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ANALYTICAL GEOMETRY

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AN ELEMENTARY TREATISE ON ANALYTICAL GEOMETRY

WITH NUMEROUS EXAMPLES

BY

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PREFACE

This book is to some extent an amplification of lecture notes; my special aim has been an easy and gradual development of the principles of the subject. An account of the elementary properties of the Ellipse, Parabola, and Hyperbola precedes the discussion of the general equation of the second degree.

I have had in view the requirements of two classes of readers.

First, a limited course of the subject is read by most students in the University Colleges of this country. Such a course, confined to selected paragraphs, with the exercises appended thereto, is indicated at the end of the table of contents.

Secondly, as chapters on Trilinears, Reciprocal Polars, and Projection are included, it is hoped that the book will be useful to candidates for mathematical honours; and especially as an introduction to the writings of Dr Salmon.

It seems superfluous to state that I am under great obligations to the works of Dr Salmon; I would also make grateful mention of Dr C. Taylor's *Conics*, and the tract on *Conics and Curves* by the Rev. W. H. Laverty.

The Exercises at the end of each chapter have been mainly selected from the examination papers of different

Universities. I have freely availed myself of Mr Miller's kind permission to extract problems from the *Educational Times.* Many problems have been manufactured to illustrate special points; and for several of the most interesting I am indebted to Professors Curtis, S.J., Genese, and Purser.

In endeavouring to simplify the treatment of the subject, I found that some novelties were unavoidable. Thus the determinant form of the equation of the join of two points is used throughout, the elements of the determinant notation being explained in § 20; my experience with beginners justifies this innovation. The generality of the usual trapezium proof of the formula for the area of a triangle in terms of the co-ordinates of its vertices is invalid without a more detailed examination than is usual of the consequences of the convention of signs. A more rigorous proof has therefore been given.

I have included the usual methods of tracing a conic whose Cartesian equation is given; but further methods, which seem to possess *some* advantages, are given in §§ 294, 296.

Professor Purser's investigation of the equation of a diameter (§§ 228, 255, 303) is the simplest that I know of.

The use of the invariants θ, θ' (§§ 443, 444) has not been emphasised in any previous textbook; this method is due to Professor W. S. Burnside. Further applications are given in his paper in vol. 8 of the *Quarterly Journal.*

I have ventured to indicate (§§ 479, 480) an elementary method of projecting metrical properties, which was suggested to me by a paper of Clifford's on Analytical Metrics. The formula of § 481 had previously been deduced from other considerations by Professor Curtis. I have merely suggested my own method; further development is easy. As an instance of its use I may mention that the equation of the conic-tangent to three conics, all the conics passing through two given points, is thereby at once deducible from Dr Casey's equation of the circle-tangent to three circles.

I have further to acknowledge my great indebtedness to my former mathematical master, Mr R. C. J. Nixon, from whom I have received many valuable suggestions, and to whose advice as to the arrangement of the book much of its value is due. My best thanks are due to my colleague Professor Genese for many important hints; and to another colleague, Mr A. W. Warrington, whose generous assistance has been of the utmost value to me in preparing the work for the Press.

W. J. JOHNSTON.

UNIVERSITY COLLEGE OF WALES, ABERYSTWYTH.
April, 1893.

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The following limited course is recommended to beginners.

I. The Straight Line and Circle in Rectangular and Polar Coordinates :—

II. The Following Additional Portions of Chapters I–VII should be read before reading the Chapters on Conics :—

III. The Central and Focal Properties of Conics ; Poles and Polars, etc. :—

ANALYTICAL GEOMETRY

CHAPTER I. CO-ORDINATES

POSITION DETERMINED BY CO-ORDINATES

§ 1. ALGEBRA may be applied to deduce relations between geometrical magnitudes.

For example, combining the statements

(1) The area of a rectangle is measured by the product of the lengths of its sides,

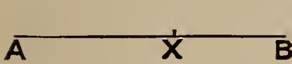
and (2) $$(a + b)(a - b) \equiv a^2 - b^2,$$

we deduce the theorem:

The rectangle contained by the sum and difference of two lines is equal to the difference of the squares on those lines.

As another example let us solve this problem:

Divide a given line AB *into two parts such that the square on one may be twice the square on the other.*



Suppose that X is the point required.

Put a for AB, and x for AX.

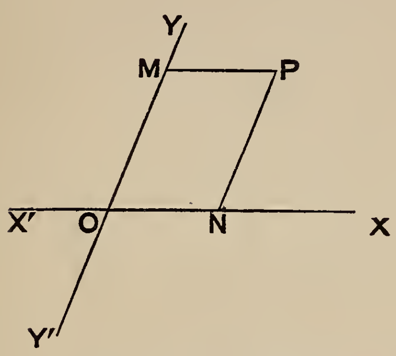
$$\text{Then} \quad x^2 = 2(a - x)^2$$

$$\therefore \quad x = \sqrt{2}(a - x)$$

$$\therefore \quad x = \frac{a\sqrt{2}}{1 + \sqrt{2}} = a(2 - \sqrt{2})$$

§ **2.** In other geometrical inquiries we have to consider position. Thus the locus of a point which moves subject to an assigned condition may be required. It is not at once obvious how Algebra may be applied to such questions. The method of **co-ordinates** is an adequate one for the purpose. This method was first introduced by Descartes in his *Geometrie*, 1637; we shall now give an account of it.

§ **3.** In a plane take two fixed lines (or *axes*), X′OX, Y′OY. Then through any point P in the plane drawing PN parallel to YOY′ to meet X′OX in N; to every position of P correspond definite lengths of ON, PN.



Conversely, if the lengths of ON, PN are given, the point P is determined.

We have merely to measure along OX the given distance ON, and then the other distance NP on a line through N parallel to OY.

Or thus, if we are given ON = a and PN = b; measure a distance ON = a along OX, and a distance OM = b along OY: parallels through N, M to the axes intersect in the point required.

§ **4.** The two fixed lines OX and OY are called the *axes of co-ordinates* or simply the *axes*; their point of intersection O is called the *origin*. OX is called the *axis of* x and OY the *axis of* y.

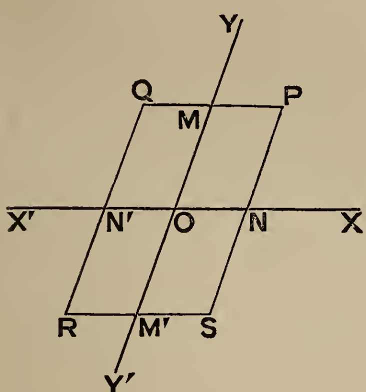
If the axes are at right angles they are said to be *rectangular*, otherwise they are *oblique*.

ON and NP are called the *co-ordinates* of P; ON is called its *abscissa* or its 'x,' and NP its *ordinate* or its 'y.'

The point whose co-ordinates are a and b may be referred to as 'the point ($x = a$, $y = b$)'; or briefly, 'the point (a, b)': the abscissa being always named first.

§ **5.** The straight lines XOX' and YOY' divide the plane into four compartments. The following convention serves to distinguish points in the several compartments.

Distances measured along OX are considered positive and along OX' negative; distances measured along OY are considered positive and along OY' negative.



Accordingly, if along OX we measure

$$ON = ON' = a$$

and $\quad OM = OM' = b,$

and through N, N', M, M' draw $\|^s$ to the axes meeting in P, Q, R, S: then

$$\text{the co-ord's of } P \text{ are } \left.\begin{matrix} x = +a \\ y = +b \end{matrix}\right\}$$

$$\text{,, \quad ,, \quad } Q \text{ ,, } \left.\begin{matrix} x = -a \\ y = +b \end{matrix}\right\}$$

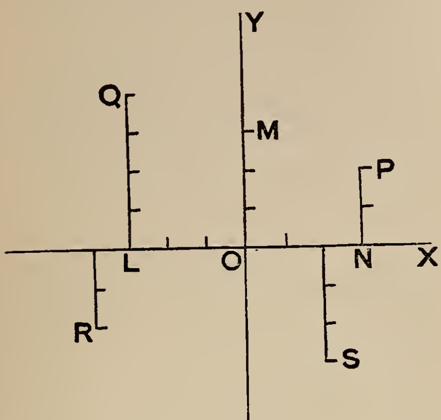
$$\text{,, \quad ,, \quad } R \text{ ,, } \left.\begin{matrix} x = -a \\ y = -b \end{matrix}\right\}$$

$$\text{,, \quad ,, \quad } S \text{ ,, } \left.\begin{matrix} x = +a \\ y = -b \end{matrix}\right\}.$$

Thus, as in Trigonometry, the abscissa of a point is positive or negative according as the point is on the right or the left of $Y'OY$, and its ordinate is positive or negative according as the point is above or below $X'OX$. We shall presently make some further remarks on the use of the signs $+$ and $-$ to indicate opposite directions of measurement along any line.

§ **6.** To illustrate this,

Suppose the axes rectangular; and let us 'plot' the points $(3, 2)$, $(-3, 4)$, $(-4, -2)$, $(2, -3)$.



Take any convenient unit of length.

Then the point $(3, 2)$ is P
" " $(-3, 4)$ " Q
" " $(-4, -2)$ " R
" " $(2, -3)$ " S

What are the co-ord's of N in the figure? *Ans.* $(3, 0)$; for to reach N from O we measure 3 units to the right so that the abscissa of N is 3: while the distance to be measured upwards or downwards is zero, so that its ordinate is 0.

What are the co-ord's of L? *Ans.* $(-3, 0)$.
What are the co-ord's of M? *Ans.* $(0, 3)$.
What are the co-ord's of the origin O? *Ans.* $(0, 0)$.

THE CONVENTION OF SIGNS

§ **7.** If distances measured in either direction along a straight line be considered positive, distances measured in the opposite direction are considered negative.

If A and B are two points on a straight line, then in Modern Geometry AB not only stands for the distance between A and B, but also implies that this distance is measured in the direction from A to B.

If then $AB = +5$, we have $BA = -5$.

Generally $AB = -BA$.

It follows that if O, A, B are three points anyhow placed on a straight line, then

$$AB = OB - OA. \qquad (1)$$

For a point may travel from O to B by two steps: first from O to A, then from A to B.

$$\text{Thus} \quad OA + AB = OB.$$

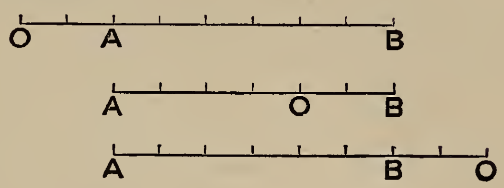
Subtract OA from both sides, $\therefore\ AB = OB - OA$ . . (2)

Cor'—Let O be the origin of co-ord's and A, B two points on the axis of x.

Let $OA = x_1$ and $OB = x_2$ so that the co-ord's of the points A, B are $(x_1, 0)$, $(x_2, 0)$.

$$\text{Then} \quad AB = x_2 - x_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

§ **8.** To illustrate this



In the first figure $AB = 6$, $OA = 2$, $OB = 8$; $6 = 8 - 2$

" second " $AB = 6$, $OA = -4$, $OB = 2$; $6 = 2 - (-4)$

" third " $AB = 6$, $OA = -8$, $OB = -2$; $6 = (-2) - (-8)$

Thus in each case the equation $AB = OB - OA$ is verified.

§ **9.** Ex. 1. If A, B, C are three points in a straight line

$$AB + BC + CA = 0.$$

For if a point travels successively from A to B, from B to C, and from C to A, the total distance traversed is zero.

Or thus: Let a, b, c be the distances of A, B, C from a fixed point O on the line; then $AB = b - a$, &c.

$$\therefore\ AB + BC + CA = (b - a) + (c - b) + (a - c) = 0.$$

Ex. 2. Let the distances of four points A, B, C, D anyhow situated on a straight line from a fixed point O on the line be a, b, c, d; then the identity

$$(d - a)(b - c) + (d - b)(c - a) + (d - c)(a - b) \equiv 0$$

$$\text{gives} \quad AD \cdot CB + BD \cdot AC + CD \cdot BA = 0.$$

Suppose now that the four points are in the order A, B, C, D.

Then since

$$CB = -BC,\ BA = -AB,$$

we obtain

$$BD \cdot AC = AD \cdot BC + CD \cdot AB,$$

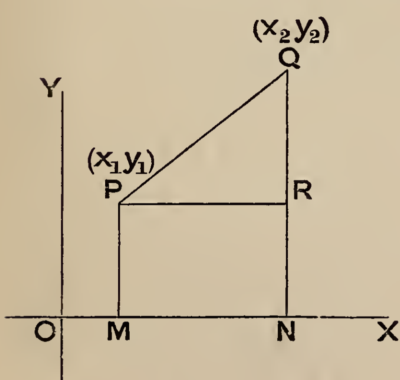
where all the lines and rectangles are positive*.

DISTANCE BETWEEN TWO POINTS

§ **10.** *To find the distance between two points whose co-ordinates are given.*

Suppose that the axes are rectangular.

Let the two points be P, $(x_1 y_1)$, and Q, $(x_2 y_2)$.



Draw PM, QN parallel to OY, to meet OX in M, N and PR parallel to OX to meet QN in R.

Then $PQ^2 = PR^2 + QR^2$.

But

$$PR = MN = ON - OM$$
$$= x_2 - x_1$$

and

$$QR = QN - PM = y_2 - y_1$$

$$\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

§ **11.** We may express this result in words:

The square of the distance between two points is equal to the square of the difference of their abscissae together with the square of the difference of their ordinates.

The order in which we take the differences is indifferent, as they are afterwards squared.

---

* A graphical proof of this theorem is given in Nixon's *Euclid Revised*: p. 104.

§ 12. The proof in the last § is perfectly general, in virtue of equation (2) of § 7. As an Exercise the learner may go through the proof for the case where P is in the first compartment and Q in the third.

§ 13. Ex. 1. Find the distance δ between the points $(2, -3)$ and $(-3, 4)$.

$$\begin{aligned}\delta^2 &= [2 - (-3)]^2 + [-3 - 4]^2 \\ &= 5^2 + (-7)^2 \\ &= 25 + 49 \\ &= 74 \\ \therefore\ \delta &= \sqrt{74}\end{aligned}$$

The learner should illustrate this by a diagram.

Ex. 2. Find the distance δ between $(x_1 y_1)$ and the origin.

$$\delta^2 = (x_1 - 0)^2 + (y_1 - 0)^2 = x_1^2 + y_1^2.$$

This is also obvious from the figure, § 10; $OP^2 = OM^2 + PM^2 = x_1^2 + y_1^2$.

Ex. 3. Find the co-ord's of a point equidistant from the three points

$$(2, -2), \quad (8, -2), \quad (8, 6).$$

Let its co-ord's be x, y.

$$\therefore\ (x - 2)^2 + (y + 2)^2 = (x - 8)^2 + (y + 2)^2 = (x - 8)^2 + (y - 6)^2.$$

Cancel $x^2 + y^2$

$$\therefore\ -4x + 4y + 8 = -16x + 4y + 68 = -16x - 12y + 100$$

Solving these we get $x = 5$, $y = 2$.

Ex. 4. Find the condition that the point (x, y) may be at a distance 3 from the point $(4, 5)$.

$$3^2 = 9 = (x - 4)^2 + (y - 5)^2$$

$$\therefore\ x^2 + y^2 - 8x - 10y + 32 = 0.$$

This equation then is satisfied by the co-ordinates of every point on a circle whose centre is $(4, 5)$ and radius 3.

Ex. 5. Express that a point $P\ (x, y)$ may be equidistant from $A\ (1, 2)$ and $B\ (3, 4)$.

$$PA^2 = (x - 1)^2 + (y - 2)^2$$

$$PB^2 = (x - 3)^2 + (y - 4)^2.$$

Equating these values and simplifying we get $x + y - 5 = 0$.

We know from Elementary Geometry that P must lie on the line which bisects AB at right angles: hence the co-ordinates of every point on this line satisfy the equation $x + y - 5 = 0$.

§ **14.** Suppose that in § 10 the axes are oblique, the angle between them being ω.

Then $\widehat{PRQ} = \widehat{ONQ} = 180^\circ - \omega$

$$\therefore \quad PQ^2 = PR^2 + QR^2 - 2\,PR\,.\,QR \cos PRQ$$
$$= PR^2 + QR^2 + 2\,PR\,.\,QR \cos \omega;$$
$$\therefore \quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$+ 2\,(x_2 - x_1)\,(y_2 - y_1) \cos \omega.$$

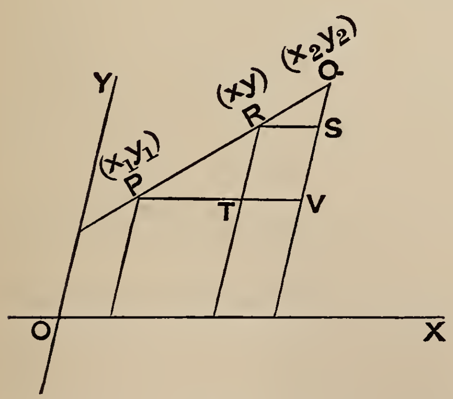
Cor'—The distance of (x, y) from the origin is

$$\sqrt{x^2 + y^2 + 2\,xy \cos \omega}.$$

Note—Oblique axes are rarely used as the formulae are more complicated; and in future we shall usually assume that the axes are rectangular.

DIVISION OF JOIN OF TWO POINTS IN A GIVEN RATIO

§ **15.** *To find the co-ordinates of the point which divides the join of* $(x_1 y_1)$ *and* $(x_2 y_2)$ *in a given ratio* $m : n$.



Let (x, y) be the co-ord's of the required point R, so that

$$PR : RQ = m : n.$$

Draw ||s to the axes through R, P, Q as in the figure.

$$\text{Then} \quad PT : TV = PR : RQ = m : n,$$
$$\text{or} \quad x - x_1 : x_2 - x = m : n$$
$$\therefore \quad n\,(x - x_1) = m\,(x_2 - x)$$

$$\therefore \quad x = \frac{mx_2 + nx_1}{m + n}$$

$$\text{Similarly} \quad y = \frac{my_2 + ny_1}{m + n}$$

These formulae are applicable whether the axes are rectangular or oblique.

Cor′—The co-ord′s of the mid point of PQ are

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

§ **16.** If PQ is to be divided externally in R so that

PR : QR = m : n; then since QR = − RQ we have

PR : RQ = m : − n.

Thus by changing n into − n in the formulae of § 15 we obtain the required co-ord′s

$$x = \frac{mx_2 - nx_1}{m - n}, \quad y = \frac{my_2 - ny_1}{m - n}.$$

§ **17.** Ex. 1. The co-ord′s of A are (3, − 1) and of B are (6, 4); find the co-ord′s of the point of trisection of AB nearest A.

Here $\quad m = 1, \quad n = 2,$

$$\therefore \quad x = \frac{x_2 + 2x_1}{3} = \frac{6 + 2(3)}{3} = 4$$

$$y = \frac{y_2 + 2y_1}{3} = \frac{4 + 2(-1)}{3} = \frac{2}{3}$$

Ex. 2. A, B being the same points as in Ex. 1, AB is produced to C so that BC = AB; find the co-ord′s of C.

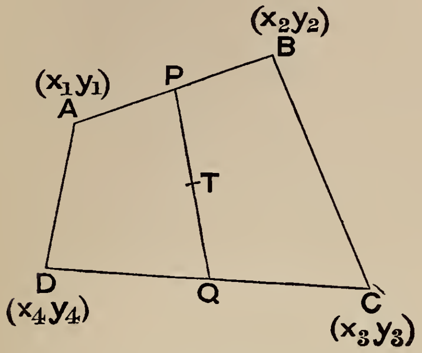
Here $\quad AC = 2\,BC = -2\,CB, \quad AC : CB = 2 : -1,$

$\therefore \quad m = 2, \quad n = -1;$ and

$$x = \frac{2x_2 - x_1}{2 - 1} = 2x_2 - x_1 = 2(6) - 3 = 9$$

$$y = \frac{2y_2 - y_1}{2 - 1} = 2y_2 - y_1 = 2(4) + 1 = 9$$

Ex. 3. Prove that the joins of the mid points of the opposite sides of a quadrilateral and the join of the mid points of its diagonals meet in one point and bisect each other.



Let ABCD be the quad′, $(x_1\, y_1)$ the co-ord′s of A, &c.

Then the co-ord′s of P the mid point of AB are

$$\tfrac{1}{2}(x_1 + x_2), \quad \tfrac{1}{2}(y_1 + y_2);$$

of Q the mid point of CD are

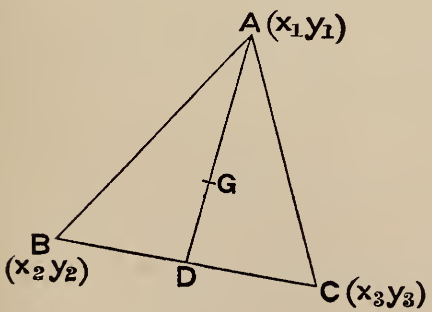
$$\tfrac{1}{2}(x_3 + x_4), \quad \tfrac{1}{2}(y_3 + y_4);$$

of T the mid point of PQ are

$$\tfrac{1}{2}\left\{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}\right\} = \tfrac{1}{4}(x_1 + x_2 + x_3 + x_4),$$

and $\tfrac{1}{4}(y_1 + y_2 + y_3 + y_4)$.

We shall get the same values for the co-ord′s of the mid points of the joins of AD, BC and of AC, BD.



Ex. 4. A, B, C are three points whose co-ord′s are given, BC is bisected in D and on AD is taken DG $= \tfrac{1}{3}$ DA. Find the co-ord′s of G.

The abscissa of D is

$$\tfrac{1}{2}(x_2 + x_3);$$

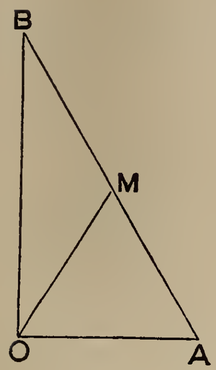
∴ the abscissa of G is

$$\frac{2 \times \tfrac{1}{2}(x_2 + x_3) + x_1}{2 + 1} = \frac{x_1 + x_2 + x_3}{3}.$$

Similarly its ordinate is $\dfrac{y_1 + y_2 + y_3}{3}$.

We see then that the three medians of a triangle meet in one point, this point being the point of trisection of each furthest from the vertex. This point G corresponds to the centre of gravity of a physical triangular lamina.

Ex. 5. The angle O of a triangle OBA is a right angle, M is the mid point of AB : prove



$$OM = \tfrac{1}{2} AB.$$

Take OA for axis of x and OB for axis of y.

Then the co-ord's of A are (a, o) and of B (o, b),

$$\therefore \text{ of M are } \frac{a}{2} \text{ and } \frac{b}{2};$$

$$\therefore\ OM = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \tfrac{1}{2}\sqrt{a^2 + b^2} = \tfrac{1}{2} AB.$$

Exercises

1. Find the length of the join of (1, − 3) and (0, −2). *Ans.* $\sqrt{2}$.

2. The vertices of a triangle are (− 1, − 1), (2, 3) and (0, 2). Plot these points and find the lengths of the sides of the triangle. *Ans.* 5, $\sqrt{5}$, $\sqrt{10}$.

3. Show that the points (1, 2), (1, 6), ($\sqrt{12} + 1$, 4) are the vertices of an equilateral triangle. *Ans.* Side = 4.

4. Find the co-ord's of the mid points of the sides of the triangle in Ex. 2. *Ans.* ($\frac{1}{2}$, 1), (1, $2\frac{1}{2}$), ($-\frac{1}{2}$, $\frac{1}{2}$).

5. Find the co-ord's of the centre of its circum circle. *Ans.* ($2\frac{1}{2}$, $-\frac{1}{2}$).

6. Find the point of trisection nearest (1, 2) of the join of (1, 2) and (7, − 13). *Ans.* (3, −3).

7. The co-ord's of P are (4, 1) and of Q (9, 8): find the co-ord's of a point R in PQ produced such that

(1) QR = PQ, (2) QR = 2 PQ.

Ans. (14, 15), (19, 22).

8. Show that (1, 2) lies on the join of (− 4, 5) and (11, − 4). In what ratio does it divide this join? *Ans.* 1 : 2.

9. Show that the following four points are the vertices of a parallelogram: (− 1, − 2), (2, − 1), (5, 2), (2, 1).

10. Show that the following four points are the vertices of a square: (−2, 5), (2, 2), (5, 6), (1, 9).

11. Prove by co-ordinates that the join of the mid points of two sides of a triangle is half the third side.

12. M is the mid point of the side BC of a triangle ABC: prove by co-ordinates that $AB^2 + AC^2 = 2\,AM^2 + 2\,BM^2$.

DETERMINANTS; AREA OF A TRIANGLE

§ 18. In the more advanced parts of the book we shall assume that the reader is familiar with the properties of determinants, which are proved in most modern books on Algebra.

For the convenience of the junior reader we shall explain here the elements of the determinant notation; this notation is often useful in abbreviating formulae, and a knowledge of it often facilitates numerical calculations.

§ 19. *Def'*—The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

means $ad - bc$, and is called a *determinant of the second order.*

Ex. $\begin{vmatrix} 3 & -2 \\ -4 & -1 \end{vmatrix} = 3(-1) - (-4)(-2) = -3 - 8 = -11.$

Again $\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} = 0 - (-6) = +6.$

N.B. Observe that the diagonal term ad always comes first.

§ 20. *Def'*—The symbol $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$

means $a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$= a\,(ek - hf) - b\,(dk - gf) + c\,(dh - ge)$$

and is called a *determinant of the third order.*

Observe that the determinants of the second order which are multiplied by a, $-b$, c are got from the original determinant by scoring out the row and column containing a, b, c respectively.

§ 21. Ex. $\begin{vmatrix} 2 & -5 & -4 \\ -1 & -15 & -5 \\ -11 & -2 & 1 \end{vmatrix}$

$$= 2\begin{vmatrix} -15 & -5 \\ -2 & 1 \end{vmatrix} - (-5)\begin{vmatrix} -1 & -5 \\ -11 & 1 \end{vmatrix} + (-4)\begin{vmatrix} -1 & -15 \\ -11 & -2 \end{vmatrix}$$

$$= 2[-15-(+10)] - (-5)[-1-(+55)] - 4[2-(+165)]$$

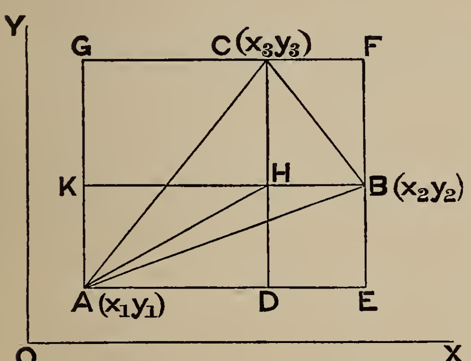
$$= -50 - 280 + 652$$

$$= 322.$$

After a little practice it will be found unnecessary to set down more than the following steps:

$$\text{The det.}' = 2(-25) + 5(-56) - 4(-163)$$
$$= -50 - 280 + 652$$
$$= 322.$$

§ 22. *To find the area of a triangle in terms of the co-ordinates of its vertices.*



Let the vertices be $A\,(x_1y_1)$, $B\,(x_2y_2)$, $C\,(x_3y_3)$

Through A, B, C draw ||s to the axes.

Then $\triangle ABC = \triangle AHB + \triangle BHC + \triangle CHA$

$$= \tfrac{1}{2} \square HDEB$$
$$+ \tfrac{1}{2} \square BHCF$$
$$+ \tfrac{1}{2} \square CHKG$$
$$= \tfrac{1}{2}[\square GAEF - \square ADHK]$$
$$= \tfrac{1}{2}(AE\,.\,AG - AD\,.\,AK)$$
$$= \tfrac{1}{2}[(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)],$$

multiplying out we get

$$2 \text{ area} = x_1(y_2-y_3) - y_1(x_2-x_3) + x_2y_3 - x_3y_2.$$

But this is the expansion of $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$\therefore \quad \text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

This formula should be remembered.

Cor′—Let the vertex A be at the origin: then $x_1 = 0$, $y_1 = 0$,

$$\text{area} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}(x_2 y_3 - x_3 y_2).$$

The learner should prove this case independently.

§ **23.** Ex. Find the area of the triangle whose vertices are (1, 2), (−2, 3), (5, 6).

$$2 \text{ area} = \begin{vmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \\ 5 & 6 & 1 \end{vmatrix}$$

$$= 1(-3) - 2(-7) + 1(-27)$$

$$= -3 + 14 - 27$$

$$= -16.$$

Thus the area contains 8 square units. In § 41 we shall account for the negative sign.

§ **24.** Ex. Find the area of the triangle (2, 3), (− 4, 1), (− 1, 2).

$$2 \text{ area} = \begin{vmatrix} 2 & 3 & 1 \\ -4 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = 2(-1) - 3(-3) + 1(-7) = -2 + 9 - 7 = 0.$$

We infer that the three points are collinear.

Similarly three points $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

Exercises

1. Evaluate the determinants :

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}, \quad \begin{vmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{vmatrix}, \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

Ans. 1, 0, 14.

2. Evaluate the determinants:

$$\begin{vmatrix} 6 & 7 & -3 \\ 5 & 2 & 2 \\ 2 & 4 & 1 \end{vmatrix}, \quad \begin{vmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 0 & 2 & 5 \end{vmatrix}, \quad \begin{vmatrix} 3 & -1 & 2 \\ -5 & 2 & 4 \\ 7 & -3 & 6 \end{vmatrix}$$

Ans. − 91, 12, 16.

3. Find the areas of the triangles whose vertices are

1°. (2, 1), (3, − 2), (− 4, − 1)

2°. The origin and (2, 3), (4, − 5)

3°. (2, 3), (4, − 5), (− 3, − 6)

Ans. 10, 11, 29.

4. Show that (4, 5), (− 2, 3), (16, 9) are collinear.

§ **25.** If the axes are oblique and inclined at an angle ω we obtain as before

$$2 \text{ area } ABC = \square GAEF - \square ADHK,$$

but area of

$$\square GAEF = AE \,.\, AG \sin GAE = (x_2 - x_1)(y_3 - y_1) \sin \omega, \text{ \&c.}$$

$$\therefore \text{ area of } \triangle ABC = \tfrac{1}{2} \sin \omega \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

LOCUS OF AN EQUATION.

§ **26.** An equation connecting the co-ordinates x and y represents a geometrical locus. The learner will see this clearly after he has considered the following instances.

§ **27.** Take for example the equation $2y = x^2 - 3x + 1$.

Here if we assign any value to x we can determine a corresponding value of y. Thus if $x = 3$ we get $y = \frac{1}{2}$.

We may form a table of corresponding values of x and y:

x	-2	-1	0	1	2	3	4	&c.
y	$+5\frac{1}{2}$	$2\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$2\frac{1}{2}$	

The co-ordinates then of the points $(-2, 5\frac{1}{2})$, &c. satisfy the equation. Let us 'plot' these points.

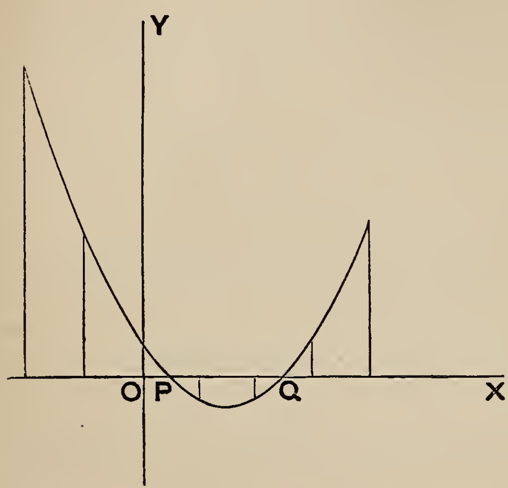
It will be seen that the points are arranged according to a continuous law, and we may draw a curve through them on which intermediate points will lie, and by plotting intermediate points, as for instance

$$x = 1\tfrac{1}{2}, \quad y = -\tfrac{5}{8},$$

we may draw the curve with greater accuracy. We see then that there exists a curve such that the co-ordinates of every point on it satisfy the equation

$$2y = x^2 - 3x + 1;$$

accordingly this equation represents that curve locus.



§ **28.** Suppose we wish to find the points P and Q where the curve cuts OX; the distances OP, OQ might be found by measurement to a close degree of approximation.

Or we may calculate OP and OQ thus. $y = 0$ for each of these points. But their co-ord's satisfy the equation $2y = x^2 - 3x + 1$.

Put then $y = 0$ in this equation.

$$\therefore \quad x^2 - 3x + 1 = 0.$$

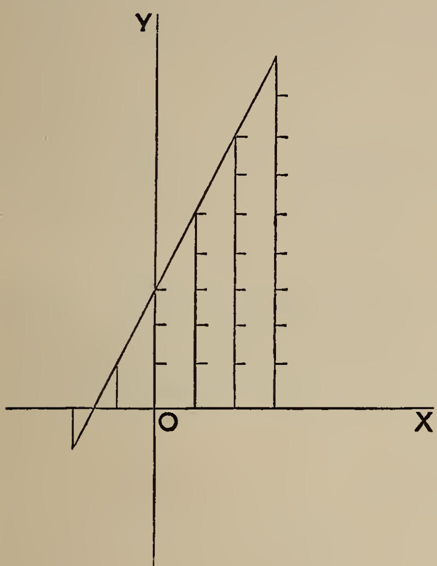
$$\therefore \quad x = \frac{3 \pm \sqrt{5}}{2}.$$

Thus $OP = \cdot 382$, $OQ = 2\cdot 618$, $PQ = \sqrt{5} = 2\cdot 236$.

§ **29.** As another example consider the equation $y = 2x + 3$. Form a table and plot the points as before:

x	−2	−1	0	1	2	3	&c.
y	−1	1	3	5	7	9	

It is seen that the points are ranged on a straight line.



Note—The equation $y = 2x + 3$ is of the first degree in x and y: it will be proved in Chap. III that such an equation always represents a straight line.

§ **30.** Generally an equation represents the locus of a point which moves so that its co-ordinates satisfy the equation.

If we assign any value a to x we may determine one or more corresponding values of y, say $y = b$, $y = b'$, $y = b''$, ...

Thus the points

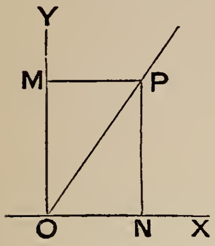
$$\left.\begin{array}{l} x = a \\ y = b \end{array}\right\}, \quad \left.\begin{array}{l} x = a \\ y = b' \end{array}\right\}, \quad \left.\begin{array}{l} x = a \\ y = b'' \end{array}\right\}, \quad \&c.$$

lie on the locus. We may thus obtain as many points on the locus as we please.

§ **31.** Exs. 4 and 5, § 13, afford further illustrations.

In each of these examples a point moves subject to an assigned condition; we have replaced the condition by a relation between the co-ordinates x, y of a point on the locus. Thus the resulting equation in Ex. 4 represents a certain circle, and that in Ex. 5 a certain straight line.

§ 32. Conversely we may give a geometrical interpretation to an equation containing x or y or both.

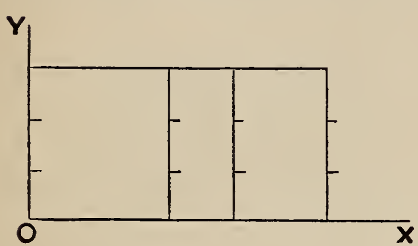


Ex. 1. Thus the equation $x = y$ expresses that if P is any point on the locus then

$$PN = ON.$$

By Euclid Book I, we may see that the locus of P is the bisector of $Y\hat{O}X$.

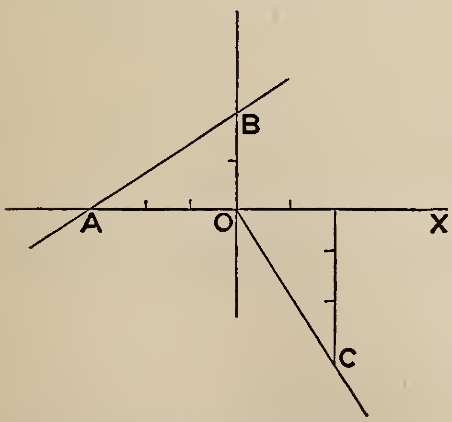
Ex. 2. $y = 3$ expresses that the ordinate of any point on the locus is 3.



Hence $y = 3$ represents a straight line parallel to OX at a distance 3 from it.

Ex. 3. Again, $x^2 + y^2 = r^2$ expresses that the distance of (x, y) from the origin is r: hence this is the equation to a circle whose centre is the origin and whose radius is r.

§ 33. Assuming, as stated in § 29, that a simple equation represents a straight line; we see that to draw a straight line whose equation is given it is sufficient to plot two points on the locus and join them. It is best to get the points where the line cuts the axes.



Thus to draw the line

$$2x - 3y + 6 = 0.$$

Putting $y = 0$ we get $x = -3$.

Thus $(-3, 0)$ is a point on the line.

This is the point A in the figure.

Putting $x = 0$ we get $y = 2$.

Thus $(0, 2)$ is a point on the line.

This is B in the figure.

Hence the required line is AB.

Again, draw the line $3x + 2y = 0$.

We see that this is satisfied if $x = 0$, $y = 0$; thus the line passes through the origin. If we give x any other value we can find y. Thus put $x = 2$, then $y = -3$. The required line is OC.

INTERSECTIONS

§ **34.** The co-ordinates of a point of intersection of two loci satisfy both their equations; accordingly we may obtain the points of intersection of two loci by solving for x, y from their equations, considered as simultaneous.

Ex. 1. Find the point of intersection of the straight lines

$$3x + 4y = 11, \quad y - x = 1.$$

Solving these, $x = 1$, $y = 2$.

Ex. 2. Find the points of intersection of

$$2y = x^2 - 3x + 1 \text{ and } 2y + x = 4.$$

Solving these we find $(-1, 2\frac{1}{2})$ and $(3, \frac{1}{2})$.

The learner should draw diagrams to illustrate these Examples.

Exercises

1. Plot the loci of the equations

$$y^2 = 4x, \quad x + y - 4 = 0, \quad 2x - 3y + 6 = 0, \quad 3x - 5 = 0,$$
$$xy = 24, \quad x^2 + y^2 = 25$$

2. Find the p′t or p′ts of inters′n of the loci

1°, $x + y - 4 = 0, \quad 3y - 2x + 3 = 0$

2°, $x + y - 4 = 0, \quad 3x - 4 = 0$

3°, $x^2 + y^2 = 25, \quad 3x + 4y = 25$

4°, $y^2 = 4x, \quad y = 6$

5°, $\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1$

Ans. $(3, 1)$; $(1\frac{1}{3}, 2\frac{2}{3})$; $(3, 4)$; $(9, 6)$; $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

3. Find the distance between the two p'ts of inters'n of

$$x^2 + y^2 = 25 \text{ and } 4x - 3y = 7.$$

Ans. $1\frac{23}{25}$.

4. Find the intercept which the locus $4(x - 2)^2 = y + 1$ cuts off on the axis of x. *Ans.* 1.

5. Find the intercepts which the locus of $x^2 + y^2 - 4x - 8y - 5 = 0$ cuts off on OX and OY. *Ans.* 6, $2\sqrt{21}$.

§ **35.** The degree of an equation is the greatest sum of the indices of x and y in any term.

Thus $2x + 3y + 4 = 0$ is of the first degree.

$xy + 5x + 6 = 0$ is of the second degree, for the sum of the indices in the term xy is $1 + 1 = 2$.

$x^2y + y^2 = x$ is of the third degree, for the sum of the indices in the term x^2y is $2 + 1 = 3$.

$\sqrt{x} + \sqrt{y} = 1$ when rationalized gives

$$x^2 - 2xy + y^2 - 2x - 2y + 1 = 0;$$

this is an equation of the second degree.

Equations are classified according to their degrees.

The general equation of the first degree is

$$ax + by + c = 0,$$

where a, b, c stand for constant numbers. It will be shown in Chap. III that this represents a straight line.

The general equation of the second degree is

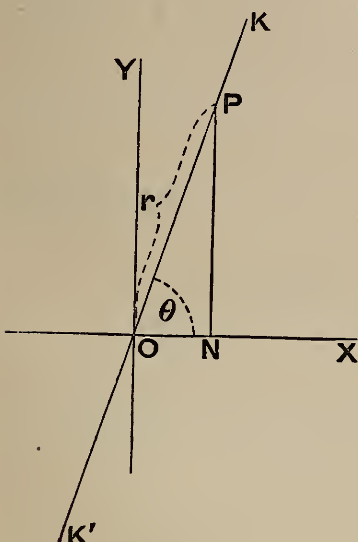
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

It will be shown hereafter that this represents a conic section.

POLAR CO-ORDINATES

§ **36.** The conception in the method of rectangular co-ordinates is that we may reach a point P by first travelling a definite distance ON along a fixed line OX, and then another definite distance NP parallel to another given direction OY.

We may reach the point P by another method.



Rotate OX through an angle O into the position OK and on OK measure off a definite length OP.

If OP $= r$ and $\widehat{POX} = \theta$, then r, θ are called the polar co-ordinates of P. These co-ordinates evidently determine the position of P. r is called its radius vector and θ its vectorial angle.

§ 37. θ is positive or negative according as we suppose that OX revolves in a counter-clockwise or in a clockwise direction.

In the figure it is positive.

r is positive if it is measured along the line which is the new position of OX after the rotation; i. e. if it is measured along OK. Negative values of r are got by measurements along OK produced backwards, i. e. along OK′. OX is called the *initial line.*

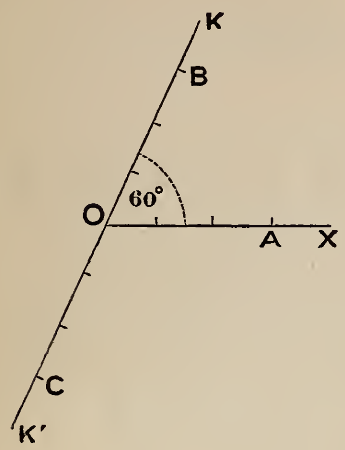
§ 38. Ex. 1. Plot the points $(3, 0)$, $\left(3, \frac{\pi}{3}\right)$, $\left(3, \frac{4\pi}{3}\right)$.

These are A, B, C in the figure. (Page 22)

Ex. 2. Plot the point $\left(-3, \frac{\pi}{3}\right)$.

This is C in the figure. (Page 22)

Here instead of measuring the distance 3 along OK it is measured along OK′.

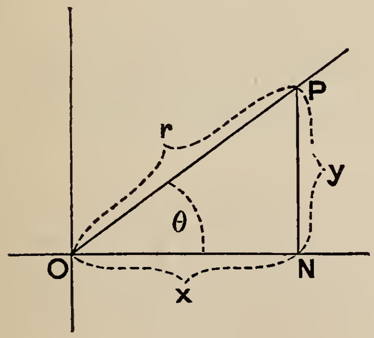


Ex. 3. Plot the point $\left(-\frac{2\pi}{3}, 3\right)$.
This is C in the figure.

Ex. 4. Plot the point $(-3, \pi)$.
Ans. A in the figure.

Ex. 5. Plot $\left(3, 2n\pi + \frac{\pi}{3}\right)$, where n is any integer.
Ans. B in the figure.

§ **39.** *To transform from rectangular to polar co-ordinates or vice versa.*



From the right-angled triangle OPN we have

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \ldots \quad (1)$$

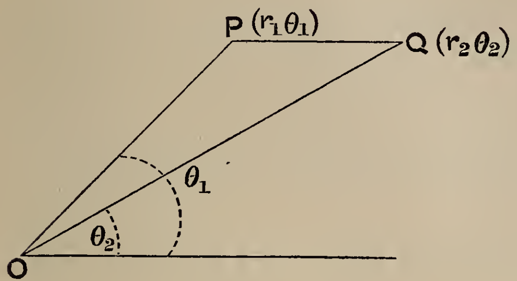
These express x, y in terms of r, θ.

$$\text{Again}\quad \left.\begin{aligned} r^2 &= x^2 + y^2 \\ \text{and}\quad \frac{y}{x} &= \tan\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\quad r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

These express r, θ in terms of x, y.

§ **40.** *To express the distance between two points in terms of their polar co-ordinates.*



Let the points be $P\ (r_1\theta_1)$ and $Q\ (r_2\theta_2)$.

$$\text{Then}\quad PQ^2 = OP^2 + OQ^2 - 2\,OP \cdot OQ \cos P\hat{O}Q,$$

$$\therefore\quad PQ^2 = r_1^2 + r_2^2 - 2\,r_1 r_2 \cos(\theta_1 - \theta_2).$$

Otherwise: let the rectangular co-ordinates of P be $(x_1 y_1)$ and of Q, $(x_2 y_2)$.

$$\text{Then}\quad \left.\begin{aligned} x_1 &= r_1 \cos\theta_1 \\ y_1 &= r_1 \sin\theta_1 \end{aligned}\right\},\qquad \left.\begin{aligned} x_2 &= r_2 \cos\theta_2 \\ y_2 &= r_2 \sin\theta_2 \end{aligned}\right\},$$

$$\begin{aligned} \therefore\quad PQ^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2 \\ &= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) \\ &\qquad - 2\,r_1 r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= r_1^2 + r_2^2 - 2\,r_1 r_2 \cos(\theta_1 - \theta_2). \end{aligned}$$

Exercises

1. Plot the points whose co-ord's are

$$\left(3, \frac{\pi}{4}\right),\quad \left(3, -\frac{\pi}{3}\right),\quad \left(-3, \frac{2\pi}{3}\right),\quad \left(-3, -\frac{\pi}{4}\right).$$

Obtain their rectangular co-ord's.

Ans. $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$; $\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$; $\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$; $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

2. Find the polar co-ord's of the p'ts whose rect' co-ord's are

$$(1, 1),\quad (-1, 2),\quad (-3, 3),\quad (-4, -4).$$

Ans. $\left(\sqrt{2}, \frac{\pi}{4}\right)$; $(\sqrt{5}, -\tan^{-1}2)$; $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$; $\left(4\sqrt{2}, \frac{5\pi}{4}\right)$.

3. Find the distance between the points $\left(4, \frac{\pi}{3}\right)$ and $\left(2\sqrt{2}, \frac{\pi}{4}\right)$.
Ans. $2(\sqrt{3}-1)$.

4. Show that the origin and the points $\left(4, \frac{7\pi}{18}\right)$ $\left(4, \frac{\pi}{18}\right)$ are the vertices of an equilateral triangle.

5. Express in polar co-ord's the eq'n, $x^2 - y^2 = a^2$. *Ans.* $r^2 \cos 2\theta = a^2$.

6. Express in rect' co-ord's the eq'n, $r^2 \sin 2\theta = 2a^2$. *Ans.* $xy = a^2$.

7. What loci are represented by the equations $r = 5$, $\theta = \frac{\pi}{3}$?

§ 41. In fig', § 40,

$$\text{area } OPQ = \tfrac{1}{2} OP \,.\, OQ \sin P\widehat{O}Q = \tfrac{1}{2} r_1 r_2 \sin(\theta_1 - \theta_2).$$

This is positive or negative according as $\theta_1 - \theta_2$ is positive or negative.

The reader will see on reflection that if we go round the triangle OPQ in the order in which the points O, P, Q are mentioned; then if this order is clockwise $\theta_1 - \theta_2$ is positive; otherwise it is negative.

We can now account for the sign of the determinant in § 22.

Take A for origin and AE for initial line in fig' of that §.

Let the polar co-ord's of B be $(r_1\,\theta_1)$, and of C $(r_2\,\theta_2)$.

Then

$$\left.\begin{array}{l} x_2 - x_1 = r_1 \cos\theta_1 \\ y_2 - y_1 = r_1 \sin\theta_1 \end{array}\right\}, \qquad \left.\begin{array}{l} x_3 - x_1 = r_2 \cos\theta_2 \\ y_3 - y_1 = r_2 \sin\theta_2 \end{array}\right\}.$$

$$\therefore \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \begin{array}{l} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \\ = r_1\cos\theta_1 \,.\, r_2 \sin\theta_2 - r_2\cos\theta_2 \,.\, r_1\sin\theta_1 \\ = -\, r_1 r_2 \sin(\theta_1 - \theta_2). \end{array}$$

Thus $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is positive or negative according as in traversing the perimeter of the triangle ABC from A to B, from B to C and from C to A the order is counter-clockwise or clockwise.

Thus $\text{area } ABC = \pm \tfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, the upper or lower sign being used according as the order is counter-clockwise or clockwise.

§ 42. The expression at the beginning of the last §,

$$\text{area OPQ} = \tfrac{1}{2}\, r_1 r_2 \sin(\theta_1 - \theta_2)$$

gives

$$\text{area OPQ} = \tfrac{1}{2}(r_1 \sin\theta_1 \,.\, r_2 \cos\theta_2 - r_1 \cos\theta_1 \,.\, r_2 \sin\theta_2)$$
$$= \tfrac{1}{2}(x_2 y_1 - x_1 y_2).$$

This gives the area of the triangle whose vertices are (o, o), $(x_1 y_1)$, $(x_2 y_2)$, and is positive if the order of (o, o), $(x_1 y_1)$, $(x_2 y_2)$ is clockwise.

§ 43. To find the area of a polygon whose vertices are $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$, $(x_4 y_4)$, &c.

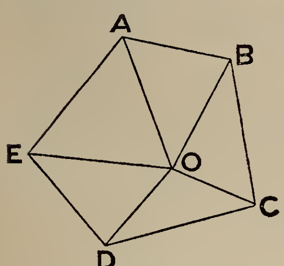


Fig. (1).

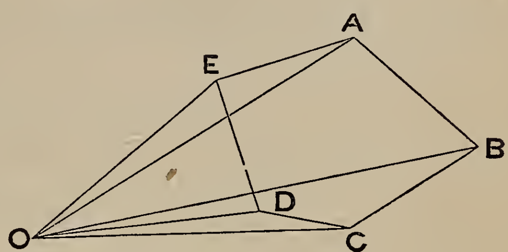


Fig. (2).

In fig. 1,

$$\text{area ABCDE} = \text{OAB} + \text{OBC} + \&c.$$
$$= \tfrac{1}{2}[x_2 y_1 - x_1 y_2 + x_3 y_2 - x_2 y_3 + \&c.]$$

In fig. 2,

$$\text{area ABCDE} = \text{OAB} + \text{OBC} - \text{OCD} - \text{ODE} + \text{OEA}$$

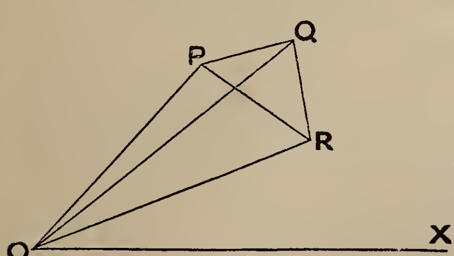
But area ODE $= -\tfrac{1}{2}(x_5 y_4 - x_4 y_5)$ and area OCD $= -\tfrac{1}{2}(x_4 y_3 - x_3 y_4)$.

Thus *in all cases*,

$$\text{area of polygon} = \tfrac{1}{2}[x_2 y_1 - x_1 y_2 + x_3 y_2 - x_2 y_3 + x_4 y_3 - x_3 y_4 + \&c.],$$

the order of the suffixes being cyclic.

§ 44. To find the area of the triangle whose vertices are $(r_1 \theta_1)$, $(r_2 \theta_2)$, $(r_3 \theta_3)$.



Let the vertices be P, Q, R.

Then

area PQR =
OPQ + OQR − OPR.

But $\Delta\,OPQ = \tfrac{1}{2}OP\,.\,OQ\sin P\hat{O}Q = \tfrac{1}{2}r_1 r_2 \sin(\theta_1-\theta_2)$,

similarly $\Delta\,OQR = \tfrac{1}{2}r_2 r_3 \sin(\theta_2-\theta_3)$,

$$\Delta\,OPR = \tfrac{1}{2}r_3 r_1 \sin(\theta_1-\theta_3),$$
$$= -\tfrac{1}{2}r_3 r_1 \sin(\theta_3-\theta_1).$$

$\therefore$ area $PQR = \tfrac{1}{2}[r_1 r_2 \sin(\theta_1-\theta_2) + r_2 r_3 \sin(\theta_2-\theta_3)$
$+ r_3 r_1 \sin(\theta_3-\theta_1)]$,

the order of the suffixes being cyclic.

Cor′—We shall obtain the condition that the three points are collinear by equating the preceding expression to zero.

Exercises on Chapter I

1. Show that (a, b), (b, a), $(3a - 2b, 3b - 2a)$ are collinear.

2. O is the origin, A is $(r_1\,\theta_1)$ and B is $(r_2\,\theta_2)$; show that the polar co-ord′s of the point where the bisector of $A\hat{O}B$ meets AB are

$$r = \frac{2\,r_1 r_2}{r_1 + r_2}\cos\tfrac{1}{2}(\theta_2 - \theta_1), \qquad \theta = \tfrac{1}{2}(\theta_1 + \theta_2).$$

3. A, B, C are collinear points: if P is any fourth point on this line, prove

$$PA^2\,.\,BC + PB^2\,.\,CA + PC^2\,.\,AB + AB\,.\,BC\,.\,CA = 0.$$

4. If P is any point *not* on the line ABC; the preceding equation is still true.

5. G is the intersection of the medians (or *centroid*) of a triangle ABC: if O is any other point, prove

$$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3\,GO^2.$$

6. Prove that $\Delta\,GBC = \tfrac{1}{3}\Delta\,ABC$.

7. AB, AC are adjacent sides of a ▱, and AD the conterminous diag′: if the sign of an area LMN be negative or positive according as the order of L, M, N is clockwise or not; prove that

$$\Delta\,PAD = \Delta\,PAC + \Delta\,PAB,$$

where P is any point in the plane of the ▱.

[*Note*—Take AB, AC for axes; let $AB = h$, $AC = k$; co-ord's of P (xy): then

$$\text{area } PAD = -\tfrac{1}{2}\sin\omega \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ h & k & 1 \end{vmatrix} = -\tfrac{1}{2}\sin\omega\,(hy - kx), \&c.$$

This gives a proof of the fundamental theorem of moments in mechanics.]

8. The condition that the point (xy) may be within the triangle $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ is that the following three det's have the same sign:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}, \quad \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}.$$

9. Find the area of the triangle whose vertices are

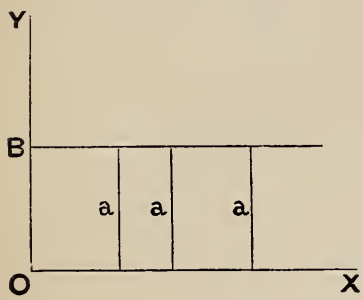
$$(a\mu_1^2), 2a\mu_1), \qquad (a\mu_2^2, 2a\mu_2), \qquad (a\mu_3^2, 2a\mu_3).$$

Ans. $a^2(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)$.

CHAPTER II. THE STRAIGHT LINE

EQUATION TO A LINE

§ **45.** *To find the equation to a straight line parallel to the axis of* x.



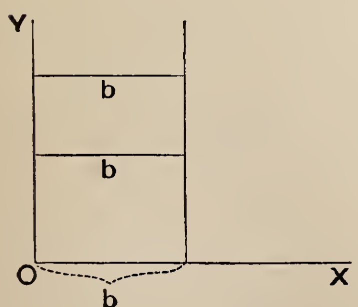
Let it cut OY in B, put OB = a.

Then the ordinate of every point in the line is a : hence its equation is

$$y = a.$$

Similarly the *equation to a straight line parallel to* OY *is*

$$x = b$$



The equation to the axis of x *is* y = o.

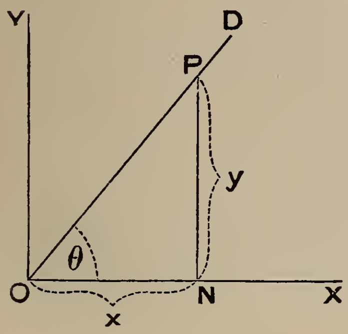
For the ordinate of every point on OX or its "y" = o.

Similarly *the equation to the axis of* y *is*

$$x = o.$$

§ **46**. *To find the equation to a straight line through the origin.*

Let OD be the line, θ its inclination to OX, $\tan\theta = m$.



Then if P be any point (x, y) on the line

$$\frac{PN}{ON} = \tan\theta = m,$$

i.e. $\dfrac{y}{x} = m$

∴ the required equation is

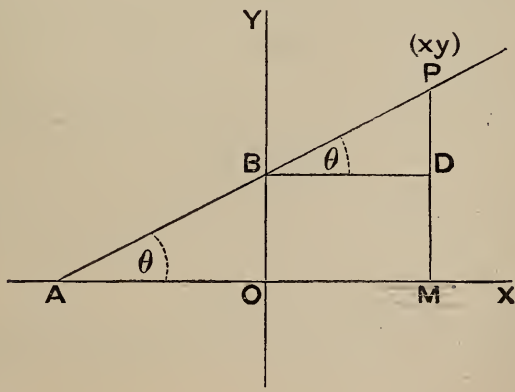
$$y = mx.$$

Example. Find the equation of the join of (4, 3) to the origin.

Here $m = \tan\theta = \frac{3}{4}$

∴ the equation is $y = \frac{3}{4}x$, or $4y = 3x$.

§ **47**. *To find the equation to any straight line.*



Let AB be the line, cutting the axes in A and B. Let the inclination of the line to OX be θ; put

$$OB = c, \quad \tan\theta = m.$$

Let P (x, y) be any point on the line.

Draw BD ∥ OX.

$$\text{Then} \quad \frac{PD}{BD} = \tan\theta$$

$$\therefore \quad PD = BD\tan\theta$$

$$\therefore \quad PM - OB = OM\tan\theta.$$

That is $y - c = x\tan\theta$, $y = x\tan\theta + c$.

Thus the required equation is

$$y = mx + c.$$

Thus the equation to any line is $y = mx + c$, *where* m *is the tangent of its inclination to the axis of* x, *and* c *its intercept on the axis of* y.

§ **48.** If two points on a line are known the line is determined. Thus two conditions determine a straight line.

If two conditions are assigned which a straight line is to fulfil these will give two equations which will enable us to find the constants m and c.

Ex. 1. Find the equation to a line through (0, 3) and inclined at 60° to OX.

Here $c = 3$, $m = \tan 60° = \sqrt{3}$; thus the required equation is

$$y = x\sqrt{3} + 3.$$

Ex. 2. Find the equation to a line through (– 1, 2) inclined at 150° to OX.

One condition is $m = \tan 150° = -\dfrac{1}{\sqrt{3}}$.

Another is got by expressing that $y = mx + c$ is to be satisfied by $x = -1$, $y = 2$. This gives

$$2 = m(-1) + c$$

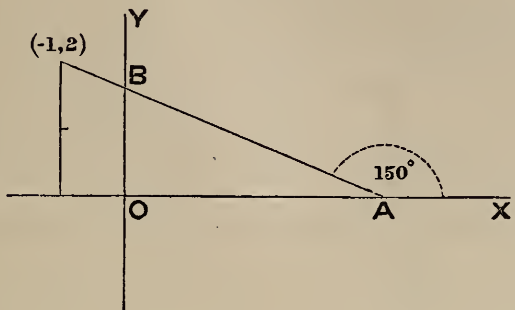
$$2 = \left(-\frac{1}{\sqrt{3}}\right)(-1) + c = \frac{1}{\sqrt{3}} + c$$

$$\therefore \quad c = 2 - \frac{1}{\sqrt{3}}$$

$$\therefore \text{ eq'n required is } \quad y = -\frac{1}{\sqrt{3}}x + 2 - \frac{1}{\sqrt{3}}$$

We have drawn fig' to illustrate; here

$$OB = 2 - \frac{1}{\sqrt{3}}$$



Ex. 3. Find the equation to the join of (1, 2), (3, 4).

Let it be $y = mx + c$.

This is to be satisfied if $x = 1,\ y = 2,\ \therefore 2 = m + c$
Also if $x = 3,\ y = 4,\ \therefore 4 = 3m + c$

Solving these simultaneous equations $m = 1$, $c = 1$. Thus the line required is

$$y = x + 1.$$

General formulae for the solution of such examples will be given presently.

§ **49.** If two lines are parallel they have the same inclination to OX; hence *parallel lines have the same* m.

Ex. Find the equation of a line through $(1, -2)$ parallel to $y = 3x - 4$. The m of the line is 3.

$$\therefore \text{ its equation is } \quad y = 3x + c.$$

To find c we express that this is satisfied by $x = 1$, $y = -2$

$$\therefore \quad -2 = 3 + c, \qquad c = -5.$$

$$\text{Thus the eq'n req'd is} \quad y = 3x - 5.$$

§ **50.** *Any simple equation* $Ax + By + C = 0$, *represents a straight line.* For the equation gives

$$By = -Ax - C.$$

I. Suppose that B is not $= 0$. Then dividing by B we get

$$y = -\frac{A}{B} \cdot x - \frac{C}{B};$$

this may be written $y = mx + c$, if we put

$$m = -\frac{A}{B}, \quad c = -\frac{C}{B}$$

Thus the equation $Ax + By + C = 0$ represents a straight line passing through the point $\left(0, -\frac{C}{B}\right)$ and inclined to OX at an angle $\tan^{-1}\left(-\frac{A}{B}\right)$.

II. Let $B = 0$: then $Ax + C = 0$,

$$\therefore \quad x = -\frac{C}{A}$$

This represents a straight line parallel to OY at a distance $-\frac{C}{A}$.

N.B. A, B, C here stand for any numerical quantities, *signs included.* Thus take the equation

$$-3y + 4x - 5 = 0.$$

Here $A = 4, \quad B = -3, \quad C = -5,$

$$\therefore \quad m = -\frac{A}{B} = -\frac{4}{-3} = \frac{4}{3}$$

$$c = -\frac{C}{B} = -\frac{-5}{-3} = -\frac{5}{3}$$

Or we may go through the process, thus

$$-3y + 4x - 5 = 0$$

$$\therefore \quad 3y = 4x - 5$$

$$\therefore \quad y = \tfrac{4}{3}x - \tfrac{5}{3}$$

Thus $-3y + 4x - 5 = 0$ represents a line through $(0, -\frac{5}{3})$, and inclined at $\tan^{-1}\frac{4}{3}$ to OX.

§ **51.** We may give another proof that $Ax + By + C = 0$ represents a straight line.

Let $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ be any three points on the locus.

Then

$$Ax_1 + By_1 + C = 0, \qquad Ax_2 + By_2 + C = 0,$$
$$Ax_3 + By_3 + C = 0.$$

We may eliminate A, B, C from these equations.

Thus, subtract the second equation from the first and the third from the second. This gives

$$A(x_1 - x_2) + B(y_1 - y_2) = 0$$
$$A(x_2 - x_3) + B(y_2 - y_3) = 0.$$

Multiply the first of these by $y_2 - y_3$ and the second by $y_1 - y_2$ and subtract; then divide by A.

$$\therefore \quad (x_1 - x_2)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_2) = 0$$

$$\text{i. e.} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Thus the area of the triangle whose vertices are $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ is zero; and

$\therefore$ the three points are collinear.

Exercises

1. Draw the lines

$x = 2$, $y = 3$, $3x + 5y = 0$, $y - x = 0$, $y + x = 0$.

2. Draw the lines

$y + x = 1$, $y = 3x + 4$, $y = 3x - 4$, $y = -3x + 4$.

3. Write the equations of lines through the origin inclined respectively at $45°$, $60°$ and $120°$ to OX. *Ans.* $y = x$, $y = x\sqrt{3}$, $y = -x\sqrt{3}$.

4. Find the equation to a line through (1, 2) at $60°$ to OX; also to a line through (0, − 3) at $45°$ to OX. *Ans.* $x\sqrt{3} - y = \sqrt{3} - 2$, $y = x - 3$.

5. Reduce the equation $2x - 3y + 4 = 0$ to the form $y = mx + c$. *Ans.* $m = \frac{2}{3}$, $c = \frac{4}{3}$.

6. Find the equation to a line through (1, 2) parallel to $y = 2x + 3$. *Ans.* $y = 2x$.

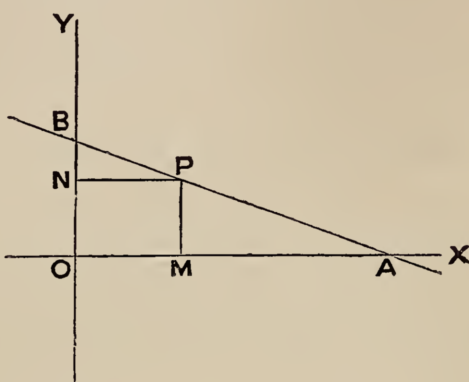
EQUATION IN TERMS OF INTERCEPTS

§ **52.** *To express the equation to a straight line in terms of its intercepts on the axes.*

Let AB be the line cutting the axes in A and B. Let

$$OA = a, \quad OB = b.$$

Let P (x, y) be any point on the line.



By similar triangles

$$\frac{PM}{MA} = \frac{OB}{OA}$$

$$\therefore \quad \frac{y}{a - x} = \frac{b}{a}$$

$$\therefore \quad ay = ab - bx$$

$$\therefore \quad ay + bx = ab$$

Divide by ab : thus $\frac{x}{a} + \frac{y}{b} = 1.$

This is the required equation.

§ **53.** Alternative Proofs :

I. Join OP.

Then $\quad 2\,\Delta\,OBA = \Delta\,OBP + \Delta\,OPA$

$$\therefore \quad OA\,.\,OB = OB\,.\,PN + OA\,.\,PM$$

$$\therefore \quad ab = bx + ay$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1.$$

II. Let the intercepts which $Ax + By + C = 0$ cuts off on the axes be a, b.

To get a put $y = 0$: this gives

$$Ax + C = 0, \text{ or } x = -\frac{C}{A}.$$

That is $\quad a = -\dfrac{C}{A}$

Similarly by putting $x = 0$ we get $b = -\dfrac{C}{B}$

Now the equation $Ax + By + C = 0$ may be written

$$-Ax - By = C$$

$$\text{or} \quad -\frac{A}{C}x - \frac{B}{C}y = 1$$

$$\text{or} \quad \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1, \quad \text{i.e. } \frac{x}{a} + \frac{y}{b} = 1.$$

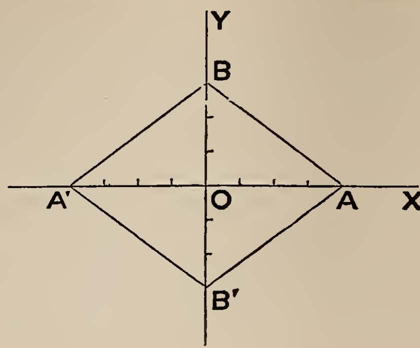
Ex. In fig′ page 36 the eq′n

To AB is $\quad \dfrac{x}{4} + \dfrac{y}{3} = 1, \quad$ or $\quad 3x + 4y = 12$

To A′B, $\quad \dfrac{x}{-4} + \dfrac{y}{3} = 1, \quad$ or $\quad 3x - 4y + 12 = 0$

To A′B′, $\quad \dfrac{x}{-4} + \dfrac{y}{-3} = 1, \quad$ or $\quad 3x + 4y + 12 = 0$

To AB′, $\quad \dfrac{x}{4} + \dfrac{y}{-3} = 1, \quad$ or $\quad -3x + 4y + 12 = 0$



Exercises

1. Find the equation to a line across the third quadrant cutting off intercepts 4, 5 from the axes. *Ans.* $5x + 4y + 20 = 0$.

2. Find the intercepts cut off on the axes by

$$3x - 4y = 12, \qquad y = x\sqrt{3} + 4, \qquad lx + my = 1.$$

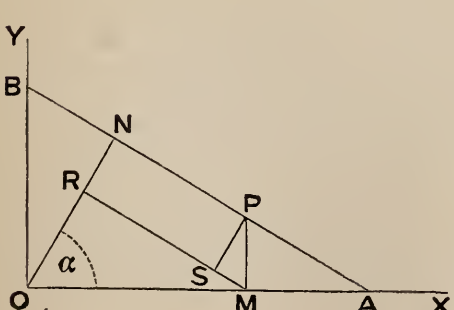
Ans. $4, -3$; $-\dfrac{4}{\sqrt{3}}, 4$; $\dfrac{1}{l}, \dfrac{1}{m}$.

3. Find the line through (3, 5) which cuts off equal intercepts on the axes. *Ans.* $x + y = 8$.

4. Find the line through (3, 3) which forms with the axes a triangle whose area is 18. *Ans.* $x + y = 6$.

STANDARD FORM

§ **54.** *To express the equation to a straight line in terms of* p *and* α *where* p *is the length of the perpendicular from the origin on the line and* α *is the angle which this perpendicular makes with the axis of* x.



Let AB be the line,

$$ON = p, \quad N\hat{O}X = \alpha.$$

Let P be any point on the line;

$$x = OM, \quad y = PM$$

its co-ord's.

Draw $MR \perp ON$, and $PS \perp MR$.

Then $P\hat{M}S = 90^\circ - RMO = \alpha$;

$$OR = OM \cos\alpha = x\cos\alpha,$$

$$NR = PS = PM \sin PMS = y \sin\alpha;$$

$$p = ON = OR + RN = x\cos\alpha + y\sin\alpha.$$

Thus the equation to the line is

$$x\cos\alpha + y\sin\alpha - p = 0.$$

§ **55.** Alternative proof.

The equation to the line is $\dfrac{x}{OA} + \dfrac{y}{OB} = 1$ (§ 52).

But $p = ON = OA\cos\alpha$

$$\therefore \quad OA = \frac{p}{\cos\alpha}$$

Also $p = ON = OB \cos NOB = OB\sin\alpha$

$$\therefore \quad OB = \frac{p}{\sin\alpha}$$

Substitute these values of OA and OB: then the equation becomes

$$\frac{x}{\left(\dfrac{p}{\cos\alpha}\right)} + \frac{y}{\left(\dfrac{p}{\sin\alpha}\right)} = 1, \quad \text{or} \quad x\cos\alpha + y\sin\alpha = p.$$

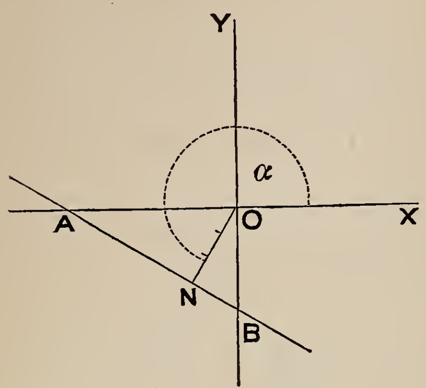
§ **56.** The form $x\cos\alpha + y\sin\alpha = p$ of the equation to a straight line is called the 'standard' form.

It will be assumed as a convention that p is always positive, and also that α is always positive, i. e. measured round in the positive direction from OX.

§ **57.** With this understanding, it will be found on examination that whatever be the position of the line the proofs in Arts. 54 and 55 are perfectly general, and that in all cases the equation is

$$x\cos\alpha + y\sin\alpha = p.$$

The generality of the proof in Art. 55 depends on this, that whatever be the magnitude of α, OA and cos α have always the same sign, and also OB and sin α have always the same sign.



Example. If in fig′ the Euclidean angle

$$XON = 120°,$$

then $\alpha = 240°$,

also $p = 3$;

and the equation to AB is

$$x \cos 240° + y \sin 240° = 3$$

or $$x\left(-\frac{1}{2}\right) + y\left(-\frac{\sqrt{3}}{2}\right) = 3$$

or $$x + y\sqrt{3} + 6 = 0.$$

§ **58.** We have obtained the equation to a straight line in the following forms,

1°, $y = mx + c$

2°, $Ax + By + C = 0$

3°, $\dfrac{x}{a} + \dfrac{y}{b} = 1$

4°, $x \cos \alpha + y \sin \alpha = p$ (the *standard* form).

If an equation is given in one of these forms we may reduce it to any of the others.

Ex. Express $\dfrac{x}{a} + \dfrac{y}{b} = 1$ in the form $y = mx + c$.

Here $\dfrac{y}{b} = 1 - \dfrac{x}{a}$, $y = -\dfrac{b}{a}x + b$.

Thus $m = -\dfrac{b}{a}$, $c = b$.

§ **59.** *To reduce the equation* $Ax + By + C = 0$ *to the form*

$$x \cos\alpha + y \sin\alpha - p = 0.$$

Divide by $\sqrt{A^2 + B^2}$; then

$$\frac{A}{\sqrt{A^2 + B^2}} \cdot x + \frac{B}{\sqrt{A^2 + B^2}} \cdot y + \frac{C}{\sqrt{A^2 + B^2}} = 0.$$

Now since the sum of the squares of

$$\frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \frac{B}{\sqrt{A^2 + B^2}}$$

is unity; if C be negative we may write

$$\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}, \qquad \sin\alpha = \frac{B}{\sqrt{A^2 + B^2}},$$

$$-p = \frac{C}{\sqrt{A^2 + B^2}}$$

If C is positive, then changing the signs of all the terms the equation may be written

$$-Ax - By - C = 0;$$

the preceding method is then applicable.

In words:

To reduce the equation of a straight line to the standard form: if necessary change the sign of every term so that the last term is negative, then divide by the square root of the sum of the squares of the coefficients of x *and* y.

Ex. 1. Reduce $y = \frac{5}{12}x + \frac{7}{12}$ to the form

$$x \cos\alpha + y \sin\alpha - p = 0.$$

Here $5x - 12y + 7 = 0$

Changing the signs, $-5x + 12y - 7 = 0$

Divide by $\sqrt{(-5)^2 + 12^2}$ or 13

$$\therefore \quad -\tfrac{9}{13}x + \tfrac{12}{13}y - \tfrac{7}{13} = 0.$$

$$\text{Thus} \quad p = \tfrac{7}{13}, \quad \cos\alpha = -\tfrac{5}{13}, \quad \sin\alpha = \tfrac{12}{13}.$$

As the cosine of α is negative and its sine positive, α must be *an angle in the second quadrant,* i.e. between 90° and 180°.

Ex. 2. Reduce $x - y\sqrt{3} + 6 = 0$ to the form

$$x\cos\alpha + y\sin\alpha - p = 0.$$

$$\text{Here} \quad -x + y\sqrt{3} - 6 = 0.$$

$$\text{Divide by} \quad \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2;$$

$$\text{thus} \quad -\tfrac{1}{2}x + \frac{\sqrt{3}}{2}y - 3 = 0.$$

Thus $p = 3$; also $\cos\alpha = -\tfrac{1}{2}$, $\sin\alpha = \dfrac{\sqrt{3}}{2}$, or $\alpha = 120°$.

The learner should draw diagrams to illustrate these examples.

Exercises

1. Express the following equations in the standard form:

$$x + y = 3, \quad x\sqrt{3} + y + 6 = 0, \quad -x\sqrt{3} + y + 6 = 0, \quad x\sqrt{3} - y + 6 = 0,$$
$$x\sqrt{3} + y - 6 = 0, \qquad 5x - 12y + 20 = 0.$$

Ans. $\alpha = 45°$, $p = \dfrac{3}{\sqrt{2}}$; $\alpha = 210°$, $p = 3$; $\alpha = 330°$, $p = 3$; $\alpha = 150°$, $p = 3$; $\alpha = 30°$, $p = 3$; $\cos\alpha = -\tfrac{5}{13}$, $\sin\alpha = \tfrac{12}{13}$, $p = \tfrac{20}{13}$.

2. Express in the standard form

$$y = mx + c, \qquad \frac{x}{a} + \frac{y}{b} = 1.$$

Ans. $$-\frac{m}{\sqrt{1+m^2}}x + \frac{1}{\sqrt{1+m^2}}y - \frac{c}{\sqrt{1+m^2}} = 0,$$

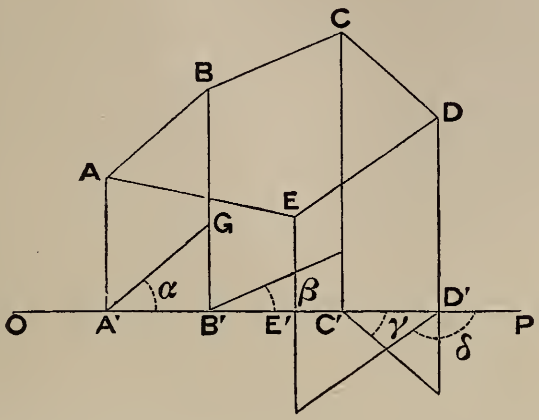
$$\frac{bx}{\sqrt{a^2+b^2}} + \frac{ay}{\sqrt{a^2+b^2}} - \frac{ab}{\sqrt{a^2+b^2}} = 0.$$

PROJECTIONS

§ 60. If the properties of projections are assumed, the statement of the proof in § 54 may be simplified.

Def'—If A' be the foot of a $\perp$ from a point A on a line OP, then A' is called the *projection* of A on OP.

Def'—If A', B' be the projections of two points A, B on OP then $A'B'$ is called the projection of AB on OP.



Now let α be the angle between OP, AB: then drawing $A'G \parallel AB$ we see that

$$A'B' = A'G \cos\alpha = AB \cos\alpha.$$

If A, E are the terminals of a broken line AB, BC, CD, DE; then

$$A'E' = A'B' + B'C' + C'D' + D'E',$$

the signs being taken into account.

Or,

projection of AE = sum of projections of AB, BC, CD, DE.

If α, β, γ, δ are the angles which AB, BC, CD, DE make with OP; then the projection of

$$AE = AB \cos\alpha + BC \cos\beta + CD \cos\gamma + DE \cos\delta.$$

In the figure it will be seen that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are positive while $\cos\delta$ is negative.

We may now prove the equation of § 54.

The projection of OP on ON = sum of projections of OM, MP.

(fig', § 54)

But the projection of OP is p, that of OM is $x \cos\alpha$, and of MP is $y \sin\alpha$

$$\therefore \quad p = x \cos\alpha + y \sin\alpha.$$

INTERSECTION OF TWO LINES

§ **61.** The co-ordinates of the point of intersection of two straight lines are got by solving the equations simultaneously for x and y.

We here remind the reader of a theorem of Algebra, which will give a general formula.

$$\text{If} \begin{cases} ax + by + cz = 0 & \dots\dots (1) \\ a'x + b'y + c'z = 0 & \dots\dots (2) \end{cases}$$

$$\text{Then} \quad \frac{x}{\begin{vmatrix} b & c \\ b' & c' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c & a \\ c' & a' \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}$$

To remember this formula we may use this mnemonic :

Imagine the coefficients written twice in succession, thus

$$\begin{matrix} a & b & c & a & b & c \\ a' & b' & c' & a' & b' & c' \end{matrix}$$

Then the second and third columns give the denominator of x; sliding forward one column we get the denominator of y; and sliding forward another we get the denominator of z.

If now we put $z = 1$, we see that the intersection of the lines

$$ax + by + c = 0, \quad a'x + b'y + c' = 0$$

is given by

$$\frac{x}{\begin{vmatrix} b & c \\ b' & c' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c & a \\ c' & a' \end{vmatrix}} = \frac{1}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}$$

Ex. The intersection of

$$2x - 3y + 6 = 0, \quad x - y + 1 = 0$$

is given by

$$\frac{x}{\begin{vmatrix} -3 & 6 \\ -1 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 6 & 2 \\ 1 & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix}}$$

$$\text{or} \quad \frac{x}{3} = \frac{y}{4} = 1,$$

$$\therefore \quad x = 3, \quad y = 4.$$

§ **62.** *To find the condition that three lines may be concurrent.*

The values of x, y got by solving two of the equations must satisfy the third.

Ex. Show that the lines

$$x + y + 1 = 0, \qquad 2x - y - 7 = 0, \qquad 4x + y - 5 = 0$$

are concurrent.

Solving the first two we get

$$x = 2, \qquad y = -3.$$

This point lies on the third line if

$$4(2) - 3 - 5 = 0, \quad \text{or} \quad 8 = 3 + 5. \qquad \text{Q.E.D.}$$

Or we may obtain a general formula.

Let the three lines be

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0,$$
$$a''x + b''y + c'' = 0.$$

From the last two equations we obtain

$$\frac{x}{\begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c' & a' \\ c'' & a'' \end{vmatrix}} = \frac{1}{\begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix}} = k \text{ say.}$$

$$\therefore \quad x = k(b'c'' - b''c'), \qquad y = k(c'a'' - c''a'),$$
$$1 = k(a'b'' - a''b')$$

Substituting these values in $ax + by + c = 0$, and dividing by k, we obtain the req'd cond'n.

$$a(b'c'' - b''c') + b(c'a'' - c''a') + c(a'b'' - a''b') = 0$$

$$\text{or} \quad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0$$

§ **63.** *To find the condition that the lines*

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0$$

may be parallel.

They must have the same m: this gives

$$-\frac{a}{b} = -\frac{a'}{b'},$$

$$\therefore \quad \frac{a}{b} = \frac{a'}{b'}, \quad \text{and} \quad ab' - a'b = 0.$$

Thus *the coeff's of* x *and* y *must be in the same ratio in both eq'ns.*

Ex. 1. $6x - 4y + 1 = 0$, $9x - 6y - 7 = 0$, are parallel, for

$$\frac{6}{-4} = \frac{9}{-6}, \quad \text{each being} \quad = -\frac{3}{2}.$$

Ex. 2. $ax + by + c = 0$, $ax + by + c' = 0$ are parallel.

Ex. 3. Find the equation of a line through $(2, -1)$ parallel to

$$3x - 5y + 1 = 0.$$

The required equation is of the form

$$3x - 5y + k = 0.$$

To find k substitute $x = 2$, $y = -1$,

$$\therefore \quad 6 + 5 + k = 0, \qquad k = -11.$$

Ans. $3x - 5y - 11 = 0$.

Exercises

1. Show that the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \frac{x}{b} + \frac{y}{a} = 1, \qquad x = y$$

are concurrent.

Ans. The point of intersection is $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$

2. Find the equation to a line through $(1, -2)$ parallel to

$$3x + 4y + 6 = 0.$$

Ans. $3x + 4y + 5 = 0$.

3. Find the area of the triangle whose sides are

$$x - y = 0, \quad x + 3y = 0, \quad x + y + 4 = 0.$$

Ans. 8.

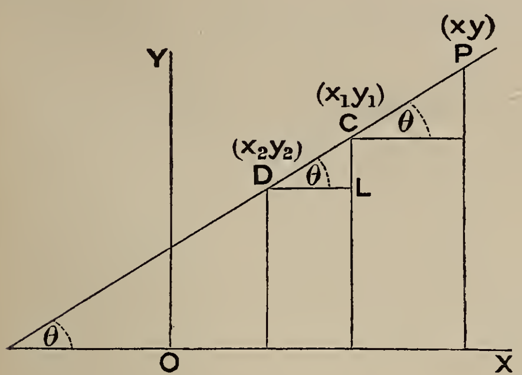
4. Find the area of the triangle whose sides are

$$3x + y - 7 = 0, \quad x + 7y + 11 = 0, \quad x - 3y + 1 = 0.$$

Ans. 10.

EQUATION TO JOIN OF TWO POINTS

§ **64.** Suppose we take two points $(x_1 y_1)$, $(x_2 y_2)$ on a straight line



The m of the line

$$= \tan\theta$$

$$= \frac{CL}{DL}$$

$$= \frac{y_1 - y_2}{x_1 - x_2}$$

Thus *the* m *of a line = the difference between the* y's *of any two points on the line divided by the difference between their* x's.

§ **65.** *To find the equation to the join of two points* $(x_1 y_1)$, $(x_2 y_2)$.

Let (x, y) be any other point on the line.

$$\text{The } m \text{ of the line} = \frac{y_1 - y_2}{x_1 - x_2} \text{ (fig', § 64)}$$

$$\text{It also} = \frac{y - y_1}{x - x_1}$$

$$\therefore \quad \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \quad \text{. . . . . (1)}$$

This is the required equation.

If we multiply up and transpose we get

$$x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - x_2 y_1 = 0$$

$$\therefore \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \quad \ldots \ldots \quad (2)$$

This is another form of the required equation.

Both forms should be remembered.

The form (1) is intuitive, if we think of the figure.

We might also write down (2) at once from the consideration that the area of the triangle whose vertices are (xy), $(x_1 y_1)$, $(x_2 y_2)$ is zero.

§ 66. Ex. 1. Find the equation to the join of (1, − 2) and (− 3, 4).

Applying (1) it is $\dfrac{y+2}{x-1} = \dfrac{-2-4}{1+3}$

or $\dfrac{y+2}{x-1} = -\dfrac{3}{2}$

or $2(y+2) + 3(x-1) = 0$

or $3x + 2y + 1 = 0$.

[*Note*—We may verify this:

We see that $3x + 2y + 1 = 0$ is a simple eq'n satisfied both when $x = 1$ $y = -2$ and when $x = -3$, $y = 4$]

Or thus, applying (2) the req'd eq'n is

$$\begin{vmatrix} x & y & 1 \\ 1 & -2 & 1 \\ -3 & 4 & 1 \end{vmatrix} = 0$$

or $x(-6) - y(+4) + 1(-2) = 0$

or $-6x - 4y - 2 = 0$

or $3x + 2y + 1 = 0$; as before.

