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[2] H. Poincaré, "Sur les déterminants d'ordre infini," *Bull. Soc. math. France*, 14, 1886.

[3] É. Borel, "Leçons sur les fonctions monogènes," Paris, 1917.

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ON THE UNIFIED FIELD THEORY. II

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Communicated November 6, 1930

In our previous note[1] we constructed a system of field equations for the combined gravitational and electromagnetic field which, on the basis of the interpretation of the quantities h_α^i as electromagnetic potentials, led to Maxwell's equations in their exact form at the origin of the local coördinates. To secure the *exact* form of Maxwell's equations in the local coordinate system was, in fact, the principal motive for the introduction of this system of field equations. Actually, however, a system of field equations constructed with primary regard to a law of conservation appears to be of deeper physical significance. This latter point of view is made the basis for the construction of a system of field equations in the present note— and the equations so obtained differ from those of Note I only by the appearance of terms quadratic in the quantities $h_{j,k}^i$. It would thus appear that we can carry over the interpretation of the h_α^i as electromagnetic potentials; doing this, we can say that Maxwell's equations hold approximately in the local coördinate system in the presence of weak electromagnetic fields.

Equations of Note I will be referred to by prefixing the numeral I before the number of the equation in question.

1. In the Einstein theory of gravitation the operation of forming the divergence of a tensor constituted an almost unavoidable generalization from the analogous operation of pre-relativity physics. When dealing with quantities of the nature of absolute invariants, however, as we shall do in the development of the present theory, it is by no means obvious how this operation can best be defined. However, actual investigation of the identities of the field theory points to a definition of the operation of divergence which is well adapted to a statement of the laws of conservation; the reasons for this particular definition will be apparent from the identities of the following sections.

DIVERGENCE RULE: *The formula*

$$\nabla_m T^i_{j\ldots m} = \sum_{m=1}^{4} e_m \, [T^i_{j\ldots m,m} + 2h^i_{m,r} \, T^r_{j\ldots m}]$$

defines the divergence of the invariant $T^i_{j\ldots m}$ *with respect to the index* m. Any covariant index, i.e., covariant under transformations of the fundamental vectors, can of course appear in place of the particular index m in the definition of the divergence operation. It should be observed that the term involving the absolute electromagnetic forces $2 \, h^i_{j,k}$ in the above formula, corresponds to those which contained the components of affine connection in the divergence formula of the earlier theory of relativity. A generalization of the above formula to include invariants T with any number of contravariant indices could obviously be made but this is not necessary for the requirements of the present note. On the basis of the divergence rule we obtain the field equations from the following

POSTULATE OF THE UNIFIED FIELD. *The divergence of the absolute electromagnetic forces is equal to zero, i.e.,*

$$\boxed{\nabla_k \, h^i_{j,k} = 0} \,. \tag{1.1}$$

There are 16 equations in the system (1.1) for the determination of the 16 electromagnetic potentials h^i_α. By reference to the field equations in what follows we shall mean equations (1.1) rather than the corresponding equations of our previous note.

2. Before proceeding further we must derive certain special identities which we shall need in the following work; this will be done in as concise a manner as possible. We have

$$h^i_{j,k,l} = \frac{\partial h^i_{j,k}}{\partial x^\alpha} \, h^\alpha_l. \tag{2.1}$$

Hence

$$h^i_{j,k,l} = \tfrac{1}{2} \, [h^i_{j,kl} - k^i_{k,jl}] + h^i_{m,j}h^m_{k,l} + h^i_{m,k}h^m_{l,j} \tag{2.2}$$

when use in made I (4.5) and I (4.10). Interchanging k,l in (2.2) and adding, we obtain a set of identities which can be reduced to the form

$$h^i_{j,kl} = \tfrac{2}{3}[h^i_{j,k,l} + h^i_{j,l,k} + h^i_{m,k} h^m_{j,l} + h^i_{m,l}h^m_{j,k}]. \tag{2.3}$$

We observe that these latter identities constitute an inverse form of the identities (2.2) in that (2.2) expresses the $h^i_{j,k,l}$ in terms of the $h^i_{j,kl}$ while (2.3) gives $h^i_{j,kl}$ in terms of the $h^i_{j,k,l}$ invariants.

