the magnitudes of the given P. E.'s, as common sense would indicate.

The proper application of the formula $r_0 = 0.6745 \sqrt{\frac{\sum wv^2}{(n-1)\sum w}}$ is to a set of direct measurements to which weights are arbitrarily assigned. It is *not* applicable to the type of problem considered in this paper, for it would give the same result if all the given P. E.'s were ten times as great or only a hundredth part as great as they actually are in any given case. The inherent character of this formula may be concisely stated in the language of biology, as follows:

The formula $r_0 = 0.6745 \sqrt{\frac{\sum wv^2}{(n-1)\sum w}}$ *takes account of the errors arising in its own generation, but takes no account of those inherited from preceding generations.*

ON CERTAIN FOURIER SERIES EXPANSIONS OF DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND[1]

By Miguel A. Basoco

Department of Mathematics, California Institute of Technology

Communicated June 26, 1929

It is a well-known fact that the Fourier series expansions of the doubly periodic functions of the first (i.e., elliptic), second and third kinds (in the sense of Hermite) yield, when subjected to appropriate methods, important results in the theory of numbers. The purpose of this paper is to indicate the derivation of such expansions for certain doubly periodic functions of the third kind of a type having a larger number of zeros than poles.[2]

Hermite[3] defines a function $\varphi(z)$ to be doubly periodic of the third kind if it is meromorphic and satisfies two periodicity relations of the form

$$\left.\begin{aligned} \varphi(z + 2w) &= e^{az + b}\, \varphi(z) \\ \varphi(z + 2w') &= e^{cz + d}\, \varphi(z) \end{aligned}\right\}, \tag{1}$$

where a, b, c, d, w, w' are constants and w'/w is a complex number $\alpha + i\beta$, $\beta \neq 0$. It may be shown that the properties of $\varphi(z)$ as defined may be obtained from those of a suitably defined function $F(z)$ which also is meromorphic and satisfies the simpler periodicity relations

$$\left.\begin{aligned} F(z+\pi) &= F(z), \\ F(z+\pi\tau) &= e^{-2miz}F(z),\ i=\sqrt{-1} \end{aligned}\right\}, \tag{2}$$

where τ is a complex number $\alpha + i\beta$, $\beta \neq 0$ and m is an integer (not zero). It can be proved that m is the excess of the number of zeros over the number of poles of the function in a period cell. Thus, the functions under consideration may be classified into two groups (A) and (B) according as m is positive or negative. As mentioned above, this paper deals with functions of type (A) only.

In a series of papers, Appell[4] has developed a theory for obtaining the Fourier expansions of these functions. He introduces a certain function $X_m(x, y)$, which in his theory plays a rôle analogous to that of the zeta function in Hermite's decomposition of an elliptic function into simple elements. It is found that his theory is applicable in a practical manner to functions for which m is less than zero, while for functions such that m is positive, the theory, while complete from a function theoretic point of view, does not lead, in general, to arithmetically useful results, since it leaves certain constants expressed in the form of definite integrals, the actual evaluation of which is quite impracticable.

Owing to this difficulty in Appell's theory when applied to functions of type (A), it was found necessary to use another method in obtaining the expansions of the functions under consideration. This method was first indicated by Liouville and has been utilized by C. Biehler[5] and G. Humbert[6] to derive similar expansions.

The members of the class of functions considered are exhibited as quotients of products of Jacobi theta functions, there being a larger number of theta factors in the numerator than in the denominator. The notation adopted is that of Jacobi,[7] except that his $\vartheta(z)$ is replaced by $\vartheta_0(z)$. In this notation the argument of the circular functions does not, as in some others, contain the factor π.

