10.6. REGULARITY OF A BOUNDARY POINT FOR ELLIPTIC EQUATIONS*

"I learned from Kellogg that the old problem of potential distribution was attracting renewed interest."
N. Wiener, "I am a Mathematician"

In recent years much attention has been given to a circle of questions grouped around the classical criterion of Wiener concerning the regularity of a boundary point relative to harmonic functions [1, 2]. According to Wiener's theorem, the continuity at the point 0, $0 \in \partial\Omega$, of the solution of the Dirichlet problem for the Laplace equation in the n-dimensional domain Ω, $n > 2$, under the condition that on $\partial\Omega$ there is given the function, continuous at 0, is equivalent to the divergence of the series

$$\sum_{k\geq 1} 2^{(n-2)k} \operatorname{cap}(C_{2^{-k}} \setminus \Omega).$$

Here $C_\rho = \{x : x \in R^n,\ \rho/2 \leq |x| \leq \rho\}$, while cap C is the harmonic capacity of the compactum C.

This statement has been extended (sometimes only in the sufficiency part) to various classes of linear and quasilinear second-order equations (the description of these investigations and references can be found in [3]). As far as equations of higher order than two are concerned, for them, up to recently, there have been no results similar to Wiener's theorem. In a recent paper of the author [4], one investigates the behavior near a boundary point of the solutions of Dirichlet problems with zero boundary conditions for the equation $\Delta^2 u = f$, where $f \in C_0^\infty(\Omega)$. It is proved in [4] that for n = 5, 6, 7 the condition

$$\sum_{k\geq 1} 2^{k(n-4)} \operatorname{cap}_2(C_{2^{-k}} \setminus \Omega) = \infty, \tag{1}$$

where cap_2 is the so-called biharmonic capacity, guarantees the continuity of the solution at the point 0. For n = 2, 3 the continuity of the solution follows from S. L. Sobolev's embedding theorem, while in the case n = 4, also examined in [4], the continuity condition has another form.

Conjecture 1. The condition $n < 8$ is not essential.

The author knows only one argument in favor of this conjecture: For all n, for any spherical sector, the solution of the problem under consideration is continuous at the vertex. The restriction $n < 8$ occurs only in one of the lemmas on which the proofs in [4] are based, but it is necessary for this lemma. The problem concerns the positivity property of the operator Δ^2 with the weight $|x|^{4-n}$. This property of weight positivity allows us to give for n = 5, 6, 7 the following estimate for the Green function of the biharmonic operator in an arbitrary domain:

$$|G(x,y)| \leq c(n)|x-y|^{4-n}, \tag{2}$$

where $x, y \in \Omega$, while c(n) is a constant depending only on n.

Conjecture 2. The estimate (2) holds also for $n \geqslant 8$.

It is clear that similar problems can be posed also for more general equations but I wish to draw the attention of the reader to an unsolved problem and to the Laplace operator. According to [5, 6], a harmonic function, whose generalized boundary values satisfy a Hölder condition at the point 0, satisfies itself the same condition at this point if

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$$\varliminf_{N\to\infty} N^{-1} \sum_{N\geq k\geq 1} 2^{k(n-2)} \operatorname{cap}(C_{2^{-k}} \setminus \Omega) > 0. \tag{3}$$

It would be interesting to prove or disprove the following assumption.

Conjecture 3. Condition (3) is necessary.

Finally, we turn to a nonlinear elliptic equation of the second order. As proved in [7], the point 0 is regular for the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)=u,\ 1<p<n$, if

$$\sum_{k\geq 1} [2^{(n-p)k} p\text{-}\operatorname{cap}(C_{2^{-k}} \setminus \Omega)]^{\frac{1}{p-1}} = \infty, \tag{4}$$

where

$$p\text{-}\operatorname{cap}(C) = \inf\{\|\nabla u\|^p_{L^p(\mathbb{R}^n)} : u \in C_0^\infty(\mathbb{R}^n),\ u\geq 1 \text{ on } C\}.$$

Recently, this result has been carried over in [8] to the very general class of equations $\operatorname{div} \vec{A}(x,u,\nabla u) = B(x,u,\nabla u)$. Since for p = 2 condition (4) coincides with Wiener's criterion, it is natural to formulate the following conjecture.

Conjecture 4. Condition (4) is necessary.

In [9] there are given examples which show that condition (4) is sharp in a certain sense. Recent results on the continuity of nonlinear potentials [10, 11] also speak in favor of Conjecture 4.

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