The invariants $h^i_{j,k,l}$ satisfy a set of identities

$$h^i_{j,k,l} + h^i_{k,l,j} + h^i_{l,j,k} = 2 [h^i_{m,j} h^m_{k,l} + h^i_{m,k} h^m_{l,j} + h^i_{m,l} h^m_{j,k}] \qquad (2.4)$$

which is easily deduced from (2.2).

Let us denote the quantities $\Lambda^\alpha_{\beta\gamma\delta}$ by $\lambda^i_{j,k,l}$ with reference to a system of local coördinates and define the invariant $\mathbf{A}^i_{jklm}$ by the formula

$$\mathbf{A}^i_{jklm} = \left(\frac{\partial \lambda^i_{jkl}}{\partial z^m}\right)_{z\,=\,0}.$$

(See Sect. 2 in Note I.) Now transform equations I (4.6) to local coördinates, differentiate, and evaluate at the origin of the local system; we thus obtain a set of identities which can be written

$$h^i_{j,kl,m} = h^i_{j,klm} - \mathbf{A}^i_{jklm} + \tfrac{1}{2} h^i_{j,t} h^t_{m,kl} + \tfrac{1}{2} h^i_{t,k} h^t_{m,jl} + \tfrac{1}{2} h^i_{t,l} h^t_{m,jk}$$
$$+ h^t_{m,j} h^i_{t,kl} + h^t_{m,k} h^i_{j,tl} + h^t_{m,l} h^i_{j,kt}. \qquad (2.5)$$

Interchanging the indices j,k in (2.5) and subtracting the resulting identities from (2.5) we obtain

$$h^i_{j,kl,m} - h^i_{k,jl,m} = h^i_{j,klm} - h^i_{k,jlm} + h^i_{j,t} h^t_{m,kl} + h^i_{t,k} h^t_{m,jl} + [h^t_{m,j}$$
$$(h^i_{t,kl} - h^i_{k,tl}) + h^t_{m,k} (h^i_{j,tl} - h^i_{t,jl}) + h^t_{m,l} (h^i_{j,kt} - h^i_{k,jt})]. \qquad (2.6)$$

To the identities (2.6) we now add those two sets of identities which result from (2.6) by permutating the indices k,l,m cyclically. By a suitable arrangement of terms the set of identities so obtained can be given the form

$$8h^i_{j,klm} = 3 [h^i_{j,kl,m} + h^i_{j,lm,k} + h^i_{j,mk,l} + h^t_{j,m} h^i_{t,kl} + h^t_{j,l} h^i_{t,mk} +$$
$$h^t_{j,k} h^i_{t,lm}] + h^i_{t,k} h^t_{j,lm} + h^i_{t,l} h^t_{j,km} + h^i_{t,m} h^t_{j,kl}. \qquad (2.7)$$

These identities express the invariants $h^i_{j,klm}$ in terms of the invariants $h^i_{j,kl,m}$ plus invariants of lower order in the derivatives of the electromagnetic potentials.

To obtain the inverse form of the set of identities (2.7) we deduce the formula

$$\mathbf{A}^i_{jklm} = -\tfrac{1}{3} h^i_{m,jkl} + \tfrac{1}{6} (h^i_{t,j} h^t_{m,kl} + h^i_{t,k} h^t_{m,lj} + h^i_{t,l} h^t_{m,jk})$$

which we use to eliminate the invariants $\mathbf{A}^i_{jklm}$ from (2.5). The result of this elimination is a set of identities which can be put into the form

$$3h^i_{j,kl,m} = 3h^i_{j,klm} + h^i_{m,jkl} + 2 h^i_{j,t} h^t_{m,kl} + h^i_{t,k} h^t_{m,jl} + h^i_{t,l} h^t_{m,jk}$$
$$+ 3 (h^t_{m,j} h^i_{t,kl} + h^t_{m,k} h^i_{j,tl} + h^t_{m,l} h^i_{j,kt}). \qquad (2.8)$$