The form (2) of the equation of the join will usually be found the most convenient in practice.

Ex. 2. The equation to the join of (a, 0) and (0, b) is

$$\begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0,$$

$$\text{or} \quad -bx - ay + ab = 0,$$

$$\text{or} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

We have thus another proof of the equation of § 52.

Exercises

1. Find the equations to the joins of

1°. The origin and $(3, 4)$

2°. The origin and $(-3, 4)$

3°. $(4, 2)$ and $(-3, 9)$

4°. $(-3, -1)$ and $(4, -8)$

5°. $(2, 3)$ and $(4, 3)$

6°. (a, b) and $(-a, -b)$

7°. (a, b) and (b, a)

Ans. 1°, $4x-3y=0$. 2°, $4x+3y=0$. 3°, $x+y-6=0$. 4°, $x+y+4=0$. 5°, $y=3$. 6°, $bx-ay=0$. 7°, $x+y=a+b$.

2. Find the equations to the sides of a triangle whose vertices are

$$(2, 1), \quad (3, -2), \quad (-4, -1).$$

Ans. $3x+y-7=0$, $x+7y+11=0$, $x-3y+1=0$.

3. Find the equations to the medians of the same triangle.

Ans. $x-y-1=0$, $x+2y+1=0$, $x-13y-9=0$.

4. Show that the following three points are collinear:

$$(-3, -9), \quad (1, -1), \quad (2, 1).$$

Find the ratio of the segments into which these points divide the line.

Ans. $4:1$.

5. Find the equations to the diagonals of the rectangle formed by the lines

$$x = a, \quad x = a', \quad y = b, \quad y = b'.$$

Ans. $(b-b')x - (a-a')y + ab' - a'b = 0$,

$(b-b')x + (a-a')y - ab + a'b' = 0$.

6. Find the equations to

1°, the join of the origin to the intersection of

$$y = 2x - 3, \qquad 3y + x + 4 = 0;$$

2°, the join of $(-4, 7)$ to the intersection of

$$y = 3x - 1, \qquad x = y - 1.$$

Ans. 1°, $5y + 11x = 0$. 2°, $x + y - 3 = 0$.

7. Find the equations of the lines through the intersection of

$$y = 3x - 1, \quad x = y - 1$$

parallel to OX and to

$$4x + 5y + 6 = 0.$$

Ans. $y = 2$, $4x + 5y = 14$.

8. Find the equation of a line joining the intersection of

$$3y + 2x + 3 = 0, \quad 2y + x + 2 = 0$$

to that of

$$y + 2x + 3 = 0, \quad 2y + 3x + 4 = 0.$$

Ans. $y + x + 1 = 0$.

LINE THROUGH A FIXED POINT

§ 67. Suppose that any line is drawn through a point (h, k); then if (x, y) be any point on the line its m is $\dfrac{y-k}{x-h}$.

$$\text{Thus} \quad \frac{y-k}{x-h} = m, \quad \text{or} \quad y - k = m(x - h)$$

denotes a line through (h, k), and inclined to OX at an angle whose tangent is m.

By giving m a suitable value the equation $y - k = m(x - h)$ may be made to represent any one of a pencil of lines through (h, k).

Ex. The line through $(-1, 2)$ at 150° to OX is

$$y - 2 = -\frac{1}{\sqrt{3}}(x + 1).$$

(See Ex. 2, § 48.)

Exercises on Chapter II

1. Find the equation of a line through (3, 4) at 75° to OX.
Ans. $y - 4 = (2 + \sqrt{3})(x - 3)$.

2. Through the vertices of the triangle whose sides are

$$x + 2y - 5 = 0, \quad 2x + y - 7 = 0, \quad y - x - 1 = 0$$

parallels are drawn to the opposite sides: find their equations.
Ans. $x - y - 2 = 0, \quad x + 2y - 8 = 0, \quad 2x + y - 4 = 0$.

3. Find the equation of the join of (a, b) to the intersection of

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1$$

Ans. $b^2x - a^2y + (a - b)ab = 0$.

4. Two lines can be drawn through (6, − 4) each of which forms with the axes a triangle whose area is $\frac{1}{6}$; find their equations.
Ans. $3x + 4y = 2, \quad 16x + 27y + 12 = 0$.

5. Show that the area of the triangle contained by the axis of y and the straight lines

$$y = m_1x + c_1, \quad y = m_2x + c_2 \quad \text{is}$$

$$\frac{(c_2 - c_1)^2}{2(m_2 - m_1)}$$

[*Note*—If lines meet in P and cut OY in B, C; 2 area = BC . abscissa of P.]

6. Deduce that the area of the triangle formed by the lines

$$y = m_1x + c_1, \quad y = m_2x + c_2, \quad y = m_3x + c_3$$

$$\text{is } \frac{1}{2}\frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{1}{2}\frac{(c_3 - c_1)^2}{m_3 - m_1} + \frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2}$$

7. Obtain also the following expression for the area:

$$\frac{\begin{vmatrix} 1 & m_1 & c_1 \\ 1 & m_2 & c_2 \\ 1 & m_3 & c_3 \end{vmatrix}^2}{2(m_1 - m_2)(m_2 - m_3)(m_3 - m_1)}$$

[*Note*—We may reason thus: Calling the determinant in the numerator Δ; then area = 0 if Δ = 0. Also area = ∞ if $m_1 = m_2$ or $m_3 = m_3$ or $m_3 = m_1$.

Further m_1, m_2, m_3 are ratios while c_1, c_2, c_3 are lines. Hence as the area must be of the second degree in c_1, c_2, c_3 we may assume it

$$= k\Delta^2 \div (m_1 - m_2)(m_2 - m_3)(m_3 - m_1)$$

where k is a numerical constant. We may determine k by taking the three lines

$$y = 0, \quad y = x, \quad y = -x + 2,$$

which form a triangle whose area is 1. We thus get $k = \frac{1}{2}$.]

8. Show that the area of the triangle formed by the three lines

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0, \quad A''x + B''y + C'' = 0$$

is $\frac{1}{2}\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}^2 \div (AB' - BA')(A'B'' - B'A'')(A''B - B''A)$

9. Show that the area of the triangle whose sides are

$$\frac{x}{a} - \frac{y}{b} = 0, \quad \frac{x}{a} + \frac{y}{b} = 0, \quad \frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

is ab.

10. Find the area of the triangle whose sides are

$$x - \mu_1 y + a\mu_1^2 = 0, \quad x - \mu_2 y + a\mu_2^2 = 0, \quad x - \mu_3 y + a\mu_3^2 = 0.$$

Ans. $\frac{a^2}{2}(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)$.

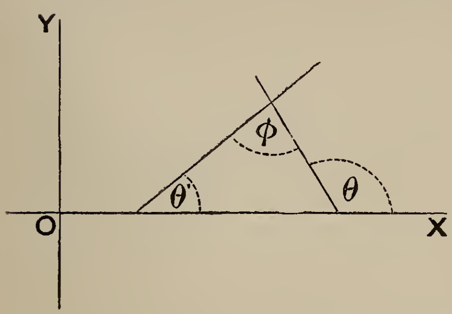
CHAPTER III

THE STRAIGHT LINE—(continued)

ANGLE BETWEEN TWO LINES

§ **68.** *To find the angle between the lines*

$$y = mx + c, \qquad y = m'x + c'.$$



Here $m = \tan\theta$

$m' = \tan\theta'$

By Euclid I. 32,

$$\theta = \phi + \theta'$$

$$\therefore \quad \phi = \theta - \theta'$$

$$\therefore \quad \tan\phi = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}$$

$$= \frac{m - m'}{1 + mm'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Cor' (1)—If the lines are $\parallel$, $\phi = 0$

$$\therefore \quad m - m' = 0 \quad \text{or} \quad m = m'$$

This agrees with § 49.

Cor' (2)—If the lines are $\perp$

$$\phi = 90^\circ, \ \tan\phi = \infty$$

$$\therefore \quad 1 + mm' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Cor' (3)—If the lines are

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0.$$

$$\text{Then} \quad m = -\frac{a}{b}, \quad m' = -\frac{a'}{b'}$$

$\therefore$ the lines are $\perp$ if

$$1 + \left(-\frac{a}{b}\right)\left(-\frac{a'}{b'}\right) = 0$$

Multiply up by bb'

$\therefore$ the lines are $\perp$ if

$$aa' + bb' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The formulae (1), (2), (3) should be carefully remembered.

From (3) we see that

Two lines are at right angles if

PRODUCT COEFF'S OF x + PRODUCT COEFF'S OF $y = 0$

Cor' (4)—Generally, if ϕ is the angle between the lines

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0$$

$$\text{then} \quad \tan\phi = \frac{m - m'}{1 + mm'}$$

Substituting for m, m' their values, we find

$$\tan\phi = \frac{a'b - ab'}{aa' + bb'}$$

§ **69.** Ex. 1. Find the angle between

$$x - y\sqrt{3} + 1 = 0, \qquad x + y\sqrt{3} - 2 = 0.$$

$$\text{Here} \quad m = \frac{1}{\sqrt{3}}, \quad m' = -\frac{1}{\sqrt{3}}$$

$$\therefore \tan\phi = \frac{m - m'}{1 + mm'} = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}} = \frac{2}{\sqrt{3}} \div \frac{2}{3} = \sqrt{3},$$

$$\therefore \phi = 60^\circ.$$

Ex. 2. $6x + 4y + 5 = 0$ and $2x - 3y + 7 = 0$ are at right angles.

For $6(2) + 4(-3) = 12 - 12 = 0$.

Ex. 3. $Ax + By + C = 0$ and $Bx - Ay + C' = 0$ are at right angles.

For $A(B) + B(-A) = AB - AB = 0$, $\therefore$ &c. [*Cor'* (3)]

Similarly

$$ax + by + c = 0 \quad \text{and} \quad \frac{x}{a} - \frac{y}{b} + c' = 0$$

are at right angles.

§ **70.** *To find the equation of a line through* (h, k) *perpendicular to* $y = mx + c$.

By § 67 the required equation is of the form

$$y - k = m'(x - h).$$

Also $1 + mm' = 0$ [§ 68, *Cor'* (2)]

This gives $m' = -\dfrac{1}{m}$

and the eq'n req'd is

$$y - k = -\frac{1}{m}(x - h)$$

Ex. Find the equation of a line through (6, 7) perpendicular to

$$3x + 2y + 4 = 0$$

Here $2y = -3x - 4$

$\therefore \quad y = -\frac{3}{2}x - 2$

$\therefore \quad m = -\frac{3}{2}$

$$m' = -\frac{1}{m} = +\frac{2}{3}$$

and the eq'n req'd is

$$y - 7 = \tfrac{2}{3}(x - 6), \quad \text{or} \quad 2x - 3y + 9 = 0.$$

§ **71.** *To find the equation of a line through* (h, k) *perpendicular to* $Ax + By + C = 0$.

This case may be reduced to the last.

$$\text{Here} \quad m = -\frac{A}{B}$$

$$\therefore \quad m' = -\frac{1}{m} = \frac{B}{A}$$

Thus the equation required is

$$y - k = \frac{B}{A}(x - h)$$

$$\text{or} \quad \frac{x - h}{A} = \frac{y - k}{B}$$

The student will find it convenient to remember this equation. Thus taking the example in § 70 we at once write down the equation of the $\perp$

$$\frac{x - 6}{3} = \frac{y - 7}{2}, \quad \text{or} \quad 2x - 3y + 9 = 0 \quad \text{as before.}$$

§ 72. Or we may proceed thus.

$Bx - Ay =$ constant represents a $\perp$ (§ 69, Ex. 3).

To find the constant express that the eq'n is satisfied by the co-ord's of the given p't (h, k);

$\therefore$ eq'n req'd is $Bx - Ay = Bh - Ak$.

Thus taking the Ex. in § 70; the eq'n req'd is

$$2x - 3y = 2(6) - 3(7), \quad \text{or} \quad 2x - 3y + 9 = 0.$$

Exercises

1. Find the angles between the pairs of lines

1°, $5y - 3x + 1 = 0$, $y - 4x + 2 = 0$

2°, $2x - 3y = 0$, $6x + 4y + 7 = 0$

3°, $y + x = 0$, $(2 + \sqrt{3})y - x = 0$

4°, $y - kx = 0$, $(1 - k)y - (1 + k)x = 0$

Ans. 45°, 90°, 60°, 45°.

2. Find the equations to

1°, a line through $(1, 2)$ perpendicular to $3x + 4y + 5 = 0$

2°, ,, ,, $(0, -1)$,, ,, $x + y = 1$

Ans. $4x - 3y + 2 = 0$, $x - y - 1 = 0$.

3. Find the equation to a line through the origin

$$\perp \; Ax + By + C = 0$$

Ans. $Bx = Ay$.

4. Find the equation of a line through $(2, 3)$

$$\perp \text{ the join of } \quad (1, 2), \qquad (-3, -14)$$

Ans. $x + 4y = 14$.

5. Find the lines through the origin inclined at $45°$ to

$$y + x\sqrt{3} = 0.$$

Ans. $y + x(2 - \sqrt{3}) = 0$, $y = x(2 + \sqrt{3})$.

6. Show that the acute angle between the lines

$$2x + 3y = 4, \qquad 4x - 5y = 6 \quad \text{is} \quad \sin^{-1}\frac{22}{\sqrt{533}}.$$

POSITIVE AND NEGATIVE SIDES OF A LINE

§ **73**. *To find the ratio in which the join of* $(x_1 y_1)$, $(x_2 y_2)$ *is divided by* $Ax + By + C = 0$.

Let the required ratio be $m : n$.

Then the values

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}$$

(§ 15) must satisfy $Ax + By + C = 0$.

Substitute these values and multiply up by $m + n$.

$$\therefore \quad A(mx_2 + nx_1) + B(my_2 + ny_1) + C(m + n) = 0$$

$$\therefore \quad m(Ax_2 + By_2 + C) = -n(Ax_1 + By_1 + C)$$

$$\therefore \quad \frac{m}{n} = -\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C}$$

Cor'—If $(x_1 y_1)$ and $(x_2 y_2)$ are on the same side of

$$Ax + By + C = 0$$

then their join is cut externally, and $m : n$ is negative (§ 16);

$$\therefore \quad Ax_1 + By_1 + C \text{ and } Ax_2 + By_2 + C$$

have the same sign. If $(x_1 y_1)$ and $(x_2 y_2)$ are on opposite sides their join is cut internally; $m : n$ is positive,

$$\therefore \quad Ax_1 + By_1 + C \quad \text{and} \quad Ax_2 + By_2 + C$$

have opposite signs.

§ **74**. Thus, *two points* $(x_1 y_1)$, $(x_2 y_2)$ *are on the same* or *opposite sides of a line* $Ax + By + C = 0$ *according as the results of substituting the co-ordinates of the points in the sinister side of the equation have like or unlike signs.*

Ex. How are the origin and the points (1, 1), (1, 3) situated with reference to the line $3x - 4y + 5 = 0$?

Substitute (0, 0), (1, 1), (1, 3) successively for (x, y) in the expression $3x - 4y + 5$: the results are $+5$, $+4$, -4.

Hence the origin and (1, 1) are on the same side of the line, and (1, 3) on the opposite side.

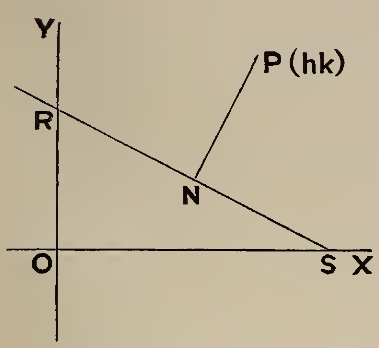
§ **75**. We see now that a line $Ax + By + C = 0$ divides the plane of the axes into two compartments such that the expression $Ax + By + C = 0$ is positive for all points in one compartment (this may be called the positive compartment), and negative for all points in the other (the negative compartment).

If we write the equation $-Ax - By - C = 0$; then the positive and negative compartments are interchanged.

LENGTH OF PERPENDICULAR

§ **76.** *To find the length of the perpendicular from* (h, k) *on*

$$Ax + By + C = 0$$



Let RS be the line

$$Ax + By + C = 0 \quad . \quad (1)$$

N the foot of the ⊥

The equation to PN is (§ 71)

$$\frac{x-h}{A} = \frac{y-k}{B} \quad . \quad . \quad (2)$$

The co-ord's x, y of N are got by solving (1), (2).

Put each member of (2) = λ

$$\therefore \quad x = \lambda A + h, \quad y = \lambda B + k \quad . \quad . \quad . \quad (3)$$

Substitute these values in (1);

$$\therefore \quad A(\lambda A + h) + B(\lambda B + k) + C = 0$$

$$\therefore \quad (A^2 + B^2)\lambda + Ah + Bk + C = 0$$

$$\therefore \quad \lambda = -\frac{Ah + Bk + C}{A^2 + B^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now $\quad PN^2 = (x-h)^2 + (y-k)^2$

$$= (\lambda A)^2 + (\lambda B)^2 \text{ by } (3)$$

$$= \lambda^2 (A^2 + B^2)$$

Hence substituting for λ its value from (4) and taking the square root,

$$PN = \frac{Ah + Bk + C}{\pm\sqrt{A^2 + B^2}}$$

§ 77. We have seen that the numerator is positive or negative according as (h, k) is one side or other of the line. (§ 74.) It is important to remember that the absolute length of the $\perp$ is

$$\frac{Ah + Bk + C}{\sqrt{A^2 + B^2}}$$

Thus *after, if necessary, bringing all the terms of the equation to one side, we substitute the co-ordinates of the point, and divide by the square root of the sum of the squares of the coefficients of* x *and* y.

Ex. 1. Find the length of the perpendicular from (2, 3) on

$$3x + 4y - 20 = 0$$

Here $p = \dfrac{3 \cdot 2 + 4 \cdot 3 - 20}{\sqrt{3^2 + 4^2}} = -\dfrac{2}{5}$.

Ans. The required length is $\dfrac{2}{5}$.

Ex. 2. Find the length of the perpendicular from the origin or

$$3x + 4y + 20 = 0$$

Here $p = \dfrac{3 \cdot 0 + 4 \cdot 0 + 20}{\sqrt{3^2 + 4^2}} = \dfrac{20}{5} = 4$

Exercises

1. Find the ratio in which the join of (1, 2), (3, −2) is cut by

$$4x + 5y - 6 = 0.$$

Ans. The join is cut in a point of trisection.

2. Show that (1, −2), (0, 1), and the origin are on one side of

$$4x + 5y - 6 = 0;$$

and (1, 1) on the opposite side.

3. Show that the origin and $(\frac{1}{2}, \frac{1}{2})$ are within the triangle whose sides are

$$5y - 4x = 1, \qquad x - 3y = 9, \qquad x + 9y = 9.$$

4. Find the lengths of the $\perp^s$ from (1, 3), (2, −3) and the origin on

$$4x - 5y + 6 = 0.$$

Ans. $\frac{5}{\sqrt{41}}, \frac{29}{\sqrt{41}}, \frac{6}{\sqrt{41}}$.

5. The length of the $\perp$ from the origin on

$$h(x+h) + k(y+k) = 0 \quad \text{is} \quad \sqrt{h^2+k^2};$$

$$\text{and on} \quad \frac{x}{h} + \frac{y}{k} = 1 \quad \text{is} \quad \frac{hk}{\sqrt{h^2+k^2}}.$$

6. The vertices of a triangle are (3, 8), (12, 2), (−4, −6); find the equations to the $\perp^s$ from the vertices on the opposite sides. Show that they cointersect in (4, 6).

Ans. $2x + y = 14$, $x + 2y = 16$, $3x - 2y = 0$.

7. Find the lengths of these $\perp^s$.

Ans. $\frac{21}{\sqrt{5}}, \frac{24}{\sqrt{5}}, \frac{56}{\sqrt{13}}$.

8. Find the equations to the $\perp^s$ to the sides of the same triangle through their mid points; show that they cointersect in $(3\frac{1}{2}, -1)$.

Ans. $2x + y = 6$, $2x + 4y = 3$, $6x - 4y = 25$.

9. Find the parallels to $12x - 5y + 1 = 0$ at a distance 2 from it.

Ans. $12x - 5y + 27 = 0$, $12x - 5y - 25 = 0$.

10. Find the points on $3x = y + 1$ at a distance 2 from

$$12x - 5y + 24 = 0.$$

Ans. (1, 2), $(\frac{55}{3}, 54)$.

11. Find the equation to the join of the feet of the $\perp^s$ from the origin on

$$3x - 4y + 25 = 0, \qquad 12x + 5y = 169;$$

find also the length of this join.

Ans. $x - 15y + 63 = 0$, $\sqrt{226}$.

12. Show that the origin is inside the triangle whose vertices are (3, 4), (2, −3), $(-2, -2\frac{1}{2})$.

§ **78.** The formula of § 76 gives the length of the $\perp$ from $x'y'$ on

$$x \cos \alpha + y \sin \alpha - p = 0,$$

viz. it is

$$\frac{x' \cos \alpha + y' \sin \alpha - p}{\sqrt{\cos^2 \alpha + \sin^2 \alpha}} = \pm (x' \cos \alpha + y' \sin \alpha - p).$$

We add an independent geometrical proof of this important formula.

§ **79.** *To find the length of the perpendicular from* $(x'y')$ *on*

$$x \cos \alpha + y \sin \alpha - p = 0$$

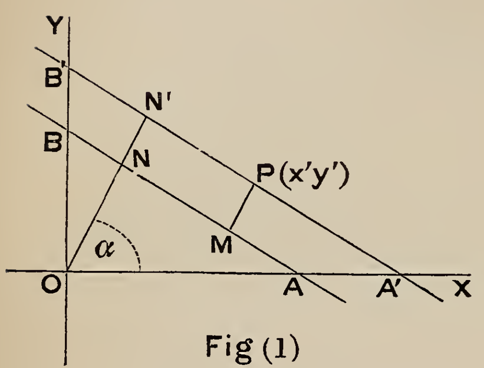


Fig (1)

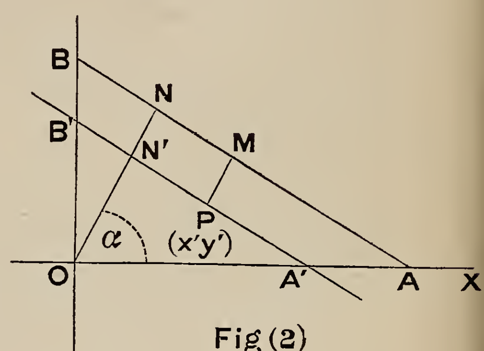


Fig (2)

Let AB be the given line,

$$ON = p, \quad N\hat{O}X = \alpha.$$

Through P, $(x'y')$ draw $A'B' \parallel AB$ and $PM \perp AB$.

If ON or ON produced meet $A'B'$ in N', then ON' is $\perp A'B'$ (Euclid I. 29).

If $p' = ON'$ the equation to $A'B'$ is

$$x \cos \alpha + y \sin \alpha - p' = 0.$$

Expressing that this equation is satisfied by the co-ord's of P, (viz. $x'y'$)

$$x' \cos \alpha + y' \sin \alpha - p' = 0$$

$$\therefore \quad p' = ON' = x' \cos \alpha + y' \sin \alpha.$$

In fig. 1,

$$PM = NN' = p' - p = x' \cos\alpha + y' \sin\alpha - p$$

In fig. 2,

$$PM = N'N = p - p' = p - x' \cos\alpha - y' \sin\alpha$$

Thus the required length is

$$\pm (x' \cos\alpha + y' \sin\alpha - p).$$

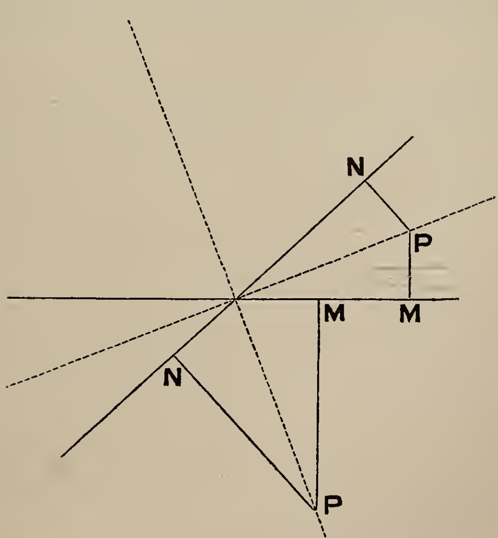
The upper or lower sign is to be used according as $(x' y')$ is on the opposite side of the line to the origin, or on the same side.

This agrees with § 74.

BISECTORS OF ANGLE

§ **80.** *To find the equations to the bisectors of the angles between*

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0.$$



The dotted lines in the diagram represent the bisectors.

Now the $\perp^s$ PM, PN from any point P(x, y) on either bisector on the lines are equal.

$\therefore$ the equation to one bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = + \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} \quad . \quad . \quad . \quad (1)$$

and to the other is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = - \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} \quad . \quad . \quad . \quad (2)$$

To distinguish between these.

Let both square roots have the positive sign; *and let the equations be so written that* C, C′ *are negative.* Then the origin is on the negative side of both lines. If now (1) is satisfied

$$Ax + By + C \quad \text{and} \quad A'x + B'y + C'$$

must have the same sign, i. e. any point which satisfies (1) lies on the positive side of *both* lines, or on the negative side of both.

Accordingly (1) represents the bisector of that angle in which the origin lies.

Ex. Find the bisectors of the angles between

$$3x + 4y - 9 = 0, \qquad 12x - 5y + 6 = 0.$$

Let the second equation be written

$$-12x + 5y - 6 = 0.$$

Then the absolute terms -9, -6 have the same sign, so that the origin is on the negative side of both lines.

The bisector of that angle in which the origin lies is

$$\frac{3x + 4y - 9}{5} = + \frac{-12x + 5y - 6}{13}$$

$$\text{or} \quad 99x + 27y - 87 = 0.$$

The other bisector is

$$\frac{3x + 4y - 9}{5} = - \frac{-12x + 5y - 6}{13}$$

$$\text{or} \quad 3x - 11y + 21 = 0.$$

Cor′—If the equations be given in the form

$$x \cos\alpha + y \sin\alpha - p = 0, \quad x \cos\alpha' + y \sin\alpha' - p' = 0;$$

then $x \cos\alpha + y \sin\alpha - p = x \cos\alpha' + y \sin\alpha' - p'$

is the bisector of that angle in which the origin lies,

and $x \cos\alpha + y \sin\alpha - p = -(x \cos\alpha' + y \sin\alpha' - p')$

is the bisector of the other angle.

Exercises

1. Find the bisectors of the angles between

$$3x - 4y + 7 = 0, \qquad 12x + 5y - 5 = 0.$$

Ans. $21x + 77y = 116$, $33x - 9y + 22 = 0$.

2. Find the lengths of the $\perp^s$ from the origin on the bisectors of the angles between

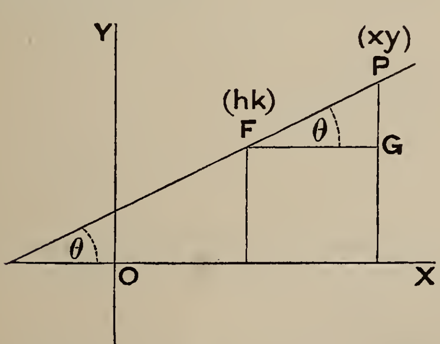
$$x \cos\alpha + y \sin\alpha = p, \qquad x \cos\beta + y \sin\beta = p'.$$

Ans. $\dfrac{p - p'}{2 \sin\frac{1}{2}(\alpha - \beta)}$, $\dfrac{p + p'}{2 \cos\frac{1}{2}(\alpha - \beta)}$.

3. Show that the origin lies on the bisector of the acute angle between the lines $3x - 4y = 5$, $5x + 12y = 13$.

LINE DRAWN IN GIVEN DIRECTION

§ **81.** The following form of the equation to the straight line is sometimes useful.



Let the line be inclined to OX at $\hat{\theta}$.

Let F (h, k) be a fixed point on the line;

P (x, y)

any other point on the line.

Let $FP = r$.

From the figure we see that

$$x - h = FG = r\cos\theta$$

$$y - k = PG = r\sin\theta.$$

Thus the equation to the line may be written in the form

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r.$$

Ex. Through the point D (2, 3) a line is drawn inclined at 60° to OX and meeting $4x + 5y + 6 = 0$ in E; find the length DE.

The equation to DE is $\dfrac{x-2}{\cos 60^\circ} = \dfrac{y-3}{\sin 60^\circ} = r$

$$\therefore\quad x = 2 + r\cos 60^\circ = 2 + \frac{r}{2},$$

$$y = 3 + r\sin 60^\circ = 3 + \frac{\sqrt{3}}{2}r \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If x, y are the co-ordinates of E, the intersection of the two lines, then the equations (1) and

$$4x + 5y + 6 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

are simultaneously true.

Substituting then the values of x, y from (1) in (2)

$$4\left(2 + \frac{r}{2}\right) + 5\left(3 + \frac{\sqrt{3}}{2}r\right) + 6 = 0.$$

This gives $DE = r = -\dfrac{29}{2 + \dfrac{5\sqrt{3}}{2}} = -\dfrac{58(5\sqrt{3} - 4)}{59}.$

§ **82.** *To find the equations to the lines through* (h, k) *which are inclined at a given angle* ϕ *to the line*

$$y = mx + c.$$

Let AB be the line

$$y = mx + c.$$

The req'd eq'ns are

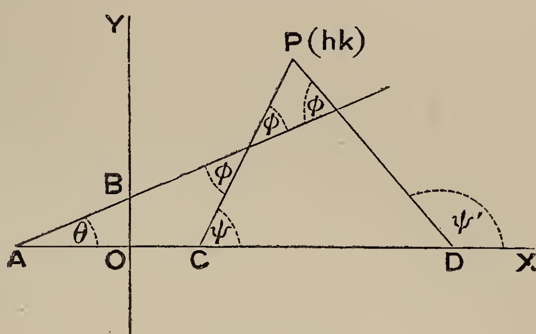
$$y - k = (x - h)\tan\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y - k = (x - h)\tan\psi' \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(vide fig'); $\tan\psi$, $\tan\psi'$ are to be determined.

By Euclid $\psi = \theta + \phi$

$$\therefore \quad \tan\psi = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{m + \tan\phi}{1 - m\tan\phi}$$



Also $\quad \psi' = \theta + (\pi - \phi) = \pi + (\theta - \phi)$

$$\therefore \quad \tan\psi' = \tan(\theta - \phi) = \frac{m - \tan\phi}{1 + m\tan\phi}$$

The req'd eq'ns are obtained by substituting these values of $\tan\psi$, $\tan\psi'$ in (1), (2).

POLAR EQUATION

§ **83**. *To find the polar equation to a straight line.*

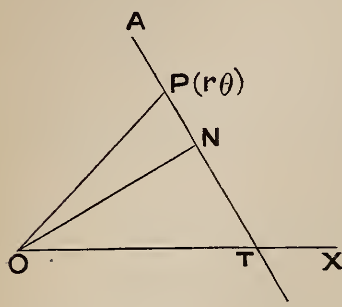
Substitute $r\cos\theta$, $r\sin\theta$ for x, y in

$$Ax + By + C = 0;$$

we thus obtain the required equation

$$A r\cos\theta + B r\sin\theta + C = 0$$

Or we may obtain the polar equation directly thus.



Let AT be the line, P the point (r, θ) on it.

Draw $ON \perp AT$

Let $ON = p$, $N\hat{O}X = \alpha$

Then $ON = OP \cos N\hat{O}P$

$= OP \cos(POX - NOX)$

i.e. $p = r \cos(\theta - \alpha)$

This is the required equation.

Cor'—The equation $p = r\cos(\theta - \alpha)$ gives

$p = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = (r\cos\theta).\cos\alpha + (r\sin\theta).\sin\alpha$

$\therefore\ p = x\cos\alpha + y\sin\alpha$

This agrees with § 54.

Exercises

1. Find the lengths of the $\perp$s from the origin and $(13, \tan^{-1}\frac{5}{12})$ on

$$\frac{1}{r} = 3\cos\theta + 4\sin\theta$$

Ans. $\frac{1}{5}$, 11.

2. Find the equations to these $\perp$s.

Ans. $\theta = \tan^{-1}\frac{4}{3}$, $\frac{33}{r} = 4\cos\theta - 3\sin\theta$

3. Find the equation to the join of

$$(r_1, \theta_1), \qquad \left(2r_1, \theta_1 + \frac{\pi}{3}\right)$$

Ans. $r_1 = r\cos(\theta - \theta_1)$.

4. Find the length of the $\perp$ from the origin on the join of

$$(r_1\,\theta_1), \qquad (r_2\,\theta_2)$$

Ans. $\dfrac{r_1 r_2 \sin(\theta_2 - \theta_1)}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}}$

5. Find the equation to the join of

$$(3\cos\alpha,\ 2\alpha),\qquad (3\cos 2\alpha,\ 3\alpha)$$

Ans. $r = 3\cos(\theta - \alpha)$.

6. Find the equation of the line through

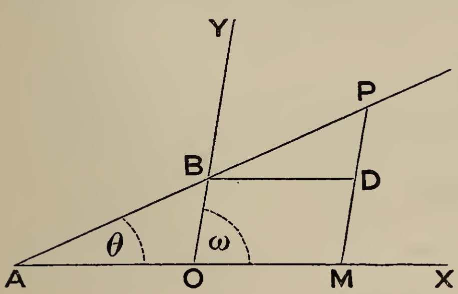
$$(r_1\,\theta_1)\ \perp\ a = r\cos(\theta - \alpha).$$

Ans. $r\sin(\alpha - \theta) = r_1\sin(\alpha - \theta_1)$.

OBLIQUE AXES

§ 84. As in Art. 45, $y = a$, $x = b$ represent straight lines parallel to the axes.

§ 85. *To find the equation to any straight line.*



Let it cut the axes in A, B.

Let θ be its inclination to OX, ω the angle between the axes.

Let P be any point in the line;

$$x = OM,\quad y = PM \text{ its co-ord's.}$$

$$\text{Let}\quad OB = c$$

$$\text{Then}\quad \frac{PD}{DB} = \frac{OB}{OA} = \frac{\sin\theta}{\sin(\omega - \theta)}$$

$$\text{That is}\quad \frac{y - c}{x} = \frac{\sin\theta}{\sin(\omega - \theta)}$$

$$\therefore\quad y - c = \frac{\sin\theta}{\sin(\omega - \theta)}\,x.$$

Putting $\dfrac{\sin\theta}{\sin(\omega-\theta)} = m$, this becomes

$$y = mx + c$$

This is the required equation. The signification of c is the same as for the case of rectangular axes, viz. it is the intercept on OY.

m no longer means $\tan\theta$, but $\dfrac{\sin\theta}{\sin(\omega-\theta)}$.

Cor'—$\tan\theta = \dfrac{m\sin\omega}{1 + m\cos\omega}$

§ **86.** As in § 50 or 51 we may show that any simple equation represents a straight line.

As in § 52 it is shown that the equation in terms of the intercepts on the axes is

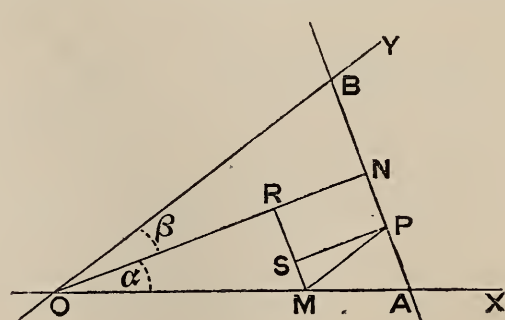
$$\frac{x}{a} + \frac{y}{b} = 1$$

As in § 53 the intercepts on the axes by

$$Ax + By + C = 0$$

are $-\dfrac{C}{A}, -\dfrac{C}{B}$

§ **87.** *To find the equation to a straight line* AB *in terms of* p, α, β *where* p *is the perpendicular on the line from the origin and* α, β *are the angles which this perpendicular makes with* OX, OY.



Let P be any point on the line;

$$OM = x, \; PM = y \text{ its co-ord's.}$$

Then, as in § 60, the projection of OP on ON = sum of projections of OM, MP on ON.

$$\therefore \quad p = x \cos \alpha + y \cos \beta$$

Or we may supply the proof in full, thus: draw MR parallel to AB and PS parallel to ON.

$$\text{Then} \quad OR = OM \cos \alpha = x \cos \alpha;$$

$$RN = PS = PM \cos S\hat{P}M = y \cos \beta$$

$$\therefore \quad x \cos \alpha + y \cos \beta = p$$

§ **88.** Or thus: The equation to AB is

$$\frac{x}{OA} + \frac{y}{OB} = 1$$

$$\text{But} \quad OA = \frac{p}{\cos \alpha}, \quad OB = \frac{p}{\cos \beta}$$

$$\therefore \quad x \cos \alpha + y \cos \beta = p, \text{ as before.}$$

§ **89.** If the axes are oblique no modification is required in the investigations of §§ 61–63.

As in § 64 we get $\dfrac{y_1 - y_2}{x_1 - x_2} = m$, where m has now the same meaning as in § 85.

Thus the formulae (1), (2) of § 65 for the equation of the join of $(x_1 y_1)$, $(x_2 y_2)$ are applicable when the axes are oblique.

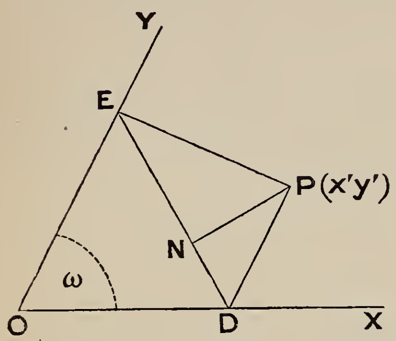
§ 90. As in § 79 it is proved that the length of the $\perp$ from $(x'y')$ on

$$x \cos \alpha + y \cos \beta - p = 0 \quad \text{is}$$

$$\pm (x' \cos \alpha + y' \cos \beta - p)$$

§ 91. *To find the length of the perpendicular from* $(x'y')$ *on*

$$ax + by + c = 0$$



Let

$$ax + by + c = 0$$

meet the axes in D, E.

$$\text{Then} \quad OD = -\frac{c}{a}, \quad OE = -\frac{c}{b}$$

$$\therefore \quad DE^2 = OD^2 + OE^2 - 2\,OD\,.\,OE \cos \omega$$

$$= \frac{c^2}{a^2} + \frac{c^2}{b^2} - \frac{2c^2}{ab} \cos \omega$$

$$\therefore \quad DE = \frac{c}{ab} \sqrt{a^2 + b^2 - 2ab \cos \omega}$$

If the required length $PN = P$, then

2 area of $\triangle$ PED $= P \,.\, DE$

$$= P \,.\, \frac{c}{ab} \sqrt{a^2 + b^2 - 2ab \cos \omega} \quad . \quad . \quad (1)$$

Again, the co-ord's of D are $\left(-\frac{c}{a}, 0\right)$, and of E are $\left(0, -\frac{c}{b}\right)$

$$\therefore \quad 2 \text{ area } PED = \pm \sin\omega \begin{vmatrix} x' & y' & 1 \\ -\frac{c}{a} & 0 & 1 \\ 0 & -\frac{c}{b} & 1 \end{vmatrix} \quad (\S\ 25)$$

$$= \pm \sin\omega \left(\frac{c}{b}x' + \frac{c}{a}y' + \frac{c^2}{ab}\right)$$

$$= \pm \sin\omega \cdot \frac{c}{ab}(ax' + by' + c) \quad \ldots \quad (2)$$

Comparing (1) and (2),

$$P = \pm \frac{(ax' + by' + c) \cdot \sin\omega}{\sqrt{a^2 + b^2 - 2ab\cos\omega}} \quad \ldots \quad (3)$$

§ **92.** To reduce $ax + by + c = 0$ to the form

$$x\cos\alpha + y\cos\beta - p = 0$$

In fig' § 91,

$$p \cdot DE = 2 \text{ area } ODE = OD \cdot OE \sin\omega,$$

$$\text{also} \quad \cos\alpha = \frac{p}{OD}, \quad \cos\beta = \frac{p}{OE}$$

Using then the values of OD, OE, DE in § 91 we shall get

$$p = \frac{c\sin\omega}{\sqrt{a^2 + b^2 - 2ab\cos\omega}}$$

$$\cos\alpha = \frac{-a\sin\omega}{\sqrt{a^2 + b^2 - 2ab\cos\omega}}$$

$$\cos\beta = \frac{-b\sin\omega}{\sqrt{a^2 + b^2 - 2ab\cos\omega}}$$

The equation is then transformed by multiplying by

$$\frac{-\sin\omega}{\sqrt{a^2 + b^2 - 2ab\cos\omega}}$$

This is also obvious from (3), § 91.

§ **93**. If ϕ be the angle between two lines

$$y = mx + c, \qquad y = m'x + c'$$

then with the notation of § 85

$$\phi = \theta - \theta', \qquad \tan\phi = \frac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}$$

Also $\tan\theta = \dfrac{m\sin\omega}{1 + m\cos\omega}, \qquad \tan\theta' = \dfrac{m'\sin\omega}{1 + m'\cos\omega}$

Substituting and reducing we obtain

$$\tan\phi = \frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}$$

Cor' (1)—The lines are $\parallel$ if $m - m' = 0$, and $\perp$ if

$$1 + (m + m')\cos\omega + mm' = 0$$

Cor' (2)—If the two lines are

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0,$$

then $m = -\dfrac{a}{b}, \qquad m' = -\dfrac{a'}{b'};$

and we may substitute these values in the preceding formulae.

This gives

$$\tan\phi = \frac{(a'b - ab')\sin\omega}{aa' + bb' - (ab' + a'b)\cos\omega}$$

Thus the lines are $\parallel$ if $a'b - ab' = 0$,

and $\perp$ if $\qquad aa' + bb' = (ab' + a'b)\cos\omega$

Exercises

1. If $\omega = 30°$, prove that $y + x = 1$ is inclined at $105°$ to OX.

2. The lines $y + x = 0$, $y - x = 0$ are at right angles, whatever be the angle between the axes.

3. If $\omega = 60^\circ$, find the equation to the $\perp$ from $(1, 1)$ on

$$2x + 3y + 4 = 0;$$

find also the length of this $\perp$.

Ans. $4x - y - 3 = 0$, $\dfrac{2\sqrt{3}}{\sqrt{7}}$.

4. Find the equation to a line through (h, k) perpendicular to the axis of x.

Ans. $x + y\cos\omega = h + k\cos\omega$

5. Prove that the lines $y = mx + c$, $y = m'x + c'$ are equally inclined to the axis of x in opposite directions if

$$\frac{1}{m} + \frac{1}{m'} = -2\cos\omega$$

Exercises on Chapter III

1. Find the length of the perpendicular from (b, a) on

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Ans. $\dfrac{a^2 + b^2 - ab}{\sqrt{a^2 + b^2}}$

2. Interpret the equation $\sin 3\theta = 1$.

Ans. Three lines through origin inclined at 30°, 150°, 270° to OX.

3. Find the centre of the inscribed circle of the triangle whose sides are

$$3x - 4y = 0, \qquad 8x + 15y = 0, \qquad x = 20.$$

Ans. $(13, 1)$.

4. The vertices of a triangle are $(1, 2)$, $(2, 5)$, $(3, 4)$; show that the co-ordinates of the centre of the inscribed circle are $(\sqrt{5}, 4)$.

5. Show that the area of the parallelogram whose sides are

$$Ax + By + C = 0, \qquad Ax + By + C + \gamma = 0,$$
$$A'x + B'y + C' = 0, \qquad A'x + B'y + C' + \gamma' = 0$$

is $\dfrac{\gamma\gamma'}{AB' - A'B}$

6. Find the lines through $(1, -2)$ which are inclined at 45° to

$$3x - 4y + 5 = 0.$$

Ans. $7x - y = 9$, $x + 7y + 13 = 0$

7. Show that the angle between

$$7y = 17x + 1, \qquad y = x + 2$$

is $\cos^{-1}\frac{12}{13}$.

8. Show that the equation to the join of the points whose polar co-ordinates are $(2c\cos\alpha, \alpha)$, $(2c\cos\beta, \beta)$ is

$$2c\cos\beta\cos\alpha = r\cos(\beta + \alpha - \theta)$$

9. Prove that the equation to the straight line through $(r_1\,\theta_1)$ perpendicular to

$$\frac{1}{r} = a\cos\theta + b\sin\theta$$

is $$\frac{r_1}{r} = \frac{b\cos\theta - a\sin\theta}{b\cos\theta_1 - a\sin\theta_1}$$

10. Find the length of the perpendicular from $(3, -4)$ on

$$4x + 2y = 7,$$

the axes being inclined at 60°.

Ans. $\frac{3}{4}$.

11. From the point $P\,(h, k)$ perpendiculars PM, PN are drawn to the axes. If the axes are inclined at ω show that

$$MN = \sin\omega\sqrt{h^2 + k^2 + 2hk\cos\omega}$$

[*Note*—$OM = h + k\cos\omega$, $ON = k + h\cos\omega$].

12. With the notation of the last question, the equation to the perpendicular from P on MN is

$$hx - ky = h^2 - k^2$$

If this perpendicular meet the bisectors of the angles between the axes in R, S, prove that

$$OR = 2(h + k)\cos\frac{\omega}{2}, \quad OS = 2(h - k)\sin\frac{\omega}{2}, \quad RS = 2\,MN\operatorname{cosec}\omega$$

CHAPTER IV

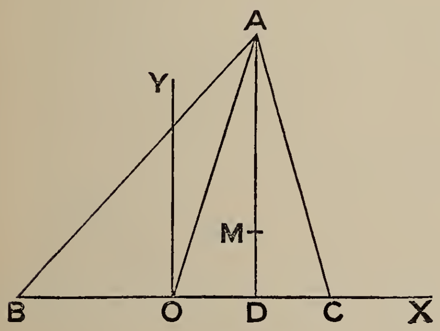
APPLICATIONS TO GEOMETRY

PROOF OF GEOMETRICAL THEOREMS

§ **94.** We shall now give some examples to illustrate the manner of applying the method of co-ordinates to the investigation of Geometrical Theorems.

§ **95.** Ex. 1. Prove that the perpendiculars from the vertices of a triangle ABC on the opposite sides are concurrent.

Take the mid point O of BC for origin and OC for axis of x.



Let the co-ord's of A be

$OD = h, \quad AD = k;$

let $BO = OC = a$

Thus C is $(a, 0)$

" B " $(-a, 0)$

" A " (h, k)

The eq'n to AC is

$$\begin{vmatrix} x & y & 1 \\ h & k & 1 \\ a & 0 & 1 \end{vmatrix} = 0, \quad \text{or} \quad kx - (h - a)y - ak = 0$$

The eq'n to the $\perp$ from B $(-a, o)$ on this is

$$\frac{x + a}{k} = \frac{y}{-(h - a)}$$

$$\text{or} \quad (h - a)x + ky + (h - a)a = o \quad . \; . \; . \; . \; . \; . \quad (1)$$

Similarly we obtain the eq'n of the $\perp$ from C on AB

$$(h + a)x + ky - (h + a)a = o \quad . \; . \; . \; . \; . \; . \quad (2)$$

The equation of the $\perp$ from A on BC, i.e. of AD is

$$x = h \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

To ascertain whether the three lines (1), (2), (3) meet in one point we must solve two of the eq'ns for x, y and ascertain if the resulting values of x, y satisfy the third.

From (2), (3) we obtain

$$\left.\begin{aligned} x &= h \\ y &= \frac{a^2 - h^2}{k} \end{aligned}\right\}$$

$\therefore$ These values satisfy (1). Q.E.D.

Cor'—Let M be the point of intersection of the perpendiculars (the *orthocentre*). We have its y, viz.

$$DM = \frac{(a + h)(a - h)}{k} = \frac{BD . DC}{AD}$$

$$\therefore \quad DM . DA = BD . DC$$

§ **96.** Ex. 2. Show that the medians of ABC are concurrent.

The mid point of AC is $\left(\frac{a + h}{2}, \quad \frac{k}{2}\right)$

The eq'n to the join of this to B $(-a, o)$ is

$$\begin{vmatrix} x & y & 1 \\ \frac{a + h}{2} & \frac{k}{2} & 1 \\ -a & o & 1 \end{vmatrix} = o$$

$$\text{or} \quad kx - (3a + h)y + ak = o \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Similarly the join of the mid point of AB to C is

$$kx + (3a - h)y - ak = o \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The third median is AO; its equation is

$$\frac{y}{x} = \frac{k}{h}, \quad \text{or} \quad hy - kx = 0 \quad \dots\dots\dots \quad (3)$$

Solving (2), (3) we get

$$x = \frac{h}{3}, \qquad y = \frac{k}{3};$$

these values satisfy (1). Q.E.D.

§ **97.** Ex. 3. The perpendiculars to the sides of ABC at their mid points are concurrent.

The ⊥ to AC through its mid point is

$$\frac{x - \frac{a+h}{2}}{k} = \frac{y - \frac{k}{2}}{a - h}$$

$$\text{or} \quad (a - h)x - ky - \frac{a^2 - h^2 - k^2}{2} = 0$$

Similarly the ⊥ to AB through its mid point is

$$(a + h)x + ky + \frac{a^2 - h^2 - k^2}{2} = 0$$

The ⊥ to BC through its mid point is

$$x = 0$$

These three lines concur in the point

$$\left.\begin{aligned} x &= 0 \\ y &= \frac{h^2 + k^2 - a^2}{2k} \end{aligned}\right\}$$

CHOICE OF AXES

§ **98.** We may select any two lines in our figure for axes of co-ordinates. Much of the elegance and brevity of a solution depends upon a judicious choice. To illustrate this remark we add two other proofs of the theorem of § 96.

§ **99.** **First Proof.** Take OC for axis of x and OA for axis of y (fig' § 95); these axes are usually oblique.

Let OC = a, OA = b.

Then C is (a, o), B is (– a, o) and A is (o, b)

The mid point of AC is $\left(\frac{a}{2}, \frac{b}{2}\right)$, and the join of this to B is

$$\begin{vmatrix} x & y & 1 \\ \frac{a}{2} & \frac{b}{2} & 1 \\ -a & 0 & 1 \end{vmatrix} = 0, \quad \text{or} \quad bx - 3ay + ab = 0 \quad . \quad . \quad . \quad (1)$$

Similarly the join of the mid point of AB to C is

$$bx + 3ay - ab = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The third median is AO; its equation is

$$x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The three lines (1), (2), (3) concur in the point

$$x = 0, \qquad y = \frac{b}{3}$$

This proof is somewhat shorter than that in § 96.

§ 100. Second Proof. Take any lines as axes of co-ordinates; let the co-ordinates of A be $(x_1\, y_1)$, of B $(x_2\, y_2)$, and of C $(x_3\, y_3)$

The mid point of BC is

$$\left(\frac{x_2 + x_3}{2}, \quad \frac{y_2 + y_3}{2}\right)$$

The join of A to this point is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2 + x_3}{2} & \frac{y_2 + y_3}{2} & 1 \end{vmatrix} = 0$$

or $$x\,(2\,y_1 - y_2 - y_3) - y\,(2\,x_1 - x_2 - x_3) + x_1\,(y_2 + y_3) \\ - y_1\,(x_2 + x_3) = 0 \quad . \quad . \quad . \quad (1)$$

By *symmetry* we may write the eq′ns to the other two medians

$$x\,(2\,y_2 - y_3 - y_1) - y\,(2\,x_2 - x_3 - x_1) + x_2\,(y_3 + y_1) \\ - y_2\,(x_3 + x_1) = 0 \quad . \quad . \quad . \quad (2)$$

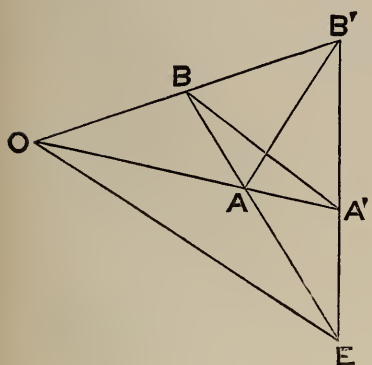
$$x\,(2\,y_3 - y_1 - y_2) - y\,(2\,x_3 - x_1 - x_2) + x_3\,(y_1 + y_2) \\ - y_3\,(x_1 + x_2) = 0 \quad . \quad . \quad . \quad (3)$$

It will be found that (1), (2), (3) concur in the point

$$\tfrac{1}{3}(x_1 + x_2 + x_3), \qquad \tfrac{1}{3}(y_1 + y_2 + y_3)$$

Although longer, this solution has the advantage of symmetry.

§ **101.** Ex. 4. The mid points of the three diagonals of a quadrilateral ABB′A′ are collinear.



Take OA, OB as axes.

Let $OA = a$, $OB = b$, $OA' = a'$, $OB' = b'$

Then A is (a, o) and B′ is (o, b')

$\therefore$ mid p′t of AB′ is $\left(\frac{a}{2}, \frac{b'}{2}\right)$

Sim′y mid p′t of A′B is $\left(\frac{a'}{2}, \frac{b}{2}\right)$

$$\left.\begin{array}{ll} \text{Again, eq'n to AB is} & \frac{x}{a} + \frac{y}{b} = 1 \\ \quad\text{,,} \quad \text{A'B' ,,} & \frac{x}{a'} + \frac{y}{b'} = 1 \end{array}\right\}$$

Solving these we get the co-ord′s of E and thence those of the mid p′t of OE, viz.

$$\frac{aa'(b' - b)}{2(ab' - a'b)}, \qquad \frac{bb'(a - a')}{2(ab' - a'b)}$$

It may now be verified as in § 24 that the three points are collinear.

Or thus: Observe that the values just obtained are the co-ord′s of the p′t which divides the join of $\left(\frac{a}{2}, \frac{b'}{2}\right)$ and $\left(\frac{a'}{2}, \frac{b}{2}\right)$ in the ratio

$$ab' : - a'b \quad (\S\ 16)$$

LOCI

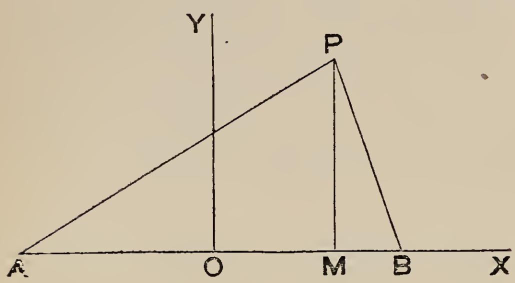
§ **102.** In § 13, Ex. 5, we have found the equation of the locus of a point equidistant from two given points (1, 2) and (3, 4). In this case we know from Elementary Geometry that the locus is the line which bisects at right angles the join of the given points.

This might be verified in the present instance by forming the equation of the line through the mid point of the join of (1, 2) and (3, 4) perpendicular to this join.

In most cases it is easy to translate a geometrical condition into an equation in x and y. We now give further examples.

§ 103. Ex. 1. A and B are two given points; find the locus of P if

$$PA^2 - PB^2 = \text{a constant} = k^2$$



Take the mid point O of AB for origin, and OB for axis of x.

Let $AB = 2a$.

Let P, (x, y) be a point on the locus.

Then A is $(-a, o)$, and B is (a, o)

$$\therefore \quad PA^2 = (x+a)^2 + y^2, \quad PB^2 = (x-a)^2 + y^2 \qquad (\S\ 10)$$

$$\therefore \quad PA^2 - PB^2 = 4ax$$

$$\therefore \text{ the equation to the locus is } \quad 4ax = k^2, \quad \text{or} \quad x = \frac{k^2}{4a}$$

Thus the locus is a straight line $\perp$ AB.

Ex. 2. Find locus of P if

$$PA^2 + PB^2 = k^2.$$

Using fig′ and notation of Ex. 1, the equation to the locus is

$$2x^2 + 2a^2 + 2y^2 = k^2$$

$$\text{or} \quad x^2 + y^2 = \frac{k^2}{2} - a^2$$

This expresses that the distance of x, y from the origin is

$$\sqrt{\frac{k^2}{2} - a^2};$$

or the locus is a circle centre O and radius

$$\sqrt{\frac{k^2}{2} - a^2}$$

Ex. 3. Find the locus of P if

$$PA = n\,PB.$$

Here $$PA^2 = n^2\,PB^2;$$

$\therefore$ eq'n to locus is $(x + a)^2 + y^2 = n^2[(x - a)^2 + y^2]$

It will be seen hereafter that this represents a circle.

Exercises

1. A point moves so that its distance from (1, 2) = its distance from (3, − 4); find its locus.

Ans. The straight line $x - 3y = 5$.

2. The co-ord's of A are (3, 0) and of B (−3, 0); a point P moves so that

$$PA^2 + PB^2 = 50;$$

find the locus of P.

Ans. The circle $x^2 + y^2 = 16$.

3. The co-ord's of A are (− a, 0) and of B are (a, 0); if

$$PA = 2\,PB$$

find the eq'n to the locus of P.

Ans. $3x^2 + 3y^2 - 10ax + 3a^2 = 0$

4. Find the eq'n to the locus of a point which moves so that its distance from the origin = twice its distance from the axis of x.

Ans. The two lines $x \pm y\sqrt{3} = 0$.

5. Find the eq'n to the locus of a point which moves so that its distance from the axis of y = its distance from the point (1, 0).

Ans. $y^2 - 2x + 1 = 0$.

§ 104. Ex. 4. Find the equation to the locus of P if

$$PA + PB = \text{constant} = 2a.$$

As before take the mid point O of AB for origin and OB for axis of x.

Let $AB = 2c$; then A is (− c, 0) and B is (c, 0)

The equation to the locus is

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

or $$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}$$

Square both sides, cancel and transpose; we get

$$4a\sqrt{(x-c)^2+y^2} = 4a^2 - 4cx$$

Divide by 4 and square again:

$$a^2(x-c)^2 + a^2y^2 = (a^2-cx)^2$$

This gives $(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$

Divide by $(a^2-c^2)a^2$, and put $a^2-c^2=b^2$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This equation represents an ellipse; a curve which will be discussed in Chap. IX.

§ 105. Ex. 5. A, B, C are three given points: find the locus of P if

$$PB^2 + PC^2 = 2PA^2.$$

Take any two lines as axes of co-ordinates.

Let the co-ord's of A be $(x_1 y_1)$, of B $(x_2 y_2)$ and of C $(x_3 y_3)$.

Let P be a point on the locus; x, y its co-ord's.

Then $PA^2 = (x-x_1)^2 + (y-y_1)^2$, $PB^2 =$ &c., $PC^2 =$ &c.

and the equation to the locus is

$$(x-x_2)^2 + (y-y_2)^2 + (x-x_3)^2 + (y-y_3)^2 = 2(x-x_1)^2 + 2(y-y_1)^2$$

or

$$(4x_1-2x_2-2x_3)x + (4y_1-2y_2-2y_3)y = 4x_1^2+4y_1^2-x_2^2-y_2^2-x_3^2-y_3^2$$

This equation being of the first degree in (x, y) represents a straight line.

Otherwise thus.

Take the mid point of BC for origin and OC for axis of x (see fig' § 95); let $BC = 2a$, $OD = h$, $AD = k$.

Then C is $(a, 0)$, B is $(-a, 0)$ and A is (h, k).

Thus

$$PB^2 = (x+a)^2 + y^2, \quad PC^2 = (x-a)^2 + y^2, \quad PA^2 = (x-h)^2 + (y-k)^2$$

and the equation to the locus is

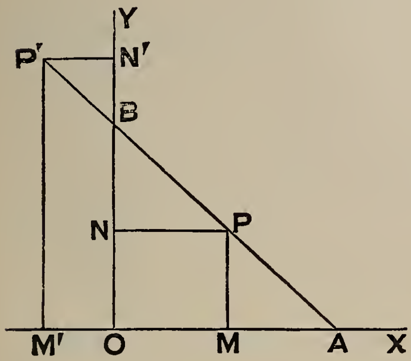
$$2x^2 + 2a^2 + 2y^2 = 2(x-h)^2 + 2(y-k)^2$$

$$\text{or} \quad 2xh + 2yk = h^2 + k^2 - a^2$$

This represents a straight line $\perp$ OA.

We have here another instance of the advantage of a judicious choice of axes.

§ **106.** Ex. 6. Find the locus of a point P such that if PM, PN be the perpendiculars on the axes the sum of PM, PN is constant and $= a$.



Here $PM = y$, $PN = x$; and *the word sum in the question being understood to mean algebraic sum* the equation to the locus is

$$x + y = a$$

This is the equation to a straight line whose intercepts on the axes are

$$OA = a, \quad OB = a.$$

As an illustration of the force of the words in italics we give the following discussion by Elementary Geometry.

On the axes cut off $OA = a$, $OB = a$, and let P be a point on AB between A and B.

Then since $OA = OB$

$$\widehat{OAB} = \widehat{OBA} = \widehat{MPA}$$

$$\therefore \quad PM = MA$$

$$\therefore \quad PM + PN = MA + OM$$

$$= OA$$

Again, taking a point P′ on AB produced

$$P'N' = BN'$$

and

$$P'M' + (-P'N') = P'M' - P'N'$$

$$= P'M' - N'B$$

$$= OB$$

Thus it is the *arithmetical difference* of the perpendiculars from P′ that is equal to a.

§ **107.** Ex. 7. Find the locus of a point whose distances from the lines

$$ax + by + c = 0, \quad a'x + b'y + c' = 0$$

are in a given ratio $m : n$.

By § 76 the locus consists of the two straight lines

$$n \cdot \frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm m \cdot \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

Ex. 8. A, B, C, D are given points; find the locus of a point P which moves so that

$$\triangle PAB + \triangle PCD = \text{constant} = k^2.$$

Draw $PM \perp AB$ and $PN \perp CD$ (the reader can easily supply the figure)

Let $AB = h, \quad CD = h'$.

Let the eq'n to AB be $x \cos \alpha + y \sin \alpha - p = 0$

" " CD " $x \cos \alpha' + y \sin \alpha' - p' = 0$

Then $$AB . PM + CD . PN = 2 k^2$$

But $PM = x \cos \alpha + y \sin \alpha - p$, $PN = x \cos \alpha' + y \sin \alpha' - p'$ (§ 78)

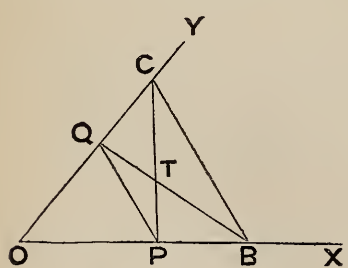
Thus the locus is the straight line

$$h (x \cos \alpha + y \sin \alpha - p) + h' (x \cos \alpha' + y \sin \alpha' - p') = 2 k^2$$

Note—As in § 106, by *sum* is meant *algebraic sum.*

CASES IN WHICH FORMULAE ARE NOT IMMEDIATELY APPLICABLE

§ 108. Ex. 9. A straight line moves parallel to the base BC of a given triangle ABC, and cuts the sides in P, Q; BQ, PC are joined: find the locus of their point of intersection T.



Take OB, OC for axes.

Let $OB = b$, $OC = c$.

We cannot as in the preceding Examples replace at once the geometrical statement by a relation between x, y the co-ordinates of T.

We may proceed thus :

Let the co-ord's of T be (α, β)

Then the eq'n to CT is

$$\begin{vmatrix} x & y & 1 \\ \alpha & \beta & 1 \\ 0 & c & 1 \end{vmatrix} = 0, \quad \text{or} \quad (\beta - c)x - \alpha y + \alpha c = 0$$

The eq'n to OB is $y = 0$.

Combining these eq'ns we get the co-ord's of P, $\left(\frac{\alpha c}{c-\beta}, 0\right)$

Similarly we get the co-ord's of Q, $\left(0, \frac{\beta b}{b-\alpha}\right)$

Thus the eq'n to PQ is

$$\frac{x(c-\beta)}{\alpha c} + \frac{y(b-\alpha)}{\beta b} = 1$$

Now express the condition that this is $\parallel$ BC, whose eq'n is $\frac{x}{b} + \frac{y}{c} = 1$. This gives

$$\frac{b(c-\beta)}{\alpha c} = \frac{c(b-\alpha)}{\beta b}$$

This is a relation between α, β the co-ordinates of T: we may now replace α, β by x, y and the equation to the locus of T is

$$\frac{b(c-y)}{cx} = \frac{c(b-x)}{by}$$

This reduces to $(cx - by)(cx + by - bc) = 0$

The factor $cx + by - bc = 0$ is the equation to BC; rejecting this irrelevant factor the locus is the straight line through O and the mid point of BC,

$$cx - by = 0$$

N.B. We may account for the irrelevant factor thus. If we take any point in BC and join this to C and B, and if the joins meet OB, OC in P, Q; then PQ coincides with BC and $\therefore$ fulfils the condition of being parallel to it. In this sense then BC is part of the locus.

LOCUS FOUND BY ELIMINATION

§ 109. We may solve the last example in another way.

Take as before OB, OC for axes. Let OB = b, OC = c.

Then putting OP = μb we have OQ = μc.

The eq'n to CP is $\frac{x}{\mu b} + \frac{y}{c} = 1$ . . . . (1)

" BQ is $\frac{x}{b} + \frac{y}{\mu c} = 1$ . . . . (2)

Now T is on CP and also on BQ; thus each of the eq'ns (1), (2) states a fact about the co-ord's of T depending on the individual value of μ.

If we eliminate μ from (1), (2) we obtain a relation between x *and* y *which is independent of μ: this relation is the equation to the locus.*

μ is eliminated by subtracting the eq'ns: this gives

$$\left(\frac{x}{b}-\frac{y}{c}\right)\left(1-\frac{1}{\mu}\right)=0$$

Thus the locus is

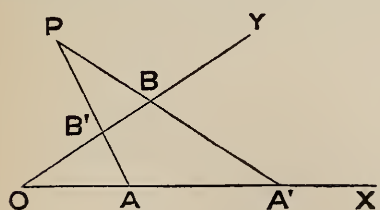
$$\frac{x}{b}-\frac{y}{c}=0$$

We shall frequently in the sequel obtain the equation to a locus by elimination: the reader should consider the above illustration carefully.

§ **110.** Ex. 10. A and B are fixed points one on each of the axes: if A′ and B′ be taken on the axes so that

$$OA' + OB' = OA + OB;$$

find the locus of the intersection of AB′, A′B.



Let OA = a, OB = b; put AA′ = k

The eq'n to AB′ is

$$\frac{x}{a}+\frac{y}{b-k}=1$$

or $$bx + ay + (a - x)k = ab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The eq'n to A′B is $\dfrac{x}{a+k}+\dfrac{y}{b}=1$

or $$bx + ay + (y - b)k = ab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

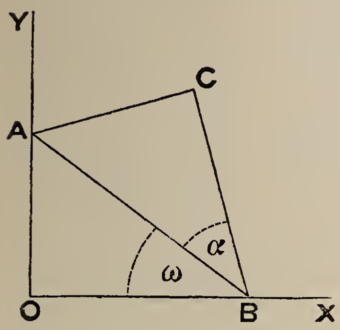
The equation to the locus is got by eliminating k from (1), (2).

(1) – (2) gives $\quad (a - x - y + b)k = 0$

$\therefore$ the locus is the straight line

$$x + y = a + b$$

Ex. 11. The extremities A, B of the hypotenuse of a right-angled triangle ABC move on two rectangular axes OX, OY; find the locus of C.



Let $AB = h$, $O\hat{B}A = \omega$, $A\hat{B}C = \alpha$.

Then $BC = h \cos \alpha$

Let x, y be the co-ordinates of C.

Project OC and OBC on OX and OY (§ 60);

$$\therefore \quad x = h \cos \omega - h \cos \alpha \cos(\omega + \alpha) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = h \cos \alpha \sin(\omega + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The eq'n to the locus is obtained by elim'g ω from (1), (2).

Now $$\omega = (\omega + \alpha) - \alpha,$$

$$h \cos \omega = h \cos(\omega + \alpha) \cos \alpha + h \sin(\omega + \alpha) \sin \alpha$$

Substitute this expression for $h \cos \omega$ in (1); thus

$$x = h \sin \alpha \sin(\omega + \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2), (3) $y = x \cot \alpha$; or the locus is a straight line through O.

POLAR CO-ORDINATES

§ 111. If a straight line revolves round a fixed point and we require the locus of a point on the revolving line whose position on that line is defined by any law; it is advantageous to use polar co-ordinates.

Ex. 1. A straight line which revolves round a fixed point O meets a given line PM in P; on OP a point Q is taken such that

$$OP \, . \, OQ = k^2, \text{ a constant;}$$

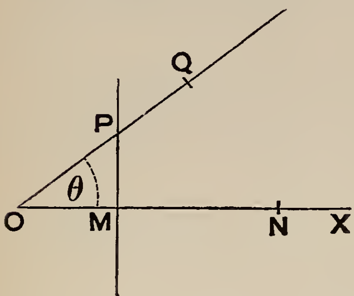
find the locus of Q.

Draw OM ⊥ the given line.

Let $OM = a$, $OQ = r$, $Q\hat{O}M = \theta$.

$$\text{Then} \quad OP \, . \, r = k^2, \qquad \therefore \; OP = \frac{k^2}{r}$$

Also $$a = OP \cos\theta = \frac{k^2}{r}\cos\theta$$



Thus the equation to the locus is

$$ar = k^2 \cos\theta$$

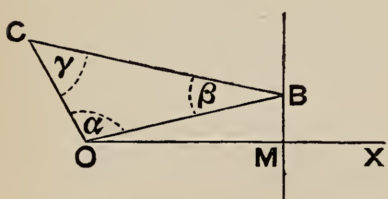
We may write this

$$ar^2 = k^2 (r\cos\theta)$$

or $$a(x^2 + y^2) = k^2 x$$

By Chap. VI. this equation represents a circle on ON as diameter, where N is a point on OM such that $ON = \dfrac{k^2}{a}$.

Ex. 2. One vertex O of a triangle OBC whose angles are given is fixed; another vertex B moves on a given line BM; find the locus of the third vertex C.



Take O as origin and $OM \perp$ the given line as initial line.

Let $OM = a$, $OC = r$, $C\hat{O}M = \theta$.

Then $a = OB \cos B\hat{O}M$.

But $$OB = OC \, . \, \frac{\sin\gamma}{\sin\beta}$$

and $$B\hat{O}M = \theta - \alpha$$

$$\therefore \text{ the eq'n to the locus is } \quad a = r\,\frac{\sin\gamma}{\sin\beta}\cos(\theta - \alpha)$$

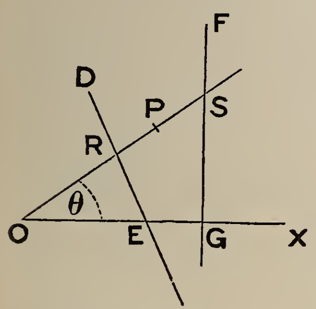
or $$\frac{a\sin\beta}{\sin\gamma} = r\cos\theta\cos\alpha + r\sin\theta\sin\alpha;$$

or $$x\cos\alpha + y\sin\alpha = a\,\frac{\sin\beta}{\sin\gamma}$$

$\therefore$ the locus is a straight line.

Ex. 3. A straight line revolves round a fixed point O and cuts two given lines DE, FG in R, S; find the locus of a point P on the revolving line such that

$$\frac{2}{OP} = \frac{1}{OR} + \frac{1}{OS}$$



Let (r, θ) be the polar co-ordinates of P.

Let the equation to DE be

$$a_1 x + b_1 y + c_1 = 0$$

and to FG

$$a_2 x + b_2 y + c_2 = 0$$

$\therefore$ the polar equation to DE is

$$a_1 r \cos\theta + b_1 r \sin\theta + c_1 = 0$$

$$\therefore \quad a_1 \cos\theta + b_1 \sin\theta + \frac{c_1}{r} = 0$$

$$\therefore \quad \frac{1}{r} = -\frac{a_1}{c_1}\cos\theta - \frac{b_1}{c_1}\sin\theta$$

That is

$$\frac{1}{OR} = -\frac{a_1}{c_1}\cos\theta - \frac{b_1}{c_1}\sin\theta$$

Similarly

$$\frac{1}{OS} = -\frac{a_2}{c_2}\cos\theta - \frac{b_2}{c_2}\sin\theta$$

Thus the polar equation to the locus of P is given by

$$\frac{2}{r} = \frac{2}{OP} = \frac{1}{OR} + \frac{1}{OS}$$

$$= -\left(\frac{a_1}{c_1} + \frac{a_2}{c_2}\right)\cos\theta - \left(\frac{b_1}{c_1} + \frac{b_2}{c_2}\right)\sin\theta$$

Multiply up by r and replace $r\cos\theta$, $r\sin\theta$ by x, y: we thus get

$$2 + \left(\frac{a_1}{c_1} + \frac{a_2}{c_2}\right)x + \left(\frac{b_1}{c_1} + \frac{b_2}{c_2}\right)y = 0$$

This may be written

$$\frac{a_1 x + b_1 y + c_1}{c_1} + \frac{a_2 x + b_2 y + c_2}{c_2} = 0$$

By Chap. V it is seen that the straight line represented by this equation passes through the intersection of the given lines.

Exercises on Chapter IV

1. ACB is a triangle in which $\hat{C}$ is a right angle. Squares ACED, BCGH are described as in the figure to Euclid I. 47. Show that BD, AH meet on the perpendicular from C on AB.

2. PQ, RS are parallels to adjacent sides of a parallelogram. Show that PR, QS meet on a diagonal.

3. The base AB of a triangle is given. Taking $(-a, 0)$ and $(+a, 0)$ as the co-ord's of A, B, find the equation to the locus of the vertex C

1°, If $\cot A + m \cot B = k$

2°, If $A - B = D$

3°, If $B = 2A$

4°, If $m\,AC^2 + n\,BC^2 = k^2$

Ans. 1°, The straight line $(1 - m)x - ky + (1 + m)a = 0$

2°, $x^2 - 2xy \cot D - y^2 = a^2$

3°, $3x^2 - y^2 + 2ax = a^2$

4°, $(m + n)(x^2 + y^2) + 2(m - n)ax + (m + n)a^2 = k^2$

4. Points P, Q are taken on the sides AB, AC of a triangle ABC such that

$$AP . AQ = BP . CQ;$$

find locus of mid point of PQ.

[Take AB, AC as axes; let $AB = a$, $AC = b$. If mid point of PQ is (xy) then $AP = 2x$, $AQ = 2y$; $\therefore 4xy = (a - 2x)(b - 2y)$

$\therefore$ locus is the straight line $2bx + 2ay = ab$]

5. Given base and sum of areas of a number of triangles with a common vertex P: show that the locus of P is a straight line.

6. From a point P perpendiculars PM, PN are drawn to the axes (which include $\hat{\omega}$); if locus of P is a straight line show that locus of mid point of MN is a straight line.

[*Note*—If P is (xy) and mid point of MN is (hk): then

$$OM = x + y \cos\omega = 2h, \quad ON = y + x \cos\omega = 2k]$$

7. With the notation of the last question, if OM + ON is given ($= k$), find the locus of P.

Ans. The straight line $(x + y)(1 + \cos\omega) = k$.

8. A parallel to the base of a triangle meets the sides in B′, C′; R, S are fixed points on the base: show that the locus of the intersection of B′R, C′S is a straight line.

Note—Let B be $(-a, 0)$, C $(+a, 0)$, A (h, k), R $(r, 0)$ and S $(s, 0)$. Let eq′n to B′C′ be $y = \lambda$.

We find that

$$\text{eq'n to } C'S \text{ is } \quad \lambda[y(h + a) + k(s - x)] = ky(s + a)$$

$$\text{,,} \quad B'R \text{ ,,} \quad \lambda[y(h - a) + k(r - x)] = ky(r - a)$$

Elim′ λ: locus is

$$(r - a)[y(h + a) + k(s - x)] = (s + a)[y(h - a) + k(r - x)]$$

9. A and B are fixed points: if PA, PB intercept a constant length c on a given line, find the equation to the locus of P.

Note—Let AB meet given line in O; take given line and OAB as axes. Let $OA = a$, $OB = b$.

Ans. $c(a - y)(b - y) = (a - b)xy$

10. A line AB of constant length ($= c$) slides between the axes, which include $\hat{\omega}$: show that the equation to the locus of the orthocentre of the triangle OAB is

$$x^2 + 2xy\cos\omega + y^2 = c^2\cot^2\omega$$

11. On a line which revolves round the origin and cuts the lines

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0$$

in P, Q a point R is taken such that

$$OR = OP + OQ:$$

show that the equation to the locus of R is

$$(ax + by + c)(a'x + b'y + c') = cc'$$

12. Show that the orthocentre, centroid and circum centre of a triangle are collinear.

13. OIJ, OLM are given straight lines; I, J are fixed points; IL, JM meet at P. Prove that P describes a straight line if

$$\frac{a}{OL} + \frac{b}{OM} = 1,$$

where a and b are constants.

14. The opposite sides of a quadrilateral meet at P, Q: if the internal bisectors of the angles P, Q are at right angles, prove that their intersection lies on the join of the mid points of the diagonals.

[*Note*—Take the bisectors as axes; then sides of quad' are

$$y = mx + c, \quad y = -mx - c, \quad x = \mu y + k, \quad x = -\mu y - k$$

We find that mid points of diag's are

$$\left(\frac{\mu c}{1 - m\mu}, \frac{mk}{1 - m\mu}\right), \quad \left(\frac{-\mu c}{1 + m\mu}, \frac{-mk}{1 + m\mu}\right);$$

these lie on the line $y/x = mk/\mu c$]

CHAPTER V

EQUATIONS WITH LINEAR FACTORS. ABRIDGED NOTATION; RANGES AND PENCILS

CASE OF FACTORS

§ 112. An equation which splits into factors of the first degree represents straight lines.

Ex. 1. What is represented by

$$x^2 - 5xy + 6y^2 = 0?$$

Here $$(x - 2y)(x - 3y) = 0$$

$$\therefore \text{ either } \quad x - 2y = 0 \quad \text{or} \quad x - 3y = 0$$

Thus the co-ord's of every point on either of the lines

$$x - 2y = 0, \qquad x - 3y = 0$$

satisfy the given equation; which $\therefore$ represents this pair of straight lines.

Ex. 2. What is represented by

$$x^3 - 6ax^2 + 11a^2x - 6a^3 = 0?$$

Here $$(x - a)(x - 2a)(x - 3a) = 0;$$

$\therefore$ the equation represents the three lines

$$x = a, \qquad x = 2a, \qquad x = 3a$$

Ex. 3. What is represented by

$$2xy - 8x + 3y - 12 = 0?$$

$$2xy - 8x + 3y - 12 \equiv y(2x + 3) - 4(2x + 3)$$
$$\equiv (y - 4)(2x + 3)$$

Ans. The two lines $y = 4$, $2x + 3 = 0$.

Ex. 4. Show that the equation

$$6x^2 - xy - y^2 - x + 3y - 2 = 0$$

represents two straight lines.

Arrange according to powers of x, and solve for x.

$$\therefore \quad 6x^2 - (y + 1)x - y^2 + 3y - 2 = 0$$

$$\therefore \quad x = \frac{y + 1 \pm \sqrt{(y + 1)^2 + 24(y^2 - 3y + 2)}}{12}$$

$$= \frac{y + 1 \pm \sqrt{25y^2 - 70y + 49}}{12}$$

$$= \frac{y + 1 \pm (5y - 7)}{12}$$

$$= \frac{y - 1}{2} \quad \text{or} \quad \frac{2 - y}{3}$$

Thus the given equation implies that either

$$x = \frac{y - 1}{2} \quad \text{or} \quad x = \frac{2 - y}{3};$$

i. e. it represents the two lines

$$2x - y + 1 = 0, \qquad 3x + y - 2 = 0.$$

Note—The quantity under the radical, viz.

$$25y^2 - 70y + 49$$

turned out to be a perfect square: had it been otherwise the equation would not represent straight lines.

§ 113. *The homogeneous equation of the second degree*

$$ax^2 + 2hxy + by^2 = 0$$

represents two straight lines through the origin.

Dividing by x^2,

$$b\left(\frac{y}{b}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

If m_1, m_2 are the roots of this quadratic in $\frac{y}{x}$

$$m_1 = \frac{-h + \sqrt{h^2 - ab}}{b} \quad \text{and} \quad m_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

Thus the equation implies that either $\frac{y}{x} = m_1$ or $\frac{y}{x} = m_2$;
i. e. it represents the two lines

$$y = m_1 x, \quad y = m_2 x.$$

§ **114.** If ϕ is the angle between the lines

$$ax^2 + 2\,hxy + by^2 = 0$$

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2} \qquad (\S\ 68)$$

$$= \frac{2\sqrt{h^2 - ab}}{b} \div \left(1 + \frac{a}{b}\right)$$

$$= \frac{2\sqrt{h^2 - ab}}{a + b}$$

Cor' (1)—The lines are $\perp$ if $a + b = 0$

Cor' (2)—Any two lines at right angles through the origin may be represented by

$$x^2 + \lambda\,xy - y^2 = 0$$

§ **115.** If the axes are oblique

$$\tan\phi = \frac{(m_1 - m_2)\sin\omega}{1 + (m_1 + m_2)\cos\omega + m_1 m_2} \qquad (\S\ 93)$$

$$= \frac{2\sin\omega\sqrt{h^2 - ab}}{a + b - 2\,h\cos\omega}$$

Cor'—The lines are $\perp$ if

$$a + b - 2\,h\cos\omega = 0$$

§ **116.** *To find the equation to the bisectors of the angles between the lines* $ax^2 + 2\,hxy + by^2 = 0$

The bisectors of the angles between

$$y - m_1 x = 0, \qquad y - m_2 x = 0$$

are

$$\frac{y - m_1 x}{\sqrt{1 + m_1^2}} - \frac{y - m_2 x}{\sqrt{1 + m_2^2}} = 0, \qquad \frac{y - m_1 x}{\sqrt{1 + m_1^2}} + \frac{y - m_2 x}{\sqrt{1 + m_2^2}} = 0$$

Multiplying, the equation to the pair of bisectors is

$$\frac{(y - m_1 x)^2}{1 + m_1^2} - \frac{(y - m_2 x)^2}{1 + m_2^2} = 0$$

or
$$(m_2^2 - m_1^2)(y^2 - x^2) + 2(m_1 m_2 - 1)(m_1 - m_2)xy = 0$$

or
$$(m_2 + m_1)(y^2 - x^2) - 2(m_1 m_2 - 1)xy = 0$$

or
$$-\frac{2h}{b}(y^2 - x^2) - 2\left(\frac{a}{b} - 1\right)xy = 0$$

$$\therefore \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

This result should be remembered.

§ 117. *To find the condition that the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent a pair of lines.

Rearrange,

$$ax^2 + 2(hy + g)x + by^2 + 2fy + c = 0$$

Solving for x,

$$ax + hy + g = \pm\sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}$$
$$= \pm\sqrt{(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac}$$

The expression under the radical is a perfect square if

$$(h^2 - ab)(g^2 - ac) = (gh - af)^2$$

or $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad . \quad . \quad . \quad (1)$

This is the required condition; it should be remembered.

§ **118.** The quantity

$$abc + 2fgh - af^2 - bg^2 - ch^2$$

is called the *discriminant* of the expression *

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

The discriminant is usually denoted by Δ.

The condition (1) then is $\Delta = 0$

We have also

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Ex. Determine λ so that

$$x^2 + \lambda xy - 8y^2 + 12y - 4 = 0$$

may represent a line-pair.

$$\text{Here} \quad a = 1 \qquad f = 6$$
$$b = -8 \qquad g = 0$$
$$c = -4 \qquad h = \tfrac{1}{2}\lambda$$
$$\therefore \quad \Delta = 32 - 36 + \lambda^2 = \lambda^2 - 4$$
$$\text{Put} \quad \Delta = 0; \qquad \therefore \quad \lambda = +2 \text{ or } -2$$

§ **119.** If $$lx + my = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad (2)$$

then evidently

$$ax^2 + 2hxy + by^2 + 2(gx + fy)(lx + my)$$
$$+ c(lx + my)^2 = 0 \quad . \quad . \quad (3)$$

* The notation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will be invariably used to represent the general equation of the second degree.

If we introduce the linear unit $z = 1$ so as to render the expression homogeneous it assumes the symmetrical form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0;$$

this remark will assist the learner to remember the notation.

Since the values of x and y which satisfy (1) and (2) also satisfy (3), the locus represented by (3) passes through the points of intersection of the loci represented by (1), (2).

Also since (3) is homogeneous and of the second degree, it represents a pair of lines through the origin: $\therefore$ (3) represents the pair of lines joining the origin to the inters'ns of the loci represented by (1), (2).

Ex. The line-pair which joins the origin to the inters'ns of $2x + y = 6$, $x^2 + y^2 = x + 5y + 6$ is

$$x^2 + y^2 = (x + 5y)\left(\frac{2x + y}{6}\right) + 6\left(\frac{2x + y}{6}\right)^2$$

or reducing, $xy = 0$; i.e. the axes.

Exercises

1. Show that $3x^2 + 2xy - 3y^2 = 0$ represents two lines at right angles; and $x^2 + 2xy + y^2 - x - y - 6 = 0$ two parallel lines.

