which the interval differs from zero by an amount equal to, or less than $l/2$, a quantity so small that the graduations will nearly, if not quite, coincide. Suppose this pair to be n' divisions from the zero of the vernier. The value of the corresponding interval will be $v - n'l = \varepsilon$, and we therefore find

$$v = n'l + \varepsilon. \qquad (69)$$

In practice we disregard ε and use

$$v = n'l. \qquad (70)$$

To determine the value of v, therefore, we count the number of vernier divisions from the zero of the vernier to the vernier graduation which most nearly coincides with a graduation of the scale. The product of this number into the least reading is the value of v. The final result is the sum of v and the reading corresponding to the last scale graduation preceding the zero of the vernier.

In practice the actual counting of the number of divisions between the zero of the vernier and the coincident pair is avoided by making use of the numbers stamped on the vernier. These give directly the values of $n'l$ corresponding to certain equidistant divisions of the vernier. Usually one or two divisions precede the zero and follow the last numbered graduation of the vernier. These do not form a part of the n divisions of the vernier, and are therefore to be disregarded in the determination of l. They are added to assist in the selection of the coincident pair when coincidence occurs near the end of the vernier.

38. Uncertainty of the result.—The error of a reading made with a perfectly constructed vernier is ε, whose maximum absolute value is $l/2$. The uncertainty of the result is therefore $l/2$.

The gain in precision resulting from the use of the vernier may be found by comparing the uncertainty of its readings with that arising when the scale alone is used. The latter may be fixed at 0.05d, as experience shows that this is approximately the uncertainty of a careful eye estimate of the magnitude of v. The inverse ratio of the two uncertainties may be taken as a measure of the increase in precision, whence we find that the result given by the vernier is approximately $n/10$ times as precise as that derived from an estimate of the fractional parts of a scale division. It appears, therefore, that a vernier is of no advantage unless the number of its divisions is in excess of ten.

The use of a magnifying lens usually shows that none of the vernier graduations exactly coincides with a graduation of the scale. With a carefully graduated instrument, it is possible, by estimating the magnitude of ε, to push the precision somewhat beyond the limit given above. To do so it is only necessary to compare ε with the interval between the next following pair of graduations. or with that of the pair immediately preceding, according as ε is positive or negative. The sum of the two intervals to be compared is l. It is therefore possible to estimate ε in fractional parts of the least reading.

The condition that n divisions of the vernier equal $n-1$ divisions of the scale must be rigorously fulfilled if reliable results are to be obtained. The matter should be tested for different parts of the scale by bringing the zero of the vernier into coincidence with a scale graduation, and then examining whether the $(n+1)$st vernier graduation stands exactly opposite graduation of the scale. Information may thus be obtained as to the accuracy with which the graduation of the instrument has been performed.

The vernier should lie, preferably, in the same plane as the scale, and, in all positions, should fit snugly against the latter. In many instruments, however, it rests on top, the plate being beveled to a knife edge where it touches the scale. With this arrangement the greatest care must be exercised in reading to keep the line of sight perpendicular to the scale. Otherwise an error due to parallax will affect the result.

THE ENGINEER'S TRANSIT

39. Historical.—The combination of a horizontal circle with a vertical arc for the measurement of azimuth and altitude is known to have been used by the Persian astronomers at Meraga in the thirteenth century, and it is possible that a similar contrivance was employed by the Arabs at an even earlier date. The principle involved did not appear in western Europe, however, until the latter half of the sixteenth century. There it found its first extensive application in the instruments of Tycho Brahe, who constructed a number of "azimuth-quadrants" for his famous observatory on the island of Hveen. The vertical arcs of Tycho's instruments were movable about the axis of the horizontal circle, and were provided with index arms fitted with sights for making the pointings. The adjustment for level was accomplished by means of a plumb line, the spirit level not yet having been invented. Magnification of the object was impossible, as a quarter of a century was still to elapse before the construction of the first telescope. The instruments were large and necessarily fixed in position; and, indeed, there was no need for moving them from place to place as they were intended solely for astronomical observations. Though primitive in design, they were constructed with the greatest care, and were capable of determining angular distances with an uncertainty of only 1′ or 2′. They are of interest not only on account of the remarkable series of results they yielded in the hands of Tycho, but also because they embody the essential principle of the modern altazimuth, the universal instrument, the theodolite, the engineer's transit, and a variety of other instruments.

None of these modern instruments is the invention of any single person, but rather a combination of inventions by various individuals at different times. The telescope, first constructed during the early years of the seventeenth century, was adapted to sighting purposes through the introduction of the reticle by Gascoigne, Auzout, and Picard. Slow motions were introduced by Hevelius. The vernier was invented in 1631, and the spirit level, by Thévenot, in 1660. All these were combined with the principle of the early azimuth-quadrant to form the altazimuth, which appears first to have been made in a portable form by John Sisson, an Englishman, about the middle of the eighteenth century. At the beginning of the nineteenth century the design and construction were greatly improved by Reichenbach, who also added the movable horizontal circle, thus making it possible to measure angles by the method of repetitions. The universal instrument was then practically complete, and the transition to the engineer's transit required only the addition of the compass and such minor modification as would meet the requirements of precision and portability fixed by modern engineering practice.

For a detailed description of the engineer's transit, the student is referred to any standard work on surveying. Certain attachments, notably the compass and the telescope level, are not required for the determination of latitude, time, and azimuth. On the other hand, it is desirable that the instrument used in the solution of these problems should possess features not always present in the modern instrument. In particular, the vertical circle should be complete, and should be provided with two verniers situated 180° apart. A diagonal prism for the observation of objects near the zenith, and shade glasses for use in solar observations are a convenience, though not an absolute necessity.

40. Influence of imperfections of construction and adjustment.—It is assumed that the student is familiar with the methods by which the engineer's transit may be adjusted, and that observations will not be undertaken until the various adjustments have been made with all possible care. But since an instrument is never perfect, it becomes of importance to determine the influence of the residual errors in construction and adjustment, and to establish precepts for the arrangement of the observing program such that this influence may be reduced to a minimum.

In the instrument fulfilling the ideal of construction and adjustment, the following conditions, among others, are satisfied:

1. The rotation axes of the horizontal circle and the alidade coincide.
2. The planes of the circles are perpendicular to the corresponding axes of rotation.
3. The centers of the circles lie in the corresponding axes of rotation, and the lines joining the zeros of the verniers pass through the axes.

4. The vertical axis of rotation is truly vertical when the plate bubbles are centered.
5. The horizontal rotation axis is perpendicular to the vertical axis.
6. The line of sight, *i.e.* the line through the optical center of the objective and the middle intersection of the threads, is perpendicular to the horizontal axis.
7. The vertical circle reads zero when the line of sight is horizontal.

It is the task of the instrument maker to see that the first three of these conditions are satisfied. The observer, on the other hand, is responsible for the remainder.

No. 1 is of importance only in the measurement of horizontal angles by the method of repetitions. The error arising in such measures from non-coincidence of the vertical axes may be eliminated by the arrangement of the observing program described in Section 47.

No. 2. It can be shown that the error due to lack of perpendicularity of the circles to the axes is of the order of the square of the deviation. In well constructed instruments it is therefore insensible.

No. 3. If the third condition is not satisfied the readings will be affected by an error called **eccentricity.**

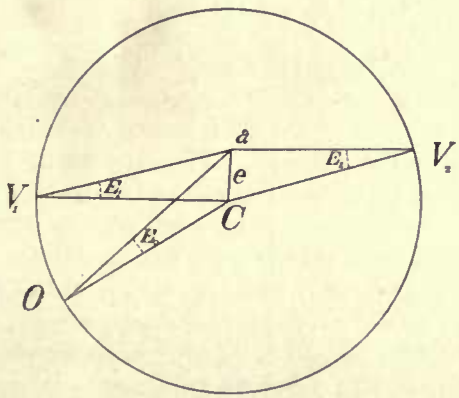

Fig. 7.

In Fig. 7 let C be the center of the graduated circle OV_1V_2; a, the point where the rotation axis intersects the plane of the circle; O, the zero of the graduations; and V_1 and V_2 the zeros of the verniers. The distance $aC=e$ is the **eccentricity of the circle.** The perpendicular distance of a from the line joining V_1 and V_2 is the **eccentricity of the verniers.** The reading of V_1 is the angle OCV_1, and of V_2, OCV_2. Denote these by R_1 and R_2, respectively. The angles through which the instrument must be rotated in order that the zeros of the verniers may move from O to the positions indicated, are $OaV_1=A_1$ and $OaV_2=A_2$, respectively. A_1 and A_2 are therefore to be regarded as the angles which determine the positions of the verniers with respect to O for the pointing in question. The relations connecting A_1 and A_2 with the vernier readings, R_1 and R_2, are

$$A_1 = R_1 + E_0 - E_1, \qquad (71)$$
$$A_2 = R_2 + E_0 - E_2, \qquad (72)$$

where E_0, E_1, and E_2 are the corrections for eccentricity for the points O, V_1, and V_2. The mean of (71) and (72) is

$$\tfrac{1}{2}(A_1 + A_2) = \tfrac{1}{2}(R_1 + R_2) + E_0 + \tfrac{1}{2}(E_2 - E_1). \qquad (73)$$

For any other pointing of the telescope, we have the analogous equation

$$\tfrac{1}{2}(A_1' + A_2') = \tfrac{1}{2}(R_1' + R_2') + E_0 + \tfrac{1}{2}(E_2' - E_1'). \qquad (74)$$

It is easily shown that $E_2 - E_1$ and $E_2' - E_1'$ are of the order of ee'/r^2, where e' is the eccentricity of the verniers and r the radius of the circle. The last terms of (73) and (74) are entirely insensible in a well constructed instrument. The difference of (73) and (74) is therefore

$$\tfrac{1}{2}(A_1' + A_2') - \tfrac{1}{2}(A_1 + A_2) = \tfrac{1}{2}(R_1' + R_2') - \tfrac{1}{2}(R_1 + R_2). \qquad (75)$$

The left member of (75) is the angular distance through which the instrument is rotated in passing from the first position to the second, and the equation shows that this angle is equal to the difference in the means of the vernier readings for the final and initial positions. The eccentricity is therefore eliminated by combining the means of the readings of *both* verniers.

It can be shown that the eccentricity will also be eliminated by combining the means of any number of verniers, greater than two, uniformly distributed about the circle. In practice it is sufficient to use the degrees indicated by the first vernier with the means of the minutes and seconds of the two readings.

Nos. 4—7. *Horizontal Angles*: In the measurement of horizontal angles an error of adjustment in No. 7 has no influence. To investigate the effect of residual errors in Nos. 4—6, let

i = inclination of the vertical axis to the true vertical,
$90° - j$ = inclination of the horizontal axis to the vertical axis,
b = inclination of the horizontal axis to the horizon plane,
$90° + c$ = inclination of the line of sight to the horizontal axis.

The quantities b and c are the errors in **level** and **collimation**, respectively. Then, in Fig. 8, which represents a projection of the celestial sphere on the plane of the horizon, let Z be the zenith, Z' the intersection with the celestial sphere of the vertical axis produced, O an object whose zenith distance is z_0, and A the intersection of the horizontal axis produced with the celestial sphere when O is seen at the intersection of the threads. The sides of the triangles ZAZ'

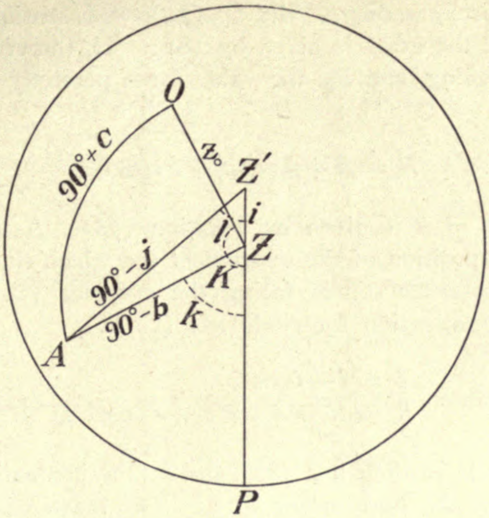

Fig. 8.

and ZAO have the values indicated in the figure. Finally, let k, K and l be the directions of ZA, ZO, and $Z'A$, respectively, referred to ZP.

Applying equations (13) and (15) to triangle ZAZ', we find

$$\sin b = \sin j \cos i + \cos j \sin i \cos l, \qquad (76)$$

$$\cos b \sin k = \cos j \sin l. \qquad (77)$$

In a carefully adjusted instrument i, j, and b are very small, and we may neglect their squares as insensible. Equations (76) and (77) thus reduce to

$$b = j + i \cos l, \qquad (78)$$

$$k = l. \qquad (79)$$

Equation (13) applied to triangle ZAO gives

$$-\sin c = \sin b \cos z_0 + \cos b \sin z_0 \cos (K - k). \qquad (80)$$

Since c and $90° - K + k$ are also very small, equation (80) may be written

$$-c = b \cos z_0 + (90° - K + k) \sin z_0$$

or

$$K - 90° - k = b \cot z_0 + c \operatorname{cosec} z_0. \qquad (81)$$

Were there no errors of adjustment, the direction of A referred to P would be $K - 90°$. The direction given by the instrument, determined by the angle through which it must be rotated to bring A from coincidence with ZP to its actual position, is l. Since the verniers maintain a fixed position with respect to A, the difference $K - 90° - l$ represents the effect of the residual errors on the horizontal circle readings. But by (79) $l = k$, sensibly, whence it follows that the amount of the error is given by (81). If, therefore, R be the actual horizontal circle reading, and R_0, the value for a perfectly adjusted instrument, we have

$$R_0 = R + b \cot z_0 + c \operatorname{cosec} z_0, \qquad \text{C. R.} \qquad (82)$$

in which the value of b is given by equation (78). Assuming that equation (82) refers to that position of the instrument for which the vertical circle is on the right as the observer stands facing the eyepiece (C. R.), we find by a precisely similar investigation for circle left (C. L.),

$$b_1 = j - i \cos l, \qquad (83)$$
$$R_0 = R_1 - b_1 \cot z_0 - c \operatorname{cosec} z_0, \qquad \text{C. L.} \qquad (84)$$

where R_1 is the circle reading less 180°, and b_1, the inclination of the horizontal axis to the plane of the horizon for C. L. The mean of equations (82) and (84) is

$$R_0 = \tfrac{1}{2}(R + R_1) + \tfrac{1}{2}(b - b_1) \cot z_0, \qquad (85)$$

or, substituting the values of b and b_1 from (78) and (83)

$$R_0 = \tfrac{1}{2}(R + R_1) + i \cos l \cot z_0. \qquad (86)$$

It therefore appears that the mean of the readings of the horizontal circle taken C. R. and C. L. for settings on any object is free from the influence of j, c, and the component of i in the direction of the line of sight, viz., $i \sin l$. Moreover, for objects near the horizon the effect of $i \cos l$, the component of i parallel to the horizontal axis, is small, for it appears in (86) multiplied by cot z_0.

If the instrument be provided with a striding level, the values of b and b_1 may be determined by observation. Their substitution into (85) will then give the horizontal circle reading completely freed from i, j, and c.

The readings may also be freed from the influence of b by combining the results of a setting on O with those obtained by pointing on the image of O seen reflected in a dish of mercury, both observations being made in the same position of the instrument, either C. R. or C. L. The reflected image, O', will be on the vertical circle through O, and as far below the horizon as O is above. Since the horizontal axis is not truly horizontal, it will be necessary to rotate the instrument slightly about the vertical axis in turning from O down to O'. A will thus move a small amount to a new position A'.

To investigate the effect of the errors for a pointing on O' we must therefore consider the triangle $A'ZO'$ in place of AZO in Fig. 8. The sides of $A'ZO'$ are $ZA' = ZA = 90° - b$, $A'O' = AO = 90° + c$, and $ZO' = 180° - z_0$. The angle at Z is $K - k'$ where k' is the direction of ZA' referred to ZP. We then find, similarly to equation (80),

$$-\sin c = -\sin b \cos z_0 + \cos b \sin z_0 \cos (K - k'),$$

whence

$$K - 90^\circ - k' = -b \cot z_0 + c \operatorname{cosec} z_0,$$

and, finally, if R' be the horizontal circle reading for the setting on O'

$$R_0 = R' - b \cot z_0 + c \operatorname{cosec} z_0. \qquad \text{C. R.} \qquad (87)$$

Equations (82) and (87) both refer to C. R. Their mean is

$$R_0 = \tfrac{1}{2}(R + R') + c \operatorname{cosec} z_0, \qquad \text{C. R.} \qquad (88)$$

By the same method we find from the reflected observation, C. L.,

$$R_0 = R_1' + b_1 \cot z_0 - c \operatorname{cosec} z_0, \qquad \text{C. L.} \qquad (89)$$

in which R_1' is the circle reading less 180° for C. L. This equation combined with (84) gives

$$R_0 = \tfrac{1}{2}(R_1 + R_1') - c \operatorname{cosec} z_0, \qquad \text{C. L.} \qquad (90)$$

Equations (88) and (90) show that the mean of the horizontal circle readings for direct and reflected observations of an object in the same position of the instrument is free from the influence of any adjustment error in level.

Finally the combination of (88) and (90) gives

$$R_0 = \tfrac{1}{4}(R + R' + R_1 + R_1'), \qquad (91)$$

in other words the mean of the readings, direct and reflected, for both positions of the instrument, is free not only from b, but from the collimation error as well.

Vertical Circle Readings: To investigate the influence of i, j and c upon the readings of the vertical circle, consider again Fig. 8. The true zenith distance of O is $ZO = z_0$; that given by the vertical circle readings is equal to the angle $Z'AO$. From the triangle ZAO we find

$$\cos z_0 = -\sin b \sin c + \cos b \cos c \cos (ZAO).$$

The squares and products of the errors of adjustment are ordinarily quite insensible, whence we find with all necessary precision.

$$z_0 = \text{Angle } ZAO.$$

Denoting the instrumental zenith distance $Z'AO$ by z, we find $z_0 - z =$ angle ZAZ', and from triangle ZAZ'

$$\cos b \sin (z_0 - z) = \sin i \sin l,$$

or, since b, $z_0 - z$, and i are very small,

$$z_0 = z + i \sin l, \qquad \text{C. R.} \qquad (92)$$

A similar investigation gives for the reversed position of the instrument

$$z_0 = z_1 + i \sin l, \qquad \text{C. L.} \qquad (93)$$

in which z_1 is the instrumental zenith distance for C. L.

The angles z and z_1 are not read directly from the circles. The ordinary engineer's transit reads altitudes, but if there is any deviation from the condition expressed in No. 7, the readings will not be the true altitudes, for they will include the effect of the **index error.** If r and r_1 be the vertical circle readings for C. R. and C. L., respectively, and I the reading when the line of sight is horizontal, we have

$$z = 90° - r + I, \qquad \text{C. R.} \qquad (94)$$
$$z_1 = 90° - r_1 - I, \qquad \text{C. L.} \qquad (95)$$

Substituting (94) and (95) into (92) and (93)

$$z_0 = 90° - r + I + i \sin l, \qquad \text{C. R.} \qquad (96)$$
$$z_0 = 90° - r_1 - I + i \sin l, \qquad \text{C. L.} \qquad (97)$$

The mean of (96) and (97) is

$$z_0 = 90° - \tfrac{1}{2}(r + r_1) + i \sin l. \qquad (98)$$

For an instrument whose vertical circle is graduated continuously from 0° to 360° it is easily shown that the equation corresponding to (98) is

$$z_0 = \tfrac{1}{2}(v_1 - v_2) + i \sin l, \qquad (99)$$

in which v_1 and v_2 are the circle readings, the subscripts being assigned so that $v_1 - v_2 < 180°$.

It therefore appears that the vertical circle readings are not sensibly affected by j, c, or the component of i parallel to the horizontal axis. The component of i in the direction of the line of sight, viz., $i \sin l$ enters with its full value, and (98) and (99) show that it cannot be eliminated even when readings taken C. R. and C. L. are combined. The formation of the mean for the two positions of the instrument does eliminate the index error, however, *i.e.* the residual error of adjustment in No. 7.