Finally the set of identities

$$5h^i_{j,kl,m} = 3\,(h^i_{j,lm,k} + h^i_{j,mk,l}) + h^i_{m,kl,j} + h^i_{m,lj,k} + h^i_{m,jk,l} + h^i_{t,l}$$

$$(3\,h^t_{m,jk} + h^i_{j,km}) + h^i_{t,k}\,(3\,h^t_{m,jl} + h^t_{j,lm}) \qquad (2.9)$$

$$+\,5\,h^i_{j,t}\,h^t_{m,kl} + h^i_{t,m}\,h^t_{j,kl} + 6\,h^t_{m,j}\,h^i_{t,kl} + h^t_{m,k}\,(8\,h^i_{j,tl} + h^i_{t,jl})$$

$$+\,h^t_{m,l}\,(8\,h^i_{j,kt} + h^i_{t,jk}) + 3\,(h^t_{j,l}\,h^i_{t,mk} + h^t_{j,k}\,h^i_{t,lm})$$

can be obtained by eliminating the invariants $h^i_{j,klm}$ between the identities (2.7) and (2.8).

3.   The identities of Sect. 2 can be used to deduce identities of especial interest for the field theory.   By use of (2.2) it readily follows that

$$\nabla_k\,h^i_{j,k} = \tfrac{3}{4}\sum_{k=1}^{4} e_k\,(h^i_{j,kk} + 4\,h^i_{r,k}\,h^r_{k,j}) \qquad (3.1)$$

identically.   Hence

$$\boxed{\sum_{k=1}^{4} e_k\,(h^i_{j,kk} + 4\,h^i_{r,k}\,h^r_{k,j}) = 0} \qquad (3.2)$$

constitutes a system of equations completely equivalent to the field equations (1.1).

Now put $l = k$ and $m = j$ in (2.9), then multiply these equations through by $e_j e_k$ and sum on the two repeated indices.   This gives

$$\sum_{j=1}^{4}\sum_{k=1}^{4} e_j\,e_k\,(h^i_{j,kk,j} + h^i_{r,j}\,h^r_{j,kk} + 2\,h^r_{k,j}\,h^i_{j,kr}) = 0 \qquad (3.3)$$

identically.   We next consider the set of identities

$$2\sum_{j=1}^{4}\sum_{k=1}^{4} e_j\,e_k\,h^r_{k,j}\,h^i_{j,kr}$$

$$= 3\sum_{j=1}^{4}\sum_{k=1}^{4} e_j\,e_k\,h^i_{j,m}\,(h^m_{j,kk} + \tfrac{8}{3}\,h^m_{r,k}\,h^r_{k,j}) + 4\sum_{j=1}^{4}\sum_{k=1}^{4} e_j\,e_k\,(h^i_{r,k}\,h^r_{k,j})_{,j}$$

which we use to eliminate the last set of terms from (3.3).   As a result we obtain the set of four identities

$$\sum_{j=1}^{4} e_j\left\{\left[\sum_{k=1}^{4} e_k\,(h^i_{j,kk} + 4\,h^i_{r,k}\,h^r_{k,j})\right]_{,j} + 2\,h^i_{j,m}\right.$$

$$\left.\left[\sum_{k=1}^{4} e_k\,(h^m_{j,kk} + 4\,h^m_{r,k}\,h^r_{k,j})\right]\right\} = 0.$$

By (3.1) this last set of identities can be given the form

$$\boxed{\nabla_j\,\nabla_k\,h^i_{j,k} = 0}\,, \qquad (3.4)$$

i.e., *the divergence of the left member of* (1.1) *vanishes identically.* The identities (3.4) are the mathematical counterpart of the laws of conservation of the physical world.[2]

4. If we fail to take into account the conditions imposed by the field equations on the structure of space, then the number of independent invariants $h^i_{j,kl_1\ldots l_r}$ of order $r + 1$ in the derivatives of the electromagnetic potentials h^i_α, i.e., the number of arbitrary values $(h^i_{j,kl_1\ldots l_r})_Q$ which these invariants can assume at a point Q, is

$$16\,K(4,\,r+1) - 4K(4,\,r+2),$$

where $K(p,q)$ denotes the number of combinations with repetitions of p things taken q at a time; this is an immediate consequence of the fact that I (4.7) and I (4.9) constitute a complete set of identities. We shall denote the above number by $N(r + 1)$ and observe that we can write

