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ON THE PROJECTIVE AND EQUI-PROJECTIVE GEOMETRIES OF PATHS

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1. *Projective Geometry of Paths.*—(Cf. H. Weyl, *Göttingen Nachrichten,* 1921, p. 99; also O. Veblen and T. Y. Thomas, *Trans. Amer. Math. Soc.,* **25,** 1923, p. 551.) In a sense the projective geometry of paths is the *true geometry of paths* for we are here concerned with properties of the paths apart from their definition by means of a particular affine connection $\Gamma^i_{\alpha\beta}$. When the affine connection $\Gamma^i_{\alpha\beta}$ of an affine geometry of paths is replaced by an affine connection $\Lambda^i_{\alpha\beta}$ in accordance with the equations

$$\Lambda^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} + \delta^i_\alpha \Phi_\beta + \delta^i_\beta \Phi_\alpha \tag{1.1}$$

where Φ_α is an arbitrary covariant vector, the set of functions $\Lambda^i_{\alpha\beta}$ defines the same system of paths as the set of functions $\Gamma^i_{\alpha\beta}$. Also any two sets of functions $\Gamma^i_{\alpha\beta}$ and $\Lambda^i_{\alpha\beta}$ defining the same system of paths are related by equations of the form (1.1).

2. *The Projective Connection.*—One of the problems of the projective geometry of paths is the problem of the projective equivalence of two affine spaces of paths, i.e., the problem of the determination of the conditions under which there exists a transformation.

$$x^i = \varphi^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n), \quad \Delta = \left|\frac{\partial x}{\partial \bar{x}}\right| \neq 0 \tag{2.1}$$

mapping two affine spaces of paths upon one another in such a way that the paths of one space are mapped upon the paths of the other. Let us denote the affine connection of two affine spaces of paths by sets of functions $\Gamma^i_{\alpha\beta}$ and $\bar{\Gamma}^i_{\alpha\beta}$, respectively. We then have that a necessary and sufficient condition for the projective equivalence of the two spaces is that there exists a relation (2.1) and vector Φ_α which satisfy the equations

$$\Gamma^i_{\alpha\beta} + \delta^i_\alpha \Phi_\beta + \delta^i_\beta \Phi_\alpha = \frac{\partial x^i}{\partial \bar{x}^\gamma}\left(\frac{\partial^2 \bar{x}^\gamma}{\partial x^\alpha \partial x^\beta} + \bar{\Gamma}^\gamma_{\mu\nu}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta}\right). \tag{2.2}$$

Eliminating the vector Φ_α from the equations (2.2) we have

$$\bar{\Pi}^\sigma_{\alpha\beta}\frac{\partial x^i}{\partial \bar{x}^\sigma} = \Pi^i_{\sigma\tau}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha}\frac{\partial x^\tau}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} - \frac{1}{n+1}\frac{\partial \log \Delta}{\partial \bar{x}^\alpha}\frac{\partial x^i}{\partial \bar{x}^\beta} - \frac{1}{n+1}\frac{\partial \log \Delta}{\partial \bar{x}^\beta}\frac{\partial x^i}{\partial \bar{x}^\alpha}, \qquad (2.3)$$

where

$$\Pi^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} - \frac{\delta^i_\alpha}{n+1}\Gamma^\sigma_{\sigma\beta} - \frac{\delta^i_\beta}{n+1}\Gamma^\sigma_{\sigma\alpha}. \qquad (2.4)$$

The functions $\bar{\Pi}^i_{\alpha\beta}$ have a similar definition in terms of the affine connection $\bar{\Gamma}^i_{\alpha\beta}$. These functions are independent of the particular affine connection by means of which the paths are defined, and constitute a normalized affine connection which I shall speak of as the *projective connection* $\Pi^i_{\alpha\beta}$. We observe that

$$\Pi^\sigma_{\sigma i} = 0 \qquad (2.5)$$

identically.

3. *The Projective Parameter.*—The set of differential equations

$$\frac{d^2 x^i}{dp^2} + \Pi^i_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} = 0 \qquad (3.1)$$

defines our system of paths. Also any set of differential equations of the paths may be put into the form (3.1) for a proper choice of the parameter p. The parameter p is thus a normalized parameter which is independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ defining the paths and which I shall call the *projective parameter* p.