To free the results from $i \sin l$ we may combine observations direct and reflected, using the mercurial horizon. Considering the triangle $A'ZO'$ previously defined, we find for the reflected observation

$$\cos(180° - z_0) = -\sin b \sin c + \cos b \cos c \cos(ZA'O')$$

whence, neglecting products and squares of the errors of adjustment, the true zenith distance of O' is $180° - z_0 =$ angle $ZA'O'$. Denoting the instrumental zenith distance of O', which is the angle $Z'A'O'$, by $180° - z'$ we find

$$\text{Angle } ZA'Z' = (180° - z_0) - (180° - z') = z' - z_0.$$

In the triangle $ZA'Z'$ the sides are $ZA' = 90° - b$, $Z'A' = 90° - j$, and $ZZ' = i$, and denoting the angle $ZZ'A'$ by l' we find

$$\cos b \sin (z' - z_0) = \sin i \sin l',$$

or with sufficient approximation

$$z_0 = z' - i \sin l', \qquad \text{C. R.} \qquad (100)$$

Now if r' be the vertical circle reading for C. R., reflected, and I the circle reading when the line of sight is horizontal, we shall have, similarly to (94),

$$z' = 90° - r' - I, \qquad \text{C. R.} \qquad (101)$$

This substituted into (100) gives

$$z_0 = 90° - r' - I - i \sin l' \qquad \text{C. R.} \qquad (102)$$

Since l' differs from l by a quantity of the order of the errors, the difference between i sin l and i sin l' will be insensible, so that when equations (96) and (102) are combined to form the mean we have simply

$$z_0 = 90° - \tfrac{1}{2}(r + r'). \qquad \text{C. R.} \qquad (103)$$

Similar considerations for observations direct and reflected, C. L., give

$$z_0 = 90° - \tfrac{1}{2}(r_1 + r_1'). \qquad \text{C. L.} \qquad (104)$$

In other words, the formation of the means of the vertical circle readings for observations direct and reflected in the same position of the instrument eliminates not only the component of i in the direction of the line of sight, but the index correction as well. The influence of $i \cos l$, j and c is insensible. So far as the errors here considered are concerned, observations direct and reflected in a single position of the instrument are sufficient. Nevertheless it is desirable that measures be made both C. R. and C. L. for in this way different parts of the vertical circle are used, thus partially neutralizing errors of graduation.

For an instrument with a vertical circle graduated continuously from 0° to 360°, it is easily shown as before that in (103) and (104) the sum of the circle readings must be replaced by their difference taken in such a way that it is less than 180°.

The preceding discussion assumes that the adjustments of the instrument remain unchanged throughout the observations. If this is not so, the elimination of the errors will, in general, be incomplete.

It is not always convenient to make use of the artificial horizon, and it is therefore desirable to be able to apply a method of elimination which does not depend upon this accessory.

It is easily shown that if the *instrument be relevelled before observing in the reversed position,* the mean of the readings C. R. and C. L., both for the horizontal and the vertical circle, will be free from the errors in all of the adjustment under Nos. 4—7, within quantities of the order of the products and squares of the errors. The same will be true, even though the plate bubbles are not accurately centered during the direct observations, provided, after reversal, they be brought to the *same position in the tubes* that they occupied before.

That such will be the case follows from a consideration of Fig. 8. The reversal and relevelling is equivalent to rotating the triangle ZAZ' about Z through the angle $180°+2c$, its dimensions remaining unchanged. A thus assumes a new position A_1, distant from O by $90°+c$, and Z' a position Z_1'. The triangle ZA_1O leads to an equation differing from (84) only in that b_1 is replaced by b. The mean of the new equation and (82) is simply

$$R_0 = \tfrac{1}{2}(R + R_1), \qquad (105)$$

where R and R_1 are the horizontal circle readings; the latter having been reduced by 180°. The result is therefore free from both b and c.

Again, from triangle ZA_1Z_1', we find for circle left analogously to (97),

$$z_0 = 90° - r_1 - I - i \sin l, \qquad \text{C. L.} \qquad (106)$$

in which the vertical circle reading r_1 is not the same as the r_1 of (97), for (106) presupposes that the instrument is relevelled after reversal, while (97) assumes that no change is made in the position of the vertical axis during the observations. The mean of (96) and (106) is

$$z_0 = 90 - \tfrac{1}{2}(r + r_1), \qquad (107)$$

which is free from b, c, and I. For a circle graduated continuously we have similarly,

$$z_0 = \tfrac{1}{2}(v_1 - v_2) \qquad (108)$$

where as before the readings are to be taken in such an order that their difference is less than 180°.

It is assumed throughout that the pointings are always made by bringing the object accurately to the intersection of the threads. It is important that this be done, even though the threads be respectively horizontal and vertical; for observing at one side of the field is equivalent to introducing an abnormal value of the collimation, while pointings above or below the horizontal thread correspond to a modification of the index error of the vertical circle.

41. Summary of the preceding section.—The preceding results may be summarized as follows:

No. 1. Non-coincidence of vertical axes enters only when the horizontal circle is used by the method of repetitions. Error eliminated by proper arrangement of observing program. See Section 47.

No. 2. Non-perpendicularity of circles to axes usually has no sensible influence on circle readings.

No. 3. Eccentricity of circles and verniers eliminated by forming means of readings of both verniers. See equation (75).

Nos. 4—7. *Horizontal circle readings*: Component of deviation of vertical axis from vertical in direction of line of sight, non-perpendicularity of axes, and collimation eliminated by forming mean of readings taken C. R. and C. L. Component of deviation from vertical which is parallel to horizontal axis appears multiplied by $\cot z_0$. See equation (86). Correction for the latter may be made by observations with the striding level. See equation (85). All errors in Nos. 4—6 eliminated by forming mean of readings direct and reflected, for both C. R. and C. L. See Equation (91). The error in No. 7—index error of vertical circle—does not enter. *Vertical circle readings*: All errors in Nos. 4—7 excepting component of deviation of vertical axis from vertical in direction of line of sight insensible or eliminated from mean of readings C. R. and C. L. See equation (98) or (99). All errors in Nos. 4—7 insensible or eliminated from mean of readings, direct and reflected, in same position of instrument. See equations (103) and (104). Desirable to observe both C. R. and C. L., however, to reduce graduation error of vertical circle.

All errors under Nos. 4—7 eliminated from mean of readings C. R. and C. L. for both horizontal and vertical angles provided plate bubbles have same position in tubes for both positions of the instrument. See equations (105) and (107) or (108).

42. The level.—The adjustment of the engineer's transit with respect to the vertical is usually made by means of the plate bubbles, any residual error being eliminated by some one of the methods of Section 40. In some cases, however, it is desirable to remove the effect of this error by measuring the inclination of the horizontal axis to the horizon and applying a suitable correction to the circle readings. This method of procedure requires a knowledge of the theory of the striding level.

The striding level is more sensitive than the plate bubbles, its tube is longer, and the scale includes a larger number of divisions. It is made in two forms, one with the zero of the scale at the middle of the tube, the other with the zero at the end. Theoretically the two forms are equivalent. The adjustment of the level tube within its mounting should be such that the bubble stands at the middle of the tube when the base line is horizontal. The scale reading of the middle of the bubble for this position is called the **horizontal reading.** Owing to residual errors of adjustment, the horizontal reading will not usually be zero, even for the form in which the zero of the scale is at the middle of the tube. Its value must be determined and applied as a correction to the scale readings, or else its influence must be eliminated. The latter is easily accomplished by combining readings made in the direct and the reversed position, reversal being made by turning the level end for end.

Let $d =$ the angular value of one division of the level scale.
$h =$ the horizontal reading.

Further, for any inclination of the base line, let m' and m'' be the readings of the middle of the bubble, and b' and b'' the corresponding observed inclinations, for the level direct and reversed, respectively. Finally, assume that all readings increasing toward the right are positive, and all toward the left, negative. We then find, whatever the position of the zero of the scale,

$$b' = (m' - h)d, \qquad (109)$$
$$b'' = (m'' - h)d. \qquad (110)$$

Since h has opposite signs for the two positions of the level, the mean of (109) and (110) is

$$b = \tfrac{1}{2}(m' + m'')d, \qquad (111)$$

in which the mean of the observed inclinations has been written equal to b. Denoting by r', l', and r'', l'', the readings of the ends of the bubble for two positions, and writing

$$D = \tfrac{1}{4}d, \qquad (112)$$

we find from (111)

$$b = (r' + l' + r'' + l'')D. \qquad (113)$$

This result depends only upon the readings of the ends of the bubble and the value of one division of the scale, and is therefore free from the horizontal reading. The convention regarding the algebraic sign is such that when b calculated from (113) is positive, the right end of the level is high.

Since b' and b'' are two observed values of the same quantity, we find from the difference of (109) and (110)

$$h = \tfrac{1}{4}(r' + l' - r'' - l''), \qquad (114)$$

which may be used for the calculation of h when a complete observation has been made.

43. Precepts for the use of the striding level.—The level is a sensitive instrument, and great care must be exercised in its manipulation if precise results are to be obtained. The inclinations to be measured should be small and the horizontal reading should correspond as closely as possible with the scale reading of the middle of the tube. The points of contact of the level with the pivots upon which it rests must be carefully freed from dust particles.

The length of the bubble, which is adjustable in the more sensitive forms, should be about one-third the length of the tube, and ample time should be allowed for the bubble to come to rest before reading. The instrument should be protected from changes in temperature, and, to this end, it should be shielded from the rays of the sun, and from the heat of the reading lamp and the person of the observer. The right end of the bubble should always be read first, careful attention being given to the algebraic sign, and the time of reversal for each observation should be noted. Mistakes in reading may be avoided by noting that $r' - l'$, the length of the bubble, must equal $r'' - l''$.

The following, in which S represents the sum of the four readings, is a convenient form for the record:

$$\begin{array}{cc} \multicolumn{2}{c}{\text{Time}} \\ r' & l' \\ r'' & l'' \\ \hline \end{array}$$

$$\left.\begin{array}{c} r' + l'' \\ r'' + l' \end{array}\right\} S, \qquad b = Sd.$$

S is most easily found by forming first the diagonal sums of the four readings written as above, for both r' and l'' and r'' and l' will be opposite in sign and approximately equal in absolute magnitude.

Example 28. The following illustrates the record and reduction of level observations. The first observation was made with a level whose zero point is in the middle of the tube; the second, with one whose zero is at the end. The values of D are $8''.16$ and $0^{s}.032$, respectively.

$$\begin{array}{cc} \multicolumn{2}{c}{\theta = 6^{h}\ 15^{m}} \\ +14.1 & -9.7 \\ +10.1 & -13.8 \\ \hline \end{array} \qquad\qquad \begin{array}{cc} \multicolumn{2}{c}{T = 9^{h}\ 12^{m}} \\ +31.0 & +16.4 \\ -20.3 & -35.0 \\ \hline \end{array}$$

$$\left.\begin{array}{c} +0.3 \\ +0.4 \end{array}\right\} +0.7, \quad b = +5''.7 \qquad\qquad \left.\begin{array}{c} -4.0 \\ -3.9 \end{array}\right\} -7.9, \quad b = -0^{s}.25$$

44. Determination of the value of one division of a level.—The observer should be familiar with the sensitiveness of all the levels of his instrument, even though he depends entirely upon a simple centering of the bubble for the adjustment. If the striding level is to be used, a knowledge of the angular value of one division of its scale is an essential.

The investigation of levels is most easily carried out with the aid of a level trier, which is an instrument consisting essentially of a rigid base carrying a movable arm whose inclination to the horizon may be varied by a known amount by means of a graduated micrometer screw. The entire transit may be mounted on the arm, or the various levels may be attached separately for the investigation. The determination of the change in the inclination of the arm of the level trier necessary to move the bubble over a given number of divisions gives at once the angular value of one division of the scale.

Example 29. The following shows part of the reduction of observations made with a level trier for the determination of the value of one division of a level. The bubble was run from the left to the right end of the tube and back again, for both level direct and reversed, by moving the micrometer head through four divisions at a time. The ends of the bubble were read for each setting of the micrometer. Column two of the table gives the micrometer settings; and columns three and four, the corresponding means of the end readings of the bubble for level direct. The fifth column contains the means of the quantities in the two preceding columns; and column six, the differences between the nth and the $(6+n)$th readings in column five. The principle used in combining the observations is the same as that employed in Examples 26 and 27. The length of the arm and the pitch of the micrometer screw are such that a rotation of the micrometer head through one division changes the inclination by 1″. Each of the displacements of the bubble in column six therefore corresponds to a change in inclination of 24″. The quotients formed by dividing the displacements into 24″ are the values of one division of the level for different portions of the tube. A similar reduction of the readings taken with the level in the reversed position gave for d the values in column eight. The means for the two series are in the last column. A glance at the results in this column is sufficient to show that the curvature of the level tube is variable.

ONE DIVISION OF A LEVEL—LEVEL TRIER

No.	Microm. Reading	Reading Middle of Bubble			Displacement	d		
		L to R	R to L	Mean		Direct	Rev'sed	Mean
1	164	12.35	12.95	12.65	14.45	1″.66	1″.73	1″.70
2	160	15.50	15.75	15.62	13.33	1.80	1.76	1.78
3	156	17.70	18.00	17.85	13.03	1.84	1.84	1.84
4	152	19.80	20.50	20.15	12.73	1.89	1.90	1.90
5	148	22.45	23.00	22.72	12.30	1.95	1.90	1.92
6	144	24.55	25.20	24.88	12.22	1.96	1.92	1.94
7	140	26.95	27.25	27.10				
8	136	28.90	29.00	28.95				
9	132	30.70	31.05	30.88				
10	128	32.60	33.15	32.88				
11	124	34.95	35.10	35.02				
12	120	37.05	37.15	37.10				

The investigation may also be carried out by a method first proposed by Comstock in which the circles of the transit are used to change the inclination of the level tube by a known amount. If the instrument is levelled, the levels themselves being in adjustment, the bubbles will remain centered when the alidade is rotated about the vertical axis. If now the vertical axis be deflected from the true vertical by a small angle i, the bubbles will not remain centered as the instrument is rotated. For any given level, however, there will be two readings of the horizontal circle, differing by 180°, for which the bubble will stand at the middle of the tube; and by rotating slightly about the vertical axis it can be brought to any desired position in the tube. The change in the inclination of the level tube corresponding to any given displacement of the bubble can be expressed in terms of i and the observed change in the reading of the horizontal circle, whence the angular value of one division of the level may be determined as before.

To express d as a function of i and the horizontal circle readings, let HC and HC' in Fig. 9 represent portions of the horizontal circles for the normal and the deflected positions of the vertical axis; L, any position of the level, which is supposed to be attached with its axis perpendicular to the radius through L and parallel to the plane of the circle; and b, the corresponding inclination. In the spherical right triangle HLC the angle H is equal to i, the

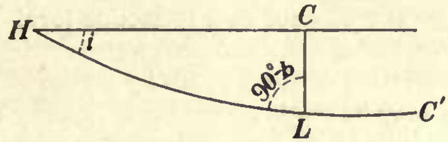

Fig. 9

deflection of the axis from the vertical, while that at L is $90° - b$. Now, if r_0 and r be the horizontal circle readings corresponding to the inclinations zero and b respectively, we find

$$\text{Arc } HL = 90° - (r - r_0),$$

whence from the triangle HLC,

$$\tan b = \tan i \sin (r - r_0). \qquad (115)$$

The angle b is very small and, for i equal two or three degrees, $r - r_0$ will never exceed one degree. We may therefore use the approximate relation

$$b = (r - r_0) \tan i, \qquad (116)$$

with an error not exceeding 0″.01.

For any other inclination, b_1, we have the analogous equation

$$b_1 = (r_1 - r_0) \tan i,$$

which, combined with (112) gives

$$b_1 - b = (r_1 - r) \tan i. \qquad (117)$$

The angle $r_1 - r$ is the change in the horizontal circle reading corresponding to the change in inclination $b_1 - b$. The latter, however, may be written equal to sd, where s is the displacement of the bubble in scale divisions, and d the angular value of one division.

We thus have finally as the expression for d

$$d = \frac{r_1 - r}{s} \tan i. \qquad (118)$$

The angle i should be two or three degrees for the investigation of the ordinary transit levels. For very sensitive levels it should be less. If the instrument be provided with a telescope level, the deflection of the vertical axis may be accomplished as follows: Level the instrument and center the telescope bubble. Then change the vertical circle reading by the angle i and, by means of the levelling screws, bring the telescope bubble back to the middle of its tube, taking care at the same time that the transverse plate bubble is also centered after the deflection. This precaution is necessary in order that the deflection may have no component perpendicular to the plane of the vertical circle. In the absence of a telescope level, level the instrument, sight on a distant object, change the vertical circle reading by i, and bring the object back to the intersection of the threads by means of the levelling screws.

The observations may be made either by displacing the bubble through a definite number of divisions and noting the corresponding change in the horizontal circle readings, or by changing the circle readings by a definite amount, say 10′, and observing the variations in the position of the bubble. For short tubes with only a few graduations the former method is more convenient, while the latter is to be preferred for the long finely graduated tubes of sensitive levels.

The bubble should be run from one end of the tube to the other and then back again, in both positions of the instrument. Such a series of readings constitutes a set.

The instrument must be as rigidly mounted as possible, preferably on a masonry pier. It is desirable to check the constancy of i by deflecting through this angle *toward* the vertical at the end of a set and noting whether the instrument is then levelled.

Example 30. Observations were made by the deflected axis method for the determination of the value of one division of the striding level of a Berger transit. The deflection was 3°. The graduations of the tube are in two groups of three each, the groups being separated

ONE DIVISION OF A LEVEL—DEFLECTED AXIS

Level Divisions	Hor. Circle Reading				Mean	$r_1 - r$	Position
	L to R		R to L				
1 and 4	341°	55′.0	341°	56′.0	55′.5	12′.5	Direct
3 and 6	342	8.0	342	8.0	8.0		Direct
3 and 6	162	6.5	162	6.3	6.4	13.4	Reversed
1 and 4	161	53.0	161	53.0	53.0		Reversed
					Mean	12.95	

$$i = 3^\circ \qquad \tan i = 8.719$$

$$r_1 - r = 12'.95 \qquad \log(r_1 - r) = 1.112$$

$$s = 2 \qquad \text{colog } s = 9.699$$

$$\log d = 9.530$$

$$d = 0'.339 = 20''.3 \quad \textit{Ans.}$$

by a space approximately equal to the length of the bubble. The horizontal circle was read when the bubble was symmetrically placed with respect to the pairs of graduations indicated in column one of the table. The circle readings themselves are in columns two and three; and the minutes of the means of corresponding settings, in column four. The differences of the readings for a displacement of the bubble through two divisions are in the fifth column. The calculation for the determination of d is in accordance with equation (118).

45. The measurement of vertical angles.—The observer will have occasion to measure the altitude not only of rapidly moving equatorial stars but also of circumpolar objects like Polaris whose positions with respect to the horizon change but slowly. The difference in motion in the two cases necessitates a difference of method in making the settings. For Polaris or any other close circumpolar object, the star should be brought to the intersection of the threads by the slow motions, the time of coincidence and the vertical circle readings being carefully noted. For stars whose altitude varies rapidly, this cannot be done with precision. The object is therefore brought into coincidence with the vertical thread near the point of intersection, and kept on the thread by slowly turning the horizontal slow motion until the instant of transit across the horizontal thread, the time and the vertical circle readings being noted as before.

Observations on the sun are most readily made with the aid of a shade of colored glass, but if this is not available, the image may be projected on a card held a few inches back of the eyepiece, by a proper focusing of the objective. In order that the threads may be seen sharply defined on the card, it is necessary that the eyepiece be drawn out a small fraction of an inch from its normal position before the solar image is focused. There are several methods by which the pointings may be made. For example, the instrument may be adjusted so that the preceding limb is near the horizontal thread and approaching the intersection. The instrument is clamped and the instant of tangency carefully noted. Then, without changing the vertical circle reading, the image is allowed to trail through the field until the transit of the following limb occurs, when the time is again noted, the instrument in the meantime being rotated by means of the horizontal slow motion so that both transits are observed at the intersection of the threads. While waiting for the second transit, the vertical circle is read. This method is open to the objection that an interval of three or four minutes separates the transits of the two limbs, which entails a considerable loss of time. The interval may be shortened by shifting the position of the telescope between the observations, but this of course requires a reading of the vertical circle for each transit. If there be more than one horizontal thread, the difficulty can be avoided by observing the transits over the extreme threads—the preceding limb over the first thread and the following limb over the last thread. The same number of settings should be made for both limbs. The mean of the readings will then correspond to the altitude of the sun's center, the influence of semidiameter being eliminated. If for any reason the program cannot be made complete in this particular, the altitude of the sun's center may still be found with the aid of the value of the semidiameter interpolated from page I of the *Ephemeris* for the instant of observation.

The arrangement of the observing program is determined by the results derived in Section 40 and summarized in Section 41. The number of settings to be made for the determination of the altitude depends upon the precision desired, the rapidity with which the observer can make the pointings and read the circle, and the position of the object. It is desirable, however, that the number should not be less than two for each position of the instrument. The maximum number to be included in a single set is limited by the fact that it is convenient to use for the reduction the means of the circle readings and the times. Since the change in the altitude of the star is not proportional to the change in the time, the two means, rigorously speaking, will not correspond to each other; but if the observing interval does not exceed a certain limit, say a quarter of an hour, no appreciable error will be introduced into results secured with the engineer's transit by treating the means as a single observation. The observing program will also depend on the method employed for the elimination of the instrumental errors i, j, c and I. Bearing in mind the various factors involved, we adopt the following as convenient arrangements for a set of observations on a star. The necessary modifications for measures on the sun will at once be suggested by the methods for making the settings described in the preceding paragraph.