The following is a brief account of the method used. Let

$$F(z) = \vartheta_\alpha^{k_1}(z)\,\vartheta_\beta^{k_2}(z)\,\vartheta_\gamma^{k_3}(z)\,\vartheta_\delta^{k_4}(z) \equiv (\alpha, \beta, \gamma, \delta;\ k_1, k_2, k_3, k_4), \tag{3}$$

where α, β, γ, δ may take any of the values 0, 1, 2, 3 and where k_1, k_2, k_3, k_4 are positive or negative integers or zero such that $m = k_1 + k_2 + k_3 + k_4$ is positive. Associate with $F(z)$ the function $G(z)$ obtained from $F(z)$ by replacing z by $z + \frac{\pi\tau}{2}$ and neglecting the exponential multiplier which appears when this substitution is made. It follows from the properties of the theta functions that both $F(z)$ and $G(z)$ have real periods. The application of Fourier's theorem to these functions now leads to the following two cases:

CASE I. *All the k_i, $(i = 1, 2, 3, 4)$ are positive.*—In this case $F(z)$ and

$G(z)$ are integral functions and are representable by sine or cosine series, depending on their oddness or evenness. The coefficients in these series are then determined by means of the solutions of certain linear difference equations which arise from a comparison of the coefficients in $F(z)$ and $G(z)$ when the relation existing between these two associated functions is taken into account. The initial constants which appear in the solutions of the difference equations may be determined either by substituting special values of the argument into the series expansions, or else, by the multiplication of known expansions of less complicated functions whose product is either $F(z)$ or $G(z)$, and the comparison of particular coefficients in the product with the corresponding coefficients in the expansion of $F(z)$ or $G(z)$, as the case may be. For $m > 3$ this process becomes quite laborious and thereby is limited in its usefulness.

CASE II. *Some of the k_i are negative; $m > 0$.*—In this case it is possible for either $F(z)$ or $G(z)$ to remain finite in a strip of the z-plane bounded by lines parallel to the axis of reals and symmetric therewith; for concreteness let $F(z)$ remain finite. Then $G(z)$ will possess poles at points occurring at regular intervals along the real axis. It follows that $G(z)$ does not fulfil the hypotheses of Fourier's theorem and cannot be expanded. However, the function $G(z) - T(z)$ has a Fourier series expansion provided $T(z)$ is a trigonometric expression having the same period, parity and Laurent expansion in the neighborhood of the pole of smallest affix $\left(\text{either } z = 0 \text{ or } z = \frac{\pi}{2}\right)$, as $G(z)$. From here on the process is the same as in the preceding case; the constants appearing in the solution of the linear difference equations are determined as before. For this purpose, the lists of known expansions, such as Biehler's,[5] Humbert's,[6] Hermite's[8] and Bell's[9] are of assistance.

It should be noticed that for certain combinations of theta functions in $F(z)$, the process described is not applicable. This may best be seen from an example. Thus, let

$$F(z) = \frac{\vartheta_3^3(z)}{\vartheta_0(z)\,\vartheta_1(z)} \text{ and } G(z) = \frac{\vartheta_2^3(z)}{\vartheta_0(z)\,\vartheta_1(z)}; \qquad (4)$$

then

$$G\left(z + \frac{\pi\tau}{2}\right) = -e^{-iz}g^{-1/4}F(z). \qquad (5)$$

Now, in a strip bounded by lines parallel to the axis of reals and passing through the points $z = \pm \frac{\pi\tau}{2}$ the following hold

$$\left.\begin{aligned}\vartheta_0\vartheta_1^1\, F(z) &= \vartheta_3^3\, cscz + \sum_0^\infty C_n \sin\,(2n+1)z,\\ \vartheta_0\vartheta_1^1\, G(z) &= \vartheta_2^3 \cot z + \sum_1^\infty B_n \sin 2nz.\end{aligned}\right\} \qquad (6)$$

However, it is not possible to make use of (5) and (6) simultaneously in order to obtain the relations between the B_n and the C_n since the argument $z + \frac{\pi\tau}{2}$ which appears in (5) may lie outside the strip in which (6) is valid. This difficulty will always appear whenever the strips in which the expansions for $F(z)$ and $G(z)$ are valid coincide, as they do in the above example. However, in this and other similar cases it is still possible to obtain the expansions sought by making use of the expansions of certain other functions which do not offer the above difficulty, in conjunction with the well-known relations which exist between the squares of the theta functions. Thus, for $F(z)$ above we have

$$\vartheta_0^2\vartheta_3^2(z) = \vartheta_3^2\vartheta_0^2(z) - \vartheta_2^2\vartheta_1^2(z), \qquad (7)$$

so that

$$\vartheta_0^2\vartheta_3^3(z)/\vartheta_0(z)\vartheta_1(z) = \vartheta_3^2\vartheta_0(z)\vartheta_3(z)/\vartheta_1(z) - \vartheta_2^2\vartheta_1(z)\vartheta_3(z)/\vartheta_0(z). \qquad (8)$$