$$N(r + 1) = 12K(4,r) + 8K(3,r) + 4K(2,r)$$

It can easily be shown that the number of independent quantities $(h^i_{j,k,l_1\ldots l_r})_Q$ is likewise given by $N(r + 1)$. To do this we transform the expression for $h^i_{j,k}$ given by 1 (4.5), to a system of local coördinates, differentiate repeatedly, and evaluate at the origin of the system. We thereby obtain a system of equations of the form

$$h^i_{j,k,l_1\ldots l_r} = \tfrac{1}{2}\,[h^i_{j,kl_1\ldots l_r} - h^i_{k,jl_1\ldots l_r}] + \bigstar, \qquad (4.1)$$

where the ★ is used to denote terms of lower order than those which have been written down explicitly. Now let P denote the operations of holding j fixed, permuting the indices $kl_1\ldots l_r$ cyclically, and adding the resulting terms. Then

$$2\,P(h^i_{j,k,l_1\ldots l_r}) = P\,(h^i_{j,kl_1\ldots l_r}) - P\,(h^i_{k,jl_1\ldots l_r}) + \bigstar$$

from (4.1) or

$$(r + 2)\,h^i_{j,kl_1\ldots l_r} = 2\,\mathrm{P}\,(h^i_{j,k,l_1\ldots l_r}) + \bigstar. \qquad (4.2)$$

In view of (4.1) and (4.2) it follows that if we consider the quantities

$$h^i_\alpha;\quad h^i_{j,k};\quad \ldots;\quad h^i_{j,kl_1\ldots l_{r-1}}$$

to have fixed values at Q, the number of independent quantities $(h^{ki}_{j,k,l_1\ldots l_r})_Q$ is equal to the number of independent quantities $(^i_{j,kl_1\ldots l_r})_Q$, i.e., the number $N(r + 1)$.

By making use of the above result it can be shown that $N(r + 1)$ gives the number of arbitrary partial derivatives of the rth order of the quantities $h^i_{j,k}$ at the point Q. The reader can easily work out the details of this demonstration.

5. When account is taken of the conditions imposed by the field equa-

tions, the number of arbitrary partial derivatives of the quantities $h^i_{j,k}$ at the point Q is decreased. A *lower bound* to this number can, however, be deduced in the following manner. Taking the absolute derivative of the left member of (1.1) we obtain

$$(\nabla_k h^i_{j,k})_{,l} = 0. \tag{5.1}$$

By (3.4) four of the equations of the set (5.1) are linearly dependent on the remaining equations (5.1) and the field equations (1.1). In fact,

$$(\nabla_k h^i_{1,k})_{,1} = \ldots,$$

where the dots have been used to denote a linear expression in certain of the left members of (5.1) plus a linear expression in the quantities in the left members of (1.1) with coefficients equal to the absolute electromagnetic forces. Assuming that the $h^i_{j,k}$ and all their partial derivatives to those of order r (≥ 1) inclusive have fixed values at the point Q, it follows that the number of partial derivatives of the quantities $h^i_{j,k}$ of order ($r + 1$) at Q, whose values are determined as a consequence of equations (1.1), is *at most* equal to

$$16K(4,r) - 4K(4,r - 1). \tag{5.2}$$

Hence the arbitrary derivatives of the $h^i_{j,k}$ of order $r + 1$ at Q cannot be less than the difference of $N(r + 2)$ and (5.2), i.e.,

$$16K(3,r + 1) + 8K(2,r + 1);$$

for $r = 0$ this expression likewise gives the number of arbitrary derivatives of the first order of the $h^i_{j,k}$. In our next note we shall apply the above lower bound to the problem of constructing the general existence theorem for the field equations.

[1] These PROCEEDINGS, **16**, 761–776 (1930).

[2] If we replace (1.1) by

$$\nabla_k h^i_{j,k} = D^i_j,$$

where D^i_j is the analogue of the vector of charge and current density of the classical theory, then (3.4) yields

$$\nabla_j D^i_j = 0.$$

These latter equations are the analogue of the equation of the classical theory which shows that electric charge is conserved. We have purposely failed to introduce the invariants D^i_j of unknown structure into our field theory as it is our wish to investigate the extent to which the physical world can be described by equations of the type (1.1).