Observations Direct	Observations Direct and Reflected	
Level.	Level.	
2 readings on star, C. R.	1 reading on star, direct.	C.R.
Reverse	2 readings on star, reflected.	C.R.
Level.	1 reading on star, direct.	C.R.
4 readings on star, C. L.	Reverse.	
Reverse.	1 reading on star, direct.	C.L.
Level.	2 readings on star, reflected.	C.L.
2 Readings on star, C. R.	1 reading on star, direct.	C.L.

With the first arrangement, which is to be used when all of the pointings are made directly on the star, the elimination of the errors depends upon the bubbles occupying the same positions in their tubes for both C.R. and C.L. The instrument must therefore be relevelled carefully after each reversal.

With the second, which will find application when the artificial horizon is employed, the elimination will be complete if the adjustments remain unchanged during the intervals separating the various direct observations and the corresponding reflected observations immediately preceding or following. After the instrument has once been levelled, therefore, the screws need not be touched until the set has been completed unless the bubbles should become displaced by a considerable amount.

Both verniers should be read for each setting of the telescope.

If only an approximate result is required, the observations may be discontinued at the middle of the set. On the other hand, if more precision is desired, additional sets may be observed, each of which, however, should be reduced separately.

The fact that for a short interval the change in the altitude is sensibly proportional to the change in the time makes it possible to test the consistency of the measures. For direct observations the quotients of the differences between the successive circle readings by the differences between the corresponding times must be sensibly equal. If this condition is not satisfied, an error has been committed. The errors most likely to occur are those involving mistakes of 10′ or 20′, or perhaps a whole degree, in the circle readings, and an exact number of minutes in the times. It is convenient to express differences of the circle readings in minutes of arc, and the time intervals in minutes and tenths. The quotients will thus express the change in the altitude in minutes of arc for one minute of time. If the artificial horizon has been used the quotients must be calculated for the direct and reflected observations separately. For observations on the sun, the combination of the data for the calculation of the quotients will depend upon the method followed in making the settings, and is easily derived in any special case. The test is usually sufficient to locate errors of the class mentioned with such certainty as to justify a correction of the original record, and should always be applied immediately after the completion of the set in order that the measures may be repeated if necessary. For circumpolar objects, a simple inspection will usually be sufficient to indicate the consistency of the observations.

Equations (103) and (104), and (107) show that for an instrument graduated to read altitudes, the apparent altitude, free from the instrumental errors, i, j, c, and I, will be given by forming the mean of the circle readings obtained in accordance with the above programs.

For an instrument with its vertical circle graduated continuously 0° to 360° the zenith distance will be given by

$$z_0 = \tfrac{1}{2}(v_1 - v_2) \qquad (119)$$

where the subscripts are assigned in such a manner that $v_1 - v_2 < 180°$. If the observations are direct, one v will represent the mean of all the circle readings C.R.; the other, the mean of all C.L. If the artificial horizon has been used, one v will represent the mean of all the direct readings; and the other, the mean of all the reflected readings.

The observed altitude, or zenith distance, thus derived must be corrected for refraction and parallax in accordance with Sections 8 and 9.

Example 31. The following is the record of partial sets of observations made with a Buff & Buff engineer's transit at the Laws Observatory, on 1908, Oct. 2, Friday P. M., for the determination of the altitudes of Polaris and Alcyone. The measures were all direct. The timepiece used was an Elgin watch.

An inspection of the readings for Polaris shows that the measures are consistent. The relatively large difference in the readings C. R. and C. L. reveals the existence of an index error of 2′ or 3′.

Polaris

Watch	Vertical Circle Ver. A	Vertical Circle Ver. B	Circle
$8^h 31^m 19^s$	39° 26′	39° 26′	R
33 25	26	26	R
36 50	33	33	L
36 11	34	34	L
$8^h 35^m 11^s$	39° 29′.8		

Alcyone

Watch	Vertical Circle Ver. A	Vertical Circle Ver. B	Circle	Rate
$9^h 35^m 21^s.4$	20° 40′	20° 40′	L	11′.2
37 1.2	20 59	20 59	L	
41 1.4	21 38	21 38	R	11.5
43 1.4	22 1	22 1	R	
$9^h 39^m 6^s.4$	21° 19′.5			

For Alcyone the close agreement of the values for the rate of change in altitude per minute of time given in the last column is evidence of the consistency of the measures.

The quantities in the fifth line are the means. The angles are the apparent altitudes corresponding to the watch times immediately preceding. To obtain the true altitudes a correction for refraction, which may be obtained from Table I, page 20, must be applied.

Example 32. The following observations were made with a Berger engineer's transit on 1908, October 15, Thursday P. M., for the determination of the altitude of the sun. The measures were all direct and were made by projecting the image of the sun on a card. The transits were observed over the middle horizontal thread, the telescope being shifted after each transit. The timepiece was the Fauth sidereal clock of the Laws Observatory.

	Fauth Clk.	Vertical Circle Ver. A	Vertical Circle Ver. B	Limb	Circle	Rate
	$16^h 37^m 51^s.2$	24° 1′	24° 1′	F	R	9′.6
	40 46.8	23 33	23 33	F	R	
	43 34.3	22 35	22 35	P	R	9.2
	45 19.1	22 19	22 19	P	R	
	47 7.5	22 26	22 27	F	L	9.0
	48 47.3	22 11	22 11	F	L	
	50 11.7	21 25	21 25	P	L	10.0
	51 29.9	21 12	21 12	P	L	
Means	$16^h 45^m 38^s.5$	22° 27′.8				

46. The measurement of horizontal angles.—It is assumed that the two objects whose difference of azimuth is to be determined are a terrestrial mark and a celestial body, either the sun or a star. The directions given in Section 45 for making settings in the measurement of vertical angles apply here with only slight and obvious modifications. The conditions determining the arrangement of the observing program are similar to those enumerated in the present section. Although the details may vary with circumstances, the following will serve to indicate the essentials. The first arrangement is intended for use when only an approximate result is required, while the second and third are designed for more precise determinations. The first two include only direct observations, while the last is arranged for measures in which the artificial horizon is employed. In direct observations care should be taken to keep the bubbles centered throughout, but when the artificial horizon is used, the levelling screws must not be touched between any direct observation and its corresponding reflected setting. For settings on the mark the zenith distance will usually be so nearly equal to 90° that the error due to the deviation of the vertical axis from the true vertical will be quite insensible, even though no special effort be made to eliminate its influence.

DIRECT OBSERVATIONS

1 setting on mark } C.R.
2 settings on star }

2 settings on star } C.L.
1 setting on mark }

DIRECT OBSERVATIONS	DIRECT AND REFLECTED OBSERVATIONS
1 setting on mark C.R.	1 setting on mark C.R.
1 setting on mark C.L.	1 setting on mark C.L.
3 settings on star C.L.	1 setting on star, direct } C.L.
3 settings on star C.R.	1 setting on star, reflected } C.L.
1 setting on mark C.R.	1 setting on star, reflected } C.R.
1 setting on mark C.L.	1 setting on star, direct } C.R.
	1 setting on mark C.R.
	1 setting on mark C.L.

Both verniers of the horizontal circle should be read for each setting, and for those made on the star, the time should be noted in addition.

The required difference of azimuth will be the difference between the means of the readings on the mark and on the star. Its value will correspond to the mean of the times. If more precision is desired than can be obtained from a single set, several sets may be observed, each of which should be reduced separately. To reduce the influence of graduation error, the horizontal circle should be shifted between the sets. If the number of sets is n, the amount of the shift between the successive sets should be $360°/n$.

Example 33. The following is the record of a simultaneous determination of the altitude of Polaris and the difference in azimuth of Polaris and a mark.

ALTITUDE OF POLARIS AND AZIMUTH OF MARK NO. 2

1908, Oct. 13, Thursday P. M. — Observer Sh.
Station No. 2 — Recorder W.

Buff & Buff Engineer's Transit No. 5606

$\Delta T_W = -38^s.7$ at $7^h\ 59^m$ P.M., and $-31^s.4$ at $9^h\ 54^m$ P.M.

Object	Watch	Hor. Circle Ver. A	Hor. Circle Ver. B	Vertical Circle Ver. A	Vertical Circle Ver. B	Circle
Mark	———	147° 23'.5	327° 23'.0	———	———	R
Polaris	$9^h\ 35^m\ 23^s$	322 10.0	142 10.0	39° 51'	39° 51'	R
Polaris	40 35	8.0	8.0	51	51	R
Polaris	44 33	142 6.5	322 6.5	59	59	L
Polaris	48 8	4.5	5.0	59	58	L
Mark	———	327 23.5	147 23.0	———	———	L
Means	$9^h\ 42^m\ 10^s$	Star 322° 7'.31 / Mark 147 23.25		39° 54'.9 = Appt. Alt.		

Difference of Azimuth $S-M = 174$ 44.06

47. The method of repetitions.—The precision of the measurement of the azimuth difference, D, of two objects, A and B, may be increased materially by making a series of alternate settings on A and B such that the rotation from A to B is always made with the upper motion of the instrument, and that from B to A with the lower motion. Assuming that the graduations of the horizontal circle increase in the direction AB, each turning from A to B will increase the reading by the angle D, while that from B back to A will produce no change since during this rotation the vernier remains clamped to the circle. If the turning from A to B is repeated n times, the difference between the circle readings for the final setting on B and the initial setting on A will be nD; and if the initial and final readings be R_1 and R_2, repectively, we shall have

$$D = \frac{R_2 - R_1}{n}. \tag{120}$$

The method of repetitions derives its advantage from the fact that the circle is not read for the intermediate settings on A and B. Not only is the observer thus spared considerable labor, but, what is of more importance, the errors which necessarily would affect the readings do not enter into the result. Consequently, that part of the resultant error of observation arising from the intermediate settings is due solely to the imperfect setting of the cross threads on the object. For instruments such as the engineer's transit, in which the uncertainty accompanying the reading of the angle is large as compared with that of the pointing on the object, the precision of the result given by (120) will be considerably greater than that of the mean of n separate measurements of the angle D, each of which requires two readings of the circle. But for instruments in which the accuracy of the readings is comparable with that of the pointings, as is the case with the modern theodolite provided with reading microscopes, the method of repetitions is not to be recommended. Although there is even here a theoretical advantage, it is offset by the fact that the peculiar observing program required for the application of the method presupposes the stability of the instrument for a relatively long interval, and hence affords an unusual opportunity for small variations in position to affect the precision of the measures. Moreover, experience has shown that there are small systematic errors dependent upon the direction of measurement, *i.e.* upon whether the initial setting is made on A or on B; and, although these may be eliminated in part by combining series measured in opposite directions, it is not certain that the compensation is of the completeness requisite for observations of the highest precision. With the engineer's transit, however, the method of repetitions may be used with advantage.

Since rotation takes place on both the upper and the lower motions, any non-parallelism of the vertical axes will affect the readings; and the observing program must be arranged to eliminate this along with the other instrumental errors. For any given setting the deviation of the axis from parallelism, p, unites with the inclination of the lower axes to the true vertical, i', and determines the value of i, the inclination of the upper axis to the vertical, for the setting in question. For different settings i will be different, for a rotation of the instrument on the lower motion causes the upper axis to describe a cone whose apex angle is $2p$ and whose axis is inclined to the true vertical by i'. But no matter what the magnitude of i may be, within certain limits easily including all values arising in practice, it may be eliminated by forming the mean of direct and reflected readings made in the same position of the instrument, provided that i is the same in direction and magnitude for both settings. This follows from the discussion on pages 66 and 67 whose result is expressed by equation (88). Hence, if after a series of n repetitions observed C.R. direct, n further repetitions be made C.R. reflected, such that the vernier readings for the corresponding settings in the two series are *approximately the same*, the instrumental errors i' and p will be eliminated. Equation (88) shows that j, the deviation of the upper vertical axis from perpendicularity with the horizontal axis, will also be eliminated. To remove the influence of the collimation, c, the entire process must be repeated C.L.; and to neutralize the systematic error dependent upon the direction of measurement, the direct and reflected series should be measured in opposite directions. We thus have the following observing program, in which A' and B' denote the reflected images of A and B, respectively:

Level on the lower motion.

Direct	Set on A and read the horizontal circle. Turn from A to B on the upper motion n times. Read the horizontal circle for last setting on B.	C.R.
Reflected	Set on B' and read the horizontal circle. Turn from B' to A' on the upper motion n times. Read the horizontal circle for last setting on A'.	C.R.

Repeat for C.L.

The circle reading for the first setting on B' must be the same, approximately at least, as that for the last setting on B.

The mean of the values of D calculated from the four series is the required azimuth difference of A and B.

Uusually one of the objects, say A, will be near the horizon, in which case reflected settings on A' will be impossible. A must then be substituted for A' in the above program. The error due to i will not be eliminated from these settings; but, owing to the presence of the factor $\cot z_0$, it may be disregarded.

When the artificial horizon is not used the program must be modified. Were i' zero, i would constantly be equal to p, although the direction of the deflection would change with a rotation of the instrument on the lower motion. If a series of n repetitions C.R. be made under these circumstances, equation (82) shows that each setting will be affected by an error of the form

$$j \cot z_0 + p \cos l \cot z_0 + c \operatorname{cosec} z_0.$$

The first and last terms of this expression will have the same values for all pointings on the same object. Equations (82), (84), and (86) show that they may be eliminated by combining with a similar series made C.L. The values of the second term will be different for each setting owing to the change in l, but their sum will be zero if the values of l are uniformly distributed throughout 360°, or any multiple of 360°. In order that this may be the case, approximately at least, it is only necessary that n be the integer most nearly equaling $k\,360°/D$, where the k is any integer, in practice usually 1 or 2.

It is also easily seen that, if after any arbitrary number of settings the instrument be reversed about the *lower* motion and the series repeated in the reverse order, the sum of the errors involving p will be zero, provided that the circle readings for corresponding settings C.R. and C.L. are the same, or approximately so. The reversal of the instrument on the lower motion changes the direction of the deflection p by 180°. The values of l for corresponding settings C.R. and C.L. will therefore differ by 180°, and the errors will be opposite in sign and will cancel when the mean of the two series is formed. The reversal also eliminates the influence of j and c as indicated in the preceding paragraph.

The above assumes that the deflection of the lower axis, i', is zero. If this is not the case, each setting will be affected by an additional error of the form $i' \cos l' \cot z_0$, in which l' is constant so long as i' remains unchanged in direction. If i' be the result of a non-adjustment of the plate bubbles, the error which it produces may be eliminated from the mean of two series, one C.R. and one C.L., by relevelling after reversal. (See page 70.) This will change the direction of i' by 180°. Consequently, the values of l' for C.R. and C.L. will differ by 180°, and the errors for the two positions will neutralize each other when the mean is formed.

The consideration of these results leads to the following arrangement of the observing program.

Level on the lower motion.

Set on A and read horizontal circle. Turn from A to B on upper motion n times. Read horizontal circle for last setting on B.	} C.R.

Reverse on *lower* motion and relevel.

Set on B and read horizontal circle. Turn from B to A on upper motion n times. Read horizontal circle for last setting on A.	} C.L.

The circle reading for the first setting on B, C.L. should be the same, approximately at least, as that for the last setting on B, C.R.

The mean of the values of D calculated from the two series is the required azimuth difference of A and B.

With this arrangement the instrumental errors i', p, j, and c will be completely eliminated, whether the settings are distributed through 360° or not, provided only that the instrumental errors remain constant during the observations. Practically, it is desirable that the value of n should be such that nD equals 360°, or a multiple of 360°, at least approximately; but when D is small this may unduly prolong the observations. The maximum number of repetitions which can be made advantageously depends upon the stability of the instrument and must be determined by experience.

If the instrument is provided with a striding level, the influence of i', p, and j may be taken into account by measuring the inclination of the horizontal axis for each setting and applying a correction to R_1 and R_2 of the form $b \cot z_0$, in which b denotes the sum of all the observed inclinations for settings on A and B respectively.

When one of the objects, say B, is a star, the time of each setting on B must be noted. The calculated value of D will then correspond sensibly to the mean of the times, provided the observing program be not too long.

Example 34. On 1909, April 9, the following observations of the difference in azimuth of Polaris and a mark were made by the method of repetitions with a Buff & Buff engineer's transit. The recorded times are those of a Fauth sidereal clock whose error was $+6^m 36^s$. After four repetitions C.R., the instrument was reversed on the lower motion, relevelled, and the series repeated in the reverse order. Since the azimuth difference is approximately 174°, 720° must be added to the readings on the star before combining them with those on the mark. The results for the two halves are derived separately, although the means for the set are also given.

Object	θ_F	Hor. Circle Ver. A	Hor. Circle Ver. B	Circle	Means
Mark	———	179° 59′.5	359° 59′.5	R	179° 59′ 30″
Polaris	$9^h\ 27^m\ 30^s$	353 32		R	
Polaris	30 48			R	
Polaris	33 36			R	
Polaris	35 49	154 13.5	334 13.5	R	874 13 30
	4) 7 43				4)694 14 0
	9 31 56	$\theta = 9^h\ 38^m\ 32^s$			$S-M=$ 173 33 30
Polaris	9 39 12	154 13.0	334 13.5	L	874 13 15
Polaris	42 28			L	
Polaris	44 9			L	
Polaris	46 20			L	
Mark	———	179 51.0	359 51.0	L	179 51 0
	4) 12 9				4)694 22 15
	9 43 2	$\theta = 9^h\ 49^m\ 38^s$			$S-M=$ 173 35 34
Final Means		$\theta = 9\ 44\ 5$			$S-M=$ 173 34 32

THE SEXTANT

48. Historical and descriptive.—The instruments typified by the engineer's transit may be used for the measurement of horizontal or vertical angles only. Simultaneously with the development of the altazimuth principle there was gradually evolved a contrivance adapted for the measurement of angles lying in any plane. Beginning with the astrolabe of the ancients, the application of various ideas gave in succession the Jacob's staff, or cross-staff, which dates apparently from the middle of the fourteenth century, the back-staff, or Davis quadrant; the sextant of Tycho, which was also used by the Arabs in the tenth century; the octants of Hooke and Fouchy, in which a mirror was used for the first time; and, finally, the reflecting octant whose principle was due to Newton, although the construction was first carried out by John Hadley about 1731. The instrument of Hadley has been improved in design, but no essential modification has been made in its principle. In its modern form it is known as the **reflecting sextant,** or more generally, simply as the **sextant.**

With the exception of the astrolabe and the large fixed sextants of Tycho, the various forms mentioned are characterized by the fact that they may be held in the hand during observations, small oscillations and variations in the position of the instrument offering no serious difficulty in the execution of the measures. These instruments have therefore played an important part in the practice of navigation, and to-day the sextant is the only instrument which can advantageously be employed in the observations necessary for the determination of a ship's position. In addition, its compactness and lightness, and the precision of the results that may be obtained with it render it one of the most convenient and valuable instruments at our command.

The modern sextant consists of a light, flat, metal frame supporting a graduated arc, usually 70° in length; a movable index arm; two small mirrors perpendicular to the plane of the arc; and a small telescope. The index arm is pivoted at the center of the arc and has rigidly attached to it one of the mirrors, the **index glass**, whose reflecting surface contains the rotation axis of the arm and the attached mirror. The position of the index glass corresponding to any setting may be read from the graduated arc by means of a vernier. The second mirror, the **horizon glass**, is firmly attached to the frame of the sextant in a manner such that when the vernier reads zero the two mirrors are parallel. Only that half of the horizon glass adjacent to the frame is silvered. The telescope, whose line of sight is parallel to the frame, is directed toward the horizon glass, and with it a distant object may be seen through the unsilvered portion. When the frame is brought into coincidence with the plane determined by the object, the eye of the observer, and any other object, a reflected image of the second object may be seen in the field of the telescope, simultaneously with the first, by giving the index arm a certain definite position depending upon the angular distance separating the objects. If the position of the arm is such that the rays of the second object reflected by the index glass to the horizon glass, and then from the silvered portion of the latter, enter the telescope parallel to the rays that pass from the first object through the unsilvered portion of the horizon glass, the two images will be seen in coincidence. This being the case, the relative inclination of the mirrors as shown below, will be one-half the angular distance separating the objects; and, since the construction is such that the inclination may be read from the graduated arc, it becomes possible to find the angular distance between the objects. The use of the instrument is simplified by graduating the arc so that the vernier reading is twice the inclination of the mirrors, and hence, directly, the angular distance of the objects. With the usual form of the instrument the maximum angle that can be measured is therefore about 140°. The two mirrors and the telescope are provided with adjusting screws, which may be used to bring them accurately into the positions presupposed by the theory of the instrument. In addition, the telescope may be moved perpendicularly back and forth with respect to the frame thus permitting an equalization of the intensity of the direct and reflected images by varying the ratio of the reflected and transmitted light that enters the telescope. Adjustable shade glasses adapt the instrument for observations on the sun.

49. The principle of the sextant.—In Fig. 10 let OV represent the graduated arc; I and H, the index glass and the horizon glass, respectively; and IV, the index arm, pivoted at the center of the arc and provided with a vernier at V. When V coincides with O, the mirrors are parallel. The position indicated in the figure is such that the two objects S_1 and S_2 are seen in coincidence, for the rays from S_1 pass through the unsilvered portion of H and enter the telescope in the direction HE, while those from S_2 falling on I are reflected to H and thence in the direction HE. The two beams therefore enter the telescope parallel.