2. Interpret the equations

$$(x-3)(y-4) = 0, \quad x^2 - 3ax + 2a^2 = 0, \quad xy = 0, \quad x^2 - y^2 = 0,$$
$$xy - 2x + 3y - 6 = 0, \quad 6xy + 2bx + 3ay + ab = 0.$$

3. Show that the angle between the lines $6x^2 - xy - y^2 = 0$ is $\frac{\pi}{4}$.

4. Show that $3x^2 + xy - 2y^2 + x + 6y - 4 = 0$ represents two lines inclined at $\tan^{-1} 5$.

5. Show that the angle between the lines $x^2 - 2xy\sec\theta + y^2 = 0$ is θ.

6. Determine λ so that

$$6xy - 2x + \lambda y + 5 = 0$$

may represent straight lines.

Ans. $\lambda = -15$.

7. Determine λ so that

$$x^2 + \lambda xy + y^2 - 5x - 7y + 6 = 0$$

may represent straight lines.

Ans. $\lambda = \frac{5}{2}$ or $\frac{10}{3}$.

8. Find the line-pair joining the origin to the intersections of

$$y + x = 2, \qquad x^2 + y^2 - 2x - 4y - 31 = 0.$$

Ans. $31x^2 + 74xy + 35y^2 = 0.$

9. Show that the lines joining the origin to the intersections of

$$y = \lambda(x - 4), \qquad y^2 = 4x$$

are at right angles.

Ans. The line-pair is $\lambda(x^2 - y^2) - xy = 0.$

10. Find the bisectors of the angle between the lines

$$3x^2 + 4xy - 5y^2 = 0.$$

Ans. $x^2 - 4xy - y^2 = 0.$

11. Find the bisectors of the angle between the lines

$$3x^2 + 8xy + 3y^2 = 0.$$

Ans. $x^2 - y^2 = 0.$

12. Show that the lines

$$ax^2 + 2hxy + by^2 = A(x^2 + y^2)$$

are equally inclined to the lines

$$ax^2 + 2hxy + by^2 = 0$$

[*Note*—They have the same bisectors of angles.]

IMAGINARY POINTS AND LINES

§ 120. Further examples.

Ex. 1. What is represented by $x^2 + y^2 = 0$?

Since the square of any positive or negative number is positive, $x^2 + y^2$ is never negative; it can only be zero if both $x = 0$, $y = 0$. It was ∴ formerly the custom to say that $x^2 + y^2 = 0$ represented the point $x = 0$, $y = 0$.

Another account may be given: $x^2 + y^2$ is the product of

$$x + y\sqrt{-1}, \qquad x - y\sqrt{-1}$$

Thus we may say that the equation represents the two imaginary straight lines

$$x + y\sqrt{-1} = 0, \qquad x - y\sqrt{-1} = 0$$

These imaginary lines intersect in the real point

$$x = 0, \qquad y = 0.$$

Ex. 2. Interpret

$$4(3x - y - 1)^2 + 3(x + y - 3)^2 = 0$$

The only real values of x, y which satisfy this are given by

$$3x - y - 1 = 0, \quad x + y - 3 = 0, \quad \text{i.e.} \quad x = 1, \quad y = 2$$

If $\therefore$ we do not admit imaginary lines the equation represents the point $(1, 2)$.

If imaginaries are admissible, then factorizing, we see that the equation represents the two imaginary lines

$$2(3x - y - 1) + \sqrt{-3}(x + y - 3) = 0$$

$$2(3x - y - 1) - \sqrt{-3}(x + y - 3) = 0$$

These intersect in the real point, $(x = 1, y = 2)$.

§ **121.** Let us now recur to the investigation of § 113.

If $h^2 - ab > 0$ the values obtained for m_1, m_2 are real and unequal.

„ $h^2 - ab = 0$ „ „ real and equal.

„ $h^2 - ab < 0$ „ „ imaginary.

Thus the two lines through the origin represented by

$$ax^2 + 2hxy + by^2 = 0$$

are real and different, coincident, or imaginary according as

$$h^2 - ab > = < 0$$

§ **122.** *A homogeneous equation of the* n^{th} *degree*

$$ay^n + bxy^{n-1} + cx^2y^{n-2} + \ldots + kx^{n-1}y + lx^n = 0 \quad . \quad . \quad . \quad (1)$$

represents n *straight lines through the origin.*

Dividing by x^n,

$$a\left(\frac{y}{x}\right)^n + b\left(\frac{y}{x}\right)^{n-1} + \ldots + l = 0$$

Let the roots of this equation in $\frac{y}{x}$ be $m_1, m_2, m_3, \ldots$

The equation then represents the n straight lines

$$y = m_1x, \quad y = m_2x, \ldots$$

Note—The sinister side of (1)

$$\equiv ax^n\left[\left(\frac{y}{x}\right)^n + \frac{b}{a}\left(\frac{y}{x}\right)^{n-1} + \dots + \frac{l}{a}\right]$$

$$\equiv ax^n\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\dots$$

$$\equiv a\,(y - m_1 x)\,(y - m_2 x)\,(y - m_3 x)\dots$$

Similarly it is proved that the equation

$$a\,(y - y_1)^n + b\,(y - y_1)^{n-1}(x - x_1) + c\,(y - y_1)^{n-2}(x - x_1)^2 + \dots + l\,(x - x_1)^n = 0$$

represents n straight lines through $(x_1 y_1)$.

LINE THROUGH INTERSECTION OF TWO GIVEN LINES

§ **123.** Consider the equation

$$Ax + By + C + \lambda\,(A'x + B'y + C') = 0 \quad . \quad . \quad (1)$$

I. This is of the first degree in x, y;

$\therefore$ it represents a straight line.

II. It is satisfied if

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0:$$

thus (1) represents a line through the inters'n of

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0$$

By giving λ a suitable value (1) may be made to represent *any* such line.

§ **124.** Thus, *to find equation to join of* $(x_1 y_1)$ *to intersection of*

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0,$$

λ is determined by

$$Ax_1 + By_1 + C + \lambda\,(A'x_1 + B'y_1 + C') = 0$$

Substitute the value of λ so determined in (1); $\therefore$ the eq'n req'd is

$$\frac{Ax + By + C}{Ax_1 + By_1 + C} = \frac{A'x + B'y + C'}{A'x_1 + B'y_1 + C'}$$

Note—The form

$$l\,(Ax + By + C) + m\,(A'x + B'y + C') = 0$$

by dividing by l and putting $\frac{m}{l} = \lambda$ is reducible to (1).

§ 125. Ex. Find equation of line $\perp$ $3x + 4y + 5 = 0$ through inters'n of

$$3x - y - 1 = 0, \qquad x + y - 3 = 0.$$

A line through this inters'n is

$$3x - y - 1 + \lambda\,(x + y - 3) = 0$$

$$\text{or} \quad (3 + \lambda)x + (\lambda - 1)y - 3\lambda - 1 = 0$$

This is $\perp$ $3x + 4y + 5 = 0$, if

$$3\,(3 + \lambda) + 4\,(\lambda - 1) = 0 \quad [\S\ 68,\ \text{Cor}'\ (3)]$$

$\therefore\ \lambda = -\frac{5}{7}$, and the eq'n req'd is

$$3x - y - 1 - \tfrac{5}{7}(x + y - 3) = 0$$

$$\text{or} \quad 4x - 3y + 2 = 0$$

Exercises

1. Find equation of join of $(1, -2)$ to inters'n of

$$3x - 4y + 5 = 0, \qquad 2x - 3y + 4 = 0$$

Ans. $x = 1$.

2. Find join of same intersection to origin.

Ans. $2x - y = 0$.

3. Find line through same inters'n $\perp$ $10x + 7y = 8$

Ans. $7x - 10y + 13 = 0$

4. Find the equation of join of origin to inters'n of

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \frac{x}{b} + \frac{y}{a} = 1$$

Ans. $x = y$.

5. Find the equation of join of origin to intersection of

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0$$

Ans. $(AC' - A'C)x + (BC' - B'C)y = 0$

6. Find the equations to diag's of ▱ whose sides are

$$3x - 2y = 1, \quad 4x - 5y = 6, \quad 3x - 2y = 2, \quad 4x - 5y = 3$$

Ans. $13x = 11y + 9$, $5x = y$.

7. Find the area of this ▱

Ans. $\frac{3}{7}$.

ABRIDGED NOTATION

§ **126.** It is often convenient to represent such an expression as $Ax + By + C$ by a single symbol.

Thus we may put

$$u \equiv Ax + By + C, \qquad v \equiv A'x + B'y + C'$$

The proposition of § 123 may then be stated thus.

If $u = 0$, $v = 0$ *represent straight lines then* $u + \lambda v = 0$ *represents a line through their intersection; and by giving a suitable value to* λ *it may be made to represent any such line.*

Thus $u + \lambda v = 0$ represents a ray of the pencil through the intersection of $u = 0$, $v = 0$.

§ **127.** Greek letters are used as abbreviations for expressions of the form $x \cos \alpha + y \sin \alpha - p$.

We put

$$\alpha \equiv x \cos \alpha + y \sin \alpha - p, \quad \beta \equiv x \cos \beta + y \sin \beta - p', \text{ \&c.}$$

Thus $\alpha + \beta = 0$, $\alpha - \beta = 0$ are the bisectors of the angles between $\alpha = 0$, $\beta = 0$ (§ 80, Cor')

No ambiguity results in practice from the double signification of the symbols α, β, &c.

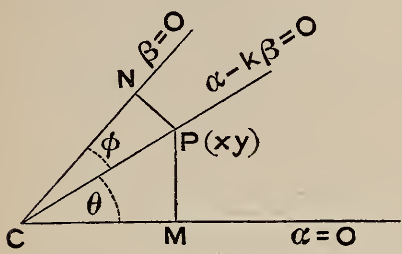
§ **128.** If we write at full length the equation

$$\alpha = k\beta$$

it becomes

$$x \cos \alpha + y \sin \alpha - p$$
$$= k(x \cos \beta + y \sin \beta - p') \quad . \quad . \quad . \quad (1)$$

Let P, (x, y) be any point on (1), and draw PM, PN respectively $\perp$ $\alpha = 0$, $\beta = 0$.



Then (1) expresses that

$$PM = k\,PN$$

$$\therefore \quad \frac{PM}{CP} = k\frac{PN}{CP}$$

$$\text{or} \quad \sin\theta = k\sin\phi$$

Thus $\alpha - k\beta = 0$ *divides the angle between* $a = 0$, $\beta = 0$ *into parts whose sines are in the ratio* $k : 1$.

Again, $u - kv = 0$ where

$$u \equiv Ax + By + C, \qquad v \equiv A'x + B'y + C'$$

may be written

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = k.\frac{\sqrt{A'^2 + B'^2}}{\sqrt{A^2 + B^2}}.\frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}$$

$$\text{or} \quad \alpha = k.\frac{\sqrt{A'^2 + B'^2}}{\sqrt{A^2 + B^2}}.\beta$$

Thus $u - kv = 0$ divides the angle between $u = 0$, $v = 0$ into parts whose sines are in the ratio

$$\frac{k\sqrt{A'^2 + B'^2}}{\sqrt{A^2 + B^2}} : 1$$

CONDITION FOR CONCURRENCE OF THREE LINES

§ 129. *If the equations to three lines* $u = 0$, $v = 0$, $w = 0$ *are connected by an identical relation* $lu + mv + nw \equiv 0$; *the three lines are concurrent.*

$$\text{For} \quad w \equiv -\frac{l}{n}.u - \frac{m}{n}.v;$$

thus the equation to the third line is

$$-\frac{l}{n}.u - \frac{m}{n}.v = 0$$

$$\text{or} \quad la + mv = 0$$

This passes through inters'n of $u = 0$, $v = 0$ (§ 123). Q.E.D.

Ex. 1. The lines

$$x - y + 1 = 0, \quad x + y - 3 = 0, \quad 2x - 3y + 4 = 0$$

are concurrent.

For multiplying the first equation by 5, the second by -1, and the third by -2, and adding, the result is identically zero,

$$\text{or} \quad 5u - v - 2w \equiv 0$$

Note—The beginner will see from this example that the proposition is sufficiently obvious. Thus multiplying the first equation by 5, and the second by -1 and adding, we get

$$4x - 6y + 8 = 0, \quad \text{or} \quad 2x - 3y + 4 = 0,$$

which is the third equation.

Thus the values of x, y which satisfy both of the first two equations must satisfy the third.

Ex. 2. The medians of the triangle $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ are concurrent.

Let u, v, w stand for the sinister sides of equations (1), (2), (3), § 100.

The medians are $u = 0$, $v = 0$, $w = 0$.

Adding the three equations we find that the sum vanishes identically;

$$\text{or} \quad u + v + w \equiv 0. \quad \text{Q.E.D.}$$

LINE THROUGH A FIXED POINT

§ **130.** We have seen that the line $u + \lambda v = 0$, where λ is a variable parameter, passes through a fixed point, viz. the intersection of $u = 0$, $v = 0$.

Thus if the equation to a line contain a variable parameter λ in the first degree, the line passes through a fixed point.

Ex. 1. A straight line moves so that the sum of the reciprocals of its intercepts on the axes is constant; show that it passes through a fixed point.

The equation to the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\text{Also} \quad \frac{1}{a} + \frac{1}{b} = \text{constant} = k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We may eliminate $\frac{1}{b}$ from (1) by means of (2); thus

$$\frac{1}{b} = k - \frac{1}{a}$$

and (1) becomes
$$\frac{x}{a} + y\left(k - \frac{1}{a}\right) = 1$$

or
$$\frac{1}{a}(x - y) + ky - 1 = 0$$

Whatever be the value of $\frac{1}{a}$ this equation is satisfied if

$$x - y = 0, \qquad ky - 1 = 0;$$

i. e. the line passes through the fixed point

$$x = \frac{1}{k}, \qquad y = \frac{1}{k}$$

Ex. 2. A straight line moves so as always to have two of the three points $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ on one side of it and the third point on the opposite side; and the sum of the $\perp$s on the movable line from the first two points = the $\perp$ from the third point: show that the line passes through a fixed point.

Let its equation in any position be

$$x \cos\alpha + y \sin\alpha - p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then the algebraic sum of the $\perp$s = 0

$$\therefore \quad (x_1 \cos\alpha + y_1 \sin\alpha - p) + (x_2 \cos\alpha + y_2 \sin\alpha - p) + (x_3 \cos\alpha + y_3 \sin\alpha - p) = 0$$

$$\text{or} \quad (x_1 + x_2 + x_3)\cos\alpha + (y_1 + y_2 + y_3)\sin\alpha - 3p = 0 \quad . \quad . \quad . \quad (2)$$

Eliminate p from (1) by means of (2); thus the equation to the movable line becomes

$$\left(x - \frac{x_1 + x_2 + x_3}{3}\right)\cos\alpha + \left(y - \frac{y_1 + y_2 + y_3}{3}\right)\sin\alpha = 0$$

$$\text{or} \quad \left(x - \frac{x_1 + x_2 + x_3}{3}\right) + \left(y - \frac{y_1 + y_2 + y_3}{3}\right)\tan\alpha = 0$$

Whatever $\tan\alpha$ may be, this passes through the fixed point

$$\tfrac{1}{3}(x_1 + x_2 + x_3), \qquad \tfrac{1}{3}(y_1 + y_2 + y_3)$$

THE LINE AT INFINITY

§ 131. The intercepts which $Ax + By + C = 0$ cuts off on the axes are $-\frac{C}{A}$, $-\frac{C}{B}$ (§ 52)

I. If A is very small, $-\frac{C}{A}$ is numerically very great; or the point where the line cuts OX is very distant from O. The line

$$Ax + By + C = 0$$

is then *nearly* parallel to OX.

Let now $A = 0$, then $\frac{C}{A} = \infty$; and the line

$$Ax + By + C = 0$$

is parallel to OX. This agrees with § 45.

II. If both A and B are very small, both the intercepts $-\frac{C}{A}$, $-\frac{C}{B}$ are numerically very large. Let now $A = 0$, $B = 0$; then the intercepts are infinite; or the line is *altogether* at infinity.

We see then that if a line cuts off infinite intercepts on both axes its equation is $0 . x + 0 . y + C = 0$.

This statement is usually abbreviated thus:

The equation to the line at infinity is

$$\text{constant} = 0.$$

§ 132. *Parallel lines intersect at infinity*

Let the equations to two lines be

$$Ax + By + C = 0 \quad \ldots\ldots\ldots \quad (1)$$

$$Ax' + By' + C' = 0 \quad \ldots\ldots\ldots \quad (2)$$

The equation to the line at infinity is

$$0 . x + 0 . y + C'' = 0 \quad \ldots\ldots\ldots \quad (3)$$

The condition that the lines (1), (2), (3) may be concurrent is

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ 0 & 0 & C'' \end{vmatrix} = 0 \quad (§\ 62)$$

$$\text{or} \quad (AB' - A'B)\, C'' = 0$$

$$\text{or} \quad AB' - A'B = 0$$

But this is cond'n that the lines (1), (2) may be $\parallel$ (§ 63)

Otherwise, solving (1), (2),

$$x = \frac{BC' - B'C}{AB' - A'B}, \qquad y = \frac{CA' - C'A}{AB' - A'B}$$

When the lines are $\parallel$,

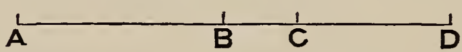
$$AB' - A'B = 0; \qquad \therefore \; x = \infty, \quad y = \infty$$

HARMONIC RANGES

§ 133. Def′—*Four collinear points* A, B, C, D *form a harmonic range if*

$$\frac{AB}{BC} = \frac{AD}{CD} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is AC is divided internally in B and externally in D in the same ratio. B and D are said to be *harmonically conjugate* to A and C.



For example, the internal and external bisectors of any angle of a triangle divide the opposite side harmonically (Euclid VI. 3).

From (1) we deduce:

I. Since $CD = -DC$, $\quad \therefore \; \dfrac{AB}{BC} \div \dfrac{AD}{DC} = -1.$

II. Multiplying up, $\qquad AD \,.\, BC = AB \,.\, CD.$

Or, *rectangle under whole line and middle segment = rectangle under extreme segments.*

III. AB, AC, AD are in harmonic progression; or $\dfrac{1}{AB}, \dfrac{1}{AC}, \dfrac{1}{AD}$ are in arithmetical progression.

For let $AB = x$, $AC = y$, $AD = z$.

Then (1) gives
$$\frac{x}{y - x} = \frac{z}{z - y}$$

$$\therefore \; x(z - y) = z(y - x)$$

Divide by xyz,

$$\therefore \; \frac{1}{y} - \frac{1}{z} = \frac{1}{x} - \frac{1}{y}$$

or
$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

That is, $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in A. P.

IV. Similarly DC, DB, DA are in H. P.

Also V. If M be the mid point of AC,

$$MB \cdot MD = MC^2.$$

This is proved in Nixon's *Euclid Revised*: page 360

VI. From V. we see that if the points A, C are given then B, D are on the same side of M; and if B approaches M indefinitely D moves off to infinity. Let ∞ denote the point at infinity on AC; then the points AMC ∞ form an harmonic range.

§ **134.** *The points on* OX *determined by*

$$ax^2 + 2hx + b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a'x^2 + 2h'x + b' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

are harmonic if $\quad ab' + a'b - 2hh' = 0.$

Let the roots of (1) be $OA = \alpha$, $OA' = \alpha'$; and of (2) $OB = \beta$, $OB' = \beta'$.

The condition is $\quad AB' \cdot BA' = AB \cdot A'B'$

or $\quad (\beta' - \alpha)(\alpha' - \beta) = (\beta - \alpha)(\beta' - \alpha')$

or $\quad 2\beta\beta' + 2\alpha\alpha' = (\alpha + \beta)(\alpha' + \beta')$

or $\quad 2\dfrac{b'}{a'} + 2\dfrac{b}{a} = \left(-\dfrac{2h}{a}\right)\left(-\dfrac{2h'}{a'}\right)$

$$\therefore \quad ab' + a'b - 2hh' = 0$$

*CROSS RATIOS

§ **135.** Def′—The *Cross Ratio* (or *Anharmonic Ratio*) in which two points A, C on a line are divided by two other points B, D on the same line is the ratio of the ratios in which AC is divided by B, D and is denoted by $\{ABCD\}$.

Thus $\quad \{ABCD\} = \dfrac{AB}{BC} \div \dfrac{AD}{DC} = \dfrac{AB \cdot DC}{BC \cdot AD}$

If $\{ABCD\} = -1$, the range is harmonic.

* The beginner may omit the rest of this Chapter.

Ex. If $\{ABCD\} = -1$, prove $\{BDAC\} = 2$. (See figure, § 133)

By Art. 9, Ex. 2,

$$AD . BC + BD . CA + CD . AB = 0$$

Divide by $AD . BC$;

$$\therefore \quad 1 - \frac{BD}{DA} \cdot \frac{CA}{BC} - \frac{AB}{BC} \cdot \frac{DC}{AD} = 0$$

or

$$1 - \{BDAC\} - \{ABCD\} = 0$$

$$\therefore \quad \{BDAC\} = 1 - \{ABCD\} = 2.$$

§ **136.** Def′—The Cross Ratio (or Anharmonic Ratio) in which two lines OA, OC of a pencil are divided by two others OB, OD is

$$\frac{\sin AOB}{\sin BOC} \div \frac{\sin AOD}{\sin DOC}$$ and is denoted by $\{O . ABCD\}$.

§ **137.** *If a pencil of four lines whose vertex is* O *be cut by any transversal in* A, B, C, D; *then the cross ratio of the range* $\{ABCD\}$ *is constant and = that of the pencil.*

Let the $\perp$ from O on $AD = p$. (fig′ page 111)

Then $$\frac{AB}{BC} = \frac{\Delta AOB}{\Delta BOC} \quad \text{(Euclid VI. 1)}$$

$$= \tfrac{1}{2} OA . OB \sin AOB \div \tfrac{1}{2} OB . OC \sin BOC$$

$$= \frac{OA}{OC} \cdot \frac{\sin AOB}{\sin BOC} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Similarly $$\frac{AD}{DC} = \frac{OA}{OC} \cdot \frac{\sin AOD}{\sin DOC} \quad . \; . \; . \; . \; . \; . \quad (2)$$

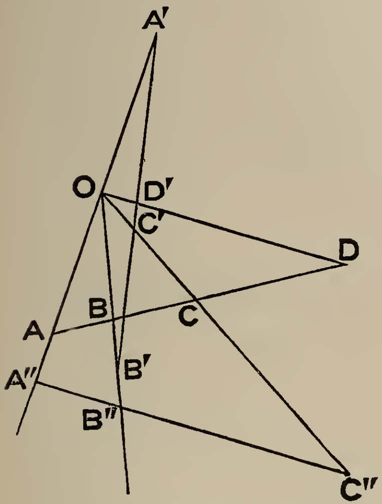
Divide by (1) by (2); thus

$$\frac{AB}{BC} \div \frac{AD}{DC} = \frac{\sin AOB}{\sin BOC} \div \frac{\sin AOD}{\sin DOC}$$

$$\therefore \quad \{ABCD\} = \{O . ABCD\}. \quad \text{Q.E.D.}$$

(See *Euclid Revised*: page 324.)

Note (1)—Since sine of an angle = sine of its supplement, it is indifferent on which side of the vertex O any ray of the pencil is cut by a transversal; e. g. in fig′



$$\{ABCD\} = \{A'B'C'D'\}.$$

Note (2)—It will be remarked that in our diagram

AD : DC and sin AOD : sin DOC

are both negative.

Note (3)—Draw a transversal A″B″C″ parallel to OD.

This will meet OD in a point at an infinite distance which we shall denote by ∞.

Thus

$$\{O.ABCD\} = \frac{A''B''}{B''C''} \div \frac{A''\infty}{\infty C''}$$

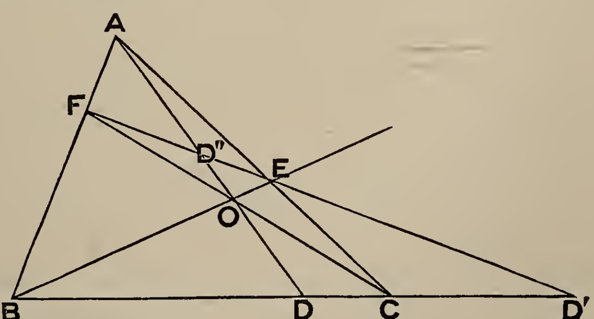
Now $\dfrac{A''\infty}{\infty C''} = -\dfrac{A''\infty}{C''\infty}$; if we put $A''C'' = p$, $C''\infty = q$,

$$\frac{A''\infty}{\infty C''} = -\frac{q+p}{q} = -\left(1 + \frac{p}{q}\right) = -1, \text{ for } \frac{p}{q} = 0$$

$$\therefore \{O.ABCD\} = -\frac{A''B''}{B''C''}.$$

Thus if the pencil is harmonic A″C″ is bisected in B″.

§ **138.** Ex. The joins of a point O to the vertices of a triangle ABC meet the opposite sides in D, E, F; FE meets BC in D′: prove that {BDCD′} is a harmonic range.



Let AO meet EF in D″.

Then $\{BDCD'\} = \{O.BDCD'\}$.

Notice the points in which the rays of this pencil are cut by the transversal EF; thus

$$\{BDCD'\} = \{ED''FD'\}$$
$$= \{A.ED''FD'\}$$
$$= \{CDBD'\}.$$

Now $$\{CDBD'\} = \frac{CD}{DB} \div \frac{CD'}{D'B} = \frac{BD'}{D'C} \div \frac{BD}{DC} = 1 \div \{BDCD'\}$$

$$\therefore \{BDCD'\}^2 = 1$$
$$\therefore \{BDCD'\} = \pm 1$$

But $\{BDCD'\}$ cannot $= +1$, for this would imply that D, D′ coincide;

$$\therefore \{BDCD'\} = -1. \quad \text{Q.E.D.}$$

§ **139.** *The cross ratio of the pencil*

$$u = 0, \quad v = 0, \quad u = kv, \quad u = k'v \quad \textit{is} \quad \frac{k}{k'}$$

Let $u \equiv ax + by + c, \qquad v \equiv a'x + b'y + c'$

Let θ, θ' be $\angle$s which $u = kv$ makes with $u = 0$, $v = 0$;

and ϕ, ϕ' $\angle$s which $u = k'v$ makes with $u = 0$, $v = 0$.

Then $$\frac{\sin\theta}{\sin\theta'} = k\sqrt{\frac{a'^2 + b'^2}{a^2 + b^2}} \quad (\S\ 128)$$

and $$\frac{\sin\phi}{\sin\phi'} = k'\sqrt{\frac{a'^2 + b'^2}{a^2 + b^2}}$$

$$\therefore \frac{\sin\theta}{\sin\theta'} \div \frac{\sin\phi}{\sin\phi'} = \frac{k}{k'}. \quad \text{Q.E.D.}$$

Cor′ (1)—The four lines

$$u = 0, \qquad v = 0, \qquad u + kv = 0, \qquad u - kv = 0$$

form a harmonic pencil.

Cor′ (2)—The lines

$$x = 0, \qquad y = 0, \qquad x + ky = 0, \qquad x - ky = 0$$

form a harmonic pencil.

As $x + ky = 0$ and $x - ky = 0$ are equally inclined to the axis of x we infer—

Cor′ (3)—If a pencil is harmonic and two alternate rays are at right angles, they bisect the angles between the other two.

§ **140.** *The cross ratio of the four lines*

$$u = lv,\ u = mv,\ u = nv,\ u = rv \quad \textit{is} \quad \frac{l-n}{n-m}\,\frac{r-m}{l-r}$$

Put $\qquad U \equiv u - lv, \qquad V \equiv u - mv, \qquad W \equiv u - nv$

Then $\qquad (n-m)\,U + (l-n)\,V + (m-l)\,W \equiv 0$

$\therefore$ $W = 0$, or $u = nv$ is equivalent to

$$(n-m)\,U + (l-n)\,V = 0$$

or

$$U + \frac{l-n}{n-m}V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly the equation $u = rv$ may be written

$$U + \frac{l-r}{r-m}V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have then the four lines $U = 0$, $V = 0$, and (1), (2) whose cross ratio is (§ 139)

$$\frac{l-n}{n-m} \div \frac{l-r}{r-m} \quad \text{Q.E.D.}$$

§ **141.** *The lines*

$$a'u^2 + 2\,h'uv + b'v^2 = 0$$

are harmonically conjugate to the lines

$$au^2 + 2\,huv + bv^2 = 0$$

if

$$ab' + a'b - 2\,hh' = 0$$

Let $\qquad au^2 + 2\,huv + bv^2 \equiv a\,(u - lv)\,(u - mv)$

$\qquad\qquad a'u^2 + 2\,h'uv + b'v^2 \equiv a'\,(u - nv)\,(u - rv)$

Then the pencil is harmonic if

$$\frac{(l-n)\,(r-m)}{(n-m)\,(l-r)} = -1 \quad (\S\ 140)$$

or if

$$2\,nr + 2\,ml = lr + ln + mr + mn$$
$$= (l+m)\,(n+r)$$

or

$$2\,\frac{b'}{a'} + 2\,\frac{b}{a} = \left(-\frac{2\,h}{a}\right)\left(-\frac{2\,h'}{a'}\right)$$

or if

$$ab' + a'b - 2\,hh' = 0. \quad \text{Q.E.D.}$$

Cor'—In general, if x is the cross ratio of the pencil

$$x = \frac{(l-n)\,(r-m)}{(n-m)\,(l-r)}$$

$$\therefore \quad \frac{x - 1}{x + 1} = \frac{(l - m)(r - n)}{(l + m)(r + n) - 2(lm + nr)}$$

Reducing, we get

$$\left(\frac{x - 1}{x + 1}\right)^2 = \frac{(h^2 - ab)(h'^2 - a'b')}{(ab' + a'b - 2hh')^2}$$

§ 142. Ex. Find the equation to the bisectors of the angles between the lines

$$ax^2 + 2hxy + by^2 = 0$$

the axes being inclined at $\hat{\omega}$

Let the required equation be

$$a'x^2 + 2h'xy + b'y^2 = 0 \quad \ldots\ldots\ldots \quad (1)$$

Since bisectors are at right angles

$$a' - 2h' \cos\omega + b' = 0 \quad \ldots\ldots\ldots \quad (2)$$

Since lines and bisectors form a harmonic pencil

$$a'b - 2hh' + ab' = 0 \quad \ldots\ldots\ldots \quad (3)$$

We can eliminate a', h', b' linearly from (1), (2), (3):

$\therefore$ req'd eq'n is
$$\begin{vmatrix} x^2 & xy & y^2 \\ 1 & -\cos\omega & 1 \\ b & -h & a \end{vmatrix} = 0$$

or
$$x^2(h - a\cos\omega) - xy(a - b) + y^2(b\cos\omega - h) = 0$$

INVOLUTION

§ 143. *Def'*—Let O be a fixed point on a straight line, and let A, A'; B, B'; C, C'; be pairs of points on the line such that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \text{\&c.} = \text{constant} = k^2:$$

then the points A, A', B, B', &c. form a system in *involution.*

The point O is the *centre* Points such as A, A' are called *conjugate* points. The point conjugate to the centre is at infinity.

A point which coincides with its conjugate is called a *double point* or a *focus.*

§ 144. If F is a focus, $OF^2 = k^2$, $\therefore OF = \pm k$. Thus there are two foci F, F' at equal distances on each side of the centre.

Since $OF^2 = OA \cdot OA'$, the range F'AFA' is harmonic (§ 133, V). Thus *any two conjugate points and the two foci form a harmonic range.*

If two conjugate points A, A′ are on opposite sides of the centre

$$k^2 = OA.OA'$$

is negative; the foci are ∴ imaginary.

§ **145.** *The cross ratio of any four points of the system is equal to that of their conjugates.*

Let A, B, C, D be any four points of the system, A′, B′, C′, D′ their conjugates.

Let $OA = \alpha$, $OB = \beta$, $OC = \gamma$, $OD = \delta$. Then $OA' = \dfrac{k^2}{\alpha}$, &c.

$$\therefore \quad \{ABCD\} = \frac{\beta - \alpha}{\gamma - \beta} \frac{\gamma - \delta}{\delta - \alpha}$$

Also

$$\{A'B'C'D'\} = \frac{\left(\frac{k^2}{\beta} - \frac{k^2}{\alpha}\right)\left(\frac{k^2}{\gamma} - \frac{k^2}{\delta}\right)}{\left(\frac{k^2}{\gamma} - \frac{k^2}{\beta}\right)\left(\frac{k^2}{\delta} - \frac{k^2}{\alpha}\right)} = \frac{(\beta - \alpha)(\gamma - \delta)}{(\gamma - \beta)(\delta - \alpha)}$$

$$\therefore \quad \{ABCD\} = \{A'B'C'D'\}.$$

Cor'—

$$\{ABCA'\} = \{A'B'C'A\}$$

This relation enables us to ascertain whether six points A, B, C, A′, B′, C′ are in involution.

§ **146.** If two pairs of conjugate points are given, the system is completely determined.

For let α, α'; β, β' be their distances from any fixed point on the line, x the required distance of the centre from this point

Then

$$(\alpha - x)(\alpha' - x) = (\beta - x)(\beta' - x)$$

$$\therefore \quad (\alpha + \alpha' - \beta - \beta')x = \alpha\alpha' - \beta\beta'$$

This equation determines the centre.

§ **147.** If a system of points in involution be joined to any point P we obtain a pencil of lines in involution. The cross ratio of any four lines of the pencil = that of their conjugates and any transversal is cut by the pencil in a system of points in involution (§ 137). There are two lines of the pencil which coincide with their conjugates; these are called *focal lines*. The two focal lines and any two conjugate rays form a harmonic pencil (§ 144).

§ **148.** *To find the condition that the three line-pairs*

$$au^2 + 2huv + bv^2 = 0, \qquad a'u^2 + 2h'uv + b'v^2 = 0,$$
$$a''u^2 + 2h''uv + b''v^2 = 0$$

should form a pencil in involution

Let the equation to the focal lines be

$$Au^2 + 2Huv + Bv^2 = 0$$

These are harmonically conjugate to each of the above:

$$\therefore \begin{cases} Ba - 2Hh + Ab = 0 \\ Ba' - 2Hh' + Ab' = 0 \\ Ba'' - 2Hh'' + Ab'' = 0 \end{cases}$$

Eliminate B, H, A: the required condition is ∴

$$\begin{vmatrix} a & h & b \\ a' & h' & b' \\ a'' & h'' & b'' \end{vmatrix} = 0$$

Exercises on Chapter V

1. The sides of a parallelogram are

$$4x + 5y = 6, \quad 4x + 5y = 9, \quad 5x + 4y = 6, \quad 5x + 4y = 9.$$

Show that its area = 1, and find equations to its diagonals.

Ans. $x = y$, $3x + 3y = 5$.

2. Show that the equation of the join of the

inters'n of $\quad a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0$

to that of $\quad a_3x + b_3y + c_3 = 0, \quad a_4x + b_4y + c_4 = 0$

is $\quad a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$

where λ is determined by

$$\lambda \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0$$

3. Show that the lines

$$bx^2 - 2hxy + ay^2 = 0$$

are at right angles to the lines

$$ax^2 + 2hxy + by^2 = 0$$

4. Show that one of the lines

$$ax^2 + 2hxy + by^2 = 0$$

will coincide with one of the lines

$$a'x^2 + 2h'xy + b'y^2$$

if

$$4(ah' - a'h)(hb' - h'b) = (ab' - a'b)^2$$

Find also the condition that a line of the first pair should be perpendicular to a line of the second pair

Ans. Write b, $-$ h, a for a, h, b in preceding cond'n (by Ex. 3).

5. Find the lines represented by

$$x^3 - 3x^2y - 3xy^2 + y^3 = 0$$

[Subst' $r\cos\theta$, $r\sin\theta$ for x, y; deduce $\tan 3\theta = 1$; $\theta = 15°, 75°$ or $135°$]

6. Find the lines represented by

$$m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$$

[*Note*—deduce $\tan 3\theta = m$]

7. Show that the lines

$$a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$$

are equally inclined to the lines

$$ax^2 + 2hxy + by^2 = 0$$

8. Show that

$$(h^2 + k^2 - 1)(x^2 + y^2 - 1) = (hx + ky - 1)^2$$

represents a line-pair.

9. Find the lines through (α, β) perpendicular to the lines

$$ax^3 + bx^2y + cxy^2 + dy^3 = 0$$

Ans. $d(x - \alpha)^3 - c(x - \alpha)^2(y - \beta)$

$$+ b(x - \alpha)(y - \beta)^2 - a(y - \beta)^3 = 0$$

10. Prove that the product of the perpendiculars from $(x'y')$ on the lines

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a - b)^2 + 4h^2}}$$

11. Prove that the distance between the points where the line

$$x \cos\alpha + y \sin\alpha = p$$

meets the lines

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\frac{2p\sqrt{h^2 - ab}}{a \sin^2\alpha - 2h \sin\alpha \cos\alpha + b \cos^2\alpha}$$

12. The line

$$lx + my + n$$

cuts the lines

$$ax^2 + 2hxy + by^2 = 0,$$

making angles α, β with them. Prove that

$$\tan\alpha + \tan\beta = 2\frac{(a-b)lm - h(l^2 - m^2)}{al^2 + 2hlm + bm^2}$$

13. Find the conditions that the polar equation

$$A \tan\theta + B \sec\theta = 1$$

should represent the bisectors of the angles between the straight lines whose polar equation is

$$A \sec\theta + B \tan\theta = 1$$

Ans. $1 + A^2 = 2B^2$, $(1 - B^2)^2 + 2AB = 0$

14. If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent straight lines, the co-ord's of their inters'n are determined by any two of the equations

$$ax + hy + g = 0, \qquad hx + by + f = 0, \qquad gx + fy + c = 0$$

[*Note*—By § 117 the lines are

$$ax + hy + g = \pm\sqrt{(\ldots)};$$

adding these

$$ax + hy + g = 0]$$

15. Find the condition that the line

$$Ax + By + C = 0$$

should pass through the intersection of the lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Ans. $\begin{vmatrix} a & h & g \\ h & b & f \\ A & B & C \end{vmatrix} = 0$

16. A straight line moves so that the algebraic sum of the perpendiculars on it from n fixed points $= 0$: show that the line passes through the fixed point

$$\frac{1}{n}(x_1 + x_2 + \ldots + x_n), \qquad \frac{1}{n}(y_1 + y_2 + \ldots + y_n)$$

[*Note*—These values may be briefly expressed

$$\frac{1}{n}\Sigma x, \qquad \frac{1}{n}\Sigma y.$$

The fixed point is the *mean centre* of the given points.]

17. More generally, if m_1 times the first $\perp$ + m_2 times the second $\perp$ + &c. $= 0$; the line passes through the fixed point

$$\left(\frac{\Sigma mx}{\Sigma m}, \frac{\Sigma my}{\Sigma m}\right)$$

[*Note*—This point is the *mean centre for the system of multiples* m_1, m_2, m_3, ... See *Euclid Revised*, p. 110.]

18. Given three fixed lines meeting in a point, if the three vertices of a triangle move one on each of these lines, and two sides of the triangle pass through fixed points, prove that the third side passes through a fixed point.

19. If the axes are inclined at $\hat{\omega}$, show that

$$x^2 + 2xy\cos\omega + y^2\cos 2\omega = 0$$

represents lines inclined at $45°$, $135°$ to the axis of x.

20. Find the equation to the locus of a point which moves so that the product of the perpendiculars from it on the four lines

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$

is constant $(= \delta^4)$.

Ans. $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = \delta^4\sqrt{(a - c + e)^2 + (b - d)^2}$

[*Note*—Let

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$$
$$\equiv e(y - m_1x)(y - m_2x)(y - m_3x)(y - m_4x) \quad . \quad . \quad . \quad (1)$$

The locus is
$$\Pi\frac{y - mx}{\sqrt{1 + m^2}} = \delta^4$$

or
$$ax^4 + bx^3y + \ldots = e\delta^4\sqrt{\Pi(1 + m^2)}$$

Again in the *identity* (1) substitute successively $(1, \sqrt{-1})$ and $(1, -\sqrt{-1})$ for x, y and multiply the results :

$$\therefore \quad (a - c + e)^2 + (b - d)^2 = e^2\,\Pi(1 + m^2)]$$

21. Determine k so that two of the lines

$$2xy(ax + by) - k(x^3 + y^3) = 0$$

may be at right angles.

Ans. $k = 0,\ a + b$.

[*Note*—We must have

$$2xy(ax + by) - k(x^3 + y^3) \equiv (x^2 - \lambda xy - y^2)(\mu x + \nu y)$$

Multiply out dexter and equate coeff's: we get eq'ns to determine μ, ν, λ, k]

22. Find the conditions that two of the lines

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0$$

should bisect the angles between the other two.

Ans. $3a + 3e + c = 0,\ 2(a - e)^2(a + e) = (b + d)(be + da)$

[*Note*—Put

$$ax^4 + bx^3y + \ldots \equiv (Ax^2 + 2Hxy + By^2)\left(x^2 - \frac{A - B}{H}xy - y^2\right)$$

Multiply out and equate coeff's; then elim' A, H, B.]

23. The condition that one of the lines

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$$

may bisect the angle between the other two is

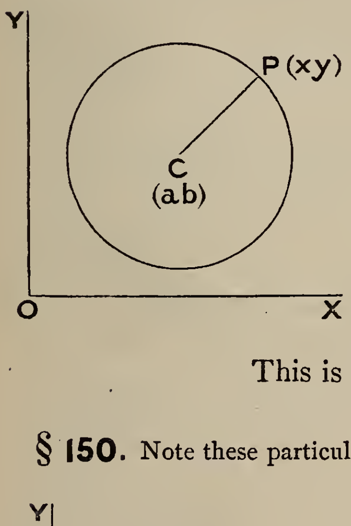
$$a(b + d)^3 - d(a + c)^3 = 3(a + c)(b + d)(b^2 + bd - ac - c^2)$$

CHAPTER VI. THE CIRCLE

EQUATION TO A CIRCLE

§ **149.** *To find the equation to a circle whose centre is* (a, b) *and radius* r.

Let $P(x, y)$ be any point on the circle, C its centre.



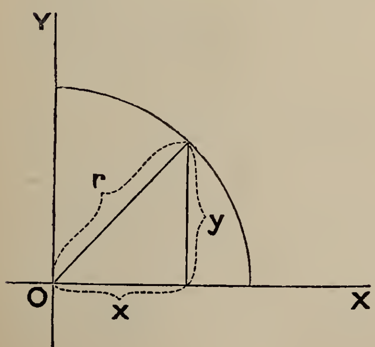
Then $CP = r$

$$\therefore \quad CP^2 = r^2$$

$$\therefore \quad (x - a)^2 + (y - b)^2 = r^2 \quad (\S\ 10)$$

This is the required equation.

§ **150.** Note these particular cases.



1°. When the centre is the origin,

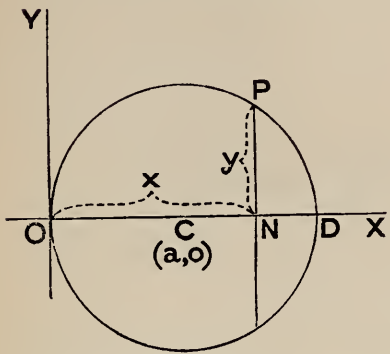
$$a = 0, \qquad b = 0.$$

The equation is

$$x^2 + y^2 = r^2$$

This is also obvious from the figure.

2°. When the circle passes through the origin and its centre is on OX.



Let the centre be (a, o)

Then the radius is a.

$\therefore$ the equation is

$$(x - a)^2 + (y - o)^2 = a^2$$

or $$(x - a)^2 + y^2 = a^2$$

or $$x^2 + y^2 = 2ax$$

Otherwise, by Euclid III. 35,

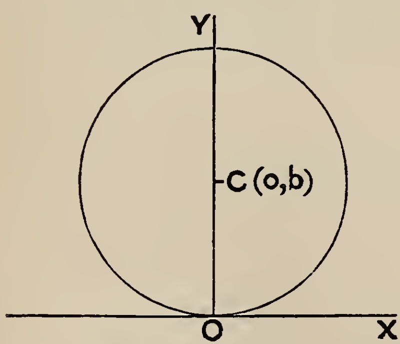
$$PN^2 = ON \,.\, ND$$

$$= ON \,.\, (2a - ON)$$

i.e. $$y^2 = x(2a - x);$$

or $$x^2 + y^2 = 2ax$$

4°. When the circle passes through the origin and its centre is on OY.



Let the centre be (o, b).

Then the equation is

$$x^2 + (y - b)^2 = b^2$$

or $$x^2 + y^2 = 2by$$

§ **151.** Ex. 1. Let the centre be $(3, -4)$ and the radius 5. The equation to the circle is

$$(x - 3)^2 + (y + 4)^2 = 5^2;$$

or $$x^2 + y^2 - 6x + 8y = 0$$

Ex. 2. What locus is represented by

$$9x^2 + 9y^2 - 42x + 36y = 59?$$

Divide by the coeff′ of x^2, viz. 9,

$$\therefore \quad x^2 + y^2 - \tfrac{14}{3}x + 4y = \tfrac{59}{9}$$

Rearrange, $$x^2 - \tfrac{14}{3}x + y^2 + 4y = \tfrac{59}{9}$$

Complete the squares (as in solving quadratics):

$$x^2 - \tfrac{14}{3}x + (\tfrac{7}{3})^2 + y^2 + 4y + 2^2 = \tfrac{59}{9} + \tfrac{49}{9} + 4$$
$$= 16$$
$$\therefore \quad (x - \tfrac{7}{3})^2 + (y + 2)^2 = 4^2$$

Thus the locus is a circle, centre $(\tfrac{7}{3}, -2)$ and radius $= 4$.

CONDITIONS THAT AN EQUATION REPRESENT A CIRCLE

§ **152.** To discuss the locus

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Rearrange, $$x^2 + 2gx + y^2 + 2fy = -c$$

Complete the squares,

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c;$$
$$\therefore \quad (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

Compare this with

$$(x - a)^2 + (y - b)^2 = r^2$$
$$\therefore \quad a = -g, \quad b = -f, \quad r^2 = g^2 + f^2 - c$$

Thus the locus is a circle, centre $(-g, -f)$ and radius

$$= \sqrt{g^2 + f^2 - c}$$

§ **153.** We notice then that *an equation of the second degree in rectangular axes represents a circle if*

1°, *there is no term in* xy

and 2°, *the coeff′s of* x^2 *and* y^2 *are equal.*

If the coeff′ of x^2 is not unity we may divide by it; thus the general form of the equation to a circle is

$$x^2 + y^2 + 2gx + fy + c = 0$$

The centre and radius of this circle are determined in § 152.

CONDITIONS DETERMINING A CIRCLE

§ **154.** Three conditions determine a circle.

For let the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Each condition gives a relation between g, f, c.

We have $\therefore$ three eq'ns to determine g, f, c.

Ex. Find the circle through $(0, 0)$, $(3, 0)$, $(3, 4)$.

Let its eq'n be $x^2 + y^2 + 2gx + 2fy + c = 0$

Express that co-ord's of each of the given points satisfy this eq'n :

$$\therefore \begin{cases} c = 0 \\ 9 + 6g + c = 0 \\ 25 + 6g + 8f + c = 0 \end{cases}$$

Solving these, $c = 0, \quad 2g = -3, \quad f = -2$

$\therefore$ req'd eq'n is $x^2 + y^2 - 3x - 4y = 0$

INTERSECTIONS

§ **155.** The points of intersection of a line and a circle are obtained by solving the simultaneous equations (§ 34).

Ex. 1. Find the inters'ns of the line $y = x + 1$ and the circle

$$x^2 + y^2 - 2x - 3y + 3 = 0$$

Eliminate y; we find $2x^2 - 3x + 1 = 0$

$$\therefore \quad x = 1 \text{ or } \tfrac{1}{2}$$

and $y = x + 1 = 2 \text{ or } \tfrac{3}{2}$.

Ans. $(1, 2)$ and $(\tfrac{1}{2}, \tfrac{3}{2})$

We have here two distinct p'ts of inters'n.

Ex. 2. Find the inters'ns of

$$x + y = 7 \quad \text{and} \quad x^2 + y^2 - 2x - 4y = 3$$

Eliminate y. We find

$$x^2 - 6x + 9 = 0, \quad \text{or} \quad (x - 3)^2 = 0$$

$$\therefore \quad x = 3, \qquad y = 7 - x = 4$$

Ans. $(3, 4)$.

Here the two p'ts of inters'n coincide: in fact the line *touches* the circle.

Ex. 3. Find the p'ts of inters'n of

$$x + y = 12 \quad \text{and} \quad x^2 + y^2 = 25$$

Solving these,

$$x = \tfrac{1}{2}(12 \pm \sqrt{-94}), \qquad y = \tfrac{1}{2}(12 \mp \sqrt{-94})$$

Here the line does not meet the circle.

It is however proper to say that it meets the circle in the two imaginary points whose co-ord's have just been assigned.

The beginner should illustrate these examples by figures.

Exercises

1. Find the equations to the following circles:

1°, centre is (1, – 2) and radius = 3

2°, centre (a, b) and radius $\sqrt{a^2 + b^2}$

3°, on join of (1, – 3), (3, 5) as diameter

4°, touching OY at O and on *left* of OY; radius = 4

5°, in first quadrant touching OX, OY; radius = c

Ans. 1°, $x^2 + y^2 - 2x + 4y = 4$

2°, $x^2 + y^2 = 2ax + 2by$

3°, $x^2 + y^2 - 4x - 2y - 12 = 0$

4°, $x^2 + y^2 + 8x = 0$

5°, $x^2 + y^2 - 2cx - 2cy + c^2 = 0$

2. Find the centre and radius of each of the circles

$$x^2 + y^2 - 4x + 4y - 1 = 0$$

$$x^2 + y^2 + 2x - 6y = 0$$

$$4x^2 + 4y^2 + 12ax - 6ay - a^2 = 0$$

$$\sec\alpha\,(x^2 + y^2) - 2cx - 2cy\tan\alpha = c$$

Ans. (2, –2), 3; (–1, 3), $\sqrt{10}$; $\left(-\frac{3a}{2}, \frac{3a}{4}\right)$, $\frac{7a}{4}$; (c cos α, c sin α), c

3. Find the circle through the origin which cuts off intercepts (3, 4) on the axes.

Ans. $x^2 + y^2 = 3x + 4y$

4. Find the circle through the three points $(0, 0)$, $(h, 0)$, $(0, k)$.

Ans. $x^2 + y^2 = hx + ky$

5. What is represented by

$$x^2 + y^2 - 6x - 8y + 25 = 0?$$

[Here $(x - 3)^2 + (y - 4)^2 = 0$; a circle centre $(3, 4)$, radius 0: i. e. a *point-circle* *]

6. Find the circle through $(0, 0)$, $(2, 3)$, $(3, 4)$.

Ans. $x^2 + y^2 = 23x - 11y$

7. Find the circle through $(2, 3)$, $(4, 5)$, $(6, 1)$.

Ans. $3x^2 + 3y^2 = 26x + 16y - 61$

8. Find the circle on the join of $(x_1 y_1)$, $(x_2 y_2)$ as diameter.

[Here centre is

$$\tfrac{1}{2}(x_1 + x_2), \quad \tfrac{1}{2}(y + y_2) \quad \text{and} \quad (\text{diam}')^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$\therefore$ eq'n is

$$[x - \tfrac{1}{2}(x_1 + x_2)]^2 + [y - \tfrac{1}{2}(y_1 + y_2)]^2 = \tfrac{1}{4}[(x_1 - x_2)^2 + (y_1 - y_2)^2]$$

or reducing, $\quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$]

9. Find the points where the line

$$3x + y = 25$$

cuts the circle $\quad x^2 + y^2 = 65$

Ans. $(7, 4)$ and $(8, 1)$.

10. Find the inters'ns of the line

$$y = x + 1$$

and the circle $\quad x^2 + y^2 = 23x - 11y$

Ans. $(2, 3)$, $(3, 4)$.

11. Find the inters'ns of

$$4x + 3y = 35, \qquad x^2 + y^2 - 2x - 4y = 20$$

Ans. The line touches the circle at $(5, 5)$.

---

* The factors of

$$(x - 3)^2 + (y - 4)^2 \text{ are } x - 3 \pm (y - 4)\sqrt{-1}:$$

$\therefore$ the given equation may also be regarded as representing two imaginary lines

$$x - 3 + (y - 4)\sqrt{-1} = 0, \qquad x - 3 - (y - 4)\sqrt{-1} = 0.$$

These intersect in the real point $(3, 4)$.

12. Find the length of the chord which

$$x^2 + y^2 - 5x - 6y + 6 = 0$$

intercepts on OX.

[Put $y = 0$, $\therefore$ $x^2 - 5x + 6 = 0$; $x = 2$ or 3.

$\therefore$ intercept $= 3 - 2 = 1$. See § 28]

13. Find the chords which the circles in Ex. 2 intercept on OX.

Ans. $2\sqrt{5}$, 2, $a\sqrt{10}$, $2c\cos\alpha$.

14. Find the chord which the circle

$$x^2 + y^2 = 4x + 2y + 20$$

intercepts on the line $3x + 4y + 5 = 0$.

[Centre of circle is $(2, 1)$ and its radius 5. Draw a figure, let C be centre and let line cut circle in P, Q. Draw $CM \perp PQ$.

Then $CM = \perp$ from $(2, 1)$ on

$$(3x + 4y + 5 = 0) = 3;$$

$$\therefore \quad PQ = 2PM = 2\sqrt{CP^2 - CM^2} = 2\sqrt{25 - 9} = 8]$$

15. Find the chord which

$$x^2 + y^2 = 2x + 3y - 3$$

intercepts on $y = x + 1$

Ans. $\dfrac{1}{\sqrt{2}}$

16. Find the chord which

$$x^2 + y^2 = 2x + 2y + 23$$

intercepts on $y = x + 3$.

Ans. $\sqrt{82}$

17. The co-ord's of A are $(-a, 0)$ and of B $(+a, 0)$; find the locus of a point P which moves so that

$$PA = 2PB.$$

[Let P be (x, y); then $PA^2 = 4PB^2$

$$\therefore \quad (x + a)^2 + y^2 = 4[(x - a)^2 + y^2]$$

or

$$3x^2 + 3y^2 - 10ax + 3a^2 = 0.$$

Ans. Locus is a circle, centre $(\frac{5}{3}a, 0)$, radius $\frac{4}{3}a$.]

18. If $PA = nPB$; find the locus of P.

Ans. A circle, centre $\left(\frac{n^2+1}{n^2-1}a, 0\right)$, radius $\frac{2n}{n^2-1}a$.

19. If $A\hat{P}B$ = a constant β; find the locus of P.

Ans. The circle $x^2 + y^2 - 2ay\cot\beta = a^2$.

20. Find the locus of a point which moves so that the sum of the squares of its distances from the sides of a square (side $= a$) is constant $= 2k^2$.

Ans. Taking two adjacent sides of square as axes, the circle

$$x^2 + y^2 - ax - ay + a^2 - k^2 = 0$$

21. Find locus of a point which moves so that sum of squares of its distances from sides of an equilateral triangle = constant = k^2.

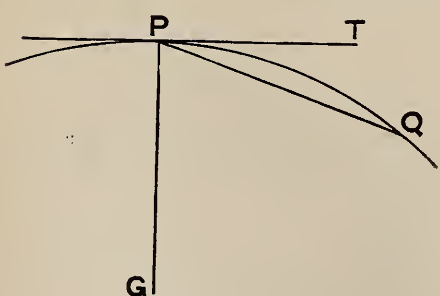
[Let two vertices of Δ be $(-a, 0)$, $(+a, 0)$ so that third vertex is $0, a\sqrt{3}$.]

Ans. Locus is the circle

$$6x^2 + 6y^2 - 4ay\sqrt{3} = 4k^2 - 6a^2$$

DEFINITION OF TANGENT

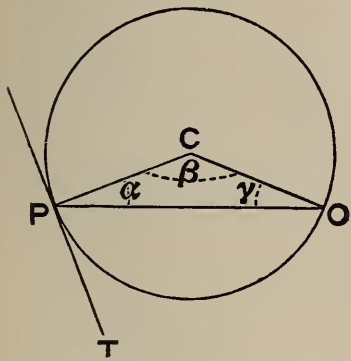
§ **156.** *Def′* (1)—If P is a point on a curve, then if P be joined to another point Q on the curve, the limiting position of the chord PQ when Q moves up to P so as ultimately to coincide with it is called the *tangent* at P.



Def′ (2)—The *normal* at P is a line PG through P perpendicular to the tangent.

The following geometrical illustration will assist the beginner to understand the preceding Def′ (1)

In the case of a circle, centre C,



$$\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 180^\circ \quad \text{(Euclid I. 32)}$$

But $$\hat{\alpha} = \hat{\gamma} \quad \text{(Euclid I. 5)}$$

$$\therefore \; 2\hat{\alpha} + \hat{\beta} = 180^\circ$$

$$\therefore \; \hat{\alpha} + \tfrac{1}{2}\hat{\beta} = 90^\circ$$

$$\therefore \; \hat{\alpha} = 90^\circ - \tfrac{1}{2}\hat{\beta}$$

Now let O move up to coincidence with P. Then ultimately

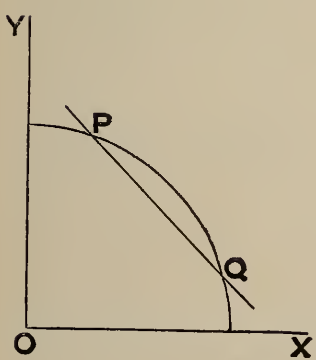
$$\hat{\beta} = 0; \qquad \therefore \; \hat{\alpha} = 90^\circ.$$

That is, the angle between CP and the tangent at P is a right angle. This agrees with Euclid III. 16

EQUATION OF TANGENT

§ **157.** *To find the equation of the tangent at* $(x'\,y')$ *to the circle*

$$x^2 + y^2 = r^2$$



Let P be the point $(x'\,y')$; Q $(x''\,y'')$ an adjacent point on the circle.

The eq'n to PQ is

$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''} \quad \cdots \quad (1)$$

If Q approaches P indefinitely, then ultimately

$$y' - y'' = 0, \qquad x' - x'' = 0$$

and the dexter side of (1) assumes the indeterminate form $\frac{0}{0}$. But if we make use of the fact that Q always remains on the circle we can find a determinate limiting value.

Since P, Q are on the circle

$$x'^2 + y'^2 = r^2, \qquad x''^2 + y''^2 = r^2$$

Subtract, $\therefore \quad x'^2 - x''^2 + y'^2 - y''^2 = 0$

$$\therefore \quad (x' - x'')(x' + x'') = -(y' - y'')(y' + y'')$$

$$\therefore \quad \frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''}$$

Thus (1) becomes

$$\frac{y - y'}{x - x'} + \frac{x' + x''}{y' + y''} = 0.$$

Now put $x'' = x'$ and $y'' = y'$, and multiply up

$$\therefore \quad y'y - y'^2 + x'x - x'^2 = 0$$

$$\therefore \quad xx' + yy' = x'^2 + y'^2$$

$$= r^2, \text{ since } (x'y') \text{ is on the circle}$$

Thus the equation of the tangent at $(x'y')$ is

$$xx' + yy' = r^2$$

Ex. The tangent to $x^2 + y^2 = 25$ at (3, 4) is

$$3x + 4y = 25$$

§ **158.** The normal is the $\perp$ to the tangent through $x'y'$; its equation is $\therefore$

$$\frac{y - y'}{y'} = \frac{x - x'}{x'}$$

or

$$x'y - y'x = 0$$

Thus the normal passes through the origin.

§ **159.** The result of § 157 may be obtained by assuming Euclid III. 16.

The normal is the join of P $(x'y')$ to the centre (0, 0).

Its equation is $\therefore$

$$\frac{y}{x} = \frac{y'}{x'}, \qquad \text{or} \qquad x'y - y'x = 0$$

The tangent is the $\perp$ to this through P $(x'y')$; its equation is $\therefore$

$$\frac{y-y'}{x'} = \frac{x-x'}{-y'}$$

or $$y'(y-y') + x'(x-x') = 0$$

or $$x'x + y'y = x'^2 + y'^2 = r^2, \quad \text{as before.}$$

§ **160.** *To find the equation of the tangent at* $(x'y')$ *to the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let Q $(x''y'')$ be an adjacent point on the circle. The equation of PQ is

$$\frac{y-y'}{x-x'} = \frac{y'-y''}{x'-x''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since P, Q are on the circle

$$x'^2 + y'^2 + 2gx' + 2fy' + c = 0$$

$$x''^2 + y''^2 + 2gx'' + 2fy'' + c = 0$$

Subtract,

$$\therefore \quad (x'-x'')(x'+x''+2g) + (y'-y'')(y'+y''+2f) = 0$$

$$\therefore \quad \frac{y'-y''}{x'-x''} = -\frac{x'+x''+2g}{y'+y''+2f}$$

Thus (1) becomes

$$\frac{y-y'}{x-x'} = -\frac{x'+x''+2g}{y'+y''+2f}$$

Now put $x'' = x'$, $y'' = y'$; the equation of the tangent at $(x'y')$ is $\therefore$

$$\frac{y-y'}{x-x'} = -\frac{x'+g}{y'+f}$$

This equation may be reduced. Multiply up,

$$\therefore \quad (x-x')(x'+g) + (y-y')(y'+f) = 0 \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad xx' + yy' + gx + fy = x'^2 + y'^2 + gx' + fy'$$

Add to both sides

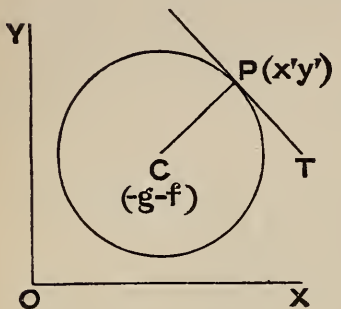
$$gx' + fy' + c$$

$$\therefore \quad xx' + yy' + g(x + x') + f(y + y') + c$$
$$= x'^2 + y'^2 + 2gx' + 2fy' + c$$
$$= 0, \text{ since } (x'y) \text{ is on the circle.}$$

Thus the equation to the tangent at $(x'y')$ is

$$xx' + yy' + g(x + x') + f(y + y') + c = 0$$

§ **161.** Otherwise, assuming Euclid III. 16.



The normal CP is the join of the centre C $(-g, -f)$ [§ 152] to $(x'y')$, $\therefore$ its eq'n is

$$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ -g & -f & 1 \end{vmatrix} = 0$$

or $\quad x(y' + f) - y(x' + g) + gy' - fx' = 0$

The tangent PT is the $\perp$ to this through $(x'y')$; its equation is $\therefore$

$$(x - x')(x' + g) + (y - y')(y' + f) = 0$$

This is equation (2) of the last §, and is reduced as before.

§ **162.** The equation of the tangent at $(x'y')$

$$xx' + yy' + g(x + x') + f(y + y') + c = 0$$

may be remembered by this rule:—

Write the equation to the circle thus.

$$xx + yy + g(x + x) + f(y + y) + c = 0;$$

then dash one letter in each term.

Ex. Find the tangent at (1, 2) to

$$x^2 + y^2 - 2x - 3y + 3 = 0$$

Write this $\quad xx + yy - (x + x) - \frac{3}{2}(y + y) + 3 = 0$

The req'd eq'n is

$$x(1) + y(2) - (x + 1) - \tfrac{3}{2}(y + 2) + 3 = 0;$$

or reducing, $\quad y = 2$

CONDITION OF TANGENCY

§ **163.** *To find the condition that the line* $y = mx + n$ *may touch the circle* $x^2 + y^2 = r^2$

Eliminate y; $\quad \therefore \quad x^2 + (mx + n)^2 = r^2$

$$\therefore \quad (1 + m^2)x^2 + 2mnx + n^2 - r^2 = 0$$

This quadratic determines the abscissae of the points of intersection.

The quadratic has equal roots, i. e. the line touches the circle, if

$$(1 + m^2)(n^2 - r^2) = m^2 n^2$$

or
$$n^2 = r^2(1 + m^2)$$

or
$$n = \pm r\sqrt{1 + m^2}$$

Cor'—Whatever m is, the line

$$y = mx + r\sqrt{1 + m^2}$$

is a tangent to the circle $\quad x^2 + y^2 = r^2$

§ **164.** The method in the last § is applicable to any curve. In the case of the circle however the following method is easier.

It follows from Euclid that a line touches a circle if

$\perp$ on line from centre = radius.

$\therefore$ condition is that

length of $\perp$ from (0, 0) on $[y = mx + n]$ is r;

i. e.
$$\frac{n}{\pm\sqrt{1 + m^2}} = r$$

or
$$n = \pm r\sqrt{1 + m^2}, \quad \text{as before.}$$

Ex. 1. Find the tangents to

$$x^2 + y^2 = 25$$

which are parallel to $\quad 3y - 4x = 0$

Any parallel is $\quad 3y - 4x + k = 0$

First method. Eliminating y,

$$x^2 + \left(\frac{4x - k}{3}\right)^2 = 25$$

or reducing $$25x^2 - 8kx + k^2 - 225 = 0$$

This quadratic in x has equal roots if

$$64k^2 = 4 \cdot 25 \cdot (k^2 - 225)$$

Solving this, $$k = \pm 25$$

and the required tangents are

$$3y - 4x + 25 = 0, \qquad 3y - 4x - 25 = 0$$

Second method.

Length of $\perp$ from centre, viz. $(0, 0)$ on $[3y - 4x + k = 0] = 5$;

$$\therefore \quad \frac{k}{\pm 5} = 5 \quad \text{and} \quad k = \pm 25, \text{ as before.}$$

Ex. 2. Find the equations of the tangents from the origin to

$$x^2 + y^2 - 4x - 4y + 7 = 0$$

Here centre is $(2, 2)$ and radius $= 1$.

Any line through the origin is $y = mx$; this will be a tangent if

$$\perp \text{ from } (2, 2) \text{ on } [y - mx = 0] = 1$$

$$\therefore \quad \frac{2 - 2m}{\pm\sqrt{1 + m^2}} = 1$$

$$\therefore \quad (2 - 2m)^2 = 1 + m^2$$

Solving this, $$m = \tfrac{1}{3}(4 \pm \sqrt{7})$$

and the required tangents are

$$y = \tfrac{1}{3}(4 + \sqrt{7})x, \qquad y = \tfrac{1}{3}(4 - \sqrt{7})x$$

LENGTH OF TANGENT

§ 165. *To find the length of either tangent from* $(x'y')$ *to*

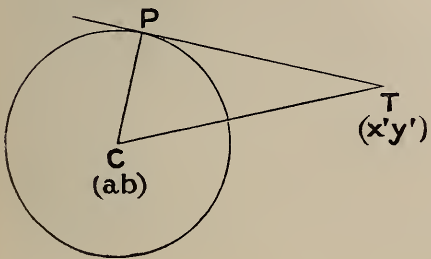
$$(x - a)^2 + (y - b)^2 - r^2 = 0$$

Let C be the centre, T the point $(x'y')$, P the point of contact.

Then $$C\hat{P}T = \text{rt}' \angle;$$

$$\therefore \quad TP^2 = TC^2 - CP^2$$
$$= TC^2 - r^2$$

But $$TC^2 = (x' - a)^2 + (y' - b)^2, \quad (\S\ 10).$$
$$\therefore\ TP^2 = (x' - a)^2 + (y' - b)^2 - r^2$$



Thus *the square of the tangent = result of substituting* the given co-ordinates in the sinister side of the equation, prepared if necessary by dividing by the coeff' of* x^2.

Ex. Find the length of the tangent from (3, 4) to
$$3x^2 + 3y^2 + 2x + y + 1 = 0$$
The 'prepared' equation is
$$x^2 + y^2 + \tfrac{2}{3}x + \tfrac{1}{3}y + \tfrac{1}{3} = 0$$
$$\text{Length req'd} = \sqrt{3^2 + 4^2 + (\tfrac{2}{3})\,3 + (\tfrac{1}{3})\,4 + \tfrac{1}{3}} = \sqrt{\tfrac{86}{3}}$$

Exercises

1. Find the equation of the tangent at (5, 4) to
$$x^2 + y^2 - 4x - 6y + 3 = 0$$
Ans. $3x + y = 19$

2. Find the tangent and normal to
$$x^2 + y^2 = 25 \quad \text{at} \quad (3, -4).$$
Ans. $3x - 4y = 25$, $4x + 3y = 0$

---

* *Def'*—The *power* of a point $(x'y')$ with regard to a curve $S = 0$ is the result of substituting x', y' instead of x, y in S.

The power of $T\,(x'y')$ with regard to the circle
$$(x - a)^2 + (y - b)^2 - r^2 = 0$$
is
$$(x' - a)^2 + (y' - b)^2 - r^2$$
This $= CT^2 - r^2$; it is $\therefore$ positive, zero, or negative according as T is outside, on, or inside the circle.

If the point T is outside, its power = square of tangent from T (§ 165).

3. Find the tangent at the origin to

$$x^2 + y^2 + 4x + 5y = 0$$

Ans. $4x + 5y = 0$

4. Find the tangent at the origin to

$$x^2 + y^2 + hx + ky = 0$$

Ans. $hx + ky = 0$

5. Find the tangent and normal at $(-2, 3)$ to

$$x^2 + y^2 - 2x - 4y = 5$$

Ans. $3x - y + 9 = 0$, $x + 3y = 7$

6. Find the tangent at $(3, 2)$ to

$$(x - 1)^2 + (y + 2)^2 = 20$$

Ans. $x + 2y = 7$

7. Find the tangents to $x^2 + y^2 = 36$
which are inclined at 60° to OX.

Ans. $y = x\sqrt{3} + 12$, $y = x\sqrt{3} - 12$

8. Find the tangents to

$$x^2 + y^2 - 2x - 2y + 1 = 0$$

which are parallel to $x = y$.

Ans. $x - y \pm \sqrt{2} = 0$

9. Show that the line $x + y = 2 + \sqrt{2}$
touches the circle $x^2 + y^2 - 2x - 2y + 1 = 0$
and determine the point of contact.

Ans. $(1 + \frac{1}{2}\sqrt{2}, 1 + \frac{1}{2}\sqrt{2})$

10. Find the tangents to

$$x^2 + y^2 - 4x - 2y + 1 = 0$$

which are parallel to OX; also the tangents from the origin.

Ans. $y = 3$, $y + 1 = 0$; $x = 0$, $3x + 4y = 0$.

11. Find the length of the tangent from $(1, -2)$ to

$$3x^2 + 3y^2 - x - 4 = 0;$$

also of the tangent from the origin to

$$2x^2 + 2y^2 + 5x + 6y + 8 = 0$$

Ans. $\sqrt{\frac{10}{3}}$, 2.

12. Find the length of the tangent from (13, 8) to

$$x^2 + y^2 + 22x - 2y = 278$$

Ans. 15.

13. If the tangent from $(h\,k)$ to

$$x^2 + y^2 = 3$$

be twice the tangent from the same point to

$$x^2 + y^2 = 3x + 6$$

prove that $$h^2 + k^2 = 4h + 7$$

14. Find the condition that the line

$$\frac{x}{h} + \frac{y}{k} = 1$$

may be a tangent to the circle $x^2 + y^2 = r^2$

Ans. $h^2 k^2 = r^2(h^2 + k^2)$

15. Any tangent to a circle whose radius is r meets the tangents at the ends of a diameter AB in Q, R: prove that

$$AQ \,.\, BR = r^2$$

TANGENTS FROM A GIVEN POINT

§ 166. We may obtain the equation to the pair of tangents from $(x'y')$ to the circle $x^2 + y^2 = r^2$ thus—

Let (hk) be a point on either tangent from $(x'y')$.

Form the eq'n to the join of $(x'y')$, (hk); it is

$$x(y' - k) - y(x' - h) + x'k - y'h = 0$$

The $\perp$ on this from the centre $(0, 0)$ is r, i.e. the join touches circle if

$$\frac{x'k - y'h}{\pm\sqrt{(y' - k)^2 + (x' - h)^2}} = r$$

Writing x, y instead of h, k we obtain the required equation,

$$r^2\left\{(x - x')^2 + (y - y')^2\right\} = (xy' - x'y)^2$$

Another method is deduced in the next §.

§ 167. We may find the ratio $m : n$ in which the join of two points $(x'y')$, $(x''y'')$ is divided by the circle $x^2 + y^2 = r^2$, thus—

The values $$x = \frac{mx'' + nx'}{m + n}, \qquad y = \frac{my'' + ny'}{m + n} \qquad (§\,15)$$

must satisfy $$x^2 + y^2 = r^2$$

$$\therefore \quad (mx'' + nx')^2 + (my'' + ny')^2 = r^2(m+n)^2$$

$$\therefore \quad m^2(x''^2 + y''^2 - r^2) + 2\,mn\,(x'x'' + y'y'' - r^2) + n^2(x'^2 + y'^2 - r^2) = 0.$$

This quadratic determines the values of $m : n$ corresponding to the two points where the join cuts the circle.

Cor'—The quadratic has equal roots, i. e. the join touches the circle if

$$(x'^2 + y'^2 - r^2)(x''^2 + y''^2 - r^2) = (x'x'' + y'y'' - r^2)^2$$

This is true if $(x''y'')$ is any point on either tangent from $(x'y')$.

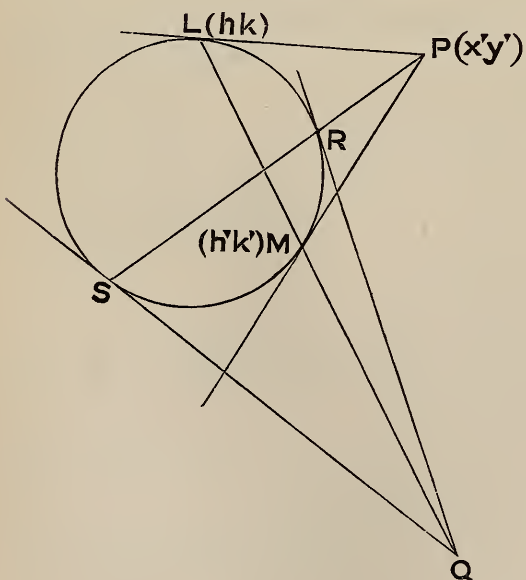
Writing then x, y instead of x'', y'', we obtain the equation to the pair of tangents from $(x'y')$, viz.

$$(x'^2 + y'^2 - r^2)(x^2 + y^2 - r^2) = (xx' + yy' - r^2)^2$$

Note—The preceding method is due to Joachimsthal.

POLES AND POLARS

§ **168.** *Def'*—If L, M are the points of contact of tangents from P, the line LM is called the *polar* of P, and P is called the *pole* of LM.



To find the equation to the polar of $(x'y')$ *with respect to the circle*

$$x^2 + y^2 = r^2.$$

Let (hk), $(h'k')$ be the points of contact of tangents from $(x'y')$.

The equation of the tangent at (hk) is

$$xh + yk = r^2.$$

But $(x'y')$ is a point on this tangent,

$$\therefore \quad x'h + y'k = r^2 \quad \dots\dots \quad (1)$$

Similarly $$x'h' + y'k' = r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now consider the straight line whose equation is

$$xx' + yy' = r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By (1), (hk) is a point on this line

By (2), $(h'k')$ „ „

$\therefore$ (3) is the equation to the join of (hk), $(h'k')$.

Therefore the required equation is

$$xx' + yy' = r^2$$

§ **169.** If $(x'y')$ are real this equation represents a real straight line; if $(x'y')$ is inside the circle the points (hk), $(h'k')$ are imaginary, but their join is the real straight line

$$xx' + yy' = r^2$$

§ **170.** The equation of the tangent at $(x'y')$ is

$$xx' + yy' = r^2,$$

with the implication $$x'^2 + y'^2 = r^2$$

The equation of the polar is of the same form as the equation of the tangent, but the values of $(x'y')$ are unrestricted. Thus if a point is on the circle its polar coincides with the tangent.

This may also be seen geometrically.

If in fig' § 168 P approaches L along the line PL (the point L remaining fixed), then M will approach L along the circle; and as P approaches to coincidence with L the polar LM approaches coincidence with the tangent LP.

§ **171.** *To find the pole of the line*

$$Ax + By + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let it be $(x'y')$; the polar of this point is

$$xx' + yy' - r^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(1) and (2) must represent the same line;

$$\therefore \quad \frac{x'}{A} = \frac{y'}{B} = -\frac{r^2}{C}$$

$$\therefore \quad x' = -\frac{Ar^2}{C}, \qquad y' = -\frac{Br^2}{C}$$

These values of x', y' are real if A, B, C are real. Thus every line has a real pole.

§ 172. *If a point* P *lie on the polar of* Q, *then* Q *will lie on the polar of* P.

Let P be $(x'y')$, and Q $(x''y'')$.

The polar of P is $\quad xx' + yy' = r^2 \quad \ldots \ldots \quad (1)$

The polar of Q is $\quad xx'' + yy'' = r^2 \quad \ldots \ldots \quad (2)$

Express that $(x'y')$ lies on (2);

$$\therefore \quad x'x'' + y'y'' = r^2$$

But this is also the condition that $(x''y'')$ lies on (1). Q.E.D.

§ 173. *Straight lines are drawn through a fixed point* P *cutting a circle in* R *and* S; *the tangents at* R *and* S *meet in* Q: *to find the locus of* Q (*see* fig', § 168).

We see here that P lies on the polar of Q;

$\therefore$ Q lies on the polar of P (§ 172)

i. e. the locus of Q is LM the polar of P.

The proposition may be otherwise stated thus:

If a line revolve round a fixed point P; *then its pole moves on a fixed line, viz. the polar of* P.

Conversely, *if a point* P *move on a fixed line, then its polar will pass through a fixed point, viz. the pole of the fixed line.*

For let Q be the pole of the fixed line.

By hypothesis P lies on the polar of Q;

$\therefore$ Q lies on the polar of P (§ 172):

i. e. the polar of P passes through the fixed point Q.

§ 174. *If the polars of* A *and* B *intersect in* C, *then* C *is the pole of* AB.

The polar of A passes through C,

$\therefore$ the polar of C passes through A (§ 172).

Similarly the polar of C passes through B:

$\therefore$ AB is the polar of C. Q.E.D.

§ **175.** To deduce a construction for the polar of a point P with respect to a circle.

Let O be the centre of the circle.

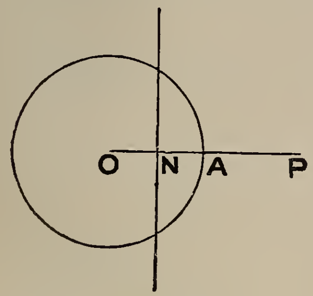


Fig. (1)

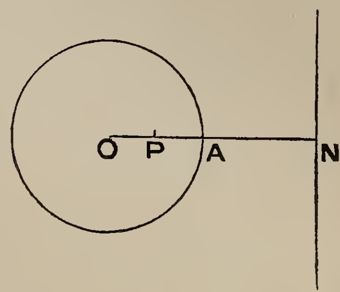


Fig. (2)

Take O for origin and OP for axis of x. Let the co-ord's of P be (x', o), and the equation to the circle

$$x^2 + y^2 = r^2$$

Then the polar of P is $\quad xx' + y.o = r^2$

or $\quad xx' = r^2$

This is a line parallel to the axis of y at a distance $\frac{r^2}{x'}$.

We have then this construction for the polar of P:

Join OP, let OP meet the circle in A.

Take a point N on OP such that

$$OP.ON = OA^2.$$

Then a $\perp$ through N to OP is the polar.

§ **176.** If the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = o,$$

it is proved as in § 158 that the polar of $(x'y')$ is

$$xx' + yy' + g(x + x') + f(y + y') + c = o.$$

This is of the same form as the equation of the tangent.

Exercises

1. Find the polars of (1, 2), (0, 1), (1, − 1) with respect to

$$x^2 + y^2 = 3$$

Ans. $x + 2y = 3$, $y = 3$, $x - y = 3$.

2. Find the poles of

$$2x - 3y = 5, \qquad x + y = 1, \qquad x\cos\alpha + y\sin\alpha = p$$

with respect to $x^2 + y^2 = 50$

Ans. $(20, -30)$, $(50, 50)$, $\left(\dfrac{50\cos\alpha}{p}, \dfrac{50\sin\alpha}{p}\right)$

3. Find the points of contact of tangents from $(1, 2)$ to

$$x^2 + y^2 = 4$$

Ans. $(0, 2)$, $(\frac{8}{5}, \frac{6}{5})$

4. Find the condition that the polar of (hk) with respect to

$$x^2 + y^2 = 9$$

may touch $x^2 + y^2 = 6y$

Ans. $h^2 + 6k = 9$

5. Find the condition that the polar of (hk) with respect to

$$x^2 + y^2 = a^2$$

may touch $(x - \alpha)^2 + (y - \beta)^2 = r^2$

Ans. $(\alpha h + \beta k - a^2)^2 = r^2(h^2 + k^2)$

RADICAL AXIS

§ **177.** Let the equations of two circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and let us interpret the equation

$$x^2 + y^2 + 2gx + 2fy + c = x^2 + y^2 + 2g'x + 2f'y + c' \quad . \quad (3)$$

Any values of x, y which satisfy (1) and (2) also evidently satisfy (3):

∴ (3) represents a locus passing through the inters'ns of (1), (2).

Again, (3) reduces to

$$2(g - g')x + 2(f - f')y + c - c' = 0 \quad . \quad . \quad . \quad (4)$$

This equation represents a straight line.

Accordingly, (3) represents the join of the points of inters'n (real or imaginary) of the circles (1), (2).

§ **178.** *Def′*—The join of the points of intersection (real or imaginary) of two circles is called their *radical axis.*

Equation (3), or (4), represents the radical axis of the circles (1), (2).

Although the circles may not intersect in real points, we see from (4) that the radical axis is always a real line.

Put
$$S \equiv x^2 + y^2 + 2gx + 2fy + c$$
$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c'$$
so that $S = 0$, $S' = 0$ are the prepared eq′ns to two circles.

Then the equation to the radical axis is
$$S - S' = 0$$

§ **179.** The equation $S - S' = 0$ expresses (§ 165) that the square of the tangent from (x, y) to $S = 0$ is equal to the square of the tangent from (x, y) to $S' = 0$: thus *tangents drawn to two circles from any point on their radical axis are equal.*

§ **180.** *The radical axis is perpendicular to the join of the centres.*

For the join of the centres is
$$\begin{vmatrix} x & y & 1 \\ -g & -f & 1 \\ -g' & -f' & 1 \end{vmatrix} = 0$$
or
$$x(f' - f) - y(g' - g) + gf' - g'f = 0$$
This is $\perp$ the radical axis,
$$2(g - g')x + 2(f - f')y + c - c' = 0 \quad \text{[§ 68, Cor′ (3)]}$$

§ **181.** One of the circles may be a point-circle.

Thus the radical axis of the circles
$$(x - a)^2 + (y - b)^2 - r^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$(x - a')^2 + (y - b')^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
is the line
$$2(a' - a)x + 2(b' - b)y + a^2 + b^2 - a'^2 - b'^2 - r^2 = 0 \quad . \quad . \quad (3)$$
Accordingly the line (3) [the radical axis of the circle (1) and the point $(a'b')$] is the locus of a point which moves so that the tangent drawn from it to the circle (1) is equal to its distance from the point $(a'b')$.

§ **182.** Suppose two circles to intersect in P, Q so that PQ is the radical axis.

Let Q approach coincidence with P: then the circles touch and PQ becomes the common tangent.

Thus *if two circles touch, their radical axis is the common tangent at the point of contact.*

Ex. Determine c so that the circles

$$x^2 + y^2 - 3x - 4y - c = 0, \qquad x^2 + y^2 - 4x - 3y - c = 0$$

may touch.

Subtracting, the radical axis is

$$x - y = 0$$

At the points where this meets the first circle

$$2x^2 - 7x - c = 0:$$

this eq'n has equal roots if

$$49 + 8c = 0, \quad \text{or} \quad c = -\tfrac{49}{8}.$$

§ **183.** Let the prepared equations to three circles be

$$S = 0, \quad S' = 0, \quad S'' = 0$$

Taking these in pairs, the radical axes are

$$S - S' = 0, \quad S' - S'' = 0, \quad S'' - S = 0$$

It is evident that any values of (x, y) which satisfy two of these equations will satisfy the third.

Thus *the three radical axes of three circles taken in pairs are concurrent.*

The point of concurrence is called the *radical centre.*

CO-AXAL CIRCLES

§ **184.** Let the equations to two circles be $S = 0$, $S' = 0$ where

$$S \equiv x^2 + y^2 + 2gx + 2fy + c$$

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c'$$

and let us interpret the equation

$$S - \lambda S' = 0$$

Writing it at full length we see that it represents a circle (§ 153).

Again, any values of (x, y) which render $S = 0$ and $S' = 0$ will also render $S - \lambda S' = 0$.

