Therefore, the matrix U_ν defined in Nr. 6 has been determined under the assumption that T is one-valued:

$$U_\nu = (-1)^{n(\nu+1)} a E_{\pi_\nu} \tag{21}$$

where E_{π_ν} is the matrix unity. From (21) and (16) formula (2) follows immediately.

[1] Cf. S. Lefschetz, (a) *Trans. Am. Math. Soc.*, **28**, pp. 1–49; (b) *Trans. Am. Math. Soc.*, **29**, pp. 429–462; particularly p. 32 of (a). The sign "$\approx$" introduced in (a) means "$\sim$" mod. zero-divisors. The fundamental sets of this paper are all with respect to the operation $\approx$.

[2] L. E. Y. Brouwer, *Mathem. Annalen*, **71**, pp. 92–115.

[3] O. Veblen, *Trans. Am. Math. Soc.*, **25**, pp. 540–550.

[4] (a), formula 1.1; (b), formula 10.5.

[5] A great deal of Nr. 6 is only a summarizing report on facts which are included in the papers of Lefschetz, quoted above.

[6] Lefschetz, (a) No. 52.

[7] Lefschetz, (a) No. 55. The proof of this formula, not explicitly given there, can be obtained easily by the same considerations as for the formulas of (a) Nos. 53, 54.

[8] Lefschetz, (b) 9.2.

HARMONY AS A PRINCIPLE OF MATHEMATICAL DEVELOPMENT

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Communicated January 27, 1928

In the preface of volume 6 of the "Collected Works" of Sophus Lie the editor remarked that these works constitute an artistic contribution which is entirely comparable with that of Beethoven. In both cases a few apparently trivial motives dominated the entire creation. In the case of Lie it was mainly the explicit use of the concept of group in domains where its dominance had not been explicitly recognized before his time. This concept was practically refused by the mathematical builders up to the beginning of the nineteenth century but became during this century "the head stone of the corner" largely through the work of Sophus Lie. While the work of Beethoven has reached and probably will continue to reach a much larger number of people than that of Lie, it is questionable whether those to whom Lie's work is actually revealed receive less inspiration therefrom or are less impressed by the marvelous new harmonies which it introduces into a wide range of mathematical developments. As Sophus Lie was a Foreign Associate of this Academy and his work has greatly influenced the development of mathematics in our country it seems appropriate to note here this tribute to his memory.

A. Speiser recently directed attention to the importance of the study of the symmetries of ancient ornaments with a view to an understanding of their influence on the development of mathematics. In particular, he expressed the view that an investigation of the geometric content of Egyptian and Arabian ornaments constitutes one of the most beautiful chapters of the history of mathematics which remains to be written, and that the development of higher mathematics began about 1000 years earlier than is now commonly assumed, having been inspired by an interest in regular figures on the part of the ancient Egyptians as early as 1500 B. C. Evidence of his faith in the fruitful contact between mathematics and ancient ornaments is exhibited by the remark that an intensive study of the noted work entitled "The Grammar of Ornament" by Owen Jones, 1856, may be urgently recommended for all who are interested in the theory of groups.[1] This new point of view is emphasized here because it tends to widen the horizon of the specialist in this field and to unite scientific efforts along lines which are commonly regarded as widely apart.

Our main present object is to exhibit the principle of harmony as a useful principle in the study of the history of mathematics. The difficulties of this study are increasing rapidly with the growth of our subject and hence it is increasingly desirable to find general coördinating and clarifying principles. It is well known that in the development of the number concept the principle of permanence of form, which has much in common with the principle of harmony, played a very fundamental rôle. The art of combining with a minimum of blind formulas a maximum of seeing thoughts, which Minkowski characterized as the true Dirichlet Principle,[2] is largely due to the principle of harmony in mathematics. H. Poincaré,[3] noted that without mathematics "we should have been ignorant forever of the internal harmony of the world, which is the only true objective reality." It is only natural that the language which is so powerful for discovering the harmonies of nature should be dominated by the same fundamental principle. This is reflected in the fact that the adjective *harmonic* is used in connection with a very large number of mathematical terms.

The ancient Greeks introduced a very fundamental note of discord into the development of mathematics when they assumed that there is no (1, 1) correspondence between the real numbers and the points on a straight line. This discord was largely removed by Newton when he defined a number as the ratio of one line segment to another. Here, as well as in various other parts of mathematics, perfect harmony was finally established by means of an explicit postulate. Since the times of the ancient Greeks mathematicians have never been in complete accord as regards some of the fundamental principles. Fortunately, there have always been also large domains of mathematics which have been free from discord.