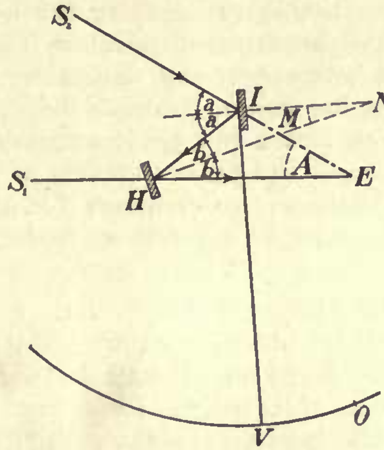

Fig. 10.

It is to be shown that the inclination of I to H is one-half the angular distance A separating the objects. IN and HN are the normals to the mirrors, and by the fundamental laws of reflection they bisect the angles S_2IH and IHE, respectively. In the triangle IHE.

$$2a = 2b + A,$$

whence

$$a = b + \tfrac{1}{2} A.$$

But in the triangle IHN

$$a = b + M.$$

Therefore,

$$M = \tfrac{1}{2} A.$$

But M, being the angle between the normals to the mirrors, measures their inclination, and is equal to the angle subtended by the arc OV, whence

$$A = 2OV. \qquad (121)$$

But since the arc is graduated so that the reading is twice the angle subtended by OV the angular distance between the two objects is given directly by the scale.

50. Conditions fulfilled by the instrument.—The following conditions, among others, must be fulfilled by the perfectly adjusted sextant.

1. The index glass must be perpendicular to the plane of the arc.
2. The horizon glass must be perpendicular to the plane of the arc.
3. The axis of the telescope must be parallel to the plane of the arc.
4. The vernier must read zero when the mirrors are parallel.
5. The center of rotation of the index arm must coincide with the center of the graduated arc.

Since the positions of the mirrors and the telescope are liable to derangement, methods must be available for adjusting the instrument as perfectly as possible. This is the more important inasmuch as it is impossible to eliminate from the measures the influence of any residual errors in the adjustments. Although elimination is impossible, it should be remarked that the errors arising in connection with Nos. 4 and 5, at least, may be determined by the methods given in Sections 52 and 53, and applied as corrections to the readings obtained with the instrument. Conditions Nos. 1–4 are within the control of the observer. No. 5 must be satisfied as perfectly as possible by the manufacturer.

51. Adjustments of the sextant.—No. 1. *Index glass.* To test the perpendicularity of the index glass, place the sextant in a horizontal position, unscrew the telescope and stand it on the arc just in front of the surface of the index glass produced. If then the eye be placed close to the mirror, the observer will see the reflected image of the upright telescope alongside the telescope itself. By carefully moving the index arm, the telescope and its image may be brought nearly into coincidence. If the two are parallel, the index glass is in adjustment. The telescope should be rotated about its axis in order to be sure that it is perpendicular to the plane of the arc. If the adjustment is imperfect, correction must be made by the screws at the base of the mirror. Some instruments are not provided with the necessary screws, and in such cases the adjustment had best be entrusted to an instrument maker.

The test can also be made by looking into the index glass as before, and noting whether the arc and its reflected image lie in the same place. If not, the position of the mirror must be changed until such is the case.

No. 2. *The horizon glass.* The adjustment of the horizon glass may be tested by directing the telescope toward a distant, sharply defined object, preferably a star, and bringing the index arm near the zero of the scale. Two images of the object will then be seen in the field of view—one formed by the rays transmitted by the horizon glass, the other, by those reflected into the telescope by the mirrors. The reflected image should pass through the direct image as the index arm is moved back and forth by the slow motion. If it does not, the horizon glass is not perpendicular to the plane of the arc, and must be adjusted until the direct and reflected images of the same object can be made accurately coincident.

No. 3. *The telescope.* The parallelism of the telescope to the frame may be tested by bringing the images of two objects about 120° apart into coincidence at the edge of the field nearest the frame. Then, without changing the reading, shift the images to the opposite side of the field. If they remain in coincidence, the telescope is in adjustment. If not, its position must be varied by means of the adjusting screws of the supporting collar until the test is satisfactory.

No. 4. *Index adjustment.* If the fourth condition is not fulfilled, an **index error** will be introduced into the angles read from the scale. To test the adjustment, bring the direct and reflected images of the same distant object into the coincidence as in the adjustment of the horizon glass. The corresponding scale reading is called the **zero reading** $= R_0$. If R_0 is zero, the adjustment is correct. If not, set the index at 0°, and bring the images into coincidence by means of the proper adjusting screws attached to the horizon glass. *It is better, however, to disregard this adjustment* and correct the readings by the amount of the index error.

It can be shown that the errors affecting the readings as a result of an imperfect adjustment of the index glass, the horizon glass, and the telescope are of the order of the squares of the residual errors of adjustment. If care be exercised in making the adjustments, the resulting errors will be negligible as compared with the uncertainty in the readings arising from other sources.

52. Determination of the index correction.—Make a series of zero readings on a distant, sharply defined object, a star if possible. If the zero of the vernier falls to the right of the zero of the scale, do not use negative readings, but consider the last degree graduation preceding the zero of the scale as 359°, and read in the direction of increasing graduations. The zero reading is what the instrument actually reads when it should read zero. The index correction, I, is the quantity which must be added algebraically to the scale readings to obtain the true reading. We therefore have

$$I = 0° - R_0, \qquad (122)$$
$$I = 360° - R_0. \qquad (123)$$

The latter expression is to be used for the determination of I when the zero of the vernier falls to the right of that of the scale for coincidence of the direct and reflected images of the same object.

When observations are to be made on the sun, the index correction should be determined from measures on this object. Since it is impossible, on account of their size, to bring the solar images accurately into coincidence, we determine the zero reading as follows: Make the two images externally tangent, the reflected being above the direct, and read the vernier. Let R_1 represent the mean of a series of such readings. Then make an equal number of settings for tangency with the reflected image below. Call the mean of the corresponding readings R_2. The mean of R_1 and R_2 will then be the value of the zero reading, and we shall have

$$I = 0° - \tfrac{1}{2}(R_1 + R_2), \qquad (124)$$
$$I = 360° - \tfrac{1}{2}(R_1 + R_2). \qquad (125)$$

The readings thus obtained will also give the value of S, the sun's semi-diameter. Since the center of the reflected image moves over a distance of four semi-diameters in shifting from the first position to the second, we have

$$S = \tfrac{1}{4}(R_1 - R_2). \qquad (126)$$

Owing to the brilliancy of the solar image, its diameter appears larger than it really is—a phenomenon known as irradiation. Should the value of S be required for the reduction of observations on the sun (see Section 55), the value calculated from equation (126) should be used rather than that derived from the Ephemeris, in order that the influence of irradiation may be eliminated.

53. Determination of eccentricity corrections.—Any defect in the fifth condition introduces an eccentricity error into the readings. Since, with the usual form of the instrument there is but a single vernier, this cannot be eliminated. Each sextant must be investigated specially for the determination of the eccentricity errors affecting the readings for different parts of the scale. These may be found by measuring a series of known angles of different magnitudes. The mean result for each angle, A, gives by (71) an equation of the form

$$A = R + I + E_0 - E, \qquad (127)$$

where R is the sextant reading for coincidence of the two objects whose angular distance is A; I, the index correction; and E_0 and E, the eccentricity corrections for those graduations of the scale which coincide with vernier graduations for the readings R_0 and R, respectively. The readings of the coinciding graduations when the vernier reads R_0 and R may be denoted by R_0' and R', respectively. $E_0 - E$ is the correction which must be applied to the sextant reading, freed from index correction, in order to obtain the true value of the angle. Denoting its value by ε, (127) may be written

$$\varepsilon = A - (R + I). \qquad (128)$$

Having determined ε from (128) for a considerable number of angles distributed as uniformly as possible over the scale, the results may be plotted as ordinates with the corresponding values of R' as abscissas. From the plot a table may be constructed giving the values of ε for equidistant values of R', from which the value of ε for any other reading, R, can then be derived. Care should be taken always to enter the table with the R' corresponding to the given R as argument. It should be noted that the usefulness of the table depends upon I remaining sensibly constant, for if the index correction changes by any considerable amount, R_0' may change sufficiently to render the tabular values of ε no longer applicable.

The chief difficulty in investigating the eccentricity of a sextant consists in securing a suitable series of known angles. A simple method is to measure with a good theodolite the angles between a series of distant objects, nearly in the horizon, care being taken to tilt the instrument so that in turning from one object to the next no rotation about the horizontal axis is necessary.

54. **Precepts for the use of the sextant.**—The following points should carefully be noted in using the sextant: Focus the telescope accurately. The image of a star should be a sharply defined point; that of the sun must show the limb clearly defined and free from all blurring. For solar observations, use, whenever possible, shade glasses attached to the eyepiece rather than those in front of the mirrors; and reduce the intensity of the images as much as is consistent with clear definition. If the use of the mirror shade glasses cannot be avoided, select those which will make the direct and reflected images of the same color, and reverse them through 180° at the middle of the observing program to eliminate the effect of any non-parallelism of their surfaces. If a roof is used to protect the surface of the mercury from wind, it also should be reversed at the middle of the program. In all cases make the direct and reflected images of the same intensity by regulating the distance of the telescope from the frame. Make the adjustments in the order in which they are given above, and always test them before beginning observations. The index correction should be determined both before and after each series of settings. Make all coincidences and contacts in the center of the field. Finally, the instrument should be handled with great care, for a slight shock may disturb the adjustment of the mirrors and change the value of the index correction.

55. **The measurement of altitudes.**—Although the sextant may be used for the measurement of angles lying in any plane, it finds its widest application in practical astronomy in the determination of the altitude of a celestial body.

At sea the observations are made by bringing the reflected image of the body into contact with the image of the distant horizon seen directly through the unsilvered portion of the horizon glass. To obtain the true reading the plane of the arc must be vertical. Practically, the matter is accomplished by rotating the instrument back and forth slightly about the axis of the telescope, which causes the reflected image to oscillate along a circular arc in the field. The index is to be set so that the arc is tangent to the image of the horizon. The corresponding reading corrected for index correction, dip of horizon, and refraction is the required altitude. The correction for dip is necessary, since, owing to the elevation of the observer, the visible horizon lies below the astronomical horizon. The square root of the altitude of the observer above the level of the sea, expressed in feet, will be the numerical value of the correction in minutes of arc. The observations are not susceptible of high precision, and the correction for eccentricity may be disregarded as relatively unimportant.

For observations on land the artificial horizon must be used. The measurement of the angular distance between the object and its mercury image gives the value of the double altitude of the object. Some practice is required in order to be able to bring the object and its mercury image into coincidence quickly and accurately. In case the object is a star, care must be taken that the images coinciding are really those of the object and its reflection in the mercury. The following is the simplest method of procedure: Stand in a position such that the mercury image is clearly visible in the center of the horizon, and direct the telescope toward the object. By bringing the index near zero the reflected image will appear in the field. The telescope is then turned slowly downward toward the mercury, the index being moved forward along the arc at the same time at a rate such that the reflected image of the object remains constantly in the field. If the plane of the sextant is kept vertical, and if the observer is careful to stand so that the mercury reflection can be seen, its image seen directly through the unsilvered portion of the horizon glass will come into the field when the telescope has been sufficiently lowered. Both images should then be visible. The varying altitude of the object will cause them to change their relative positions. The index is set so that the images are approaching and clamped. When they become coincident the time is noted and the vernier read. The instant of coincidence is best determined by giving the instrument a slight oscillatory motion about the axis of the telescope and noting the time when the reflected image in its motion back and forth across the field passes through the direct image.

To obtain an accurate value of the altitude, a series of such settings should be taken in quick succession, the time and the vernier reading being noted for each. It is not necessary to use the method described above for bringing the images into the field for any of the settings but the first; for if, after reading, the index be left clamped and the telescope be directed toward the mercury image, the plane of the arc being held vertical, the reflected image will also be in the field. If it is not at once seen, a slight rotation about the axis of the telescope will bring it into view, unless too long an interval has elapsed.

Measures for altitude may also be made by setting the zero of the vernier accurately on one of the scale divisions so that the images are near each other and approaching a coincidence. The time of coincidence and the vernier reading are noted. The index is then moved 20′ so that the images will again be approaching coincidence. The time and the reading are noted as before and the process is repeated until a sufficient number of measures has been secured.

The consistency of the measures should always be tested, as in the case of the engineer's transit (see page 79) by calculating the rate of change of the readings per minute of time. If however, the observations have been made by noting the times of coincidence for equidistant readings of the vernier, the constancy of the time intervals between the successive settings will be a sufficient test.

If R denote the mean of the vernier readings, the apparent double altitude of the object will be given by

$$2h' = R + I + \varepsilon, \qquad (129)$$

in which I is the index correction, and ε the correction for eccentricity. The true altitude corresponding to the mean of the observed times is found from

$$h = h' - r,$$

where the refraction, r, may be derived from Table I, page 20, or if more accurate results are required, by equation (3), page 18. If the zenith distance is desired instead of the altitude, we calculate z' from

$$z' = 90^\circ - h', \tag{130}$$

and z from

$$z = z' + r. \tag{131}$$

For measures on the sun coincidences are not observed, but, instead, the instants when the images are externally tangent. To eliminate the influence of semidiameter, the same number of contacts should be observed for both images approaching and images receding. If for any reason this cannot be done, a correction for semidiameter must be applied. Let

n_a = number of settings for images approaching,
n_r = number of settings for images receding,
n = total number of settings,
S = the semidiameter of the sun calculated by equation (126).

We shall then have for solar observations

$$2h' = R \pm \frac{n_a - n_r}{n} S + I + \varepsilon, \quad \left\{ \begin{array}{l} \text{Upper sign, altitude decreasing.} \\ \text{Lower sign, altitude increasing.} \end{array} \right\} \tag{132}$$

in which h' is the apparent altitude of the sun's center corresponding to the mean of the observed times; and the term involving S, the correction for semidiameter. The true altitude and zenith distance are then given by

$$h = h' - r + p, \tag{133}$$
$$z = z' + r - p, \tag{134}$$

The solar parallax, p, may be obtained from columns four and eight of Table I, page 20. For approximate results $r - p$ may be taken from the fifth and tenth columns of this same table.

Example 35. On 1909, April 10, the following sextant observations of the altitude of the sun were made at the Laws Observatory near the time of meridian transit. The error of the timepiece was $\Delta\theta_p = + 6^m\ 37^s$. The observations will be reduced later for the determination of latitude.

Readings on Sun for Index Correction					
R_1			R_2		
0°	31′	30″	359°	28′	0″
	31	40		28	20
	32	0		28	0
	32	10		28	10
0	31	50	359	28	8

Zero Reading = 359 59 59
Index Corr. = + 1
$R_1 - R_2$ = 1 3 42
Semidiameter = 15 56

θ_F			Reading			Limb
0^h	59^m	22^s	117°	18′	50″	Lower
1	1	28	118	23	10	Upper
	2	29	118	25	50	Upper
	4	9	117	24	0	Lower
	5	19	117	24	30	Lower
	6	53	118	28	40	Upper
	8	27	118	29	0	Upper
	12	8	117	24	10	Lower

CHAPTER V

THE DETERMINATION OF LATITUDE

56. Methods.—On page 34 it was shown that if the zenith distance or altitude of a star of known right ascension and declination be measured at a known time, the latitude of the place of observation can be determined by means of equation (31). The preceding chapter indicates the methods that may be employed for the measurement of the zenith distance. It is the purpose of the present chapter to determine the most advantageous method of using the fundamental equation and to develop the formulæ necessary for the practical solution of the problem.

To establish a criterion for the use of equation (31), it is to be noted that the resultant error of observation in φ will depend upon the errors affecting δ, z, θ, and α. Star positions are so accurately known, however, that the errors in α and δ are insignificant as compared with those occurring in z and θ; and we need concern ourselves only with those affecting the latter two quantities. It is particularly important to know the influence of an error in the time, for since this quantity is assumed to be known, it is desirable to be able to specify *how accurately* it must be given in order to obtain a definite degree of precision in the latitude.

The relation connecting small variations in z and θ with changes in φ is found by differentiating (31), z, $t = \theta - \alpha$, and φ being considered variable. (Num. Comp. p. 11.) We thus find

$$-\sin z\,dz = \sin\delta\cos\varphi\,d\varphi - \cos\delta\sin\varphi\cos t\,d\varphi - \cos\delta\cos\varphi\sin t\,dt, \qquad (135)$$

which by means of (32) and (33) reduces to

$$dz = \cos A\,d\varphi + \sin A\cos\varphi\,dt.$$

Writing $dt = d\theta$ and solving for $d\varphi$

$$d\varphi = \sec A\,dz - \tan A\cos\varphi\,d\theta. \qquad (136)$$

Assuming now that the differentials of z and θ represent the errors in these quantities, the resultant error in φ will be given by (136). In order that this may be a minimum, $\sec A$ and $\tan A$ must have their minimum absolute values, which will occur when A is 0° or 180°. Since these quantities increase as the azimuth deviates from 0° or 180°, the object observed for the determination of latitude should be as near the meridian as possible. Even with this limitation there will be considerable variety in the procedure depending upon the position of the star and the circumstances of the observations; and we now proceed to the consideration of the following five cases in which the given data are, respectively,

1. The zenith distance of an object when on the meridian,
2. The difference of the meridian zenith distances of two stars,
3. A series of zenith distances when the object is near the meridian,
4. The zenith distance of an object at any hour angle,
5. The altitude of Polaris at any hour angle.

I. MERIDIAN ZENITH DISTANCE

57. Theory.—The hour angle of an object on the meridian is zero. For this case equation (31) reduces to

$$\cos z_0 = \cos (\varphi - \delta), \qquad (137)$$

whence

$$\pm z_0 = \varphi - \delta,$$

or

$$\varphi = \delta \pm z_0. \qquad (138)$$

Equation (138) may also be derived geometrically by means of Fig. 4, p. 24 whence it is seen that the upper sign must be used for objects south of the zenith; and the lower, for objects between the zenith and the pole. For lower culmination the fundamental relation becomes

$$\varphi = 180^\circ - \delta - z_0. \qquad (139)$$

58. Procedure.—For the instant of observation we have by (35) $\theta = a$. If $\Delta\theta$ be the error of the timepiece, the clock time of transit will be

$$\theta' = a - \Delta\theta, \qquad (140)$$

where a, along with δ, is to be interpolated from the *Ephemeris* for the instant of observation. The true zenith distance is then to be determined by some one of the methods of Section 45 or 55 for the clock time θ'. Equation (138) or (139) will then give the required value of the latitude.

If a mean timepiece is used, the sidereal time of transit must be converted into the corresponding mean time, T, by equations (62) and (41), pp. 49 and 39, respectively. The clock time of obervation is then given by

$$T' = T - \Delta T. \qquad (141)$$

In case the error of the timepiece is uncertain, the observer will bring the image to the intersection of the threads, or the direct and reflected images into coincidence if the sextant is used, a little before the time of transit and follow with the slow motion until it becomes necessary to reverse the direction in which the tangent screw is turned in order to keep the image on the thread. This instant marks the time of meridian passage. The corresponding reading, properly corrected, then gives the altitude as before.

Example 36. On 1909, April 10, an observation was made at the Laws Observatory with a sextant for the determination of the latitude by a meridian altitude of the sun. The reading on the upper limb at the calculated time of transit was 118° 29′ 10″. The error of the clock, the index correction, and the semidiameter to be used are those of Ex. 35. The calculation of the clock time of transit is in the left hand column. The reduction of the observation for the determination of the latitude is in the second column.

Gr. A. T. of Col. A. N. $= 6^h\ 9^m\ 18^s = 6^h.155$	$R = 118°\ 29'\ 10''$
Sun's α at Col. A. N. $= 1\ \ 14\ \ 42$	$I = \qquad + 1$
$\Delta\theta_F = +6\ \ 37$	$\epsilon =$ Unknown
$\theta' = 1\ \ 8\ \ 5$	$2h' = 118\ \ 29\ \ 11$
	$h' = 59\ \ 14\ \ 36$
The true value of the latitude is	$z' = 30\ \ 45\ \ 24$
known to be 38° 56′ 52″	$r - p = \qquad 30$
	$S = \quad 15\ \ 56$
	$z = 31\ \ 1\ \ 50$
	$\delta = +7\ \ 54\ \ 29$
	$\varphi = 38\ \ 56\ \ 19$

2. DIFFERENCE OF MERIDIAN ZENITH DISTANCES

TALCOTT'S METHOD

59. Theory.—From equation (138) we have

$$\varphi = \delta_S + z_S,$$
$$\varphi = \delta_N - z_N,$$

where the subscripts indicate the position of the stars with respect to the zenith. One-half the sum of these two equations gives

$$\varphi = \tfrac{1}{2}(\delta_S + \delta_N) + \tfrac{1}{2}(z_S' - z_N') + \tfrac{1}{2}(r_S - r_N), \qquad (142)$$

in which the true zenith distances have been replaced by $z_N' + r_N$ and $z_S' + r_S$, respectively. The declinations are given by the *Ephemeris*, and the difference of the refractions is readily calculated. If therefore the difference between the apparent zenith distances of two stars be measured, the latitude can be calculated by (142).