Thus $S - \lambda S' = 0$ represents a circle passing through the points of intersection of $S = 0$, $S' = 0$. If we choose λ suitably $S - \lambda S' = 0$ will represent any such circle.

§ **185.** *Def′*—A system of circles passing through two fixed points is called a *co-axal system.*

The points may be real or imaginary. Their join is the common radical axis of any two circles of the system.

If $S = 0$, $S' = 0$ are two circles of the system, any other is $S - \lambda S' = 0$ (§ 184).

§ **186.** $S - \lambda S' = 0$ signifies (§ 165) that the square of the tangent from $(x\,y)$ to $S = 0$ is λ times the square of the tangent from $(x\,y)$ to $S' = 0$. Therefore if a point move so that the tangents from it to two given circles are in a constant ratio, its locus is a co-axal circle.

§ **187.** If $S = 0$ be one circle of the system and $u = 0$ the radical axis, then any other circle of the system is

$$S + \lambda u = 0 \ldots \qquad (1)$$

where λ is a numerical constant.

For the radical axis of (1) and $S = 0$ is $\lambda u = 0$, or $u = 0$.

Q.E.D.

§ **188.** A co-axal system may be simply represented thus.

Take the join of the fixed points for axis of y, and the mid point of this join for origin. Let the fixed points be $(0, \pm c)$.

The common radical axis is $x = 0$.

$x^2 + y^2 - c^2 = 0$ is evidently one of the circles.

Therefore by § 187 the equation

$$x^2 + y^2 + 2\lambda x - c^2 = 0 \qquad (1)$$

represents any circle of the system.

c is the same for all the circles; λ varies from circle to circle.

L

§ 189. Equation (1) of the last § may be written

$$(x + \lambda)^2 + y^2 = \lambda^2 + c^2$$

If $\lambda^2 + c^2 = 0$, this reduces to a point-circle.

If $\therefore \lambda = \pm c\sqrt{-1}$, the circle reduces to one of the points $(\pm c\sqrt{-1}, 0)$.

These points are called the *limiting points* of the system.

If the circles intersect in real points, c is real and the limiting points are imaginary.

If the circles intersect in imaginary points, i. e. if

$$c = \pm k\sqrt{-1},$$

then the limiting points are real, viz. they are $(\pm k, 0)$.

Ex. The polar of a limiting point is the same for every circle of the system.

For the polar of $(k, 0)$ with respect to

$$x^2 + y^2 + 2\lambda x + k^2 = 0$$

is

$$kx + \lambda(x + k) + k^2 = 0$$

or

$$(x + k)(\lambda + k) = 0, \quad \text{or} \quad x + k = 0:$$

i. e. the line through the other limiting point || radical axis.

Exercises

1. Find the radical axis of the circles

$$x^2 + y^2 + 4x + 5y = 6 \quad \text{and} \quad x^2 + y^2 + 5x + 4y = 9$$

Ans. $x - y = 3$.

2. Show that the circles

$$x^2 + y^2 + 4x + y = 3, \quad x^2 + y^2 - x - y = 1, \quad x^2 + y^2 + 14x + 5y = 7$$

are co-axal.

3. Find the radical centre of the circles

$$x^2 + y^2 + 3x = 0, \quad x^2 + y^2 + 5y = 1, \quad x^2 + y^2 + 2x + 2y = 1$$

Ans. (3, 2)

4. Find the radical axis of the circles

$$(x - a)^2 + (y - b)^2 = c^2, \quad (x - b)^2 + (y - a)^2 = c^2;$$

and determine c so that the circles may touch.

Ans. $x - y = 0$, $\frac{1}{\sqrt{2}}(a - b)$.

5. Find the length of the common chord of the circles

$$x^2 + y^2 - 4x - 2y - 20 = 0, \qquad x^2 + y^2 + 8x + 14y = 0.$$

[The radical axis is

$$12x + 16y + 20 = 0, \quad \text{or} \quad 3x + 4y + 5 = 0.$$

The chord which the first circle intercepts on this line is obtained, Exercise 14, page 127.]

Ans. 8.

6. Find the length of the common chord of the circles in Ex. 4.

Ans. $\sqrt{4c^2 - 2(a-b)^2}$

7. Find the equation to a circle through the points of intersection of the circles in Ex. 1 and through the point (1, 2).

[The req'd eq'n is of the form

$$x^2 + y^2 + 4x + 5y - 6 + \lambda(x^2 + y^2 + 5x + 4y - 9) = 0.$$

Expressing that this is satisfied by

$$x = 1, \qquad y = 2, \qquad 13 + 9\lambda = 0.]$$

Ans. $4x^2 + 4y^2 + 29x + 7y - 63 = 0$

8. Find the circle through the origin and the inters'ns of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \qquad x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

Ans. $(c' - c)(x^2 + y^2) + 2(gc' - g'c)x + 2(fc' - f'c)y = 0$

9. Find the locus of a point which moves so that tangents from it to

$$x^2 + y^2 = 3, \qquad x^2 + y^2 = 3x + 6$$

are in the ratio 2 : 3.

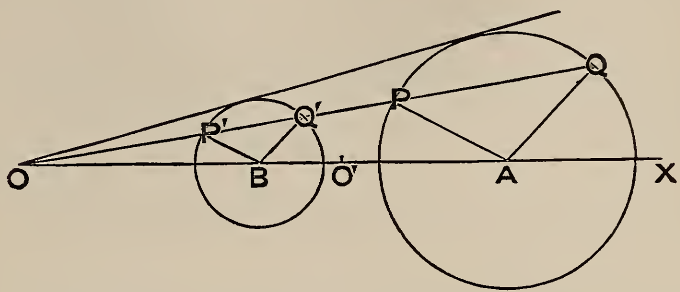
Ans. The circle $5(x^2 + y^2) + 12x = 3$

CENTRES OF SIMILITUDE

§ 190. *Def'*—If the join AB of the centres of two circles is divided internally in O′ and externally in O in the ratio of the radii: then O′ is their *internal centre of similitude*, and O their *external centre of similitude.*

The following properties are easily proved by Pure Geometry (see *Euclid Revised*, p. 343). Analytical proofs are indicated in the Exercises.

(1) The two direct common tangents pass through O and the two transverse common tangents through O′.



(2) Any line through either centre of similitude is cut similarly by the circles.

Thus if a line through O cut the circles in P, Q, P′, Q′ then

$$OP : OP' = OQ : OQ' = r : r',$$

where r, r′ are the radii.

(3) The six centres of sim′de of three circles taken in pairs lie in threes on four straight lines (called *axes of similitude*).

§ **191.** Ex. 1. Find the external common tangents to the circles

$x^2 + y^2 - 26x - 12y + 201 = 0, \qquad x^2 + y^2 - 12x - 4y + 39 = 0$

The centres are (13, 6), (6, 2) and the radii are 2, 1.

∴ the external centre of similitude is (§ 16)

$$\frac{2.6 - 1.13}{2 - 1}, \quad \frac{2.2 - 1.6}{2 - 1}$$

or $$(-1, -2)$$

Any line through this is $\qquad y + 2 = m(x + 1)$

If this touch the first circle, the ⊥ on it from (13, 6) = 2

$$\therefore \quad \frac{8 - 14m}{\pm\sqrt{1 + m^2}} = 2$$

Solving this, $\qquad m = \frac{3}{4}$ or $\frac{5}{12}$

∴ the common tangents are

$$y + 2 = \tfrac{3}{4}(x + 1), \qquad y + 2 = \tfrac{5}{12}(x + 1)$$

or $$3x - 4y = 5, \qquad 5x - 12y = 19$$

Exercises

1. Find the direct common tangents to the circles

$$x^2 + y^2 - 4x - 2y + 4 = 0, \qquad x^2 + y^2 + 4x + 2y - 4 = 0$$

Ans. $y = 2$, $4x - 3y = 10$.

2. Find the transverse common tangents to the same circles.

Ans. $x = 1$, $3x + 4y = 5$.

3. Find the direct common tangents to the circles

$$x^2 + y^2 - 6x - 8y = 0, \qquad x^2 + y^2 - 4x - 6y = 3$$

Ans. $x + 2 = 0$, $y + 1 = 0$.

4. Show that the tangents from the origin to the circle

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

touch the circle

$$(x - \lambda\alpha)^2 + (y - \lambda\beta)^2 = (\lambda r)^2$$

5. Also, if any line through the origin cut these circles in P, Q, P', Q', prove

$$OP : OP' = OQ : OQ' = 1 : \lambda$$

[Writing $\rho \cos\theta$, $\rho \sin\theta$ for x, y in the equation to the first circle it becomes

$$\rho^2 - 2\rho\,(\alpha \cos\theta + \beta \sin\theta) + \alpha^2 + \beta^2 - r^2 = 0$$

Solve this quadratic in ρ; its roots are OP, OQ; &c.]

Note—Exercises 4, 5 afford proofs of properties (1), (2), § 190.

6. The centres of three circles are $(\alpha\,\beta)$, $(\alpha'\,\beta')$, $(\alpha''\,\beta'')$ and their radii are r, r', r''. If these circles are taken in pairs, show that the three external centres of similitude lie on the line

$$\begin{vmatrix} r & r' & r'' \\ \beta & \beta' & \beta'' \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} r & r' & r'' \\ \alpha & \alpha' & \alpha'' \\ 1 & 1 & 1 \end{vmatrix} y = \begin{vmatrix} r & r' & r'' \\ \beta & \beta' & \beta'' \\ \alpha & \alpha' & \alpha'' \end{vmatrix}$$

ANGLE OF INTERSECTION

§ 192. If two circles whose centres are A, B intersect in P, then since the tangents at P are $\perp$ AP, BP the angle of intersection is APB.

Let $\widehat{APB} = \alpha$, $AB = \delta$, $AP = r$, $BP = r'$

then $$\delta^2 = r^2 + r'^2 - 2\,rr'\cos\alpha$$

If the circles be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

then $$r^2 = g^2 + f^2 - c, \qquad r'^2 = g'^2 + f'^2 - c',$$

$$\delta^2 = (g - g')^2 + (f - f')^2$$

$$\therefore \quad 2\,rr'\cos\alpha = 2\,gg' + 2\,ff' - c - c'$$

Cor'—The circles cut orthogonally if

$$2\,gg' + 2\,ff' - c - c' = 0$$

§ **193.** *If a variable circle cut the two circles* $S = 0$, $S' = 0$ *at constant angles* α, β *then it cuts any co-axal circle* $S + kS' = 0$ *at a constant angle* γ.

Let the variable circle be

$$x^2 + y^2 + 2\,Gx + 2\,Fy + C = 0,$$

R its radius.

$S + kS' = 0$ when 'prepared' is

$$x^2 + y^2 + 2\,\frac{g + kg'}{1 + k}x + 2\,\frac{f + kf'}{1 + k}y + \frac{c + kc'}{1 + k} = 0$$

Let r'' be its radius. The formula of § 192 gives

$$2\,R\,r''\cos\gamma = 2\,G\,\frac{g + kg'}{1 + k} + 2\,F\,\frac{f + kf'}{1 + k} - \frac{c + kc'}{1 + k} - C$$

$$\therefore \quad 2\,R\,r''\cos\gamma\,(1 + k = 2\,Gg + 2\,Ff - c - C$$
$$+ k[2\,Gg' + 2\,Ff' - c' - C]$$
$$= 2\,Rr\cos\alpha + 2\,kRr'\cos\beta$$

$$\therefore \quad (1 + k)\,r''\cos\gamma = r\cos\alpha + kr'\cos\beta$$

This formula shows that γ is constant.

§ **194.** *A system of circles which cuts orthogonally two given circles has a common radical axis.*

Let the given circles be

$$x^2 + y^2 + 2\,gx + 2\,fy + c = 0, \qquad x^2 + y^2 + 2\,g'x + 2\,f'y + c' = 0$$

and $$x^2 + y^2 + 2Gx + 2Fy + C = 0 \quad \ldots\ldots \quad (1)$$
one of the orthogonal circles.

$$\therefore \quad 2gG + 2fF - c - C = 0 \quad \ldots\ldots \quad (2)$$
$$2g'G + 2f'F - c' - C = 0 \quad \ldots\ldots \quad (3)$$

Eliminate G, F linearly from (1), (2), (3).

Thus (1) is replaced by

$$\begin{vmatrix} x & y & x^2+y^2+C \\ g & f & -c-C \\ g' & f' & -c'-C \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y & x^2+y^2 \\ g & f & -c \\ g' & f' & -c' \end{vmatrix} + C \begin{vmatrix} x & y & 1 \\ g & f & -1 \\ g' & f' & -1 \end{vmatrix} = 0$$

If C varies this represents a circle of a co-axal system whose common radical axis is

$$\begin{vmatrix} x & y & 1 \\ g & f & -1 \\ g' & f' & -1 \end{vmatrix} = 0$$

i. e. the join of the centres of the given circles.

§ **195.** Otherwise thus:

Let L, M be the limiting points of the system determined by the given circles S, S′.

Let P be any point on the radical axis. Then PL = PM = tangent from P to any of the circles.

$\therefore$ a circle, centre P, radius PL, passes through M and cuts each of the circles orthogonally.

The system passing through the two fixed points L, M is $\therefore$ orthogonal to S, S′.

Cor′—The centre of the circle which cuts three given circles orthogonally is their radical centre.

§ **196.** Ex. Find the circle which cuts orthogonally the three circles

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$
$$x^2 + y^2 + 2g''x + 2f''y + c'' = 0$$

Let the orthotomic circle be

$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \quad \ldots\ldots \quad (1)$$

Then
$$2gG + 2fF - c - C = 0 \quad \ldots\ldots \quad (2)$$
$$2g'G + 2f'F - c' - C = 0 \quad \ldots\ldots \quad (3)$$
$$2g''G + 2f''F - c'' - C = 0 \quad \ldots\ldots \quad (4)$$

From (1), (2), (3), (4) we may eliminate linearly G, F, C.

The required equation is $\therefore$

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ -c & g & f & -1 \\ -c' & g' & f' & -1 \\ -c'' & g'' & f'' & -1 \end{vmatrix} = 0$$

POLAR CO-ORDINATES

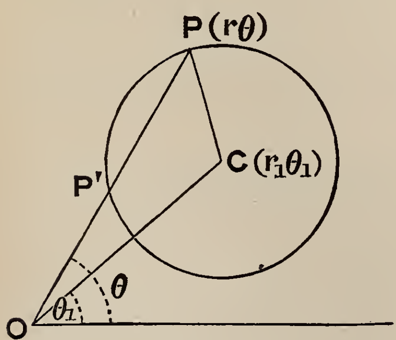
§ **197.** *To find the polar equation to a circle.*

In $$x^2 + y^2 + 2gx + 2fy + c = 0,$$
substitute $r\cos\theta$, $r\sin\theta$ for x, y.

This gives the required equation, viz.
$$r^2 + 2r(g\cos\theta + f\sin\theta) + c = 0$$

If the circle pass through the origin $c = 0$; dividing by r the equation becomes
$$r + 2(g\cos\theta + f\sin\theta) = 0$$

§ **198.** Another method. To find the polar equation to a circle, centre $(r_1\,\theta_1)$, radius $= \delta$.



Let C be the centre, P any point $(r\,\theta)$ on the circle.

Then $CP^2 = OP^2 + OC^2 - 2\,OP\,.\,OC\cos COP$;

$$\therefore \quad \delta^2 = r^2 + r_1^2 - 2\, rr_1 \cos(\theta - \theta_1) \quad . \quad . \quad . \quad (1)$$

This is the equation required.

§ 199. The equation (1), viz.

$$r^2 - 2\, rr_1 \cos(\theta - \theta_1) + r_1^2 - \delta^2 = 0$$

is a quadratic in r.

Thus for each value of θ it determines two values OP, OP' of r.

By the theory of quadratics the product of the roots is $r_1^2 - \delta^2$; this is independent of θ.

Thus $\qquad OP \,.\, OP' = r_1^2 - \delta^2 = OC^2 - CP^2.$

This agrees with Euclid III. 35, 36.

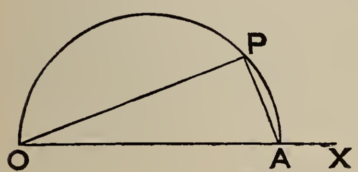
Ex. 1. What locus is represented by $r = a \cos\theta$?

Multiply by r; $\qquad \therefore \quad r^2 = a\,(r\cos\theta)$

That is, $x^2 + y^2 = ax$; thus the locus is a circle.

Otherwise thus. Along the initial line measure $OA = a$.

Let P be a point on the locus; join PA.



Then $\qquad OP = OA \cos\theta$

$\therefore \quad O\hat{P}A = \text{rt.} \angle$

$\therefore \quad$ locus is a circle on diameter OA.

Ex. 2. Show that

$$r = 2\, a \cos\theta + 2\, b \sin\theta$$

represents a circle; find its radius and the polar co-ordinates of its centre.

Multiply by r, $\quad \therefore \quad r^2 = 2\, a\,(r\cos\theta) + 2\, b\,(r \sin\theta)$

$$\therefore \quad x^2 + y^2 = 2\, ax + 2\, by$$

$$\therefore \quad (x - a)^2 + (y - b)^2 = a^2 + b^2$$

This represents a circle, centre (a, b), and radius $= \sqrt{a^2 + b^2}$

By § 39, (2), the polar co-ordinates of the centre are

$$\left(r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\frac{b}{a}\right)$$

Ex. 3. Two circles intersect in O; through O any line is drawn cutting the circles in P, Q; find locus of mid point of PQ.

Take O for origin; let the equations of the circles be

$$x^2 + y^2 + 2\, gx + 2\, fy = 0, \qquad x^2 + y^2 + 2\, g'x + 2\, f'y = 0$$

Or, in polar co-ordinates (§ 197)

$$r + 2g\cos\theta + 2f\sin\theta = 0, \qquad r + 2g'\cos\theta + 2f'\sin\theta = 0$$

Now let OPQ be inclined at $\hat{\theta}$ to OX

$$\therefore \quad OP = -2g\cos\theta - 2f\sin\theta, \qquad OQ = -2g'\cos\theta - 2f'\sin\theta$$

Let R be mid point of PQ; let OR = r.

Then $$2r = OP + OQ.$$

Thus the locus is

$$2r = -2(g+g')\cos\theta - 2(f+f')\sin\theta$$

or $$r^2 + (g+g')(r\cos\theta) + (f+f')(r\sin\theta) = 0$$

or $$x^2 + y^2 + (g+g')x + (f+f')y = 0;$$

another circle through O.

OBLIQUE CO-ORDINATES

§ **200.** *To find the equation to a circle whose centre is* (a, b) *and radius* r; *the axes being inclined at* $\hat{\omega}$.

If (x, y) be any point on the circle, the distance between the points (x, y) and (a, b) is r.

Thus by § 14 the required equation is

$$(x-a)^2 + (y-b)^2 + 2(x-a)(y-b)\cos\omega = r^2 \quad . \quad . \quad (1)$$

If this be expanded, we see that the terms of the second degree are

$$x^2 + 2xy\cos\omega + y^2.$$

Thus the most general equation of the second degree representing a circle is

$$h(x^2 + 2xy\cos\omega + y^2) + 2gx + 2fy + c = 0 \quad . \quad . \quad (2)$$

We may find the co-ordinates of the centre and the radius of (2).

Thus divide (2) by h, and comparing its coefficients with those of (1) we get

$$\frac{g}{h} = -a - b\cos\omega, \qquad \frac{f}{h} = -b - a\cos\omega,$$

$$\frac{c}{h} = a^2 + b^2 + 2ab\cos\omega - r^2$$

Solving these we obtain the values of a, b, r.

CO-ORDINATES EXPRESSED BY A SINGLE PARAMETER

§ **201.** $(r\cos\theta,\ r\sin\theta)$ is evidently a point on the circle

$$x^2 + y^2 = r^2.$$

We may call this 'the point θ.'

Ex. 1. The equation to the chord joining the points α, β is

$$\begin{vmatrix} x & y & 1 \\ r\cos\alpha & r\sin\alpha & 1 \\ r\cos\beta & r\sin\beta & 1 \end{vmatrix} = 0$$

or reducing,

$$x\cos\tfrac{1}{2}(\alpha+\beta) + y\sin\tfrac{1}{2}(\alpha+\beta) = r\cos\tfrac{1}{2}(\alpha-\beta)$$

Ex. 2. Put $\beta = \alpha$ in the preceding: the tangent at α is $\therefore$

$$x\cos\alpha + y\sin\alpha = r$$

Ex. 3. Find locus of mid points of chords of

$$x^2 + y^2 = r^2$$

drawn through a fixed point (hk).

Let $(r\cos\alpha,\ r\sin\alpha)$, $(r\cos\beta,\ r\sin\beta)$ be the extremities of one of these chords. We have to eliminate α, β from

$$\left.\begin{aligned} h\cos\tfrac{1}{2}(\alpha+\beta) + k\sin\tfrac{1}{2}(\alpha+\beta) &= r\cos\tfrac{1}{2}(\alpha-\beta) \\ 2x &= r(\cos\alpha+\cos\beta) \\ 2y &= r(\sin\alpha+\sin\beta) \end{aligned}\right\}$$

The result is $\qquad x^2 + y^2 = hx + ky$

MISCELLANEOUS PROPOSITIONS

§ **202.** If O is the centre of a circle, and A, B two points, then

OA : OB = $\perp$ from A on polar of B : $\perp$ from B on polar of A

(*Salmon*)

Let the circle be $x^2 + y^2 = r^2$, let A be $(x_1 y_1)$ and B $(x_2 y_2)$.

Let the $\perp$s be AM, BN.

The polar of B is $\qquad xx_2 + yy_2 = r^2$

$$\therefore\ AM = (x_1x_2 + y_1y_2 - r^2)\big/\sqrt{x_2^2 + y_2^2}$$

$$\therefore\ AM.OB = x_1x_2 + y_1y_2 - r^2$$

$$= BN.OA \text{ by symmetry; } \therefore \text{ \&c.}$$

§ **203.** If the polars of the vertices of a $\triangle$ ABC form a $\triangle$ A′B′C′; then AA′, BB′, CC′ are concurrent.

Let the circle be $$x^2 + y^2 = r^2.$$

Let A be $(x_1 y_1)$, B $(x_2 y_2)$, C $(x_3 y_3)$.

The equation to A′B′ is $xx_3 + yy_3 = r^2$ . . . . (1)

" " A′C′ " $xx_2 + yy_2 = r^2$ . . . . (2)

As in § 124, the equation of the join of $(x_1 y_1)$ to inters′n of (1), (2), is

$$(xx_3 + yy_3 - r^2)\big/(x_1 x_3 + y_1 y_3 - r^2) = (xx_2 + yy_2 - r^2)\big/(x_1 x_2 + y_1 y_2 - r^2)$$

or

$$(x_1 x_2 + y_1 y_2 - r^2)(xx_3 + yy_3 - r^2) - (x_1 x_3 + y_1 y_3 - r^2)(xx_2 + yy_2 - r^2) = 0$$

say $u = 0$.

If by symmetry we write down the equations of BB′, $v = 0$ and of CC′, $w = 0$, we observe that

$$u + v + w = 0 \quad \textit{identically}$$

$\therefore$ &c. (§ 129)

Exercises on Chapter VI

1. Find the circle through the origin and the inters′ns of the line

$$2x + 3y + 4 = 0$$

and the circle $$x^2 + y^2 + 3x + 4y + 2 = 0;$$

also the circle through (1, 2) and the same intersections.

Ans. $2x^2 + 2y^2 + 4x + 5y = 0$, $2x^2 + 2y^2 = y + 8$

2. Find the circle whose diameter is the common chord of the circles

$$x^2 + y^2 - 23x + 11y = 0, \qquad x^2 + y^2 - 12x + 11 = 0$$

Ans. $x^2 + y^2 - 5x - 7y + 18 = 0$

3. Given base of a triangle, and $ab \sin(C - \alpha)$, where α is a given angle, find locus of vertex.

[Let $ab \sin(C - \alpha) = \delta^2$; take side AB as axis of x, A being origin]

Ans. The circle $x^2 + y^2 - cx - cy \cot\alpha + \delta^2 \operatorname{cosec}\alpha = 0$

4. A and A′ are two points moving along a given line connected by a rod of length a, B and B′ two other points connected by a rod of length b, moving along a line at right angles to the first. Show that if A and B are connected by a third rod of length c, the middle point of A′ B′ will describe a circle.

Ans. Taking the given lines as axes, the equation to the locus is

$$(2x - a)^2 + (2y - b)^2 = c^2$$

5. Two segments AB, CD of a given line subtend equal angles at P; find the locus of P.

Ans. Taking the given line as axis of x; if $\alpha, \beta, \gamma, \delta$ are the distances of A, B, C, D from origin, locus is the circle

$$(\alpha - \beta - \gamma + \delta)(x^2 + y^2) + 2(\beta\gamma - \alpha\delta)x + \gamma\delta(\alpha - \beta) - \alpha\beta(\gamma - \delta) = 0$$

6. A and B are fixed points; R, S are variable points on AB such that

$$AR^2 + RS^2 + SB^2 = c^2 = \text{constant.}$$

If PRS is an equilateral triangle find the locus of P.

Ans. Taking A as origin, AB as axis of x, AB = a;

$$2x^2 + 2y^2 - 2ax - \frac{2ay}{\sqrt{3}} = c^2 - a^2$$

7. Prove that the diameter of the circle through the origin and $(\rho_1\theta_1)$, $(\rho_2\theta_2)$ is

$$\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2)}\Big/\sin(\theta_1 - \theta_2)$$

8. A point moves so that the square of its distance from the base of an isosceles triangle = rectangle under its distances from the sides: show that its locus is a circle.

[Take mid point of base for origin and base for axis of x.

Let vertices be (a, o), (– a, o), (o, h).]

Ans. $h(x^2 + y^2) + 2a^2y = ha^2$

9. A, B, C, &c. are given points; a point P moves so that

$$PA^2 + PB^2 + PC^2 + \&c. = \text{constant}:$$

show that its locus is a circle whose centre is the mean centre of the given points.

10. If $\quad m_1 PA^2 + m_2 PB^2 + m_3 PC^2 + \&c. = \text{constant},$

the locus of P is a circle whose centre is the mean centre of the points A, B, C, &c. for the system of multiples m_1, m_2, &c.

11. A point moves so that the sum of the squares of its distances from the sides of a regular polygon is constant: show that its locus is a circle.

[Equation to locus is

$$(x \cos\alpha + y \sin\alpha - p)^2$$

$$+ \left[x \cos\left(\alpha + \frac{2\pi}{n}\right) + y \sin\left(\alpha + \frac{2\pi}{n}\right) - p\right]^2 + \dots = \text{constant}$$

Then

$$\text{coeff}' \text{ of } xy = \sin 2\alpha + \sin\left(2\alpha + \frac{4\pi}{n}\right) + \sin\left(2\alpha + \frac{8\pi}{n}\right) + \dots = 0 \,*$$

Also

$$\text{coeff}' \text{ of } x^2 - \text{coeff}' \text{ of } y^2$$

$$= \cos 2\alpha + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \cos\left(2\alpha + \frac{8\pi}{n}\right) + \dots = 0]$$

12. On a line which revolves round a fixed point O and meets a given circle in P a point Q is taken such that

$$OP.OQ = \text{constant} = k^2;$$

show that the locus of Q is a circle except in the case when O is on the given circle, when the locus is a straight line.

[Proceed as in § 111, Ex. 1. The locus of Q is the *inverse* (see *Euclid Revised*, p. 351) of the given circle. Accordingly the inverse of a circle is a circle, unless the centre of inversion is on the given circle, when the inverse is a straight line.]

13. Show that the inverse of a straight line is a circle (see § 111).

14. A point moves so that the square of its distance from a fixed point varies as its distance from a fixed line: show that its locus is a circle.

15. The co-ord's of the ends of the base of a triangle are $(a, 0)$, $(-a, 0)$: if the vertical angle C is given, show that the locus of the orthocentre is the circle

$$x^2 + y^2 + 2ay \cot C = a^2$$

16. A and B are fixed points on rectangular axes such that

$$OA = OB = a;$$

a point P moves so that $\widehat{OPA} = \widehat{OPB}$:

show that the equation to the locus of P is

$$(x - y)(x^2 + y^2 - ax - ay) = 0$$

Interpret this.

* See Nixon's *Trigonometry*, page 328.

17. Show that the equation to the tangents from the origin to the circle

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

is

$$(\beta x - \alpha y)^2 = r^2 (x^2 + y^2)$$

18. Show that the equation to a segment of a circle through $(x_1 y_1)$, $(x_2 y_2)$ containing an angle α is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = \pm \cot \alpha \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

[Express that joins of (hk) to $(x_1 y_1)$ and to $(x_2 y_2)$ include $\hat{\alpha}$: then write x, y for h, k.]

19. The area of the triangle formed by the two tangents from (hk) to

$$x^2 + y^2 = r^2$$

and their chord of contact is

$$\frac{r(h^2 + k^2 - r^2)^{\frac{3}{2}}}{h^2 + k^2}$$

20. The circle whose diameter is the join of $(r_1 \theta_1)$, $(r_2 \theta_2)$ is

$$r^2 = r_1 r \cos(\theta - \theta_1) + r_2 r \cos(\theta - \theta_2) - r_1 r_2 \cos(\theta_1 - \theta_2)$$

[Express that $PA^2 + PB^2 = AB^2$, where P is $(r\theta)$, A $(r_1 \theta_1)$, B $(r_2 \theta_2)$.]

21. Find the equation of the chord joining the points

$$(2a\cos\alpha, \alpha), \qquad (2a\cos\beta, \beta)$$

on the circle $\qquad r = 2a\cos\theta$

Ans. $2a\cos\beta\cos\alpha = r\cos(\beta + \alpha - \theta)$

22. Find the tangent to the circle

$$r = 2a\cos\theta$$

at the point $\qquad (2a\cos\alpha, \alpha)$.

Ans. $2a\cos^2\alpha = r\cos(2\alpha - \theta)$

[Put $\beta = \alpha$ in preceding result]

23. Find the condition that the line

$$\frac{1}{r} = a\cos\theta + b\sin\theta$$

may touch the circle $\qquad r = 2c\cos\theta$

Ans. $b^2c^2 + 2ac = 1$

24. If
$$x^2 + xy + y^2 + 2x + 3y = 0$$
represent a circle, show that $\omega = 60^\circ$. Determine the centre and radius.

Ans. $(-\frac{1}{3}, -\frac{4}{3})$; $\frac{1}{3}\sqrt{21}$

25. Find the equation to a circle whose centre is $(1, -2)$ and radius 5, the axes being inclined at 120°.

Ans. $x^2 - xy + y^2 = 4x - 5y + 18$

26. The axes being inclined at an angle ω, find the equation to the circle through the origin which intercepts lengths α, β on the axes.

Ans. $x^2 + 2xy\cos\omega + y^2 = \alpha x + \beta y$

27. If the intercepts in the last question are connected by the relation
$$l\alpha + m\beta = 1$$
where l, m are constants, show that the circle passes through the fixed point
$$\left(\frac{l}{l^2 + 2lm\cos\omega + m^2}, \frac{m}{l^2 + 2lm\cos\omega + m^2}\right)$$

28. A circle touches the axes (which include $\hat{\omega}$) at points A, B such that
$$OA = OB = \delta:$$
show that its equation is
$$(x + y - \delta)^2 = 4xy\sin^2\frac{\omega}{2}$$

29. Show that
$$\frac{x}{h} + \frac{y}{k} = 1$$
touches the circle in the last question if
$$\left(1 - \frac{\delta}{h}\right)\left(1 - \frac{\delta}{k}\right) = \sin^2\frac{\omega}{2}$$

[Combining eq'ns of line and circle
$$x + y - \delta = \pm 2\sin\frac{\omega}{2}\sqrt{xy}$$
we get
$$x + y \pm 2\sin\frac{\omega}{2}\sqrt{xy} = \delta\left(\frac{x}{h} + \frac{y}{k}\right)$$
or
$$x\left(1 - \frac{\delta}{h}\right) \pm 2\sin\frac{\omega}{2}\sqrt{xy} + y\left(1 - \frac{\delta}{k}\right) = 0:$$
Express that the roots of this quadratic for $\sqrt{x} : \sqrt{y}$ are equal.]

30. Find the equation to the locus of a point whose polar with regard to

$$x^2 + y^2 = a^2$$

touches

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

Ans. $(\alpha x + \beta y - a^2)^2 = r^2(x^2 + y^2)$

31. Show that the polar of a point with regard to any circle of a co-axal system passes through a fixed point.

[The polar of $x'y'$ with respect to

$$x^2 + y^2 + 2\lambda x = c^2 \quad (\S\ 188)$$

is

$$xx' + yy' + \lambda(x + x') = c^2$$

The fixed point is determined by

$$x + x' = 0, \qquad xx' + yy' = c^2]$$

32. If AB is a diameter of a circle, prove that the polar of A with respect to any circle which cuts the first orthogonally will pass through B.

33. Show that the orthotomic circle of

$$x^2 + y^2 = a^2, \qquad (x - b)^2 + y^2 = a^2, \qquad x^2 + (y - c)^2 = a^2$$

is

$$x^2 + y^2 - bx - cy + a^2 = 0$$

[*Note*—Centre of req'd circle is radical centre; its radius = tangent from radical centre to one of the circles.]

34. Find the circles through (1, 2), (1, 18) touching the axis of x.

Ans. $x^2 + y^2 + 10x - 20y + 25 = 0$, $x^2 + y^2 - 14x - 20y + 49 = 0$

35. Find the circle through (2, 0) which cuts

$$x^2 + y^2 = 4 \quad \text{and} \quad x^2 + y^2 = 2y + 8$$

at right angles.

Ans. $x^2 + y^2 - 4x + 4y + 4 = 0$

36. Show that a circle can be inscribed in the quadrilateral whose sides are

$$x = 0, \quad y = 0, \quad x\cos\alpha + y\sin\alpha = p, \quad x\cos\beta + y\sin\beta = p',$$

if

$$p(1 + \sin\beta + \cos\beta) = p'(1 + \sin\alpha + \cos\alpha)$$

37. Find the system of circles which cut orthogonally each of the circles

$$x^2 + y^2 + 2\lambda x - c^2 = 0$$

where λ is a variable parameter.

Ans. $x^2 + y^2 + 2\mu y + c^2 = 0$, where μ is a variable parameter.

38. Show that any common tangent to two circles is bisected by their radical axis, also that the common tangent subtends a right angle at either limiting point.

[*Note*—If A, B are p'ts of contact, L a limiting p't, C point where AB meets radical axis: then

$$CA = CB = CL \quad (\S\ 179)]$$

39. The equation to the circle whose diameter is the join of the centres of similitude of two circles $S = 0$, $S' = 0$ is

$$\frac{S}{r^2} - \frac{S'}{r'^2} = 0$$

where r, r' are the radii of the circles.

[*Note*—This is the *circle of similitude* of the two circles. See *Euclid Revised,* page 336. We see then that tangents to S, S' from any point on their circle of similitude are in the constant ratio $r : r'$.]

40. Find the locus of a point P at which two circles subtend equal angles.

Ans. The circle of similitude.

41. Show that the circles

$$\frac{S}{r} \pm \frac{S}{r'} = 0$$

cut at right angles.

[*Note*—The centres of these circles are the centres of similitude.]

42. Show that the circles in the last question bisect the angles between $S = 0$, $S' = 0$.

43. Show that the equation to the circle through the three points

$$(x_1 y_1), \quad (x_2 y_2), \quad (x_3 y_3)$$

is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

[*Note*—Let circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Then $\quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0, \ \ldots = 0, \ \ldots = 0.$ Elim'g f, c.]

44. Find the condition that $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$, $(x_4 y_4)$ may be concyclic; interpret geometrically.

Ans. (1) Write x_4, y_4 instead of x, y in preceding result.

(2) If A, B, C, D are four concyclic points and O any other point,

$$OA^2 . \text{area } BCD + OC^2 . \text{area } ABD = OB^2 . \text{area } ACD + OD^2 . \text{area } ABC$$

45. Show that the Nine-points' circle of the triangle whose vertices are

$$\left(2a, \frac{2}{a}\right), \quad \left(2b, \frac{2}{b}\right), \quad \left(2c, \frac{2}{c}\right)$$

passes through the origin.

46. Show that the radical axis of a circle and a point bisects the distance between the point and its polar with regard to the circle.

47. A and B are points on the circle

$$x^2 + y^2 = a^2:$$

if AB subtend a right angle at a fixed point $(x'y')$, show that the locus of the mid point of AB is the circle

$$(x - x')^2 + (y - y')^2 + x^2 + y^2 = a^2$$

[*Note*—Let A be $(a \cos \alpha, a \sin \alpha)$ and B $(a \cos \beta, a \sin \beta)$. Eliminate α, β from

$$(x' - a \cos \alpha)(x' - a \cos \beta) + (y' - a \sin \alpha)(y' - a \sin \beta) = 0$$

(Page 126, Ex. 8)

$$2x = a \cos \alpha + a \cos \beta, \qquad 2y = a \sin \alpha + a \sin \beta]$$

48. Show also that locus of the foot of perpendicular from centre of

$$x^2 + y^2 = a^2$$

on AB is the *same* circle; and that locus of intersection of tangents at A, B is the circle

$$(x'^2 + y'^2 - a^2)(x^2 + y^2) - 2a^2x'x - 2a^2y'y + 2a^4 = 0$$

49. Two chords of a circle cut at right angles; show that tangents at their extremities form a quadrilateral whose vertices are concyclic: and deduce that the problem 'to inscribe a quadrilateral in a circle whose sides shall touch another given circle' is either indeterminate or impossible.

[*Note*—This follows from Ex. 48]

50. Show that the points where the line

$$Ax + By + C = 0$$

cuts the circle $\qquad x^2 + y^2 + 2gx + 2fy + c$

are concyclic with the points where the axis of x cuts the circle

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

if $\qquad 2C(g - g') = A(c - c')$

[*Note*—The lines $\quad y = 0, \quad Ax + By + C = 0$ must meet on radical axis (§ 183)]

51. The equation to the real common tangents to the circles

$$x^2 + y^2 = 2\,ax, \qquad x^2 + y^2 = 2\,by$$

is $$2\,ab\,(x^2 + y^2 - 2\,ax) = (by - ax + ab)^2$$

52. Show that the locus of the centre of a circle which cuts two fixed circles orthogonally is their radical axis.

53. The centres of three circles are A, B, C and their radii r_1, r_2, r_3: show that the circles are co-axal if

$$BC \,.\, r_1^2 + CA \,.\, r_2^2 + AB \,.\, r_3^2 + BC \,.\, CA \,.\, AB = 0$$

54. Prove that if t_1, t_2, t_3 are the tangents from any point to three co-axal circles whose centres are A, B, C, then

$$BC \,.\, t_1^2 + CA \,.\, t_2^2 + AB \,.\, t_3^2 = 0$$

55. A variable circle cuts two given circles at constant angles α, β; show that it cuts their radical axis at a constant angle γ given by

$$\delta \cos\gamma = r\cos\alpha - r'\cos\beta$$

where r, r' are the radii of the given circles and δ the distance between their centres.

56. Deduce that the perpendicular p from the centre of the variable circle on the radical axis varies as its radius R.

[*Note*—$p = R\cos\gamma$]

57. Show that three circles are cut at equal imaginary angles by one of their axes of similitude.

[*Note*—Observe that $\perp$s from centres on an axis of sim′de are proportional to radii.]

58. A circle which cuts two given circles at constant angles touches two fixed circles.

59. Find locus of centre of a circle which cuts three given circles at equal angles.

Ans. A perpendicular from radical centre on one of the axes of similitude.

[*Note*—Obtain eq′n of locus by elim′n from three eq′ns like that in § 192; observe that radical centre, being centre of orthotomic circle, is a point on locus, and that axis of sim′de is a circle of infinite radius cutting given circles at equal angles.]

60. One vertex of a rectangle is fixed $(x'y')$; two other vertices move on the circle

$$x^2 + y^2 = a^2:$$

find the locus of the fourth vertex.

Ans. The circle $x^2 + y^2 = 2a^2 - x'^2 - y'^2$

61. Find locus of intersection of two straight lines at right angles, each of which touches one of the two circles

$$(x - a)^2 + y^2 = b^2, \qquad (x + a)^2 + y^2 = c^2,$$

and prove that the bisectors of the angles between the straight lines always touch one or other of two fixed circles.

[*Note*—Two $\perp$ tangents are

$$(x - a)\cos\theta + y\sin\theta = b, \qquad y\cos\theta - (x + a)\sin\theta = c;$$

elim′ θ, locus is

$$(x^2 + y^2 - a^2)^2 = (bx + cy + ab)^2 + (by - cx + ac)^2$$

The bisectors are

$$(x - a \pm y)\cos\theta + (y \mp x \mp a)\sin\theta = b \pm c,$$

which touch $\quad (x - a \pm y)^2 + (y \mp x \mp a)^2 = (b \pm c)^2,$

or $\quad x^2 + (y - a)^2 = \frac{1}{2}(b + c)^2, \qquad x^2 + (y + a)^2 = \frac{1}{2}(b - c)^2$]

62. The vertices of a triangle are $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$. If the lengths of its sides are a, b, c, show that the co-ordinates of its incentre are

$$\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \qquad \frac{ay_1 + by_2 + cy_3}{a + b + c}$$

[*Note*—Let bisector of $\hat{A}$ meet BC in D, and bisector of $\hat{C}$ meet AD in O. Then $BD : DC = c : b$, and

$$AO : OD = b : CD = b : \frac{ab}{b + c} = b + c : a; \text{ apply § 15.}]$$

63. Show that the co-ordinates of the limiting points of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \qquad x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

are given by the equations

$$x = -\frac{g + \lambda g'}{1 + \lambda}, \qquad y = -\frac{f + \lambda f'}{1 + \lambda}$$

where λ is either root of the equation

$$(g + \lambda g')^2 + (f + \lambda f')^2 = (1 + \lambda)(c + \lambda c')$$

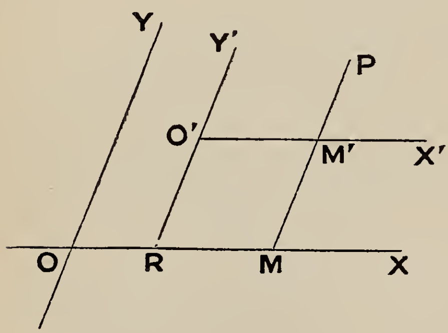
CHAPTER VII

TRANSFORMATION OF CO-ORDINATES

TRANSFERENCE OF ORIGIN

§ **204.** It is sometimes useful to refer points to a new pair of axes.

To transform to parallel axes through a given point (hk).



Let O' be the new origin;

$$OR = h, \; RO' = k$$

its co-ord's.

Let $OM = x$, $PM = y$ be the co-ord's of any point P referred to the old axes;

$$O'M' = x', \; PM' = y'$$

its co-ord's referred to the new axes.

We see from the figure that

$$x = OM = O'M' + OR = x' + h$$
$$y = PM = PM' + RO' = y' + k$$

Thus if, in the equation to a curve which expresses a relation between x and y, we substitute $x' + h$ for x and $y' + k$ for y we obtain a relation between x' and y'; i. e. this is the equation to the curve referred to the new axes.

Ex. The equation to a curve is

$$5x^2 + 6xy + 5y^2 - 38x - 42y + 93 = 0$$

What is its equation referred to parallel axes through $(2, 3)$?

Substitute $x' + 2$ for x and $y' + 3$ for y and reduce; then

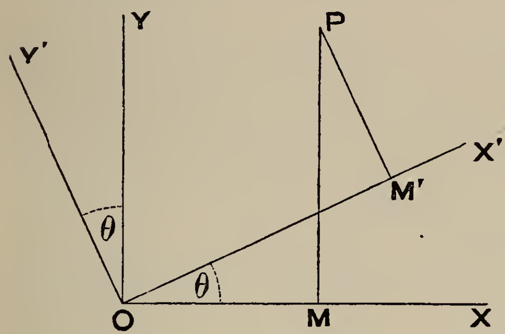
$$5x'^2 + 6x'y' + 5y'^2 = 8$$

We may now suppress the accents: and the req'd eq'n is

$$5x^2 + 6xy + 5y'^2 = 8$$

RECTANGULAR AXES TURNED THROUGH AN ANGLE

§ **205.** *To change from a set of rectangular axes to another set through the same origin inclined to the first set at an angle* θ.



Let the old co-ord's of a point P be

$$OM = x, \quad PM = y;$$

and the new co-ord's

$$OM' = x', \quad PM' = y'.$$

Project the broken line $OM'P$ on OX; this projection = the projection of OP, ∴ (§ 60)

$$x = x'\cos\theta - y'\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, project $OM'P$ on OY,

$$\therefore \quad y = x'\sin\theta + y'\cos\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(1) and (2) are the required equations expressing the old co-ordinates in terms of the new.

206. The equations (1), (2) may also be obtained thus.

Join OP; let $OP = r$, $P\hat{O}X' = \phi$

Then $\quad x = OM = r\cos POM = r\cos(\theta + \phi)$

$$\therefore \quad x = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$$

$\therefore \quad x = x'\cos\theta - y'\sin\theta,$ which is (1)

Also $\quad y = PM = r \sin POM = r \sin(\theta + \phi)$

$$\therefore \quad y = r(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= (r\cos\phi)\sin\theta + (r\sin\phi)\cos\theta$$

or $\quad y = x'\sin\theta + y'\cos\theta,$ which is (2)

Ex. The equation to a curve is

$$5x^2 + 6xy + 5y^2 = 8;$$

what is the equation referred to rect′ axes through the same origin, inclined at 45° to the old axes?

Here $\quad x = x'.\frac{1}{\sqrt{2}} - y'.\frac{1}{\sqrt{2}}, \qquad y = x'.\frac{1}{\sqrt{2}} + y'.\frac{1}{\sqrt{2}}$

$$\therefore \quad 5\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 6\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + 5\left(\frac{x'+y'}{\sqrt{2}}\right)^2 = 8;$$

this reduces to $\quad 4x'^2 + y'^2 = 4.$

We may now suppress the accents: and the req′d eq′n is

$$4x^2 + y^2 = 4$$

Note—After a little practice the learner will find it unnecessary to use accented letters.

Thus the preceding example would be worked thus

Substitute
$$\left.\begin{array}{ll} \frac{x-y}{\sqrt{2}} & \text{instead of } x \\ \frac{x+y}{\sqrt{2}} & \quad ,, \quad ,, \quad y \end{array}\right\}$$

$\therefore$ eq′n becomes $\quad \frac{5}{2}(x-y)^2 + \frac{6}{2}(x^2-y^2) + \frac{5}{2}(x+y)^2 = 8$

or reducing, $\quad 4x^2 + y^2 = 4$

Exercises

1. If the origin is transferred to $(-3, 4)$, what does the equation

$$x^2 + y^2 + 6x - 8y = 0 \quad \text{become?}$$

Ans. $x^2 + y^2 = 25$

2. What do the equations

$$x^2 - y^2 = 2, \qquad x^2 + 3xy + y^2 = 2$$

become when the axes are turned through 45°?

Ans. $xy = -1, \; 5x^2 - y^2 = 4$

3. What do the equations

$$2x^2 + \sqrt{3}\,xy + y^2 = 7, \qquad 3x^2 + 2\sqrt{3}\,xy + y^2 = 5$$

become when the axes are turned through $30°$?

Ans. $5x^2 + y^2 = 14$, $4x^2 = 5$.

4. What does the equation

$$8x^2 + 24xy + y^2 + 32x - 20y = 37$$

become when referred to axes through $(1, -2)$ inclined at $\tan^{-1}\frac{3}{4}$ to the original axes?

Ans. $17x^2 - 8y^2 = 1$.

5. What does the equation

$$x^2 + 2xy\tan 2\beta - y^2 = 2c^2$$

become when the axes are turned through an angle β?

Ans. $x^2 - y^2 = 2c^2\cos 2\beta$.

6. With the notation of § 205 verify the relation

$$x'^2 + y'^2 = x^2 + y^2$$

7. If by the transformation of § 205

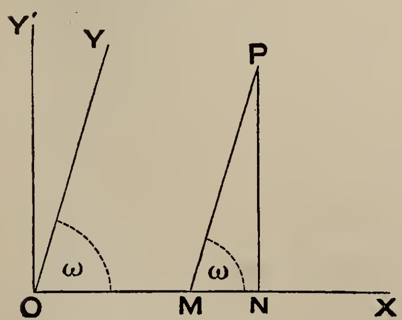
$$ax^2 + 2hxy + by^2$$

becomes $$a'x'^2 + 2h'x'y' + b'y'^2,$$

verify the relations

$$a' + b' = a + b, \qquad a'b' - h'^2 = ab - h^2$$

§ **207.** *To transform from oblique to rectangular axes, retaining the origin and axis of* x.



Let $OM = x$, $PM = y$ be the old co-ord's of a point P;

$ON = x'$, $PN = y'$ the new co-ord's.

We see from fig' that

$$\left.\begin{aligned} x' &= x + y\cos\omega \\ y' &= y\sin\omega \end{aligned}\right\}$$

Solving these equations, for x, y we have the required formulae expressing the old co-ord's in terms of the new:

$$\left.\begin{aligned} x &= x' - y' \cot \omega \\ y &= y' \operatorname{cosec} \omega \end{aligned}\right\}$$

§ **208.** Suppose that we take the point (h, k) as origin of polar co-ordinates, the initial line being parallel to OX.

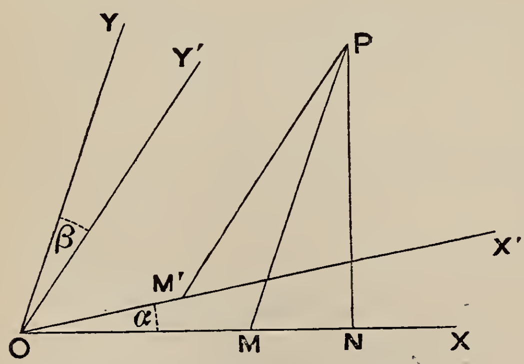
By drawing a figure we see at once that

$$\left.\begin{aligned} x &= h + r \cos \theta \\ y &= k + r \sin \theta \end{aligned}\right\}$$

These formulae are sometimes useful.

The above are the most useful cases of transformation. The formulae investigated in the next § are rarely used.

§ **209.** *To transform from one set of oblique axes to another, retaining the same origin.*



Let OX, OY be the old axes, OX′, OY′ the new axes.

Let $X\hat{O}Y = \omega$.

Also let $X\hat{O}X' = \alpha$,

$YOY' = \beta$.

Let the old co-ord's of a point P be

$$x = OM, \qquad y = PM;$$

and its new co-ord's $\quad x' = OM', \qquad y' = PM'$.

Draw PN ⊥ OX.

The broken lines OMP, OM′P have equal projections on any line.

Project these broken lines on PN;

$$\therefore \quad y \sin \omega = x' \sin \alpha + y' \sin (\omega - \beta)$$

Similarly by projecting on the $\perp$ from P to OY we obtain

$$x \sin \omega = x' \sin(\omega - \alpha) + y' \sin \beta$$

Cor'—If we change to axes through the point (h, k) parallel to OX', OY'; then from § 204 it follows that the equations expressing the old co-ordinates in terms of the new are of the form

$$x = h + lx' + my', \qquad y = k + l'x' + m'y'$$

DEGREE OF TRANSFORMED EQUATION

§ **210.** We have just seen that however the axes are changed the old co-ordinates are expressed in terms of the new by equations of the form

$$x = h + lx' + my', \qquad y = k + l'x' + m'y'$$

The degree of an equation is unaltered by transformation of co-ordinates.

For when the above expressions for x, y are substituted any term $Ax^r y^s$ (whose degree is $r + s$) becomes

$$A(h + lx' + my')^r (k + l'x' + m'y')^s;$$

and if this be expanded no term is of a degree higher than $r + s$. Thus *the degree cannot be raised* by transformation. *Neither can it be depressed:* for if it could the degree of the new equation could be raised by returning to the original axes.

INVARIANTS

§ **211.** Suppose that we are transforming from axes inclined at an angle ω to other axes through the same origin inclined at an angle ω', and that on making the substitutions of § 209 (which are of the form

$$x = lx' + my', \qquad y = l'x' + m'y'),$$

the expression $\quad ax^2 + 2hxy + by^2$

becomes $\quad a'x'^2 + 2h'x'y' + b'y'^2;$

we proceed to investigate relations between the coefficients a, h, b, a', h', b'.

There are two expressions for the square of the distance of any point from the origin; if P is the point whose old co-ordinates are x, y and new co-ordinates x', y',

$$OP^2 = x^2 + 2xy \cos\omega + y^2,$$

and $\quad OP^2 = x'^2 + 2x'y' \cos\omega' + y'^2.$

Therefore when the substitutions are made

$$x^2 + 2xy\cos\omega + y^2$$

must become $$x'^2 + 2x'y'\cos\omega' + y'^2$$

Also $$ax^2 + 2hxy + by^2$$

becomes $$a'x'^2 + 2h'x'y' + b'y'^2$$

Therefore

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + 2xy\cos\omega + y^2) \quad . \quad . \quad (1)$$

becomes

$$a'x'^2 + 2h'x'y' + b'y'^2 + \lambda(x'^2 + 2x'y'\cos\omega' + y'^2) \quad . \quad (2)$$

Whatever be the value of λ the expression (1) is identically equal to (2); if we suppose λ a constant and its value so chosen that (1) is a perfect square, then (2) is a perfect square for the same value of λ.

(1) is a perfect square if

$$(a + \lambda)(b + \lambda) = (h + \lambda\cos\omega)^2$$

or $$\lambda^2 + \frac{a + b - 2h\cos\omega}{\sin^2\omega}\lambda + \frac{ab - h^2}{\sin^2\omega} = 0$$

Similarly (2) is a perfect square if

$$\lambda^2 + \frac{a' + b' - 2h'\cos\omega'}{\sin^2\omega'}\lambda + \frac{a'b' - h'^2}{\sin^2\omega'} = 0$$

Since these quadratics have the same roots, we see that

$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} = \frac{a' + b' - 2h'\cos\omega'}{\sin^2\omega'}$$

and $$\frac{ab - h^2}{\sin^2\omega} = \frac{a'b' - h'^2}{\sin^2\omega'}$$

If both sets of axes are rectangular, $\omega = \omega' = 90^\circ$

$$\therefore \quad a + b = a' + b'$$

and $$ab - h^2 = a'b' - h'^2$$

§ **212.** Functions of the coefficients which are equal to the same functions of the new coefficients obtained by transformation are called *invariants*. We have then obtained two invariants of the expression

$$ax^2 + 2\,hxy + by^2,$$

viz. $$\frac{a + b - 2\,h\cos\omega}{\sin^2\omega} \quad\text{and}\quad \frac{ab - h^2}{\sin^2\omega}$$

When the axes are rectangular the invariants are

$$a + b \quad\text{and}\quad ab - h^2$$

Miscellaneous Exercises

1. The equation to a line referred to rectangular axes OX, OY is

$$y = 3x + 2;$$

find its equation referred to OX, OY′ where OY′ makes an angle of 60° with OX.

Ans. $6x + (3 - \sqrt{3})y + 4 = 0$

2. If x, y are the co-ord′s of a point referred to rectangular axes; find its co-ord′s referred to the two lines whose equation is

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

Ans. $\frac{1}{2}\sqrt{a^2 + b^2}\left(\frac{x}{a} - \frac{y}{b}\right)$, $\frac{1}{2}\sqrt{a^2 + b^2}\left(\frac{x}{a} + \frac{y}{b}\right)$

3. Show that

$$(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2 = k^2(x\cos\alpha + y\sin\alpha - a)^2$$

represents two straight lines; and that the bisectors of the angle they include are

$$y = x\tan\alpha, \qquad x\cos\alpha + y\sin\alpha = a$$

4. If the formulae for transformation to a new pair of axes with the same origin be

$$x = mx' + ny', \qquad y = m'x' + n'y';$$

prove that $$nn'(m^2 + m'^2 - 1) = mm'(n^2 + n'^2 - 1)$$

[*Note*—

$x'^2 + 2\,x'y'\cos\omega' + y'^2 = x^2 + 2\,xy\cos\omega + y^2 = (mx' + ny')^2 + \&c.$

Equate coeff′s of x'^2, y'^2 and elim′ $\cos\omega$.]

5. If $$lx + my + n = 0, \qquad Lx + My + N = 0$$

represent the same line, referred to axes with a common origin but inclined at angles ω, Ω, prove that

$$(l^2 + m^2 - 2\, lm \cos\omega) \sin^2\Omega = (L^2 + M^2 - 2\, LM \cos\Omega) \sin^2\omega$$

[*Note*—$n = N$; compare expressions for length of $\perp$ on line from origin.]

6. If a straight line meet the sides of a triangle ABC in X, Y, Z, then the product of the ratios

$$(BX : XC)\,(CY : YA)\,(AZ : ZB) = -1$$

[*Note*—Let A be $(x_1 y_1)$, B $(x_2 y_2)$ and C $(x_3 y_3)$; and let the equation to XYZ be

$$ax + by + c = 0$$

Then $$\frac{BX}{XC} = -\frac{ax_2 + by_2 + c}{ax_3 + by_3 + c} \quad (\S\ 73), \text{ \&c.}$$

This is Menelaus' Theorem. See *Euclid Revised*, p. 321]

7. The three sides of a triangle pass each through one of three collinear points, and two of its vertices move on fixed lines: show that the third vertex describes a straight line.

8. Show that the lines

$$(ah' - a'h)\,x^2 + (ab' - a'b)\,xy + (hb' - h'b)\,y^2 = 0$$

are harmonic conjugates of the lines

$$ax^2 + 2\,hxy + by^2 = 0$$

and also of the lines

$$a'x^2 + 2\,h'xy + b'y^2 = 0$$

9. If the circles

$$(y - b)^2 + (x - a)(x - a') = 0$$
$$(y - B)^2 + (x - A)(x - A') = 0$$

touch each other, prove that either

$$(B - b)^2 + (A - a)(A' - a') = 0$$

or $$(B - b)^2 + (A - a')(A' - a) = 0$$

10. Determine the equation of the circle intersecting the circle

$$x^2 + y^2 + 3x + 5y + 2 = 0$$

in the chord $$x + 2y - 3 = 0$$

and the circle $$x^2 + y^2 - x + y - 2 = 0$$
in the chord $$2x + y + 6 = 0;$$
and examine under what circumstances such a problem is possible.

Ans. $3x^2 + 3y^2 + 5x + 7y + 18 = 0$

[*Note*—The two lines must meet on the radical axis of the circles.]

11. Determine λ so that

$$\lambda(x^2 + y^2 - a^2) + \left[x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} - a\cos\frac{\alpha-\beta}{2}\right]$$
$$\left[x\cos\frac{\alpha+\gamma}{2} + y\sin\frac{\alpha+\gamma}{2} - a\cos\frac{\alpha-\gamma}{2}\right]$$

may represent a pair of straight lines.

Ans. $\lambda = \sin\dfrac{\alpha-\beta}{2}\sin\dfrac{\gamma-\alpha}{2}$

12. The equations of two circles taking a centre of similitude as origin may be written

$$x^2 + y^2 - 2ax + a^2\cos^2\gamma = 0, \qquad x^2 + y^2 - 2a'x + a'^2\cos^2\gamma = 0:$$

find that of the circle passing through the four points of contact of the common tangents from the origin.

Ans. $x^2 + y^2 - (a + a')x + aa'\cos^2\gamma = 0$

13. If $$2hxy + 2gx + 2fy + c = 0$$
represent two straight lines, show that
$$2fg = ch,$$
and that these lines and the axes form a parallelogram whose diagonals are
$$\frac{x}{f} - \frac{y}{g} = 0, \qquad \frac{x}{f} + \frac{y}{g} + \frac{1}{h} = 0.$$

14. Find the locus of a point such that the two pairs of tangents drawn from it to the circles
$$x^2 + y^2 + 2gx = 0, \qquad x^2 + y^2 + 2fy = 0$$
may form a harmonic pencil.

Ans. The two parallel lines $gx - fy = \pm fg$.

15. If $S = 0$, $S' = 0$ be two circles whose radii are r, r', prove that their internal centre of similitude is the centre of

$$\frac{S}{r} + \frac{S'}{r'} = 0,$$

and their external centre of similitude the centre of

$$\frac{S}{r} - \frac{S'}{r'} = 0$$

Thence infer that the six centres of similitude of three circles lie three by three on four right lines.

16. A, B, C are three points on a circle; show that the feet of the perpendiculars from any point O on the circle on the sides of the triangle ABC are collinear.

[Take O as origin of polar co-ord's; let eq'n of circle be

$$r = 2a \cos \theta$$

Let α, β, γ be the vectorial $\wedge$s of A, B, C.

The eq'n of BC is

$$2a \cos \beta \cos \gamma = r \cos (\beta + \gamma - \theta)$$

The eq'n of the $\perp$ on this line from O is

$$0 = r \sin (\beta + \gamma - \theta)$$

The co-ord's of foot of $\perp$ from O on BC are $\therefore$

$$(2a \cos \beta \cos \gamma,\ \beta + \gamma)$$

Similarly the feet of the other $\perp$s are

$$(2a \cos \gamma \cos \alpha,\ y + \alpha) \quad \text{and} \quad (2a \cos \alpha \cos \beta,\ \alpha + \beta)$$

These points evidently lie on the line

$$2a \cos \alpha \cos \beta \cos \gamma = r \cos (\alpha + \beta + \gamma - \theta)$$

This line is *Simson's Line*; see *Euclid Revised*, p. 168].

17. One diagonal of a complete quadrilateral is altogether fixed, the second diagonal is a segment of a fixed right line; show that if one extremity of the third diagonal describe a right line, the other extremity describes another right line.

CHAPTER VIII

THE PARABOLA

§ **213**. A *Conic Section* or a *Conic* is the locus of a point which moves so that its distance from a fixed point is to its distance from a fixed line in a constant ratio $e : 1$. Thus if (see fig' § 214) S is the fixed point, KK' the fixed line, and PM the $\perp$ on this line from a point P on the curve; then

$$SP = e\,PM$$

The fixed point S is called the *focus* and the fixed line KK' the *directrix*; e is called the *eccentricity*.

The curve is called a *parabola* if $e = 1$, an *ellipse* if $e < 1$, and a *hyperbola* if $e > 1$.

The properties of the curves obtained by cutting a cone on a circular base by a plane were first investigated by the Greek Geometers, Apollonius, &c.; any such section, as will be seen hereafter, is one of the curves just defined.

Ex. 1. Find eq'n to a parabola whose focus is $(1, 2)$ and directrix

$$3x + 4y + 5 = 0$$

Express that distance of a point (x, y) on the curve from focus = its distance from directrix.

$$\therefore \quad \sqrt{(x-1)^2 + (y-2)^2} = \frac{3x + 4y + 5}{\pm 5}$$

Squaring and reducing, the req'd eq'n is

$$16x^2 - 24xy + 9y^2 - 80x - 140y + 100 = 0$$

N

Ex. 2. The eq′n to an ellipse whose focus is (1, 0), directrix

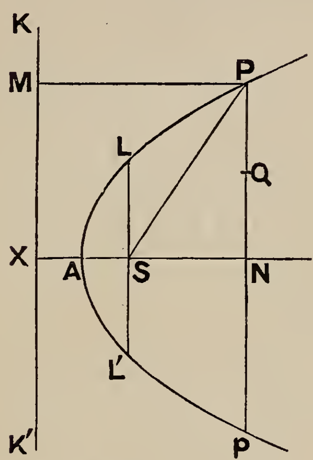
$$x - 4 = 0,$$

and eccentricity $\frac{1}{2}$, is $\sqrt{(x-1)^2 + y^2} = \frac{1}{2}(x-4)$;

or reducing, $3x^2 + 4y^2 = 12$

EQUATION TO PARABOLA

§ **214.** Let S be the focus, KK′ the directrix, P a point on the curve.



Draw PM, SX ⊥ KK′.

Bisect SX in A; then by def′ A is a point on the curve.

Take AS and the ⊥ to AS through A as axes of co-ord′s.

Put $AS = a$,

and let

$$AN = x, \quad PN = y$$

be the co-ord′s of P.

Then by def′ $$SP = PM$$

$$\therefore \quad SP = NX$$

$$\therefore \quad SP^2, \text{ or } SN^2 + NP^2 = NX^2$$

$$\text{i.e.} \quad (x-a)^2 + y^2 = (x+a)^2$$

$$\therefore \quad y^2 = 4ax$$

This is the simplest form of the equation to a parabola.

§ **215.** *Defs′*—The point A is called the *vertex*, and the line SX ⊥ the directrix is called the *axis* of the curve.

Cor′s—The co-ord′s of the focus are (a, 0).

The equation to the directrix is

$$x = -a, \quad \text{or} \quad x + a = 0$$

Again, $$SP = PM = NX = AN + XA$$

$$\therefore \quad SP = x + a$$

FIGURE OF THE CURVE

§ 216. From $$y^2 = 4ax$$

we deduce $$y = \pm 2\sqrt{ax}$$

Thus to each value of x correspond values of y which are equal in magnitude and of opposite sign.

If then p is the image of P with respect to the axis, p is also a point on the curve.

$\therefore$ the curve is symmetrical with respect to the axis.

If x is negative, y is imaginary.

Thus the curve lies wholly on the same side of the axis of y as the point S.

If x is very great so is y.

The curve $\therefore$ widens out indefinitely.

The axis of y is the tangent at the vertex. This is proved in § 221.

§ 217. The double ordinate LSL' through the focus is called the *latus rectum.*

The latus rectum $= 4a$.

For by def′ $SL = \perp$ from L on directrix $= SX = 2a$;

$$\therefore \quad LL' = 2SL = 4a$$

INTERNAL AND EXTERNAL POINTS

§ 218. If (x, y) are the co-ord′s of an internal point Q (see (fig′ § 214).

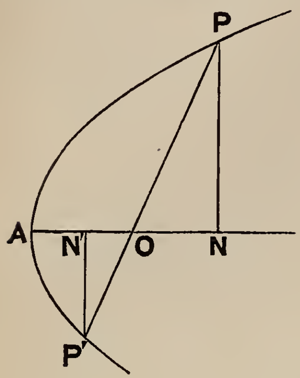
$$y^2 - 4ax = QN^2 - 4a \cdot AN$$
$$= QN^2 - PN^2, \quad \text{which is } \textit{negative.}$$

Thus *the* $y^2 - 4ax$ *of an internal point is negative.*

Similarly *the* $y^2 - 4ax$ *of an external point is positive.*

Of course the $y^2 - 4ax$ of a point on the curve is zero.

§ 219. Ex. 1. If a chord PP' passes through a fixed point O on the axis and PN, $P'N'$ are drawn $\perp$ the axis. Prove that $AN \cdot AN'$ is constant.



Let $AO = h$, $\therefore$ co-ord's of O are $(h, 0)$

The eq'n to any line PP' through O is

$$y = m(x - h)$$

If we combine this equation with

$$y^2 = 4ax$$

we obtain the co-ord's of P, P'.

Eliminating y, $\qquad m^2(x - h)^2 = 4ax$

$$\therefore \quad x^2 - \left(2h + \frac{4a}{m^2}\right)x + h^2 = 0$$

If the roots of this are x_1, x_2 then

$$x_1 x_2 = h^2$$

i.e. $\qquad AN \cdot AN' = AO^2$

Ex. 2. Trace the curve $\qquad x^2 = -4ay.$

The curve passes through the origin.

Giving any value to y there are two values of x, viz.

$$\pm\sqrt{-4ay}$$

That these may be real, y must be negative.

The curve is evidently a parabola of which the origin is the vertex, and axis of x tangent at the vertex; the curve lies *below* the axis of x.

The co-ord's of the focus are $(0, -a)$ and the equation to the directrix is

$$y = a$$

The learner should draw the figure.

Ex. 3. Trace the curve $\quad (x - 1)^2 = 2(y + 3).$

Changing to parallel axes through $(1, -3)$ this becomes

$$x^2 = 2y$$

As in Ex. 2 we see that this eq'n represents a parabola having for tangent at vertex the new axis of x, and its axis along the positive direction of the new axis of y.

Its latus rectum or $4a = 2$, $\therefore a = \frac{1}{2}$.

The co-ord's of the focus with reference to the new axes are $(0, \frac{1}{2})$ and the equation to the directrix is

$$y = -\frac{1}{2}$$

Returning to the original axes, the focus is $(1, -2\frac{1}{2})$ and the directrix is

$$y = -3\frac{1}{2}$$

These details will be clear from a figure, which the learner should draw.

Exercises

1. Is the point $(1, 2)$ inside or outside the parabola

$$y^2 = 8x?$$

2. Find the equation to a parabola whose focus is $(1, 0)$ and directrix

$$3x = 4y$$

Ans. $(4x + 3y)^2 = 50x - 25$.

3. The three vertices of an equilateral triangle are on the parabola

$$y^2 = 4ax,$$

one of them being the vertex: find the length of its side.

Ans. $8a\sqrt{3}$

4. Find the points where the line

$$y = 2x - 4a$$

cuts the parabola $y^2 = 4ax$

Ans. $(a, -2a)$, $(4a, 4a)$

5. Show that the line $y = x + 3$

touches the parabola $y^2 - 8x - 8 = 0$

Ans. The point of contact is $(1, 4)$.

6. Find the length of the chord which the parabola

$$y^2 = 8x$$

intercepts on the line $2x + y = 8$

Ans. $6\sqrt{5}$.

7. Find the co-ordinates of the focus, the equation of the directrix, and the length of the latus rectum in each of the parabolas

$$y^2 + 4ax = 0, \quad x^2 = -4ay, \quad (y - 1)^2 = 4(x - 2), \quad x^2 + 4y + 8 = 0.$$

Ans. $(-a, 0)$, $x = a$, $4a$; $(0, -a)$, $y = a$, $4a$; $(3, 1)$, $x = 1$, 4;
$(0, -3)$, $y + 1 = 0$, 4.

8. Show that $x^2 + 2ax + 4by + a^2 + 4bc = 0$
represents a parabola whose focus is

$$(-a, \; -c-b),$$

and directrix

$$y + c = b.$$

9. PSQ is a focal chord; PA meets the directrix in M. Prove that QM is parallel to the axis of the parabola.

10. A chord PQ of the parabola

$$y^2 = 4ax$$

subtends a right angle at the vertex. Prove that PQ passes through the fixed point $(4a, 0)$.

11. The vertices of a triangle are three points on the parabola

$$y^2 - 4ax = 0$$

whose ordinates are y_1, y_2, y_3.

Show that its area is

$$\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

12. Show that the polar equation of a parabola, the vertex being pole, is

$$r \sin^2\theta = 4a\cos\theta$$

13. Two chords through the vertex, whose lengths are r, r', are at right angles. Prove that

$$r^{\frac{4}{3}} r'^{\frac{4}{3}} = 16a^2 \left(r^{\frac{2}{3}} + r'^{\frac{2}{3}}\right)$$

EQUATION OF TANGENT

§ 220. *To find the equation of the tangent at* $(x'y')$.

If $(x'y')$, $(x''y'')$ are two points on the curve, the equation of their join is

$$\frac{y-y'}{x-x'} = \frac{y'-y''}{x'-x''} \quad \ldots\ldots\ldots \quad (1)$$

But since $(x'y')$, $(x''y'')$ satisfy the equation to the curve

$$y'^2 = 4ax' \quad \text{and} \quad y''^2 = 4ax''$$

$$\therefore \quad y'^2 - y''^2 = 4a(x' - x'')$$

$$\therefore \quad \frac{y' - y''}{x' - x''} = \frac{4a}{y' + y''}$$

$\therefore$ by substitution (1) becomes

$$\frac{y - y'}{x - x'} = \frac{4a}{y' + y''}$$

or $$y(y' + y'') - 4ax = y'^2 + y'y'' - 4ax'$$

$$= y'y'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since $$y'^2 = 4ax'$$

In (2) put $y'' = y'$, $x'' = x'$: it becomes

$$2yy' - 4ax = y'^2$$

$$= 4ax'$$

Thus the required equation is

$$yy' = 2a(x + x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This equation should be remembered.

§ **221.** *Cor'* (1)—The tangent at the vertex (0, 0) is

$$0 = 2ax, \quad \text{or} \quad x = 0;$$

i.e. the axis of y.

Cor' (2)—The equation of the chord obtained above, viz.

$$y(y' + y'') = 4ax + y'y''$$

is sometimes useful.

Cor' (3)—The normal is the line through $(x'y')$

$$\perp \; yy' = 2a(x + x')$$

Its equation is $\therefore$ $$\frac{y - y'}{y'} = \frac{x - x'}{-2a}$$

or $$2a(y - y') + y'(x - x') = 0$$

§ **222.** If we eliminate y between

$$y = mx + c \quad \text{and} \quad y^2 = 4ax$$

we get $$(mx + c)^2 = 4ax$$

or $$m^2x^2 + (2cm - 4a)x + c^2 = 0 \quad . \quad . \quad . \quad (1)$$

This equation determines the abscissae of the points where the line $y = mx + c$ cuts the curve; and since the equation is a quadratic we see that the line meets the curve in two points; these may be real, coincident, or imaginary.

If the points are coincident (1) has equal roots: the condition for this is

$$(cm - 2a)^2 = c^2 m^2$$

or

$$4a^2 - 4acm = 0$$

or

$$c = a/m$$

Again, substituting this value of c in (1) it reduces to

$$(mx - a/m)^2 = 0$$

$\therefore$ the abscissa of the point of contact is given by

$$mx - a/m = 0 \quad \text{or} \quad x = a/m^2.$$

Substitute this in $\qquad y^2 = 4ax$

$$\therefore \quad y = 2a/m$$

Thus *whatever be the value of* m, *the line*

$$y = mx + \frac{a}{m}$$

touches the parabola at the point $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

This result should be remembered.

§ **223.** Otherwise thus. Compare

$$yy' = 2a(x + x')$$

or

$$y = \frac{2a}{y'}x + \frac{2ax'}{y'}$$

and

$$y = mx + c$$

This gives $\qquad m = \dfrac{2a}{y'}, \qquad c = \dfrac{2ax'}{y'}$

These enable us to express c, x', y' in terms of m.

Thus $\qquad m = \dfrac{2a}{y'} \quad \therefore \quad y' = \dfrac{2a}{m}$

$$x' = \frac{y'^2}{4a} = \frac{a}{m^2}$$

$$c = \frac{2ax'}{y'} = \frac{2a^2}{m^2} \Big/ \frac{2a}{m} = \frac{a}{m}$$

the same results as before.

§ **224.** Or thus. Since

$$\left(\frac{2a}{m}\right)^2 = 4a\left(\frac{a}{m^2}\right)$$

we see that $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is a point on the parabola

$$y^2 = 4ax;$$

and substituting $\frac{a}{m^2}, \frac{2a}{m}$ for x', y' in the equation of the tangent

$$yy' = 2a(x + x')$$

it reduces to
$$y = mx + \frac{a}{m}$$

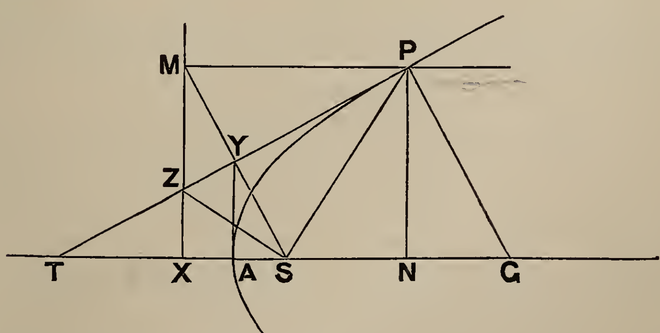
We may call the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ on the parabola 'the point m.'

GEOMETRICAL PROPERTIES

§ **225.** We shall now deduce some properties of the parabola.

Def'—If N is the foot of the ordinate at any point P of a curve, and the tangent and normal at P meet the axis of x in T, G respectively, then NT is called the *subtangent* and NG the *sub-normal.*

I. *The subtangent is bisected at the vertex.*



Let $(x'y')$ be the co-ord's of P.

$\therefore$ tangent at P is $\quad yy' = 2a(x + x')$

To get the intercept of this line on the axis of x put $\ y = 0$

$$\therefore \quad x + x' = 0$$

or $$x' = -x$$

$$\text{i.e.} \quad TA = AN$$

Cor'—We deduce this construction for the tangent at P :—
Draw PN $\perp$ the axis, measure AT = NA and join PT.

II. The *subnormal is constant.*

Put $y = 0$ in eq'n of normal

$$2a(y - y') + y'(x - x') = 0;$$

thus its intercept on the axis of x is given by

$$-2ay' + y'(x - x') = 0$$

or $$x - x' = 2a$$

$$\text{i.e.} \quad AG - AN, \quad \text{or} \quad NG = 2a$$

III. $$SP = ST$$

For $$ST = AT + a = AN + a \quad \text{by I.};$$

$$\therefore \quad ST = x' + a = SP \quad (\S\, 215)$$

IV. *If* PM *is drawn* $\perp$ *the directrix then the tangent bisects* $S\hat{P}M$.

We have just proved $$SP = ST$$

$$\therefore \quad S\hat{P}T = S\hat{T}P \quad \text{(Euclid I. 6)}$$

$$= T\hat{P}M \quad \text{(Euclid I. 29)}$$

V. *The foot of the* $\perp$ *from the focus on the tangent lies on the tangent at the vertex.*

Let SM, PT meet in Y.

Then in the Δs SPY, MPY we have

$$\left.\begin{aligned} SP &= PM \\ PY &= PY \\ S\hat{P}Y &= M\hat{P}Y \end{aligned}\right\}$$

$$\therefore \quad SY = MY \quad \text{and} \quad S\hat{Y}P = M\hat{Y}P = \text{a right angle};$$

i.e. Y is foot of $\perp$ from focus on tangent.

Also since $SY = MY$ and $SA = AX$, AY is $\parallel$ XM.

VI. *If the tangents at* Q, Q′ *intersect in* T, *a parallel to the axis through* T *bisects* QQ′.

Let the ∥ to the axis meet QQ′ in V (fig′ § 231).

Let the co-ord′s of Q be $(x'y')$ and of Q′ $(x''y'')$.

The equation to QT is

$$yy' = 2a(x + x') \quad \ldots \ldots \quad (1)$$

The equation to Q′T is

$$yy'' = 2a(x + x'') \quad \ldots \ldots \quad (2)$$

The co-ord′s of T are got by solving these eq′ns for x, y.

By subtraction

$$y(y' - y'') = 2a(x' - x'')$$
$$= \tfrac{1}{2}(y'^2 - y''^2)$$
$$\therefore \quad y = \tfrac{1}{2}(y' + y'')$$

$\therefore$ ordinate of V = ordinate of T $= \tfrac{1}{2}(y' + y'')$

$\therefore$ V is mid point of QQ′.

Exercises

1. Find the equations of the tangent and normal at L (fig′, § 214).

Ans. $y = x + a$, $y + x = 3a$

2. The tangent at P meets the directrix in Z; prove that PZ subtends a right angle at S.

3. Y is the foot of perpendicular from focus on tangent at P; prove that

$$SY^2 = SA \,.\, SP$$

4. Find co-ord′s of the inters′n of tangents at $(x'y')$, $(x''y'')$.

Ans. $y'y''/4a$, $(y' + y'')/2$

5. If J, J′ are points on the axis equidistant from the focus, the difference of squares of ⊥s from J, J′ on any tangent is constant, and $= 2a \,.\, JJ'$.

6. P $(x'y')$ and Q $(x''y'')$ are two points on the parabola

$$y^2 = 4ax:$$

prove that their join passes through the focus if

$$y'y'' + 4a^2 = 0$$

7. Show that tangents at the extremities of any focal chord meet at right angles on the directrix.

8. Show that the tangent at any point meets the directrix and latus rectum at points equidistant from the focus.

9. Find the locus of the foot of the perpendicular from the focus on the normal.

Ans. The parabola $y^2 = a(x - a)$.

10. Prove that the parabolas

$$y^2 = ax, \qquad x^2 = by$$

cut at an angle $$\tan^{-1} \frac{3 a^{\frac{1}{3}} b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}$$

11. Tangents are drawn to the parabola

$$y^2 = 4ax$$

at points whose abscissae are in the ratio $\mu : 1$; show that the locus of their intersection is the parabola

$$y^2 = \left(\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}}\right)^2 ax$$

§ 226. Ex. 1. To find locus of foot of $\perp$ from focus on tangent. (See § 225, V.)

Any tangent is $$y = mx + \frac{a}{m}$$

The $\perp$ to this from focus $(a, 0)$ is

$$y = -\frac{1}{m}(x - a)$$

To eliminate m, subtract the eq'ns; this gives

$$\left(m + \frac{1}{m}\right) x = 0$$

$$\therefore \quad x = 0$$

i.e. the locus is the tangent at the vertex.

Ex. 2. To find locus of inters'n of tangents at right angles.

Let the tangents be

$$y = mx + \frac{a}{m}, \qquad y = m'x + \frac{a}{m'};$$

then $$mm' = -1$$

We have to eliminate m, m' from these three eq'ns.

Subtract second eq'n from first and divide by $m - m'$

$$\therefore \quad 0 = x - \frac{a}{mm'}$$

Using then the third eq'n, $\quad x = -a$

i. e. the locus is the directrix.

TANGENTS FROM A GIVEN POINT

§ 227. Any tangent is

$$y = mx + \frac{a}{m}$$

This will pass through a given point $(x'y')$ if

$$y' = mx' + \frac{a}{m}$$

or

$$m^2x' - my' + a = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This quadratic gives two values of m, and $\therefore$ in general two tangents can be drawn from a given point $(x'\,y')$.

If the roots of (1) are m, m' then the two tangents are

$$y = mx + \frac{a}{m}, \quad y = m'x + \frac{a}{m'}$$

The roots of (1) are real, coincident, or imaginary, according as

$$y'^2 - 4ax' > = < 0$$

The two tangents are $\therefore$ (§ 218) real, coincident, or imaginary according as $(x'y')$ is outside, on, or inside the parabola.

Ex. 1. Find $\hat{\phi}$ between tangents from $(x'y')$

From (1)

$$\left.\begin{aligned} m + m' &= \frac{y'}{x'} \\ mm' &= \frac{a}{x'} \end{aligned}\right\}$$

$$\therefore \quad (m - m')^2 = (m + m')^2 - 4mm'$$

$$= \frac{y'^2 - 4ax'}{x'^2}$$

$$\therefore \quad \tan\phi = \frac{m - m'}{1 + mm'}$$

$$= \frac{\sqrt{y'^2 - 4ax'}}{a + x'}$$

Ex. 2. Find locus of point of inters'n of tangents which cut at $\hat{\phi}$.

Writing x, y instead of x', y' in the last result, we obtain the equation to the locus, viz.

$$y^2 - 4ax = (a + x)^2 \tan^2 \phi.$$

This may be written

$$y^2 + (x - a)^2 = (x + a)^2 \sec^2 \phi$$

This evidently (§ 213) represents a hyperbola having the same focus and directrix as the parabola and whose eccentricity = $\sec \phi$.

Exercises

1. Find co-ord's of the point of inters'n of tangents at the points m, m'.

Ans. $\dfrac{a}{mm'}$, $a\left(\dfrac{1}{m} + \dfrac{1}{m'}\right)$

2. Find the co-ord's of the point where the directrix is intersected by the $\perp$ from the inters'n of

$$y = m_1 x + \frac{a}{m_1}, \qquad y = m_2 x + \frac{a}{m_2}, \qquad \text{on} \quad y = m_3 x + \frac{a}{m_3}.$$

Ans. $x = -a$, $y = a\left(\dfrac{1}{m_1} + \dfrac{1}{m_2} + \dfrac{1}{m_3} + \dfrac{1}{m_1 m_2 m_3}\right)$

3. Deduce that the orthocentre of the triangle formed by any three tangents lies on the directrix.

[The symmetry of the above result shows that each $\perp$ intersects the directrix in the same point.]

4. Two tangents to the parabola

$$y^2 = 4ax$$

make angles θ, θ' with its axis. Find locus of their intersection,

1°, If $\tan\theta \tan\theta' = \text{constant} = \lambda$

2°, If $\cot\theta + \cot\theta' = \text{constant} = \lambda$

3°, If $\sin\theta \sin\theta' = \text{constant} = \lambda$

Ans. 1°, the straight line $a = \lambda x$

2°, the straight line $y = \lambda a$

3°, the circle $x^2 + y^2 - 2ax = a^2(1 - \lambda^2) / \lambda^2$

DIAMETERS

§ **228.** *To find the locus of the mid points of a system of parallel chords.*

Let the chords be inclined to the axis at $\hat{\theta}$.

Let $(x'y')$, $(x''y'')$ be the extremities of one of the chords; (xy) its mid point.

Then $$2x = x' + x'', \quad 2y = y' + y'' \quad \ldots \quad (1)$$

Also $$\frac{y' - y''}{x' - x''} = \tan\theta \quad \ldots \quad (2)$$

Since $(x'y')$, $(x''y'')$ are on the parabola

$$y'^2 = 4ax', \quad y''^2 = 4ax''$$

By subtraction $y'^2 - y''^2 = 4a(x' - x'')$

$$\therefore \quad (y' - y'')(y' + y'') = 4a(x' - x'')$$

Divide by $x' - x''$; then from (1), (2) we see that

$$2y\tan\theta = 4a$$

$$\therefore \quad y = 2a\cot\theta$$

This is the eq'n to the locus. It is $\therefore$ a straight line $\parallel$ the axis.