Among the many fundamental developments in elementary mathematics which seem to have been largely influenced by the principle of harmony we may note here the slow but continued progress towards supplanting the proportion by the equation. It is well known that the proportion played a much more prominent rôle in Greek mathematics than the equation, and that even in the eighteenth century some of the textbooks considered the theory of proportion at undue length. It is clear that the equation naturally implies a higher degree of knowledge than the proportion, and hence it implies greater intellectual harmony. The long struggle towards supplanting the proportion by the equation is not even now at an end, but progress in this direction has been very marked. A similar movement towards harmony may be seen in the prolonged effort towards replacing the definitions of the trigometric functions, depending both on the angle and the length of the line segments involved, by definitions depending only on the former. Fortunately, in this case the victory towards greater harmony in the formulas is now complete.

Developments in mathematics have frequently been made more harmonious by the introduction of ideal elements. As a conspicuous instance we may cite the use of imaginary numbers from about the middle of the sixteenth century to the close of the eighteenth, when the use of these numbers began to be placed on the basis of reals by the work of Caspar Wessell and others. Early in the seventeenth century Kepler noted that parallel lines may be regarded as having a point at infinity in common, and this use of the point at infinity later became general through the work of Desargnes and others. As a much later instance of the introduction of ideal elements we may refer to the ideals introduced by Kummer for the study of algebraic numbers. It is true that the introduction of such ideal elements can also be explained as having been actuated by the principle of generalizations, but the more general principle of harmony is clearly also involved therein. As harmony in music makes an almost universal appeal to the human ear so harmony in scientific matters makes an almost universal appeal to the human intellect, and many of the developments in mathematics can be more fully understood if this principle is duly emphasized.

It is commonly recognized that the mathematical results are more interdependent than those of any other subject and hence *the history of mathematics is naturally more scientific than any other history.* The principle of harmony is not only a source of inspiration for the investigator but also a guide as regards the novelty of the results obtained. Moreover, just as an increase in the accuracy of measurement usually discloses greater discrepancies, so an increase in accuracy of knowledge usually discloses new elements of discord. Hence, the principle of harmony tends to become more effective with the advance of knowledge and to dominate the energies of the investigator more powerfully as he attacks problems of a more fundamental nature.

[1] Speiser, A., "Die Theorie der Gruppen von Endlicher Ordung," **1927**, p. 77.
[2] Klein, F., "Vorlesungen über die Enlivicklung der Mathematik im 19. Jahrhundert," **1926**, p. 97.
[3] Poincaré, H., "La Valeur de la Science," **1908**, p. 7.

NOTE ON THE BEHAVIOR OF CERTAIN POWER SERIES ON THE CIRCLE OF CONVERGENCE WITH APPLICATION TO A PROBLEM OF CARLEMAN

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Communicated February 11, 1928

1. The first example of a continuous periodic function whose associated Fourier series $\sum_{-\infty}^{+\infty} c_\nu e^{i\nu x}$ is such that $\sum |c_\nu|^\rho$ converges for no $\rho < 2$ was given by Carleman.[1] Landau,[2] who simplified the example, called attention to a power series studied by Fabry[3] and Hardy[4] which series is continuous on the unit circle, the series $\sum |c_\nu|^{2-\delta}$ being divergent for a fixed but arbitrarily small $\delta > 0$. Hardy had to use a powerful machinery in his study of the singularities of this function; if one is satisfied, however, with determining merely the properties of convergence of the series on the unit-circle, a simpler argument can be used which applies to a much larger class of series. The method which I use for this purpose is based on some recent applications of Weyl's ideas regarding equi-distributed point-sets to number-theoretic questions due to van der Corput.[5] A fairly simple solution of Carleman's problem is obtained in this manner.

2. We shall study a class of power series

$$\sum_{n_0}^{\infty} f(n) \exp[2\pi i a(n)]\, z^n \qquad (1)$$

where $f(n)$ and $a(n)$ are subjected to one of the following three sets of conditions:

I. (*i*) $a(u)$ is a real differentiable function when $u \geqq n_0$ and $\lim a(u) = +\infty$;

(*ii*) $a'(u)$ is positive, never increasing and $\lim a'(u) = 0$;

(*iii*) $\sum |f(n) - f(n+1)|$ converges and $\lim f(n) = 0$.