By limiting the application of the equation to those cases in which the zenith distances are nearly equal, a considerable increase in precision will be obtained as compared with that resulting from meridian zenith distances. Since the measures are differential, instrumental errors affecting the two observations equally will be eliminated. In the case of measures with the sextant, for example, the index correction and the eccentricity will be eliminated and need not, therefore, be determined. But what is of more importance, so far as precision is concerned, is the fact that the errors of observation which would affect these instrumental corrections, were they determined, do not enter into the result. A similar condition exists in the case of the refraction, for the difference of two refractions corresponding to nearly equal zenith distances can be calculated with a higher degree of precision than is possible in the determination of the total refraction. Finally, the fact that the quantity to be observed is small, makes it possible to introduce other and more precise methods of measurement than those which depend upon the use of a graduated circle. For example, with the engineer's transit small differences of zenith distance may be measured more accurately with the gradienter screw than with the vertical circle.

The method under discussion was first proposed by Horrebow, the director of the Observatory of Copenhagen about the middle of the eighteenth century, and was given extensive practical application in the work of the United States Coast and Geodetic Survey about a century later by Captain Talcott, from which circumstance it is commonly known as Talcott's method. It reaches its highest precision when used in connection with the zenith telescope, an instrument of the altazimuth type fitted with an accurately constructed micrometer eyepiece and a very sensitive altitude level. The level enables the observer to give the line of sight the same inclination to the vertical during both observations, while the micrometer affords a very precise determination of the required difference in zenith distance of the two stars.

If the method is to be used in connection with the engineer's transit, the angular value of one revolution of the gradienter screw should first be determined by measuring a small angle whose value is known. The observations should be made and reduced in a way such that any irregularity in the screw will be revealed. To this end a process analogous to that used in Examples 26 and 29 may be employed.

Since the correction for refraction will always be small, we may assume

$$r_S - r_N = \frac{dr}{dz'}(z'_s - z').$$

From (4) we find

$$\frac{dr}{dz} = 57'' \sec^2 z' \sin 1^\circ,$$

which expresses the rate of change of r per 1° of change in z'. Denoting this quantity by C, the correction for refraction in seconds of arc becomes

$$\tfrac{1}{2}(r - r_N)'' = \tfrac{1}{2}(z_s' - z_N')^\circ C \qquad (143)$$

in which the difference of the zenith distances must be expressed in degrees. The value of C may be taken from Table V with the mean zenith distance of the two stars as argument.

TABLE V

z'	C	z'	C
10°	1″.0	50°	2″.4
20	1.1	55	3.0
30	1.3	60	4.0
40	1.7	65	5.6
50	2.4	70	8.5

60. Procedure.—Select two stars culminating within 15^m or 20^m of each other whose declinations satisfy as nearly as possible the condition

$$2\varphi = \delta_S + \delta_N \qquad (144)$$

and calculate the clock time of meridian transit by (140) or (141).

If a sextant is used, measure the double altitudes of the two stars at the instants of transit. Let R_s and R_N be the corresponding sextant readings. The second term of (142) will then be given by

$$\tfrac{1}{2}(z_s' - z_N') = \tfrac{1}{4}(R_N - R_s). \qquad (145)$$

If the engineer's transit is employed, level carefully and bring the star culminating first to the intersection of the threads at the instant of its transit. Read the gradienter screw, reverse, relevel, bring the second star to the intersection of the threads at the instant of transit by means of the screw, and note the reading as before. The vertical circle should be firmly clamped when the setting on the first star is made, and must not be disturbed thereafter until the second star has been observed. If the two screw readings be denoted by m_s and m_n, and if G be the value of one-half a revolution of the screw, we shall have

$$\tfrac{1}{2}(z'_s - z'_n) = \pm G(m_s - m_n), \qquad (146)$$

in which the upper sign is to be used when the screw readings increase with increasing zenith distance.

In levelling, special attention should be given to the altitude level. Unless the bubble has the same position for both observations, an error will be introduced into the result. If the level is a sensitive one, it will be better to omit the levelling after reversal and apply a correction to the result given by (146). If o and e be the readings of the object and eye ends of the bubble, respectively, and if readings increasing toward the north be recorded as positive while those increasing toward the south are entered as negative, the correction to be added algebraically to the result given by (146) will be

$$(o_s + e_s + o_n + e_n)D, \qquad (147)$$

in which D is one-fourth the angular value of one division of the level. The bubble readings should be taken as near the times of transit as possible.

The last term of (142) is given by (143), the value of C being derived from Table V. The declinations are to be taken from the list of apparent places in the *Ephemeris* for the instant of observation. In case the northern star is observed at lower culmination, its declination in (142) must be replaced by $180° - \delta_N$.

3. CIRCUMMERIDIAN ALTITUDES

61. Theory.—The zenith distance to be used in equations (138) and (139) is that of the object when on the meridian. Since only a single determination of this quantity can be made at any given transit, it is desirable for the sake of precision to modify the method described under No. 1 so as to permit a multiplication of the settings.

The change in the zenith distance during an interval immediately preceding or following the instant of transit is small and its value is easily and accurately calculated. The meridian zenith distance may therefore be found by observing when the object is near the meridian and applying to the measured value of the coördinate the amount of the change during the interval separating the instant of observation from that of culmination. A series of such measures reduced to the meridian gives a precise value of z_0 which can then be substituted into (138) or (139) for the determination of the latitude. It is of course immaterial whether the quantity measured be zenith distance or altitude. The method is commonly known as that of circummeridian altitudes.

The development of the formulæ to be used for the calculation of the reduction to the meridian is as follows: Equation (31) may be written in the form

$$\cos z = \cos (\varphi - \delta) - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t. \qquad (148)$$

Let z be the observed value of the coördinate, z_0 the meridian value, and Z the reduction to the meridian. We then have

$$z + Z = z_0. \qquad (149)$$

Substituting into (148) we find

$$\cos (z_0 - Z) = \cos z_0 - 2 \cos \varphi \cos \delta \sin^2 \tfrac{1}{2} t. \qquad (150)$$

To express Z explicitly we may replace the left member of (150) by its expansion by Taylor's theorem. Since Z is small the convergence will be rapid. Introducing at the same time

$$A = \cos \varphi \cos \delta \operatorname{cosec} z_0, \qquad m = 2 \sin^2 \tfrac{1}{2} t, \qquad (151)$$

and neglecting terms in Z^3 we find

$$Z = - Am + \tfrac{1}{2} Z^2 \cot z_0. \qquad (152)$$

Squaring, we have to the same degree of approximation

$$Z^2 = A^2 m^2.$$

Substituting into (152), and writing

$$B = A^2 \cot z_0, \qquad n = \tfrac{1}{2} m^2 = 2 \sin^4 \tfrac{1}{2} t, \qquad (153)$$

we have finally for the reduction to the meridian.

$$Z = - Am + Bn. \qquad (154)$$

Since the observations may be arranged so that Z will not exceed 15′ or 20′, the error in (154) will be insensible.

Combining equations (138), (149), and (154), the expression for the latitude becomes

$$\varphi = \delta \pm z \mp Am \pm Bn, \qquad (155)$$

in which the upper sign is to be used for southern stars; and the lower, for those culminating between the zenith and the pole.

For an object observed near lower culmination, t in (31) must be replaced by $180° + t$. The resulting value of the reduction to the meridian substituted into (139) gives

$$\varphi = 180° - \delta - z - Am - Bn. \qquad (156)$$

Equations (155) and (156), in which the last terms are to be calculated by (151) and (153), express the solution of the problem. For observations with the engineer's transit the term Bn will usually be insensible when the hour angle is less than 15^m or 20^m.

It will be observed that A and B depend upon the latitude — the quantity to be determined. A value of φ sufficiently accurate for the calculation of these coefficients may be obtained by (138) or (139) from the value of z observed nearest the time of transit. It will be noted further that A and B are constant for any given series of observations and need be calculated but once. The factors m and n, on the other hand, are different for each setting. Since they depend only upon the hour angle, their values may be tabulated with t as argument. Tables VI and VII may be used for all ordinary observations with the transit or sextant.

TABLE VI

t	m		t	m	
0^m	0″	2	10^m	196″	42
1	2	6	11	238	45
2	8	10	12	283	49
3	18	13	13	332	53
4	31	18	14	385	57
5	49	22	15	442	60
6	71	25	16	502	65
7	96	30	17	567	69
8	126	33	18	636	72
9	159	37	19	708	77
10	196		20	785	

TABLE VII

t	n
0^m	0″.0
5	0.0
10	0.1
15	0.5
16	0.6
17	0.8
18	1.0
19	1.2
20	1.5

62. Procedure.—Calculate the clock time of transit, θ' or T', of the object to be observed. Beginning a few minutes before this instant, make a series of observations for the determination of the zenith distance by some one of the methods of Section 45 or 55, noting the time for each. Correct the apparent zenith distance for refraction in the case of a star, and for refraction and parallax in the case of the sun. Form the hour angle, t, corresponding to each observation by subtracting the clock time of transit from the time of observation. For a star, t must be expressed in sidereal units; for the sun, in solar units. Calculate A and B from (151) and (153), using the value of the zenith distance observed nearest the time of transit for z_0 and for the determination of an approximate value of φ, both of which are required for the computation. Finally, calculate the latitude for each observation by means of (155) or (156). The declination to be used is that corresponding to the instant of observation.

The final result may also be obtained by applying the mean of all the values of Am and of Bn to the mean of all the zenith distances in accordance with equations (155) or (156). This method, however, gives no indication as to the consistency of the observations, and it is better to reduce the results separately, or, at least, to reduce separately the means of not more than two or three consecutive measures.

The method of circummeridian altitudes may advantageously be combined with that of Talcott. When this is done there will be given a series of values of $\frac{1}{2}(z'_S - z'_N)$ derived from observations made near the meridian. Each of these must be reduced to the meridian by adding to (142) the term $\frac{1}{2}(Z_S - Z_N)$, in which Z_S and Z_N are to be calculated by (154).

Example 37. The reduction of the circummeridian altitudes given in Ex. 35, p. 93, is as follows: To eliminate the semidiameter the means are formed for the 1st and 2nd, 3rd and 4th, 5th and 6th, and the 7th and 8th observations. These results are in the first and sixth lines of the calculation below. The eccentricity corrections are unknown. The index correction found in Ex. 35 is + 1″. In Ex. 36, p. 96, the clock time of transit was found to be $1^h\ 8^m\ 5^s$.

θ_F	$1^h\ 0^m\ 25^s$	$1^h\ 3^m\ 19^s$	$1^h\ 6^m\ 6^s$	$1^h\ 10^m\ 18^s$
t (sidereal)	−7 40	−4 46	−1 59	+2 13
t (solar)	−7 39	−4 45	−1 59	+2 13
m	115″	44″	8″	10″
n	0	0	0	0
R	117° 51′ 0″	117° 54′ 55″	117° 56′ 35″	117° 56′ 35″
h'	58 55 30	58 57 28	58 58 18	58 58 18
z'	31 4 30	31 2 31	31 1 42	31 1 42
$r-p$	+30	+30	+30	+30
δ	+7 54 22	+7 54 25	+7 54 27	+7 54 31
$-Am$	−2 52	−1 6	−0 12	−0 15
φ	38 56 30	38 56 20	38 56 27	38 56 28

From the 3rd column			
φ	38° 57′	$\cos\varphi$	9.8908
δ	+7 54	$\cos\delta$	9.9959
z_0	31 2	$\operatorname{cosec} z_0$	0.2877
		$\log A$	0.1744

The mean of the four values of φ is 38° 56′ 26″, which is 26″ less than the known true latitude. This fact taken in connection with the close agreement of the individual values suggests the existence of an eccentricity correction of about −50″ for the part of the scale used in the observations.

4. ZENITH DISTANCE AT ANY HOUR ANGLE

63. Theory.—It is desirable to be able to determine the latitude from a zenith distance measured when the object is so far from the meridian that the formulæ for circummeridian altitudes no longer give convergent results. This is readily accomplished by using the fundamental equation (31) in the form

$$\cos z = n \cos (\varphi - N). \tag{157}$$

Equation (157) is the last of equations (34), the auxiliaries n and N being defined by the first and second of this group.

64. Procedure.—Having determined the true zenith distance of the object, calculate the hour angle by

$$t = \theta - \alpha, \tag{158}$$

in which θ is the true sidereal time of observation. Then determine n and N by

$$\begin{aligned} n \sin N &= \sin \delta, \\ n \cos N &= \cos \delta \cos t, \end{aligned} \tag{159}$$

and $\varphi - N$ from

$$\cos (\varphi - N) = \frac{\cos z}{n}. \tag{160}$$

A reference to the fourth of (34) shows that $\sin (\varphi - N)$ must have the same algebraic sign as $\cos A$. This together with the sign of $\cos (\varphi - N)$ from (159) determines the quadrant of $\varphi - N$. The latitude is then given by

$$\varphi = (\varphi - N) + N. \tag{161}$$

Equations (158)-(161) are rigorous and apply to all values of the hour angle, but care should be taken to observe as near the meridian as possible in order that errors in z and θ may not appear multiplied in the result. (See Section 56.) A sufficient number of decimal places must be employed to offset the fact that the angle $\varphi - N$ is determined from its cosine.

Example 38. On 1908, Oct. 2, at watch time $8^h\ 35^m\ 11^s$ P.M. the altitude of Polaris was found to be 39° 29'.8. (See Ex. 31, p. 79.) The error of the watch on C.S.T. was $+ 1^m\ 45^s$. Find the latitude by equations (158)-(161).

C.S.T.	$8^h\ 36^m\ 56^s$	$\cos \delta$	8.3150
Columbia θ	21 13 14	$\cos t$	9.6498
α	1 27 10	$n \cos N$	7.9648
t	19 46 4	$n \sin N$	9.9999
t	296° 31'.0	$\tan N$	2.0351
z'	50 30.2	N	89° 28'.3
r	1.1	$\sin N$	0.0000
z	50 31.3	$\log n$	9.9999
δ	88 49.0	$\cos z$	9.8033
		$\cos (\varphi - N)$	9.8034
The calculated φ is larger		$\varphi - N$	—50° 30'.8
than the true value by 0'.6		φ	38 57.5 *Ans.*

The C.S.T. is converted into the corresponding Columbia θ by (41) and (58). α and δ are from p. 321 from the *Ephemeris*. The value of t shows that Polaris was east of the meridian at the time of the observation, whence $\cos A$ and $\sin(\varphi - N)$ are negative. Since $\cos(\varphi - N)$ is positive, $\varphi - N$ is in the fourth quadrant.

5. ALTITUDE OF POLARIS

65. Theory.—The peculiar location of Polaris with respect to the pole makes it possible to simplify the fundamental latitude equation for use in connection with this object. Since the latitude is by definition equal to the altitude of the north celestial pole, the problem may be solved by finding an expression for the difference in altitude of the pole and Polaris. The polar distance of Polaris is about $1° 11'$, consequently, the required difference will always be a small angle. To this fact is due the possibility of a simplification of equation (31). (See *Num. Comp.* pp. 14 and 16.)

Replacing z and δ in (31) by the altitude, h, and the north polar distance, π, respectively, we find

$$\sin h = \cos\pi \sin\varphi + \sin\pi \cos\varphi \cos t. \qquad (162)$$

If H be the difference in altitude of Polaris and the pole, we shall have

$$\varphi = h + H. \qquad (163)$$

Writing $h = \varphi - H$ in (162), and expanding and solving for $\sin H$

$$\sin H = -\sin\pi \cos t + \tan\varphi(\cos H - \cos\pi). \qquad (164)$$

Since at a maximum

$$H = \pi = 1° 11',$$

we may replace $\sin H$ and $\sin\pi$ in (164) by H and π, respectively, with an error not exceeding $0''.3$. At the same time we may write

$$\cos H = 1 - \tfrac{1}{2}H^2, \qquad \cos\pi = 1 - \tfrac{1}{2}\pi^2,$$

with errors which are still smaller, thus obtaining

$$H = -\pi\cos t + \tfrac{1}{2}\tan\varphi(\pi^2 - H^2). \qquad (165)$$

Neglecting terms involving π^3,

$$H^2 = \pi^2\cos^2 t,$$

and substituting H^2 into (165) we have

$$H = -\pi\cos t + \tfrac{1}{2}\pi^2\tan\varphi\sin^2 t. \qquad (166)$$

Finally, by (163)

$$\varphi = h - \pi \cos t + K, \tag{167}$$

in which

$$K = \tfrac{1}{2}\pi^2 \tan \varphi \sin^2 t. \tag{168}$$

The error in the latitude calculated from (167) due to the approximate form of the equation will usually be less than 2″.

The calculation of K requires a knowledge of φ — the quantity to be determined; but, since the coefficient $\frac{1}{2}\pi^2$ is only about 0.02, a rough approximation for the latitude will answer. The values of K may be derived from Table VIII with an approximate latitude and the hour angle as arguments. The table is based on the value $\pi = 1^\circ\ 11'\ 0''$.

TABLE VIII $K = \frac{1}{2}\pi^2 \tan \varphi \sin^2 t$

t	$\varphi = 30^\circ$	$\varphi = 35^\circ$	$\varphi = 40^\circ$	$\varphi = 45^\circ$	$\varphi = 50^\circ$	t
0^h	0′.00	0′.00	0′.00	0′.00	0′.00	12^h
1	0.03	0.03	0.04	0.05	0.06	11
2	0.11	0.13	0.15	0.18	0.22	10
3	0.21	0.26	0.31	0.37	0.44	9
4	0.32	0.38	0.46	0.55	0.66	8
5	0.40	0.48	0.57	0.68	0.82	7
6	0.42	0.51	0.62	0.73	0.87	6

For values of t greater than 12^h enter the table with $24^h - t$ as argument.

In rough work, the values of H may be taken directly from Table IV at the end of the *Ephemeris* with t, or $24^h - t$, as argument. This table is calculated with a mean value of φ equal to 45°. The interpolated H added to the true altitude of Polaris will give the latitude of points within 10° or 15° of the mean latitude of the table with an error not exceeding a few tenths of a minute of arc.

66. Procedure.—Having determined the true altitude, h, of Polaris, calculate

$$\begin{aligned} t &= \theta - \alpha, \\ \pi &= 90^\circ - \delta, \\ \varphi &= h - \pi \cos t + K. \end{aligned} \tag{169}$$

in which θ is the sidereal time of observation, and α and δ the apparent right ascension and declination, taken from the *Ephemeris*, pp. 312-323. Interpolate K from Table VIII with t, or $24^h - t$, and an approximate value of φ as arguments. π is conveniently expressed in minutes of arc.

Example 39. Find the latitude by equations (169) from the data of Ex. 33, p. 81.

T_W	$9^h\ 42^m\ 10^s$	δ	$88°\ 49'.1$	h'	$39°\ 54'.9$
ΔT_W	-32	π	70.9	r	1.1
C. S. T.	9 41 38	$\log \pi$	1.8506	$\pi \cos t$	57.0
Columbia θ	23 1 29	$\cos t$	9.9056	K	0.2
α	1 27 13	$\log \pi \cos t$	1.7562	φ	38 57.0 *Ans.*
t	21 34 16				
t	$323°\ 34'.0$				

The calculated φ is larger than the true value by $0'.1$.

The application of equations (169) to the data of Ex. 38 gives $\varphi = 38°\ 57'.5$, which agrees exactly with the result obtained by the formulæ of Section 64.

Example 40. Find the latitude by means of Table IV of the *Ephemeris* from the data of Exs. 38 and 39.

The hour angle is to be calculated as before. Its value in both cases is greater than 12^h. Consequently, H is to be interpolated from Table IV, *Ephemeris*, p. 595, with $24^h - t$ as argument. We then find

	Ex. 38	Ex. 39
$24^h - t$	$4^h\ 13^m.9$	$2^h\ 25^m.7$
h	$39°\ 28'.7$	$39°\ 53'.8$
H	-31.2	-57.0
φ	38 57.5	38 56.8

67. Influence of an error in time.—We may now examine more closely the influence of an error in the time upon the calculated latitude. The change is φ produced by a small change in θ is by (136)

$$d\varphi = -\tan A \cos \varphi\, d\theta. \qquad (170)$$

For all of the preceding methods but No. 4, A will be small, a few degrees at most, and we may write $\sin A$ in place of $\tan A$ in (170) with sufficient accuracy for the present purpose. Substituting for $\sin A$ its value from (33), and writing z equal to the meridian zenith distance, z_0, (170) becomes

$$d\varphi = \pm \cos \varphi \cos \delta \operatorname{cosec} z_0 \sin t\, d\theta. \qquad (171)$$

in which the upper sign refers to southern stars. For circummeridian altitudes (171) reduces by (151) to

$$d\varphi = \pm A \sin t\, d\theta. \qquad (172)$$

For Polaris we have with sufficient approximation

$$z_0 = 90° - \varphi, \qquad \cos \delta = \pi = 0.02,$$

whence

$$d\varphi = 0.02 \sin t\, d\theta. \qquad (173)$$

Equations (172) and (173) may be obtained directly from (153) and (157) by differentiating with respect to t and introducing $dt = d\theta$, the small terms Bn and K being disregarded.