§ **229.** *Def's*—The locus of mid points of a system of parallel chords of a conic is called a *diameter*. The chords which a diameter bisects are called its *ordinates*.

We have proved then that the diameters of a parabola are straight lines parallel to the axis.

§ **230.** *The tangent at the end of a diameter is parallel to its ordinates.*

In fig' § 231 QQ' is one of the chords bisected by the diameter PV

Let QQ' move parallel to itself until V comes to P.

Then ultimately the bisected chord becomes the tangent at P. Q.E.D

Or thus.

By § 228 $y = 2a\cot\theta$ is diameter bisecting chords $\parallel$ $y = x\tan\theta$.

The co-ord's of its extremity satisfy the equations

$$y = 2a\cot\theta, \qquad y^2 = 4ax$$

∴ they are $$(a\cot^2\theta,\ 2a\cot\theta)$$

The tangent at this point is

$$y . 2a\cot\theta = 2a(x + a\cot^2\theta)$$

which is ∥ $$y = x\tan\theta.$$ Q.E.D.

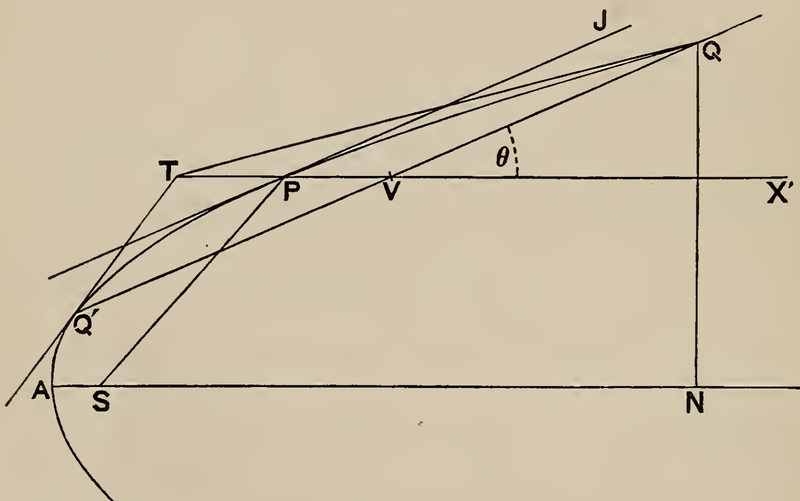
§ **231.** *To find the equation of a parabola referred to a diameter and the tangent at its extremity as axes.*

Let the new axes be PX′, PJ.

Let QVQ′ be one of the chords bisected by the diameter, so that the co-ord's of Q are

$$PV = x, \qquad QV = y.$$

Draw QN ⊥ AS.



Let P be the point m, (§ 224). Then

$$m = \tan\theta$$

where $$\theta = \widehat{JPX'};$$

and the co-ord's of P referred to the old axes are

$$a/m^2 = a\cot^2\theta$$

and $$2a/m = 2a\cot\theta$$

Project the broken line QVPA on AN and NQ

$$\therefore \left.\begin{aligned} AN &= y\cos\theta + x + a\cot^2\theta \\ NQ &= y\sin\theta + 2a\cot\theta \end{aligned}\right\}$$

But $$QN^2 = 4a \, . \, AN$$

$$\therefore \quad (y \sin\theta + 2a \cot\theta)^2 = 4a\,(y\cos\theta + x + a\cot^2\theta)$$

which reduces to $$y^2 \sin^2\theta = 4ax \quad \ldots\ldots\ldots\ldots \quad (1)$$

Again, $SP = a + a\cot^2\theta$ (§ 215) $= a/\sin^2\theta$.

Putting $SP = a'$, (1) becomes

$$y^2 = 4a'x$$

This is the required equation.

Cor'—As in § 220 the tangent at $(x'y')$ is

$$yy' = 2a'(x + x')$$

Putting $y = 0$ we get intercept on axis of x:

$$x + x' = 0$$

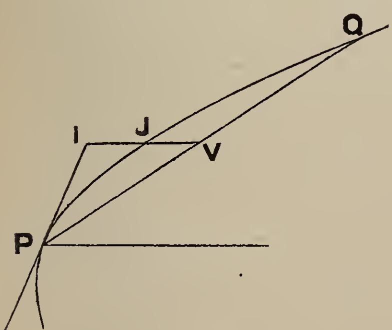
$$\therefore \quad x = -x'$$

i. e. if Q be $(x'y')$ we deduce (see fig')

$$TP = PV$$

§ **232.** Ex. Any diameter of the parabola meets the tangent at P, a chord PQ and the curve in I, V, J respectively: prove that

$$IV : IJ = QP : VP$$



Take for axes the diameter through P and the tangent at P; let the eq'n to the parabola be

$$y^2 = 4a'x.$$

Let eq'n to PQ be

$$y = mx,$$

and to IJ, $\quad y = c.$

Then co-ord's of Q are determined by $y^2 = 4a'x$, $\; y = mx$

„ „ V „ „ $y = c$, $\; y = mx$

„ „ J „ „ $y = c$, $\; y^2 = 4a'x$

$\therefore$ co-ord's of Q are $(4a'/m^2,\ 4a'/m)$

„ „ V „ $(c/m,\ c)$

„ „ J „ $(c^2/4a',\ c)$

Evidently $$\frac{QP}{VP} = \frac{y \text{ of } Q}{y \text{ of } V} = \frac{4a'}{mc}$$

Also $$\frac{IV}{IJ} = \frac{x \text{ of } V}{x \text{ of } J} = \frac{4a'}{mc}; \quad \therefore \text{ \&c.}$$

POLES AND POLARS

§ **233.** *Def'*—If P, Q are the points of contact of tangents from T, then PQ is called the *polar* of T, and T is called the *pole* of PQ.

To find the equation of the polar of $(x' y')$.

Let (hk) be the point of contact of *either* tangent from $(x' y')$ to

$$y^2 = 4ax$$

Express that the tangent at (hk) viz.

$$ky = 2a(x + h)$$

passes through $(x' y')$

$$\therefore \quad ky' = 2a(x' + h)$$

This equation expresses that (hk) lies on the straight line

$$yy' = 2a(x + x')$$

As *both* points of contact lie on this line, it is ∴ their join.

The equation of the polar of $(x' y')$ is ∴

$$yy' = 2a(x + x')$$

§ **234.** *If* P *lies on the polar of* Q, *then* Q *lies on the polar of* P.

Let P be $(x_1 y_1)$ and Q $(x_2 y_2)$.

The polar of Q is $\quad yy_2 = 2a(x + x_2)$

P $(x_1 y_1)$ lies on this if

$$y_1 y_2 = 2a(x_1 + x_2)$$

The symmetry of this equation shows that it is also the condition that Q lies on the polar of P.

Cor'—The polar of the focus (a, o) is

$$x + a = o,$$

i. e. the directrix; ∴ tangents at the ends of a focal chord intersect on the directrix.

NORMALS

§ **235.** If we substitute a/m^2, $2a/m$ for x', y' in the equation of § 221, *Cor'* (3), we obtain, after reduction

$$m^3 y + (x - 2a) m^2 - a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation to the normal at the point m.

This may be expressed differently.

If the normal is $\qquad y = \mu x + c$

then $\qquad m = -\frac{1}{\mu}$

since normal is $\perp$ tangent.

Substituting this value of m in (1) the equation of the normal becomes

$$y = \mu x - 2a\mu - a\mu^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ex. Find locus of point of inters'n of normals at right angles.

A normal through (hk) is

$$y = \mu x - 2a\mu - a\mu^3$$

where μ is one of the roots of the cubic

$$k = \mu h - 2a\mu - a\mu^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the roots of this cubic be μ_1, μ_2, μ_3.

Then $\qquad \mu_1 \mu_2 \mu_3 = -k/a$

But $\qquad \mu_1 \mu_2 = -1,$

if two of the normals are at right angles.

$$\therefore \quad \mu_3 = k/a$$

Substitute this value of μ_3 instead of μ in (1): then writing x, y for h, k we obtain the locus required, viz.

$$y^2 = ax - 3a^2$$

§ **236.** Suppose we wish to find the normals which can be drawn through a given point (hk).

Let $(x_1 y_1)$ be the foot of one of the normals.

Express that normal at $(x_1 y_1)$ passes through (hk):

$$\therefore \quad 2a(k - y_1) + y_1(h - x_1) = 0$$

$$\therefore \quad 2a(k - y_1) + y_1(h - y_1^2/4a) = 0$$

or

$$y_1^3 + 4a(2a - h)y_1 - 8a^2 k = 0$$

This cubic in y_1 has three roots y', y'', y'''; $\therefore$ three normals can be drawn.

Cor'—As the term in y_1^2 is absent,

$$y' + y'' + y''' = 0$$

Ex. If chords are drawn parallel to

$$y = mx,$$

find locus of intersection of normals at their extremities.

Let $(x'y')$, $(x''y'')$ be the extremities of one of the chords.

Let the normals at these points meet in (hk): let $(x'''y''')$ be the foot of the third normal from (hk).

Then [§ 221, *Cor'* (2)] $\qquad y' + y'' = 4a/m$

∴ (by preceding *Cor'*) $\qquad y''' = -4a/m$

Thus $(x'''y''')$ is a fixed point; and the locus is the normal at $(x'''y''')$.

Exercises on the Parabola

[Unless otherwise implied, the equation of the parabola in these questions is

$$y^2 = 4ax.]$$

1. If p, p' are perpendiculars from the extremities of a focal chord on tangent at vertex, prove

$$pp' = \text{constant} = a^2$$

2. Find equation of locus of intersection of tangents inclined at 60°.

Ans. $3x^2 - y^2 + 10ax + 3a^2 = 0$

3. Find equation of chord joining the points m, m'.

Ans. $2(mm'x + a) = y(m + m')$

4. Find locus of intersection of tangents inclined at complementary angles to the axis.

Ans. The latus rectum.

5. The join of a point P on the parabola to the vertex cuts the perpendicular from the focus on the tangent at P in R; find equation of locus of R.

Ans. $y^2 + 2x^2 = 2ax$

6. The equation of the parabola referred to its axis and latus rectum as axes of co-ordinates is

$$y^2 = 4a(x + a);$$

and any tangent is

$$x\cos\alpha + y\sin\alpha + a/\cos\alpha = 0$$

7. Show that the line

$$y = m(x + a) + a/m$$

touches the parabola $\quad y^2 = 4a(x + a)$

8. Find locus of intersection of rectangular tangents to the confocal parabolas

$$y^2 = 4a(x + a), \qquad y^2 = 4a'(x + a')$$

Ans. The straight line $x + a + a' = 0$

9. If R is the mid point of a chord PQ, show that the polar of R is parallel to PQ.

[*Note*—Taking diam′ through R and tangent at its vertex as axes; if co-ord′s of R are $(h, 0)$, then eq′n of PQ is $x = h$: and the polar of R is

$$x + h = 0\text{*}.]$$

10. The equation of the tangents from (hk) may be written in either of the forms

$$h(y - k)^2 - k(y - k)(x - h) + a(x - h)^2 = 0$$

or $$(k^2 - 4ah)(y^2 - 4ax) = [ky - 2a(x + h)]^2$$

[*Note*—Proceed as in §§ 166, 167.]

11. If the intercept of the tangents from P on tangent at vertex is constant: prove that locus of P is an equal parabola.

[*Note*—If P is (hk), intercept $= \sqrt{k^2 - 4ah}$.]

12. Find equation of chord whose mid point is (hk).

Ans. $(y - k)k = 2a(x - h)$

[*Note*—By Ex. 9, chord is $\parallel$ $yk = 2a(x + h)$.]

13. Find locus of mid points of chords through $(x'y')$.

Ans. The parabola $y(y - y') = 2a(x - x')$

[*Note*—Express that chord whose eq′n is obtained in Ex. 12 passes through $(x'y')$; then write x, y for h, k.]

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* This may also be seen geometrically; the polar of R is the join of the poles of PQ and the diameter through R: the latter pole is the point at infinity on PQ.

14. Given base and area of a triangle; find locus of orthocentre.

Ans. If $(+a, o)$, $(-a, o)$ are extremities of base and

$$\text{area} = ap/2;$$

locus is the parabola $x^2 + yp = a^2$

15. If P, Q, R are three points on the parabola whose abscissae are in geometrical progression; prove that the tangents at P, R intersect on the ordinate of Q.

16. The area of the triangle formed by three tangents is half that of the triangle formed by joining their points of contact.

17. SY is the perpendicular from the focus on the tangent at P: show that locus of centre of circum circle of the triangle SYP is the parabola

$$y^2 = a(2x - a)$$

18. Show that the length of the chord of contact of tangents from (hk) is

$$\sqrt{(k^2 + 4a^2)(k^2 - 4ah)}/a$$

19. Find equation of lines joining vertex to points of contact of tangents from (hk).

Ans. $hy^2 = 2x(ky - 2ax)$

20. Find locus of mid points of focal chords.

Ans. The parabola $y^2 = 2a(x - a)$

21. PQ is a double ordinate of a parabola; the join of P to the foot of the directrix cuts the curve in P'. Show that $P'Q$ passes through the focus.

22. Find locus of mid point of PG.

Ans. The parabola $y^2 = a(x - a)$

23. Find locus of intersection of normals inclined at complementary angles to the axis.

Ans. The parabola $y^2 = a(x - a)$

24. Show that the locus of the mid point of the intercept on a variable tangent between two fixed tangents is a straight line.

25. $(x_1 y_1)$, $(x_2 y_2)$, ... are the vertices of a re-entrant quadrilateral whose sides touch the parabola. Prove that

$$x_1 x_3 = x_2 x_4 \quad \text{and} \quad y_1 + y_3 = y_2 + y_4$$

26. Normals are drawn at two points on opposite sides of the axis whose abscissae are in the ratio 1 : 4. Show the locus of their intersection is the curve

$$27\,ay^2 = 4\,(x - 2\,a)^3$$

27. A perpendicular $A\,p$ from the vertex to the tangent at P meets the curve in q. Prove that

$$A\,p\,.\,A\,q = 4\,a^2$$

28. The tangents at P, P' meet in T. Prove that

$$TP^2 : TP'^2 = SP : SP' \quad \text{and} \quad ST^2 = SP\,.\,SP'$$

29. If the normals at P, Q meet on the curve; prove that PQ passes through a fixed point on the axis of the parabola.

[*Note*—Let normals at P, Q meet in R: let R, P, Q be the points m_1, m_2, m_3.

The m's of the normals through (xy) are the roots of

$$m^2\,(my - 2\,a) + m^2 x - a = 0$$

In this eq'n write a/m_1^2, $2\,a/m_1$ for x, y; $\therefore$ m_1, m_2, m_3 are the roots of

$$2\,m^2\left(\frac{m}{m_1} - 1\right) + \frac{m^2}{m_1^2} - 1 = 0$$

Hence m_2, m_3 are the roots of

$$2\,m^2 + \frac{m}{m_1} + 1 = 0;$$

$\therefore$ $m_2\,m_3 = \frac{1}{2}$, and join of m_2, m_3 [Ex. 3, p. 196] passes through $(-2\,a, 0)$.]

30. The circle whose diameter is the join of the points m_1, m_2 meets the parabola again in the points m_3, m_4: show that

$$\frac{1}{m_3\,m_4} = \frac{1}{m_1\,m_2} + 4$$

[*Note*—The circle is (Ex. 8, p. 126)

$$(x - a/m_1^2)\,(x - a/m_2^2) + (y - 2\,a/m_1)\,(y - 2\,a/m_2) = 0$$

$\therefore$ m_1, m_2, m_3, m_4 are the roots of

$$(a/m^2 - a/m_1^2)\,(a/m^2 - a/m_2^2)$$
$$+ (2\,a/m - 2\,a/m_1)\,(2\,a/m - 2\,a/m_2) = 0 :$$

$\therefore$ m_3, m_4 are the roots of

$$\left(\frac{1}{m} + \frac{1}{m_1}\right)\left(\frac{1}{m} + \frac{1}{m_2}\right) + 4 = 0]$$

31. The circle whose diameter is a chord PQ meets the parabola again in R, S: prove that if PQ passes through a fixed point on the axis so does RS.

[*Note*—Deduce from Exercises 3, 30 that distance between fixed points is $4a$.]

32. The sides of a quadrilateral inscribed in a parabola are respectively inclined at angles α, β, γ, δ to the axis: prove that

$$\cot\alpha + \cot\gamma = \cot\beta + \cot\delta$$

Hence infer that if three of the sides are parallel to given lines the fourth side is parallel to a given line.

[*Note*—Let parameters of vertices be m_1, m_2, m_3, m_4. It follows from Ex. 3, page 196 that

$$2\cot\alpha = \frac{1}{m_1} + \frac{1}{m_2}; \quad \&c.]$$

33. Find the co-ordinates of the second point in which the normal at the point m meets the parabola.

Ans. $\dfrac{a(2m^2+1)^2}{m^2}$, $\dfrac{-2a(2m^2+1)}{m}$

[*Note*—Elim′ x from

$$y^2 = 4ax, \qquad m^3y + m^2(x - 2a) - a = 0 \quad (\S\ 235)$$

This gives $\quad m^2y^2 + 4am^3y - 4a^2(2m^2+1) = 0;$

a quadratic in y the product of whose roots is

$$-4a^2(2m^2+1)/m^2.$$

One root is $2a/m$; &c.]

34. Find equation of locus of mid points of normal chords.

Ans. $y^4 + 8a^4 = 2ay^2(x - 2a)$

35. A circle cuts the parabola

$$y^2 = 4ax$$

in four points whose ordinates are y_1, y_2, y_3, y_4; prove that

$$y_1 + y_2 + y_3 + y_4 = 0$$

[*Note*—Elim′ x from

$$y^2 = 4ax, \qquad (x-\alpha)^2 + (y-\beta)^2 = r^2;$$

this gives a biquadratic in y which wants the term in y^3.]

36. The circle through the feet of the three normals from any point passes through the vertex.

[*Note*—Let y_1, y_2, y_3 be ordinates of feet of normals; y_4 the ordinate of fourth point in which circle meets parabola. Then (§ 236, *Cor.*)

$$y_1 + y_2 + y_3 = 0:$$

from preceding Ex. it follows then that

$$y_4 = 0; \quad \&c.]$$

37. If P, Q, R are the feet of the normals from T: prove that

$$SP.SQ.SR = a\,ST^2.$$

[*Note*—Let T be (h, k): the normal at (x′y′) passes through (hk) if

$$2a(k - y') + y'(h - x') = 0$$

Write x, y for x′, y′; we infer that feet of normals lie on the curve

$$2a(k - y) + y(h - x) = 0$$

But they also lie on $\quad y^2 = 4ax.$

Eliminate y; we get $\quad (x + 2a - h)^2 x = ak^2$

Now if $SP = \rho$ and x is abscissa of P,

$$\rho = x + a:$$

$$\therefore \quad (\rho + a - h)^2 (\rho - a) = ak^2$$

The roots of this cubic in ρ are SP, SQ, SR;

$$\therefore \quad SP.SQ.SR = a[k^2 + (a - h)^2] = a\,ST^2]$$

38. Find the equations of the common tangents to the parabola

$$y^2 = 4ax$$

and the circle $\quad 2(x^2 + y^2) = 9ax.$

Ans. $12y \pm 16x \pm 9a = 0$

39. Show that the polar of any point on the circle

$$x^2 + y^2 = ax$$

with respect to the circle $\quad x^2 + y^2 = 2ax$

touches the parabola $\quad y^2 = 4ax$

40. Find locus of poles of tangents to the parabola

$$y^2 = 4a'x$$

with respect to the parabola $\quad y^2 = 4ax$

Ans. The parabola $a'y^2 = 4a^2x$

41. Two parabolas have a common axis, but different vertices; show that the portion of a tangent to either intercepted by the other is bisected at the point of contact.

42. Find locus of poles of chords of the parabola whose mid points lie on a fixed line

$$Ax + By + C = 0$$

Ans. The parabola $A(y^2 - 2ax) + 2a(By + C) = 0$

43. Find locus of intersection of tangents which form with tangent at vertex a triangle of constant area Δ.

Ans. The curve $x^2(y^2 - 4ax) = 4\Delta^2$

44. The locus of the intersection of equal chords of a parabola drawn in fixed directions is a straight line.

45. A chord PQ of a parabola, which is normal at P, meets the axis in G; y is the ordinate of P: prove that

$$\text{area SPQ} = PG^4/(4ay)$$

46. PQ is a normal chord of the parabola: prove that the locus of the centroid of SPQ is the curve

$$36ay^2(3x - 5a) - 81y^4 = 128a^4$$

47. Prove that the locus of a point such that normals to the parabola

$$y^2 = 4ax$$

at its intersections with the polar of the point meet on the parabola is the straight line

$$x = 2a$$

48. If the normals at three points P, Q, R on the parabola are concurrent; and if P′, Q′, R′ are three other points on the parabola such that PP′, QQ′, RR′ are respectively parallel to QR, RP, PQ: show that the normals at P′, Q′, R′ are also concurrent.

49. If $(\alpha\beta)$, $(\alpha'\beta')$, $(\alpha''\beta'')$ are the feet of the normals from (hk) to the parabola, prove the relation

$$\beta'\beta''(\alpha' - \alpha'') + \beta''\beta(\alpha'' - \alpha) + \beta\beta'(\alpha - \alpha') = 0$$

50. TP, TQ are tangents to a parabola; TL is a perpendicular to the axis and the perpendicular from T on PQ meets the axis in M. Prove that

$$2\,LM = \text{latus rectum.}$$

51. L, M, N are the feet of the normals from O (hk) to the parabola

$$y^2 = 4ax:$$

prove that $\quad OL^2 . OM^2 . ON^2 = (k^2 - 4ah)^2 [k^2 + (h - a)^2]$

[*Note*—Let L, M, N be the points m_1, m_2, m_3; let $OL = \rho_1$, $OM = \rho_2$, $ON = \rho_3$. Then

$$\rho_1 / \sqrt{1 + m_1^2} = \text{proj'n of OL on a} \parallel \text{to axis of } y = \frac{2a}{m_1} - k;$$

$$\therefore \quad \rho_1 = \sqrt{1 + m_1^2}\left(\frac{2a}{m_1} - k\right); \quad \text{similarly for } \rho_2, \rho_3.$$

Also m_1, m_2, m_3 are the roots of the cubic in m

$$m^3 k + m^2 (h - 2a) - a = 0$$

We have $\therefore$ the *identity*

$$m^3 k + m^2 (h - 2a) - a \equiv k (m - m_1)(m - m_2)(m - m_3)$$

In this identity substitute successively $\pm \sqrt{-1}$ for m, and multiply;

$$\therefore \quad (h - a)^2 + k^2 = k^2 (1 + m_1^2)(1 + m_2^2)(1 + m_3^2)$$

Again, $$\frac{2a}{m_1} - k = \left(\frac{2a}{k} - m_1\right) \Big/ \frac{m_1}{k};$$

the value of $\left(\frac{2a}{m_1} - k\right)$ (...) (...) is then easily deduced by subs'g $2a/k$ for m in preceding identity.]

52. If n_1, n_2, n_3 are the lengths of the normals and t_1, t_2 the lengths of the tangents from any point to the parabola

$$y^2 = 4ax$$

prove that $$n_1 n_2 n_3 = a t_1 t_2$$

53. The tangents at P, Q meet in T and the normals at P, Q meet in T′: if the co-ordinates of T are h, k show that those of T′ are

$$\left(2a + \frac{k^2}{a} - h, \; -\frac{hk}{a}\right)$$

If T describes a straight line $\quad y = mx + c$,

show that the equation of the locus of T′ is

$$(a - cm)[acx + a(2mc - a)y - c(c^2 + 2a^2)]$$
$$= m[a(x + my) - c^2 - 2a^2]^2$$

[*Note*—It will be seen hereafter (Chap. XI.) that this eq'n represents a parabola whose axis is $\perp$ given line.]

54. On a chord PP′ inclined at a constant angle θ to the axis of a parabola a point Q is taken such that

$$PQ \, . \, QP' = \text{constant} = \delta^2;$$

find the locus of Q.

[*Note*—Let co-ord's of Q be h, k. Take Q as origin of polar co-ord's; then the equation of the parabola is (§ 208)

$$(k + r \sin \theta)^2 = 4a(h + r \cos \theta);$$

$$\therefore \quad \delta^2 = - QP \, . \, QP' = - \text{product of roots of this quadratic in } r$$
$$= (4ah - k^2) / \sin^2 \theta.$$

The req'd locus is ∴ the parabola

$$y^2 - 4ax + \delta^2 \sin^2 \theta = 0.]$$

55. Three tangents to a parabola whose focus is S intersect in A, B, C; SA, SB, SC meet BC, CA, AB in A′, B′, C′ respectively; show that the perpendiculars from A, B, C to the other tangents from A′, B′, C′ respectively are concurrent.

CHAPTER IX

THE ELLIPSE

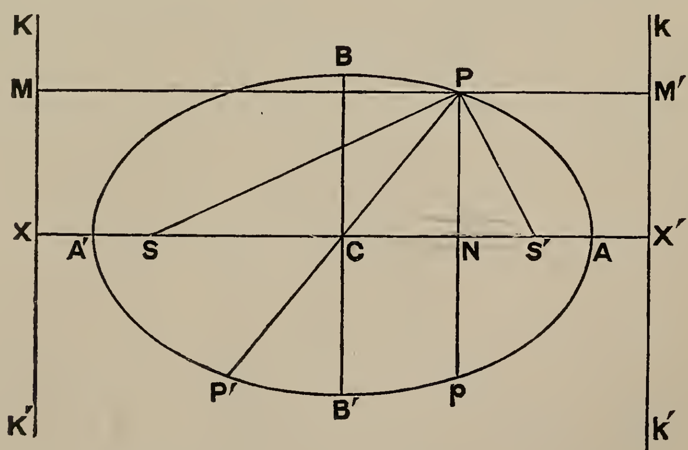
EQUATION TO ELLIPSE

§ **237.** We have defined the ellipse in § 213; viz. if P is a point on the curve then

$$\mathsf{SP} = \mathsf{e}\,\mathsf{PM},$$

S being the focus, PM the $\perp$ from P on the directrix $\mathsf{KK'}$, and e (the eccentricity) < 1.

The equation to the ellipse may be obtained in a simple form thus.



Draw $\mathsf{SX} \perp$ directrix.

Divide SX internally in $\mathsf{A'}$ and externally in A in the given ratio $\mathsf{e} : 1$.

Then $$SA' = e\,A'X \qquad (1)$$
$$SA = e\,AX \qquad (2)$$

By *def'* A, A' are points on the curve.

Bisect AA' in C; let $AA' = 2a$

Adding (1) and (2) we find

$$2\,CA = e \,.\, 2\,CX$$

$$\text{i.e.} \quad 2a = 2e\,.\,CX$$

$$\therefore \quad CX = \frac{a}{e}$$

Subtracting (1) from (2) we find

$$2\,CS = e\,.\,2\,CA$$

$$\therefore \quad CS = ae$$

Through C draw $BCB' \perp CA$; and take CA, CB as axes.

Let the co-ord's of P be $CN = x$, $PN = y$.

The co-ord's of S are $(-ae, 0)$

$$\therefore \quad SP^2 = (x + ae)^2 + y^2 \quad (\S\ 10)$$

Also $$PM = CN + XC$$
$$= x + \frac{a}{e}$$

Now $$SP^2 = e^2\,PM^2$$

$$\therefore \quad (x + ae)^2 + y^2 = (ex + a)^2$$

$$\therefore \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Put $$a^2(1 - e^2) = b^2$$

The equation to the ellipse is $\therefore$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we put $x = 0$ in this equation we get $y = \pm b$

$\therefore$ if along the axis of y we measure $CB = CB' = b$, then B, B' are points on the curve.

Def's—The point C is called the *centre*, the line AA' the *major axis*, and the line BB' the *minor axis*.

FIGURE OF THE CURVE

§ 238. Let $(x'y')$ be the co-ord's of a point P on the curve. [Fig', § 237.]

$$\therefore \quad x'^2/a^2 + y'^2/b^2 = 1$$

$$\therefore \quad (-x')^2/a^2 + (-y')^2/b^2 = 1$$

i. e. the point $(-x', -y')$ or P', the image of P with respect to the centre is on the curve.

$\therefore$ all chords through the centre are bisected.

Again, $$x'^2/a^2 + (-y')^2/b^2 = 1;$$

$\therefore$ $(x', -y')$ or p, the image of P with respect to the major axis is a point on the curve.

Thus the curve is symmetrical with respect to the major axis.

Similarly it is symmetrical with respect to the minor axis.

We infer that there is another focus S' and a corresponding directrix $kX'k'$, images respectively of S and KXK' with respect to the minor axis.

Again, if we write $r\cos\theta$, $r\sin\theta$ for x, y in the equation of the ellipse, we obtain its polar equation referred to the centre as pole, viz.

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

or $$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$$

From this we deduce

$$r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{a^2b^2}{b^2 + (a^2 - b^2)\sin^2\theta}$$

Now $b^2 + (a^2 - b^2)\sin^2\theta$ is least when $\theta = 0$, and then $r^2 = a^2$; thus the greatest value of r is a.

The curve is $\therefore$ bounded in all directions.

§ 239. *Def'*—The double ordinate through the focus is called the *latus rectum.*

To find its length, substitute $-ae$ for x in the equation of the ellipse.

$$\therefore \quad y^2 = b^2(1 - e^2) = \frac{b^4}{a^2}$$

$$\therefore \quad \text{latus rectum} = 2\frac{b^2}{a}$$

FOCAL DISTANCES

§ 240. We may express SP, $S'P$ in terms of x, the abscissa of P (fig', § 237).

$$SP = e\,PM = e\,NX = e\,(CX + CN) = e\left(\frac{a}{e} + x\right)$$

$$\therefore \quad SP = a + ex$$

Also $\quad S'P = e\,PM' = e\,(CX' - CN) = e\left(\frac{a}{e} - x\right)$

$$\therefore \quad S'P = a - ex$$

We infer that $\qquad SP + S'P = 2a$

Thus the ellipse is the locus of a point P which moves so that the sum of its distances from two fixed points S, S' = a constant $2a$.

(Compare § 104.)

This gives a method of describing the ellipse mechanically.

If the ends of a thread are fastened at two fixed points S, S', then a pencil moved about so as to keep the thread always stretched will describe an ellipse whose foci are S, S' and whose major axis = length of thread.

INTERNAL AND EXTERNAL POINTS

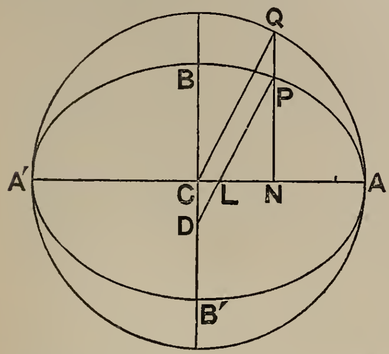
§ 241. As in § 218 we can prove that if (x, y) is a point inside the ellipse the function

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

of its co-ord's is negative, and if (x, y) is outside the ellipse the function is positive. Of course if (x, y) is on the ellipse the function is zero.

AUXILIARY CIRCLE

§ **242.** *Def′*—The circle whose diameter is the major axis AA′ is called the *auxiliary circle.*



Let the ordinate PN of the point P (x, y) on the ellipse meet the circle in Q.

From
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we deduce
$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

i.e.
$$PN = \frac{b}{a}\sqrt{CQ^2 - CN^2}$$

$$= \frac{b}{a} QN$$

$$\therefore \quad PN : QN = b : a$$

Thus the ordinates of the ellipse and circle at corresponding points P, Q are in a constant ratio.

Cor′—Draw PLD ∥ CQ, cutting the axes in L, D.

Then
$$PD = CQ = a$$

Also, by similar Δs,

$$PL : CQ = PN : QN = b : a$$

$$\therefore \quad PL = b$$

Hence, if we suppose PLD to be a ruler having pins at L and D, and if AA′, BB′ be grooves in which these pins run, a pencil at the point P will trace an ellipse whose semi-axes are PD, PL. This is the principle of the *elliptic compass.*

Exercises

1. Find the equation of the ellipse whose focus is $(1, 0)$, directrix $x + y = 0$, and eccentricity $\frac{1}{2}$.

Ans. $7x^2 - 2xy + 7y^2 - 16x + 8 = 0$

2. Show that the point $(1\frac{1}{2}, 1\frac{1}{2})$ is inside the ellipse

$$4x^2 + 9y^2 = 30$$

3. Find the eccentricity, the length of the latus rectum, and the co-ordinates of the foci of each of the ellipses

$$4x^2 + 9y^2 = 36, \qquad 9x^2 + 4y^2 = 36$$

Ans. $\frac{1}{3}\sqrt{5}$, $2\frac{2}{3}$, $(\pm\sqrt{5}, 0)$; $\frac{1}{3}\sqrt{5}$, $2\frac{2}{3}$, $(0, \pm\sqrt{5})$

[*Note*—Observe that major axis of second ellipse is along axis of y.]

4. Determine the same particulars for the ellipses

$$2(x-1)^2 + 3(y+2)^2 = 1, \qquad 4x^2 + 5y^2 - 8x - 20y = 0$$

Ans. $\frac{1}{\sqrt{3}}$, $\frac{2\sqrt{2}}{3}$, $\left(1 \pm \frac{1}{\sqrt{6}}, -2\right)$; $\frac{1}{\sqrt{5}}$, $\frac{8\sqrt{6}}{5}$, $\left(1 \pm \sqrt{\frac{6}{5}}, 2\right)$

5. Show that the line $\qquad y = x + 2$

touches the ellipse $\qquad x^2 - xy + y^2 + 2x + 4y + 4 = 0$

at the point $(-4, -2)$

6. Find the eccentricity of the ellipse

$$2x^2 + y^2 = 3x;$$

also its foci and directrices.

Ans. $\frac{1}{\sqrt{2}}$; $(\frac{3}{4}, \pm\frac{3}{4})$; $y = \pm\frac{3}{2}$

7. The joins of a point P on the ellipse to the extremities of the minor axis meet the major axis in p, q: prove that

$$Cp \cdot Cq = CA^2$$

8. If CP, CQ are two semi-diameters at right angles, prove that

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

[*Note*—Use polar equation (see § 238).]

9. The major axis of an ellipse is divided into two parts equal to the focal distances of a point P on the ellipse: prove that the distance of the point of division from either end of the minor axis is equal to the distance of P from the centre.

10. Find the co-ordinates of the mid point of the chord which the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

intercepts on the line $\qquad Ax + By + C = 0$

Ans. $-ACa^2/(a^2A^2 + b^2B^2), \; -BCb^2/(a^2A^2 + b^2B^2)$

11. A line AB of given length moves with its extremities on two rectangular axes OX, OY: find the locus traced by a fixed point P on the line.

[*Note*—Draw $PN \perp OX$ so that $ON = x$, $PN = y$ are the co-ord's of P; let

$$O\hat{A}B = \theta.$$

Then $\qquad \sin\theta = PN/PA = y/PA, \qquad \cos\theta = x/PB$

Square and add, ∴ locus of P is the ellipse

$$x^2/PB^2 + y^2/PA^2 = 1.$$

Compare *Cor'*, § 242.]

ECCENTRIC ANGLE

§ **243.** *Def'*—The angle QCA (fig', § 242) is called the *eccentric angle* of the point P.

If the eccentric angle of P is ϕ we may express its co-ord's (x, y) in terms of ϕ.

For $\qquad x = CN = CQ\cos\phi = a\cos\phi$

and $\qquad y = PN = \frac{b}{a}QN = \frac{b}{a}(CQ\sin\phi) = \frac{b}{a}(a\sin\phi)$

$$= b\sin\phi$$

These values $\qquad \left.\begin{aligned} x &= a\cos\phi \\ y &= b\sin\phi \end{aligned}\right\}$

evidently satisfy the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We can thus express the co-ord's of any point of the ellipse in terms of the single parameter ϕ.

§ **244.** *To find the equation of the chord joining the points whose eccentric angles are* α, β.

The eq'n to the join of

$$\left.\begin{matrix} a\cos\alpha \\ b\sin\alpha \end{matrix}\right\},\qquad \left.\begin{matrix} a\cos\beta \\ b\sin\beta \end{matrix}\right\}$$

is

$$\begin{vmatrix} x & y & 1 \\ a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \end{vmatrix} = 0$$

or

$$bx(\sin\alpha - \sin\beta) - ay(\cos\alpha - \cos\beta) = ab\sin(\alpha - \beta)$$

or

$$2bx\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} + 2ay\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$= 2ab\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Divide by $2ab\sin\frac{\alpha-\beta}{2}$: the required equation is $\therefore$

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

This result is often useful.

EQUATION OF TANGENT

§ **245.** *To find the equation of the tangent at* $(x'y')$

If $(x'y')$, $(x''y'')$ are two points on the curve

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad \text{and} \quad \frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1$$

By subtraction

$$\frac{(x'-x'')(x'+x'')}{a^2} + \frac{(y'-y'')(y'+y'')}{b^2} = 0$$

$$\therefore \quad \frac{y'-y''}{x'-x''} = -\frac{b^2}{a^2}\frac{x'+x''}{y'+y''}$$

Hence the equation to the join of the two points, which is

$$\frac{y-y'}{x-x'} = \frac{y'-y''}{x'-x''}$$

becomes
$$\frac{y-y'}{x-x'} = -\frac{b^2}{a^2}\frac{x'+x''}{y'+y''}$$

In this put $x''=x',\ y''=y'$

$\therefore$ the equation of the tangent at $(x'y')$ is
$$\frac{y-y'}{x-x'} = -\frac{b^2}{a^2}\frac{x'}{y'}$$

or
$$a^2yy' + b^2xx' = a^2y'^2 + b^2x'^2$$

or
$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2}$$

The dexter of this equation $=$ 1, since $(x'y')$ is on the curve.

The equation of the tangent at $(x'y')$ is $\therefore$
$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

This equation should be remembered.

Cor' (1)—The tangents at the ends of either axis are $\perp$ that axis.

Thus the tangent at A (a, o) is
$$\frac{ax}{a^2} = 1, \quad \text{or} \quad x = a, \text{ \&c.}$$

Cor' (2)—The normal is the line through $(x'y')$
$$\perp \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

Its equation is $\therefore$ (§ 71)
$$\frac{x-x'}{\left(\frac{x'}{a^2}\right)} = \frac{y-y'}{\left(\frac{y'}{b^2}\right)}$$

§ **246.** In the result of § 244 put $\beta = \alpha$: thus the equation to the tangent at the point whose eccentric angle is α is
$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The normal is the $\perp$ to this through $(a\cos\alpha,\ b\sin\alpha)$: its equation is $\therefore$
$$\frac{x-a\cos\alpha}{\left(\frac{\cos\alpha}{a}\right)} = \frac{y-b\sin\alpha}{\left(\frac{\sin\alpha}{b}\right)}$$

$$\frac{ax}{\cos\alpha} - \frac{by}{\sin\alpha} = a^2 - b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These results are often useful.

§ **247.** The abscissae of the points where the line

$$y = mx + c$$

cuts the ellipse $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

are determined by the equation

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

or $\qquad (a^2 m^2 + b^2)\, x^2 + 2\, mc\, a^2 x + a^2 (c^2 - b^2) = 0$

As this equation is a quadratic it has two roots, which may be real, coincident, or imaginary.

Thus *every line meets an ellipse in two points, which may be real, coincident, or imaginary.*

The roots of the quadratic are equal, i. e. the line $y = mx + c$ touches the ellipse if

$$(a^2 m^2 + b^2)\, a^2 (c^2 - b^2) = m^2 c^2 a^4$$

or $\qquad c^2 = a^2 m^2 + b^2$

or $\qquad c = \pm\sqrt{a^2 m^2 + b^2}$

We can thus draw two tangents ∥ a given line $y = mx$, viz.

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

and $\qquad y = mx - \sqrt{a^2 m^2 + b^2}$

Note—That two tangents can be drawn in a given direction may be seen otherwise thus.

In fig', § 237, if co-ord's of P are $(x'y')$, then those of P′ are $(-x', -y')$.

The tangent at P′ is ∴ (§ 245)

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = -1$$

which is parallel to the tangent at P.

Exercises

1. Find the tangent and normal to the ellipse

$$3x^2 + 4y^2 = 12$$

at the end of the latus rectum in the first quadrant.

Ans. $x + 2y = 4,\ 4x - 2y = 1$

2. Determine c so that $\quad y + x = c$

may be a tangent to the ellipse

$$2x^2 + 3y^2 = 1$$

Ans. $c = \pm \frac{1}{6}\sqrt{13}$

3. Find the tangent and normal to

$$x^2/a^2 + y^2/b^2 = 1$$

at the extremity of the latus rectum in the first quadrant.

Ans. $y + ex = a,\ ey - x + ae^3 = 0$

4. Find the tangents to the ellipse

$$3x^2 + 4y^2 = 12$$

which cut off equal intercepts on the axes.

Ans. $y \pm x = \pm\sqrt{7}$

5. If the tangent at the end of the latus rectum pass through a point of trisection of the minor axis, prove that the eccentricity of the ellipse is determined by the equation

$$9e^4 + e^2 = 1$$

6. Find co-ord's of inters'n of tangents at two points on the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

whose eccentric angles are α, β.

Ans. $a\cos\frac{\alpha+\beta}{2}\Big/\cos\frac{\alpha-\beta}{2},\quad b\sin\frac{\alpha+\beta}{2}\Big/\cos\frac{\alpha-\beta}{2}$

[*Note*—Tangents are

$$\left.\begin{aligned}\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha &= 1\\ \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta &= 1\end{aligned}\right\}$$

Determine $\quad \frac{x}{a} : \frac{y}{b} : 1\quad$ by § 61.]

7. P is a point on the ellipse whose eccentric angle is α; find equations of AP, A′P.

Ans. $\frac{x}{a}\cos\frac{\alpha}{2} + \frac{y}{b}\sin\frac{\alpha}{2} = \cos\frac{\alpha}{2}, \quad -\frac{x}{a}\sin\frac{\alpha}{2} + \frac{y}{b}\cos\frac{\alpha}{2} = \sin\frac{\alpha}{2}$

[*Note*—Ecc′ $\wedge$ of A is 0 and of A′ is π; use eq′n of § 244.]

8. P and Q are two points on an ellipse; AP, A′Q meet at J, and A′P, AQ meet at J′. Prove that JJ′ is perpendicular to AA′.

9. P is a point on an ellipse; find locus of inters′n of A′P with perpendicular to AP through A.

Ans. The straight line $x(a^2 - b^2) = a(a^2 + b^2)$

10. P, Q, R are three points on the ellipse; p, q, r the corresponding points on the auxiliary circle. Prove that

$$\triangle PQR : \triangle pqr = b : a$$

[*Note*—Let α, β, γ be eccentric $\wedge$s of P, Q, R: then co-ord′s of P are $(a\cos\alpha, b\sin\alpha)$ and of p are $(a\cos\alpha, a\sin\alpha)$, &c.; use § 22.]

11. The vertices of a triangle are three points on the ellipse whose eccentric angles are α, β, γ. Show that its area

$$= 2ab\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$$

12. P is any point on the curve; the perpendiculars through P to AP, A′P meet AA′ in L, M. Prove that

$$LM = \text{latus rectum}$$

13. P is any point on the ellipse, y its ordinate. Prove that

$$\cot APA' \propto y$$

GEOMETRICAL PROPERTIES

§ 248. Let the tangent at P meet the axes in T, t and let the normal meet the major axis in G.

Let $(x'y')$ be the co-ord′s of P.

The tangent at P is

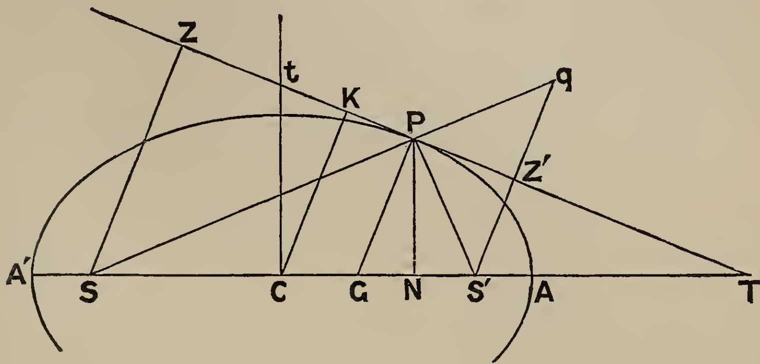
$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

To get its intercept on CA, put $y = 0$

$$\therefore \quad xx' = a^2$$

$$\text{i.e.} \quad CN \,.\, CT = CA^2 \quad \ldots\ldots \quad \text{(I)}$$

Similarly $$PN \,.\, Ct = CB^2 \quad \ldots\ldots \quad \text{(II)}$$



Again, putting $y = 0$ in the normal's equation

$$\frac{x - x'}{\left(\dfrac{x'}{a^2}\right)} = \frac{y - y'}{\left(\dfrac{y'}{b^2}\right)}$$

its intercept on CA is determined by

$$x - x' = -\frac{b^2}{a^2}x'$$

or $$x = x'\left(1 - \frac{b^2}{a^2}\right) = e^2 x'$$

$$\text{i.e.} \quad CG = e^2 . CN \quad \ldots\ldots \quad \text{(III)}$$

Hence $$SG = ae + e^2 x' = eSP \quad (\S\ 240)$$

and $$S'G = ae - e^2 x' = eS'P$$

$$\therefore \quad SG : S'G = SP : S'P$$

$$\therefore \quad PG \text{ bisects } S\hat{P}S' \quad \text{(Euclid VI. 3)}$$

Accordingly,

The normal bisects the angle between the focal distances, and the tangent bisects the supplementary angle . . . . (IV)

Again, since the co-ord's of P are $(x'y')$ and of G are $(e^2x', 0)$

$$\therefore \quad PG^2 = (x' - e^2x')^2 + y'^2$$

$$= x'^2 \frac{b^4}{a^4} + y'^2$$

$$\therefore \quad PG = b^2 \sqrt{\frac{x'^2}{a^4} + \frac{y'^2}{b^4}}$$

Also, if CK is central $\perp$ on tangent

$$CK = \perp \text{ from origin on } \frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0$$

$$\therefore \quad CK = 1 \Big/ \sqrt{\frac{x'^2}{a^4} + \frac{y'^2}{b^4}}$$

From these we deduce

$$PG \, . \, CK = b^2 \quad \ldots \ldots \quad \text{(V)}$$

Further, let SZ, $S'Z'$ be the focal $\perp$s on tangent; and let $S'Z'$ meet SP in q.

It follows from (IV) and Euclid I. 26, that

$$\triangle PS'Z' \equiv \triangle PqZ'$$

Accordingly,

$$S'Z' = qZ' \quad \text{and} \quad PS' = Pq$$

$$\therefore \quad Sq = SP + PS' = 2a$$

Also

$$S'Z' : S'q = 1 : 2$$

$$= S'C : S'S$$

$$\therefore \quad CZ' \text{ is } \parallel Sq, \text{ and}$$

$$CZ' = \tfrac{1}{2} Sq = a$$

Similarly

$$CZ = a$$

Thus *the feet of the focal perpendiculars on any tangent lie on the auxiliary circle* . . . . (VI)

This is proved analytically in § 249.

Further, if the eccentric angle of P is ϕ, the equation to the tangent at P is (§ 246)

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

$\therefore \quad S'Z' = \perp$ on this from (ae, o)

$$= (1 - e\cos\phi) \Big/ \sqrt{\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}}$$

and $\quad SZ = \perp$ on same from (— ae, o)

$$= (1 + e\cos\phi) \Big/ \sqrt{\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}}$$

$$\therefore \quad SZ \,.\, S'Z' = (1 - e^2\cos^2\phi) \Big/ \left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)$$

Now $\quad 1 - e^2\cos^2\phi = 1 - \dfrac{a^2 - b^2}{a^2}\cos^2\phi$

$$= \sin^2\phi + \frac{b^2}{a^2}\cos^2\phi$$

$$= b^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)$$

Hence $\quad SZ \,.\, S'Z' = b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad$ (VII)

The learner will find it easy to prove this geometrically. Another analytical proof is given in § 250.

Again, *tangents at the ends of a chord intersect on the diameter which bisects the chord* . . . . (VIII)

Subtracting the equations of the tangents at P $(x_1 y_1)$ and Q $(x_2 y_2)$, viz.

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{and} \quad \frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$$

we get .

$$\frac{x(x_1 - x_2)}{a^2} + \frac{y(y_1 - y_2)}{b^2} = 0$$

This eq'n represents a line through the inters'n of the tangents.

But the centre (o, o) and the mid point of the chord

$$\left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}\right)$$

are also obviously points on this line, $\therefore$ &c.

CONDITIONS OF TANGENCY

§ **249.** The condition in § 247 may also be obtained thus.

The equation to the pair of lines joining the origin to the inters'ns of

$$y = mx + c \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is (§ 119)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{y - mx}{c}\right)^2$$

If these lines are coincident the points of inters'n coincide

$$\therefore \quad y = mx + c \text{ is a tangent if}$$

$$\left(\frac{1}{a^2} - \frac{m^2}{c^2}\right)\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = \frac{m^2}{c^4}$$

which reduces to $c = \pm\sqrt{a^2m^2 + b^2}$, as before.

The result may be usefully stated thus—

Whatever be the value of m, *the line*

$$y = mx + \sqrt{a^2m^2 + b^2}$$

is a tangent to the ellipse.

Ex. Find locus of foot of $\perp$ from focus on tangent.

Any tangent is $\qquad y - mx = \sqrt{a^2m^2 + b^2}$

The equation to the $\perp$ on this from S $(-ae, 0)$ is

$$my + x = -ae$$

To eliminate m, square and add: this gives

$$(1 + m^2)(x^2 + y^2) = a^2m^2 + b^2 + a^2e^2$$
$$= a^2(1 + m^2)$$
$$\therefore \quad x^2 + y^2 = a^2$$

The locus is $\therefore$ the auxiliary circle. (Compare § 248, VI.)

§ **250.** *To find the condition that*

$$lx + my = n \quad \dots\dots\dots\dots \quad (1)$$

may be a tangent.

This may be deduced as in § 247, or § 249. For variety we give another method.

The tangent at $(x'y')$ is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If this represent the same line as (1)

$$\frac{x'}{a^2 l} = \frac{y'}{b^2 m} = \frac{1}{n}$$

$$\therefore \quad \frac{\left(\frac{x'}{a}\right)}{al} = \frac{\left(\frac{y'}{b}\right)}{bm} = \frac{1}{n}$$

Hence, as

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

we obtain

$$a^2 l^2 + b^2 m^2 = n^2$$

the condition required. This result is often useful.

Cor' (1)—$x \cos\alpha + y \sin\alpha = p$ is a tangent if

$$p^2 = a^2 \cos^2\alpha + b^2 \sin^2\alpha$$

This important result might of course be proved independently.

Cor' (2)—The equation to any tangent may be written

$$x \cos\alpha + y \sin\alpha = \sqrt{a^2 \cos^2\alpha + b^2 \sin^2\alpha}$$

Ex. 1. Prove $\quad SZ . S'Z' = b^2 \quad$ (fig' § 248)

Let the tangent at P be

$$x \cos\alpha + y \sin\alpha - p = 0$$

Then $\quad S'Z' = \perp$ on this from $\quad (ae, 0) = p - ae \cos\alpha$

$\quad SZ = \perp$ on it from $(-ae, 0) = p + ae \cos\alpha$

$$\begin{aligned} \therefore \quad SZ . S'Z' &= p^2 - a^2 e^2 \cos^2\alpha \\ &= (a^2 \cos^2\alpha + b^2 \sin^2\alpha) - (a^2 - b^2) \cos^2\alpha \\ &= b^2 \end{aligned}$$

Compare § 248, VII.

§ 251. Ex. 2. Find locus of point of intersection of tangents at right angles to each other.

Substitute $\dfrac{\pi}{2} + \alpha$ for α in

$$x \cos\alpha + y \sin\alpha = \sqrt{a^2 \cos^2\alpha + b^2 \sin^2\alpha} \quad . \quad . \quad . \quad . \quad (1)$$

The tangent at right angles to the tangent (1) is $\therefore$

$$-x\sin\alpha + y\cos\alpha = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} \quad \ldots \quad (2)$$

By squaring and adding (1) and (2), α is eliminated.

The required locus is $\therefore$ the circle

$$x^2 + y^2 = a^2 + b^2$$

Def′—This circle, which is the locus of the inters′n of rectangular tangents, is called the *director circle.*

Exercises

1. CP, CQ are semi-diameters at right angles; prove that PQ touches a fixed circle whose centre is C.

[*Note*—Let $\perp$ from C on PQ = p; let α = angle which this $\perp$ makes with CP. We have to prove that p is constant.

Now $$\frac{\cos\alpha}{p} = \frac{1}{CP}, \quad \frac{\sin\alpha}{p} = \frac{1}{CQ}; \text{ square and add}$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} + \frac{1}{b^2} \quad \text{(page 210, Ex. 8).]}$$

2. Find the polar and rectangular equations of locus of foot of perpendicular from centre on tangent.

Ans. $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$, $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

[*Note*—Use result of § 250, *Cor′* (1).]

3. If LSL′ be the latus rectum, and any ordinate PN be produced to meet the tangent at L in Q, prove that SP = QN.

4. Any tangent to an ellipse meets the tangent at A in V and the minor axis in t. Prove that tV = tS.

5. The normal at P meets the axes in G, g. Prove that

$$PG \cdot Pg = SP \cdot S'P.$$

6. Find the condition that the line

$$x\cos\alpha + y\sin\alpha = p$$

may be normal to the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

Ans. $p^2(a^2\sin^2\alpha + b^2\cos^2\alpha) = (a^2 - b^2)^2\sin^2\alpha\cos^2\alpha$

7. P is any point on the ellipse. If $SP = r$ and perpendicular from S on tangent at $P = p$, find the relation between r and p.

Ans. $\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$

8. A line through C parallel to the tangent at P meets the focal distances of P in d, d′. Prove that

$$Pd = Pd' = a$$

9. An ellipse slides between two lines at right angles. Find the locus of its centre.

[*Note*—Take given lines as axes; use result of § 251. The req'd locus is the circle

$$x^2 + y^2 = a^2 + b^2]$$

10. Find locus of intersection of perpendicular from focus on any tangent, with join of centre to point of contact.

Ans. The corresponding directrix.

11. Points J, J′ are taken on the minor axis such that

$$CJ = CJ' = CS;$$

p and p′ are the perpendiculars from J, J′ on any tangent. Prove that

$$p^2 + p'^2 = 2a^2$$

12. The sum of the eccentric angles of two points P, Q on the ellipse is constant ($= 2\gamma$); find locus of intersection of tangents at P, Q.

Ans. The straight line $ay = bx \tan\gamma$

13. Show that the equation of the tangents from $(x'y')$ to the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

is
$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{x'x}{a^2} + \frac{y'y}{b^2} - 1\right)^2$$

[*Note*—Proceed as in § 167.]

TANGENTS FROM A GIVEN POINT

§ **252.** Any tangent is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

This will pass through a given point $(x'y')$ if

$$y' = mx' + \sqrt{a^2m^2 + b^2}$$

or
$$(y' - mx')^2 = a^2m^2 + b^2$$

or
$$(a^2 - x'^2)m^2 + 2x'y'm + b^2 - y'^2 = 0 \quad . \quad . \quad (1)$$

This quadratic gives two values of m, and $\therefore$ in general two tangents can be drawn from a given point $(x'y')$.

The roots of (1) are real, coincident, or imaginary, according as

$$x'^2y'^2 > = < (a^2 - x'^2)(b^2 - y'^2)$$

i. e. according as

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 > = < 0$$

The two tangents are $\therefore$ (§ 241) real, coincident, or imaginary, according as the given point $(x'y')$ is outside, on, or inside the ellipse.

§ **253.** *Cor'*—If (xy) is any point on either tangent from $(x'y')$

$$(y - y')/(x - x') = m$$

If we substitute this value of m in (1) we get a relation between x and y.

The equation to the pair of tangents from $(x'y')$ is $\therefore$

$$(a^2 - x'^2)(y - y')^2 + 2x'y'(x - x')(y - y') + (b^2 - y'^2)(x - x')^2 = 0$$

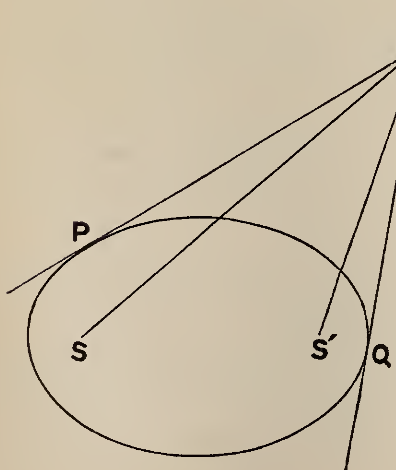
This equation reduces to

$$a^2(y - y')^2 + b^2(x - x')^2 = (xy' - x'y)^2$$

Note—Another form of this eq'n is given, Ex. 13, page 223.

§ **254.** *If* TP, TQ *are the tangents from* T, *then*

$$S\hat{T}P = S'\hat{T}Q$$



Let p, q be the $\perp$s from S on TP, TQ, and p', q' the $\perp$s from S' on TP, TQ.

Then $pp' = b^2$ (§ 248, VII.)

$= qq'$

$$\therefore \frac{p}{q} = \frac{q'}{p'}$$

But $$\frac{p}{q} = \frac{\sin STP}{\sin STQ} \quad \text{and} \quad \frac{q'}{p'} = \frac{\sin S'TQ}{\sin S'TP}$$

$\therefore$ ST, S′T divide $\widehat{PTQ}$ into parts having the same ratio of sines; and the proposition is evident.

Note—The proposition may also be proved thus.

It is evidently true if the line-pairs (TP, TQ) and (TS, TS′) have the same bisectors of angles; and $\therefore$ if ||s through the origin to these line-pairs have the same bisectors.

Retaining only the terms of the second degree in the equation of § 253, the line-pair through the origin || TP, TQ is

$$(a^2 - x'^2)\, y^2 + 2\, x'y'xy + (b^2 - y'^2)\, x^2 = 0$$

Again, the ||s through the origin to TS, TS′ are

$$y\,(x' + ae) - y'x = 0 \quad \text{and} \quad y\,(x' - ae) - y'x = 0$$

The product of these is

$$y^2\,(x'^2 - a^2 + b^2) - 2\, x'y'xy + y'^2 x^2 = 0$$

Each of these line-pairs is bisected (§ 116) by the line-pair

$$x'y'\,(x^2 - y^2) = (x'^2 - a^2 + b^2 - y'^2)\, xy$$

Q.E.D.

DIAMETERS

§255. *To find the locus of the mid points of a system of parallel chords.*

Let the chords be || $y = mx$.

Let $(x_1 y_1)$, $(x_2 y_2)$ be the extremities of one of the chords; (xy) its mid point.

Then $$2x = x_1 + x_2, \quad 2y = y_1 + y_2 \quad \ldots\ldots \quad (1)$$

Also $$\frac{y_1 - y_2}{x_1 - x_2} = m \quad \ldots\ldots\ldots \quad (2)$$

Now $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1$$

By subtraction $$\frac{x_1^2 - x_2^2}{a^2} + \frac{y_1^2 - y_2^2}{b^2} = 0$$

$$\therefore \quad \frac{(x_1 - x_2)(x_1 + x_2)}{a^2} + \frac{(y_1 - y_2)(y_1 + y_2)}{b^2} = 0$$

Divide by $x_1 - x_2$; then from (1), (2) we see that

$$\frac{x}{a^2} + \frac{my}{b^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the equation to the locus. It is ∴ a straight line passing through the centre.

Cor′ (1)—All diameters of the ellipse pass through the centre. (See def′, § 229.)

Cor′ (2)—If we write the equation to the locus (3) in the form

$$y = m'x$$

then $$m' = -b^2/(ma^2)$$

$$\therefore \quad mm' = -b^2/a^2$$

The symmetry of this relation proves that

$$y = mx$$

is the locus of mid points of chords

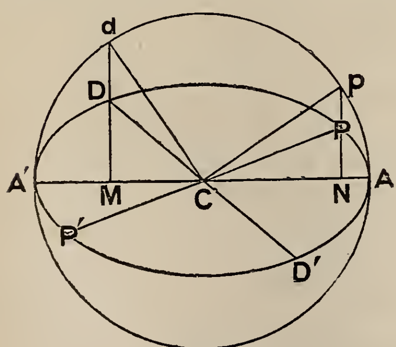
$$\parallel \quad y = m'x$$

Hence *if one diameter of an ellipse bisects chords parallel to a second, the second bisects all chords parallel to the first.* Two such diameters are said to be *conjugate.*

Cor′ (3)—Two diameters $y = mx$, $y = m'x$ are conjugate if

$$mm' = -\frac{b^2}{a^2}$$

§ 256. If PCP′, DCD′ are conj′ diam′rs there is a simple relation between the eccentric ∧s ϕ, ϕ' of P, D.



The co-ord′s of P are

$$(a \cos \phi, \quad b \sin \phi)$$

∴ if eq′n to CP is $y = mx$

then $$m = b \sin \phi / (a \cos \phi)$$

Similarly, if eq′n to CD is

$$y = m'x$$

then $$m' = b \sin \phi' / (a \cos \phi')$$

But [§ 255, *Cor′* (3)] $$mm' = -b^2/a^2$$

$$\therefore \quad b^2 \sin\phi \sin\phi' / (a^2 \cos\phi \cos\phi') = -b^2/a^2$$

$$\therefore \quad \cos\phi \cos\phi' + \sin\phi \sin\phi' = 0$$

$$\therefore \quad \cos(\phi' - \phi) = 0$$

$$\therefore \quad \phi' - \phi = 90^\circ$$

Cor' (1)—Let p, d be the points on the auxiliary circle corresponding to P, D.

Then $$\phi' - \phi = d\hat{C}A - p\hat{C}A = d\hat{C}p$$

$\therefore$ Cp, Cd are at right angles.

Cor' (2)—Since $$\phi' = \phi + 90^\circ$$

the equation to the tangent at D, which is

$$\frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = 1 \quad (\S\ 246)$$

becomes $$-\frac{x}{a}\sin\phi + \frac{y}{b}\cos\phi = 1$$

or $$bx\sin\phi - ay\cos\phi + ab = 0$$

Again, the eq'n to CP is

$$y/x = b\sin\phi/(a\cos\phi)$$

or $$bx\sin\phi - ay\cos\phi = 0$$

$\therefore$ tangent at D is parallel to CP.

Hence *the tangent at either end of any diameter is parallel to the chords which that diameter bisects.*

This may also be proved geometrically, exactly as in § 230.

§ 257. *If* PCP′, DCD′ *are conjugate diameters, then*

$$CP^2 + CD^2$$

is constant; and the area of the parallelogram whose sides are the tangents at P, P′, D, D′ *is constant.*

Let the eccentric angle of P be ϕ; then that of D is $\phi + 90^\circ$ [fig' § 256.]

Let the co-ord's of P be $(x'y')$ and of D $(x''y'')$.

Then $$x' = a\cos\phi, \quad y' = b\sin\phi$$

Also $$x'' = a\cos(\phi + 90^\circ) = -a\sin\phi$$

and $$y'' = b \sin(\phi + 90°) = b \cos\phi$$

$$\therefore \quad CP^2 = x'^2 + y'^2 = a^2\cos^2\phi + b^2\sin^2\phi$$

and $$CD^2 = x''^2 + y''^2 = a^2\sin^2\phi + b^2\cos^2\phi$$

$$\therefore \quad CP^2 + CD^2 = a^2 + b^2$$

Again, area of $\square$ who sides are the tangents at P, P′, D, D′

$= 4$ area of $\square$ whose adjacent sides are CP, CD

$= 8$ area of $\triangle$ CPD

$= 4\,(x'y'' - x''y')$ [*Cor′*, § 22]

$= 4\,ab\,(\cos^2\phi + \sin^2\phi)$

$= 4\,ab.$

Cor′ (1)—Denote the lengths of two conjugate semi-diameters CP, CD by a', b' and the angle between them by ω.

Since area of $\triangle$ CPD $= \frac{1}{2} a'b' \sin\omega$

$$\left.\begin{aligned} \therefore \quad a'b'\sin\omega &= ab \\ \text{Also} \quad a'^2 + b'^2 &= a^2 + b^2 \end{aligned}\right\}$$

If CP, CD are given in position and magnitude, i.e. if a', b' and ω are given, we can determine a and b by solving these equations.

Cor′ (2)—From the previous equations we get

$$4\,a^2b^2\,\mathrm{cosec}^2\,\omega = 4\,a'^2 b'^2 = (a'^2 + b'^2)^2 - (a'^2 - b'^2)^2$$

$$\therefore \quad 4\,a^2b^2\,\mathrm{cosec}^2\,\omega = (a^2 + b^2)^2 - (a'^2 - b'^2)^2$$

Hence cosec ω is greatest, and $\therefore$ sin ω is least when $a' = b'$.

SUPPLEMENTAL CHORDS

§ 258. Let Q be any point on the ellipse, PCP′ any diameter; then QP, QP′ are called *supplemental* chords.

The diameters parallel to supplemental chords QP, QP′ *are conjugate.*

For diam′ ∥ QP′ bisects QP, and diam ∥ QP bisects QP′ [Euclid VI. 2].

The diameters are $\therefore$ conjugate, since each bisects a chord parallel to the other.

The proposition may also be proved analytically thus.

Let $y = mx$, $y = m'x$ be eq'ns of diam'rs ∥ QP and QP'.

Let α = ecc' ∧ of P and β = ecc' ∧ of Q; then ecc' ∧ of P' is $\pi + \alpha$.

The eq'n of PQ is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2} \qquad (\S\ 244)$$

$$\therefore \quad m = -\frac{b}{a}\cot\frac{\alpha+\beta}{2}$$

Again, writing $\pi + \alpha$ for α in this eq'n we get

$$m' = -\frac{b}{a}\cot\frac{\pi+\alpha+\beta}{2}$$

$$= +\frac{b}{a}\tan\frac{\alpha+\beta}{2}$$

$$\therefore \quad mm' = -\frac{b^2}{a^2}, \quad \text{and} \therefore \text{ \&c. } [\S\ 255,\ \textit{Cor}'\ (3)]$$

EQUAL CONJUGATE DIAMETERS

§ 259. The diameters ∥ AB, A'B (fig' § 237) are equal by symmetry; and by § 258 they are conjugate.

The equi-conjugate diameters are ∴ the diagonals of the rectangle whose sides are the tangents at A, B, A', B'; and their equations are

$$y = +\frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

To find their lengths; since

$$CP^2 + CD^2 = a^2 + b^2 \quad [\S\ 257]:$$

if $$CP = CD = a'$$

we find $$2\,a'^2 = a^2 + b^2$$

POLES AND POLARS

§ 260. *Def'*—If P, Q are the points of contact of tangents from T, then PQ is called the *polar* of T, and T is called the pole of PQ.

To find the equation to the polar of $(x'y')$.

Let (hk) be the point of contact of *either* tangent from $(x'y')$.

Express that the tangent at (hk), viz.

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$$

passes through $(x'y')$

$$\therefore \quad \frac{x'h}{a^2} + \frac{y'k}{b^2} = 1$$

This equation expresses that (hk) lies on the straight line

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

As *both* points of contact lie on this line, it is $\therefore$ their join.

The equation of the polar of $(x'y')$ is $\therefore$

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

Cor' (1)—It is proved exactly as in § 234 that *if* P *lies on the polar of* Q *then* Q *lies on the polar of* P.

Cor' (2)—The polar of the focus $(ae, 0)$ is

$$xe = a, \quad \text{or} \quad x = \frac{a}{e},$$

i. e. the corresponding directrix;

$\therefore$ tangents at the ends of a focal chord intersect on the directrix.

Exercises

1. If CP, CD are conjugate semi-diameters, prove that

$$SP.S'P = CD^2.$$

2. Find locus of mid point of PD.

Ans. The ellipse $2x^2/a^2 + 2y^2/b^2 = 1$

3. Find locus of intersection of tangents at P, D.

Ans. The ellipse $x^2/a^2 + y^2/b^2 = 2$

4. CP, CD and CP′, CD′ are pairs of conjugate semi-diameters. Prove that

$$\triangle PCP' = \triangle DCD'.$$

5. CP, CD meet the tangent at A in V, W. Prove that AV.AW is constant.

6. The normal at P meets the major axis in G, and the diameter conjugate to CP in F. Prove that

$$PG.PF = b^2.$$

7. SJ, SJ′ are perpendiculars from a focus on a pair of conjugate diameters: prove that JJ′ meets the major axis at a fixed point.

8. The perpendiculars from S, S′ on a pair of conjugate diameters meet at Q. Find the locus of Q.

Ans. The ellipse $b^2 y^2 + a^2 x^2 = a^2 (a^2 - b^2)$

9. If the perpendicular from the centre on the tangent at $P = p$, and $CP = r$, prove that

$$p^2 (a^2 + b^2 - r^2) = a^2 b^2$$

10. The normals at P, D meet in J. Prove that JC is perpendicular to PD.

11. The perpendicular from P on its polar meets the major axis in γ; n is the foot of the ordinate of P. Prove that

$$C\gamma = e^2 Cn$$

12. V is the mid point of a chord PQ; prove that the polar of V is parallel to PQ.

[Proceed as in *Note*, Ex. 9, page 197.]

13. Find the equation of the chord whose mid point is (hk).

Ans. $\dfrac{h}{a^2}(x - h) + \dfrac{k}{b^2}(y - k) = 0$

[*Note*—Use result of Ex. 12.]

14. Chords of the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

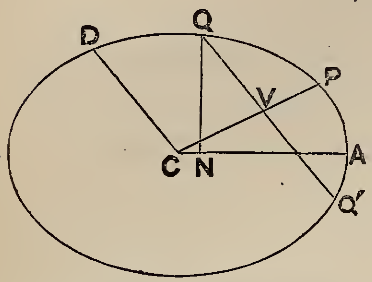
are drawn through (x′y′); find locus of their mid points.

Ans. The ellipse $x^2/a^2 + y^2/b^2 = xx'/a^2 + yy'/b^2$

[See *Note*, Ex. 13, page 197.]

EQUATION REFERRED TO CONJUGATE DIAMETERS

§ **261.** *To find the equation of the ellipse referred to any conjugate diameters* CP, CD *as axes.*



Let $CP = a'$, $CD = b'$.

Let QVQ' be a chord $\parallel$ CD; so that $x = CV$, $y = QV$ are the co-ord's of Q.

Draw $QN \perp CA$.

Put also

$$P\hat{C}A = \theta, \qquad D\hat{C}A = \theta'$$

Project the broken line CVQ on CA and CB.

$$\therefore \left.\begin{aligned} CN &= x\cos\theta + y\cos\theta' \\ QN &= x\sin\theta + y\sin\theta' \end{aligned}\right\}$$

But $$CN^2/a^2 + QN^2/b^2 = 1$$

$$\therefore\ (x\cos\theta + y\cos\theta')^2/a^2 + (x\sin\theta + y\sin\theta')^2/b^2 = 1$$

$$\therefore\ x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) + 2xy\left(\frac{\cos\theta\cos\theta'}{a^2} + \frac{\sin\theta\sin\theta'}{b^2}\right)$$
$$+ y^2\left(\frac{\cos^2\theta'}{a^2} + \frac{\sin^2\theta'}{b^2}\right) = 1$$

The coeff′ of $2xy$ in this $= 0$

$$\because\ \tan\theta\tan\theta' = -b^2/a^2 \quad [\text{§ 255, } Cor' \text{ (3)}]$$

Coeff′ of $$x'^2 = 1/CP^2 = 1/a'^2 \quad [\text{§ 238}]$$

Coeff′ of $$y'^2 = 1/CD^2 = 1/b'^2$$

The required equation is $\therefore$

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$$

§ **262.** As this equation is of the same form as that referred to CA, CB, several of the previous §§ are applicable when any pair of conj′ diam′rs are chosen as axes.

Thus the tangent at $(x'y')$ is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1;$$

and as in § 248 we deduce that the tangents at Q, Q' meet CP in the same point t such that

$$CV.Ct = CP^2$$

Again, $$lx + my = n$$

is a tangent if $$a'^2l^2 + b'^2m^2 = n^2; \quad \&c.$$

Ex. If the tangent at P meet any two conjugate diameters in t, t', prove that

$$Pt.Pt'$$

is constant.

Take the conjugate diameters CP, CD as axes of co-ordinates.

Let any two conj′ diam′rs be

$$y = mx, \qquad y = m'x;$$

then $$mm' = -b'^2/a'^2$$

The eq′n of tangent at P is

$$x = a'$$

The co-ord′s of t are determined by

$$\left.\begin{aligned} x &= a' \\ y &= mx \end{aligned}\right\}$$

$$\therefore \quad y, \text{ i.e. } Pt = ma'$$

Similarly $$Pt' = m'a'$$

$$\therefore \quad Pt.Pt' = mm'a'^2 = -b'^2$$

PARABOLA A LIMITING FORM OF ELLIPSE

§ 263. Referring to § 237 we may express a, b in terms of e and SX.

For $\quad SX = CX - CS = a/e - ae = a(1 - e^2)/e$

$$\therefore \quad a = e\,SX/(1 - e^2)$$

Also $\quad b^2 = a^2(1 - e^2) = e^2 SX^2/(1 - e^2)$

If now we suppose the focus S and the distance SX to remain fixed while e becomes 1, we see that a and b become infinite.

The latus rectum $p = 2e\,SX$ and $\therefore$ remains finite.

Thus the parabola is an ellipse whose axes are infinite, while its latus rectum is finite.

§ **264.** The same conclusion may be deduced thus.

If we change to || axes through the vertex A′ the equation of the ellipse is

$$(x - a)^2/a^2 + y^2/b^2 = 1$$

or
$$y^2/b^2 = 2x/a - x^2/a^2$$

or
$$y^2 = px - px^2/2a \qquad (1)$$

where
$$p = 2b^2/a = \text{latus rectum}$$

If $a = \infty$ while p remains finite, (1) becomes

$$y^2 = px$$

§ **265.** Ex. 1. What does the property of § 248, I., viz.

$$CN \,.\, CT = CA^2$$

become for the parabola?

Put
$$CN = \delta$$

$$\therefore \quad \delta(\delta + AT + NA) = (\delta + NA)^2$$

$$\therefore \quad \delta(AT - NA) = NA^2$$

$$\therefore \quad AT - NA = NA^2/\delta$$

Now put $\delta = \infty$; $\quad \therefore \quad AT = NA$ (see § 225)

Ex. 2. If in § 254 we suppose the focus S′ to move off to infinity; we deduce the following property of the parabola—

$$Q\hat{T}X' = S\hat{T}Q' \quad \text{[see fig', § 231]}$$

Thus *if tangents are drawn from a point* T *to a parabola, one of them makes the same angle with the axis that the other does with* ST.

Exercises on the Ellipse.

[Unless otherwise implied, in these questions the ellipse is referred to its axes of figure, its equation being

$$x^2/a^2 + y^2/b^2 = 1$$

The notation of the figures in the chapter is retained throughout; thus S, S′ are the foci, G the foot of the normal at P, P and D extremities of conjugate diameters, &c.]

1. Prove that SZ′, S′Z meet at the mid point of PG. (See fig′, p. 217.)

2. If ϕ is the angle between the tangents from $(x'y')$, prove that

$$(x'^2 + y'^2 - a^2 - b^2) \tan\phi = 2\sqrt{b^2x'^2 + a^2y'^2 - a^2b^2}$$

3. If α, β are the eccentric angles of the extremities of a focal chord, then

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{e-1}{e+1}$$

4. Prove that the sum of the squares of the perpendiculars from P, D on a fixed diameter is constant.

5. Find the co-ordinates of the pole of the line

$$x\cos\alpha + y\sin\alpha = p$$

Ans. $\dfrac{a^2\cos\alpha}{p}, \dfrac{b^2\sin\alpha}{p}$

6. A chord PQ parallel to AB meets CA, CB in p, q. Prove that

$$Pp = Qq$$

Also, if m and n are the feet of the ordinates at P and Q, prove that

$$2\,Am \, . \, An = Ap^2$$

7. Given base of a triangle ABC and also

$$\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B;$$

show that locus of vertex C is an ellipse whose foci are A, B.

[*Note*—By Trigonometry

$$\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B = (s-c)/s; \quad \therefore \text{ s is given.}]$$

8. P is any point on the curve; find locus of centre of circle inscribed in triangle SPS′.

Ans. The ellipse $(1-e)x^2 + (1+e)y^2 = e^2(1-e)a^2$

[*Note*—Use result of Ex. 62, p. 165.]

9. P is a point on the curve; find locus of intersection of ordinate of P with perpendicular from centre on tangent at P.

Ans. The ellipse $a^2x^2 + b^2y^2 = a^4$

10. Any tangent to the ellipse meets the director circle

$$x^2 + y^2 = a^2 + b^2$$

in p, d; show that Cp, Cd are conjugate diameters of the ellipse.

[*Note*—Let the tangent be [§ 250, *Cor'* (2)]

$$x \cos \alpha + y \sin \alpha = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha};$$

then the eq'n to the line-pair C p, C d is (§ 119)

$$(a^2 + b^2)(x \cos \alpha + y \sin \alpha)^2 - (a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)(x^2 + y^2) = 0$$

If the factors of this are

$$y - mx, \quad y - m'x$$

then $$mm' = -b^2/a^2; \quad \therefore \text{ \&c. (§ 255, } Cor' \text{ 3).]}$$

11. P is any point on the ellipse; the perpendiculars to AP, A′P at their mid points meet AA′ in J, J′. Prove that

$$JJ' = (a^2 - b^2)/a$$

[*Note*—This gives a method of describing the ellipse mechanically. See a paper by Professor Genese in Milne's *Companion*, p. 210.]

12. The tangents from P make angles θ, θ' with the major axis. Find the locus of P if

$$\tan \theta \tan \theta' = \text{constant } (= k).$$

Ans. $y^2 - b^2 = k(x^2 - a^2)$; this equation represents an ellipse if k is negative and an hyperbola if k is positive.

13. Show that normals at the ends of a focal chord intersect on a parallel to the major axis through the mid point of the chord.

14. Find locus of a point whose polars with respect to the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

touch the circle $$x^2 + y^2 = r^2$$

Ans. The ellipse $x^2/a^4 + y^2/b^4 = 1/r^2$

15. An ellipse is cut by a series of parallel chords; α, β are the eccentric angles of the extremities of one of the chords. Prove that $\alpha + \beta$ is constant.

16. Three sides of a quadrilateral inscribed in an ellipse are parallel to given lines. Show that the fourth side is parallel to a given line.

[*Note*—Use result of Ex. 15.]

17. If the perpendiculars from the centre on the tangents at P, D meet these tangents in K, K′ and the ellipse in k, k′, prove that

$$\frac{1}{CK^2 . Ck^2} + \frac{1}{CK'^2 . Ck'^2} = \frac{1}{a^4} + \frac{1}{b^4}$$

18. If PD is inclined to the major axis at an angle ϕ, and δ is the perpendicular from C on PD, prove that

$$2\,\delta^2 = a^2 \cos^2 \phi + b^2 \sin^2 \phi$$

19. If ϕ, ϕ' are the eccentric angles of the points of contact of tangents from (hk), prove that

$$\tan \frac{\phi - \phi'}{2} = \sqrt{\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1}$$

20. A parallel to PD meets CP, CD in p, d and the ellipse in q. Prove that

$$qp^2 + qd^2 = PD^2$$

21. Any diameter meets parallels through P, D to any tangent in p, d, and the tangent in r; prove that

$$Cp^2 + Cd^2 = Cr^2$$

22. Any line through the vertex A meets the ellipse in q and the minor axis in r; CP is the semi-diameter parallel to A q. Prove that

$$Aq\,.\,Ar = 2\,CP^2$$

[*Note*—To determine A q use polar eq'n referred to A as origin.]

23. p is any point on the auxiliary circle; A p, A′p meet the ellipse in q, q′. Prove that

$$Ap/Aq + A'p/A'q' = (a^2 + b^2)/b^2$$

24. PN, PM are the perpendiculars from a point P on two given oblique axes; if $PN^2 + PM^2$ is constant ($= k^2$), show that the locus of P is an ellipse, and find its eccentricity.

[Eq'n of locus is $(x^2 + y^2) \sin^2 \omega = k^2$;

this represents an ellipse referred to its equi-conjugates. The semi-axes a, b are determined by

$$k^2/\sin^2 \omega = (a^2 + b^2)/2, \quad b/a = \cot \frac{\omega}{2}; \quad \therefore\ e^2 = 1 - \cot^2 \frac{\omega}{2}.]$$

25. PCP′ is any diameter of an ellipse. Any line through P cuts the ellipse in q and the tangent at P′ in r; CQ is the semi-diameter parallel to P q. Prove that

$$Pq\,.\,Pr = 4\,CQ^2$$

26. PCP′, DCD′ are conjugate diameters; O is any point on a circle concentric with the ellipse whose radius is r. Prove that

$$OP^2 + OP'^2 + OD^2 + OD'^2 = 4\,r^2 + 2\,(a^2 + b^2)$$

27. Show that the joins of the vertices of an ellipse to opposite ends of any segment of the directrix which subtends a right angle at the focus meet on the curve.

28. Points Q, Q′ are taken on the normal at P such that

$$PQ = PQ' = CD;$$

show that $CQ = a - b, \qquad CQ' = a + b$

[*Note*—If ecc′ $\wedge$ of P is α, deduce that co-ord′s of Q, Q′ are

$$(a \mp b) \cos \alpha, \qquad (b \mp a) \sin \alpha.]$$

29. TP, TQ are tangents from T. If PQ subtend a right angle at the centre, find the locus of T.

[*Note*—Let (hk) be co-ord′s of T. The eq′n to the line-pair CP, CQ is (§ 119)

$$\left(\frac{xh}{a^2} + \frac{yk}{b^2}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

These lines are at right angles [§ 114, *Cor′* (1)] if

$$h^2/a^4 + k^2/b^4 = 1/a^2 + 1/b^2$$

The locus req′d is ∴ the ellipse

$$x^2/a^4 + y^2/b^4 = 1/a^2 + 1/b^2.]$$

30. Find locus of intersection of normals at extremities of a chord parallel to one of the equi-conjugates.

[Take equi-conj′s as axes; let eq′n to ellipse be

$$x^2 + y^2 = c^2.$$

If the axes are inclined at $\hat{\omega}$ the normal at $(x'y')$ is

$$(x - x')(y' - x' \cos \omega) = (y - y')(x' - y' \cos \omega)$$

This intersects normal at $(-x', y')$ on the line

$$y + x \cos \omega = 0;$$

the locus is ∴ the diameter ⊥ other equi-conjugate.]

31. p is any point on the auxiliary circle; the tangent at p meets AA′ in t. If pA, pA′ meet the ellipse again in d, e, show that the chord de passes through t.

32. p is any point on the auxiliary circle; the perpendiculars from S, S′ on the tangent at p meet the ellipse in q, q′. Show that pq. pq′ are tangents to the ellipse.

33. If θ is the angle between the tangents from T show that

$$2\,ST.S'T\cos\theta = ST^2 + S'T^2 - 4a^2$$

34. The tangent at P intersects parallels to the major axis at a distance b/e from it in q, r; prove that Pq, Pr subtend equal angles at the centre.

35. Deduce the following form of the equation of the tangents from $(x'y')$, viz.

$$a^2(y-y')^2 + b^2(x-x')^2 = (xy'-x'y)^2$$

by expressing the condition that the join of (xy), $(x'y')$ should touch the ellipse.

36. Find the locus of a point such that, tangents being drawn from it to the ellipse, area of triangle which these form with major axis = area of triangle which they form with minor axis.

Ans. The equi-conjugate diameters.

37. Two ellipses have the same focus and eccentricity and their major axes coincide in direction. If PN, P′N′ are the ordinates of the larger ellipse which are tangents to the smaller, prove that

$$SP - SN = SP' - SN'$$

38. Show that the normal at P passes through the mid point of the join of the feet of the perpendiculars from P on the equi-conjugate diameters.

39. Prove that the length of the chord joining the points whose eccentric angles are α, β is

$$2\,b'\sin\frac{\alpha-\beta}{2}.$$

where b' is the parallel semi-diameter.

40. Find the area of the triangle formed by the tangents at the points whose eccentric angles are α, β, γ.

Ans. $ab\tan\frac{1}{2}(\alpha-\beta)\tan\frac{1}{2}(\beta-\gamma)\tan\frac{1}{2}(\gamma-\alpha)$

41. If P, Q are the points of contact of tangents from T (hk); show that

$$\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)SP.SQ = ST^2$$

[*Note*—If abscissa of P is x, SP = ρ; then

$$x = (\rho - a)/e \quad (\S\ 240)$$

Elim′ y between

$$x^2/a^2 + y^2/b^2 = 1, \qquad hx/a^2 + yk/b^2 = 1;$$

then subst′ preceding value of x. This gives a quadratic in ρ.]