Hence,

$$\vartheta_0\vartheta_1^1F(z) = \vartheta_3^3\varphi(z) - \vartheta_2^3\psi(z), \qquad (9)$$

where

$$\varphi(z) = \vartheta_2\vartheta_0(z)\vartheta_3(z)/\vartheta_1(z) \quad \text{and} \quad \psi(z) = \vartheta_3\vartheta_1(z)\vartheta_3(z)/\vartheta_0(z).$$

The expansions for $\varphi(z)$ and $\psi(z)$ have been given by Hermite and may be found in Biehler's thesis (loc. cit.).

Using the methods indicated above, the writer has calculated some two hundred expansions which are believed to be new. In a paper of this sort it is clearly impossible to list them in full and hence only the general types will be indicated. To do this, the following notation is introduced. Let

$$\begin{aligned}F(z) &= \vartheta_\alpha^{k_1}(z)\vartheta_\beta^{k_2}(z)\vartheta_\gamma^{k_3}(z)\vartheta_\delta^{k_4}(z) \equiv (\alpha,\,\beta,\,\gamma,\,\delta;\; k_1,\,k_2,\,k_3,\,k_4)\\ &\equiv (k_1,\,k_2,\,k_3,\,k_4).\end{aligned}$$

Thus, the set of twenty-four functions of which $F(z) = \vartheta_1^3(z)\vartheta_3(z)/\vartheta_0(z)\vartheta_2^2(z)$ is one, is represented by the symbol $(3, 1, -1, -2)$.

The expansions of the functions represented by the following symbols have been calculated:

(2,0,0,0); (1,1,0,0); (1,1,1,0); (3,0,0,0); (2,1,0,0); (3,–2,0,0); (3,–1,–1,0); (2,1,–1,–1); (2,1,1,–3); (3,1,–3,0); (3,1,–1,–2); (4,–3,0,0); (2,2,–3,0); (2,2,–1,–2); (3,1,–1,–1); (3,1,–2,0); (2,2,–2,0).

The complete details of these expansions may be found in a dissertation by the writer which is deposited in the library of the California Institute of Technology.

[1] Presented to the American Mathematical Society, Oct. 27, 1928.

[2] In a California Institute dissertation, Mr. J. D. Elder has obtained expansions for the case where the functions have more poles than zeros.

[3] Hermite, *Comptes Rendus*, 1861, 1862; *Crelle*, **100**; *Oeuvres*, tome II, p. 109; tome IV, p. 223.

[4] Appell, *Ann. Sci. l'École Normale*, Sup., **1, 2, 3, 5**, series III, 1884 to 1888; *Acta Mathematica*, **42**, 1920.

[5] Biehler, C., *Thèse de Doctorat de la Faculté des Sciences de Paris;* Gauthier-Villars, 1879.

[6] Humbert, G., *J. d. Math. Pures et Appl.*, Series VI, **3**, 1907.

[7] Jacobi, *Werke*, Bd. I, p. 501.

[8] Hermite, *Oeuvres*, t. 2, p. 245.

[9] Bell, E. T., *Messenger Math.*, **53**, 1924; *Quarterly J.*, **49**, 1923.

AUTOMORPHISM COMMUTATORS

By G. A. Miller

Department of Mathematics, University of Illinois

Communicated June 21, 1929

Any automorphism of any group G may be obtained by making each operator of G correspond to itself multiplied by some operator of G. These multipliers will be called in what follows *automorphism commutators* of G, and unless the contrary is stated it will be assumed that all of them appear on the same side of the operators of G and relate to a single automorphism of G. They will be called right or left automorphism commutators as they appear on the right or on the left of the operators of G to which they relate. When G is abelian some of the fundamental properties of these automorphism commutators have been noted at various places and a necessary and sufficient condition which these commutators must satisfy has been formulated.[1] In particular, in this case they always constitute a subgroup of G and no commutator is the inverse of the operator to which it relates except when this operator is the identity. The latter condition must also obviously be satisfied when G is non-abelian, but for such a group the former condition is not necessarily satisfied as will be seen later.

All of the operators of G for which the automorphism commutators are