Equations (170)–(173) may be used for the calculation of $d\varphi$ when $d\theta$ is known, or for the determination of the accuracy with which the time must be known in order to obtain φ with a given degree of precision. If $d\theta$ is expressed in seconds of time, the factor 15 must be introduced into the right members of the various equations in order that $d\varphi$ may be expressed in seconds of arc.

It is evident that, aside from the dependence of $d\varphi$ upon t, it also depends upon the zenith distance and declination of the star, and that an error in the time has the least influence upon the calculated latitude for stars near the pole. For Polaris the effect of $d\theta$ is always small, and if t be near 0^h or 12^h, it will be very slight indeed, even though $d\theta$ be large.

This fact taken in connection with the simplicity of the reductions renders the last of the above methods the most useful of all the various processes that may be employed for the determination of latitude. The greatest precision, however, is attained only by the method of a Talcott when used in connection with the zenith telescope.

Example 41. What is the error in the latitude calculated from the first of the circum-meridian altitudes of Ex. 37, p 102, on the assumption that the watch correction used was incorrect by 20^s?

By (172) we find, taking the values of A and t from Ex. 37

$$d\theta = 20^s = 300'' \qquad t = 7^m\,39^s = 1^\circ\,54'\,45''$$

log A	0.1744	log $d\varphi$	1.1849
sin t	8.5334	$d\varphi = 15''$	*Ans.*
log 300	2.4771		

Example 42. How accurately must the time be known in order that the altitude of Polaris given in Ex. 33, p. 81, may yield a value of the latitude uncertain by not more than $0'.1$?

By (173) and the data in Ex. 39, p. 105, we find

t	$323^\circ\,34'.0$	$d\varphi$	$0'.1$
sin t	0.595	0.02 sin t	0.012

$$d\theta = 8'.33 = 33^s \quad \textit{Ans.}$$

CHAPTER VI

THE DETERMINATION OF AZIMUTH

68. Methods.—The azimuth of a terrestrial mark may be found by observing the difference in azimuth of the mark and a celestial object and applying to this difference the calculated azimuth of the object corresponding to the instant of observation. The methods to be employed for the observational part of the process have been discussed in detail in Chapter IV. We have now to examine the means by which the azimuth of the celestial body may be computed.

A rigorous and general method of procedure leading to the fundamental equation

$$\tan A = -\frac{\cos\delta \sin t}{\sin\delta\cos\varphi - \cos\delta\sin\varphi\cos t} \qquad (174)$$

was outlined on page 34. Before proceeding to the adaptation of this equation to the purposes of calculation it is desirable to investigate the conditions under which it may most advantageously be employed. The calculated azimuth will depend upon the right ascension and declination of the star, the time, and the latitude of the place of observation. The first two quantities may be assumed to be known with precision, but the last are likely to be affected by relatively large uncertainties. To determine the influence of these upon the calculated azimuth, and thus derive a precept for the choice of objects to be observed, we differentiate (33), A, z, and t being considered variable, and substitute for dz its value from page 95. Writing at the same time $dt = d\theta$ we find after simplification.

$$dA = -\cot z \sin A\, d\varphi + (\sin z \sin\varphi + \cos z \cos\varphi \cos A)\,\text{cosec}\, z\, d\theta.$$

If in Fig. 6 we denote the angle at O by q, the expression in parenthesis reduces by the second of the fundamental formulae of spherical trigonometry to $\cos\delta\cos q$, whence

$$dA = -\cot z \sin A\, d\varphi + \cos\delta \cos q\, \text{cosec}\, z\, d\theta. \qquad (175)$$

In order that dA may be small it is necessary that the object should not be near the zenith. Otherwise, the factors $\cot z$ and $\text{cosec}\, z$ will produce a multiplication of both $d\varphi$ and $d\theta$. Further, it is desirable that the azimuth should be near 0° or 180°, for when this is the case an error in the assumed latitude will produce but little effect upon the calculated azimuth. When the object is near the pole, $\cos\delta$ will be small and the influence of $d\theta$ will be slight; and if, at the same time, it be near elongation, $\cos q$ will also be small, and the effect of $d\theta$ will still further be minimized.

A close circumpolar star at any hour angle satisfies these conditions with sufficient closeness to render the influence of any ordinary errors in φ and θ quite insensible. Should the clock correction be very uncertain, however, it may be desirable to observe for the determination of the azimuth difference of the mark and the star at or near the time of elongation in order that the coefficient of $d\theta$ in (175), already small through the presence of $\cos\delta$, may be made still smaller by the introduction of a value of q near 90°.

Far less satisfactory will be the result in the case of observations on the sun, although this object may be used when the latitude is known with some precision, provided care be taken to observe as far from the meridian as possible. With this precaution the coefficients in (175) depending on z and q will have the values best adapted for a minimization of the errors in φ and θ, especially that of the latter, which in all cases is most to be feared.

Besides the fundamental equation (174) there is another which is sometimes useful, namely, (26). If the zenith distance of the celestial body be measured simultaneously with the determination of the azimuth difference, the azimuth of the body may be calculated by this equation, whence the azimuth of the mark can be found as before. With this method of procedure the latitude of the place must be known, but the time does not enter into the problem except as it may be required for the interpolation of the declination of the object for the instant of observation.

To determine the conditions under which this method may be used with advantage, differentiate (26) considering φ, z, and A as variables. We thus find after simplification

$$\cos\varphi\, dA = \cos q \operatorname{cosec} t\, dz - \cot t\, d\varphi. \qquad (176)$$

From this it appears that errors in z and φ will have the least influence when t and q are as near 90° or 270° as possible. These conditions cannot both be fulfilled at the same time. But for circumpolar stars observed near elongation the magnitude of $\cos q$ and $\cot t$ in (176) will be such that errors in z and φ will have only an insignificant influence on the calculated azimuth.

The consideration of the preceding results indicates that we shall need adaptations of the fundamental azimuth equations designed for the calculation of

1. The azimuth of the sun.
2. The azimuth of a circumpolar star at any hour angle.
3. Azimuth from an observed zenith distance.

I. AZIMUTH OF THE SUN

69. Theory.—The first four equations of (34) are the equivalent of (32) and (33) from which the fundamental equation (174) was derived. By their combination we find the following group which for the purposes of calculation replaces (174).

$$\tan N = \frac{\tan\delta}{\cos t}, \qquad (177)$$

$$\tan A = \frac{\cos N}{\sin(\varphi - N)} \tan t. \qquad (178)$$

The quadrants of N and A are determined by noting that $\sin N$ and $\sin A$ have the same algebraic signs as $\sin\delta$ and $\sin t$, respectively.

70. Procedure.—If a sidereal timepiece is used, calculate t from

$$t = \theta - \alpha,$$

in which θ is the true sidereal time of observation, and α the sun's apparent right ascension. If a mean solar timepiece is employed, calculate the apparent solar time for which the azimuth difference has been measured. This is directly the hour angle of the sun. Interpolate δ, and α when required for the calculation of t, for the instant of observation. Finally compute A from (177) and (178). Azimuth determinations from solar observations should be made only when the sun is far from the meridian.

2. AZIMUTH OF A CIRCUMPOLAR STAR AT ANY HOUR ANGLE

71. Theory.—Dividing the numerator and denominator of (174) by $\sin\delta\cos\varphi$ and writing $\delta = 90° - \pi$, we find

$$\tan A = -\frac{\tan\pi\sec\varphi\sin t}{1 - \tan\pi\tan\varphi\cos t}, \qquad (179)$$

This equation may be replaced by the following group which is arranged with reference to the requirements of calculation.

$$\begin{aligned} g &= \tan\pi\sec\varphi, \\ h &= \tan\pi\tan\varphi = g\sin\varphi, \\ G &= \frac{1}{1 - h\cos t}, \\ \tan A &= -gG\sin t. \end{aligned} \qquad (180)$$

The quadrant of A is determined by the fact that $\sin A$ must have the same algebraic sign as $\sin t$.

The factors g and h are constant for any given night, and in approximate work they may be considered as such for a series of nights. Moreover h is small because of the factor $\tan\pi$. G therefore differs but little from unity, and the values of $\log G$ may be tabulated with $\log h\cos t$ as argument. Such a table, sufficient for all practical requirements is given in *Rept. Supt. U. S. Coast and Geodetic Survey*, 1897-8, pp. 399-407.

In case tables for $\log G$ are not accessible its values may be calculated as follows: G has the form $1/(1+v)$ or $1/(1-v)$, in which $v = h\cos t$, according as $\cos t$ is negative or positive. The latter expression may be written in the form

$$G = 1/(1-v) = (1+v)(1+v^2)(1+v^4)\ldots. \qquad (181)$$

Since v is small, the parentheses after the second or third in the last member of (181) will sensibly be equal to unity. To find the value of $\log G$, therefore, we must find the logarithms of one or more factors of the form $(1+b)$. For this purpose we use the addition logarithmic table. Since $a = 1$, the formulæ are (*Num. Comp.* p. 10).

$$A = \log b, \qquad \log(1+b) = B,$$

where B is to be interpolated from the table with A as argument. Hence

For $\cos t$ negative,

$$A = \log(h \cos t), \qquad \log G = -B.$$

For $\cos t$ positive, (182)

$$A_1 = \log(h \cos t), \quad A_2 = \log(h \cos t)^2, \quad A_3 = \log(h \cos t)^4 \ldots.$$
$$\log G = B_1 + B_2 + B_3 + \ldots.$$

Equations (180) used in connection with tables for $\log G$, or with formulæ (182) afford a convenient and precise method of calculating the azimuth of any of the close circumpolar stars whose apparent places are given in the *Ephemeris*, pp. 312-323.

Equations (180) are rigorous, however, and for approximate results they may be simplified, especially if the circumpolar observed is Polaris. For this object π at the present time is 1° 11′, and for latitudes less than 60°, its azimuth will always differ from 180° by less than 2° 3′. We may therefore write

$$180° - A = \pi G \sec \varphi \sin t \qquad (183)$$

with an error not exceeding 2″. For latitudes of 45° or less the error will always be less than 1″.

TABLE IX

t	$\log G$	$\log G \sec \varphi$	t
0^h	0.0075	0.1167	24^h
1	0.0073	0.1165	23
2	0.0065	0.1156	22
3	0.0053	0.1145	21
4	0.0037	0.1129	20
5	0.0019	0.1111	19
6	0.0000	0.1092	18
7	9.9981	0.1072	17
8	9.9963	0.1055	16
9	9.9948	0.1039	15
10	9.9936	0.1028	14
11	9.9928	0.1020	13
12	9.9926	0.1018	12

Further, $\log G$ may be tabulated, as in Table IX, for a mean value of the latitude with t as argument. Since φ enters into G through h, which contains the factor $\tan \pi$, and since G itself appears multiplied by π in (183), any difference between the value of φ assumed for the calculation of the table and that corresponding to the place of observation will have only a slight influence on the azimuth derived from (183). It is to be understood, however, that the local value of $\sec\varphi$ must be used, and that the value of the coefficient π appearing in (183) must correspond to the date of observation. In case a number of azimuth determinations are to be made at a given station, the corresponding local value of $\log\sec\varphi$ may conveniently be combined with the mean values of $\log G$. One can then interpolate $\log G\sec\varphi$ directly from the table.

The values of $\log G$ in Table IX are based upon $\varphi=40°$, and $\pi=1°\ 10'$, the latter of which is the mean north polar distance for 1910.0. The maximum absolute errors in the azimuth resulting from the use of this table for various latitudes are

Latitude	30°	35°	40°	45°	50°
Error in A	0'.24	0'.12	0'.00	0'.15	0'.38

The values of $\log G\sec\varphi$ in the third column of Table IX refer to the latitude of the Laws Observatory, which is 38° 57'.

72. Procedure.—Interpolate α and δ for the instant of observation from the list of apparent places of circumpolar stars, *Ephemeris*, pp. 312-323. Calculate

$$\pi=90°-\delta, \qquad t=\theta-\alpha,$$

where θ is the true sidereal time of observation for which the azimuth difference of the star and the mark has been measured. Then:

For a precise azimuth, calculate A from (180). The value of $\log G$ may be taken from *Rept. Supt. U. S. Coast and Geodetic Survey*, 1897-8, pp. 399-407, or from some similar table, with the argument $\log h\cos t$; or it may be calculated by means of (182).

For an approximate azimuth from Polaris, interpolate $\log G$, or $\log G\sec\varphi$, from table IX with t as argument. Then calculate A from

$$A=180°-\pi G\sec\varphi\sin t. \qquad (184)$$

If π be expressed in minutes of arc, the last term of (184) will also be given in minutes of arc.

Example 43. Determine the azimuth of the mark from the data given in Ex. 34, p. 85. The latitude of the place of observation is 38° 56′ 52″.

Equations (180) are used for the calculation, the results for the two positions of the instrument being reduced separately. The azimuth of the mark is found by subtracting the difference $S-M$, taken from p. 85, from the calculated azimuth of Polaris. The difference of the two values of M is not to be taken as an indication of the precision of the result, as these quantities are affected by instrumental errors whose influence is not eliminated until the mean is formed.

			C. R.	C. L.
α	$1^h\ 25^m\ 19^s$	θ	$9^h\ 38^m\ 32^s$	$9^h\ 49^m\ 38^s$
π	$1^\circ\ 10'\ 46''$	t	8 13 13	8 24 19
φ	38 56 52	t	$123^\circ\ 18'\ 15''$	$126^\circ\ 4'\ 45''$
sec φ	0.10918	cos t	9.73964_n	9.77005_n
tan π	8.31362	h cos t = A	7.96082_n	7.99123_n
tan φ	9.90756	B = log G	9.99605	9.99577
log g	8.42280	sin t	9.92209	9.90752
log h	8.22118	tan A	8.34094_n	8.32609_n
		A	$178^\circ\ 44'\ 38''$	$178^\circ\ 47'\ 10''$
		$S-M$	173 33 30	173 35 34
		M	5 11 8	5 11 36
		Mean	$5^\circ\ 11'\ 22''$	

3. AZIMUTH FROM AN OBSERVED ZENITH DISTANCE

73. Theory.—Equation (26) rewritten in the form

$$-\cos A = \frac{\sin \delta - \cos z \sin \varphi}{\sin z \cos \varphi} \qquad (185)$$

$A < 180^\circ$ when the object is west of the meridian

expresses the azimuth as a function of δ, z, and φ. If the zenith distance of an object of known declination be measured at a place of known latitude, the azimuth of the object can be calculated. We have seen from the differential relation (176) that the most advantageous use of (185) requires that q and t be as near 90° or 270° as possible, a condition best fulfilled by circumpolar stars. For these objects the azimuth will be near 180°. In such cases the solution of (185) will be affected by a large error of calculation owing to the fact that A is derived from its cosine. On this account it is desirable to transform the equation so that the azimuth may be determined from its tangent or cotangent. This transformation has already been made and the results are collected under (37) along with the formulæ for the determination of t and the parallactic angle, q, from the three sides of the triangle PZO shown in Fig. 6. Selecting those relating to the azimuth we find.

$$\begin{gathered} a = z, \quad b = 90^\circ - \delta, \quad c = 90^\circ - \varphi, \\ s = \tfrac{1}{2}(a + b + c) \\ \text{Check:} \quad (s-a) + (s-b) + (s-c) = s, \\ \cot^2 \tfrac{1}{2} A = \frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)} \end{gathered} \qquad (186)$$

$\tfrac{1}{2} A$ is to be taken in the 1st or 2nd quadrant according as the object is west or east of the meridian.

74. Procedure.—It is to be remembered that the object observed should satisfy the conditions $q = 90^\circ$ or 270° and $t = 6^h$ or 18^h as closely as possible. Since the error of the measured zenith distance usually will be larger than that affecting the latitude, the first of the above conditions is the more important of the two. See equation (176).

The zenith distance determined simultaneously with the measurement of the azimuth difference, the declination for the instant of observation, and the latitude of the place of observation constitute the data necessary for the calculation of the azimuth.

For objects whose azimuths are not so near 0° or 180° as to render the error of calculation for (185) large, we may calculate A by this equation. But for circumpolar stars, which are best adapted for use with the method in question, it will be desirable to derive A by means of (186). In any case, however, (185) and (186) will serve as a mutual control for testing the accuracy of the calculated azimuth.

75. Azimuth of a mark.—Having measured $S-M$, the azimuth difference of the object and the mark, and having determined the azimuth of the object by some one of the above methods we calculate M, the azimuth of the mark, by

$$M=A-(S-M). \tag{186a}$$

Example 44. Find the azimuth of the mark from the data given in Ex. 33, p. 81.

Since both the time and the altitude of Polaris corresponding to the instant of measurement of the azimuth difference of the star and the mark are known, the reduction may be made by the third as well as by the second method. The first column contains the calculation by (184); the second, that by (186). The value of t required for the first part is taken from Ex. 39, p. 106.

t	21^h 34^m 16^s	h'	39° 54′.9
t	323° 34′.0	r	1.1
$\log \pi$	1.8506	$a=z$	50 6.2
$G \sec \varphi$	0.1151	$b=\pi$	1 10.9
$\sin t$	9.7737n	$c=90-\varphi$	51 3.1
$\log A_N$	1.7394	s	51 10.1
A_N	+54′.9	$s-a$	1 3.9
A	180° 54′.9	$s-b$	49 59.2
$S-M$	174 44.1	$s-c$	0 7.0
M	6 10.8	$\sin(s-c)$	7.30882
		$\sin(s-a)$	8.26920
		$\text{cosec}(s-b)$	0.11583
		$\text{cosec}\, s$	0.10847
		$\cot \frac{1}{2}A$	7.90116
		A	180° 54.8 Ck.

76. Influence of an error in the time.—An uncertainty in the clock correction, or any error in noting the time of the measurement of $S-M$, will introduce an error into the final result, for the calculated azimuth of the object will not correspond to the observed azimuth difference. The magnitude of this error for any given given error in θ depends upon the position of the star. Its value may be calculated from the differential relation between A and θ,

$$dA=\cos\delta \cos q \,\text{cosec}\, z \, d\theta, \tag{187}$$

which is derived from (175).

Similarly, when the azimuth of the object is calculated from measures of its zenith distance there will be an uncertainty in the result due to the error affecting z. The relation in this case is, by (176),

$$dA = \cos q \sec \varphi \operatorname{cosec} t\, dz. \qquad (188)$$

Equations (187) and (188) may be used to estimate the uncertainty in A corresponding to a given uncertainty in θ and z, or they may be used to determine the accuracy with which the time or the zenith distance must be known in order to secure a given degree of precision in A.

Usually z and t may be estimated with sufficient precision for the derivation of dA. The parallactic angle q may be calculated from

$$\sec \varphi \sin q = \sin A \sec \delta = \sin t \operatorname{cosec} z. \qquad (189)$$

For circumpolar stars (187) and (188) may be simplified as follows: Since the azimuth of such an object is always a small angle, the spherical excess of the triangle PZO, Fig. 6, page 26, is small and we shall have approximately $q = 180° - t$, whence

$$\cos q = - \cos t. \qquad (190)$$

Further, we have with sufficient approximation $z = 90° - \varphi$. Substituting these results into (187) and (188) and writing $\cos \delta = \pi$, we find

$$dA = - \pi \sec \varphi \cos t\, d\theta, \qquad (191)$$
$$dA = - \sec \varphi \cot t\, dz. \qquad (192)$$

The first of these can also be derived from (184) by differentiating and writing $G = 1$.

Example 45. The altitude of the sun and the difference of its azimuth and that of a mark were measured with an engineer's transit at the Laws Observatory on 1909, April 27. The results were $T_W = 4^h\ 1^m\ 11^s.0$, P.M., $\Delta T_W = -1^m\ 44^s.5$ (referred to C.S.T.), $h' = 33°\ 19'.6$, $S - M = 81°\ 24'.7$. Find the azimuth of the mark, calculating the azimuth of the sun both by method 1 and method 3.

The computation of the solar azimuth by (177) and (178) is in the first column; that for (186), in the second. In the latter instance the time is required only with such precision as as may be necessary for the interpolation of declination from the *Ephemeris* for the instant of observation.