42. Find locus of mid point of normal PG.

Ans. The ellipse $4b^2x^2 + 4(1 + e^2)^2 a^2 y^2 = a^2 b^2 (1 + e^2)^2$

43. Find locus of mid points of chords of the ellipse which touch the concentric circle

$$x^2 + y^2 = r^2$$

Ans. $(x^2/a^2 + y^2/b^2)^2 = r^2(x^2/a^4 + y^2/b^4)$

44. Two chords meet the axis major at points equi-distant from the centre; if α, β and γ, δ are the eccentric angles of their extremities, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = 1$$

45. Any chord PQ is drawn through the fixed point (d, o) on the major axis. If the abscissae of P, Q are x_1, x_2, prove that

$$2d(x_1 x_2 + a^2) = (a^2 + d^2)(x_1 + x_2)$$

Hence show that the product of the perpendiculars from P, Q on a fixed line parallel to the minor axis is constant.

[*Note*—If $\qquad \lambda = (a^2 + d^2)/(2d)$

the preceding eq′n gives

$$(x_1 - \lambda)(x_2 - \lambda) = \text{constant}$$

The line required is ∴

$$2dx = a^2 + d^2.]$$

46. P, Q, R are three points on the ellipse: the diameter bisecting QR meets PQ, PR and the curve in m, n, d: show that

$$Cm \, . \, Cn = Cd^2$$

47. The three points P, Q, R on an ellipse are the vertices of a triangle whose area is a maximum.

Show that QR is parallel to the tangent at P.

[*Note*—By Ex. 10, page 216, Δ PQR is maximum when Δ pqr is maximum. It follows by Geometry that Δ pqr is equilateral. The point P may be anywhere on the ellipse; and if α is its ecc′ $\wedge$, that of Q is

$$\alpha + 2\pi/3,$$

and that of R is $\alpha + 4\pi/3.$

The required result follows from §§ 244, 246; or Geometrically.]

48. Any two conjugate diameters subtend angles λ, λ' at a fixed point on the ellipse; show that

$$\cot^2\lambda + \cot^2\lambda'$$

is constant.

49. Find the equation of the locus of the pole of a normal chord.

Ans. $a^6/x^2 + b^6/y^2 = a^4 e^4$

50. Find locus of intersection of normals at the ends of a chord through the focus (ae, o).

Ans. The ellipse

$$a^4(1 + e^2)^2 y^2 + a^2 b^2 (x + ae)^2 = a^3 b^2 e (1 + e^2)(x + ae)$$

[*Note*—This elimination is worked out in Nixon's *Trigonometry*, p. 158.]

51. Two vertices A, B of a *given* triangle ABC move one on each of two given lines OP, OQ; show that the third vertex C describes an ellipse.

[This may be solved geometrically thus.

Let join of C to centre of circum circle of $\triangle$ ABO meet this circle in R and S. Then diam′ of circle = AB/sin AOB, and is ∴ given; and we see that circle and points R, S are fixed relatively to triangle ABC.

Hence $\widehat{ROB} = \widehat{RAB}$ = a given angle;

∴ line OR is fixed. We have then a line RS of constant length whose extremities move on two fixed rect′ axes OR, OS, and ⌣ a fixed point on this line; ∴ &c., by Ex. 11, page 211.

An analytical solution may be obtained; see Nixon's *Trigonometry*, p. 153.]

52. PCP′, QCQ′ are any two diameters; if α, β are the eccentric angles of P, Q, prove that the area of the parallelogram formed by the tangents at P, P′, Q, Q′ is

$$4ab/\sin(\alpha - \beta)$$

53. P, Q are the points of contact of tangents from T (hk); prove that

area of triangle CPQ $= a^2 b^2 \sqrt{b^2 h^2 + a^2 k^2 - a^2 b^2}/(b^2 h^2 + a^2 k^2)$

and area of quadrilateral TPCQ $= \sqrt{b^2 h^2 + a^2 k^2 - a^2 b^2}$

54. A chord PP' passes through the fixed point (hk); if CD, CD' are the semi-diameters conjugate to CP, CP', prove that DD' passes through the fixed point $\left(-\frac{ak}{b}, \frac{bh}{a}\right)$

55. Find locus of mid points of chords of constant length $2c$.

Ans. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{c^2(a^2y^2 + b^2x^2)}{a^4y^2 + b^4x^2} = 1$

56. PGQ, QG' are normals to an ellipse at P and Q, meeting the axis in G and G'. Prove that

$$PG = \text{projection of } QG' \text{ on } QP$$

57. K is the foot of the perpendicular from the centre on the tangent at P; from K another tangent is drawn to touch the ellipse in Q. Show that the eccentric angles θ, ϕ of P and Q are connected by the relation

$$b^2 \tan\frac{\theta + \phi}{2} = a^2 \tan\theta;$$

and that the other extremity of the diameter through Q lies on the normal at P.

58. If PQ is a focal chord, and if R is the intersection of the tangent at P and the normal at Q, prove that QR is bisected by the minor axis.

59. P and P' are two points on the ellipse whose eccentric angles are ϕ, ϕ'; the circle on PP' as diameter meets the ellipse again in Q, Q'. Show that the equation to QQ' is

$$(a^2 - b^2)\left[\frac{x}{a}\cos\frac{\phi + \phi'}{2} - \frac{y}{b}\sin\frac{\phi + \phi'}{2}\right] = (a^2 + b^2)\cos\frac{\phi - \phi'}{2}$$

If PP' pass through a fixed point (hk) then QQ' passes through the fixed point

$$[h(a^2 + b^2)/(a^2 - b^2), \quad -k(a^2 + b^2)/(a^2 - b^2)]$$ (*Prof' J. Purser.*)

60. Prove that the length of the chord intercepted by the line

$$lx/a + my/b = 1$$

is

$$\frac{2}{\sqrt{l^2 + m^2}}\sqrt{(l^2 + m^2 - 1)(a^2l^2 + b^2m^2)}$$

Prove also that if θ_1, θ_2 are the angles at which the chord cuts the ellipse, and p_1, p_2 the perpendiculars from the centre on the tangents at its extremities

$$\frac{\sin\theta_1}{p_1} = \frac{\sin\theta_2}{p_2} = \frac{\sqrt{l^2 + m^2 - 1}}{\sqrt{a^2m^2 + b^2l^2}}$$

61. ab is a fixed chord of an ellipse, c, d two fixed points on the curve, both on the same side of the chord, P a variable point on the curve on the opposite side; show that if the lines Pc, Pd meet the chord in x and y, the length xy is greatest when ax = by. (*Profr J. Purser.*)

62. A triangle is inscribed in an ellipse; if its centroid is a fixed point, show that the locus of its orthocentre is an ellipse.

63. A parallelogram circumscribes the ellipse. If two opposite corners trace out the curve

$$f(x, y) = 0$$

show that the other two trace out the curve

$$f\left(-\frac{ay}{bu}, \frac{bx}{au}\right) = 0, \quad \text{where} \quad u^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

64. Show that the locus of centres of equilateral triangles whose sides touch the ellipse is

$$9(x^2 + y^2)^2 - 2(5a^2 + 3b^2)x^2 - 2(3a^2 + 5b^2)y^2 + (a^2 - b^2)^2 = 0$$

CHAPTER X

THE HYPERBOLA

EQUATION TO THE HYPERBOLA

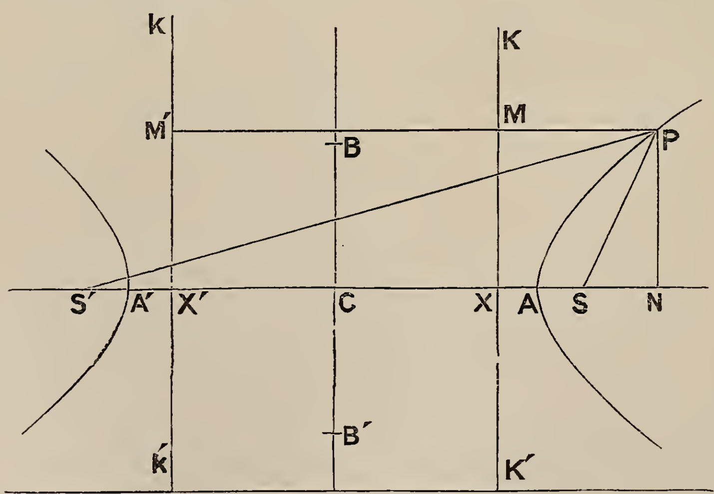
§ **266.** We have defined the hyperbola in § 213; viz. if P is a point on the curve then

$$SP = e PM,$$

S being the focus, PM the $\perp$ from P on the directrix KK′; and e (the eccentricity) > 1.

To find its equation.

Draw SX $\perp$ directrix.



Take points A, A′ on SX such that

$$SA = e AX \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$SA' = e A'X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then A, A′ are points on the curve.

Bisect AA′ in C; let AA′ = 2a.

Adding (1) and (2) we find

$$2\,CS = e \,.\, 2\,CA$$

$$\therefore \quad CS = ae$$

Subtracting (1) from (2) we find

$$2\,CA = e \,.\, 2\,CX$$

$$\therefore \quad CX = a/e$$

Through C draw BCB′ ⊥ CA; and take CA, CB as axes.

Let the co-ord's of P be CN = x, PN = y.

The co-ord's of S are (ae, 0).

$$\therefore \quad SP^2 = (x - ae)^2 + y^2$$

Also
$$PM = CN - CX$$
$$= x - a/e$$

Now
$$SP^2 = e^2\,PM^2$$

$$\therefore \quad (x - ae)^2 + y^2 = (ex - a)^2$$

$$\therefore \quad x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$$

Divide by $a^2(e^2 - 1)$, and put

$$a^2(e^2 - 1) = b^2$$

The equation to the hyperbola is $\therefore$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If we put $x = 0$ in this equation,

$$y = \pm\, b\sqrt{-1}$$

$\therefore$ the hyperbola does not meet the axis of y in real points.

It is however convenient to measure

$$CB = CB' = b.$$

Def's—The point C is called the *centre*, the line AA′ the *transverse axis*, and the line BB′ the *conjugate axis*.

§ 267. Put $x = ae$ ($= CS$) in the equation

$$x^2/a^2 - y^2/b^2 = 1$$

$$\therefore \quad y^2 = b^2(e^2 - 1) = b^4/a^2$$

$\therefore$ latus rectum (the double ordinate through focus) $= 2\,b^2/a$

§ 268. If $(x'y')$ is a point on the curve, the points

$$(-x', \; -y'), \qquad (x', \; -y'), \qquad (-x', \; y')$$

are evidently on the curve.

The curve is $\therefore$ symmetrical with respect to the centre (so that all chords through the centre are bisected); and also with respect to the axes. Since it is symmetrical with respect to the axis of y, there is another focus S' and a corresponding directrix $kX'k'$, images respectively of S and KXK' with respect to the axis of y.

FOCAL DISTANCES

§ 269. In fig', § 266,

$$SP = e\,PM = e(CN - CX) = ex - a$$

$$S'P = e\,PM' = e(CN + CX') = ex + a$$

Cor'—

$$S'P - SP = 2a$$

FIGURE OF THE CURVE

§ 270. Let $\|$s to the axes through A, B meet in J and through A, B' in J'.

Now if we write $r\cos\theta$, $r\sin\theta$ for x, y in the equation of the hyperbola, we obtain its polar equation, C being pole, viz.

$$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1$$

or

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

From (1) we deduce

$$r^2 = \frac{a^2 b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{a^2 b^2}{b^2 - (a^2 + b^2)\sin^2\theta}$$

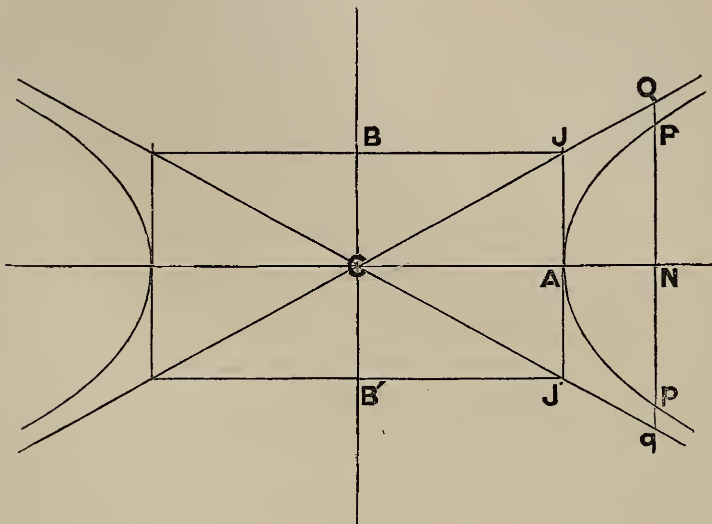
It follows from this that

$$r = a \quad \text{when} \quad \theta = 0;$$

and as θ increases r increases; and $r = \infty$ when

$$(a^2 + b^2)\sin^2\theta = b^2$$

or
$$\theta = \widehat{JCA}$$



After θ passes this value r^2 becomes negative, and $\therefore$ r imaginary.

Remembering then the symmetry of the curve (§ 268) we see that it consists of two infinite branches inclosed between CJ, CJ′ as in the figure.

ASYMPTOTES

§ **271.** Let the ordinate of a point P on the curve meet CJ in Q (fig′, § 270).

Let
$$CN = x$$

Then
$$x^2/a^2 - y^2/b^2 = 1$$

gives
$$y \text{ or } PN = \frac{b}{a}\sqrt{x^2 - a^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, the equation to CJ is

$$y = \frac{b}{a}x$$

$$\therefore \quad QN = \frac{b}{a}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus PN is always < QN; and their difference

$$PQ = \frac{b}{a}(x - \sqrt{x^2 - a^2})$$
$$= ab/(x + \sqrt{x^2 - a^2})$$

Thus PQ may be made < any assignable length, by taking x large enough.

The curve ∴ continually approaches the lines CJ, CJ′, but never meets them.

The lines CJ, CJ′ are the *asymptotes* of the hyperbola.

A general definition of asymptotes will be given later.

Cor′—From (1), (2) we deduce

$$QN^2 - PN^2 = \text{constant} = b^2$$

RECTANGULAR HYPERBOLA

§ **272.** If $a = b$ the hyperbola is called *rectangular* or *equilateral.* Its equation is

$$x^2 - y^2 = a^2$$

Since $$b^2 = a(e^2 - 1), \quad (\S\ 266)$$

$$\therefore \quad e = \sqrt{2}$$

The semi-angle between the asymptotes, being

$$\tan^{-1} b/a$$

is in this case 45°. The asymptotes of a rectangular hyperbola are ∴ at right angles.

§ **273.** Most of the work of the preceding chapter is applicable to the hyperbola, if throughout the proofs b^2 is replaced by $-b^2$.

Some of the results for the hyperbola are—

The tangent at $(x'y')$ is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$$

The polar of $(x'y')$ is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$$

Whatever be the value of m,

$$y = mx + \sqrt{a^2 m^2 - b^2}$$

is a tangent.

The line $$lx + my = n$$

is a tangent if $$a^2 l^2 - b^2 m^2 = n^2$$

The line $$x \cos \alpha + y \sin \alpha = p$$

is a tangent if $$p^2 = a^2 \cos^2 \alpha - b^2 \sin^2 \alpha$$

The locus of inters'n of rectangular tangents (the director circle) is

$$x^2 + y^2 = a^2 - b^2$$

This circle is a point-circle if $a = b$; it is imaginary if $a < b$.

Exercises

1. P is any point on a rectangular hyperbola; prove that

$$SP.S'P = CP^2$$

2. P is a point on a rectangular hyperbola; CZ is perpendicular to the tangent at P. Prove that

$$CP.CZ = a^2$$

3. Pp is a chord of the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

parallel to its minor axis; find locus of intersection of AP, A'p.

Ans. The hyperbola $x^2/a^2 - y^2/b^2 = 1$

4. P is any point on a rectangular hyperbola. Show that the angles PAA', PA'A differ by $\frac{\pi}{2}$; and that the bisectors of the angle APA' are parallel to the asymptotes.

5. Find the equation of a conic having axes coincident with the axes of co-ordinates, and passing through the points $(2\sqrt{2}, \sqrt{3})$, $(4, 3)$. What is its eccentricity?

Ans. $3x^2 - 4y^2 = 12$; $\frac{1}{2}\sqrt{7}$

6. Two rods whose lengths are a, b slide along two rectangular axes in such a way that their extremities are always concyclic; find the locus of the centre of the circle.

Ans. The hyperbola $4(x^2 - y^2) = a^2 - b^2$

7. AOB, COD are two lines which bisect one another at right angles; find the locus of a point P which moves so that

$$PA \cdot PB = PC \cdot PD$$

[Take the given lines as axes; let

$$AB = 2a, \qquad CD = 2b.]$$

Ans. The rectangular hyperbola $2(x^2 - y^2) = a^2 - b^2$

8.

$$x \cos \alpha + y \sin \alpha = p$$

is a normal to

$$x^2/a^2 - y^2/b^2 = 1$$

if

$$p^2 (a^2 \sin^2 \alpha - b^2 \cos^2 \alpha) = (a^2 + b^2)^2 \sin^2 \alpha \cos^2 \alpha$$

9. P, Q, R are three points on a rectangular hyperbola. If the angle PQR is a right angle, prove that the normal at Q is parallel to PR.

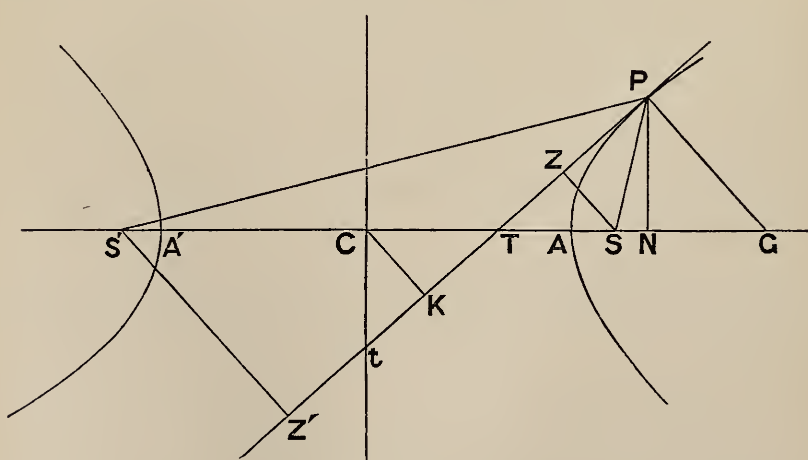
10. A perpendicular from the centre on any tangent to the hyperbola

$$x^2 - y^2 = a^2$$

meets the tangent in Z and the curve in Q; prove that

$$CZ \cdot CQ = a^2$$

§ **274.** The subjoined figure corresponds to that in § 248. All the propositions of that § are applicable to the hyperbola. (IV.) however is slightly



modified: the normal to the hyperbola bisects the exterior $\wedge$ between SP, S′P and the tangent bisects $\wedge$ SPS′.

CONJUGATE HYPERBOLA

§ **275.** The hyperbola which has BB' for its transverse axis, and AA' for its conjugate axis, is called the *conjugate* hyperbola.

Let p be a point on the new curve (fig' § 279); draw $pn \perp CB$.

Then $$Cn^2/CB^2 - pn^2/CA^2 = 1$$

The equation of the conjugate hyperbola is ∴

$$y^2/b^2 - x^2/a^2 = 1$$

§ **276.** It should be noticed that the equations to the hyperbola, the conjugate hyperbola, and the asymptotes are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (3)$$

the sinisters being identical.

§ **277.** Let a line through C inclined at $\hat{\theta}$ to the axis of x meet the conjugate hyperbolas in P, P'. We shall show that if P is real, P' is imaginary and *vice versa*.

The polar equations (see § 270) give

$$\frac{1}{CP^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} \quad \text{and} \quad \frac{1}{CP'^2} = \frac{\sin^2\theta}{b^2} - \frac{\cos^2\theta}{a^2}$$

$$\therefore \quad CP'^2 = -CP^2$$

$$\therefore \quad CP' = CP\sqrt{-1}$$

Note—If the rect' co-ord's of P are (xy), those of P' are $(x\sqrt{-1}, y\sqrt{-1})$.

DIAMETERS

§ **278.** The locus of mid points of chords parallel to $y = mx$ is (§ 255) $y = m'x$ where

$$mm' = \frac{b^2}{a^2} \quad \ldots\ldots\ldots \quad (1)$$

This is the condition that $y = mx$, $y = m'x$ may be conjugate diameters.

It follows from (1) that if $m < b/a$ then $m' > b/a$. But the semi-angle between the asymptotes is $\tan^{-1} b/a$.

It is ∴ evident that of two conjugate diameters one only meets the curve in real points.

(1) may be written

$$\left(\frac{1}{m}\right)\left(\frac{1}{m'}\right) = \frac{a^2}{b^2}$$

But this is the condition that

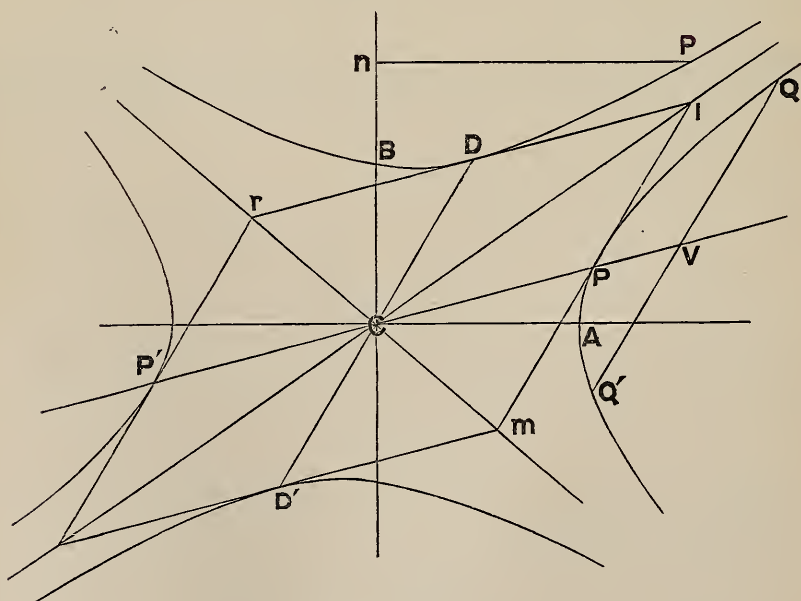
$$y = mx, \qquad y = m'x$$

should be conjugate with respect to

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Accordingly two lines which are conjugate diameters of a hyperbola are conjugate diameters of the conjugate hyperbola.

§ **279.** *If the diameter conjugate to a diameter* PCP′ *meet the conjugate hyperbola in* D, D′, *then the tangents at* P, D *form a*



parallelogram with CP, CD, *of which one of the asymptotes is a diagonal.*

We may assume for the co-ord's of P

$$\left.\begin{array}{l} x = a\sec\phi \\ y = b\tan\phi \end{array}\right\} \quad \ldots\ldots\ldots \quad (1)$$

For these values satisfy

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then the co-ord's of D are

$$\left.\begin{array}{l} x = a\tan\phi \\ y = b\sec\phi \end{array}\right\} \quad \ldots\ldots\ldots \quad (2)$$

For, 1°, the values (2) satisfy eq'n of conj' hyperb'

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

And 2°, the joins of the centre to the points (1), (2) are

$$y = \frac{b\tan\phi}{a\sec\phi}x \quad \ldots\ldots\ldots \quad (3)$$

and

$$y = \frac{b\sec\phi}{a\tan\phi}x \quad \ldots\ldots\ldots \quad (4)$$

And these satisfy cond'n for conj' diameters (§ 278)

$$mm' = b^2/a^2$$

The eq'n to tangent at P is (§ 273)

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1 \quad \ldots\ldots \quad (5)$$

This is ∥ the line (4), i. e. to CD.

Similarly the tangent at D,

$$\frac{x}{a}\tan\phi - \frac{y}{b}\sec\phi = -1 \quad \ldots\ldots \quad (6)$$

is ∥ CP.

If we add eq'ns (5) and (6) we get

$$\left(\frac{x}{a} - \frac{y}{b}\right)(\sec\phi + \tan\phi) = 0$$

or
$$\frac{x}{a} = \frac{y}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$\therefore$ the tangents (5) and (6) meet on the asymptote (7).

Cor′ (1)— $\quad CP^2 - CD^2 = a^2 - b^2$

For from eq′ns (1), (2)

$$CP^2 = a^2 \sec^2 \phi + b^2 \tan^2 \phi$$
$$CD^2 = a^2 \tan^2 \phi + b^2 \sec^2 \phi$$
$$\therefore \quad CP^2 - CD^2 = (a^2 - b^2)(\sec^2 \phi - \tan^2 \phi)$$
$$= a^2 - b^2$$

Cor′ (2)—The area of the $\square$ whose sides are the tangents at P, P′, D, D′ is constant.

For area of this $\square$ = 8 area of Δ PCD

$$= 4(a \sec \phi \,.\, b \sec \phi - a \tan \phi \,.\, b \tan \phi), \quad (\S\ 22)$$
$$= 4ab$$

Cor′ (3)—PD is bisected by one asymptote and is ∥ the other asymptote.

For the mid point of PD is

$$x = \frac{a}{2}(\sec \phi + \tan \phi), \qquad y = \frac{b}{2}(\sec \phi + \tan \phi)$$

which lies on the asymptote

$$\frac{x}{a} = \frac{y}{b}$$

Also, the equation to PD is

$$\begin{vmatrix} x & y & 1 \\ a \sec \phi & b \tan \phi & 1 \\ a \tan \phi & b \sec \phi & 1 \end{vmatrix} = 0$$

or
$$(bx + ay)(\sec \phi - \tan \phi) = ab$$

which is ∥ the asymptote

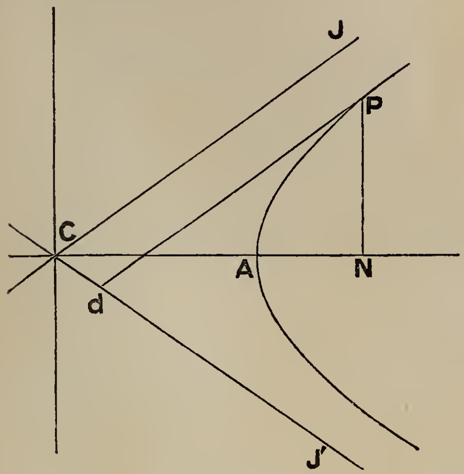
$$\frac{x}{a} + \frac{y}{b} = 0$$

Cor′ (4)—The portion of a tangent intercepted by the asymptotes is bisected at the point of contact.

For
$$PI = CD \quad \text{(Euclid I. 34)}$$
$$= CD'$$
$$= Pm$$
$$\therefore \quad Im \text{ is bisected in } P$$

EQUATION REFERRED TO ASYMPTOTES

§ **280.** *To find the equation of a hyperbola referred to its asymptotes as axes.*



Let $Cd = x$, $Pd = y$ be the co-ord's of a point P on the curve referred to the asymptotes CJ′, CJ.

Draw $PN \perp CA$;

let $\widehat{JCA} = \beta$.

Project the broken line CdP on CA, CB.

$$\therefore \quad \left.\begin{aligned} CN &= x\cos\beta + y\cos\beta \\ PN &= y\sin\beta - x\sin\beta \end{aligned}\right\}$$

$$\text{But} \quad CN^2/a^2 - PN^2/b^2 = 1$$

$$\therefore \quad (x+y)^2\cos^2\beta/a^2 - (y-x)^2\sin^2\beta/b^2 = 1 \quad . \quad . \quad (1)$$

But $\tan\beta = b/a$

and ∴ $\cos^2\beta = a^2/(a^2+b^2)$, $\sin^2\beta = b^2/(a^2+b^2)$

∴ (1) becomes $4xy = a^2 + b^2$

This is the required equation.

Cor′ (1)—If the co-ord's of P (fig′ § 279) referred to the asymptotes are (x, y), those of D are $(-x, y)$. [§ 279, *Cor′* (3)].

The equation to the conjugate hyperbola (the locus of D) is ∴

$$4xy = -(a^2 + b^2)$$

Cor′ (2)—If p_1, p_2 are the ⊥s from P on CJ′, CJ (vide fig′) then

$$p_1 = Pd\sin 2\beta = y\sin 2\beta$$

Similarly $$p_2 = x \sin 2\beta$$

$$\therefore\quad p_1 p_2 = xy \sin^2 2\beta$$
$$= \text{constant},$$

since xy is constant.

Thus *the product of the perpendiculars from any point of a hyperbola on its asymptotes is constant.*

Cor' (3)—Hence the equation to a hyperbola whose asymptotes are

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0$$

is
$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = \text{constant}$$

or
$$(Ax + By + C)(A'x + B'y + C') = k$$

where k is some constant.

The equation of the hyperbola conjugate to this is

$$(Ax + By + C)(A'x + B'y + C') = -k$$

The equation of the asymptotes is of course

$$(Ax + By + C)(A'x + B'y + C') = 0$$

Thus, whatever the axes of co-ordinates may be, the equations of two conjugate hyperbolas differ from that of the asymptotes only by constants, whose values are equal and opposite for the two hyperbolas.

§ **281.** *To find the equation of the tangent at* $(x'y')$ *to the hyperbola*

$$4xy = a^2 + b^2$$

It is convenient to put $(a^2 + b^2)/4 = k^2$, so that the equation to the hyperbola is

$$xy = k^2$$

If $(x'y')$, $(x''y'')$ are two points on the curve, the equation to their join is

$$x(y' - y'') - y(x' - x'') + x'y'' - x''y' = 0$$

Also $$x'y' = x''y'' = k^2$$

If we write the equation to the join in the form

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1$$

then
$$\alpha = \frac{x''y' - x'y''}{y' - y''}$$
$$= \frac{x''k^2/x' - x'k^2/x''}{k^2/x' - k^2/x''}$$
$$= \frac{x''^2 - x'^2}{x'' - x'}$$
$$= x' + x''$$

Similarly $\beta = y' + y''$

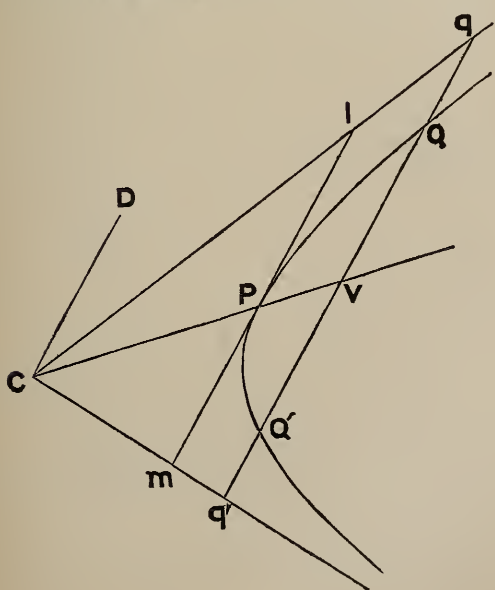
The eq'n to the chord joining $(x'y')$, $(x''y'')$ is $\therefore$

$$\frac{x}{x' + x''} + \frac{y}{y' + y''} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Put now $x'' = x'$, $y'' = y'$; the equation to the tangent at $(x'y')$ is $\therefore$

$$\frac{x}{x'} + \frac{y}{y'} = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Cor' (1)—*If any line cut the hyperbola in* Q, Q' *and the asymptotes in* q, q', *then* Qq = Q'q'.



Let Q be $(x'y')$ and Q' $(x''y'')$.

The chord QQ' [eq'n (1)] meets the axes in the points

q' $(x' + x'', 0)$

and q $(0, y' + y'')$

Thus QQ' and qq' have the same mid point, viz.

$$\left(\frac{x' + x''}{2}, \frac{y' + y''}{2}\right);$$

$\therefore$ &c.

Cor' (2)—*The intercept of any tangent between the asymptotes is bisected at the point of contact; and the area of the triangle which the tangent forms with the asymptotes is constant.*

Let the tangent at $P\ (x'y')$ meet the axes in l, m.

Its equation is $$\frac{x}{x'} + \frac{y}{y'} = 2$$

Its intercepts on the axes are $\therefore$

$$Cm = 2x', \qquad Cl = 2y'.$$

$\therefore$ m is $(2x', 0)$, l is $(0, 2y')$.

$\therefore$ mid point of lm is $(x'y')$, i. e. P. [Compare § 279, *Cor'* (4).]

Again, $$\text{area of } \triangle\, lCm = \tfrac{1}{2}\, Cl\,.\,Cm \sin 2\beta$$
$$= 4x'y' \sin\beta \cos\beta$$
$$= (a^2 + b^2) \sin\beta \cos\beta$$
$$= ab$$

Cor' (3)—*Any two conjugate diameters* CP, CD *form a harmonic pencil with the asymptotes.*

For the parallel lm to CD is bisected at the point where it meets CP; $\therefore$ &c. [§ 137, *Note* (3).]

Exercises

1. The eccentricities of two conjugate hyperbolas are e, e'; prove that

$$1/e^2 + 1/e'^2 = 1$$

2. Q, R are fixed points on a hyperbola; P is a variable point on the hyperbola. The joins PQ, PR meet an asymptote in q, r. Show that qr is constant.

[*Note*—Take the asymptotes as axes; use eq'n (1), § 281.]

3. P, Q are two points on a hyperbola; parallels through P, Q to the asymptotes meet in L, M. Prove that LM passes through C.

4. Show that the tangents at the points $(x_1 y_1)$, $(x_2 y_2)$ on the hyperbola

$$xy = k^2$$

meet at the point $$\left(\frac{2x_1x_2}{x_1 + x_2}, \frac{2y_1y_2}{y_1 + y_2}\right)$$

5. TP, TQ are tangents to a hyperbola. A parallel to an asymptote drawn through T meets PQ in V. Show that TV is bisected by the curve.

6. Find locus of mid point of a straight line which moves so as to cut off a constant area $(= k^2)$ from the corner of a square.

Ans. Taking sides of square as axes, the hyperbola

$$2xy = k^2$$

7. PM, PN are perpendiculars from P on two fixed lines meeting at O. If area of quadrilateral OMPN is constant ($= k^2$), find locus of P.

Ans. Taking given lines as axes, the hyperbola

$$x^2 + 2\,xy \sec\omega + y^2 = 4\,k^2/\sin 2\,\omega$$

8. Find the asymptotes of the hyperbola in the last question.

[*Note*—The asymptotes are the factors of

$$x^2 + 2\,xy \sec\omega + y^2 = 0\text{; see § 280, } Cor' \text{ (3).]}$$

Ans. $y/x = -\sec\omega \pm \tan\omega$

9. Find the asymptotes of the hyperbolas

$$x^2 + 2\,xy + 3\,x = 4, \qquad xy = hx + ky$$

Ans. $x = 0,\ x + 2\,y + 3 = 0$; $x - k = 0,\ y - h = 0$

10. Find the hyperbola conjugate to the hyperbola

$$xy = hx + ky$$

Ans. $xy - hx - ky + 2\,hk = 0$

11. Find the co-ordinates of the foci of the hyperbola

$$xy = k^2$$

Ans. $\left(\pm\ k \sec\frac{\omega}{2},\ \pm\ k \sec\frac{\omega}{2}\right)$; the axes being inclined at $\hat{\omega}$.

12. Through a fixed point (hk) a line is drawn cutting the axes in A, B; find locus of mid point of AB.

Ans. The hyperbola $2\,xy = hy + kx$

13. Find the asymptotes of the preceding hyperbola.

Ans. $2\,x = h,\ 2\,y = k$

DIGRESSION ON QUADRATIC EQUATIONS

§ 282. The roots of $ax^2 + 2\,bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

$$= \frac{c}{-b \mp \sqrt{b^2 - ac}}$$

If a is very small compared with b and c, the radical is nearly $= b$; $\therefore$ one root is very large and the other is nearly $= -c/2\,b$.

If a and b are both very small compared with c, the denominator

$$-b \mp \sqrt{b^2 - ac}$$

is very small; $\therefore$ both roots are very large.

These results are abbreviated thus:

The roots of $0 . x^2 + 2bx + c = 0$ are ∞ and $-c/2b$

The roots of $0 . x^2 + 0 . x + c = 0$ are both ∞

POINTS AT INFINITY

§ 283. If the tangent

$$y = mx + \sqrt{a^2m^2 - b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

pass through the centre

$$0 = \sqrt{a^2m^2 - b^2}$$

$$\therefore \quad m = \pm b/a$$

and (1) becomes

$$y = \pm \frac{b}{a} x$$

The asymptotes are $\therefore$ the tangents from the centre.

In § 284 it is shown that the points of contact are at infinity.

§ 284. The abscissae of the points where the line

$$y = mx + c$$

meets the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are determined by

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

or $$x^2(b^2 - a^2m^2) - 2a^2mcx - a^2(c^2 + b^2) = 0$$

One root of this is ∞ if

$$b^2 - a^2m^2 = 0, \quad \text{or} \quad m = \pm b/a$$

Hence *a parallel to either asymptote cuts the curve in one point at a finite distance and in one point at infinity.*

Both roots are ∞ if also $2a^2mc = 0$, i.e. if $c = 0$.

This proves the statement at the end of § 283.

EQUATION REFERRED TO CONJUGATE DIAMETERS

§ **285.** *To find the equation of the hyperbola referred to* CP, CD *as axes.*

Let $$CP = a', \qquad CD = b' \quad \text{(fig}' \text{ § 279)}$$

Let QVQ′ be a chord ∥ CD; so that

$$x = CV, \qquad y = QV$$

are the co-ord′s of Q.

Put $$\widehat{PCA} = \theta, \qquad \widehat{DCA} = \theta'.$$

Suppose that h, k are the co-ord′s of Q referred to CA, CB.

Project the broken line CVQ on CA and CB

$$\therefore \quad \left.\begin{aligned} h &= x\cos\theta + y\cos\theta' \\ k &= x\sin\theta + y\sin\theta' \end{aligned}\right\}$$

But $$h^2/a^2 - k^2/b^2 = 1$$

$$\therefore \quad (x\cos\theta + y\cos\theta')^2/a^2 - (x\sin\theta + y\sin\theta')^2/b^2 = 1$$

$$\therefore \quad x^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) + 2xy\left(\frac{\cos\theta\cos\theta'}{a^2} - \frac{\sin\theta\sin\theta'}{b^2}\right)$$
$$- y^2\left(\frac{\sin^2\theta'}{b^2} - \frac{\cos^2\theta'}{a^2}\right) = 1$$

That is (§§ 270, 278) $$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$$

This is the required equation.

Cor′ (1)—Remarks similar to those in § 262 apply here.

Cor′ (2)—Since the substitution of

$$x\cos\theta + y\cos\theta', \qquad x\sin\theta + y\sin\theta'$$

for x, y transforms equation (1) of § 276 into

$$x^2/a'^2 - y^2/b'^2 = 1$$

the same substitution will transform equation (2) of § 276 into

$$x^2/a'^2 - y^2/b'^2 = -1$$

This is ∴ the equation of the conjugate hyperbola referred to CP, CD.

Similarly equation (3) of § 276 is transformed into

$$x^2/a'^2 - y^2/b'^2 = 0$$

This is ∴ the equation of the asymptotes referred to CP CD.

LINE-PAIR CONIC

§ 286. Suppose the directrix of a conic to *pass through* the focus.

Take focus as origin; let equation of directrix be

$$x \cos \alpha + y \sin \alpha = 0$$

The equation of the conic is (def' § 213)

$$x^2 + y^2 = e^2(x \cos \alpha + y \sin \alpha)^2$$

The conic is ∴ (§ 113) two straight lines, which are real, coincident, or imaginary, according as

$$e^4 \sin^2\alpha \cos^2\alpha > = < (1 - e^2 \cos^2\alpha)(1 - e^2 \sin^2 a)$$

i. e. as $$e^2 > = < 1$$

Note—It is obvious geometrically that if a and b are very small the form of the hyperbola approximates to that of its asymptotes. Putting

$$a = 0, \qquad b = 0$$

in the equation referred to the asymptotes, viz.

$$4xy = a^2 + b^2,$$

it reduces to $$xy = 0$$

Thus a line-pair is a self-asymptotic hyperbola.

Exercises on the Hyperbola

1. A circle meets the hyperbola

$$xy = k^2$$

in four points $(x_1 y_1)$, $(x_2 y_2)$ ... ; prove that

$$x_1 x_2 x_3 x_4 = k^4 = y_1 y_2 y_3 y_4$$

2. PQ is a chord of a rectangular hyperbola, normal at P; prove that

$$PQ \propto CP^3$$

3. Any tangent meets the asymptotes in L, L'; show that L, L', S, S' are concyclic.

4. Show that the polar of any point T is parallel to the joins of the points in which the tangents from T meet the asymptotes.

5. Prove that the circles described on parallel chords of a rectangular hyperbola as diameters are co-axal.

6. Q is any point on a rectangular hyperbola; PCP′ is any diameter. Prove that the bisectors of the angle PQP′ are parallel to the asymptotes.

7. A chord of a rectangular hyperbola subtends angles at the extremities of any diameter which are either equal or supplementary.

8. Find the equation of the tangents from $(x'y')$ to the hyperbola

$$xy = k^2$$

Ans. $$4(x'y' - k^2)(xy - k^2) = (x'y + y'x - 2k^2)^2;$$

or $$(x'y - y'x)^2 + 4k^2(x - x')(y - y') = 0$$

9. Show that the director circle of the hyperbola

$$xy = k^2$$

is $$x^2 + 2xy\cos\omega + y^2 = 4k^2\cos\omega$$

[*Note*—Express cond′n that lines in Ex. 8 are at right angles.]

10. The tangent at P meets the conjugate hyperbola in Q, Q′; L is the mid point of PQ, and CL meets the hyperbola in R. Prove that CQ, CR are conjugate.

11. A straight line PQ subtends a right angle at each of two fixed points A and B. If P describes a straight line, show that Q describes a hyperbola whose asymptotes are perpendicular to AB and the given line respectively.

12. P and Q are two points on an asymptote of a hyperbola, such that

$$CP = 2PQ$$

and T is the point of contact of the tangent from P. Show that if the parallelogram PTP′Q be completed its diagonals intersect on the curve.

13. From points on the circle

$$x^2 + y^2 = a^2$$

tangents are drawn to the hyperbola

$$x^2 - y^2 = a^2;$$

show that the locus of the mid points of the chords of contact is the curve

$$(x^2 - y^2)^2 = a^2(x^2 + y^2)$$

14. Show that the equation to a rectangular hyperbola which cuts the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

at an angle ϕ, and whose asymptotes are the axes of figure of the ellipse, is

$$xy = a^2 b^2 \cos\phi / \sqrt{(a^2 + b^2)^2 \sin^2\phi + 4a^2 b^2 \cos^2\phi}$$

15. Find locus of poles of normal chords of the hyperbola

$$x^2 - y^2 = a^2$$

Ans. The curve $a^2(y^2 - x^2) = 4x^2y^2$

16. Show that any tangent to the hyperbola

$$x^2 - y^2 = 2a^2$$

is cut harmonically by the two hyperbolas

$$x^2 + xy = a^2, \qquad y^2 + xy = -a^2$$

17. A line through the centre C of an hyperbola meets the curve in P, and parallels to the asymptotes through the vertex A in T and T′. Prove that

$$CP^2 = CT \, . \, CT'$$

(*Prof' Genese*).

CHAPTER XI

GENERAL EQUATION OF THE SECOND DEGREE

§ **287.** *The general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad . \quad (1)$$

represents a conic.

We shall suppose the axes rectangular; for an equation referred to oblique axes may be transformed to rectangular axes (§ 207) and its degree is unaltered (§ 210).

The equation to a conic whose focus is $(x_1 y_1)$, directrix

$$x \cos \alpha + y \sin \alpha - p = 0$$

and eccentricity e, is (def′ § 213)

$$(x - x_1)^2 + (y - y_1)^2 = e^2 (x \cos \alpha + y \sin \alpha - p)^2 \quad . \quad (2)$$

Comparing coeff′s, this will represent the same locus as (1) if

$$\frac{1 - e^2 \cos^2 \alpha}{a} = \frac{-e^2 \cos \alpha \sin \alpha}{h} = \frac{1 - e^2 \sin^2 \alpha}{b}$$

$$= \frac{e^2 p \cos \alpha - x_1}{g} = \frac{e^2 p \sin \alpha - y_1}{f} = \frac{x_1^2 + y_1^2 - e^2 p^2}{c} \quad . \quad (3)$$

We have thus five equations, which are sufficient to determine the five quantities x_1, y_1, α, p, e.

Q. E. D.

Better practical methods of determining the particular conic represented by (1) will be given presently.

Cor′—Let $e = 1$ so that the curve is a parabola.

The first two of equations (3) become

$$\frac{\sin^2\alpha}{a} = \frac{-\cos\alpha\sin\alpha}{h} = \frac{\cos^2\alpha}{b}$$

$$\therefore\ ab = h^2$$

Thus if the general equation (1) represents a parabola, its highest terms

$$ax^2 + 2hxy + by^2$$

form a perfect square.

§ **288.** If the terms of the first degree are absent, the equation is

$$ax^2 + 2hxy + by^2 + c = 0$$

If $(x'y')$ is a point on this curve, $(-x', -y')$ is also evidently a point on the curve. Every chord through the origin is ∴ bisected; or the origin is the centre.

DETERMINATION OF CENTRE

§ **289.** If we transform the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

to ∥ axes through $(x'y')$ it becomes (§ 204)

$$a(x+x')^2 + 2h(x+x')(y+y') + b(y+y')^2 + 2g(x+x') + 2f(y+y') + c = 0 \quad . \ . \ . \quad (\alpha)$$

or

$$ax^2 + 2hxy + by^2 + 2x(ax' + hy' + g) + 2y(hx' + by' + f) + c' = 0 \quad . \ . \ . \quad (\beta)$$

where

$$c' = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c \quad . \ . \quad (\gamma)$$

The new origin is the centre (§ 288), if the coeff's of x and y are zero, i. e. if

$$\left.\begin{aligned} ax' + hy' + g &= 0 \\ hx' + by' + f &= 0 \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (\delta)$$

These equations determine the centre.

Solving them, the co-ord's of the centre are

$$x' = \frac{hf - bg}{ab - h^2}, \quad y' = \frac{gh - af}{ab - h^2}$$

These values are infinite if $ab = h^2$; this case will be considered separately.

Again, the equation (β) becomes now

$$ax^2 + 2hxy + by^2 + c' = 0$$

The value of c' [equation (γ)] may be written

$$c' = x'(ax' + hy' + g) + y'(hx' + by' + f) + gx' + fy' + c$$

Hence, using (δ) we get

$$c' = gx' + fy' + c \quad \ldots\ldots\ldots\ldots \quad (\epsilon)$$

$$= g\frac{hf - bg}{ab - h^2} + f\frac{gh - af}{ab - h^2} + c$$

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}$$

$$= \frac{\Delta}{ab - h^2} \qquad [\S\ 118.$$

Note—The preceding investigation holds if the axes are oblique.

The equations (δ) and (ϵ) should be remembered.

It is useful to notice that the coeff's in (δ) and (ϵ) are the letters in order which occur in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

The student who is acquainted with the Differential Calculus will remark that the sinisters of (δ) are half the derived functions of

$$\phi(x', y') \equiv ax'^2 + 2hx'y' + \&c.$$

with respect to x' and y'.

POINTS AT INFINITY

§ 290. If we substitute $r\cos\theta$, $r\sin\theta$ for x, y, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad (1)$$

becomes

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0$$

One root of this quadratic in r is infinite (§ 282) if

$$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta = 0$$

or

$$a + 2h\tan\theta + b\tan^2\theta = 0$$

This equation determines the lines through the origin which meet the curve at infinity.

Substituting y/x for $\tan\theta$, we see that the equation to these lines is

$$ax^2 + 2hxy + by^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation ∴ represents two lines parallel to the asymptotes. [This also follows evidently from § 280, *Cor′* (3).]

(1) If $h^2 - ab > 0$, the lines (2) are real; the curve (1) is ∴ an hyperbola.

(2) If $h^2 - ab < 0$, the lines (2) are imaginary. No real line through the origin meets the curve at infinity, or the curve is bounded in all directions. It is ∴ an ellipse.

(3) If $h^2 - ab = 0$ the lines (2) which meet the curve at infinity coincide. The curve is a parabola (§ 287, *Cor′*)

§ 291. Two roots of the quadratic in r are infinite, if also

$$g = 0 \quad \text{and} \quad f = 0$$

The asymptotes of $\quad ax^2 + 2hxy + by^2 + c = 0$

are ∴ $\quad ax^2 + 2hxy + by^2 = 0$

This also follows evidently from § 280, *Cor′* (3).

Note—The student will observe that an ellipse has imaginary asymptotes. Thus the asymptotes of

$$x^2/a^2 + y^2/b^2 = 1$$

are $\quad x^2/a^2 + y^2/b^2 = 0, \quad$ or $\quad x/a \pm y\sqrt{-1}/b = 0$

CENTRAL CONICS

§ 292. We resume the discussion of the reduced equation of § 289, viz.

$$ax^2 + 2hxy + by^2 + c' = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Let us turn the axes through $\widehat{\theta}$: we must $\therefore$ substitute (§ 205)

$x \cos\theta - y \sin\theta$ for x and $x \sin\theta + y \cos\theta$ for y

The equation becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 + c' = 0 \quad . \quad . \quad (2)$$

The coeff′ of xy in this is

$$-2a\cos\theta\sin\theta + 2h(\cos^2\theta - \sin^2\theta) + 2b\cos\theta\sin\theta\ *$$

Put this $= 0$;

$$\therefore \quad \tan 2\theta = 2h/(a-b) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we give θ a value which satisfies this, (2) assumes the form

$$\alpha x^2 + \beta y^2 + c' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This represents a conic referred to its axes, unless $c' = 0$, when it represents two straight lines.

The values of α, β are easily determined thus.

Equate the invariants of (1) and (4), (§ 212)

$$\therefore \quad \left.\begin{aligned} \alpha + \beta &= a + b \\ \alpha\beta &= ab - h^2 \end{aligned}\right\}$$

$\therefore$ α, β are the roots of the quadratic in t

$$t^2 - (a+b)t + ab - h^2 = 0$$

Note—This method of determining α, β leaves it uncertain whether the reduced equation is

$$\alpha x^2 + \beta y^2 + c' = 0 \quad \text{or} \quad \beta x^2 + \alpha y^2 + c' = 0;$$

this must be settled by other considerations. Another method, which is free from ambiguity, is described in the next §.

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* This eq′n may be written

$$h(1 - \tan^2\theta) = (a-b)\tan\theta$$

Writing y/x for $\tan\theta$, we obtain the equation of the axes

$$h(x^2 - y^2) = (a-b)xy$$

This equation is obtained otherwise in § 293.

§ **293.** The asymptotes of

$$ax^2 + 2hxy + by^2 + c' = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

are (§ 291)

$$ax^2 + 2hxy + by^2 = 0$$

The axes bisect the angles between the asymptotes; the equation to the axes is ∴ (§ 116)

$$h(x^2 - y^2) - (a - b)xy = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the factors of this be $y - \lambda x = 0$, $y - \mu x = 0$; the axes are ∴

$$y = \lambda x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$y = \mu x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the equation of the conic referred to its axes is

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1,$$

r_1 and r_2 are easily obtained by finding the points where (1) meets (3) and (4).

Thus at the points of inters'n of (1), (3)

$$x^2(a + 2h\lambda + b\lambda^2) + c' = 0$$

$$\therefore \ r_1^2 = x^2 + y^2 = x^2 + \lambda^2 x^2 = -c'(1 + \lambda^2)/(a + 2h\lambda + b\lambda^2)$$

§ **294.** A simpler expression for r_1^2 may be found thus :—

(2) may be written

$$y(ax + hy) = x(hx + by)$$

$$\therefore \ y^2(ax + hy) = xy(hx + by)$$

Add to both sides $\quad x^2(ax + hy)$;

$$\therefore \ (x^2 + y^2)(ax + hy) = x(ax^2 + 2hxy + by^2)$$

$$\therefore \ r_1^2(ax + hy) = -c'x, \quad \text{by } (1)$$

Substitute in this λx for y; then divide by x

$$\therefore \ r_1^2(a + h\lambda) = -c'$$

$$\therefore \ r_1^2 = -c'/(a + h\lambda)$$

Similarly $\quad r_2^2 = -c'/(a + h\mu)$

The equation of the conic referred to its axes is ∴

$$x^2(a + h\lambda) + y^2(a + h\mu) + c' = 0$$

THE PARABOLA

§ 295. We shall now consider the case when $h^2 = ab$.

As $ax^2 + 2hxy + by^2$ is a perfect square, the equation is of the form

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$$

This may be written

$$(\alpha x + \beta y + k)^2 = 2x(\alpha k - g) + 2y(\beta k - f) + k^2 - c \quad . \quad (1)$$

Now let Y, X denote the distances of a point from the lines

$$\alpha x + \beta y + k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$2x(\alpha k - g) + 2y(\beta k - f) + k^2 - c = 0 \quad . \quad . \quad (3)$$

Then (1) expresses that

$$Y^2 \text{ varies as } X$$

Thus whatever k is, the line (2) is a diameter and (3) the tangent at its vertex.

The lines (2), (3) are at right angles if

$$\alpha(\alpha k - g) + \beta(\beta k - f) = 0$$

$$\therefore \quad k = (\alpha g + \beta f) / (\alpha^2 + \beta^2)$$

Introducing this value of k, (2) or $Y = 0$ is the axis; and (3) or $X = 0$ is the tangent at the vertex.

Now $$Y = (\alpha x + \beta y + k) / \sqrt{\alpha^2 + \beta^2}$$
and
$$X = [2x(\alpha k - g) + 2y(\beta k - f) + k^2 - c] / \sqrt{4(\alpha k - g)^2 + 4(\beta k - f)^2}$$

Thus (1) becomes

$$[Y\sqrt{\alpha^2 + \beta^2}]^2 = 2X\sqrt{(\alpha k - g)^2 + (\beta k - f)^2}$$

or
$$Y^2 = pX \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $$p = 2\sqrt{(\alpha k - g)^2 + (\beta k - f)^2} / (\alpha^2 + \beta^2)$$

Introducing the value of k, we obtain the latus rectum p; and (4) is the equation of the parabola referred to its axis and tangent at vertex.

Note—As Y^2 is positive, and p is positive, the equation (4) shows that X is positive: the parabola lies ∴ on the positive side of the line (3).

§ 296. The focus and directrix of the parabola

$$(x \cos\alpha + y \sin\alpha)^2 = -2qx - 2ry - s \quad . \quad . \quad . \quad . \quad (1)$$

may be obtained directly thus:—

Add $(x \sin\alpha - y \cos\alpha)^2$ to both sides of (1)

$$\therefore \quad x^2 + y^2 = (x \sin\alpha - y \cos\alpha)^2 - 2qx - 2ry - s$$

$$\therefore \quad (x \sin\alpha - y \cos\alpha)^2 = x^2 + y^2 + 2qx + 2ry + s$$

$$\therefore \quad (x \sin\alpha - y \cos\alpha - k)^2$$

$$= x^2 - 2(k \sin\alpha - q)x + y^2 + 2(k\cos\alpha + r)y + k^2 + s \quad . \quad . \quad (2)$$

The dexter of this will $\equiv (x - x')^2 + (y - y')^2$

if $\qquad x' = k \sin\alpha - q, \qquad y' = -(k\cos\alpha + r)$

and $\qquad (k \sin\alpha - q)^2 + (k\cos\alpha + r)^2 = k^2 + s$

$$\therefore \quad 2k(q \sin\alpha - r\cos\alpha) = q^2 + r^2 - s$$

This determines k; the focus is $(x'y')$ and the directrix

$$x \sin\alpha - y\cos\alpha - k = 0$$

RECTANGULAR HYPERBOLA

§ 297. Since $\qquad ax^2 + 2hxy + by^2 = 0$

represents a pair of lines parallel to the asymptotes of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad . \quad . \quad (1)$$

the condition that (1) may be a rectangular hyperbola is [§ 114, *Cor'* (1)]

$$a + b = 0$$

If the axes are oblique the condition is (§ 115, *Cor'*)

$$a + b - 2h\cos\omega = 0$$

ASYMPTOTES

§ 298. The asymptotes of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad . \quad . \quad (1)$$

are $\qquad ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0 \quad . \quad . \quad . \quad (2)$

provided the value of λ is such that (2) represents a pair of straight lines [§ 280, *Cor'* (3)].

This gives (§ 118) $\qquad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+\lambda \end{vmatrix} = 0$

Expanding this, $\lambda\,(ab - h^2) + \Delta = 0$

Accordingly, if the conic (1) is denoted by $S = 0$, its asymptotes are

$$S = \Delta/(ab - h^2)$$

Cor'—The equation to the conjugate hyperbola is [§ 280, *Cor'* (3)]

$$S + 2\lambda = 0$$

or

$$S = 2\Delta/(ab - h^2)$$

INTERSECTIONS WITH CONCENTRIC CIRCLE.

§ **299.** The following is another method of determining the axes of the conic

$$ax^2 + 2hxy + by^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At the intersections of the conic and the circle

$$x^2 + y^2 = r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we obtain, by dividing (2) by r^2 and subtracting the result from (1)

$$\left(a - \frac{1}{r^2}\right)x^2 + 2hxy + \left(b - \frac{1}{r^2}\right)y^2 = 0 \quad . \quad . \quad (3)$$

We see, as in § 119, that (3) is the equation of the common diameters of the conic and circle.

If CP, CQ are two common semi-diameters, then since

$$CP = CQ = r,$$

CP and CQ are equally inclined to the axes of the ellipse.

Hence P, Q coincide only at the extremities of an axis.

If the lines (3) are coincident

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The squares of the semi-axes are the values of r^2 deduced from (4).

As (3) is now a perfect square, if we multiply (3) by $a - 1/r^2$ and take the square root, we find

$$\left(a - \frac{1}{r^2}\right)x + hy = 0$$

This is the equation of the axis whose length is $2r$.

In working numerical examples the learner may either use this method, or that of §§ 293, 294.

§ 300. Ex. 1. Trace the conic

$$5x^2 + 6xy + 5y^2 - 16x - 16y + 8 = 0$$

Here $$h^2 - ab = 9 - 25 = -16;$$

the curve is ∴ an ellipse (§ 290)

The equations to find the centre are (§ 289)

$$\left.\begin{aligned} 5x + 3y - 8 &= 0 \\ 3x + 5y - 8 &= 0 \end{aligned}\right\}$$

$$\therefore \quad x = 1, \qquad y = 1$$

Next, $$c' = (-8)(1) - 8(1) + 8 = -8$$ [§ 289, (ϵ).]

The equation referred to ∥ axes through centre is (§ 289)

$$5x^2 + 6xy + 5y^2 - 8 = 0$$

The axes are (§ 293)

$$3x^2 - 3y^2 = 0$$

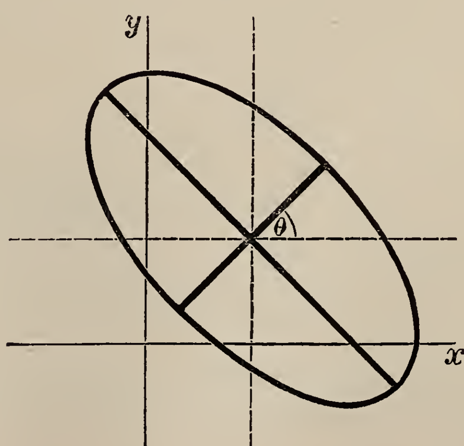
or $$y = -x \quad \text{and} \quad y = +x$$

The equation of the conic referred to its axes is (§ 294)

$$x^2(a + h\lambda) + y^2(a + h\mu) + c' = 0$$

i.e. $$2x^2 + 8y^2 - 8 = 0$$

or $$x^2 + 4y^2 = 4$$



Ex. 2. Trace the conic

$$x^2 - 4xy + 3y^2 - 4x + 7y + \tfrac{23}{4} = 0$$

Here $$h^2 - ab = 1;$$

the conic is ∴ an hyperbola.

For the centre
$$\left.\begin{aligned} x - 2y - 2 &= 0 \\ -2x + 3y + \tfrac{7}{2} &= 0 \end{aligned}\right\}$$

$$\therefore \quad x = 1, \qquad y = -\tfrac{1}{2}$$

Next, $$c' = -2 - \tfrac{7}{4} + \tfrac{23}{4} = 2$$

The equation referred to ∥ axes through centre is

$$x^2 - 4xy + 3y^2 + 2 = 0$$

The axes are

$$-2(x^2 - y^2) - (-2)xy = 0$$

Solving this quadratic for y/x the axes are

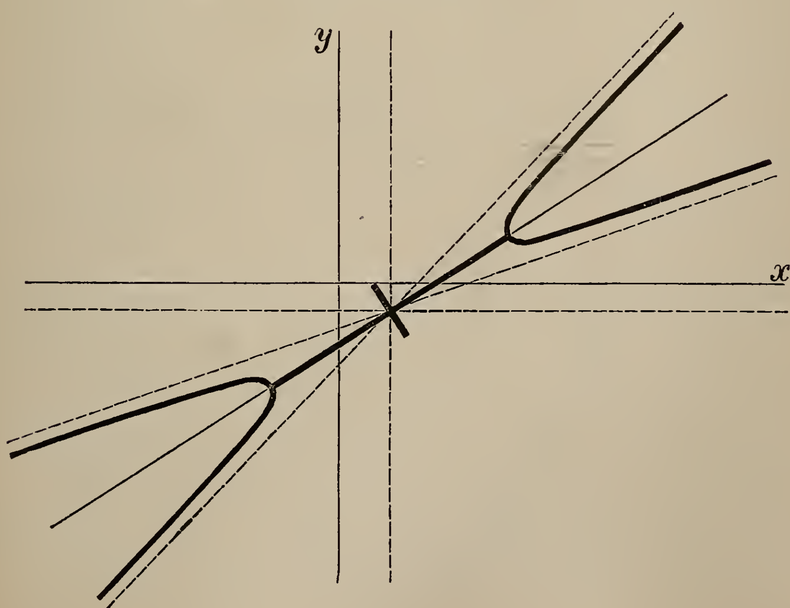
$$y/x = (-1 \pm \sqrt{5})/2$$

The equation referred to the axes is (§ 294)

$$x^2(2 - \sqrt{5}) + y^2(\sqrt{5} + 2) + 2 = 0$$

or $$x^2(\sqrt{5} - 2) - y^2(\sqrt{5} + 2) = 2$$

or $$\frac{x^2}{(2\cdot9)^2} - \frac{y^2}{(\cdot63)^2} = 1$$



Ex. 3. Trace the parabola

$$(3x + 4y)^2 - 8x + 156y - 36 = 0$$

Here

$$(3x + 4y + k)^2 = x(6k + 8) + y(8k - 156) + k^2 + 36$$

Let Y, X be the distances of (x, y) from the lines

$$3x + 4y + k = 0 \quad \dots \dots \dots \quad (1)$$

$$x(6k + 8) + y(8k - 156) + k^2 + 36 = 0 \quad \dots \dots \quad (2)$$

These lines are at right angles if

$$18k + 24 + 32k - 624 = 0, \qquad \therefore \quad k = 12$$

(1) becomes $$3x + 4y + 12 = 0 \quad \dots \dots \dots \quad (3)$$

(2) becomes

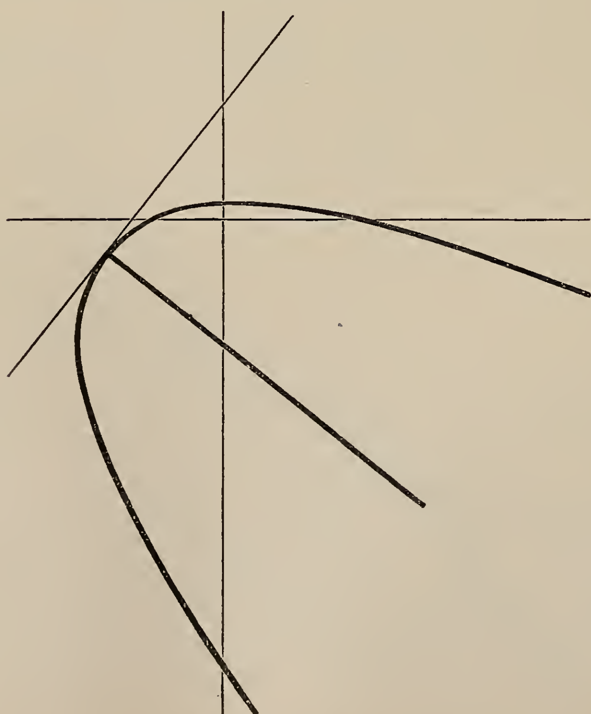
$$80x - 60y + 180 = 0, \quad \text{or} \quad 4x - 3y + 9 = 0 \quad \dots \quad (4)$$

$$\therefore \quad Y = (3x + 4y + 12)/5, \qquad X = (4x - 3y + 9)/5$$

$$\therefore \quad (5Y)^2 = 20(5X)$$

$$\therefore \quad Y^2 = 4X$$

The line (3) is the axis; (4) is tangent at vertex.



Exercises

1. Find the centres of the conics

1°, $x^2 + xy + y^2 = 3x + 3y$

2°, $xy - \beta x - \alpha y = 0$

3°, $5x^2 + 4xy + 8y^2 - 18x - 36y + 9 = 0$

4°, $4x^2 - 24xy + 11y^2 + 40x + 30y - 105 = 0$

Ans. $(1, 1)$; (α, β); $(1, 2)$; $(4, 3)$

2. Find the equations of the preceding conics referred to parallel axes through the centres.

Ans. 1°, $x^2 + xy + y^2 = 3$

2°, $xy = \alpha\beta$

3°, $5x^2 + 4xy + 8y^2 = 36$

4°, $4x^2 - 24xy + 11y^2 + 20 = 0$

3. Trace the preceding conics.

Determine the inclination of the greater axis of each conic to the axis of x, and the lengths of its semi-axes.

Ans. 1°, An ellipse; $135°$; $\sqrt{6}$, $\sqrt{2}$

2°, An hyperbola; $45°$; $\sqrt{2\alpha\beta}$, $\sqrt{2\alpha\beta}$

3°, An ellipse; $\tan^{-1}(-\frac{1}{2})$; 3, 2

4°, An hyperbola; $\tan^{-1}(\frac{3}{4})$; 2, 1

4. Find the equation of the ellipse

$$2x^2 + y^2 - 2xy - 2x = 0$$

when referred to its axes.

Ans. $(3 - \sqrt{5})x^2 + (3 + \sqrt{5})y^2 = 2$

5. Trace the following parabolas; and determine for each the equations of the axis and the tangent at the vertex. Obtain also the equation of each parabola referred to its axis and tangent at vertex.

1°, $(3x + 4y)^2 + 22x + 46y + 9 = 0$

2°, $4(x - y)^2 = 4(x + y) - 1$

3°, $(x - 2y)^2 - 2(x + 2y) + 1 = 0$

Ans. 1°, $3x + 4y + 5 = 0$, $4x - 3y + 8 = 0$; $5y^2 = 2x$

2°, $x - y = 0$, $4(x + y) - 1 = 0$; $y^2\sqrt{2} = x$

3°, $5x - 10y + 3 = 0$, $10x + 5y = 2$; $5\sqrt{5}y^2 = 8x$

6. Show that the equation

$$\sqrt{x/a} + \sqrt{y/b} = 1$$

represents a parabola; and find its latus rectum.

Ans. $4a^2b^2/(a^2+b^2)^{\frac{3}{2}}$

7. Find the latus rectum of the parabola

$$(a^2+b^2)(x^2+y^2) = (bx+ay-ab)^2$$

Ans. $2ab/\sqrt{a^2+b^2}$

8. Trace the hyperbola

$$y = x + a^2/x$$

Show that the product of its semi-axes is $2a^2$.

9. Find the equation to the hyperbola which passes through the point (1, 2) and whose asymptotes are

$$x + 2y - 1 = 0, \qquad 3x - y + 1 = 0$$

Find also the equation of the conjugate hyperbola.

Ans. $3x^2 + 5xy - 2y^2 - 2x + 3y - 9 = 0$,
$3x^2 + 5xy - 2y^2 - 2x + 3y + 7 = 0$

10. Find the centre of the conic

$$5x^2 + 11xy + 2y^2 - 13x + 10y - 28 = 0$$

Ans. $(-2, 3)$

[*Note*—This conic is a pair of lines whose intersection is the centre.]

11. Find the asymptotes of the hyperbola

$$6x^2 - xy - y^2 - x + 3y + 2 = 0$$

Find also the equation of the conjugate hyperbola.

Ans. $2x - y + 1 = 0$, $3x + y - 2 = 0$;
$6x^2 - xy - y^2 - x + 3y - 6 = 0$

12. Trace the conics

$$2x^2 = 3x + 4y + 5, \qquad y^2 = 3x + 4y, \qquad (x-y)^2 = a^2 - x^2$$

13. What curve is represented by the equation

$$(6x + 2y - 1)^2 + (3x - 9y + 2)^2 = 360\,?$$

Ans. An ellipse, the lengths of whose semi-axes are 3, 2; and their equations are

$$3x - 9y + 2 = 0, \qquad 6x + 2y - 1 = 0$$

14. Find the equation of the axes of the ellipse

$$2x^2 + y^2 = 2xy + 2y$$

Ans. $x^2 + xy - y^2 - 4x + 3y - 1 = 0$

EQUATION OF TANGENT

§ 301. We shall sometimes use abbreviations for the expression

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

It may be denoted by S, or by $\phi(x, y)$.

Thus $S = 0$ means the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Of course $\phi(x', y')$ means the result of substituting x', y' for x, y in $\phi(x, y)$: or

$$\phi(x', y') \equiv ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c$$

§ 302. *To find the equation of the tangent at* $(x'y')$ *to the conic*

$$S = 0$$

Let $(x'y')$, $(x''y'')$ be two points on the curve.

Consider the locus represented by

$$a(x-x')(x-x'') + 2h(x-x')(y-y'') + b(y-y')(y-y'')$$
$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c \quad . \quad . \quad . \quad (1)$$

The terms in x^2, xy, y^2 cancel; so that the equation is of the first degree and ∴ represents a straight line.

If we substitute x', y' for x, y in the sinister of (1) it vanishes identically. If the same substitution is made in the dexter, it vanishes also, since $(x'y')$ is on the curve.

Thus $(x'y')$ is a point on (1).

Similarly $(x''y'')$ is a point on (1).

Hence (1) is the equation of the chord joining $(x'y')$, $(x''y'')$.

Now put $x'' = x'$, $y'' = y'$; the equation of the tangent at $(x'y')$ is ∴

$$a(x - x')^2 + 2h(x - x')(y - y') + b(y - y')^2$$
$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

or, expanding and cancelling

$$2ax'x + 2h(y'x + x'y) + 2by'y + 2gx + 2fy + c$$
$$= ax'^2 + 2hx'y' + by'^2$$

This may be simplified: add

$$2gx' + 2fy' + c$$

to both sides; then dexter vanishes, since $(x'y')$ is on the curve.

The equation of the tangent at $(x'y')$ is $\therefore$

$$ax'x + h(y'x + x'y) + by'y + g(x + x') + f(y + y') + c = 0$$

In the preceding investigation (due to Prof' W. S. Burnside) the axes may be oblique or rectangular. Investigations which are applicable when the axes are oblique will be indicated in future by the sign (Ω).

The equation of the tangent at $(x'y')$ may be remembered thus:—

Imagine equation of conic written as follows

$$axx + h(xy + xy) + byy + g(x + x) + f(y + y) + c = 0;$$

then dash one letter in each term.

DIAMETERS

§ **303.** *To find the locus of the mid points of chords of* $S = 0$ *which are parallel to* $y = mx$.

We proceed as in § 228.

Let $(x'y')$, $(x''y'')$ be the extremities of one of the chords; (xy) its mid point.

Then

$$2x = x' + x'', \quad 2y = y' + y'', \quad m = (y' - y'')/(x' - x'') \quad . \ (1)$$

Now since $(x'y')$, $(x''y'')$ are on the curve

$$\therefore \quad ax'^2 + 2hx'y' + \ldots = 0, \quad ax''^2 + 2hx''y'' + \ldots = 0$$

Subtract these equations;

$$\therefore \quad a(x'^2 - x''^2) + 2h(x'y' - x''y'') + \ldots = 0$$

This may be written

$$a(x'+x'')(x'-x'') + h[(x'+x'')(y'-y'') + (x'-x'')(y'+y'')]$$
$$+ b(y'+y'')(y'-y'') + 2g(x'-x'') + 2f(y'-y'') = 0$$

Divide this by $x' - x''$; then from (1) we see that

$$ax + h(mx + y) + bmy + g + fm = 0$$

or $$ax + hy + g + m(hx + by + f) = 0 \quad . \quad . \quad . \quad (2)$$

This is the equation required.

This proof is due to Prof' J. Purser.

Cor'—If (2) is parallel to $y = m'x$, then

$$m' = -(a + hm)/(h + bm)$$

$$\therefore \quad bmm' + h(m + m') + a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega)$$

This is the condition that $y = mx$, $y = m'x$ may be parallel to conjugate diameters.

§ **304.** *To find the condition that the lines*

$$a'x^2 + 2h'xy + b'y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may be conjugate diameters of the conic

$$ax^2 + 2hxy + by^2 + c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let $a'x^2 + 2h'xy + b'y^2 \equiv b'(y - mx)(y - m'x)$

$$\therefore \quad m + m' = -2h'/b', \quad mm' = a'/b'$$

But $y - mx = 0$, $y - m'x = 0$ are conjugate if

$$bmm' + h(m + m') + a = 0 \quad [\textit{Cor'}, \S\ 303.]$$

$$\therefore \quad \frac{ba'}{b'} - \frac{2hh'}{b'} + a = 0$$

The required condition is $\therefore$

$$ab' + a'b - 2hh' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega)$$

Note—This equation expresses (§ 141) that the lines (1) are harmonically conjugate to the asymptotes of the conic (2). We have thus another proof of the proposition of § 281, *Cor'* (3).

§ **305.** Ex. 1. Find the equation to the axes of the conic

$$ax^2 + 2hxy + by^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega)$$

Let the required equation be

$$a'x^2 + 2h'xy + b'y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the axes are conjugate,

$$\therefore\ ba' - 2hh' + ab' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And since the axes are at right angles

$$\therefore\ a' - 2h'\cos\omega + b' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2) and (3),

$$\frac{a'}{h - a\cos\omega} = \frac{-2h'}{a - b} = \frac{b'}{b\cos\omega - h}$$

The required equation is $\therefore$

$$(h - a\cos\omega)x^2 + (b - a)xy + (b\cos\omega - h)y^2 = 0$$

Note—This may be expressed otherwise. Eliminate linearly a', $-2h'$, b' from (1), (2), (3). The equation to the axes is $\therefore$

$$\begin{vmatrix} x^2 & -xy & y^2 \\ b & h & a \\ 1 & \cos\omega & 1 \end{vmatrix} = 0$$

Ex. 2. To find the equation to the common conjugate diameters of the conics

$$ax^2 + 2hxy + by^2 = 1, \qquad a'x^2 + 2h'xy + b'y^2 = 1 \quad . \quad . \quad . \quad (\Omega)$$

Let the required equation be

$$a''x^2 + 2h''xy + b''y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore\ ba'' - 2hh'' + ab'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$b'a'' - 2h'h'' + a'b'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminate a'', $-2h''$, b'' from (1), (2), (3). This gives the required equation, viz.

$$\begin{vmatrix} x^2 & -xy & y^2 \\ b & h & a \\ b' & h' & a' \end{vmatrix} = 0$$

This proves that any two concentric conics have a common pair of conjugate diameters; and the equations of two such conics can $\therefore$ be expressed in the forms

$$ax^2 + by^2 = 1, \qquad a'x^2 + b'y^2 = 1$$

POLES AND POLARS

§ **306.** It is proved, exactly as in § 168 or § 233, that the polar of $(x'y')$ with respect to $S = 0$ is

$$ax'x + h(y'x + x'y) + by'y + g(x + x') + f(y + y') + c = 0 \quad . \quad . \quad . \quad . \quad (\Omega)$$

This is of the same form as the equation of the tangent.

The preceding equation may be written

$$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0$$

Cor' (1)—Put $x' = 0$, $y' = 0$: the polar of the origin is $\therefore$

$$gx + fy + c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega)$$

Cor' (2)—It is proved as in § 172 that *if* P *lies on the polar of* Q, *then* Q *lies on the polar of* P.

Cor' (3)—*If the polars of* A *and* B *meet in* C, *then* C *is the pole of* AB.

This is proved as in § 174.

Cor' (4)—*If a line revolve round a fixed point* P; *then its pole describes a straight line, viz. the polar of* P.

This is proved as in § 173.

§ 307. The tangents at the extremities of a diameter are parallel to the conjugate diameter; the pole of a diameter is $\therefore$ the point at infinity on the conjugate diameter.

The polar of the centre is [§ 306, *Cor'* (3)] the join of the poles of any two lines through the centre.

Hence the polar of the centre is the line at infinity. This is otherwise evident: the points of contact of the tangents from the centre, viz. the asymptotes, are at infinity.

Def'—Two points such that each lies on the polar of the other are called *conjugate* points.

If O, R are conjugate points it will be shown in § 308 that OR is cut harmonically by the conic.

Def'—Two lines such that each passes through the pole of the other are called *conjugate* lines.

Two conjugate diameters are conjugate lines; the pole of each is the point at infinity on the other.

§ 308. *Any line drawn through a point* O *is cut harmonically by the conic and the polar of* O.

Let any line through O meet the conic in P, Q, and the polar of O in R.

Take O for origin; let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The polar of O is [§ 306, *Cor′* (1)]

$$gx + fy + c = 0$$

Writing $r\cos\theta$, $r\sin\theta$ for x, y, these become

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$$
$$+ 2r(g\cos\theta + f\sin\theta) + c = 0 \quad . \quad . \quad . \quad . \quad (1)$$
$$r(g\cos\theta + f\sin\theta) + c = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Let OPQ make $\hat{\theta}$ with the axis of x.

(1) is a quadratic in r; its roots are OP, OQ.

$$\therefore \frac{1}{OP} + \frac{1}{OQ} = -\frac{2(g\cos\theta + f\sin\theta)}{c}$$

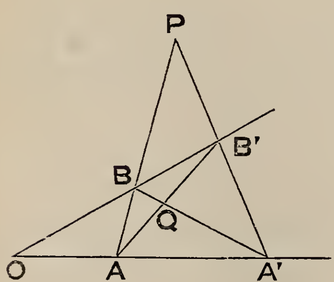
Also (2) gives

$$\frac{1}{OR} = \frac{1}{r} = -\frac{g\cos\theta + f\sin\theta}{c}$$

$$\therefore \frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OR}$$

i.e. OP, OR, OQ are in harmonic progression.

§ **309**. *If two lines through a point* O *meet a conic in* A, A′ *and* B, B′, *and the direct and transverse joins of these points meet in* P, Q; *then* PQ *is the polar of* O.



Take the fixed lines for axes; let

$$OA = \alpha, \quad OA' = \alpha',$$
$$OB = \beta, \quad OB' = \beta',$$

Let the conic be

$$ax^2 + 2hxy + by^2$$
$$+ 2gx + 2fy + c = 0$$

Putting $y = 0$ in this, we see that α, α' are the roots of the equation

$$ax^2 + 2gx + c = 0$$

$$\therefore \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = -\frac{2g}{c} \quad \ldots \ldots \quad (1)$$

Similarly
$$\frac{1}{\beta} + \frac{1}{\beta'} = -\frac{2f}{c} \quad \ldots \ldots \quad (2)$$

The equation to PQ is

$$\frac{x}{\alpha} + \frac{x}{\alpha'} + \frac{y}{\beta} + \frac{y}{\beta'} - 2 = 0 \quad \ldots \ldots \quad (3)$$

for (see § 126) this line passes through the inters'n of

$$\frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0, \quad \frac{x}{\alpha'} + \frac{y}{\beta'} - 1 = 0$$

and also of

$$\frac{x}{\alpha} + \frac{y}{\beta'} - 1 = 0, \quad \frac{x}{\alpha'} + \frac{y}{\beta} - 1 = 0$$

Using (1) and (2), (3) becomes

$$-\frac{2g}{c}x - \frac{2f}{c}y - 2 = 0$$

or
$$gx + fy + c = 0;$$

but this is the polar of O [§ 306, *Cor'* (1)].

Cor'—Similarly OQ is the polar of P; and $\therefore$ [§ 306, *Cor'* (3)] OP is the polar of Q.

A triangle such as OPQ, in which each vertex is the pole of the opposite side with respect to a conic, is said to be *self-conjugate* or *self-polar.*

Note—The geometrical proof of the preceding proposition given in *Euclid Revised*, p. 367, is applicable to conics.

Exercises

1. Find the latus rectum of the parabola

$$\sqrt{x} + \sqrt{y} = 3$$

also the equation of the tangent to it at the point (4, 1).

Ans. $3\sqrt{2}$; $x + 2y = 6$

2. Deduce the equations of the tangents at $(x'y')$ to the curves

$$y^2 = 4ax, \qquad x^2/a^2 + y^2/b^2 = 1, \qquad xy = k^2$$

by the method of § 302.

3. Find the equation of the tangent at the origin to the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$$

Ans. $gx + fy = 0$

4. Find the diameters of $S = 0$ which are conjugate to the axes of x and y respectively.

Ans. $ax + hy + g = 0$, $hx + by + f = 0$

5. Find equation to chord of $S = 0$ which is bisected at origin.

Ans. $gx + fy = 0$

6. Show that the equation to the chord which is bisected at $(x'y')$ is

$$(ax' + hy' + g)(x - x') + (hx' + by' + f)(y - y') = 0$$

7. Prove that one of the asymptotes of $S = 0$ will pass through the origin if

$$af^2 + bg^2 = 2fgh$$

8. Through a fixed point O are drawn two lines at right angles, meeting a given conic in P, P' and in Q, Q': prove that

$$\frac{1}{OP.OP'} + \frac{1}{OQ.OQ'}$$

is constant.

[*Note*—Take O as origin; let equation of conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If OP is inclined at $\hat{\theta}$ to axis of x, OP and OP′ are the roots of the following equation in r:

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0$$

$$\therefore \quad \frac{1}{OP.OP'} = \frac{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}{c}$$

Changing θ into $\pi/2 + \theta$,

$$\frac{1}{OQ.OQ'} = \frac{a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta}{c}$$

$$\therefore \quad \frac{1}{OP.OP'} + \frac{1}{OQ.OQ'} = \frac{a+b}{c}.]$$

9. On a line which revolves round a fixed point O and cuts a conic in P, Q, a point R is taken such that

$$OR^2 = OP.OQ:$$

find the locus of R.

[*Note*—Choose O as origin, and equation of conic as in Ex. 8; if PQ is inclined to axis of x at $\hat{\theta}$,

$$OR^2 = OP.OQ = c/(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta);$$

and locus required is the conic

$$ax^2 + 2hxy + by^2 = c]$$

10. A parallel through any point O to an asymptote meets the conic in P and the polar of O in R: prove that

$$OP = PR$$

[*Note*—The second point in which OP meets the conic is at infinity; ∴ (§ 308) the line OR is cut harmonically in the points P, ∞; ∴ &c. See § 133, VI.]

11. If the equation $S = 0$ is transformed to any axes through the same origin, show that $g^2 + f^2$ is unaltered.

[*Note*—Length of ⊥ from origin on its polar is unaltered.]

12. Trace the curves

$$y = x - 3x^2, \qquad \frac{x}{1 + 2x} + y = 0$$

Show that they intersect at right angles at the origin; and elsewhere at the angles $\tan^{-1}(7/6)$ and $\tan^{-1}(7/22)$.

SEGMENTS OF CHORD THROUGH GIVEN POINT

§ 310. Let a line through T $(x'y')$ inclined at $\hat{\theta}$ to the axis of x cut the conic $S = 0$ in P, Q. We can find the lengths TP, TQ.

As in § 208 we transform to polar co-ord's referred to T by substituting $x' + r\cos\theta$, $y' + r\sin\theta$ for x, y.

Thus $\qquad S = 0 \quad$ becomes

$$a(x' + r\cos\theta)^2 + 2h(x' + r\cos\theta)(y' + r\sin\theta)$$
$$+ b(y' + r\sin\theta)^2 + 2g(x' + r\cos\theta) + 2f(y' + r\sin\theta) + c = 0$$

or $\quad r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$

$$+2r[(ax'+hy'+g)\cos\theta + (hx'+by'+f)\sin\theta]$$
$$+\phi(x', y') = 0 \quad . \quad . \quad . \quad (1)$$

TP, TQ are the roots of this quadratic in r. Hence

$$TP.TQ = \phi(x', y')/(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) \quad . \quad (2)$$

§ **311.** We can draw some important inferences from the preceding equation (2).