Under the arbitrary transformation (2.1) the projective parameter p becomes $\bar{p}$, and there exists a relation

$$p = f(\bar{p}) \qquad (3.2)$$

between the parameters p and $\bar{p}$ along any particular path. In general the relation (3.2) will depend on the particular path and may contain certain arbitrary constants expressing the indeterminateness of the parameters. We shall wish to find the explicit form of the relation (3.2).

With little difficulty we may show that

$$\frac{\left[\bar{\Pi}^i_{\sigma\tau} - \left(\Pi^\lambda_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\sigma}\frac{\partial x^\beta}{\partial \bar{x}^\tau} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\sigma \partial \bar{x}^\tau}\right)\frac{\partial \bar{x}^i}{\partial x^\lambda}\right]\frac{d\bar{x}^\sigma}{d\bar{p}}\frac{d\bar{x}^\tau}{d\bar{p}}}{\frac{d\bar{x}^i}{d\bar{p}}} = \frac{\frac{d^2 p}{d\bar{p}^2}}{\left(\frac{dp}{d\bar{p}}\right)^2}, \qquad (3.3)$$

in which we know that the bracket

$$\bar{\Pi}^i_{\sigma\tau} - \left(\Pi^\lambda_{\alpha\beta}\frac{\partial x^\alpha}{\partial\bar{x}^\sigma}\frac{\partial x^\beta}{\partial\bar{x}^\tau} + \frac{\partial^2 x^\lambda}{\partial\bar{x}^\sigma\partial\bar{x}^\tau}\right)\frac{\partial\bar{x}^i}{\partial x^\lambda}$$
$$= \delta^i_\sigma\,\Phi_\tau + \delta^i_\tau\,\Phi_\sigma \tag{3.4}$$

for some vector Φ_σ. Putting $i = \tau$ in (3.4) we have

$$(n+1)\Phi_\sigma = -\frac{\partial}{\partial\bar{x}^\sigma}\log\Delta. \tag{3.5}$$

Also we easily find

$$2\Phi_\sigma\frac{\partial\bar{x}^\sigma}{\partial x^\tau}\frac{d\bar{x}^\tau}{dp} = \frac{d}{dp}\log\left(\frac{d\bar{p}}{dp}\right). \tag{3.6}$$

The equations (3.5) and (3.6) now give

$$\frac{dp}{d\bar{p}} = \Delta^N, \qquad N = 2/(n+1) \tag{3.7}$$

the constant of integration being absorbable into the parameters. In (3.7) Δ is to be regarded as a function of $\bar{p}$ obtained by eliminating the variables $(\bar{x})$ occurring in Δ by means of the parametric equations of the path. Integrating (3.7) with the initial condition $p = 0$ when $\bar{p} = 0$ we have

$$p = \int_0^{\bar{P}}\Delta^N(\bar{p})d\bar{p} \tag{3.8}$$

as the explicit form of (3.2).

4. *Equivalence Theorem for the Projective Geometry of Paths.*—Purely analytically the problem of the projective equivalence of two affine spaces is simply the problem of the determination of the integrability conditions of the equations (2.3). The determination of these integrability conditions in satisfactory form appears to me to be the *fundamental problem of the projective geometry of paths.* Comparison with the affine case would lead us to expect that if algebraic conditions of integrability are attainable that we shall have in these conditions the analytic material for the development of the projective geometry of paths corresponding to the development of the affine geometry of paths. But the solution of this problem remains for the present open.

5. *Equi-Projective Geometry of Paths.*—Let us now consider those transformations (2.1) which satisfy the condition

$$\Delta = 1 \tag{5.1}$$

Geometrically this means that we shall consider transformations leaving volume unchanged. Thus the set of vectors $\xi^i_{(\alpha)}$ (i, $\alpha = 1, 2, \ldots, n$) define the volume $|\xi^i_{(\alpha)}|$ and this volume is invariant under transformations satisfying (5.1). As there seems to be no single word in the English language descriptive of this class of transformations I shall here take the liberty of introducing the word *equi-transformation* for an arbitrary transformation (2.1) satisfying (5.1). We shall also find it convenient to speak of *equi-projective* properties of the paths as those properties independent of any particular affine connection $\Gamma^i_{\alpha\beta}$ which are invariant under equi-transformations. The body of theorems expressing equi-projective properties of the paths will constitute the *equi-projective geometry of paths*. (It is also clear what would be meant by the *equi-affine geometry of paths*. We may note that the word "equi-affine" which here appears does not have the same meaning as the word "equiaffine" used by O. Veblen, these PROCEEDINGS, **9** 1923, p. 3.)