C.S.T.	$3^h\ 59^m\ 26^s.5$	h'	$33°\ 19'.6$
Col. M.S.T.	3 50 8.2	$r - p$	1.3
E	2 24.7	$a = z$	56 41.7
$t =$ Col. A.S.T.	3 52 32.9	$b = \pi$	76 8.9
t	$58°\ 8'.2$	$c = 90 - \varphi$	51 3.1
δ	+13 51.1	s	91 56.9
$\tan \delta$	9.39196	$\sin (s - c)$	9.81604
$\cos t$	9.72255	$\sin (s - a)$	9.76132
$\tan N$	9.66941	$\operatorname{cosec} s$	0.00025
N	$25°\ 2'.2$	$\operatorname{cosec} (s - b)$	0.56498
φ	38 56.9	$\cot \frac{1}{2} A$	0.07130
$\varphi - N$	13 54.7	A	$80°\ 38'.2$ Ck.
$\cos N$	9.95715	$S - M$	81 24.7
$\tan t$	0.20651	M	359 13.5
$\operatorname{cosec} (\varphi - N)$	0.61902		
$\tan A$	0.78268		
A	$80°\ 38'.1$		

CHAPTER VII

THE DETERMINATION OF TIME

77. Methods.—The determination of time means, practically, finding the error of a timepiece. To accomplish this the true time θ or T is calculated from observations on a star or the sun and compared with the clock time at which the observations were made. The required error is given by

$$\Delta\theta = \theta - \theta', \tag{193}$$

or

$$\Delta T = T - T', \tag{194}$$

according as the timepiece is sidereal or mean solar, θ' and T' being the clock values of the time of observation.

The fundamental equation for the determination of time is

$$\theta = \alpha + t. \tag{195}$$

Applied to any celestial object this equation gives the sidereal time, from which the mean solar or apparent solar time may be derived by the transformation processes of Chapter III. For the sun, however, the hour angle t is directly the apparent solar time, and, in case of observations on this object, the mean solar time may be found from (42) written in the form

$$T = t + E. \tag{196}$$

When the timepiece is solar the use of (196) is simpler than that of (195).

Since α and E may be regarded as known, the problem is reduced to the determination of the hour angle of the object for the instant of observation. As indicated on page 34 this may be accomplished by measuring the zenith distance of the object at a place of known latitude and using equation (38) or (39).

The problem can also be solved by determining the clock time θ_0' of the instant for which the hour angle of the object is zero. For this case the fundamental equation reduces to

$$\theta = \alpha, \tag{197}$$

and

$$\Delta\theta = \alpha - \theta_0'. \tag{198}$$

In outlining the methods that may be employed for the determination of θ_0' it will be assumed that the object is a star and that the timepiece used is sidereal. The modifications necessary for the removal of these limitations will be considered in connection with the discussion of the details presented in the following sections.

To determine θ_0' we may note the time θ_1 when a star has a certain zenith distance, or altitude, east of the meridian, and, again, the time θ_2 when it has the same zenith distance west of the meridian. Since the celestial sphere rotates uniformly, we shall have

$$\theta_0' = \tfrac{1}{2}(\theta_1 + \theta_2). \qquad (199)$$

The method is known as that of **equal altitudes.**

The clock time of meridian transit, θ_0', may also be determined by noting the instant of passage of an object across the vertical thread of a transit instrument mounted so that the line of sight of the telescope lies in the plane of the meridian. This is the **meridian method** of time determination.

Finally, θ_0' may be found by observing the transit of an object across the vertical thread of an instrument *nearly* in the plane of the meridian. The application of a small correction to the observed time depending upon the displacement of the instrument from the meridian gives the clock time for which $t = 0$. In practice the deviation of the instrument is such that the line of sight lies in the plane of the vertical circle passing through Polaris at a definite instant. The process is accordingly known as the **Polaris vertical circle method** of time determination. It is of special interest on account of the fact that it is readily adapted to a simultaneous determination of time and azimuth.

There are other methods of determining the true time, but those outlined afford a sufficient variety to meet the conditions arising in practice. We therefore proceed to a detailed consideration of

1. The zenith distance method.
2. The method of equal zenith distances or altitudes.
3. The meridian method.
4. The Polaris vertical circle method.

I. THE ZENITH DISTANCE METHOD

78. Theory.—The formulæ necessary for the calculation of t from δ, φ, and z, were developed in connection with the discussion of coördinate transformations and are given in (38) and (39).

The resultant error of observation will depend upon the errors affecting α, δ, φ, and z. Those in α and δ we may disregard as relatively insignificant. From (136) we find

$$d\theta = \text{cosec}\, A \sec \varphi\, dz - \cot A \sec \varphi\, d\varphi. \qquad (200)$$

Assuming that dz and $d\varphi$ represent the errors in z and φ, and $d\theta$ the resultant error of observation in θ, it appears that for a given latitude the time will be least affected by uncertainties in z and φ when the azimuth of the object is near 90° or 270°. Care should be taken, therefore, to select for observation only those objects which are near the prime vertical.

79. Procedure.—Having found the true zenith distance corresponding to the clock time, calculate t by (38) or (39). The latter equation should not be used when the object is so near the meridian that the interpolation of t from its cosine is rendered uncertain.

Observations on a star: If the timepiece is sidereal, calculate θ by (195), and $\Delta\theta$ by (193); if solar, convert the sidereal time derived from (195) into the corresponding mean solar time T, and determine ΔT from (194), taking care that T is reduced to the meridian to which the clock time refers.

Observations on the sun: If the timepiece is sidereal, we may proceed as in the case of a star using (195) and (193), or we may convert the value of T derived from (196) into the corresponding sidereal time and then use (193). If the timepiece is solar, calculate T from (196), reduce its value to the meridian to which the clock time refers, and calculate ΔT from (194).

Owing to the change in the right ascension and declination of the sun, a knowledge of the approximate time is necessary for the reduction of solar observations. Should the error of the timepiece be unknown, the interpolation of α and δ, or E, may be made with the Greenwich mean time corresponding to the clock time of observation. The resulting data will give an approximation for the error of the clock which, in general, will be sufficient for a precise interpolation of the coördinates of the sun. A repetition of the calculation then gives the final value of the clock correction.

Example 46. Find the error of the watch from the measured altitude of Alcyone given in Ex. 31, p. 79.

We have

h'	21°	19′	30″	t	18ʰ	35ᵐ	16ˢ.7
r		2	30	α	3	42	3.2
h	21	17	0	θ	22	17	19.9
δ	+23	49	23	C.S.T.	9	40	51.4
φ	38	56	52	Watch	9	39	6.4
				ΔT_w		+1	45.0 *Ans.*

The solution of (38) gives $t = 18^h 35^m 16^s.8$. From (39), as a control, we find $18^h 35^m 16^s.6$. The value used for t is the mean of these. The conversion of θ into the corresponding C.S.T. is accomplished by (62) and (41).

2. THE METHOD OF EQUAL ALTITUDES

80. Theory.—If θ_1 and θ_2 be the sidereal clock times when a star has the same altitude, or zenith distance, east and west of the meridian, respectively, the clock time of meridian transit will be given by (199), whence by (198)

$$\Delta\theta = \alpha - \tfrac{1}{2}(\theta_1 + \theta_2). \tag{201}$$

If a solar timepiece is used we shall have

$$\Delta T = T - \tfrac{1}{2}(T_1 + T_2), \tag{202}$$

where T is the mean solar time corresponding to $\theta = \alpha$.

If the object observed is the sun, the above equations are not applicable on account of the change in the declination during the interval separating the measures. This influence may be included, however, by reducing the observed times to what they would have been had the declination been constant and equal to its value at the instant of meridian transit. Since the change in δ is small, the required corrections may be found from the differential relation connecting changes in δ with corresponding changes in t. From (31)

$$dt = (\tan\varphi \operatorname{cosec} t - \tan\delta \cot t)\, d\delta, \tag{203}$$

in which t is one-half the interval between the two observations *expressed in solar units*, δ the declination for apparent noon, and $d\delta$ the change in δ during the interval t. Both the observed times will be too late by the quantity dt. Hence, for solar observations made with a sidereal timepiece,

$$\varDelta\theta = \alpha - \tfrac{1}{2}(\theta_1 + \theta_2) + dt. \tag{204}$$

If the timepiece is solar, we have from (196) and (202), since $t = 0$ for the instant of meridian transit,

$$\varDelta T = E - \tfrac{1}{2}(T_1 + T_2) + dt. \tag{205}$$

It is sometimes convenient to combine afternoon observations with others made on the following morning. In this case the mean of the observed times corrected for the change in declination is the clock time of lower culmination. The quantity t in (203) is one-half the interval between the observations expressed in solar units as before; but δ must be interpolated from the *Ephemeris* for the instant of the sun's lower transit, and the resulting value of dt must be *added* to the clock times of observation. The expressions for the clock correction are

$$\varDelta\theta = 12^h + \alpha - \tfrac{1}{2}(\theta_1 + \theta_2) - dt, \tag{206}$$
$$\varDelta T = 12^h + E - \tfrac{1}{2}(T_1 + T_2) - dt, \tag{207}$$

in which the values of α and E refer to the instant of lower culmination.

81. Procedure.—The object observed should be near the prime vertical. When three or four hours east of the meridian note the time of transit across the horizontal thread of the transit for a definite reading of the vertical circle, most conveniently an exact degree or half degree. Change the reading by 10′ or 20′ and note the time of transit as before. Repeat a number of times, always changing the reading by the same amount. After meridian passage observe the times of transit over the horizontal thread for the same readings of the vertical circle as before, but in the reverse order. If the sextant is used, note the times of contact of the direct and reflected images for the same series of equidistant readings of the vernier before and after meridian passage. Denote the means of the two series of times by θ_1 and θ_2, or T_1 and T_2, according as the timepiece is sidereal or mean solar. For a star the error of the clock will be given by (201) or (202). For the sun, calculate dt by (203), and the clock error by (204) or (205) in case the observations are made in the morning and afternoon of the same day, or by (206) and (207) when they are secured in the afternoon and on the following morning.

Care must be taken not to disturb the instrumental adjustments between the two sets of measures. If these remain unchanged no correction need be applied for index error, eccentricity, refraction, parallax or semidiameter. This fact taken in connection with the simplicity of the reductions constitutes the chief advantage of the method. It is subject, however, to the serious objection that an interval of several hours must elapse before the observing program can be completed, with the danger that clouds may interfere with the second series of measures.

When the engineer's transit is used for the observations, all the measures should be made in the same position of the verticle circle, and the angles should all be set from the same vernier.

As in the case of the zenith distance method of time determination, an approximate knowledge of the time is necessary when the object observed is the sun. If the clock correction is quite unknown, this may be derived from the observations themselves as before. It is only necessary to interpolate the sun's right ascension, or the equation of time, as may be required, on the assumption that the clock error is zero. This approximate result will lead to an approximation for the error of the timepiece with which the calculation may be repeated for the determination of the final value.

3. THE MERIDIAN METHOD

82. Theory.—The meridian method of time determination requires a transit instrument mounted so that, when perfectly adjusted, the line of sight lies constantly in the plane of the meridian, whatever the elevation of the telescope. In order that this may be the case, the horizontal axis must coincide with the intersection of the planes of the prime vertical and the horizon, and the line of sight must be perpendicular to the horizontal axis. The instant of a star's transit across the vertical thread will then be the same as that of its meridian passage. Denoting the clock time of this instant by θ_0' the error of the timepiece will be given by

$$\Delta\theta = \alpha - \theta_0'. \qquad (208)$$

In general, however, the conditions of perfect adjustment will not be satisfied. The horizontal axis will not lie exactly in the plane of the prime vertical, nor will it be truly horizontal. When produced it will cut the celestial sphere in a point A, Fig. 8, page 65, whose azimuth referred to the east point and whose altitude we may denote by a and b, respectively. Further, the line of sight will not be exactly perpendicular to the horizontal axis, but will form with it an angle $90° + c$. The quantities a, b, and c are known as the **azimuth, level,** and **collimation constants,** respectively. In general, therefore, the star will not be on the meridian at the instant of its transit across the vertical thread, but will have a small hour angle t whose value will depend upon the magnitude of the instrumental constants a, b, and c and the position of the star. To obtain the clock time of meridian transit we must subtract t from the clock time of observation, θ', whence

$$\theta_o' = \theta' - t, \tag{209}$$

and by (208)

$$\Delta\theta = \alpha - \theta' + t. \tag{210}$$

The values of a, b, and c can always be found. Consequently $\Delta\theta$ can be determined by (210) when t has been expressed as a function of the instrumental constants. For this purpose we make use of equations (82), (89), and (33). The last two terms of (82) express the influence of the level and collimation constants, b and c, upon the reading of the horizontal circle of the engineer's transit for C. R., or, what amounts to the same thing, the amount by which the azimuth difference of the point A and the object O, when on the vertical thread, exceeds 90°. The last two terms of (89) express the corresponding quantity for C. L. These results may be applied directly to the meridian transit to determine the azimuth of the star at the instant of its transit across the vertical thread. For, denoting this azimuth by A_s, and assuming that a, the azimuth of the point A referred to the east point, is measured positive toward the south, we have at once

$$A_s = a + b \cot z \pm c \operatorname{cosec} z, \tag{211}$$

in which the upper sign refers to C. R., and the lower to C. L. In the present case, however, the positions of the instrument are less ambiguously expressed by circle west (C. W.) and circle east (C. E.), respectively. We may now use (33) to determine the hour angle of the star when its azimuth is equal to A_s. Replacing A in (33) by A_s and writing A_s and t instead of their sines, which we may do since both are very small angles, we find

$$t \cos \delta = A_s \sin z \tag{212}$$

whence by (211)

$$t \cos \delta = a \sin z + b \cos z \pm c. \tag{213}$$

Equations (211) and (212) become indeterminate for $z = 0$, on account of the presence of A, but the conditions of the problem show that there can be no such discontinuity in the expression which gives t as a function of a, b, and c. Equation (213) is therefore valid for $z = 0$, and becomes inapplicable only for stars very near the pole. Since the star is near the meridian at the instant of observation, z in (213) may be replaced by the meridian zenith distance $\varphi - \delta$. Writing

$$A = \sin(\varphi - \delta) \sec \delta, \quad B = \cos(\varphi - \delta) \sec \delta, \quad C = \sec \delta, \tag{214}$$

and substituting for t in (210) we find

$$\Delta\theta = \alpha - \theta' + aA + bB \pm cC. \tag{215}$$

Equations (214) and (215) give the value of $\Delta\theta$ when the time of transit θ' across the vertical thread has been observed, provided the instrumental constants a, b, and c are known. The quantities A, B, and C are called the **transit factors.** Their values depend only upon the position of the star and, for any given latitude, may be tabulated with δ as argument. They may also be tabulated with the double argument δ and z. Tables of the latter sort are to be found in *Rept. Supt. U. S. Coast and Geodetic Survey* 1897-8, pp. 308-319. These are applicable for all points of observation.

There remains still the determination of the constants, a, b, and c. The second of these can be made equal to zero by a careful adjustment and levelling of the instrument, or its value may be measured in case a striding level is available. The azimuth and collimation constants are best determined from the observations themselves. Assuming that b has been made equal to zero, or that its value has been determined, there remain in (215) only three unknowns, $\Delta\theta$, a, and c. The observation of any three stars will afford three equations of condition involving these quantities from which, theoretically, their values may be determined. Practically, however, the solution is simplified and rendered more precise by proceeding as follows:

Suppose that the transits of a number of stars of various declinations have been observed, the instrument having been used in both positions. Consider the results for two of these having the same declination as nearly as possible, one observed C. W., the other, C. E. Writing

$$\Delta\theta' = \alpha - \theta' + bB, \tag{216}$$

we have from (215)

$$\begin{aligned} \Delta\theta &= \Delta\theta'_{W} + aA_{W} + cC_{W} \\ \Delta\theta &= \Delta\theta'_{E} + aA_{E} - cC_{E} \end{aligned} \tag{216a}$$

Since it is assumed that the two declinations are nearly equal, we may suppose $A_W = A_E$, whence we find

$$c = \frac{\Delta\theta'_{E} - \Delta\theta'_{W}}{C_E + C_W}, \tag{217}$$

which determines the collimation constant. Should there be more than one pair of stars of equal declination, (217) may be applied to each. The mean of the resulting values of c will then be accepted as the final value.

Next, consider two stars observed in the same position of the instrument whose declinations differ as widely as possible. One of these should be a northern star, a circumpolar preferably, the other, a southern star. Writing

$$\Delta\theta'' = \Delta\theta' \pm cC \tag{218}$$

we find for these objects from (215)

$$\Delta\theta = \Delta\theta''_{N} + aA_{N}$$
$$\Delta\theta = \Delta\theta''_{S} + aA_{S}$$

whence

$$a = \frac{\Delta\theta''_{S} - \Delta\theta''_{N}}{A_{N} - A_{S}}. \tag{219}$$

Inasmuch as there is danger of a change in the azimuth constant during the reversal, a should be determined by (219) for both positions of the instrument.

The value of $\Delta\theta$ is then to be calculated by

$$\Delta\theta = \Delta\theta'' + aA. \tag{220}$$

The mean of all such values is the final value of the clock correction.

The chief advantage of the meridian method of time determination is to be found in the fact that the results do not depend upon a reading of the circles. Since the uncertainty of an observed transit is considerably less than that of an angle measured with a graduated circle, the precision is relatively high. It is the standard method of determining time in observatories. When carried out with a large and stable instrument mounted permanently in the plane of the meridian, with the inclusion of certain refinements not considered in the preceding sections, it affords results not surpassed by those of any other method, either in precision or in the amount of labor involved in the reductions.

83. Procedure.—To place the instrument in the meridian we may make use of a distant object of known azimuth. Set off the value of the azimuth on the horizontal circle and bring the object on the vertical thread by rotating on the lower motion. Having clamped the lower motion, rotate on the upper motion until the reading is zero. The line of sight will then be approximately in the plane of the meridian.

In case no object of known azimuth is available, Polaris may be used instead. In this case the star is brought on the vertical thread at an instant for which its azimuth has previously been calculated by (184). With the exception that the setting must be made at a definite instant, the procedure is the same as that for a distant terrestrial object. The determination of the azimuth of Polaris requires a knowledge of the approximate time, but (191) shows that if θ be known within two or three minutes, the azimuth will not be in error by more than one or two minutes of arc, which is sufficiently accurate. In case the clock correction is entirely unknown, an approximation may be derived as follows: Set on Polaris and clamp in azimuth. Then rotate the telescope on the horizontal axis and observe the transit across the vertical thread of a southern star of small zenith distance. Denoting the sidereal clock time of transit by θ', the approximate error of the timepiece will be given by

$$\Delta\theta = \alpha - \theta'. \tag{221}$$

Since the azimuth of Polaris differs but little from 180°, the line of sight will not deviate greatly from the plane of the meridian, especially when directed toward points near the zenith. If the zenith distance of the time star is not more than 25° or 30° the error in θ will not, ordinarily, exceed two or three minutes, and this, as stated above, is sufficient for the calculation of the azimuth of Polaris with the precision necessary for the orientation of the instrument.

The program will include the observation of four or five stars in each position of the instrument, reversal being made at the middle of the series. Each group should contain one northern star to be used for the determination of the azimuth constant. The remaining objects should be southern stars culminating preferably between the zenith and the equator. In order that there may be sufficient data for the determination of the collimation constant, care should be taken to observe at least one pair of stars, one C. W., the other, C. E., whose declinations are equal or nearly so.

For an instrument whose vertical circle reads altitudes, the settings which will give the telescope the proper elevation to bring the stars into the field at the time of culmination are to be calculated by.

$$\text{Setting} = 90° \pm (\varphi - \delta), \qquad (222)$$

in which the upper sign refers to northern stars.

The star list with the setting for each object should be prepared in advance. This having been done, the instrument is to be levelled and adjusted in azimuth. Three or four minutes before the transit of the first star, which will occur at the clock time $\alpha - \varDelta\theta$, set the vertical circle at the proper reading, and as the star comes into the field adjust in altitude until it moves along the horizontal thread. Note the time of its transit across the vertical thread to the nearest tenth of a second. After one-half the stars have been observed in this manner, reverse the instrument about the vertical axis through 180° and proceed with the observation of the remaining stars.

Observations with the striding level for the determination of b should be made at frequent intervals throughout the observing program. Level readings increasing toward the east should be recorded as positive; toward the west, as negative. If a striding level is not available, the plate levels, especially the transverse level, should be very carefully adjusted before beginning the observations and the bubbles should be kept centered during the measures.

The reduction is begun by collecting the right ascension, the declination, and the transit factors for each star. The coördinates are to be interpolated for the instant of observation from the list of apparent places in the *Ephemeris*. The transit factors may be computed by (214), or better still, they may be interpolated from the transit factor tables. (See page 122.) If the inclination of the horizontal axis has been measured, the values of b are to be computed by (113). The value of $\varDelta\theta'$ is then to be calculated for each star by (216). Then select two stars of equal or nearly equal declination, one observed C. W., the other C. E., and calculate c by (217). Compute as many such values of c as there are pairs of stars of equal declination, and form the mean of all. With the mean value of c calculate $\varDelta\theta''$ for each star by (218). Then determine a for each position of the instrument by (219), using for this purpose the stars of extreme northern and southern declination. Finally calculate $\Delta\theta$ for each object by (220). The mean of all such values of $\Delta\theta$ is the final value of the clock correction corresponding to the mean of the observed clock times of transit.