Let another line through T inclined at $\hat{\theta'}$ cut the conic in R, S.

Then

$$TR.TS = \phi(x', y')/(a\cos^2\theta' + 2h\cos\theta'\sin\theta' + b\sin^2\theta')$$

Hence the ratio TP.TQ : TR.TS is independent of the position of T, if the lines TP, TR are drawn in fixed directions.

Thus if lines through another point T′ parallel to TP and TR meet the curve in P′, Q′ and R′, S′, then

$$TP.TQ : TR.TS = T'P'.T'Q' : T'R'.T'S'.$$

Cor′ (1)—Let T′ coincide with the centre; then we see that the rectangles

TP.TQ and TR.TS

are in the ratio of the squares of the parallel semi-diameters.

Cor′ (2)—Let P, Q coincide and also R and S; and let Cp, Cr be semi-diameters parallel to the tangents TP, TR. We infer that

$$TP : TR = Cp : Cr$$

Cor′ (3)—*The joins of two pairs of points of intersection of a circle and a conic are equally inclined to an axis of the conic.*

For let the circle cut the conic in P, Q, R, S; let PQ, RS meet in T.

Then $\qquad TP.TQ = TR.TS$ (Euclid III. 35, 36)

Hence by *Cor′* (1) the semi-diameters parallel to PQ, RS are equal; and are ∴ equally inclined to an axis.

Ex. 1. If a circle intersect an ellipse in four points whose ecc′ $\wedge$s are $\alpha, \beta, \gamma, \delta$, then $\qquad \alpha + \beta + \gamma + \delta = 2n\pi$

For the joins of α, β and of γ, δ, viz.

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$$

and

$$\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}=\cos\frac{\gamma-\delta}{2}$$

are [by *Cor'* (3)] equally inclined to the axis.

$$\therefore \quad \tan(\alpha+\beta)/2 = -\tan(\gamma+\delta)/2$$

$$\therefore \quad (\alpha+\beta)/2 = n\pi - (\gamma+\delta)/2; \quad \text{and} \therefore \text{ \&c.}$$

Ex. 2. If a conic cut the sides of a triangle ABC in a, a′; b, b′; c, c′ respectively, then

$$Ba.Ba'.Cb.Cb'.Ac.Ac' = Ca.Ca'.Ab.Ab'.Bc.Bc'$$

[*Carnot's Theorem.*]

Let α, β, γ be the semi-diameters parallel to the sides of the triangle. Then

$$Ba.Ba'/Bc.Bc' = \alpha^2/\gamma^2$$

$$Ac.Ac'/Ab.Ab' = \gamma^2/\beta^2$$

$$Cb.Cb'/Ca.Ca' = \beta^2/\alpha^2$$

The result follows by multiplication.

§ **312**. From equation (2), § 310, we infer that if θ is given,

$$\phi(x', y') \text{ varies as } TP.TQ$$

Thus *the power of a point* (see footnote p. 135) *with respect to a conic varies as the rectangle under the segments of a chord drawn in a fixed direction through the point.*

Ex. If l is the length of one of the tangents from (h, k) to the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

and δ the length of the parallel semi-diameter, then

$$l = \delta\sqrt{h^2/a^2 + k^2/b^2 - 1}$$

For $\quad l^2 : \delta(-\delta) :: \text{power of } (hk) : \text{power of centre } (0, 0)$

$$:: h^2/a^2 + k^2/b^2 - 1 : -1$$

$$\therefore \quad l^2 = \delta^2(h^2/a^2 + k^2/b^2 - 1)$$

EQUATION REFERRED TO TANGENT AND NORMAL

§ **313.** If the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad (1)$$

pass through the origin it is satisfied by $x = 0$, $y = 0$; $\therefore$ $c = 0$.

The points where (1) meets the line $y = 0$ are given by

$$ax^2 + 2gx = 0$$

i. e. $$x = 0 \quad \text{or} \quad -2g/a$$

These points coincide if $g = 0$.

We may if we please take $f = -1$, as the particular conic represented depends only on the ratios of the coeff's a, h, ...

The general equation of a conic touching the axis of x at the origin is $\therefore$

$$ax^2 + 2hxy + by^2 = 2y \quad . \quad . \quad . \quad . \quad . \quad (\Omega)\ (2)$$

If now we suppose the axes rectangular, (2) is the equation of a conic referred to the tangent and normal at any point as axes of co-ordinates.

Ex. If through a given point O on a conic any two lines at right angles are drawn cutting the curve in P and Q; then PQ will pass through a fixed point on the normal. (*Fregier's Theorem.*)

Take the tangent and normal at O as axes; so that (2) is the equation of the conic.

The line-pair OP, OQ are adequately represented [§ 114, *Cor'* (2)] by the equation

$$x^2 + \lambda xy - y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiply (3) by a and subtract the result from (2). Hence

$$(2h - a\lambda)xy + (b + a)y^2 - 2y = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the eq'n of a locus passing through all the intersections of (2), (3).

But (4) splits into the factors $y = 0$, i. e. the tangent at O; and

$$(2h - a\lambda)x + (b + a)y - 2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence (5) is the equation of PQ. To find its intercept on the axis of y put $x = 0$ in (5);

$$\therefore \quad \text{intercept of PQ on normal} = 2/(b + a),$$

which is independent of λ. Q.E.D.

INTERSECTIONS OF CONICS

§ **314**. *Two conics intersect in general in four points.*

Let the conics be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

Then it is shown in treatises on the Theory of Equations that there are in general four sets of values of x and y which satisfy both these equations.

Cor'—Two conics have a common self-conjugate triangle.

Let the conics intersect in A, B, A′, B′ (fig′, § 309); then O, P, Q, the intersections of the connectors of different pairs of these points, are (§ 309, *Cor'*) the vertices of a triangle self-conjugate with respect to both conics.

Note—Two or four of the points of intersection may be imaginary. Such points occur in pairs of the form

$$\left.\begin{aligned} x_1 &= \alpha + \beta\sqrt{-1} \\ y_1 &= \gamma + \delta\sqrt{-1} \end{aligned}\right\}, \qquad \left.\begin{aligned} x_2 &= \alpha - \beta\sqrt{-1} \\ y_2 &= \gamma - \delta\sqrt{-1} \end{aligned}\right\}$$

On forming the equation of the join of these points $(x_1 y_1)$, $(x_2 y_2)$, it is found to be a real straight line.

CONICS THROUGH FOUR FIXED POINTS

§ **315**. As in § 184, $S - \lambda S' = 0$ represents a conic passing through the intersections of the conics

$$S = 0, \quad S' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega)$$

Ex. 1. *Every conic which passes through the points of intersection of two rectangular hyperbolas is a rectangular hyperbola.*

For

$$S - \lambda S' \equiv (a - \lambda a')x^2 + 2(h - \lambda h')xy + (b - \lambda b')y^2 + \ldots$$

Hence (§ 297) $\qquad S - \lambda S' = 0$

represents a rectangular hyperbola if

$$a - \lambda a' + b - \lambda b' = 0$$

But this is satisfied if

$$a + b = 0 \quad \text{and} \quad a' + b' = 0$$

i.e. if $\qquad S = 0, \qquad S' = 0$

are rectangular hyperbolas.

Ex. 2. *Every rectangular hyperbola circumscribing a triangle* ABC *passes through the orthocentre.*

Let one of the hyperbolas meet the $\perp$ from A on BC in D.

Then this hyperbola and the line-pair AD, BC are two rectangular hyperbolas; and the line-pair BD, AC is a conic through their intersections.

The latter conic is $\therefore$ (by Ex. 1) a rectangular hyperbola, i. e. BD is $\perp$ AC. Hence D is the orthocentre.

It follows also that every conic through A, B, C, D is a rectangular hyperbola.

(See *Messenger of Mathematics*, Vol. I., page 77.)

§ 316. The solutions supplied of the following two examples, although not the neatest possible, will be useful as a guide to the learner in solving similar problems.

Ex. 1. *Every rectangular hyperbola passing through the vertices of a triangle passes through the orthocentre.*

Take the side BC and the perpendicular from A on BC as axes (fig', page 75); let $DC = \alpha$, $DA = \beta$, $DB = \alpha'$ (so that α' is negative). The co-ord's of the vertices are then

$$(0, \beta), \qquad (\alpha', 0), \qquad (\alpha, 0)$$

The equation of any rect' hyperbola is (§ 297)

$$x^2 + 2hxy - y^2 + 2gx + 2fy + c = 0$$

Express that co-ord's of vertices satisfy this equation:

$$\therefore \quad -\beta^2 + 2f\beta + c = 0, \qquad \alpha'^2 + 2g\alpha' + c = 0, \qquad \alpha^2 + 2g\alpha + c = 0$$

If we solve these eq'ns for g, f, c, we find

$$2g = -(\alpha + \alpha'), \qquad c = \alpha\alpha', \qquad 2f = (\beta^2 - \alpha\alpha')/\beta;$$

and the equation of the hyperbola becomes

$$x^2 + 2hxy - y^2 - (\alpha + \alpha')x + y(\beta^2 - \alpha\alpha')/\beta + \alpha\alpha' = 0 \quad . \quad . \quad (1)$$

The curve represented by this equation, which involves the single indeterminate h, meets the axis of y in the points determined by

$$\beta y^2 - (\beta^2 - \alpha\alpha')y - \beta\alpha\alpha' = 0$$

Solving this, $\qquad y = \beta \quad \text{or} \quad -\alpha\alpha'/\beta$

Thus the curve meets AD in the fixed point

$$\left(0, -\frac{\alpha\alpha'}{\beta}\right)$$

To determine the fixed point, notice that the line-pair consisting of AB and the $\perp$ from C on AB is a rect′ hyperbola passing through the vertices of A, B, C. The fixed point is $\therefore$ the orthocentre.

Ex. 2. *The locus of centres of rectangular hyperbolas passing through the vertices of a triangle is the nine-point circle.*

Retaining the notation of Ex. 1, the equations to determine the centre of the conic (1) are

$$2x + 2hy - (\alpha + \alpha') = 0, \qquad 2hx - 2y + (\beta^2 - \alpha\alpha')/\beta = 0$$

Eliminating h, we find that locus required is the circle

$$2x^2 + 2y^2 - (\alpha + \alpha')x - y(\beta^2 - \alpha\alpha')/\beta = 0$$

This curve meets the axis of x in the points

$$(0, 0) \quad \text{and} \quad \left(\frac{\alpha + \alpha'}{2}, 0\right)$$

i.e. it passes through mid point of BC and foot of $\perp$ from A on BC.

The locus must evidently pass through the similar points on CA, AB : it is $\therefore$ the nine-point circle.

§ **317**. *To find the equation to a conic passing through two given points* A, A′ *on the axis of* x, *and two others* B, B′ *on the axis of* y ... (Ω).

Let $OA = \alpha$, $OA' = \alpha'$, $OB = \beta$, $OB' = \beta'$ (fig′ § 309).

Then the line-pair AB, A′B′ whose equation is

$$\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)\left(\frac{x}{\alpha'} + \frac{y}{\beta'} - 1\right) = 0$$

forms one such conic.

The line-pair consisting of the axes, whose equation is

$$xy = 0$$

forms another.

Hence, if a suitable value is given to λ, the equation

$$\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)\left(\frac{x}{\alpha'} + \frac{y}{\beta'} - 1\right) = \lambda xy \quad . \quad . \quad . \quad (1)$$

will represent any conic through the four given points A, B, A′, B′.

Cor' (1)—If we put $\alpha' = \alpha$, $\beta' = \beta$ we obtain the equation of a conic touching the axes at A, B. Hence

$$\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)^2 = \lambda xy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the equation of a conic touching the axes at the points (α, 0), (0, β).

Cor' (2)—If (2) represents a parabola, then (§ 287, *Cor'*)

$$\left(\frac{1}{\alpha\beta} - \frac{\lambda}{2}\right)^2 = \frac{1}{\alpha^2\beta^2}$$

$$\therefore \quad \lambda = 0 \quad \text{or} \quad 4/\alpha\beta$$

Neglecting the value $\lambda = 0$ which gives a pair of coincident lines, (2) becomes

$$\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)^2 = \frac{4\,xy}{\alpha\beta}$$

Take the square root

$$\therefore \quad \frac{x}{\alpha} + \frac{y}{\beta} - 1 = \pm 2\sqrt{\frac{xy}{\alpha\beta}}$$

$$\therefore \quad \left(\sqrt{\frac{x}{\alpha}} \pm \sqrt{\frac{y}{\beta}}\right)^2 = 1$$

Take the square root again; then, as a square root may be either positive or negative, we may write the result

$$\sqrt{\frac{x}{\alpha}} + \sqrt{\frac{y}{\beta}} = 1$$

This is $\therefore$ the equation to a parabola touching the axes.

§ 318. *To find the condition that the line*

$$\frac{x}{h} + \frac{y}{k} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may touch the parabola

$$\sqrt{\frac{x}{\alpha}} + \sqrt{\frac{y}{\beta}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \ (\Omega)$$

At the points of intersection of (1) and (2),

$$\frac{x}{h} + \frac{y}{k} = \left(\sqrt{\frac{x}{\alpha}} + \sqrt{\frac{y}{\beta}}\right)^2$$

or

$$x\left(\frac{1}{h} - \frac{1}{\alpha}\right) - 2\sqrt{\frac{xy}{\alpha\beta}} + y\left(\frac{1}{k} - \frac{1}{\beta}\right) = 0$$

This quadratic in $\sqrt{x}/\sqrt{y}$ has equal roots if

$$\left(\frac{1}{h} - \frac{1}{\alpha}\right)\left(\frac{1}{k} - \frac{1}{\beta}\right) = \frac{1}{\alpha\beta};$$

reducing, the required condition is

$$\frac{h}{\alpha} + \frac{k}{\beta} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

§ **319.** *To find the equation of the tangent at* $(x'y')$ *to the parabola*

$$\sqrt{\frac{x}{\alpha}} + \sqrt{\frac{y}{\beta}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \ (\Omega)$$

If $(x'y')$, $(x''y'')$ are two points on the curve,

$$\sqrt{\frac{x'}{\alpha}} + \sqrt{\frac{y'}{\beta}} = 1, \qquad \sqrt{\frac{x''}{\alpha}} + \sqrt{\frac{y''}{\beta}} = 1$$

$$\therefore \quad \frac{\sqrt{x'} - \sqrt{x''}}{\sqrt{\alpha}} + \frac{\sqrt{y'} - \sqrt{y''}}{\sqrt{\beta}} = 0$$

Hence the equation of the join of the points, which is

$$\frac{y - y'}{x - x'} = \frac{y' - y''}{x' - x''} \left(= \frac{\sqrt{y'} - \sqrt{y''}}{\sqrt{x'} - \sqrt{x''}} \frac{\sqrt{y'} + \sqrt{y''}}{\sqrt{x'} + \sqrt{x''}} \right)$$

becomes

$$\frac{y - y'}{x - x'} = - \frac{\sqrt{\beta}}{\sqrt{\alpha}} \frac{\sqrt{y'} + \sqrt{y''}}{\sqrt{x'} + \sqrt{x''}}$$

Now put $y'' = y'$, $x'' = x'$ and multiply up; the equation of the tangent at $(x'y')$ is $\therefore$

$$y \sqrt{\alpha x'} + x \sqrt{\beta y'} = y' \sqrt{\alpha x'} + x' \sqrt{\beta y'}$$

Divide by $\sqrt{\alpha x' \beta y'}$; then remembering that $(x'y')$ satisfies the equation of the curve, we obtain

$$\frac{x}{\sqrt{\alpha x'}} + \frac{y}{\sqrt{\beta y'}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the required equation.

Note—Eq'n (2) is *not* the equation of the polar: to obtain the polar of $(x'y')$ we must use the rationalized form of (1).

Exercises

1. A conic touches the sides of a triangle ABC in a, b, c respectively; show that A a, B b, C c are concurrent.

[*Note*—Prove, as in Ex. 2, § 311, that

$$Ac \, . \, Ba \, . \, Cb = cB \, . \, aC \, . \, bA]$$

2. O is any point on a conic; OP, OQ any two chords equally inclined to the normal at O. Prove that PQ meets the tangent at O in a fixed point.

3. Find the centre of the conic

$$ax^2 + 2hxy + by^2 = 2y$$

Ans. $[h/(h^2 - ab), \; -a/(h^2 - ab)]$

4. Find the equation to the diameter of this conic which passes through the origin.

Ans. $ax + hy = 0$

5. O is a point on a conic; P is any point on the tangent at O. Prove that the perpendiculars from P on its polar and on the diameter through O intercept a constant length on the normal at O.

6. Find the parabolas touching the axis of x at (4, 0) and meeting the axis of y at (0, 2), (0, 8).

Ans. $(x \pm y)^2 - 8x - 10y + 16 = 0$

7. Find locus of centres of conics touching the axes at $(\alpha, 0)$, $(0, \beta)$.

Ans. The straight line $\beta x = \alpha y$

8. Find the equation of a parabola referred to the tangents at the end of its latus rectum as axes.

Ans. $\sqrt{x} + \sqrt{y} = \sqrt{a\sqrt{8}}$, where $4a$ is the latus rectum.

9. The tangents to a parabola at P, Q meet in T: any other tangent meets TP, TQ in p, q: prove that

$$Tp.Tq = Pp.Qq$$

10. Show that the joins of the origin to the other intersections of the conics

$$ax^2 + 2hxy + by^2 + 2fy = 0, \qquad a'x^2 + b'y^2 + 2f'y = 0$$

are at right angles if

$$f'(a + b) = f(a' + b')$$

11. If θ is variable, show that the locus of the point

$$x = a\tan(\theta + \alpha), \qquad y = b\tan(\theta + \beta)$$

is an hyperbola whose asymptotes are

$$x + a\cot(\alpha - \beta) = 0, \qquad y - b\cot(\alpha - \beta) = 0$$

12. Find the condition that a circle may be described through the four points of intersection of the conics

$$ax^2 + 2hxy + by^2 = 1, \qquad a'x^2 + 2h'xy + b'y^2 = 1$$

Ans. $(a - b)h' = (a' - b')h$

13. Show that the equation of a conic referred to an axis and the tangent at its vertex is

$$y^2 = px + qx^2;$$

the conic being a parabola if $q = 0$, an hyperbola if $q > 0$, and an ellipse if $q < 0$.

[*Note*—The names *parabola, ellipse,* and *hyperbola* were originally derived from this property.]

CONDITIONS DETERMINING A CONIC

§ **320.** *Five points determine a conic.*

If it is required to find the equation of a conic passing through five given points (x_1y_1), (x_2y_2), ... (x_5y_5); then assuming that the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we have

$$ax_1^2 + 2hx_1y_1 + \dots = 0, \quad ax_2^2 + \dots = 0, \dots$$

We have then five simple equations which suffice to determine the five ratios a/c, h/c, b/c, g/c, f/c.

Thus there is one conic, and only one, which passes through the five given points.

[If three of the points lie on a line, then, since a line can only meet a conic in two points, the conic is the line-pair containing the five points.]

In practice it is better to proceed as in the following example.

Ex. Find the conic through the five points

$$(-1, 3), \quad (0, 5), \quad (-2, 4), \quad (0, 10), \quad (1, 2)$$

Form eq′ns to joins of two pairs of four of the points; thus the line-pairs

$$x(x + y - 2) = 0 \quad \text{and} \quad (2x - y + 5)(3x - y + 10) = 0$$

are conics passing through the first four points.

The eq′n of any conic through these four points is ∴

$$x(x + y - 2) = \lambda(2x - y + 5)(3x - y + 10)$$

Expressing now that this eq′n is satisfied by the co-ord′s of the fifth point, we find $\lambda = \frac{1}{55}$.

§ **321.** In general, *five-conditions determine a conic.*

For each condition leads to an equation connecting the five unknown ratios a/c, b/c, ...

There may be more than one conic satisfying the conditions: for if the equation to determine one of the ratios obtained by elimination is of the nth degree, there are n conics.

The number of conditions is not necessarily equal to the number of verbal statements: thus to be given the centre $(x'y')$ is equivalent to two conditions; for we have two equations

$$ax' + hy' + g = 0, \qquad hx' + by' + f = 0$$

Ex. *Two* conics can be described to pass through four points and touch a given line.

Take the joins of two pairs of the given points as axes; so that the conic is represented by eq'n (1), § 317.

The reader will find it easy to work out the condition that this conic may touch a given line

$$x/h + y/k = 1:$$

the resulting equation is a quadratic in λ; ∴ &c.

NINE-POINT CONIC

§ **322.** *To find the locus of the centre of a conic passing through four fixed points.*

Such a conic is represented by the equation of § 317, viz.

$$\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right)\left(\frac{x}{\alpha'} + \frac{y}{\beta'} - 1\right) = \lambda xy \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The centre of this conic is determined by the equations

$$\frac{1}{\alpha}\left(\frac{x}{\alpha'} + \frac{y}{\beta'} - 1\right) + \frac{1}{\alpha'}\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right) = \lambda y$$

$$\frac{1}{\beta}\left(\frac{x}{\alpha'} + \frac{y}{\beta'} - 1\right) + \frac{1}{\beta'}\left(\frac{x}{\alpha} + \frac{y}{\beta} - 1\right) = \lambda x$$

The indeterminate λ is eliminated by multiplying the eq'ns by x and y respectively, and subtracting. Reducing, we find that the locus required is the conic

$$2\beta\beta' x^2 - 2\alpha\alpha' y^2 - \beta\beta'(\alpha + \alpha')x + \alpha\alpha'(\beta + \beta')y = 0 \quad . \quad . \quad (2)$$

Putting $y = 0$ we find that the locus passes through the origin [as we might have foreseen, O being the centre of the line-pair conic AA′, BB′ (fig′, page 284)]; and through $\left(\frac{\alpha + \alpha'}{2}, 0\right)$, i. e. through the mid point of AA′.

As the joins of *any* two pairs of the points might have been chosen as axes, we see that—*Given four points* A, B, C, D; *then the six mid points of their joins* AB, AC, AD, BC, BD, CD *and the three intersections of opposite connectors* AB *and* CD, AC *and* BD, AD *and* BC *lie on a conic.*

Cor′—If $\alpha\alpha' = -\beta\beta'$ the centre locus (2) is a circle, and the conics (1) are rectangular hyperbolas. The line-pairs consisting of the joins of opposite connectors are special conics through the four given points; these line-pairs are ∴ rectangular. Hence the four points are so related that each is the orthocentre of the triangle formed by the other three. Accordingly, we infer that—*If* D *is the orthocentre of a triangle* ABC, *then the feet of the perpendiculars and the mid points of* AB, BC, CA, AD, BD, CD *lie on a circle.* This circle is called the nine-point circle.

[An exhaustive discussion of the nine-point conic will be found in Clifford's *Mathematical Papers*, pp. 579–583.]

TANGENTIAL EQUATION

§ **323.** *To find the condition that the line*

$$\lambda x + \mu y + \nu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may touch the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad (2)$$

The equation of the lines joining the origin to the inters′ns of (1), (2) is (§ 119)

$$\nu^2(ax^2 + 2hxy + by^2) - 2\nu(\lambda x + \mu y)(gx + fy) + c(\lambda x + \mu y)^2 = 0$$

or

$$x^2(a\nu^2 - 2g\nu\lambda + c\lambda^2) + 2xy(h\nu^2 - f\lambda\nu - g\mu\nu + c\lambda\mu) + y^2(b\nu^2 - 2f\mu\nu + c\mu^2) = 0$$

These lines ∴ coincide if

$$(a\nu^2 - 2g\lambda\nu + c\lambda^2)(b\nu^2 - 2f\mu\nu + c\mu^2) = (h\nu^2 - f\lambda\nu - g\mu\nu + c\lambda\mu)^2$$

This condition reduces to

$$(bc - f^2)\lambda^2 + (ca - g^2)\mu^2 + (ab - h^2)\nu^2 + 2(gh - af)\mu\nu + 2(hf - bg)\nu\lambda + 2(fg - ch)\lambda\mu = 0 \quad . \quad . \quad (\Omega)$$

In future this equation will be written

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0 \quad . \quad . \quad (\Omega)$$

The coefficients A, B, C, $2F$, $2G$, $2H$ are the derived functions of the discriminant

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$$

taken respectively with regard to a, b, c, f, g, h.

The co-ordinates of the centre (§ 289) are

$$G/C, \quad F/C$$

PAIR OF TANGENTS

§ 324. The points where the join of $(x'y')$, $(x''y'')$ meets the conic $S = 0$ are determined by substituting

$$\frac{mx'' + nx'}{m+n}, \quad \frac{my'' + ny'}{m+n}$$

for x, y in the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

This gives

$$a(mx'' + nx')^2 + 2h(mx'' + nx')(my'' + ny') + b(my'' + ny')^2$$
$$+ 2g(m+n)(mx'' + nx') + 2f(m+n)(my'' + ny') + c(m+n)^2 = 0$$

or

$$m^2(ax''^2 + 2hx''y'' + by''^2 + 2gx'' + 2fy'' + c)$$
$$+ 2mn[ax'x'' + h(x'y'' + x''y') + by'y'' + g(x' + x'') + f(y' + y'') + c]$$
$$+ n^2(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c) = 0$$

This quadratic in m/n has equal roots if

$$(ax''^2 + 2hx''y'' + \ldots)(ax'^2 + 2hx'y' + \ldots) = [ax'x'' + h(x'y'' + x''y') + \ldots]^2$$

Now the quadratic has equal roots if $(x''y'')$ is any point on *either* tangent from $(x'y')$: writing then x, y instead of x'', y'' in the preceding equation, we obtain the equation of the pair of tangents from $(x'y')$.

The equation of the pair of tangents from $(x'y')$ is $\therefore$

$$(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c)(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$= [axx' + h(xy' + x'y) + byy' + g(x + x') + f(y + y') + c]^2 \quad . \quad . \quad . \quad (\Omega)$$

§ 325. *To find the equation of the director circle of* $S = 0$.

The equation of the tangents from $(x'y')$ is

$$\phi(x', y')\,\phi(x, y) - [axx' + h(xy' + x'y) + \ldots]^2 = 0$$

If $(x'y')$ is on the director circle these tangents are at right angles; the condition for this (if we suppose the axes rectangular) is

$$\text{coeff}' \text{ of } x^2 + \text{coeff}' \text{ of } y^2 = 0$$

Express this condition and write x, y for x', y'; reducing, we obtain the equation of the director circle

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0$$

Here $C = ab - h^2$, &c., as in § 323.

Cor'—If the curve is a parabola, $C = 0$. The equation of the directrix (which is locus of inters'n of rectangular tangents) is ∴

$$2Gx + 2Fy = A + B$$

Exercises on Chapter XI

1. Find the equation of the diameters of the conic

$$ax^2 + 2hxy + by^2 = 1$$

passing through its intersections with the concentric circle

$$x^2 + 2xy\cos\omega + y^2 = \rho^2 \quad \ldots\ldots \quad (\Omega)$$

Ans. $(a\rho^2 - 1)x^2 + 2(h\rho^2 - \cos\omega)xy + (b\rho^2 - 1)y^2 = 0$

2. Show that the length of a diameter of the same conic bisecting the angle between its axes is 2ρ, where

$$(a + b - 2h\cos\omega)\rho^2 = 2\sin^2\omega$$

[*Note*—Express cond'n that line-pair in Ex. 1 is rectangular.]

3. If 2ρ is the length of an equi-conjugate diameter of the conic

$$ax^2 + 2hxy + by^2 = 1$$

then
$$2(ab - h^2)\rho^2 = a + b - 2h\cos\omega;$$

and the equation to the equi-conjugate diameters is

$$(a + b - 2h\cos\omega)(ax^2 + 2hxy + by^2) = 2(ab - h^2)(x^2 + 2xy\cos\omega + y^2)$$

4. Show that the points of intersection of the conics

$$ax^2 + 2hxy + by^2 = 1, \qquad a'x^2 + 2h'xy + b'y^2 = 1$$

are on conjugate diameters of the former conic if

$$ab' + a'b - 2hh' = 2(ab - h^2)$$

5. Show that the following function of the coefficients in the equations of two conics is unaltered by change of axes:

$$\frac{ab' + a'b - 2hh'}{\sin^2 \omega}$$

6. The major axes of two conics are parallel: show that their four points of intersection are concyclic.

7. The polars of a point with respect to two given conics intersect at right angles; prove that the point describes a conic, and that this conic is a circle if the given conics are rectangular hyperbolas.

[*Note*—Choose asymptotes of one conic as axes; so that eq'ns of given conics are

$$xy = \delta^2, \qquad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.]$$

8. Chords of a rectangular hyperbola at right angles to each other subtend right angles at a fixed point O. Show that they intersect on the polar of O.

[*Note*—Choose O as origin, and parallels to axes of figure of hyperbola as axes of co-ord's; so that its eq'n is

$$x^2 - y^2 + 2gx + 2fy + c = 0.]$$

9. Prove that two parabolas can be drawn through four given points; and that their axes are parallel to the asymptotes of the conic which is the locus of centres of conics passing through the four points.

10. Find equation of tangents from origin to $S = 0$.

Ans. $Bx^2 - 2Hxy + Ay^2 = 0$

[*Note*—This may be deduced from § 324; or directly thus.

Substitute $r\cos\theta$, $r\sin\theta$ for x, y in $S = 0$; express condition that the quadratic in r has equal roots. Then replace $\sin\theta/\cos\theta$ by y/x.]

11. Tangents from P to a conic intercept a given length on a fixed tangent; prove that the locus of P is a conic.

[*Note*—Take given tangent and corresponding normal as axes.]

12. If tangents from P intercept a given length on a fixed line, the locus of P is in general a curve of the fourth degree.

[Take given line as axis of x; the conic may be represented by the general equation $S = 0$.]

13. If tangents from P meet a fixed tangent at points equidistant from the point of contact; prove that the locus of P is a straight line, viz. the diameter conjugate to the given tangent.

[*Note*—Choose axes as in *Note*, Ex. 11. The abscissae of points where tangents from P meet given tangent are determined by a quadratic; put sum of its roots = 0.]

14. PQ is one of a system of parallel chords of $S = 0$; R is a point on PQ such that

$$\frac{1}{RP} + \frac{1}{RQ} = \text{constant} \left(= \frac{1}{\delta}\right)$$

If the chords are inclined to the axis of x at an angle θ, prove that the locus of R is the conic

$$S + 2\delta\,[(ax + hy + g)\cos\theta + (hx + by + f)\sin\theta] = 0$$

[*Note*—Let co-ord's of R be x', y'; then RP, RQ are the roots of eq'n (1), § 310.]

15. PQ is one of a system of parallel chords of an ellipse; R is a point on PQ such that

$$PR^2 + RQ^2 + PR\,.\,RQ = \text{constant.}$$

Show that the locus of R is a conic.

16. Find the equation to the axis of the parabola

$$\sqrt{x/\alpha} + \sqrt{y/\beta} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\Omega)$$

Ans. $\dfrac{x}{\alpha} - \dfrac{y}{\beta} + \dfrac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2 + 2\alpha\beta\cos\omega} = 0$

[*Note*—Rationalize eq'n; and proceed as in § 295. Use cond'n of perpendicularity given in § 93, *Cor'* (2).]

17. Find the equation to the directrix of the same parabola.

Ans. $\dfrac{x}{\beta} + \dfrac{y}{\alpha} + \cos\omega\left(\dfrac{x}{\alpha} + \dfrac{y}{\beta} - 1\right) = 0$

[*Note*—Express cond'n that tangents from $(x'y')$ are at right angles.]

18. Find the focus of the same parabola.

[*Note*—The line $\qquad x/h + y/k = 1$

is a tangent if $\qquad h/\alpha + k/\beta = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

The circum-circle of the triangle which this tangent forms with the axes is

$$x^2 + 2xy\cos\omega + y^2 = hx + ky \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminate k from (2) by means of (1); eq'n of circle becomes

$$\alpha(x^2 + 2xy\cos\omega + y^2 - \beta y) = h(\alpha x - \beta y)$$

This eq'n contains the single parameter h in the first degree; and whatever be the value of h the circle passes through the fixed point determined by

$$x^2 + 2xy\cos\omega + y^2 = \alpha x = \beta y$$

This fixed point is the focus. (§ 330, Ex. 3).]

19. A parabola touches the sides of a given triangle; show that each of the chords of contact passes through a fixed point.

20. The equation to determine the eccentricity of $S = 0$ is

$$\left(\frac{e^2}{2 - e^2}\right)^2 = \frac{(a-b)^2 + 4h^2}{(a+b)^2}$$

[*Note*—Suppose eq'n reduced by successive changes of axes to

$$ax^2 + 2hxy + by^2 + c' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\alpha x^2 + \beta y^2 + c' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equate the invariants (§ 212);

$$\therefore \quad \alpha + \beta = a + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\alpha\beta = ab - h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now if (2) is written

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

then $$A^2 = -c'/\alpha, \qquad B^2 = -c'/\beta$$

and $$e^2 = \frac{A^2 - B^2}{A^2}; \qquad \therefore \quad e^2 = \frac{\beta - \alpha}{\beta} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eliminate α, β from (3), (4), (5).

It will be observed that the equation to determine e^2 is a quadratic; the existence of a second eccentricity will be accounted for in Chap. XIII.]

21. Prove that the equation of the join of $(x'y')$ to the centre of $S = 0$ is

$$(ax + hy + g)(hx' + by' + f) = (hx + by + f)(ax' + hy' + g)$$

[*Note*—The lines

$$ax + hy + g = 0, \qquad hx + by + f = 0$$

pass through the centre (§ 289).]

22. Prove that the equation to the axes of $S = 0$ is

$$h(u^2 - v^2) = (a - b)\,uv$$

where $u \equiv ax + hy + g, \qquad v \equiv hx + by + f$

[*Note*—If $(x'y')$ is a point on an axis of the conic, the join of $(x'y')$ to centre is $\perp$ polar of $(x'y')$.]

23. PQ is a chord of a conic; Pq, Pr are chords equally inclined to PQ. Show that qr passes through a fixed point, viz. the intersection of the tangents at P and Q.

[*Note*—When Pq coincides with PQ so does Pr; the tangent at Q is $\therefore$ one position of qr.]

24. Conics are drawn through four fixed points. Prove that the polars of a given point pass through a fixed point.

[Let two of the conics be

$$S \equiv ax^2 + 2hxy + \ldots = 0, \qquad S' \equiv a'x^2 + 2h'xy + \ldots = 0$$

The polar of $(x'y')$ with respect to

$$S + \lambda S' = 0$$

is $$ax'x + h(y'x + x'y) + \ldots + \lambda[a'x'x + h'(y'x + x'y) + \ldots] = 0$$

This passes through the intersection of

$$ax'x + h(y'x + x'y) + \ldots = 0, \qquad a'x'x + h'(y'x + x'y) + \ldots = 0]$$

25. Find locus of poles of a given straight line with respect to conics passing through four fixed points.

[Let the given line be

$$lx + my + n = 0$$

Express that this coincides with polar of $(x'y')$ with respect to

$$S + \lambda S' = 0:$$

$$\therefore \quad \frac{ax' + hy' + g + \lambda(a'x' + h'y' + g')}{l} = \frac{\ldots\ldots}{m} = \frac{\ldots}{n} = \mu, \text{ say.}$$

Multiply up and eliminate λ, μ. Locus is the conic

$$\begin{vmatrix} ax + hy + g & a'x + h'y + g' & l \\ hx + by + f & h'x + b'y + f' & m \\ gx + fy + c & g'x + f'y + c' & n \end{vmatrix} = 0.]$$

26. A system of conics passes through the angular points of a square. Tangents are drawn from a given point on one diagonal. Prove that the locus of the points of contact is a rectangular hyperbola.

[*Note*—Choose parallels to sides of square through its centre as axes; so that system of conics is
$$x^2 - a^2 + \lambda (y^2 - a^2) = 0]$$

27. Conics are drawn through four points: prove that the pole of the join of any two of them describes a straight line.

28. The lengths of the tangents from $(x'y')$ to the ellipse
$$x^2/a^2 + y^2/b^2 = 1$$
are the roots of the equation in l,
$$ay' \sqrt{l^2 - b^2 S'} = bx' \sqrt{a^2 S' - l^2} + ab \sqrt{(a^2 - b^2) S'}$$
where
$$S' = x'^2/a^2 + y'^2/b^2 - 1$$

[*Note*—Let δ = length of parallel semi-diameter; $\tan^{-1} m$ its inclination to axis of x.

Then $$l = \delta \sqrt{S'} \quad \text{[Ex., § 312]}$$
and $$y' = mx' + \sqrt{a^2 m^2 + b^2} \quad \text{[§ 249]}$$
Express δ in terms of m (§ 238); then eliminate m.]

29. Find the locus of the *Fregier Points* (see Ex., § 313) corresponding to points on the ellipse
$$x^2/a^2 + y^2/b^2 = 1$$
Ans. $x^2/a^2 + y^2/b^2 = (a^2 - b^2)^2/(a^2 + b^2)^2$

[*Note*—Let P be a point on the ellipse; draw chords PQ, PR parallel to the axes. The corresponding *Fregier Point* is inters'n of diam' QR and normal at P. Express eq'ns of these lines in terms of α, the ecc' $\wedge$ of P; then eliminate α.]

30. Find the condition that the line
$$x/h + y/k = 1$$
may touch the conic
$$(x/\alpha + y/\beta - 1)^2 = 2\lambda xy$$
Ans. $\lambda = 2\left(\frac{1}{\alpha} - \frac{1}{h}\right)\left(\frac{1}{\beta} - \frac{1}{k}\right)$

[*Note*—Express cond'n that the following eq'n in x/y has equal roots:
$$(x/\alpha + y/\beta - x/h - y/k)^2 = 2\lambda xy]$$

31. Prove that the locus of centres of conics inscribed in a quadrilateral is the straight line through the mid points of its diagonals.

[Take diag′s as axes; let eq′ns to sides be

$$x/p + y/q = 1, \qquad x/p' + y/q = 1, \; \ldots .$$

The cond′ns that these lines touch the general conic are

$$A/p^2 + B/q^2 + C - 2F/q - 2G/p + 2H/(pq) = 0 \quad . \quad . \quad (1)$$

$$A/p'^2 + B/q^2 + \ldots = 0 \quad \ldots\ldots\ldots \quad (2)$$

$$A/p^2 + B/q'^2 + \ldots = 0 \quad \ldots\ldots\ldots \quad (3)$$

$$A/p'^2 + B/q'^2 + \ldots = 0 \quad \ldots\ldots\ldots \quad (4)$$

Subtract (2) from (1) and divide by

$$1/p - 1/p'$$

$$\therefore \quad A(1/p + 1/p') - 2G + 2H/q = 0$$

So from (3) and (4)

$$A(1/p + 1/p') - 2G + 2H/q' = 0$$

Whence $\qquad H = 0, \qquad A = 2pp'G/(p + p')$

Similarly $\qquad B = 2qq'F/(q + q')$

Substituting in (1), and simplifying, we get

$$C - 2G/(p + p') - 2F/(q + q') = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But co-ord′s of centre are

$$\alpha = G/C, \qquad \beta = F/C$$

We deduce then from (5)

$$2\alpha/(p + p') + 2\beta/(q + q') = 1$$

i.e. $(\alpha\beta)$ is a point on

$$x \Big/ \frac{p + p'}{2} + y \Big/ \frac{q + q'}{2} = 1$$

But this is the join of mid points of diagonals.

The cond′n $\quad = 0$ may be interpreted to show that the diagonals are conjugate with respect to the conic, i.e. the pole of each lies on the other.

This solution is by Prof′ Genese.

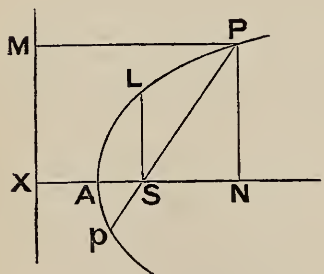
It may be seen *à priori* that the mid points of diagonals are points on the locus; for a diagonal is the limiting form of a thin inscribed ellipse.

The question may also be solved by taking two sides of quad′ as axes, and using the cond′n of Ex. 30.]

CHAPTER XII

POLAR EQUATION OF A CONIC REFERRED TO FOCUS; CONFOCAL CONICS

§ **326.** *To find the polar equation of a conic referred to the focus.*



From a point P on the curve draw PN $\perp$ axis, and PM $\perp$ directrix;

let $\quad SP = r, \quad P\hat{S}N = \theta.$

Let the semi-latus rectum

$$SL = l$$

By def^n $\quad SL = e\,SX$

also $\quad SP = e\,PM$

$$= e\,(XS + SN)$$

$$= SL + e\,SN$$

i. e. $\quad r = l + er\cos\theta$

or $\quad r\,(1 - e\cos\theta) = l$

The required equation is $\therefore$

$$\frac{l}{r} = 1 - e\cos\theta$$

Cor' (1)—The equation of the directrix is

$$r \cos (\pi - \theta) = SX = \frac{1}{e};$$

or

$$\frac{1}{r} = - e \cos \theta$$

Cor' (2)—Let PS p be a focal chord; then

$$\frac{1}{SP} = 1 - e \cos \theta$$

$$\frac{1}{Sp} = 1 - e \cos (\pi + \theta) = 1 + e \cos \theta$$

$$\therefore \quad \frac{1}{SP} + \frac{1}{Sp} = 2$$

or

$$\frac{1}{SP} + \frac{1}{Sp} = \frac{2}{l}$$

Thus *the semi-latus rectum is a harmonic mean between the segments of any focal chord.*

Note—If the initial line make an angle α with SN, then as $P\hat{S}N$ is now $\theta + \alpha$, the equation of the conic is

$$l/r = 1 - e \cos (\theta + \alpha)$$

§ **327**. It is a useful exercise to deduce the figure of the curve from the equation

$$l/r = 1 - e \cos \theta$$

by giving θ a series of values increasing from 0 to 2π.

The cases $e = 1$, $e < 1$, $e > 1$, should be separately considered.

The last case is very instructive; the following summary will guide the learner in discussing the question fully.

When $\theta = 0$, $r = l/(1 - e)$: this is negative and $=$ SA′ (fig′, § 266).

As θ increases, r continues negative (the extremity of the radius vector describing the lower portion of the further branch of the hyperbola), until

$$\cos \theta = 1/e, \quad \text{when} \quad r = -\infty.$$

The nearer branch is next described in the order PA p (fig′, § 270); and lastly the upper portion of the further branch.

Note—Observe that if a line through S cut two opposite branches of an hyperbola in P and p, then the vectorial $\wedge$s of P and p are not the same; they differ by π.

EQUATIONS OF CHORD AND TANGENT

§ **328.** Consider the locus represented by

$$\frac{1}{r} = \cos\left(\theta - \frac{\alpha+\beta}{2}\right)\sec\frac{\alpha-\beta}{2} - e\cos\theta \quad . \quad . \quad (1)$$

This may be written

$$1 = \left[r\cos\theta\cos\frac{\alpha+\beta}{2} + r\sin\theta\sin\frac{\alpha+\beta}{2}\right]\sec\frac{\alpha-\beta}{2} - er\cos\theta$$

or

$$1 = \left(x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2}\right)\sec\frac{\alpha-\beta}{2} - ex$$

Thus (1) represents a straight line.

Again, if we put $\theta = \alpha$ in (1) it becomes

$$\frac{1}{r} = 1 - e\cos\alpha$$

∴ (1) passes through the point on the conic whose vectorial angle is α.

Similarly it passes through the point whose vect′ ∧ is β.

Hence (1) is the equation of the chord joining the two points on the conic whose vect′ ∧s are α, β.

Cor′—Put $\beta = \alpha$ in (1); the equation of the tangent at the point whose vectorial angle is α is ∴

$$\frac{1}{r} = \cos(\theta - \alpha) - e\cos\theta \quad . \quad . \quad . \quad . \quad (2)$$

§ **329.** The equation of the chord may also be obtained thus.

Assume that the required equation is

$$\frac{1}{r} = A\cos\theta + B\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Express that this is satisfied by the co-ord′s of the given points:

$$\therefore \quad \left.\begin{aligned} 1 - e\cos\alpha &= A\cos\alpha + B\sin\alpha \\ 1 - e\cos\beta &= A\cos\beta + B\sin\beta \end{aligned}\right\}$$

If we solve these two equations for A, B and substitute their values in (3), we obtain after reduction the same equation as before. The student should work this out as an exercise.

§ 330. Ex. 1. *If the tangent at* P *meet the directrix in* Z, *then* PZ *subtends a right angle at* S.

Let vect′ $\wedge$ of P be α; the equation of the tangent at P is

$$1/r = \cos(\theta - \alpha) - e\cos\theta$$

The equation of directrix is

$$1/r = -e\cos\theta$$

At the point Z where these intersect we have ∴

$$\cos(\theta - \alpha) = 0$$

$$\therefore \quad \theta - \alpha = \pm\pi/2, \qquad \text{i.e. } P\hat{S}Z = \pi/2$$

Ex. 2. *If the tangents at* P, Q *intersect in* T, *then* ST *bisects the angle* PSQ.

Let vect′ $\wedge$ of P be α, and of Q, β.

At the inters′n of the tangents

$$1/r = \cos(\theta - \alpha) - e\cos\theta$$

$$1/r = \cos(\theta - \beta) - e\cos\theta$$

we have $$\cos(\theta - \alpha) = \cos(\theta - \beta) \quad \dots\dots\dots\dots \quad (1)$$

$$\therefore \quad \theta - \alpha = -(\theta - \beta)$$

$$\therefore \quad \theta = (\alpha + \beta)/2; \quad \therefore \ \&c.$$

If, however, P and Q are on opposite branches of an hyperbola, then ST does not lie between SP, SQ. The proper deduction from (1) is now

$$\theta - \alpha = 2\pi - (\theta - \beta)$$

$$\therefore \quad \theta = \pi + (\alpha + \beta)/2$$

In this case ST bisects the angle supplemental to PSQ.

Ex. 3. *The circum-circle of the triangle formed by three tangents to a parabola passes through the focus.*

Let the equation to the parabola be

$$1/r = 1 - \cos\theta$$

The tangents at the points whose vect′ $\wedge$s are α, β, γ are

$$1/r = \cos(\theta - \alpha) - \cos\theta \quad \dots\dots\dots\dots \quad (1)$$

$$1/r = \cos(\theta - \beta) - \cos\theta \quad \dots\dots\dots\dots \quad (2)$$

$$1/r = \cos(\theta - \gamma) - \cos\theta \quad \dots\dots\dots\dots \quad (3)$$

At p′t of inters′n of (1) and (2), $\theta = \frac{1}{2}(\alpha + \beta)$, $1/r = 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

" " (2) " (3), $\theta = \frac{1}{2}(\beta + \gamma)$, $1/r = 2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

" " (3) " (1), $\theta = \frac{1}{2}(\gamma + \alpha)$, $1/r = 2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}$

It is evident that these three points all lie on the circle

$$\frac{1}{r}\sin\left(\frac{\alpha + \beta + \gamma}{2} - \theta\right) = 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

The equation to the circle shows that it passes through the focus.

Ex. 4. A chord PQ of a conic subtends a constant angle 2γ at the focus. Find the locus of T the intersection of tangents at P and Q. Find also the envelope of PQ.

Let the conic be

$$1/r = 1 - e\cos\theta$$

Let the vect′ $\wedge$s of P, Q be α, β; then

$$\alpha - \beta = 2\gamma$$

The tangents at P, Q are

$$1/r = \cos(\theta - \alpha) - e\cos\theta$$

$$1/r = \cos(\theta - \beta) - e\cos\theta$$

Where these intersect we have

$$\theta = \frac{\alpha + \beta}{2}, \qquad 1/r = \cos\frac{\alpha - \beta}{2} - e\cos\theta$$

The equation to the locus of T is $\therefore$

$$1/r = \cos\gamma - e\cos\theta$$

or

$$(l\sec\gamma)/r = 1 - e\sec\gamma\cos\theta$$

The locus of T is $\therefore$ a conic having the same focus and directrix as the given conic, and whose eccentricity $= e\sec\gamma$.

Again, the equation to PQ is (§ 328)

$$l/r = \cos\left(\theta - \frac{\alpha+\beta}{2}\right)\sec\gamma - e\cos\theta$$

or

$$(l\cos\gamma)/r = \cos\left(\theta - \frac{\alpha+\beta}{2}\right) - e\cos\gamma\cos\theta$$

Comparing this with the equation of the tangent [eq'n (2), § 328] we see that PQ touches the conic

$$(l\cos\gamma)/r = 1 - e\cos\gamma\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at the point on that conic whose vect' $\wedge$ is

$$(\alpha + \beta)/2$$

The conic (1) is the envelope. It has the same focus and directrix as the given conic, and its eccentricity $= e\cos\gamma$.

Ex. 5. Find the equation of the polar of $(r_1\ \theta_1)$ with respect to

$$l/r = 1 - e\cos\theta$$

The equation of the conic is

$$l = r - er\cos\theta$$

or

$$l = \sqrt{x^2 + y^2} - ex$$

or

$$x^2 + y^2 = (l + ex)^2$$

or

$$x^2(1 - e^2) + y^2 - 2lex - l^2 = 0$$

The polar of $(r_1\cos\theta_1,\ r_1\sin\theta_1)$ is then (§ 306)

$$xr_1\cos\theta_1(1 - e^2) + yr_1\sin\theta_1 - le(x + r_1\cos\theta_1) - l^2 = 0$$

Writing $r\cos\theta$, $r\sin\theta$ for x, y we obtain the req'd eq'n, viz.

$$rr_1\cos\theta\cos\theta_1(1 - e^2) + rr_1\sin\theta\sin\theta_1 = le(r\cos\theta + r_1\cos\theta_1) + l^2$$

This may be written

$$rr_1\cos(\theta - \theta_1) = (l + er\cos\theta)(l + er_1\cos\theta_1)$$

or

$$\cos(\theta - \theta_1) = (l/r + e\cos\theta)(l/r_1 + e\cos\theta_1)$$

Exercises

[Unless otherwise implied, the equation of the conic in these questions is

$$l/r = 1 - e\cos\theta]$$

1. PS p, QS q are focal chords at right angles. Prove that the sum of the reciprocals of P p, Q q is constant.

2. A chord PQ subtends a right angle at the focus. Prove that

$$\left(\frac{1}{SP}-\frac{1}{l}\right)^2+\left(\frac{1}{SQ}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}$$

3. Prove that a focal chord varies as the square of the parallel semi-diameter.

4. PS p, QS q are focal chords. Prove that PQ, pq meet on the directrix.

5. Prove that tangents at the ends of a focal chord meet on the directrix.

6. Tangents at P and Q meet in T. If PQ meet the directrix in Z, prove that SZ is at right angles to ST.

7. Find the equation to the normal at $(r_1\, \theta_1)$.

Ans. $r\,[\sin(\theta_1-\theta)+e\sin\theta]=er_1\sin\theta_1$

8. Find the equation to the director circle.

Ans. $r^2(e^2-1)+2\,ler\cos\theta+2\,l^2=0$

9. A chord PQ passes through a fixed point O. Prove that

$$\tan\tfrac{1}{2}\,PSO\,.\,\tan\tfrac{1}{2}\,QSO$$

is constant.

10. P is a point on a hyperbola; the tangent at P meets an asymptote in T. Prove that

$$P\hat{S}T=C\hat{T}S$$

[*Note*—TP and the asymptote are the tangents from T. Use result of Ex. 2, § 330.]

11. A chord of a rectangular hyperbola subtends a right angle at the focus; prove that it envelopes a parabola.

12. If TP, TQ are tangents to a parabola, then

$$ST^2=SP.SQ;$$

and the triangles SPT, STQ are similar.

[*Note*—See § 330, Ex. 3; use Euclid VI. 6.]

13. Show that the equations

$$l/r=1-e\cos\theta,\qquad l/r=-1-e\cos\theta$$

represent the same conic.

[*Note*—If PSQ is a focal chord whose vect′ $\wedge$ is θ, one of these eq′ns determines SP and the other determines SQ.]

14. The equations of two conics having a common focus are

$$S \equiv l/r - 1 + e\cos\theta = 0, \qquad S' \equiv l'/r - 1 + e'\cos(\theta + \beta) = 0;$$

prove that $\quad S - S' = 0, \quad S + S' + 2 = 0$

are the equations to one pair of common chords.

CONFOCALS

§ 331. The co-ordinates of the foci of

$$x^2/a^2 + y^2/b^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

are $\quad (0, \pm\sqrt{a^2 - b^2})$

These are unaltered if we substitute $a^2 - \lambda$, $b^2 - \lambda$ for a^2, b^2.

Accordingly, if different values are given to λ, the equation

$$x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

represents a system of conics confocal with (1).

If $\lambda > b^2$, the curve represented by (2) is an hyperbola.

Note—We may briefly refer to the ellipse represented by (1) as 'the ellipse (a, b).' Thus (2) represents 'the ellipse $(\sqrt{a^2 - \lambda}, \sqrt{b^2 - \lambda})$.'

§ 332. Put

$$c^2 = a^2 - b^2, \qquad a_1^2 = a^2 - \lambda, \qquad b_1^2 = b^2 - \lambda;$$

then a_1, b_1 are the semi-axes of the confocal (2); and

$$a_1^2 - b_1^2 = a^2 - b^2 = c^2$$

Equation (2) may also be written in the forms

$$x^2/a_1^2 + y^2/b_1^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$x^2/a_1^2 + y^2/(a_1^2 - c^2) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$x^2/(b_1^2 + c^2) + y^2/b_1^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

§ 333. Looking at eq′n (5) we see that—

If b_1^2 is very small and positive, then a_1^2 is nearly $= c^2$; and the confocal is a thin ellipse nearly coincident with the join of the foci SS′.

If b_1^2 is very small and negative, the confocal is a thin hyperbola nearly coincident with the portions of the axis extending from S, S′ to infinity in opposite directions.

Thus, corresponding to

$$b_1^2 = 0, \quad \text{or} \quad \lambda = b^2,$$

we have as limits of the confocal system both the line-ellipse (the join of the foci), and the line-hyperbola (the complement of this join).

§ **334.** If the confocal pass through a given point (hk), then by eq′n (5)

$$h^2/(b_1^2 + c^2) + k^2/b_1^2 = 1$$

or

$$h^2 b_1^2 + k^2 (b_1^2 + c^2) = b_1^2 (b_1^2 + c^2)$$

$$\therefore \quad b_1^4 - (h^2 + k^2 - c^2)\, b_1^2 - k^2 c^2 = 0$$

This is a quadratic to determine b_1^2.

Its roots are both real; one is positive and the other negative.

Thus *two conics of a confocal system can be drawn through a given point; one is an ellipse and the other an hyperbola.*

§ **335.** We deduce similarly from eq′n (4) the quadratic in a_1^2,

$$a_1^4 - a_1^2 (h^2 + k^2 + c^2) + h^2 c^2 = 0 \quad . \; . \; . \; . \; . \; . \quad (6)$$

Let a_1^2, a_2^2 be the roots of this equation;

$$\therefore \quad h^2 c^2 = a_1^2 a_2^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Similarly

$$-k^2 c^2 = b_1^2 b_2^2 = (a_1^2 - c^2)(a_2^2 - c^2) \quad . \; . \; . \; . \quad (8)$$

The semi-axes major of the two confocals which pass through a point are called by Lamé its *elliptic co-ordinates.* The preceding eq′ns (7), (8) give expressions for the rect′ co-ord′s h, k of a point in terms of its elliptic co-ord′s a_1, a_2.

Cor′—We see also from (6) that

$$a_1^2 + a_2^2 = h^2 + k^2 + c^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

§ **336.** *Confocal conics cut at right angles.*

Suppose that (hk) is a point of intersection of the conics

$$x^2/a^2 + y^2/b^2 = 1, \qquad x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$$

Then

$$h^2/a^2 + k^2/b^2 = 1, \qquad h^2/(a^2 - \lambda) + k^2/(b^2 - \lambda) = 1$$

By subtraction

$$\frac{h^2}{a^2(a^2 - \lambda)} + \frac{k^2}{b^2(b^2 - \lambda)} = 0$$

But this is the condition that the tangents at (hk) viz.

$$xh/a^2 + yk/b^2 = 1, \qquad xh/(a^2 - \lambda) + yk/(b^2 - \lambda) = 1$$

should be at right angles.

Note—This proposition and that of the last § are obvious geometrically.

If P is the given point and S, S' the given foci: then one curve is the ellipse whose foci are S, S' and major axis $= SP + S'P$; the other is the hyperbola whose foci are S, S' and transverse axis $= SP - S'P$*.

The tangents at P bisect $S\hat{P}S'$ and the supplemental angle; they are ∴ at right angles.

§ **337.** The line $x \cos\alpha + y \sin\alpha = p$ will touch the conic

$$x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$$

$$(a^2 - \lambda)\cos^2\alpha + (b^2 - \lambda)\sin^2\alpha = p^2$$

[§ 250, *Cor'* (1).

$$\therefore \quad \lambda = a^2\cos^2\alpha + b^2\sin^2\alpha - p^2$$

Thus *there is one conic, and only one, of a confocal system which touches a given line.*

§ **338.** *If* p, p' *are perpendiculars from the centre on parallel tangents to two confocals; then* $p^2 - p'^2$ *is constant.*

Let $\qquad x\cos\alpha + y\sin\alpha = p, \qquad x\cos\alpha + y\sin\alpha = p'$

be parallel tangents to the confocals

$$x^2/a^2 + y^2/b^2 = 1, \qquad x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$$

Then
$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

$$p'^2 = (a^2 - \lambda)\cos^2\alpha + (b^2 - \lambda)\sin^2\alpha$$

$$\therefore \quad p^2 - p'^2 = \lambda = \text{constant}.$$

§ **339.** *To find the locus of the intersection of tangents to two confocals which cut at right angles.*

The line

$$x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \quad \ldots \ldots \quad (1)$$

* It follows that the elliptic co-ord's of P are

$$a_1 = \tfrac{1}{2}(SP + S'P), \qquad a_2 = \tfrac{1}{2}(SP - S'P).$$

is a tangent to the conic

$$x^2/a^2 + y^2/b^2 = 1$$

If we change a^2, b^2 into $a^2 - \lambda$, $b^2 - \lambda$ and α into $\pi/2 + \alpha$, we obtain the line at right angles to (1) which touches the confocal

$$x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$$

The tangent to this confocal at right angles to (1) is ∴

$$-x \sin\alpha + y \cos\alpha = \sqrt{(a^2 - \lambda)\sin^2\alpha + (b^2 - \lambda)\cos^2\alpha} \quad . \quad . \quad (2)$$

α is eliminated from (1), (2) by squaring and adding; the required locus is ∴ the circle

$$x^2 + y^2 = a^2 + b^2 - \lambda$$

§ **340.** *To find the locus of the pole of a given line with respect to a system of confocals.*

Let the given line be

$$x/h + y/k = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $(x'y')$ is the pole of this line with respect to

$$x^2/(a^2 - \lambda) + y^2/(b^2 - \lambda) = 1$$

we must have

$$\frac{1}{h} = \frac{x'}{a^2 - \lambda}, \quad \frac{1}{k} = \frac{y'}{b^2 - \lambda}$$

$$\therefore \quad hx' - ky' = a^2 - b^2$$

The required locus is ∴ the line

$$hx - ky = a^2 - b^2;$$

which is perpendicular to the given line.

Further, the point of contact of the given line with the conic of the system which it touches is a point on the locus.

The required locus is ∴ the normal to that conic at its point of contact with the given line.

Cor'—If the tangent at a point P on a conic intersect a confocal in p, q; then the tangents at p, q intersect on the normal at P.

§ **341.** *If a chord of a conic touch a confocal, its length varies as the square of the parallel semi-diameter.* (*Prof' Burnside.*)

Let δ be the length of a chord of the ellipse

$$x^2/a^2 + y^2/b^2 = 1;$$

α, β the ecc′ $\wedge$s of its extremities; R the length of parallel semi-diam′, θ its inclination to the axis of x.

The equation of the chord is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2} \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad \tan\theta = -b\cos\frac{\alpha+\beta}{2}\Big/\left(a\sin\frac{\alpha+\beta}{2}\right)$$

Hence, as

$$\frac{1}{R^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$$

we deduce

$$R^2 = a^2\sin^2\frac{\alpha+\beta}{2} + b^2\cos^2\frac{\alpha+\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, $\delta^2 = (a\cos\alpha - a\cos\beta)^2 + (b\sin\alpha - b\sin\beta)^2$

Reducing this, and using (2), we find

$$\delta = 2R\sin\frac{\alpha-\beta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, (1) touches the confocal

$$x^2/(a^2-\lambda) + y^2/(b^2-\lambda) = 1$$

if

$$\frac{a^2-\lambda}{a^2}\cos^2\frac{\alpha+\beta}{2} + \frac{b^2-\lambda}{b^2}\sin^2\frac{\alpha+\beta}{2} = \cos^2\frac{\alpha-\beta}{2}$$

Using (2), this reduces to

$$\sin^2\frac{\alpha-\beta}{2} = \frac{\lambda R^2}{a^2b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eliminate $\sin(\alpha-\beta)/2$ from (3), (4);

$$\therefore \quad \delta = \frac{2R^2\sqrt{\lambda}}{ab} \qquad \text{Q.E.D.}$$

§ **342.** *Def′*—Points on two ellipses which have the same eccentric angle are called *corresponding points.*

Thus $(a\cos\alpha,\ b\sin\alpha)$ and $(a'\cos\alpha,\ b'\sin\alpha)$ are corresponding points on the ellipses (a, b), (a', b').

Some properties of corresponding points on confocals are given in the Exercises which follow.

Exercises on Chapter XII

1. If tangents are drawn to a confocal system from a point in the major axis, the locus of the points of contact is a circle.

2. Find the confocal hyperbola through the point on the ellipse (a, b) whose eccentric angle is α.

Ans. $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = a^2 - b^2$

3. Prove that corresponding points on a system of confocal ellipses lie on an hyperbola.

4. TP, TQ are tangents to two confocals. If TP, TQ are at right angles, prove that the join of T to the centre bisects PQ.

5. Show that the locus of the intersection of rectangular tangents to the confocal parabolas

$$y^2 = 4\alpha(x + \alpha), \qquad y^2 = 4\beta(x + \beta)$$

is the line

$$x + \alpha + \beta = 0$$

Prove that the parabolas cut at right angles in two points at a finite distance; and that these points are imaginary if α and β have the same sign.

6. Parallel tangents are drawn to a confocal system; prove that the locus of the points of contact is a rectangular hyperbola.

7. Tangents from T to an ellipse meet a confocal in R, R′; S, S′. Prove that

$$\frac{1}{TR} - \frac{1}{TR'} = \frac{1}{TS} - \frac{1}{TS'}$$

[*Note*—$\dfrac{1}{TR} - \dfrac{1}{TR'} = \dfrac{RR'}{TR.TR'}$; use § 341 and *Cor′* (1), § 311.]

8. P and Q are any two points on an ellipse; p and q are the corresponding points on a confocal. Prove that

$$Pq = Qp$$

(*Ivory's Theorem.*)

9. Prove that the equation in elliptic co-ordinates of the director circle of the ellipse (a, b) is

$$a_1^2 + a_2^2 = 2a^2$$

10. Prove that the square of the semi-diameter of the ellipse (a, b) parallel to the tangent at its intersection with a confocal is λ, where λ is the parameter of that confocal.

11. Tangents to the ellipse (a, b) from a point whose elliptic co-ordinates are a_1, a_2 include an angle ϕ: prove that

$$\tan \tfrac{1}{2}\phi = \sqrt{\frac{a^2 - a_2^2}{a_1^2 - a^2}}$$

[*Note*—This follows from Ex. 33, page 239, and foot-note, page 317.]

12. Prove that the equation of the tangents in the last question referred to the normals to the confocals through the point as axes is

$$x^2/(a_1^2 - a^2) + y^2/(a_2^2 - a^2) = 0$$

13. The tangents to two confocals from a point P which describes a confocal ellipse are inclined at angles ψ, ψ' to the tangent at P.

Prove that the ratio $\sin\psi : \sin\psi'$ is constant.

14. If λ, λ' are the parameters of the confocals which pass through any point on a directrix of the ellipse (a, b), prove that

$$\lambda\lambda' = a^2(\lambda + \lambda')$$

15. If a point describe the director circle of the ellipse (a, b) its polar envelopes a confocal.

[*Note*—Write eq'n of polar of (x'y'), viz.

$$xx'/a^2 + yy'/b^2 = 1$$

in the form

$$lx + my = 1 \quad \ldots\ldots\ldots \quad (1)$$

$$\therefore \quad x' = a^2 l, \qquad y' = b^2 m$$

Express now that (x', y') is a point on director circle

$$\therefore \quad a^4 l^2 + b^4 m^2 = a^2 + b^2$$

The line (1) is $\therefore$ a tangent (§ 250) to the ellipse

$$\left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}}\right).]$$

16. P and Q are points on the ellipse (a, b); P', Q' are the corresponding points on the ellipse (a', b'). If the tangents at P', Q meet in T, prove that T is the pole of PQ' with respect to the ellipse ($\sqrt{aa'}$, $\sqrt{bb'}$).

Prove also that the tangents at P', Q are at right angles if the tangents at P, Q' are at right angles.

(*Mr. R. Russell.*)

17. Prove that the join of the points of contact of rectangular tangents to two confocal ellipses (a, b), (a_1, b_1) envelopes the confocal

$$\left(\frac{aa_1}{\sqrt{a^2 + b_1^2}}, \frac{bb_1}{\sqrt{a^2 + b_1^2}}\right).$$

[*Note*—If the conics in Ex. 16 are now assumed to be confocal; then if tangents at P, Q′ are at right angles, T describes the circle

$$x^2 + y^2 = a^2 + b_1^2.$$

Finish solution as in *Note*, Ex. 15.]

18. The equations to the asymptotes of the hyperbola

$$l/r = 1 - e\cos\theta$$

are
$$abe = r\,(a\sin\theta \pm b\cos\theta)$$

19. Two conics have a common focus S; a variable line through S meets the conics in P and Q. Prove that the intersection of the tangents at P and Q describes a straight line.

20. Two conics have a common focus; prove that two of their common chords pass through the intersection of their directrices.

21. A, A′; B, B′; C, C′ are three pairs of opposite summits of a quadrilateral which circumscribes a parabola whose focus is S. Prove that

$$SA \,.\, SA' = SB \,.\, SB' = SC \,.\, SC'$$

22. Two parabolas have a common focus. Show that the locus of the intersection of two tangents, one to each, cutting at a constant angle, is a parabola.

(*Prof′ Purser.*)

23. Prove that the polar equation to the normal to the parabola

$$r = a\sec^2\frac{\theta}{2}$$

at the point α is

$$a = r\cot\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\left(\theta - \frac{\alpha}{2}\right)$$

If θ be the angular co-ordinate of the point at which the normals at α, β, γ intersect, prove that

$$2\theta = \alpha + \beta + \gamma$$

24. A circle is described with the focus S of a conic as centre; a line through S meets the circle in P and the conic in Q.

Prove that the tangents at P and Q intersect on a common chord of the circle and conic.

25. Show that the locus of the extremities of the latera recta of parabolas which have a common focus and a common tangent consists of two circles.

26. Given the focus and directrix of a conic; show that the polar of a given point with respect to it passes through a fixed point.

27. A circle circumscribes the triangle formed by tangents to a parabola at three points A, B, C. If ρ is its radius, prove that

$$2\rho^2 l = SA.SB.SC$$

where S is the focus of the parabola, and $2l$ its latus rectum.

28. Two conics have a common focus; then axes are inclined at an angle β. Show that the conics touch if

$$(l - l')^2 = l^2 e'^2 + l'^2 e^2 - 2ee'll' \cos\beta$$

where e, e′ are the eccentricities, and l, l′ the semi latera recta.

[*Note*—If α is the vect′ ∧ of a point of inters′n of the conics

$$l/r = 1 - e\cos\theta, \qquad l'/r = 1 - e'\cos(\theta + \beta)$$

the tangents at that point are

$$l/r = \cos(\theta - \alpha) - e\cos\theta, \qquad l'/r = \cos(\theta - \alpha) - e'\cos(\theta + \beta)$$

These equations may be written in the form

$$l/r = A\cos\theta + B\sin\theta$$

These tangents ∴ coincide if

$$(\cos\alpha - e)/l = (\cos\alpha - e'\cos\beta)/l'$$

and $$\sin\alpha/l = (\sin\alpha + e'\sin\beta)/l'$$

The result is now obtained by eliminating α.]

29. Two conics have a common focus, about which one is turned; show that the common chords envelope a conic having a focus at the given focus.

(*S. Roberts*, *Educ′ Times*, xxxix.)

30. Two ellipses have a common focus; one revolves about this focus while the other remains fixed. Prove that the locus of the point of intersection of their common tangents is a circle.

CHAPTER XIII

ABRIDGED NOTATION; MISCELLANEOUS PROPOSITIONS

THE STRAIGHT LINE

§ **343.** In §§ 126-130, 139–141, 148, we have given some account of the abridged notation of the straight line; we shall now give some other propositions.

We shall (as agreed on in § 127) use Greek letters α, β, &c. as abbreviations for expressions of the form

$$x \cos \alpha + y \sin \alpha - p;$$

and English letters u, v, w, or L, M, N, for expressions such as

$$Ax + By + C$$

§ **344.** *If* $u = 0$, $v = 0$, $w = 0$ *are the equations to three given straight lines which form a triangle; then the equation*

$$lu + mv + nw = 0$$

may, by giving suitable values to the ratios $l : m : n$, *be made to represent any straight line whatever.*

Let the given lines form a triangle ABC (fig' § 138); and let any other line meet the sides of this triangle in D', E, F.

The line BE passes through B, the intersection of $u = 0$, $w = 0$; it may ∴ be represented (§ 126) by the equation

$$lu + nw = 0$$

The line FE passes through E, the intersection of BE and $v = 0$; it may ∴ be represented by the equation

$$(lu + nw) + mv = 0, \quad \text{or} \quad lu + mv + nw = 0 \qquad \text{Q.E.D.}$$

§ **345.** The preceding general result is of course applicable to the particular case when the equations of the given lines are expressed in the standard form.

Thus $\alpha = 0$, $\beta = 0$, $\gamma = 0$ being the equations to the sides of a fixed triangle in the plane of reference, the equation to any other line may be expressed in the form

$$l\alpha + m\beta + n\gamma = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

This equation expresses a relation between the lengths of the perpendiculars from any point of the line on the sides of the given triangle (which may be called the *triangle of reference*). *Trilinear* co-ordinates are thus suggested; the trilinear co-ordinates of a point are its perpendicular distances from the sides of the triangle of reference. Looking at equation (1) we see that any homogeneous equation of the first degree in trilinear co-ordinates represents a straight line.

§ **346.*** More generally, let the equations to the sides of the triangle of reference be

$$u \equiv Ax + By + C = 0, \qquad v \equiv A'x + B'y + C' = 0,$$
$$w = A''x + B''y + C'' = 0$$

Let u_1, v_1, w_1 denote the values of u, v, w respectively when we substitute x_1, y_1 for x and y; and u_2, v_2, w_2 their values when we substitute x_2, y_2 for x and y.

Then (§ 76)

$$u_1 = \sqrt{A^2 + B^2} \times \text{perpendicular from } (x_1 y_1) \text{ on } u = 0; \text{ \&c.}$$

Thus we may speak of the u, v, w of a point as its co-ordinates; these co-ordinates being constant multiples of its perpendicular distances from the sides of the triangle of reference.

§ **347.** *To find the equation to the join of the points* $(u_1 v_1 w_1)$, $(u_2 v_2 w_2)$.

Let the required equation be

$$lu + mv + nw = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

* The beginner may omit §§ 346–349, 354, 356, 357 until after he has read the early part of Chap. XIV.

Then we must have

$$\left.\begin{aligned} lu_1 + mv_1 + nw_1 = 0 \\ lu_2 + mv_2 + nw_2 = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The eq'ns (2) determine the ratios $l : m : n$; viz.

$$\frac{l}{v_1 w_2 - v_2 w_1} = \frac{m}{w_1 u_2 - w_2 u_1} = \frac{n}{u_1 v_2 - u_2 v_1}$$

If these values are substituted in (1) we obtain the required equation. The result may of course be written

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0$$

§ **348.** It should be noticed that a homogeneous equation

$$lu + mv + nw = 0$$

in reality expresses a relation between *two* of the ratios $u : v : w$; for the equation may be written

$$l\frac{u}{w} + m\frac{v}{w} + n = 0$$

This is unaltered if we substitute ku, kv, kw for u, v, w.

§ **349.** Ex. 1. We may apply these principles to prove the theorem of § 138.

Let the equations to BC, CA, AB and FE be

$$u = 0, \qquad v = 0, \qquad w = 0, \qquad lu + mv + nw = 0$$

Then

$$mv + nw = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the eq'n to a line through the inters'n of $v = 0$, $w = 0$, i. e. through A.

But (1) may be written

$$(lu + mv + nw) - lu = 0$$

The line (1) ∴ passes through the inters'n of

$$lu + mv + nw = 0 \quad \text{and} \quad u = 0$$

i. e. through D′.

Hence (1) is the eq'n to AD′.

Similarly the equations to BE, CF are

$$lu + nw = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$lu + mv = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Subtracting (2) from (3) we see that the equation

$$mv - nw = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

represents a line through the inters'n of BE, CF; i. e. through O.

Hence (4) is the equation to AO.

But the lines (1), (4) form a harmonic pencil with $v = 0$, $w = 0$ [§ 139, *Cor'* (1)].

Ex. 2. ABC, A′B′C′ are two triangles. BC, B′C′ meet in P; CA, C′A′ in Q and AB, A′B′ in R. If P, Q, R are collinear, show that AA′, BB′, CC′ are concurrent.

Let the equations to BC, CA, AB, and PQR be

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 0, \qquad l\alpha + m\beta + n\gamma = 0$$

By suitably choosing l', the equation

$$l'\alpha + m\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

will represent any line through the inters'n of

$$\alpha = 0, \qquad l\alpha + m\beta + n\gamma = 0$$

Hence we may take (1) as the equation to B′C′.

Similarly the eq'ns to C′A′, A′B′ are

$$l\alpha + m'\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$l\alpha + m\beta + n'\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Subtracting in pairs the eq'ns (1), (2), (3) we obtain the equations to AA′, BB′, CC′; viz.

$$(m - m')\beta = (n - n')\gamma, \qquad (n - n')\gamma = (l - l')\alpha,$$
$$(l - l')\alpha = (m - m')\beta$$

The third of these equations is a consequence of the other two; and ∴ the three lines co-intersect.

Exercises

1. What is represented by the equation

$$\alpha + C = 0?$$

Ans. A straight line parallel to $\alpha = 0$.

2. If $u = 0$, $v = 0$ represent parallel straight lines; show that

$$u + v = 0$$

represents a parallel midway between them.

3. If $u = 0$, $v = 0$, $w = 0$ are the eqnations of three parallel straight lines; then the equation

$$lu + mv + nw = 0$$

represents a straight line parallel to them.

4. With the notation of Ex. 1, § 349, find the equations to DF, DE.

Ans. $lu + mv - nw = 0$, $lu - mv + nw = 0$

5. Prove that the lines

$$v = lw, \qquad w = mu, \qquad u = nv$$

co-intersect if

$$lmn = 1$$

6. The joins of the vertices of a triangle ABC to a point O meet the opposite sides in A′, B′, C′: if the equations of BC, CA, AB are

$$u = 0, \qquad v = 0, \qquad w = 0$$

prove that AO, BO, CO may be represented by the equations

$$mv = nw, \qquad nw = lu, \qquad lu = mv$$

Prove also that BC, B′C′; CA, C′A; and AB, A′B′ meet on the line

$$lu + mv + nw = 0$$

[*Note*—Show that equation to B′C′ is

$$-lu + mv + nw = 0; \text{ \&c.}]$$

7. Two triangles are such that the perpendiculars from the vertices of one on the sides of the other are concurrent; prove that the perpendiculars from the vertices of the second on the sides of the first are concurrent.

[*Note*—Let sides be α, β, γ, α', β', γ'. Denote $\wedge$ between α, β by $(\alpha\beta)$. Then eq'n of $\perp$ from $(\alpha\beta)$ on γ' is (§ 128)

$$\alpha \cos(\beta\gamma') - \beta \cos(\alpha\gamma') = 0; \text{ \&c.}$$

The condition of concurrence is found to be

$$\cos(\alpha\beta') \cos(\beta\gamma') \cos(\gamma\alpha') = \cos(\alpha'\beta) \cos(\beta'\gamma) \cos(\gamma'\alpha).]$$

CASES OF $S - \lambda S' = 0$

§ 350. If $S = 0$, $S' = 0$ are the equations of two conics; we have seen (§ 315) that

$$S - \lambda S' = 0$$

represents a conic through their intersections.

We shall now consider some important cases of this equation.

I. Suppose that one of the conics is a line-pair; it follows that—

The equation

$$S - \lambda LM = 0$$

represents a conic passing through the points in which the conic $S = 0$ *is met by the lines* $L = 0$, $M = 0$.

Ex. If TP, TQ, $T'P'$, $T'Q'$ are tangents to a conic, the six points T, P, Q, T', P', Q' lie on a conic.

Let the given conic be

$$x^2/a^2 + y^2/b^2 = 1;$$

let $(x'y')$, $(x''y'')$ be the co-ord's of T, T'.

Then the equation

$$\left(\frac{x'x''}{a^2} + \frac{y'y''}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{x'x}{a^2} + \frac{y'y}{b^2} - 1\right)\left(\frac{x''x}{a^2} + \frac{y''y}{b^2} - 1\right)$$

represents a conic which evidently passes through the points

$$(x = x',\ y = y') \quad \text{and} \quad (x = x'',\ y = y'');$$

and also (by the preceding principle) through the inters'ns of $S = 0$ with their polars.

(Wolstenholme, *Educ' Times*, 6103.)

§ **351.** Again, let P, Q be the points in which the conic $S = 0$ is met by the line $L = 0$, and P', Q' the points in which it is met by the line $M = 0$; and let us now suppose that P, Q move up to coincidence with P', Q' respectively.

Then the chords PP', QQ' ultimately become tangents; and $M = 0$ becomes ultimately the same line as $L = 0$. We see then that—

II. *The equation*

$$S - \lambda L^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

represents a conic touching $S = 0$ *at the points where it is met by the line* $L = 0$.

That is, (1) represents a conic having double contact with $S = 0$, "along the line L."

Note—Instead of (1) we may write

$$S - L^2 = 0;$$

for the multiplier λ may be supposed to be implicitly included in L.

Ex. 1. We may deduce the equation of the tangents from $(x'y')$ to

$$\phi(x, y) = 0$$

Let $$P \equiv ax'x + h(x'y + y'x) + \ldots$$
so that $P = 0$ is the eq'n to the polar of $(x'y')$.

Then $$\phi(x, y) = \lambda P^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
is a conic which has double contact with
$$\phi(x, y) = 0$$
at its intersections with the polar of $(x'y')$; and the two tangents are such a conic determined by the condition that it is to pass through $(x'y')$.

Expressing then that (1) is satisfied by $x = x'$, $y = y'$,
$$\phi(x', y') = \lambda [\phi(x', y')]^2$$

This determines λ; and the required equation is
$$\phi(x', y')\, \phi(x, y) = P^2 \quad \text{(Compare § 324)}$$

§ 352. Ex. 2. *If two conics have double contact with a third, then two of their common chords and the two chords of contact meet in one point and form a harmonic pencil.*

For, subtracting the equations
$$S - L^2 = 0, \qquad S - M^2 = 0$$
we obtain a conic through the intersections of these conics; viz.
$$L^2 - M^2 = 0$$

But this conic is a line-pair. Hence two of the common chords are
$$L + M = 0, \qquad L - M = 0$$

But these lines form a harmonic pencil with
$$L = 0, \qquad M = 0 \quad \text{[§ 139, } Cor' \text{ (1)]}$$

Ex. 3. *The two diagonals of an inscribed quadrilateral, and of the quadrilateral whose sides are the tangents at its vertices, meet in a point and form a harmonic pencil.*

This is a particular case of Ex. 2; the conics
$$S - L^2 = 0, \qquad S - M^2 = 0$$
being now supposed to reduce to line-pairs.

§ 353. Again, as a particular case of II. it follows that—

III. *The equation*
$$LM = R^2$$
represents a conic touching the lines $L = 0$, $M = 0$ *at the points where it is met by the line* $R = 0$.

If we suppose the equations to the lines written in the standard form and interpret the equation

$$\alpha\beta = \lambda\gamma^2$$

we deduce the theorem—

The product of the perpendiculars from any point of a conic on two tangents varies as the square of the perpendicular from the point on the chord of contact.

§ **354.** The equation of a conic referred to two tangents and their chord of contact

$$\mathrm{LM} = \mathrm{R}^2$$

is evidently satisfied by the co-ordinates ($\mathrm{L'M'R'}$) of a point, if these co-ordinates are in the ratios

$$\frac{\mathrm{L'}}{1} = \frac{\mathrm{M'}}{\mu^2} = \frac{\mathrm{R'}}{\mu}$$

This may be called 'the point μ.'

The equation to the join of the points μ, μ' on the conic is (§ 347)

$$\begin{vmatrix} \mathrm{L} & \mathrm{M} & \mathrm{R} \\ 1 & \mu^2 & \mu \\ 1 & \mu'^2 & \mu' \end{vmatrix} = 0$$

This reduces to

$$\mu\mu'\mathrm{L} - (\mu + \mu')\mathrm{R} + \mathrm{M} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Putting $\mu' = \mu$ we get the equation of the tangent at μ, viz.

$$\mu^2\mathrm{L} - 2\mu\mathrm{R} + \mathrm{M} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

§ **355.** Both the conics $\mathrm{S} = 0$, $\mathrm{S'} = 0$ may be line-pairs.

We have then the theorem—

IV. The equation

$$\alpha\gamma = \lambda\beta\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

represents a conic circumscribing the quadrilateral whose sides are

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0$$

In fact the eq'n is evidently satisfied by the co-ord's of the vertices of the quadrilateral, which are the points of inters'n of the lines

$$(\alpha = 0,\ \beta = 0), \quad (\beta = 0,\ \gamma = 0), \quad (\gamma = 0,\ \delta = 0),$$
$$(\delta = 0,\ \alpha = 0).$$

Let these points be A, B, C, D; and let π_1, π_2, π_3, π_4 be the perpendiculars from any point P of the conic on the lines

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0.$$

Then (1) expresses that

$$\frac{\pi_1 \pi_3}{\pi_2 \pi_4} = \text{constant} = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, $\quad 2 \text{ area PAB} = \pi_2 . \text{AB} = \text{PA} . \text{PB} \sin \text{APB}$

$$\therefore \quad \pi_2 = \text{PA} . \text{PB} \sin \text{APB}/\text{AB}$$

Similar expressions may be obtained for π_3, π_4, π_1.

When these values are substituted in (2) it reduces to

$$\frac{\sin \text{APD} . \sin \text{CPB}}{\sin \text{APB} . \sin \text{CPD}} = \lambda$$

Hence (§ 136)—

The joins of a variable point on a conic to four fixed points on the conic form a pencil, whose cross ratio is constant.

§ 356. We may now interpret the equation

$$\lambda \text{L}^2 + \mu \text{M}^2 + \nu \text{N}^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This may be written

$$-\nu \text{N}^2 = (\text{L} \sqrt{\lambda} + \text{M} \sqrt{-\mu})(\text{L} \sqrt{\lambda} - \text{M} \sqrt{-\mu})$$

The lines

$$\text{L} \sqrt{\lambda} + \text{M} \sqrt{-\mu} = 0, \qquad \text{L} \sqrt{\lambda} - \text{M} \sqrt{-\mu} = 0 \quad . \quad . \quad . \quad (2)$$

are $\therefore$ tangents, and $\text{N} = 0$ is their chord of contact (§ 353).

The pole of $\text{N} = 0$ is $\therefore$ the inters'n of the lines (2), i. e. it is the inters'n of $\text{L} = 0$, $\text{M} = 0$.

Similarly with reference to the poles of $\text{L} = 0$, $\text{M} = 0$.

The equation (1) $\therefore$ represents a conic which is such that the lines $\text{L} = 0$, $\text{M} = 0$, $\text{N} = 0$ form a triangle which is self-conjugate with respect to it.

§ 357. If $u = 0$, $v = 0$, $w = 0$ are the equations of the sides of a given triangle, the equation to any conic may be written in the form

$$au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv = 0 \quad . \quad . \quad . \quad (1)$$

For let $(x_1 y_1)$, $(x_2 y_2)$, ... $(x_5 y_5)$ be five points on the conic; then expressing that the co-ord's of these points satisfy (1), and using the notation of § 346, we get five simple equations

$$au_1^2 + bv_1^2 + \ldots = 0, \qquad au_2^2 + \ldots = 0, \; . \; . \; . \; .$$

These equations suffice to determine uniquely the values of the five ratios

$$a/h, \quad b/h, \quad c/h, \quad f/h, \quad g/h. \qquad \text{Q.E.D.}$$

Cor′—Suppose that the conic (1) circumscribes the triangle

$$(u = 0,\ v = 0,\ w = 0)$$

If $(x_1 y_1)$ are the co-ord′s of the inters′n of $u = 0$, $v = 0$, the values of u, v, w for this point are $0, 0, w_1$. As these values satisfy (1) we must have

$$cw_1{}^2 = 0; \qquad \therefore \quad c = 0$$

Similarly $$a = 0, \qquad b = 0$$

The general equation of a conic circumscribing the triangle

$$(u = 0,\ v = 0,\ w = 0)$$

is $\therefore$ $$fvw + gwu + huv = 0$$

Exercises

1. An ellipse touches the asymptotes of an hyperbola; prove that two of the common chords of the ellipse and hyperbola are parallel.