For the equi-projective geometry of paths the equations (2.3) become

$$\bar{\Pi}^\sigma_{\alpha\beta}\frac{\partial x^i}{\partial \bar{x}^\sigma} = \Pi^i_{\sigma\tau}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha}\frac{\partial x^\tau}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}. \tag{5.2}$$

The equations of transformation on the projective connection $\pi^i_{\alpha\beta}$ in the equi-projective geometry of paths have therefore the same form as the equations of transformation of the affine connection $\Gamma^i_{\alpha\beta}$ in the affine geometry of paths. Also (3.8) gives

$$p = \bar{p}, \tag{5.3}$$

for the equi-projective geometry of paths.

6. *Equi-Projective Normal Coördinates.*—Normal coördinates may be erected for the equi-projective geometry of paths in the same manner as for the affine geometry of paths. Thus the equations (3.1) possesses a unique solution of the form

$$x^i = q^i + \Gamma^i(q, \xi p) \tag{6.1}$$

(Cf. O. Veblen and T. Y. Thomas, *Trans. Amer. Math. Soc.*, **25**, 1923, p. 551) which satisfies a set of initial conditions

$$x^i = q^i \text{ when } p = 0; \quad \frac{dx^i}{dp} = \xi^i \text{ when } p = 0. \tag{6.2}$$

Introducing the substitution

$$z^i = \xi^i p \tag{6.3}$$

into the equations (6.1) these equations become

$$x^i = q^i + \Gamma^i (q, z) \tag{6.4}$$

and define a set of *equi-projective normal coördinates* (z). The equations of a path passing through the origin of the equi-projective normal coördinate system (z) are given by (6.3). Under an arbitrary equi-transformation (2.1) of the general (x) coördinates, the equi-projective normal coordinates (z) which are defined by the (x) coördinates and a point (q) suffer a transformation of the form.

$$z^i = a^i_\alpha \bar{z}^\alpha \tag{6.5}$$

where the a's are constants such that

$$a^i_\alpha = \left(\frac{\partial x^i}{\partial \bar{x}^\alpha}\right)_q \tag{6.6}$$

evaluated at the point (q). It follows from (6.6) that (6.5) is an equi-transformation. We have also that

$$\left|\frac{\partial x}{\partial z}\right| = \left|\frac{\partial \bar{x}}{\partial \bar{z}}\right| \tag{6.7}$$

for the equi-projective geometry of paths.

7. *Equi-Tensors and Equi-Projective Extension.*—By an *equi-tensor* will be meant a set of quantities $T^{lm \ldots n}_{ij \ldots k}$ $(i, j, \ldots, k, l, m, \ldots, n = 1, 2, \ldots, n)$ which transforms as an ordinary tensor under equi-transformations. We immediately see that the ordinary process of extension applied to a tensor or equi-tensor but with the projective connection $\Pi^i_{\alpha\beta}$ replacing the affine connection $\Gamma^i_{\alpha\beta}$ gives rise to an equi-tensor of higher order. This proces of extension involving the projective connection $\Pi^i_{\alpha\beta}$ will be called *equi-projective extension.*

8. *Equi-Projective Normal Tensors and the Equi-Projective Curvature Tensor.*—By replacing the affine connection $\Gamma^i_{\alpha\beta}$ in the expressions for the ordinary normal tensors by the projective connection $\Pi^i_{\alpha\beta}$ we obtain the equi-projective normal tensors which I shall denote by $\mathfrak{A}^i_{\alpha\beta\gamma\ldots\tau}$. In a similar manner we obtain the equi-projective curvature tensor $\mathfrak{B}^i_{\alpha\beta\gamma}$.

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