In case the rate of the timepiece is large, each observed θ' should be corrected for rate before forming the values of $\Delta\theta'$, the corrections being applied in such a way that each θ' becomes what it would have been had all the observations been made at the same instant. The epoch to which the values of θ' are reduced is usually the exact hour or half-hour nearest the middle of the series.

Example 47. On 1909, May 19, Wed. P. M., the error of the Fauth sidereal clock of the Laws Observatory was determined by the meridian method, the instrument used being a Buff & Buff engineer's transit.

The error of the clock was known to be approximately $+ 7^m 0^s$. The azimuth of Polaris calculated by (184) for the clock time $11^h 51^m 0^s$ was 179° 26′.5. Vernier A of the horizontal circle was set at this value, and at the clock time indicated Polaris was brought to the intersection of the threads by means of the lower motion. After clamping, the upper motion was released and vernier A was made to read 0°. The instrument having thus been placed in the meridian, the transits of four stars were observed. The reversal was then made by changing the reading of vernier A from 0° to 180°, after which four more stars were observed. The plate levels were carefully adjusted at the beginning, and the bubbles were kept centered throughout the observations.

The first of the tables gives the observing program and the data of observation. The various columns contain, respectively, the number, name, magnitude or brightness, and the apparent right ascension and declination of the stars; the setting of the vertical circle, the the observed clock time of transit, and the position of the circle. The settings were obtained by adding the colatitude 51° 3′ to the values of the declination. For northern stars this sum must be subtracted from 180°.

The second table contains the reduction and the value of the clock correction derived from each star. The values of $\Delta\theta'$ are obtained by subtracting each θ' from the corresponding a in accordance with (216). The third and fourth columns contain the values of the transit factors interpolated from the tables of the Laws Observatory. None of the pairs of stars observed are suitable for the determination of the collimation by (217). To avoid this difficulty, approximate values of the azimuth constant are derived by (219) from stars 1 and 4, and 6 and 8, $\Delta\theta''$ being replaced by $\Delta\theta'$ for this purpose. The results are $a_w = + 2^s.2$ and $a_e = + 4^s.4$. These values are uncertain owing to the fact that the influence of the collimation has been neglected in deriving them, but they are sufficiently accurate for a determination of c by (216a), provided we use for this calculation stars whose declinations differ as little as possible. Substituting the numerical values of a, A, and C into (216a) for stars 3 and 5, and 2 and 8 we find

$$\Delta\theta = + 7^m 4^s.4 + 1.04c \qquad \Delta\theta = + 7^m 4^s.4 + 1.05c$$
$$\Delta\theta = + 7\ \ 4.6 - 1.00c \qquad \Delta\theta = + 7\ \ 5.0 - 1.00c$$

These two sets of equations give for c, $+ 0^s.10$ and $+ 0^s.27$, respectively. The mean, $+ 0^s.19$, is accepted as the value of the collimation constant. Multiplying this by the value of C for each star gives the corrections for collimation contained in the fifth column. The combination of these with the value of $\Delta\theta'$ gives the quantities in the column headed $\Delta\theta''$. It should be noted that the algebraic sign of the collimation correction changes with the reversal of the instrument. The azimuth constant is now redetermined for each position of the circle, using for this purpose the value of $\Delta\theta''$ for stars 1 and 4, and 6 and 8. The results are $a_w = + 2^s.38$ and $a_e = + 4^s.16$. From these we find the values of the azimuth corrections aA, which, added to the values of $\Delta\theta''$ in accordance with (220), give $\Delta\theta$, the clock correction for each star.

The last column contains the weight assigned to each result in forming the mean value of the clock correction. The mean $\Delta\theta$ for the southern stars is the same for each position of the instrument, which shows that the influence of the collimation has been satisfactorily eliminated. It should be noted, as a control upon the calculation of the azimuth constant, that the values of $\Delta\theta$ for each pair of azimuth stars must agree within one unit of the last place of decimals. In the present case the agreement is exact for both pairs.

No.	Star	Mag.	α	δ	Setting	θ'	Circle
1	ε Corvi	3.2	$12^h\ 5^m 27^s.3$	$-22°\ 7'$	28° 56′ S	$11^h 58^m 25^s.8$	W
2	γ Corvi	2.7	11 8.2	$-17\ 2$	34 1 S	12 4 5.7	W
3	δ^2 Corvi	3.1	25 10.1	$-16\ 1$	35 2 S	18 7.6	W
4	κ Draconis	3.8	29 39.0	$+70\ 17$	58 40 N	22 32.0	W
5	γ Virginis	2.9	37 3.9	$-0\ 57$	50 6 S	30 2.1	E
6	32² Camelop.	5.2	48 36.1	$+83\ 54$	45 3 N	41 2	E
7	ε Virginis	3.1	12 57 39.8	$+11\ 27$	62 30 S	50 37.1	E
8	θ Virginis	4.6	13 5 15.2	$-5\ 3$	46 0 S	58 13.3	E

No.	$\Delta\theta'$	A	C	cC	$\Delta\theta''$	aA	$\Delta\theta$	Wt.
1	$+7^m 1^s.5$	+0.94	+1.08	$+0^s.2$	$+7^m 1^s.7$	$+2^s.2$	$+7^m 3^s.9$	1
2	2.5	+0.87	1.05	+0.2	2.7	+2.1	4.8	1
3	2.5	+0.85	1.04	+0.2	2.7	+2.0	4.7	1
4	7.0	−1.54	2.96	+0.6	7.6	−3.7	3.9	0
5	1.8	+0.64	1.00	−0.2	1.6	+2.7	4.3	1
6	34.1	−6.65	9.41	−1.8	32.3	−27.7	4.6	0
7	2.7	+0.47	1.02	−0.2	2.5	+2.0	4.5	1
8	1.9	+0.70	+1.00	−0.2	1.7	+2.9	4.6	1

At $\theta' = 12^h.5$ $\quad \Delta\theta_P = +7^m 4^s.47$

4. THE POLARIS VERTICAL CIRCLE METHOD

SIMULTANEOUS DETERMINATION OF TIME AND AZIMUTH

84. Theory.—In the method now to be discussed the transits of stars are observed across the vertical circle passing through Polaris, the instrument being adjusted with reference to the plane of this circle by bringing Polaris on the vertical thread immediately before each transit. Since the azimuth of Polaris is always a small angle, that of each time star at the instant of its observation will also be small. The conditions do not therefore differ essentially from those in the meridian method, and the clock correction may be calculated by (215) as before. The only question to be considered is whether the approximations introduced in deriving this equation are justifiable in view of the fact that the value of a in the vertical circle method may amount to 1° or 2°, while with the meridian method it need not exceed 1′ or 2′. It can be shown that, when it is a question of hundredths of a second of time in the final result, (215) is insufficient; but for those cases in which an uncertainty of one or two tenths of a second is permissible, the approximation is ample.

In the meridian method both a and c are determined from the observations themselves. Here we determine c as before, but a is to be calculated from the known position of Polaris. The azimuth constant will nearly equal the azimuth of Polaris measured from the north point positive toward the east at the instant of setting, but not exactly, owing to the presence of the instrumental constants b and c. If a_0 represent the azimuth of Polaris defined as above, we have by (82) and (89)

$$a = a_0 + b \cot z_0 \pm c \operatorname{cosec} z_0,$$

z_0 being the zenith distance of Polaris. Since b and c are very small, z_0 may be replaced by $90 - \varphi$, whence

$$a = a_0 + b \tan \varphi \pm c \sec \varphi. \tag{223}$$

Substituting (223) into (215) and writing

$$B' = A \tan \varphi + B, \qquad C' = A \sec \varphi + C, \tag{224}$$

we have

$$\varDelta\theta = \alpha - \theta' + a_0 A + bB' \pm cC', \tag{225}$$

where, as before, the upper sign refers to C. W. Equation (225) is the same in form as (215); but its solution is slightly different, for (184) gives

$$a_0 = -\pi G \sec \varphi \sin t_0, \tag{226}$$

which may be used for the calculation of a_0. This leaves in (225) only two unknowns, $\varDelta\theta$ and c, and the observation of any two time stars therefore affords the data necessary for a complete solution of the problem. For the sake of precision one of these should be observed C. W., the other, C. E. To determine c write

$$\varDelta\theta' = \alpha - \theta' + a_0 A + bB'. \tag{227}$$

We then find from (225)

$$\varDelta\theta = \varDelta\theta'_{\mathrm{W}} + cC'_{\mathrm{W}}$$
$$\varDelta\theta = \varDelta\theta'_{\mathrm{E}} - cC'_{\mathrm{E}}$$

whence

$$c = \frac{\varDelta\theta'_{\mathrm{E}} - \varDelta\theta'_{\mathrm{W}}}{C'_{\mathrm{E}} + C'_{\mathrm{W}}}. \tag{228}$$

There is here no necessity for an equality in declination of the two stars as in the case of the meridian method, for the influence of the azimuth is in this case included in $\Delta\theta'$. Having found c from (228) we calculate $\Delta\theta$ from (225) written in the form

$$\Delta\theta = \Delta\theta' \pm cC'. \tag{229}$$

The factor A in (227) is the same as that in (215), but it must be more accurately known than in the meridian method, on account of the magnitude of a_0. The quantities B' and C' are easily reduced by (214) to

$$B' = \sec\varphi, \qquad C' = E + \tan\varphi, \tag{230}$$

in which

$$E = \sec\delta - \tan\delta. \tag{231}$$

The values of E may be taken from Table X with δ as argument, whence C' may be found by the simple addition of $\tan\varphi$. For any given latitude C' itself may be tabulated with δ as argument. The third column of Table X contains such a series of values for the latitude of the Laws Observatory, viz., 38° 57'.

The vertical circle method is easily adapted to a simultaneous determination of time and azimuth. If the horizontal circle be read at the instant of setting on Polaris, and if in addition, readings be taken on a mark, the azimuth of the mark will be given at once; for the azimuth of Polaris is calculated in the course of the reduction of the observations for time, and the horizontal circle readings give the azimuth difference of the star and the mark. Since a_0 is measured from the north point positive toward the east, the azimuth of the mark measured in the conventional manner will be

$$A_m = M - S + a_0 - 180° \tag{232}$$

in which S and M are the means of the horizontal circle readings on the star and the mark, respectively; and a_0, the mean of the calculated azimuths of Polaris.

The vertical circle method of time determination, like that of the meridian method, is not dependent upon the reading of graduated circles, and in consequence, yields results of a relatively high degree of precision. It possesses the further advantage that no preliminary adjustment in the plane of the meridian is necessary. It is especially valuable for use with unstable instruments, for the constancy of the quantities a, b, and c is assumed for only a very short interval, much less than in the meridian method. It is necessary that the azimuth and level constants remain unchanged only during the interval separating the setting on Polaris and the transit of the time star immediately following, and this need not exceed two or three minutes. Moreover, each set of two time stars is complete in itself and gives a complete determination of the error of the timepiece.

The instrument used should be carefully constructed, however, for any irregularity in the form of the pivots is likely to produce serious errors in the results.

85. Procedure.—The instrument is carefully levelled, and three or four minutes before the transit of a southern star across the vertical circle through Polaris, the telescope is turned to the north, and Polaris itself is brought to the intersection of the vertical and horizontal threads. The instrument is clamped in azimuth and the sidereal time of setting, θ_0, is noted. The telescope is then rotated about the horizontal axis until its position is such that the southern or time star will pass through the field of view. The transit of the time star is observed, and the entire process is then repeated for a second time star, with the instrument in the reversed position. The data thus obtained constitute a set and permit a determination of the error of the timepiece.

If a simultaneous determination of time and azimuth is required, the program for a set will be

Set on the mark and read the H. C. Set on Polaris, note the time, and read the H. C. Observe the transit of the time star.	C. W.
Set on Polaris, note the time, and read the H. C. Observe the transit of the time star. Set on the mark and read the H. C.	C. E.

in which C. W. and C. E. are to be interpreted as meaning that if the instrument be turned from the mark to the north by rotating about the vertical axis, the vertical circle will then be west or east, respectively. The plate levels should be carefully watched, and if there is any evidence of creeping, the instrument should be relevelled.

The observing list with the settings for the time stars should be prepared in advance. It is also desirable, in order to save time in observing and to avoid errors in the identification of the stars, to calculate in advance the approximate times of transit. Disregarding the errors in level and collimation we have from (225)

$$\theta' = \alpha + a_0 A - \Delta\theta \tag{233}$$

in which $\Delta\theta$ represents an approximate value of the clock correction. To derive a value for the term $a_0 A$ we combine equation (226) with the value of A from (214), and write

$$\pi = 1^\circ\, 10' = 4^{m}.7, \qquad G = 1, \qquad t_0 = \theta_0 - \alpha_0 = \theta_0 - 1^{h}\, 30^{m}.$$

We thus find

$$a_0 A = P(\tan\delta - \tan\varphi), \tag{234}$$

in which

$$P = 4^{m}.7 \sin(\theta_0 - 1^{h}\, 30^{m}). \tag{235}$$

Since $a_0 A$ need be known only very roughly, we may use a constant value for θ_0, choosing for this purpose the sidereal time corresponding approximately to the middle of the observing program.

P having been calculated from (235) we find the value of $a_0 A$ for each time star from (234) by introducing the corresponding value of δ. One or two places of decimals are ample for the calculation.

The observations having been secured, the first step in the reduction is the determination of an approximation for the clock correction of sufficient accuracy for the calculation of the azimuth of Polaris. Neglecting errors in level and collimation we have from (225)

$$\Delta\theta_0 = \alpha - \theta' + a_0 A, \qquad (236)$$

which applied to the time star transiting nearest the zenith will give the required approximation. For the term $a_0 A$ we may introduce the value calculated by (234) in preparing the observing list. Collecting results we have the following notation and formulæ:

α_0, π, and α, δ are the coördinates of Polaris and the time star, respectively;

θ_0 and θ', the sidereal clock times, respectively, of their observation;

S and M, the readings of the horizontal circle for settings on Polaris and the mark, respectively;

A_m, the azimuth of the mark measured from the south, positive toward the west;

$\Delta\theta$, the error of the timepiece, and $\Delta\theta_0$, an approximation for this quantity.

$$\begin{gathered}
t_0 = \theta_0 + \Delta\theta_0 - \alpha_0, \qquad a_0 = -\pi G \sec\varphi \sin t_0, \\
A = \sin(\varphi - \delta)\sec\delta, \qquad C' = \tan\varphi + E, \\
\Delta\theta' = \alpha - \theta' + a_0 A + b\sec\varphi \qquad (237) \\
c = \frac{\Delta\theta'_e - \Delta\theta'_w}{C'_e + C'_w}, \\
\Delta\theta = \Delta\theta'_w + cC'_w = \Delta\theta'_e - cC'_e.
\end{gathered}$$

Log G or log $G\sec\varphi$ is to be taken from Table IX, which is reprinted here for convenience, with the argument t_0; E or C', from Table X with the argument δ. The subscripts w and e refer to observations made circle west and circle east, respectively. Finally, calculate

$$A_m = \tfrac{1}{2}\left[M_e - (S - a_0)_e + M_w - (S - a_0)_w\right] - 180^\circ, \qquad (238)$$

where the subscripts attached to M refer to settings made with the instrument in such a position that if turned toward the north by rotation about the vertical axis, the circle would then be west or east, respectively, according to the subscript.

For the determination of the error of the clock, a_0 should be expressed in seconds of time; for the determination of the azimuth, in minutes of arc. The values of A are needed to four places of decimals, and when once obtained, should be preserved, since, for a given latitude, they may be used unchanged for several months. If the collimation is known to be small and the declinations of the two time stars do not differ too greatly, it will be sufficient to take the mean of the values of $\Delta\theta'$ for C. W. and C. E. as the error of the timepiece.

TABLE IX, 1910.0

t_0	log G	log G sec φ	t_0
0^h	0.0075	0.1167	24^h
1	0.0073	0.1165	23
2	0.0065	0.1156	22
3	0.0053	0.1145	21
4	0.0037	0.1129	20
5	0.0019	0.1111	19
6	0.0000	0.1092	18
7	9.9981	0.1072	17
8	9.9963	0.1055	16
9	9.9948	0.1039	15
10	9.9936	0.1028	14
11	9.9928	0.1020	13
12	9.9926	0.1018	12

TABLE X

δ	E	C'
$+30°$	0.58	1.39
+25	0.64	1.45
+20	0.70	1.51
+15	0.77	1.58
+10	0.84	1.65
+5	0.92	1.72
0	1.00	1.81
−5	1.09	1.90
−10	1.19	2.00
−15	1.30	2.11
−20	1.43	2.24
−25	1.57	2.38
−30	1.73	2.54

Example 48. On 1909, May 19, immediately after securing the meridian observations given in Ex. 47, a simultaneous determination of time and azimuth was made by the Polaris vertical circle method, the instrument used being the same as that employed for the meridian observations. The stars observed were

Object	Mag.	R. A.	Dec.	Setting
Polaris	2.2	$1^h\ 25^m\ 34^s$	$+88°\ 49'\ 3''$	———
α Virginis	1.1	13 20 24.9	−10 41	40 22
ζ Virginis	3.6	13 30 4.4	−0 8	50 55

During the observations the vertical circle of Polaris was so nearly in coincidence with the meridian that no special calculation of the instant of transit of the time stars across this circle was necessary. The record of the observations is as follows:

Object	Fauth Clk.	Horizontal Circle Ver. A	Ver. B	Circle
Mark	———	$7°\ 46'.0$	187 46'.0	W
Polaris	13 10 20	179 56.5	359 56.5	W
α Virginis	13 10.5	———	———	W
Polaris	16 2	359 58.5	179 58.5	E
ζ Virginis	22 57.2	———	———	E
Mark	———	187 45.5	7 45.5	E

We have $\pi = 70'.95$, $\log \pi = 1.8510$. For the calculation of the azimuth of Polaris we use the approximate clock correction $\Delta\theta_0 = +7^m 0^s$, whence $\alpha_0 - \Delta\theta_0 = 1^h 18^m 34^s$. The combination of this with θ_0 in accordance with (237) gives t_0.

The azimuth a_0 whose logarithm is given in the fourth line is expressed in minutes of arc. Since the correction a_0A must be expressed in seconds of time the logarithm of 4, viz., 0.6020, is also included when log a_0 and the two logarithms immediately following it are added to form log a_0A. The final value of the clock correction is in satisfactory agreement with that found in Ex. 47.

	α Virginis, C. W.	ζ Virginis, C. E.
t_0	$11^h\ 51^m\ 46^s$	$11^h\ 57^m\ 28^s$
$\sin t_0$	8.5553	8.0435
$G \sec \varphi$	0.1018	0.1018
$\log a_0$	0.5081_n	9.9963_n
$\sin(\varphi - \delta)$	9.8819	9.7996
$\sec \delta$	0.0076	0.0000
$\log a_0A$	0.9996_n	0.3979_n
a_0A	$-10^s.0$	$-2^s.5$
$\alpha - \theta'$	+7 14.4	+7 7.2
$\Delta\theta'$	+7 4.4	+7 4.7
C'	2.0	1.8
cC'	+0.2	−0.1
$\Delta\theta$	$+7^m 4^s.6$	
a_0	$-3'.2$	$-1'.0$
$S - a_0$	179 59.7	359 59.5
M	7 46.0	187 45.5
A_m	$+7^\circ\ 46'.15$	

ERRATA

PAGE	LINE	
3,	29,	Many nebulae show continuous spectra, indicating that they may not be wholly gaseous in constitution.
28,	14 of arguments,	*for* cos z co φ *read* cos z cos φ.
31,	14,	*for* or *read* and.
37,	11,	*for* 0 to 24 *read* 0 to 23.
37,	17,	*for* 0 to 24 *read* 0 to 23.
37,	17,	*for* 0 to 12 *read* 1 to 12.
37,	18,	*for* 0 to 12 *read* 1 to 12.
39,	last,	the equation number refers to both equations.
40,	2, Ex. 11,	*for* lime *read* time.
41,	2, Ex. 13,	*for* apparant *read* apparent.
42,	4 and 5, Sec. 26,	*interchange* I_s and I_m.
60,	last,	*for* 0.05d *read* 0.05d.
64,	eq. (72),	*for* $-E_2$ *read* $+E_2$.
70,	17,	*for* $180^\circ + c$ *read* 180°, approximately.
73,		in form for record of level observations: in last column, for r' and r'', read l' and l''; for, Sd, read SD.
75,	prec. eq. (117),	*for* (112) *read* (116).
80,		in Ex. 31, *add:* Alcyone was east of the meridian at the time of observation.
81,	4, Ex. 33,	*for* Thursday *read* Tuesday.
85,		in Ex. 34, *add:* The observations were made at the Laws Observatory.
96,	prec. eq. (141),	*for* obervation *read* observation
98,		first eq., *for* z' *read* z'_n.
98,		eq. (143), *for* r *read* r_s.

INDEX

(The Numbers Refer to Pages.)