2. Find the equation of an ellipse passing through the centre of the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

and touching it at two adjacent extremities A, B of its axes.

Ans. $b^2x^2 + abxy + a^2y^2 = ab\,(bx + ay)$

3. Find the circle having double contact with the same ellipse at the ends of its latus rectum.

Ans. $x^2 + y^2 - 2\,ae^3x = a^2(1 - e^2 - e^4)$

4. A circle has double contact with an ellipse; prove that the chord of contact is parallel to one of its axes.

5. A circle has double contact with an ellipse at the extremities of a parallel to the minor axis; prove that the tangent to the circle from any point on the ellipse is to the distance of the point from the chord of contact in the constant ratio $e : 1$.

6. Two circles have double contact with an ellipse, the chords of contact being parallel; prove that the sum or difference of the tangents drawn to the circles from any point on the ellipse is constant.

7. A, B, C, D are given points on a conic. On a line which moves parallel to itself and cuts AB, CD in P, Q and the conic in P′, Q′ a point O is taken such that

$$OP.\,OQ = \lambda\, OP'.\,OQ'$$

where λ is constant; prove that the locus of O is a conic passing through A, B, C, D.

[*Note*—Let $S = 0$, $S' = 0$ be eq'ns of line-pair AB, CD and given conic; use § 312.]

THE FOCOIDS; RELATIONS OF CONICS TO THE LINE AT INFINITY

§ 358. Since parallels meet at infinity (§ 132) lines parallel to

$$y = x\sqrt{-1}$$

pass through a fixed point on the line at infinity.

The fixed points in which the line at infinity is met by the lines

$$y - x\sqrt{-1} = 0, \quad y + x\sqrt{-1} = 0 \quad . \quad . \quad . \quad (1)$$

are called the *circular points at infinity*.

The shorter term *focoids* has been suggested by Dr. C. Taylor.

These points are also often referred to as the points I, J (Salmon, *Higher Plane Curves*).

The lines (1) have been called the *isotropic lines* through the origin; they are evidently parallel to the isotropic lines through any other point $(x' y')$, viz.

$$y - y' = \pm (x - x')\sqrt{-1}$$

We shall now prove some useful properties of the focoids.

§ 359. *All circles pass through the focoids.*

For the equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

may be written

$$(y + x\sqrt{-1})(y - x\sqrt{-1})$$
$$= (-2gx - 2fy - c)(0 \cdot x + 0 \cdot y + 1)$$

It follows (§ 355) that the intersections of

$$y \pm x\sqrt{-1} = 0 \quad \text{and} \quad 0 \cdot x + 0 \cdot y + 1 = 0$$

are points on the circle.

§ **360.** *Every right angle is divided harmonically by the isotropic lines through its vertex.*

Take the arms OA, OB of a right angle AOB as axes; then the equations of the line-pairs (OA, OB) and (OI, OJ) are

$$xy = 0 \quad \text{and} \quad x^2 + y^2 = 0$$

These form a harmonic pencil (§ 141).

§ **361.*** If the axes of co-ordinates are oblique, the circle

$$x^2 + 2\,xy\cos\omega + y^2 + 2\,gx + 2\,fy + c = 0$$

meets the line at infinity on the lines

$$x^2 + 2\,xy\cos\omega + y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence equation (1), or its factors

$$y + x\,(\cos\omega + \sqrt{-1}\sin\omega) = 0, \quad y + x\,(\cos\omega - \sqrt{-1}\sin\omega) = 0 \quad . \quad (2)$$

represent the isotropic lines through the origin.

§ **362.** We can determine the cross ratio ϕ in which any angle is divided by the isotropic lines through its vertex.

Take the arms of the angle as axes; then the rays of the pencil are $x = 0$, $y = 0$ and the lines given by eq'n (2) of the last §.

Hence (§ 139)

$$\phi = (\cos\omega + \sqrt{-1}\sin\omega)\big/(\cos\omega - \sqrt{-1}\sin\omega)$$

$$\therefore \quad \phi = \cos 2\,\omega + \sqrt{-1}\sin 2\,\omega$$

§ **363.** *The tangents to a conic from a focus are isotropic lines.*

Take the focus as origin; let the eccentricity be e and

$$x\cos\alpha + y\sin\alpha - p = 0$$

the equation of the directrix. Then the conic is

$$x^2 + y^2 = e^2(x\cos\alpha + y\sin\alpha - p)^2$$

Factorize the sinister of this equation; it follows (§ 353) that

$$x + y\sqrt{-1} = 0, \quad x - y\sqrt{-1} = 0$$

are tangents whose chord of contact is

$$x\cos\alpha + y\sin\alpha - p = 0$$

Thus the isotropic lines through the focus are tangents whose chord of contact is the directrix.

* The beginner may omit §§ 361, 362, 366, 367, 369, 371, 372.

§ **364.** *All parabolas touch the line at infinity.*

For the equation of any parabola may be written

$$LM = R^2$$

where

$$L \equiv lx + my + n, \qquad M \equiv 0.x + 0.y + 1, \qquad R \equiv \alpha x + \beta y$$

The tangents at the ends of the chord $R = 0$ are $\therefore$ (§ 353) $L = 0$ and $M = 0$; the latter being the line at infinity.

DETERMINATION OF FOCI

§ **365.** *A central conic has two imaginary foci on its minor axis.*

The foci of the conic

$$x^2/a^2 + y^2/b^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

may be determined by the process of § 287.

Suppose then that the preceding eq'n (1) is equivalent to eq'n (2) of § 287.

Compare coeff's of xy; $\therefore \cos\alpha \sin\alpha = 0$

$$\therefore \text{ either } \cos\alpha = 0 \quad \text{or} \quad \sin\alpha = 0$$

Let $\cos\alpha = 0$; then comparing coeff's of x we get $x_1 = 0$.

Equation (2) of § 287 becomes now

$$x^2 + (y - y_1)^2 = e^2 (y - p)^2 \quad . \quad . \quad . \quad . \quad (2)$$

Compare corresponding coeff's in (1) and (2).

$$\therefore \quad a^2 = b^2(1 - e^2) = e^2p^2 - y_1^2 \quad \text{and} \quad y_1 = pe^2$$

From these equations we deduce

$$y_1 = \pm \sqrt{b^2 - a^2}, \quad e = y_1/b$$

Thus the points $(0, \pm \sqrt{b^2 - a^2})$ are foci.

The other alternative $\sin\alpha = 0$ gives the known foci

$$(\pm \sqrt{a^2 - b^2}, 0).$$

If $- b^2$ is written instead of b^2, this proof answers for the hyperbola.

§ **366.** *A parabola has three foci at infinity.*

This follows from §§ 365, 263.

§ **367.** From § 363 we deduce the following general conception of foci. Draw tangents from the focoids I, J: their intersections form a quadrilateral, two opposite vertices S, S′ of which are the real foci and the other two σ, σ' are the imaginary foci of the conic.

Thus the conic is inscribed in the quadrilateral $S\,\sigma\,S'\,\sigma'$.

Cor′—A system of confocals is inscribed in a fixed quadrilateral $S\,\sigma\,S'\,\sigma'$.

§ **368.** *To find the foci of the conic*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

If $(x'y')$ is a focus, the isotropic line through it

$$y - y' = (x - x')\sqrt{-1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is a tangent (§ 363).

We must then substitute $-\sqrt{-1}$, 1, $x'\sqrt{-1} - y'$ for λ, μ, ν in the condition of § 323.

Suppressing the accents, this gives

$$-A + B + C(-x^2 + y^2 - 2xy\sqrt{-1}) + 2F(x\sqrt{-1} - y)$$
$$+ 2G(x + y\sqrt{-1}) - 2H\sqrt{-1} = 0 \quad . \quad . \quad . \quad (2)$$

The equation of the other isotropic line through $(x'y')$ is obtained by changing $\sqrt{-1}$ into $-\sqrt{-1}$ in (1); the condition that it should be a tangent differs ∴ from (2) only in the sign of $\sqrt{-1}$. We may ∴ equate to zero the real and imaginary parts of the sinister of (2).

The foci are ∴ determined as the intersections of the two loci

$$C(x^2 - y^2) + 2Fy - 2Gx + A - B = 0,$$
$$Cxy - Fx - Gy + H = 0$$

These equations represent equilateral hyperbolas, unless $C = 0$. In the latter case the conic is a parabola; and the preceding equations represent straight lines.

§ **369.** It may be deduced from the last § that the equations to determine the foci are

$$\left.\begin{aligned}(\mathrm{C}x - \mathrm{G})^2 - (\mathrm{C}y - \mathrm{F})^2 &= \Delta\,(a - b)\\ (\mathrm{C}x - \mathrm{G})(\mathrm{C}y - \mathrm{F}) &= \Delta\,h\end{aligned}\right\}$$

where Δ is the discriminant. The reader will verify this without difficulty.

§ **370.** The process of § 368 is equivalent to this rule—

To find the foci of a conic whose equation is given :

Determine c so that

$$y = mx + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $m = \sqrt{-1}$ may be a tangent ; then equate real and imaginary parts on both sides of (1).

§ **371.** If the axes are oblique the same rule is applicable; but the value of m is (§ 361)

$$-(\cos\omega + \sqrt{-1}\sin\omega)$$

Ex. Find the focus of the parabola

$$\sqrt{x/\alpha} + \sqrt{y/\beta} = 1$$

Using the condition of § 318, we find that

$$y = -x(\cos\omega + \sqrt{-1}\sin\omega) + c \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is a tangent if

$$c = \alpha\beta\big/(\alpha + \beta\cos\omega - \beta\sqrt{-1}\sin\omega)$$

Substitute this value of c in (2) and equate real and imaginary parts. This gives the co-ord's of the focus

$$x = \alpha\beta^2\big/(\alpha^2 + 2\,\alpha\beta\cos\omega + \beta^2), \qquad y = \alpha^2\beta\big/(\alpha^2 + 2\,\alpha\beta\cos\omega + \beta^2)$$

[Compare the solution, *Note*, Ex. 18, page 303.]

§ **372.** We shall describe another method of solving the problem of § 368.

(1) *To find the equation of the tangents to* $S = 0$ *which can be drawn parallel to a given line* $y = mx$.

These tangents are drawn from the point at infinity on the given line, viz. $(\lambda, m\lambda)$ where $\lambda = \infty$.

Substitute $\therefore$ λ, $m\lambda$ for x', y' in the equation of § 324 ; and neglect the terms which do not contain λ^2. We can then divide out by λ^2 ; and the required equation is

$$(a + 2\,hm + bm^2)\,\phi(x, y) = [ax + hy + g + m(hx + by + f)]^2 \quad . \quad . \quad (1)$$

(2) The tangents from the focoids are derived from this by writing $\pm\sqrt{-1}$ for m; and as the foci lie on the tangents from both focoids it follows that their co-ordinates satisfy the equations obtained by writing $\pm\sqrt{-1}$ for m in (1) and then equating real and imaginary parts on both sides of the equation.

The equations to determine the foci are $\therefore$

$$\left.\begin{array}{r}(ax + hy + g)^2 - (hx + by + f)^2 = (a - b)\,\phi(x, y) \\ (ax + hy + g)(hx + by + f) = h\,\phi(x, y)\end{array}\right\}$$

Exercises

1. Find the real foci of the conic

$$20x^2 - 32xy + 20y^2 = 9$$

Ans. (1, 1), (− 1, − 1)

2. Find the real foci of the conic

$$32x^2 - 24xy - 20x + 12y + 11 = 0$$

Ans. (0, − 1), (1, 2)

3. Prove that the foci of the conic

$$x^2/a^2 + y^2/b^2 = 1$$

are the intersections of the loci

$$xy = 0, \qquad x^2 - y^2 = a^2 - b^2$$

Prove also that the latter rectangular hyperbola is the locus of a point such that the tangents drawn from it to the conic are equally inclined to the bisectors of the angles between the axes.

4. Prove that the foci of the conic

$$ax^2 + 2hxy + by^2 = 1$$

are determined by the equations

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} = \frac{1}{h^2 - ab}$$

5. The real foci of an ellipse are S, S'; its imaginary foci are σ, σ'. If P is any point, prove that

$$PS\,.\,PS' = P\sigma\,.\,P\sigma'$$

6. If the conic

$$ax^2 + 2hxy + by^2 = 2y$$

has a given centre $(x'y')$; the locus of its foci is the hyperbola

$$xy = x'y + y'x$$

NORMALS

§ **373**. The normals at $(x'y')$ to the conic

$$x^2/a^2 + y^2/b^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

will pass through (hk), [§ 245, *Cor'* (2)] if

$$(h - x')\Big/\frac{x'}{a^2} = (k - y')\Big/\frac{y'}{b^2}$$

or

$$(a^2 - b^2)x'y' + kb^2x' - ha^2y' = 0$$

The feet of the normals from (hk) are ∴ the inters'ns of the hyperbola

$$(a^2 - b^2)xy + kb^2x - ha^2y = 0 \quad . \quad . \quad . \quad (2)$$

with the given conic (1). The hyperbola (2) is rectangular; and it passes through the centre and the given point (hk).

Again, let two of the common chords of the conics (1), (2) be

$$lx + my = 1, \qquad l'x + m'y = 1$$

Then (§ 350, I.) the equation

$$x^2/a^2 + y^2/b^2 - 1 - \lambda(lx + my - 1)(l'x + m'y - 1) = 0 \quad . \quad . \quad (3)$$

will, if λ is suitably chosen, represent the same locus as (2).

As the constant term, and the terms in x^2, y^2 are absent in (2), we must have, if eq'n (3) is identical with (2),

$$\lambda = -1; \quad \text{and} \quad ll'a^2 = mm'b^2 = -1$$

It follows that the lines

$$lx + my = 1, \qquad \frac{x}{a^2 l} + \frac{y}{b^2 m} + 1 = 0$$

meet the conic in points the normals at which are concurrent.

§ **374**. The normals which pass through (hk) may also be found thus.

The ecc' ∧s $\alpha, \beta, \gamma, \delta$ of their feet are [§ 246, eq'n (2)] the roots of the equation in ϕ

$$ah/\cos\phi - bk/\sin\phi = a^2 - b^2 \quad . \quad . \quad . \quad (1)$$

Let

$$\mu = \cos\phi + \sqrt{-1}\sin\phi$$

$$\therefore \quad 2\cos\phi = \mu + 1/\mu, \quad 2\sqrt{-1}\sin\phi = \mu - 1/\mu$$

Substituting these values in (1) we obtain a biquadratic in μ:

$$(a^2 - b^2)(\mu^4 - 1) - 2(ah - bk\sqrt{-1})\mu^3 - 2(ah + bk\sqrt{-1})\mu = 0$$

If the roots of this are $\mu_1, \mu_2, \mu_3, \mu_4$ then

$$\mu_1\mu_2\mu_3\mu_4 = -1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\Sigma\mu_r\mu_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2)

$$\cos(\alpha + \beta + \gamma + \delta) + \sqrt{-1}\sin(\alpha + \beta + \gamma + \delta) = -1$$

$$\therefore \quad \alpha + \beta + \gamma + \delta = (2n + 1)\pi \quad . \quad . \quad . \quad (4)$$

Again, using (2), (3) becomes

$$\mu_1\mu_2 - \frac{1}{\mu_1\mu_2} + \mu_2\mu_3 - \frac{1}{\mu_2\mu_3} + \mu_3\mu_1 - \frac{1}{\mu_3\mu_1} = 0$$

$$\therefore \quad \sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0 \quad . \quad . \quad (5)$$

The relations (4), (5) are often useful.

§ **375.** The feet of the normals from (hk) to the parabola

$$y^2 - 4ax = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

are (§ 236) its intersections with the hyperbola

$$2a(k - y) + y(h - x) = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Ex. Find the circle through the feet of the normals from (hk).

Multiply (1) by $2y - k$, and (2) by $2x$ and subtract; the resulting equation evidently represents a locus passing through all the intersections of (1), (2).

But this equation breaks up into factors, one of which

$$y = 0$$

does not pass through the feet of the normals which are at a finite distance. The other factor

$$2(x^2 + y^2) - 2x(h + 2a) - ky = 0$$

is the circle required.

Note—The proper multipliers $2y - k$ and $2x$ could be found by trial; or by assuming expressions with undetermined coefficients.

CURVATURE

§ **376.** Two conics intersect (§ 314) in four points.

If two of these points coincide the curves touch.

If three of them coincide the contact is closer. It is called *contact of the second order*; and each of the curves is in this case said to *osculate* the other. Since three points determine a circle, there is a circle which osculates a conic at any point. This circle is called the *circle of curvature* at the point. If A is the point of contact the circle will meet the conic in another distinct point D: the chord AD is called the *chord of curvature.*

If all four of the points of intersection of two conics coincide the contact is the closest possible: it is called *contact of the third order.*

§ **377.** If a circle meets a conic in the four points A, B, C, D; then if we suppose that B, C coincide with A it follows from *Cor′* (3), § 311, that the chord AD and the tangent at A are equally inclined to an axis.

Thus *the chord of curvature and the tangent at any point are equally inclined to an axis of the conic.*

§ **378.** If α, β, γ, δ are the ecc′ ∧s of the intersections of a circle and a conic, then (§ 311, Ex. 1)

$$\alpha + \beta + \gamma + \delta = 2n\pi$$

If the points β, γ coincide with α we have ∴

$$3\alpha + \delta = 2n\pi$$

This equation determines the eccentric angle of the point in which the circle of curvature at α meets the conic again.

Ex. If δ is given there are three values of α, viz.

$$(2\pi - \delta)/3, \quad (4\pi - \delta)/3, \quad (6\pi - \delta)/3$$

Thus there are three points on the conic the circles of curvature at which pass through the point δ; and since the sum of δ and the three values of α is 4π, the three points lie on a circle through δ.

It is easily verified that the centroid of the triangle formed by the three points coincides with the centre of the conic.

§ 379. Let $L = 0$ be the equation of the tangent at $(x'y')$ to a conic $S = 0$. The intersections of this conic with the conic

$$S - LM = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

are (§ 350, I.) the points in which S is met by the lines L, M; and as two of these points coincide, (1) is the equation of a conic touching S at $(x'y')$.

Again, if $$M \equiv lx + my - lx' - my'$$

i. e. if $M = 0$ is a line through the point of contact $(x'y')$, then three intersections of the conics coincide with $(x'y')$.

The equation of a conic osculating S at $(x'y')$ is ∴

$$S - L(lx + my - lx' - my') = 0 \quad . \quad . \quad . \quad (2)$$

If further the line M coincide with L, then all four intersections of the conics coincide with $(x'y')$; the equation of a conic having contact of the third order with S at $(x'y')$ is ∴

$$S - \lambda L^2 = 0$$

Ex. 1. The equation of a conic osculating the conic

$$ax^2 + 2hxy + by^2 = 2y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at the origin is

$$ax^2 + 2hxy + by^2 - 2y = y(lx + my) \quad . \quad . \quad . \quad . \quad (4)$$

The conic (4) is a circle if

$$a = b - m \quad \text{and} \quad 2h - l = 2a\cos\omega$$

The circle of curvature of the conic (3) at the origin is ∴

$$a(x^2 + 2xy\cos\omega + y^2) = 2y$$

Ex. 2. The conic

$$ax^2 + 2hxy + by^2 - 2y - \lambda y^2 = 0$$

which has contact of the third order at the origin with (3), is a rectangular hyperbola if

$$a + b - \lambda = 2h\cos\omega$$

The equation to the rectangular hyperbola having four-pointic contact with the conic (3) at the origin is ∴

$$ax^2 + 2hxy + (2h\cos\omega - a)y^2 = 2y$$

Exercises

1. Find the circle of curvature at the point $(x'y')$ on the rectangular hyperbola

$$x^2 - y^2 = a^2$$

Ans. $a^2(x^2 + y^2) - 4x'^3x + 4y'^3y + 3a^2(x'^2 + y'^2) = 0$

2. Find the equation of the chord of curvature through the point on an ellipse whose eccentric angle is α; the ellipse being referred to its axes.

Ans. $bx \cos\alpha - ay \sin\alpha = ab \cos 2\alpha$

3. The radius of curvature at any point P on an ellipse is b'^2/p, where b' is the semi-diameter parallel to the tangent at P and p is the central perpendicular on this tangent.

4. Find the co-ordinates of the centre of curvature corresponding to a point $(x'y')$ on an ellipse referred to its axes.

Ans. $[(a^2 - b^2)x'^3/a^4, \ -(a^2 - b^2)y'^3/b^4]$

5. At the intersection of two confocals the centre of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.

SIMILAR CONICS

§ **380.** If two lines which are always inclined at the same angle revolve round two fixed points O, O'; and if P, P' are points on the lines such that

$$O'P' = \lambda OP$$

where λ is constant, the points P, P' describe curves, one of which is a magnified representation of the other.

Two such curves are said to be *similar*.

To any point Ω rigidly connected with the first curve there corresponds a point Ω' similarly connected with the second curve. For taking a point Ω' such that

$$P\hat{O}\Omega = P'\hat{O}'\Omega' \quad \text{and} \quad O'\Omega' = \lambda O\Omega:$$

it follows by Euclid VI. 6, that

$$\Omega'P' = \lambda\Omega P;$$

and the lines ΩP, $\Omega' P'$ are inclined at the same angle as OP, $O'P'$. The curves might $\therefore$ be similarly generated by radii vectores revolving round Ω, Ω'.

If the revolving lines OP, $O'P'$ are parallel, the two curves are said to be *similar and similarly placed.* They are also said to be *homothetic.* If O' coincide with O, and the revolving lines are also coincident, the point O is a *centre of similitude.*

§ **381.** The polar equations of two parabolas referred to their vertices are

$$r \sin^2 \theta = 4 a \cos \theta, \quad r \sin^2 \theta = 4 a' \cos \theta$$

If $\therefore$ ρ, ρ' are radii vectores inclined to the respective axes at the same angle θ,

$$\rho / \rho' = a / a' = \text{constant.}$$

Hence *all parabolas are similar.*

§ **382.** Let the equations of two central conics referred to their respective axes of figure be

$$x^2/a^2 - y^2/b^2 = 1, \quad x^2/a'^2 - y^2/b'^2 = 1$$

If then ρ, ρ' are radii vectores inclined to the transverse axes at the same angle θ,

$$\rho'^2/\rho^2 = \left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) \Big/ \left(\frac{\cos^2\theta}{a'^2} - \frac{\sin^2\theta}{b'^2}\right)$$

This ratio is independent of θ if

$$a/b = a'/b'$$

Hence *two conics are similar if the ratio of the axes is the same for both.*

Cor' (1)—*Similar conics have the same eccentricity.*

For $\quad e^2 = 1 + b^2/a^2, \quad e'^2 = 1 + b'^2/a'^2; \therefore e = e'$

Cor' (2)—*The asymptotes of similar conics include the same angle.*

For this angle $= 2 \tan^{-1} b/a = 2 \tan^{-1} b'/a'$

Cor′ (3)—*The asymptotes of homothetic conics are parallel.*

For an asymptote is inclined to the transverse axis at an angle

$$\tan^{-1} b/a;$$

and the axes of the conics are parallel and proportional; ∴ &c.

§ **383.** If two conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$a'x^2 + 2h'xy + \dots = 0$$

are similar they have the same eccentricity [*Cor′* (1), § 382].

$$\therefore \frac{(a-b)^2 + 4h^2}{(a+b)^2} = \frac{(a'-b')^2 + 4h'^2}{(a'+b')^2} \quad \text{[Ex. 20, p. 304.]}$$

The condition that the two conics may be similar is ∴

$$\frac{ab - h^2}{(a+b)^2} = \frac{a'b' - h'^2}{(a'+b')^2}$$

Note—This condition may be obtained otherwise. The line-pairs through the origin parallel to the asymptotes, viz.

$$ax^2 + 2hxy + by^2 = 0, \qquad a'x^2 + 2h'xy + b'y^2 = 0$$

include equal angles; this gives at once (§ 114) the required condition.

§ **384.** If two conics are homothetic, the line-pairs through the origin parallel to their asymptotes are coincident.

The conditions ∴ that two conics

$$ax^2 + 2hxy + \dots = 0, \qquad a'x^2 + 2h'xy + \dots = 0$$

may be homothetic are

$$a/a' = h/h' = b/b'$$

Cor′—The equations of two homothetic conics may be so written that the terms in x^2, xy, y^2 are the same in both.

§ **385.** Corresponding radii vectores of two conjugate hyperbolas are (§ 277) in the constant ratio $1 : \sqrt{-1}$; these curves are not of the same shape.

That two similar curves may have the same shape, the ratio $1 : \lambda$ of corresponding radii vectores must be real.

§ **386.** The equations of two homothetic conics are

$$S = 0, \qquad S' \equiv S + lx + my + n = 0 \quad [Cor', \S\ 384.]$$

$$\therefore \quad S' \equiv S + (lx + my + n)(0 \cdot x + 0 \cdot y + 1)$$

Hence (§ 350, II.) *two homothetic conics have the line at infinity for a common chord.*

Note—This proposition is evident when the curves are hyperbolas. The reader will see this on drawing a figure: the curves continually approach the asymptotes; and since the asymptotes are parallel they meet at a point on the line at infinity. The result of § 359 is a particular case of this proposition.

§ **387.** If two conics are homothetic and concentric, then taking the centre as origin their equations are

$$S \equiv ax^2 + 2hxy + by^2 + c = 0, \qquad S' \equiv ax^2 + 2hxy + by^2 + c' = 0$$

$$\therefore \quad S' \equiv S + (0 \cdot x + 0 \cdot y + 1)^2 (c' - c)$$

Hence (§ 351, II.) *two homothetic and concentric conics have double contact along the line at infinity.*

Note—As the curves have the same asymptotes the proposition is evident when the curves are hyperbolas; the reader will see this by drawing a figure.

Cor'—Concentric circles have double contact along the line at infinity.

Exercises

1. A chord of an ellipse which touches a concentric and homothetic ellipse is bisected at the point of contact.

2. A point P describes a conic; O is a fixed point. Prove that the locus of the mid point of OP is an homothetic conic.

3. Two homothetic and concentric conics intercept equal segments on any line.

4. TP, TQ are tangents to an ellipse whose centre is C; prove that an homothetic ellipse can be drawn through the points T, P, Q, C.

INVARIANTS

§ **388.** Some problems are readily solved by using the invariants.

Ex. Find the equation of the equi-conjugate diameters of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Let $(\xi\eta)$ be the co-ord's of a point referred to the axes of the conic.

Equating two expressions for the distance of a point from the centre,

$$\xi^2 + \eta^2 \equiv (x - G/C)^2 + (y - F/C)^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also (see § 292)

$$S \equiv \alpha\xi^2 + \beta\eta^2 + \Delta/(ab - h^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$a + b = \alpha + \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$ab - h^2 = \alpha\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now the equation of the equi-conjugates is

$$\alpha\xi^2 - \beta\eta^2 = 0$$

This may be written

$$(\alpha + \beta)(\alpha\xi^2 + \beta\eta^2) = 2\alpha\beta(\xi^2 + \eta^2)$$

Substituting from (1), (2), (3), (4) we obtain the req'd eq'n, viz.

$$(a + b)[S - \Delta/(ab - h^2)] = 2(ab - h^2)[(x - G/C)^2 + (y - F/C)^2]$$

ONE-ONE CORRESPONDENCE

§ 389. If two variables x, x' are so related that to any value of either corresponds one and only one value of the other we may assume that they are connected by a relation of the form

$$xx' + \mu x + \nu x' + \rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This principle is often useful.

Ex. TP, TQ are tangents to a conic; any tangent meets TP in L and a parallel tangent meets TQ in M. Prove that TL . TM is constant.

(*Prof' Genese.*)

It is evident that each of the points L, M determines the other uniquely. The lengths TL, TM have ∴ a (1, 1) correspondence; and if $TL = x$, $TM = x'$ we may assume that x and x' are connected by the relation (1).

But when L is at T, M is at infinity; ∴ $x' = \infty$ when $x = 0$.

Substituting these values in (1), $\nu = 0$; similarly $\mu = 0$.

Thus (1) becomes

$$xx' + \rho = 0$$

Exercises

1. PP′ is a given diameter of a conic; Q is any point on the curve. If PP′, QP, QP′ meet a fixed parallel to the tangent at P in O, p, p′, prove that

$$Op \,.\, Op'$$

is constant.

2. An ellipse is inscribed in a parallelogram; any tangent meets the sides opposite to an angle L in P, Q. Prove that the area of the triangle LPQ is constant. (*Prof' Genese.*)

§ 390. If two variable points P, Q on a given line have a (1, 1) correspondence; their distances x, x' from a fixed point O on the line are connected by a relation of the form

$$xx' + \mu x + \nu x' + \rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If further P and Q are interchangeable, then $\mu = \nu$;

$$\therefore \quad xx' + \mu(x + x') + \rho = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad (x + \mu)(x' + \mu) = \text{constant}$$

Hence if M be a point on the line such that $OM = -\mu$,

$$MP \, . \, MQ = \text{constant}$$

Thus the points P, Q determine an involution whose centre is M.

Cor'—Put $x = x'$ in (2); the foci of the involution are $\therefore$ determined by the eq'n

$$x^2 + 2\mu x + \rho = 0$$

§ 391. *A system of conics passing through four fixed points determines an involution on any transversal.* (*Desargues' Theorem.*)

For let one of the conics meet the transversal in P, P′. Then P, P′ have a (1, 1) correspondence and are interchangeable; $\therefore$ &c.

The theorem may also be proved thus.

Take the transversal as axis of x; let two of the conics be $S = 0$, $S' = 0$.

The points in which the conic

$$S + kS' = 0$$

meets the axis of x are determined by the equation

$$ax^2 + 2gx + c + k(a'x^2 + 2g'x + c') = 0 \quad . \quad . \quad . \quad . \quad (1)$$

But the points determined by

$$ax^2 + 2gx + c = 0, \qquad a'x^2 + 2g'x + c' = 0$$

are in involution with the points (1). This is proved as in § 148.

Cor' (1)—The foci of the involution are the points of contact of the transversal with the two conics of the system which touch it.

Cor' (2)—As the line-pairs through the four fixed points are conics of the system, we see that—

The sides and diagonals of a quadrilateral determine an involution on any transversal.

PLANE PROJECTION

§ 392. The principle of § 242 may be extended to points not on the ellipse: thus to any point $P\ (x_1 y_1)$ in the plane of the ellipse corresponds a point P', viz. $(x_1, \lambda y_1)$ where $\lambda = a/b$.

It is convenient to call P' the *projection* of P; if P describe a straight line or curve, P' describes a straight line or curve which is the *projection* of the locus of P.

Thus the projection of the curve $\phi(x, y) = 0$ is the curve

$$\phi(x, y/\lambda) = 0$$

This use of the term projection was suggested by Dr. C. Taylor; its justification is that the theorems thereby deducible are identical with those obtained by the method of orthogonal projection. (See Taylor's *Geometry of Conics*, § 90.)

§ 393. If P', Q', R' are the projections of P, Q, R, then

$$\Delta\, P'Q'R' = \lambda\, \Delta\, PQR.$$

For

$$\begin{vmatrix} x_1 & \lambda y_1 & 1 \\ x_2 & \lambda y_2 & 1 \\ x_3 & \lambda y_3 & 1 \end{vmatrix} = \lambda \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It follows that the proj'ns P', Q', R' of three collinear points P, Q, R are collinear; or *the projection of a straight line is a straight line.*

§ 394. Let P, Q be two points on a curve; P', Q' their projections.

Then P' coincides with Q' if P coincides with Q.

Hence the projection of the tangent at P is the tangent at P'.

§ 395. We have seen [*Cor'* (1), § 256] that conjugate diameters of the ellipse project into rectangular diameters of the auxiliary circle.

§ 396. If we have four points P, Q, R, S whose co-ord's are $(x_1 y_1)$, $(x_2 y_2)$, ... : then PQ is $\parallel$ RS if

$$(y_1 - y_2)/(x_1 - x_2) = (y_3 - y_4)/(x_3 - x_4) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and in that case

$$PQ/RS = (y_1 - y_2)/(y_3 - y_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But the cond'n (1) and the dexter of eq'n (2) are unaltered if we change y_1, y_2, ... into λy_1, λy_2, ...

Hence *parallel lines project into parallel lines;* and *the ratio of segments measured on the same line or on parallel lines is unaltered by projection.*

Ex. We shall apply these principles to prove the theorem, Ex. 39, page 239.

Let P, Q be the points whose ecc′ $\wedge$s are α, β; Cp the semi-diam′ parallel to PQ.

Let P′, Q′, p′ be the projections of P, Q, p.

Then
$$PQ/Cp = P'Q'/Cp' = P'Q'/a$$

But
$$P'Q' = 2\,CP' \sin\tfrac{1}{2}\,P'CQ' = 2\,a \sin\tfrac{1}{2}\,(\alpha - \beta)$$
$$\therefore\quad PQ = 2\,b' \sin\tfrac{1}{2}\,(\alpha - \beta)$$

Exercises

1. Deduce from properties of the circle the theorems, Ex. 20, 25, page 237; Ex. 46, page 240; Ex. 52, 53, page 241.

2. Show that the projection of a parabola is a parabola.

3. A triangle is inscribed in the ellipse (a, b); b′, b″, b‴ are the semi-diameters parallel to the sides. If R is the circum-radius of the triangle, prove that
$$R = b'b''b'''/(ab)$$

Miscellaneous Exercises

1. A circle intercepts segments of given length on two fixed lines: show that the locus of its centre is a rectangular hyperbola.

2. The tangents to a parabola at P, Q meet in T; any other tangent meets TP, TQ in p, q. Find the locus of the intersection of Pq, Qp.

Ans. An ellipse touching TP, TQ at P, Q.

3. P, Q are two points on an hyperbola; parallels through P, Q to the asymptotes meet in T. If PQ passes through a fixed point, prove that the locus of T is an homothetic hyperbola.

4. A parabola whose latus rectum is $4a$ slides between two rectangular axes; find the curve traced by its focus.

Ans. $x^2 y^2 = a^2 (x^2 + y^2)$

[*Note*—Find lengths of $\perp$s on

$$y = mx + a/m, \qquad y = -x/m - am$$

from (a, o); call these lengths x and y. Then eliminate m.]

5. Find also the curve traced by the vertex.

Ans. $x^{\frac{4}{3}} y^{\frac{2}{3}} + x^{\frac{2}{3}} y^{\frac{4}{3}} = a^2$

6. If one of the common chords of a circle and a rectangular hyperbola is a diameter of the circle; prove that another common chord is a diameter of the hyperbola.

7. Prove that if a parabola be described with a point on an ellipse as focus, and the tangent at the corresponding point on the auxiliary circle as directrix, it passes through the foci of the ellipse.

(*Prof' J. Purser.*)

8. A series of parabolas whose axes are parallel have a common tangent at a given point. Prove that if parallel tangents are drawn to the parabolas the points of contact lie on a straight line through the given point.

9. If p, p' are the perpendiculars from the foci of an ellipse upon any chord, R the semi-diameter parallel to the chord, prove that

$$\text{length of chord} = \frac{2R^2}{a}\sqrt{1 - \frac{pp'}{b^2}}$$

10. CP, CD are conjugate radii of an ellipse, AA', BB' the axes.

Prove (1) that $$PA \cdot PA' = DB \cdot DB';$$

and (2) the bisectors of the angles APA', BDB' are at right angles.

(*Prof' Genese.*)

11. Given a focus, a tangent and the eccentricity of a conic; prove that the locus of the other focus is a circle.

12. Tangents from any point T meet an ellipse in P and Q, so that, S being a focus, $SP \cdot SQ \propto ST^2$. Prove that the locus of T is a similar ellipse.

13. Given a focus and two tangents to a conic; prove that the chord of contact passes through a fixed point.

14. If a parabola always touches a given straight line and has double contact with a fixed circle, then the chord of contact passes through a fixed point.

15. Show that the normals at the points of intersection of the ellipse (a, b) with the polars of the points $(x'y')$, $(x''y'')$ are concurrent if

$$x'x''/a^2 = y'y''/b^2 = -1$$

16. The normals at the points P, Q, R, S on the ellipse (a, b) are concurrent. Prove that two parabolas can be drawn through P, Q, R, S, and that the angle between their axes is

$$2 \tan^{-1} b/a.$$

17. Four normals to an ellipse are concurrent. Prove that perpendiculars to these normals from an extremity of the axis major meet the ellipse in four points which lie on a circle.

[*Note*—The ecc′ $\wedge$s of the extremities of the chord $\perp$ normal at the point whose ecc′ $\wedge$ is α are 0, 2α; use eq′n (4), § 374 and § 311, Ex. 1.]

18. Show that the equation of the normal to the conic

$$xy = k^2$$

at the point $(k\lambda, k/\lambda)$ is

$$(\lambda^3 - \lambda \cos \omega)\, x + (\lambda^3 \cos \omega - \lambda)\, y = k\,(\lambda^4 - 1)$$

where ω is the angle between the axes of co-ordinates.

If the normals at the points P, Q, R of the above conic meet at a point on the curve, prove that the centroid of the triangle PQR lies upon the conic

$$9\,xy = k^2 \cos^2 \omega$$

19. If the normals to a conic at L, M, N, R meet in O, and S is a focus, then

$$e^2\, SL \,.\, SM \,.\, SN \,.\, SR = b^2 \,.\, SO^2$$

(*Prof′ Burnside, Educ′ Times*, 1708.)

20. Show that the product of the three normals that can be drawn from the point $(x_1 y_1)$ on the ellipse (a, b) is

$$2\, ab\, (a^2 - e^2 x_1^2)^{\frac{3}{2}}/(a^2 - b^2)$$

21. The sum of the squares of the four normals from a point to an ellipse is constant; prove that the locus of the point is a conic.

22. If the circle of curvature of the ellipse (a, b) at the point whose eccentric angle is ϕ passes through the centre of the ellipse, prove that

$$\tan^2 \phi = (a^2 - 2b^2)/(2a^2 - b^2)$$

23. If ρ, ρ' are the radii of curvature at the extremities of two conjugate diameters of the ellipse (a, b), then

$$\rho^{\frac{2}{3}} + \rho'^{\frac{2}{3}} = (a^2 + b^2)/(ab)^{\frac{2}{3}}$$ (*Prof' Curtis.*)

24. Any chord of curvature of a parabola is divided by the axis in the ratio 1 : 3. (*Prof' Curtis.*)

[*Note*—The sum of the ordinates y_1, y_2, y_3, y_4 of the intersections of a circle with the parabola is zero; put

$$y_2 = y_3 = y_4, \quad \&c.]$$

25. Find the equation of the parabola which meets the ellipse (a, b) in four coincident points (x′y′); and show that the equation of its axis is

$$x/x' - y/y' = (a^2 - b^2)/(x'^2 + y'^2)$$

26. Prove that the centres of curvature at the points $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ on the parabola $y^2 = 4ax$ are collinear if

$$y_1 y_2 + y_2 y_3 + y_3 y_1 = 0$$

27. From O, the centre of curvature at any point on the ellipse (a, b), the other normals OQ, OR are drawn to the ellipse; prove that the locus of the mid point of QR is

$$(x^2/a^2 + y^2/b^2)^3 = x^2 y^2/(a^2 b^2)$$

28. Show that the latus rectum of the parabola which has contact of the third order at P with an ellipse, whose centre is C and axes 2 a, 2 b, is

$$2a^2 b^2/CP^3$$

29. The locus of the pole of a given straight line with respect to one of a given system of co-axal circles is a hyperbola whose asymptotes cut the line of centres in points equidistant from the radical axis, and which becomes two straight lines, if the given line passes through one of the limiting points.

30. Two circles have double internal contact with an ellipse, and a third circle passes through the four points of contact. If t, t′, T be the tangents from any point on the ellipse to these circles, then

$$T^2 = tt'$$

(*Prof' Crofton, Educ' Times*, 1994.)

31. Show that a tangent to the conic

$$x^2/(b + b') + y^2/(a + a') = 1/(ab' + a'b)$$

is cut harmonically by the conics

$$ax^2 + by^2 = 1, \qquad a'x^2 + b'y^2 = 1$$

32. If the normals to the conic

$$l/r = 1 - e\cos\theta$$

at the points 2α, 2β, 2γ meet on the curve, then

$$(1 + e)^2 \cos(\alpha + \beta + \gamma) = 2e \cos\alpha \cos\beta \cos\gamma$$

33. Prove that all conics through the extremities of the principal axes of the ellipse (a, b) are cut orthogonally by the hyperbola

$$x^2/a^2 - y^2/b^2 = (a^2 - b^2)/(a^2 + b^2)$$

(*Prof' Crofton, Educ' Times*, 1602.)

34. A series of parabolas have AB, the hypotenuse of a given right-angled triangle ABC, for a common chord, whilst their axes are parallel to the side AC. Prove that their foci all lie on a hyperbola of which A and B are the foci; and show that if the triangle is isosceles, the hyperbola will be equilateral.

35. The centre of a conic and two tangents are given; prove that the locus of the foci is a rectangular hyperbola.

36. The rectangle under the tangents from $(x_1\, y_1)$ to the ellipse (a, b) is

$$\frac{S_1}{S_1 + 1} \sqrt{(x^2 - y^2 - a^2 + b^2)^2 + 4x^2y^2}$$

where
$$S_1 \equiv x_1^2/a^2 + y_1^2/b^2 - 1$$

Hence find the locus of a point such that the rectangle under the tangents from it to a given conic is equal to the rectangle under the tangents to a given confocal conic. (*Prof' J. Purser.*)

37. From a fixed point O a tangent OT is drawn to one of a system of confocal conics, and a point P is taken on the tangent, such that $OP.\,OT$ is constant; prove that the locus of P is a rectangular hyperbola.

(*Prof' J. Purser.*)

38. PQ is the chord of contact of tangents OP, OQ to a conic whose centre is C, and foci S, S′, and OC cuts PQ in V; prove that

$$OP.OQ : OS.OS' = OV : OC$$

39. A circle passes through the focus of a parabola and cuts the parabola at angles whose sum is constant. Prove that the locus of its centre is a straight line.

40. Two fixed tangents OP, OQ to a parabola are met by a variable tangent in P and Q; prove that the locus of the circum-centre of the triangle OPQ is a straight line.

41. Prove that normals drawn to the ellipse (a, b) from a point on either of the straight lines

$$a^2x^2 - b^2y^2 = 0$$

meet the curve in points, a pair of whose connectors are parallel.

42. To a rectangular hyperbola with centre C and focus S normals are drawn from a point P. Show that, if these normals make angles θ_1, θ_2, ... with one of the asymptotes,

$$\Sigma \operatorname{cosec} 2\theta = CP^2/CS^2$$

43. If from any point on a given normal to a conic three other normals be drawn; prove that the circle through their feet belongs to a fixed co-axal system. (*Mr. F. Purser.*)

44. Through a fixed point O within an ellipse is drawn any pair of conjugate lines, and parallel to these are drawn a pair of tangents meeting them in R and R′ and one another in T. Prove that the locus of T is an ellipse, and that the locus of R and R′ is a similar ellipse which has double contact with the given ellipse at the extremities of the diameter through O.

45. If the line joining P, the intersection of two tangents to an ellipse to the intersection of the corresponding normals is cut by the axes of the ellipse in a constant anharmonic ratio, the locus of P is a concentric conic. (*Prof′ Purser.*)

46. If three sides of a quadrilateral inscribed in a conic pass through three fixed points in a straight line, prove that the fourth side passes through a fixed point in the same straight line.

[*Note*—The given line is cut by the conic and sides of quad′ in six points in involution (§ 391); and as five of these points are fixed, so is the sixth.]

47. If

$$\overline{12} \equiv \frac{x}{a} \cos \tfrac{1}{2} (\alpha_1 + \alpha_2) + \frac{y}{b} \sin \tfrac{1}{2} (\alpha_1 + \alpha_2) - \cos \tfrac{1}{2} (\alpha_1 - \alpha_2), \quad \&c.$$

prove that the equation

$$\overline{12} \cdot \overline{34} \cdot \overline{56} - \overline{23} \cdot \overline{45} \cdot \overline{61} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

includes as part of its locus the ellipse (a, b). What is the remaining part of the locus? (*Prof' Purser.*)

[*Note*—We find that the sinister of (1) vanishes identically if we substitute $a \cos \theta$, $b \sin \theta$ for x, y; we infer that

$$x^2/a^2 + y^2/b^2 - 1$$

is a factor. The other part of the locus (1) is a straight line through the points

$$(\overline{12}, \overline{45}), \quad (\overline{34}, \overline{61}), \quad (\overline{56}, \overline{23})$$

This affords a proof of Pascal's Theorem (see § 431).]

48. Interpret the equation

$$(\alpha + l\delta)(\beta + m\delta)(\gamma + n\delta) = (\alpha - l\delta)(\beta - m\delta)(\gamma - n\delta)$$

Ans. The equation represents the line $\delta = 0$ and the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta + lmn\delta^2 = 0$$

This conic passes through the six points

$$(\alpha + l\delta = 0, \quad \beta - m\delta = 0), \ldots$$

49. TP, TQ are tangents to a conic. The circle through T, P, Q cuts the conic again in P′, Q′; tangents at P′, Q′ meet in T′. Show that CT, CT′ are equally inclined to the axis, and that

$$CT \,.\, CT' = CS^2,$$

C being the centre, and S a focus of the conic.

(*Leudesdorf, Educ' Times*, 6103.)

[*Note*—If T is $(x'y')$, and T $(x''y'')$, the equation of the conic through T, P, Q, T′, P′, Q′ is given, Ex., § 350. Express cond'ns that this eq'n represents a circle.]

50. TP, TQ are tangents to a parabola; the circle through T, P, Q meets the parabola again in P′, Q′, and the tangents at P′, Q′ meet in T′. Prove that TT′ passes through the focus.

51. T and T′ are points external to a parabola; TP, TQ, T′P′, T′Q′ are tangents to the parabola. If TT′ is bisected by the parabola, prove that the six points T, P, Q, T′, P′, Q′ lie on another parabola.

52. A conic is drawn through a point P and the feet of the normals from it to an ellipse. Show that its centre is the centroid of the points in which a circle of any radius meets the ellipse, the centre of the circle being at P.

53. PQ, P′Q′ are chords of a conic, normals at P and P′; the tangents at P and P′ meet in T, and the tangents at Q and Q′ meet in T′. Prove that if T describes a straight line, so does T′.

54. If the axis of a parabola which touches two fixed lines passes through a fixed point, prove that the focus lies on a rectangular hyperbola.

55. If the chords PQ, P′Q′ of a conic are such that each contains the pole of the other, prove that the points P, Q, P′, Q′ subtend a harmonic pencil at every point of the conic.

[*Note*—Let tangents at P, Q meet in T; let PQ meet P′Q′ in T′. Take P for the fifth point on the conic; then one ray of the pencil is the tangent PT.

Then $\{P.PP'QQ'\} = \{TP'T'Q'\} = -1$ (§ 308).]

56. If the origin is within the quadrilateral formed by the lines α, β, γ, δ, determine the value of k in order that the equation

$$\alpha\beta = k\gamma\delta$$

may represent a circle.

Ans. $k = -1$

57. A line through the focus of a conic meets the tangents at the ends of the transverse axis in P, Q: prove that the circle whose diameter is PQ has double contact with the conic. (*Prof′ Genese.*)

58. The directrices are common chords of a conic and its director circle. (*Prof′ Genese.*)

59. A circle cuts a rectangular hyperbola in P, Q, R, S. Prove that the join of their centres is bisected by the centroid of P, Q, R, S. (*F. D. Thomson, Educ′ Times*, 5995.)

60. If five points lie on a circle, radius r, prove that the centres of the five rectangular hyperbolas which pass through them, taken four and four together, lie on another circle whose radius is $r/2$. (*J. Griffiths, Educ′ Times Reprint*, V., page 56.)

61. An ellipse has double contact with each of two concentric circles; show that the loci of its centre and its foci are circles.

62. Two supplemental chords of an ellipse PQ, $P'Q$ meet the tangents at P' and P in T', T respectively; prove that

$$PT \,.\, P'T'$$

is constant.

63. A conic is drawn passing through two given points and having double contact with a given conic; prove that the chords of contact pass through one or other of two fixed points.

64. A point describes a straight line; prove that the locus of the intersection of its polars with respect to two fixed conics is a third conic.

65. If the common tangents of the curves

$$y^2 = 4ax, \qquad x^2 + (y - b)^2 = r^2$$

make angles α, β, γ, δ with the axis of x, prove that

$$\Sigma\,(\tan\alpha) = 0$$

66. Show that the equation

$$c\tan^2\alpha = r\,(\sec\alpha + \cos\theta)$$

where α is a variable parameter, represents a system of confocals.

67. A fixed tangent to an ellipse meets the major axis in T; Q and Q' are two points on the tangent equidistant from T. Show that the second tangents which can be drawn from Q, Q' to the ellipse meet on a fixed straight line parallel to the major axis.

68. A variable circle whose diameter is D passes through a fixed point O and intersects a given conic in L, M, N, R: prove that

$$OL \,.\, OM \,.\, ON \,.\, OR \propto D^2$$

69. Two concentric circles are cut by a conic in the points M, N, P, Q and M', N', P', Q' respectively: show that a conic can be described through M, N, P, Q whose asymptotes are $M'N'$, $P'Q'$.

70. A line through P cuts a conic in Q, Q'. If N is the foot of the perpendicular from P on the polar of P, prove that NQ, NQ' make equal angles with NP.

71. An ellipse of semi-axes a, b slides between two rectangular axes. Prove that the equation to the locus of its real foci is

$$y^2(x^2 - b^2)^2 + x^2(y^2 - b^2)^2 + 2xy\,(x^2 - b^2)(y^2 - b^2) = 4x^2y^2(a^2 - b^2)$$

72. PP' is a diameter of a conic. If from points on a fixed chord through P' tangents be drawn meeting the tangent at P in R, R': prove that

$$PR + PR'$$

is constant.

73. A system of conics passes through four fixed points, of which P, Q are two; PX, QY are two fixed lines meeting any conic of the system in X, Y. Prove that XY passes through a fixed point.

74. The polar of a point P with respect to an ellipse always touches a fixed circle whose centre is on the major axis, and which passes through the centre of the ellipse. Prove that the locus of P is a parabola whose latus rectum is a third proportional to the diameter of the circle and the latus rectum of the ellipse.

75. Prove that the polar of the origin with respect to the general conic $S = 0$ is a normal if

$$f^2(f^2 - bc) + 2fg(fg - hc) + g^2(g^2 - ac) = 0$$

76. Prove that the two conics

$$ax^2 + 2hxy + by^2 = 1, \qquad a'x^2 + 2h'xy + b'y^2 = 1$$

may be placed so as to be confocal if

$$\frac{(a-b)^2 + 4h^2}{(ab-h^2)^2} = \frac{(a'-b')^2 + 4h'^2}{(a'b'-h'^2)^2}$$

77. O is a given point on a conic; OP, OQ are any chords through O. Parallels through P, Q to OQ, OP meet the conic in R, S. Prove that RS is parallel to the tangent at O.

78. Two similar conics have a common focus. Prove that a pair of their common chords intersect at right angles.

79. Through a fixed point O on a conic a chord OP is drawn; the circle whose diameter is OP cuts the curve again in Q, Q'. Prove that QQ' passes through a fixed point.

80. With the notation of Ex. 22, page 305, the equation of the joins of the extremities of the axes of $S = 0$ is

$$2h(u^2 - v^2) - 2(a-b)uv = \pm (ab - h^2)^{\frac{1}{2}} \left\{(a-b)^2 + 4h^2\right\}^{\frac{1}{2}}.S$$

81. Prove that the locus of the extremities of the principal axes of conics through the four points $(\pm a, 0)$, $(0, \pm b)$ is the curve

$$(x^2/a^2 - y^2/b^2)(x^2 + y^2) = x^2 - y^2$$

82. Prove that the product of the latera recta of the two parabolas which can be described through four concyclic points is

$$\tfrac{1}{8}(d_1^2 - d_2^2)\sin\omega$$

where d_1, d_2 are the diagonals of the quadrilateral; ω the angle between them.

83. Prove that an ellipse which has double contact with two fixed confocals has a fixed director circle.

84. Tangents are drawn to the parabola

$$2a = r(1 + \cos\theta)$$

at the points 2α, 2β, 2γ, 2δ: circles are described about the triangles formed by the tangents taken three at a time: and circles are described through the centres of these circles taken four at a time: show that the five centres of these circles lie on the circle

$$4r\cos\alpha\cos\beta\cos\gamma\cos\delta = a\cos(\theta - \alpha - \beta - \gamma - \delta)$$

85. The third diagonal of a quadrilateral inscribed in a circle and circumscribed to a parabola passes through the focus of the parabola.

86. Four rectangular hyperbolas osculate each other at a certain point. Prove that any other conic osculating these at the same point is cut by them in four other points which have a fixed anharmonic ratio.

(*Prof' Curtis.*)

87. P is a point on a rectangular hyperbola whose centre is C. A circle is described with centre P and radius PC. Prove that an infinite number of triangles can be inscribed in the circle whose sides touch the hyperbola.

88. An infinite number of triangles can be inscribed in the ellipse (a, b) and circumscribed to the ellipse (a′, b′) provided

$$a'/a \pm b'/b \pm 1 = 0$$

(*Wolstenholme*, *Educ' Times Reprint*, Vol. XV., p. 43.)

CHAPTER XIV

TRILINEAR CO-ORDINATES

§ 397. In the plane of reference take a fixed triangle; this is called the *triangle of reference*. As in Trigonometry, its sides are denoted by a, b, c and its angles by A, B, C. We shall denote its area by D, and the radius of its circum-circle by R. The *trilinear co-ordinates* α, β, γ of a point P are its perpendicular distances from the sides of the triangle of reference: α is positive or negative according as P is on the same side of BC as A, or on the opposite side; and there is a similar agreement with respect to the signs of β, γ.

If P is within the triangle ABC,

$$\triangle PBC + \triangle PCA + \triangle PAB = \triangle ABC$$

$$\therefore \quad a\alpha + b\beta + c\gamma = 2D \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By examining figures in which P is outside the triangle ABC, it is seen as a consequence of the agreement as to the signs of α, β, γ, that the trilinear co-ordinates of any point whatever are connected by the relation (1).

The relation (1) may be written in the form

$$\alpha \sin A + \beta \sin B + \gamma \sin C = D/R = \text{constant} \quad . \quad . \quad (2)$$

Note (1)—Any equation may be rendered homogeneous by means of (1). For example, the equation

$$\alpha\beta^2 = k\gamma$$

becomes

$$4D^2\alpha\beta^2 = k\gamma(a\alpha + b\beta + c\gamma)^2$$

Note (2)—The equations used in trilinears are always homogeneous. We are $\therefore$ in general concerned only with the *ratios* $\alpha : \beta : \gamma$.

Ex. The equations to the perpendiculars of the triangle ABC are (§ 128)

$$\alpha \cos A = \beta \cos B, \quad \beta \cos B = \gamma \cos C, \quad \gamma \cos C = \alpha \cos A$$

These lines obviously co-intersect in the point

$$\alpha / \sec A = \beta / \sec B = \gamma / \sec C \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations give the ratios of the co-ordinates of the ortho-centre.

Again, by a familiar algebraic theorem, each of the fractions in (3)

$$= (a\alpha + b\beta + c\gamma)/(a \sec A + b \sec B + c \sec C)$$

$$= 2D/(a \sec A + b \sec B + c \sec C) = l \text{ say};$$

the trilinear co-ord's of the ortho-centre are $\therefore$ $l \sec A$, $l \sec B$, $l \sec C$.

§ 398. We have seen in § 345 that trilinear co-ordinates are suggested by the methods of abridged notation.

It is easy to transform from trilinear to Cartesian co-ordinates.

Thus, taking the origin of rectangular axes inside the triangle of reference, so that the equations of its sides are

$$\alpha \equiv x \cos \alpha + y \sin \alpha - p = 0, \quad \beta \equiv \ldots = 0, \quad \gamma \equiv \ldots = 0;$$

then since the trilinear co-ordinates of a point inside the triangle are positive, we have the following equations connecting the trilinear co-ordinates α, β, γ of any point with its Cartesian co-ordinates x, y:

$$\left.\begin{aligned} \alpha &= p - x \cos \alpha - y \sin \alpha \\ \beta &= p' - x \cos \beta - y \sin \beta \\ \gamma &= p'' - x \cos \gamma - y \sin \gamma \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Further, since α, β, γ in the dexters of (1) are the angles made by perpendiculars on the sides of the triangle ABC with a fixed line, these angles being measured by the amount of rotation in a definite direction from that line; it will be seen from a figure that we may take

$$\beta - \alpha = \pi - C, \quad \gamma - \beta = \pi - A, \quad \alpha - \gamma = -(\pi + B)$$

Cor'—

$$\cos(\beta - \gamma) = -\cos A, \quad \cos(\gamma - \alpha) = -\cos B, \quad \cos(\alpha - \beta) = -\cos C$$

§ 399. There is another method of transforming to Cartesians.

Let (x, y) be the co-ord's of the point (α, β, γ) referred to two sides CA, CB of the triangle of reference as axes. We see at once from a figure that

$$\alpha = x \sin C, \quad \beta = y \sin C$$

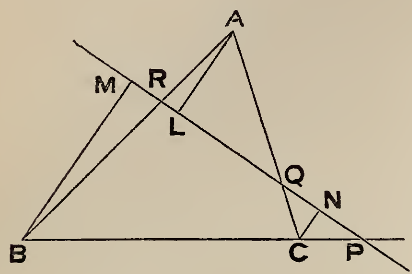
It follows then from eq'n (2), § 397, that

$$\gamma = D/(R \sin C) - x \sin A - y \sin B$$

§ **400.** The general equation to a straight line in trilinears is (§ 345)

$$l\alpha + m\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is easy to assign the geometric meanings of the ratios $l : m : n$.



Let the line (1) meet the sides of the triangle of reference in P, Q, R; let the ⊥s AL, BM, CN on the line be denoted by p, q, r.

We shall assume that ⊥s from points on different sides of the line have opposite signs; thus the signs of q, r in our figure are opposite to that of p.

The co-ord's of R are determined by

$$\gamma = 0, \qquad l\alpha + m\beta + n\gamma = 0$$

$$\therefore \quad l/m = -\beta/\alpha = -(\text{RA} \sin \text{A})/(\text{BR} \sin \text{B}) = (\text{AL} \sin \text{A})/(\text{BM} \sin \text{B});$$

$$\therefore \quad l/m = ap/(bq);$$

and a similar value is found for n/m.

The equation to a line in terms of the perpendiculars p, q, r on the line from the vertices of the triangle of reference is ∴

$$ap\alpha + bq\beta + cr\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Cor'—The ratios of the ⊥s from A, B, C on a line very distant from the triangle of reference are nearly = 1; hence for the line at infinity (see page 107)

$$p = q = r$$

The equation to the line at infinity is ∴

$$a\alpha + b\beta + c\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$\alpha \sin \text{A} + \beta \sin \text{B} + \gamma \sin \text{C} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

§ **401.** *To find the equation to the join of two points* $(\alpha_1 \beta_1 \gamma_1)$, $(\alpha_2 \beta_2 \gamma_2)$.

Let the required equation be

$$l\alpha + m\beta + n\gamma = 0$$

Then

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0, \qquad l\alpha_2 + m\beta_2 + n\gamma_2 = 0$$

Eliminating l, m, n we obtain (as in § 347) the required equation, viz.

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0$$

Cor′—The condition that three points $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$ may be collinear is

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0$$

§ **402.** The co-ordinates of the point of intersection of two lines

$$l\alpha + m\beta + n\gamma = 0, \qquad l'\alpha + m'\beta + n'\gamma = 0$$

are determined (see § 61) by the equations

$$\frac{\alpha}{mn' - m'n} = \frac{\beta}{nl' - n'l} = \frac{\gamma}{lm' - l'm}$$

These equations give the ratios of the α, β, γ of the point of intersection; their actual values may (if required) be deduced as in Ex., § 397.

§ **403.** *To find the condition that three lines*

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0, \quad l''\alpha + m''\beta + n''\gamma = 0$$

may be concurrent.

Eliminating α, β, γ we obtain the required condition, viz.

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = 0$$

§ **404.** It follows from eq′n (2), § 397, that the equation of a line parallel to

$$l\alpha + m\beta + n\gamma = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

is $l\alpha + m\beta + n\gamma = k(\alpha \sin A + \beta \sin B + \gamma \sin C)$

Cor′—The equation of the line through $(\alpha_1\beta_1\gamma_1)$ parallel to (1) is

$$\frac{l\alpha + m\beta + n\gamma}{l\alpha_1 + m\beta_1 + n\gamma_1} = \frac{\alpha \sin A + \beta \sin B + \gamma \sin C}{\alpha_1 \sin A + \beta_1 \sin B + \gamma_1 \sin C}$$

§ 405. Suppose that the third line in § 403 is the line at infinity. This gives (§ 132) the condition that the lines

$$l\alpha + m\beta + n\gamma = 0, \qquad l'\alpha + m'\beta + n'\gamma = 0$$

may be parallel; viz.
$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ \sin A & \sin B & \sin C \end{vmatrix} = 0$$

§ 406. *To find the length of the perpendicular from* $(\alpha_1\beta_1\gamma_1)$ *on the line*

$$l\alpha + m\beta + n\gamma = 0$$

Transforming to Cartesians by the substitutions of § 398, eq'n (1), the equation of the line becomes

$$(l\cos\alpha + m\cos\beta + n\cos\gamma)x$$
$$+ (l\sin\alpha + m\sin\beta + n\sin\gamma)y - lp - mp' - np'' = 0$$

The result of substituting the Cartesian co-ordinates x_1, y_1 of the point $(\alpha_1\beta_1\gamma_1)$ for x, y in the sinister of the preceding equation is evidently

$$l\alpha_1 + m\beta_1 + n\gamma_1$$

The length of the $\perp$ is (§ 76) the quotient of this by the square root of

$$(l\cos\alpha + m\cos\beta + n\cos\gamma)^2 + (l\sin\alpha + m\sin\beta + n\sin\gamma)^2$$

This expression reduces (*Cor'*, § 398) to

$$l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$

The length of the perpendicular is $\therefore$

$$\frac{l\alpha_1 + m\beta_1 + n\gamma_1}{\sqrt{l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C}} \quad . \quad . \quad (1)$$

This result should be remembered.

§ 407. *To find the condition of perpendicularity of two lines*

$$l\alpha + m\beta + n\gamma = 0, \qquad l'\alpha + m'\beta + n'\gamma = 0$$

Transform to Cartesians as in § 406; the lines are at right angles [§ 68, *Cor'* (3)] if

$$(l\cos\alpha + m\cos\beta + n\cos\gamma)(l'\cos\alpha + m'\cos\beta + n'\cos\gamma)$$
$$+ (l\sin\alpha + m\sin\beta + n\sin\gamma)(l'\sin\alpha + m'\sin\beta + n'\sin\gamma) = 0$$

Using the results of *Cor'*, § 398, this reduces to

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B$$
$$- (lm' + l'm)\cos C = 0$$

This is the required condition; it should be remembered.

§ **408.** If δ is the distance between the points $(\alpha_1 \beta_1 \gamma_1)$, $(\alpha_2 \beta_2 \gamma_2)$ it follows from § 14 and § 399 that

$$\delta^2 \sin^2 C = (\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + 2(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)\cos C$$

We may deduce a symmetrical expression for δ.

By § 397, $\quad a(\alpha_1 - \alpha_2) + b(\beta_1 - \beta_2) + c(\gamma_1 - \gamma_2) = 0$

$$\therefore \quad a(\alpha_1 - \alpha_2)^2 = -b(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) - c(\alpha_1 - \alpha_2)(\gamma_1 - \gamma_2)$$

Hence $(\alpha_1 - \alpha_2)^2$, and similarly $(\beta_1 - \beta_2)^2$ may be expressed in terms of the binary products of the differences

$$\alpha_1 - \alpha_2, \qquad \beta_1 - \beta_2, \qquad \gamma_1 - \gamma_2$$

We may $\therefore$ assume

$$\delta^2 = \lambda(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) + \mu(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2) \\ + \nu(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) \quad . \quad . \quad . \quad (1)$$

where λ, μ, ν are constants.

Substitute in (1) the co-ord's of the points B, C whose distance is a;

$$\therefore \quad a^2 = \lambda(2D/b)(-2D/c)$$

$$\therefore \quad \lambda = -a^2 bc/(4D^2);$$

and μ, ν are similarly obtained.

We have $\therefore$

$$\delta^2 = -\frac{abc}{4D^2}[a(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) + b(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2) \\ + c(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)] \quad . \quad . \quad . \quad (2)$$

AREAL CO-ORDINATES

§ **409.** It is sometimes convenient to use the following system of co-ordinates.

If ABC is the triangle of reference, the *areal co-ordinates* x, y, z of a point P are the ratios of the triangles PBC, PCA, PAB to the triangle ABC. They are connected by the relation

$$x + y + z = 1$$

If x, y, z are the areal co-ordinates of any point, and α, β, γ its trilinear co-ordinates, we have the relations

$$x = a\alpha/(2D), \qquad y = b\beta/(2D), \qquad z = c\gamma/(2D)$$

Ex. Substituting the values of α, β, γ deduced from these relations in the trilinear equation of the circum-circle of the triangle of reference (§ 424), its equation in areal co-ordinates is found to be

$$a^2 yz + b^2 zx + c^2 xy = 0$$

§ **410.** It follows from eq'n (2), § 400, that the equation to a straight line in areal co-ordinates is

$$px + qy + rz = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p, q, r are the perpendiculars from A, B, C on the line.

The length of the perpendicular from the point whose areal co-ordinates are x', y', z' *on the line* (1) *is*

$$px' + qy' + rz'$$

The denominator of the expression (1), § 406, is a constant independent of $\alpha_1, \beta_1, \gamma_1$; we infer that the length of the perpendicular from $(x'\, y'\, z')$ on the line (1) must be

$$\lambda\,(px' + qy' + rz')$$

where λ is a constant independent of x', y', z'.

The length of the $\perp$ from A, (1, 0, 0) is $\therefore\ \lambda p$; but this length is p; $\therefore\ \lambda = 1$, and $\therefore$ &c.

§ **411.** Since the ratios of the $\perp$s from A, B, C on the line at infinity are each $= 1$; the equation of the line at infinity in areal co-ordinates is

$$x + y + z = 0$$

§ **412.** *If the areal co-ordinates of the vertices of a triangle are* $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$, $(x_3\, y_3\, z_3)$ *its area is*

$$D \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The area vanishes only when the three points are collinear; the condition for this is that the determinant should vanish. We may $\therefore$ assume that the area is λ times the determinant where λ is constant.

But taking the vertices of the triangle of reference, (1, 0, 0), (0, 1, 0), (0, 0, 1) as the three points, we find $\lambda = D$; and $\therefore$ &c.

§ **413.** *To find the relation connecting the perpendiculars* p, q, r *from* A, B, C *on any line.*

The trilinear equation of the line is (§ 400)

$$ap\,\alpha + bq\,\beta + cr\,\gamma = 0$$

By § 406 the length of the $\perp$ from A, $(2\,D/a, 0, 0)$ is

$$2\,p\,D\Big/\sqrt{a^2 p^2 + b^2 q^2 + \&c.}$$

Put this $= p$; this gives the required relation, viz.

$$a^2 p^2 + b^2 q^2 + c^2 r^2 - 2\,bcqr\cos A - 2\,carp\cos B - 2\,abpq\cos C = 4\,D^2$$

Exercises

[The triangle ABC in these questions is the triangle of reference.]

1. Find the trilinear co-ordinates of the mid point L of BC, and of the circum-centre.

Ans. $(0, \text{D}/\text{b}, \text{D}/\text{c})$; $(\text{R}\cos \text{A}, \text{R}\cos \text{B}, \text{R}\cos \text{C})$

2. Find the equation of the median AL.

Ans. $\text{b}\beta = \text{c}\gamma$

3. Prove that the three medians co-intersect in the point

$$\alpha \sin \text{A} = \beta \sin \text{B} = \gamma \sin \text{C}$$

4. The internal bisectors of the angles co-intersect in the point

$$\alpha = \beta = \gamma = \text{D}/\text{s}$$

[*Note*—The bisectors are

$$\beta - \gamma = 0, \qquad \gamma - \alpha = 0, \qquad \alpha - \beta = 0.]$$

5. The external bisectors of the angles B, C meet on the internal bisector of the angle A.

[*Note*—The bisectors are

$$\gamma + \alpha = 0, \qquad \alpha + \beta = 0, \qquad \beta - \gamma = 0.]$$

6. Find the equation to the join of the in-centre and circum-centre; and show that it is perpendicular to the line

$$\alpha + \beta + \gamma = 0$$

Ans. $\alpha(\cos \text{B} - \cos \text{C}) + \beta(\cos \text{C} - \cos \text{A}) + \gamma(\cos \text{A} - \cos \text{B}) = 0$

7. Find the equation of the parallel to BC through A.

Ans. $\text{b}\beta + \text{c}\gamma = 0$

8. Find the equation to the join of the mid points of AB, AC.

Ans. $\text{b}\beta + \text{c}\gamma - \text{a}\alpha = 0$

9. The joins of the vertices of ABC to any point O meet the opposite sides in A′, B′, C′. BC, B′C′ meet in A″; CA, C′A′ in B″; and AB, A′B′ in C″. Prove that A″, B″, C″ are collinear.

10. What is the equation of the line A″B″C″ in the last question, 1° if O is the in-centre; 2° if it is the ortho-centre?

Ans. $\alpha + \beta + \gamma = 0$; $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$

11. Find the equation of the line bisecting BC at right angles.

Ans. $\beta \sin B - \gamma \sin C + \alpha \sin (B - C) = 0$

12. The in-circle touches BC in D; find the equation of the join of the mid point of AD to the mid point of BC, and show that it passes through the in-centre.

Ans. $(b - c)\alpha = b\beta - c\gamma$

13. Show that the lines

$$\alpha = k\beta, \qquad k\alpha = \beta$$

are equally inclined to the internal bisector of the angle C.

14. If three lines drawn through the vertices of a triangle co-intersect; the three lines drawn through the same vertices, equally inclined to the bisectors of the angles, also co-intersect.

[*Note*—If the first point is

$$\alpha/l = \beta/m = \gamma/n,$$

the second is $$l\alpha = m\beta = n\gamma.]$$

15. Any line meets the sides BC, CA, AB in P, Q, R respectively; P′, Q′, R′ are points on these sides equidistant with P, Q, R from their mid points. Prove that P′, Q′, R′ are collinear.

16. If δ is the distance between the points $(\alpha_1 \beta_1 \gamma_1)$, $(\alpha_2 \beta_2 \gamma_2)$, prove that

$$\delta^2 = \frac{abc}{4D^2}[a \cos A (\alpha_1 - \alpha_2)^2 + b \cos B (\beta_1 - \beta_2)^2 + c \cos C (\gamma_1 - \gamma_2)^2]$$

17. Prove that the relation of § 413 may be expressed thus:

$$a^2(p - q)(p - r) + b^2(q - r)(q - p) + c^2(r - p)(r - q) = 4D^2$$

18. Find the area of a triangle in terms of the trilinear co-ordinates of its vertices.

Ans. $$\frac{abc}{8D^2}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

[*Note*—See § 412; for another method see Nixon's *Trigonometry*, page 210.]

GENERAL EQUATION OF THE SECOND DEGREE

§ **414.** The general equation of a conic in trilinears is (§ 357)

$$\phi(\alpha, \beta, \gamma) \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \quad (1)$$

The equation of the chord joining two points $(\alpha' \beta' \gamma')$, $(\alpha'' \beta'' \gamma'')$ on the conic is

$$\begin{aligned} &a(\alpha - \alpha')(\alpha - \alpha'') + b(\beta - \beta')(\beta - \beta'') \\ &+ c(\gamma - \gamma')(\gamma - \gamma'') + 2f(\beta - \beta')(\gamma - \gamma'') \\ &+ 2g(\gamma - \gamma')(\alpha - \alpha'') + 2h(\alpha - \alpha')(\beta - \beta'') = \phi(\alpha, \beta, \gamma) \end{aligned}$$

For the locus represented by this equation evidently passes through the points $(\alpha' \beta' \gamma')$, $(\alpha'' \beta'' \gamma'')$; and the equation is really of the first degree in α, β, γ.

Now put $\alpha'' = \alpha'$, $\beta'' = \beta'$, $\gamma'' = \gamma'$; the equation of the tangent at $(\alpha', \beta', \gamma')$ is $\therefore$

$$\begin{aligned} &a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' \\ &+ f(\beta'\gamma + \gamma'\beta) + g(\gamma'\alpha + \alpha'\gamma) + h(\alpha'\beta + \beta'\alpha) = 0 \quad (2) \end{aligned}$$

As in § 168, if the point $(\alpha' \beta' \gamma')$ is not restricted to lie on the curve, (2) is the equation of its polar.

We shall use the abbreviation P for the sinister of (2).

With the notation of Differentials,

$$2P \equiv \alpha' \frac{d\phi}{d\alpha} + \beta' \frac{d\phi}{d\beta} + \gamma' \frac{d\phi}{d\gamma} \equiv \alpha \frac{d\phi}{d\alpha'} + \beta \frac{d\phi}{d\beta'} + \gamma \frac{d\phi}{d\gamma'}$$

§ **415.** It is proved as in § 15 that the point which divides the join of $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ in the ratio m : n is

$$\left(\frac{m\alpha'' + n\alpha'}{m + n}, \frac{m\beta'' + n\beta'}{m + n}, \frac{m\gamma'' + n\gamma'}{m + n}\right)$$

Hence, as in § 324, it is proved that the equation of the pair of tangents from $(\alpha'\beta'\gamma')$ is

$$\phi(\alpha', \beta', \gamma')\,\phi(\alpha, \beta, \gamma) = P^2$$

This may also be proved as in Ex. 1, § 351.

§ **416.** The condition that the conic may be a line-pair is found as in § 117. It is

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

§ **417.** If the line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

coincide with the line represented by eq'n (2), § 414, then

$$\frac{a\alpha' + h\beta' + g\gamma'}{\lambda} = \frac{h\alpha' + b\beta' + f\gamma'}{\mu} = \frac{g\alpha' + f\beta' + c\gamma'}{\nu} \quad . \quad . \quad . \quad (2)$$

The pole $(\alpha'\beta'\gamma')$ of the line (1) is $\therefore$ determined by the eq'ns (2).

We may thus determine the centre, which is the pole of the line at infinity

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0$$

Again, if the line (1) is a tangent, its pole $(\alpha'\beta'\gamma')$ is on the line. The condition of tangency is $\therefore$ obtained by eliminating α', β', γ' between equations (2) and

$$\lambda\alpha' + \mu\beta' + \nu\gamma' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Put each member of eq'ns (2) $= -\rho$

$$\therefore \quad a\alpha' + h\beta' + g\gamma' + \lambda\rho = 0, \ \ldots, \ \ldots \quad . \quad . \quad . \quad (4)$$

We can eliminate α', β', γ', ρ linearly from (3) and (4).

The tangential equation is $\therefore$

$$\begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \nu \\ \lambda & \mu & \nu & 0 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The condition of tangency may also be deduced by the method of § 323 in the form

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0 \quad . \quad . \quad . \quad (6)$$

where $$A = bc - f^2, \quad \&c.,$$

as in § 323.

Note—We shall use the abbreviation Σ for the sinister of (6).

§ **418.** The line at infinity touches a parabola (§ 364); it meets an ellipse in imaginary points and an hyperbola in real points. If we eliminate γ between the equations of the conic and the line at infinity we obtain a quadratic in α/β; from this we can infer the nature of the conic.

The condition that the conic may be a parabola is [see eq′n (6), § 417]

$$\mathsf{A}\sin^2\mathsf{A} + \mathsf{B}\sin^2\mathsf{B} + \mathsf{C}\sin^2\mathsf{C} + 2\mathsf{F}\sin\mathsf{B}\sin\mathsf{C} + 2\mathsf{G}\sin\mathsf{C}\sin\mathsf{A} + 2\mathsf{H}\sin\mathsf{A}\sin\mathsf{B} = 0$$

Note—The letters A, B, C have here double significations; it is assumed that there is no risk of confusion.

§ 419. If we transform to Cartesians (§ 398), and apply the criterion of § 297, we shall obtain the condition that the conic may be a rectangular hyperbola. It is

$$a + b + c - 2f\cos A - 2g\cos B - 2h\cos C = 0$$

§ 420. The equation

$$\phi(\alpha, \beta, \gamma) = k(\lambda\alpha + \mu\beta + \nu\gamma)^2 \quad . \; . \; . \; . \; . \; . \quad (1)$$

represents (§ 351) a conic having double contact with the given conic at its intersections with the line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The conic (1) is a line-pair (§ 416), if

$$\begin{vmatrix} a - k\lambda^2 & h - k\lambda\mu & g - k\lambda\nu \\ h - k\lambda\mu & b - k\mu^2 & f - k\mu\nu \\ g - k\lambda\nu & f - k\mu\nu & c - k\nu^2 \end{vmatrix} = 0$$

If this equation is expanded the terms in k^2 and k^3 vanish identically; and we shall obtain the following simple equation to determine k:

$$\Delta - k\Sigma = 0,$$

where Σ has the meaning assigned, *Note*, § 417.

The equation of tangents to the given conic at its intersections with the line (2) is ∴

$$\Sigma\phi(\alpha, \beta, \gamma) = \Delta(\lambda\alpha + \mu\beta + \nu\gamma)^2 \quad . \; . \; . \; . \; . \quad (3)$$

Cor′—Since the asymptotes are the tangents to the conic at its intersections with the line at infinity, the equation to the asymptotes is derived by substituting sin A, sin B, sin C for λ, μ, ν in (3).

§ 421. Let α, β, γ be the co-ordinates of the extremity of a semi-diameter ρ_a parallel to BC; $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ the centre. Then projecting ρ_a on perpendiculars to the sides of ABC, we see that

$$\alpha = \bar{\alpha}, \qquad \beta = \bar{\beta} + \rho_a \sin C, \qquad \gamma = \bar{\gamma} - \rho_a \sin B$$

$$\therefore \; \phi(\bar{\alpha}, \bar{\beta} + \rho_a \sin C, \bar{\gamma} - \rho_a \sin B) = 0$$

This equation is a quadratic in ρ_a; and since the sum of its roots is zero, the equation when expanded becomes

$$\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) + \rho_a{}^2 \phi(0, \sin C, -\sin B) = 0$$

or $$\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) + \rho_a{}^2 (b \sin^2 C + c \sin^2 B - 2 f \sin B \sin C) = 0$$

This determines ρ_a; and ρ_b, ρ_c are similarly expressed.

If the conic is a circle, $\rho_a = \rho_b = \rho_c$;

the conditions for a circle are $\therefore$

$$\phi(0, \sin C, -\sin B) = \phi(\sin C, 0, -\sin A) = \phi(\sin B, -\sin A, 0)$$

CIRCUMSCRIBING CONIC

§ **422.** The equation of a conic circumscribing the triangle of reference is (*Cor'*, § 357)

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Writing this in the form

$$l\beta\gamma + \alpha(m\gamma + n\beta) = 0$$

we see that the line $$m\gamma + n\beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

meets the conic in the points where this line meets the lines

$$\beta = 0, \qquad \gamma = 0;$$

i. e. in two points which coincide with A.

Hence (2) is the equation of the tangent at A.

§ **423.** If we eliminate γ between the equation

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the equation of the conic, we get

$$n\nu\alpha\beta = (l\beta + m\alpha)(\lambda\alpha + \mu\beta)$$

This equation (which is a quadratic in α/β) represents the lines joining C to the intersections of the line (1) with the conic.

These lines coincide if

$$(\lambda l + \mu m - \nu n)^2 = 4\lambda\mu l m$$

This reduces to

$$\sqrt{l\lambda} + \sqrt{m\mu} + \sqrt{n\nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the radicals of course admitting of a double sign. Hence (2) is the condition that the line (1) may be a tangent.

§ **424.** If equation (1) of § 422 is transformed to Cartesians by the substitutions of § 399, it becomes

$$-\,m \sin A \sin C x^2 - l \sin B \sin C y^2 + (\ldots) xy + \ldots = 0$$

If ∴ the equation represent a circle, we must have [§ 200, eq'n (2)]

$$m \sin A \sin C = l \sin B \sin C$$

$$\therefore \quad l/\sin A = m/\sin B$$

$$= n/\sin C, \text{ by symmetry.}$$

The equation of the circum-circle of the triangle of reference is ∴

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0 \quad . \quad . \quad (1)$$

Cor' (1)—Let PL, PM, PN be ⊥s from a point P on BC, CA, AB respectively; then

$$2 \text{ area PMN} = \beta\gamma \sin A, \qquad 2 \text{ area PNL} = \gamma\alpha \sin B,$$

$$2 \text{ area PLM} = \alpha\beta \sin C$$

Hence (1) expresses that the area of the triangle LMN is zero; or the locus of P when L, M, N are collinear is the circum-circle. [Compare *Note*, Ex. 16, page 176.]

Cor' (2)—If the area of the triangle LMN is constant (= k), the equation to the locus of P is

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 2k;$$

or P describes a circle concentric with the circum-circle.

§ **425.** Since the terms of the second degree are the same in the equations of all circles, if $S = 0$ is the equation to a circle, any circle whatever may be represented by an equation of the form

$$S + l\alpha + m\beta + n\gamma = 0$$

Hence [§ 397, eq'n (2)] the equation of any circle may be thrown into the form

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C$$
$$+ (l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0 \quad . \quad . \quad . \quad (1)$$

Cor' (1)—The radical axis of the circle (1) and the circum-circle is

$$l\alpha + m\beta + n\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Cor' (2)—The radical axis of two circles written in the form (1) is

$$l\alpha + m\beta + n\gamma = l'\alpha + m'\beta + n'\gamma$$

Cor' (3)—Suppose that the general equation of the second degree represents a circle; a comparison of its coeff's with those of equation (1) gives (k being some constant)

$$ka = l\sin A, \quad kb = m\sin B, \quad kc = n\sin C \quad . \quad . \quad . \quad . \quad (3)$$

and $\quad 2kf = \sin A + m\sin C + n\sin B, \quad 2kg = \ldots, \quad 2kh = \ldots \quad . \quad . \quad (4)$

It is easy to eliminate k, l, m, n from eq'ns (3), (4); this gives the conditions for a circle already obtained (§ 421).

Further, from *Cor'* (1) and eq'ns (3) we see that the radical axis of this circle and the circum-circle is

$$a\alpha/\sin A + b\beta/\sin B + c\gamma/\sin C = 0$$

Exercises

[The conic in these questions is the circumscribed conic

$$l/\alpha + m/\beta + n/\gamma = 0.]$$

1. Prove that the chord joining $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ is

$$l\alpha/(\alpha'\alpha'') + m\beta/(\beta'\beta'') + n\gamma/(\gamma'\gamma'') = 0$$

2. Prove that the equation of the tangent at $(\alpha'\beta'\gamma')$ is

$$l\alpha/\alpha'^2 + m\beta/\beta'^2 + n\gamma/\gamma'^2 = 0$$

3. Deduce from Ex. 2 the condition of tangency given in § 423.

4. A triangle is inscribed in a conic; prove that the intersections of each side with the tangent at the opposite vertex are collinear.

5. Prove that the equation of a conic circumscribing the triangle of reference is

$$\beta\gamma\sin A/b'^2 + \gamma\alpha\sin B/b''^2 + \alpha\beta\sin C/b'''^2 = 0$$

where b', b'', b''' are the semi-diameters parallel to the sides.

6. If the centroid of ABC is the centre of a circumscribing conic; find its equation in areal co-ordinates.

Ans. $yz + zx + xy = 0$

INSCRIBED CONIC

§ **426.** The intersections of BC with the general conic [§ 414, eq'n (1)] are obtained by putting $\alpha = 0$; they are ∴ determined by

$$b\beta^2 + c\gamma^2 + 2f\beta\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

These intersections coincide, i. e. BC is a tangent if

$$bc = f^2$$

The sinister of (1) is in this case a perfect square.

We see then that the equation

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0 \quad (2)$$

represents a conic inscribed in the triangle of reference: for, if α, β, or γ is put $= 0$ in its sinister the remaining terms form a perfect square. The following equation, which is equivalent to (2), is ∴ the general equation of an inscribed conic:

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

§ **427.** If we eliminate γ from the equations

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0 \quad . \quad . \quad . \quad . \quad (1)$$

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we get $\quad \nu(\sqrt{l\alpha} + \sqrt{m\beta})^2 + n(\lambda\alpha + \mu\beta) = 0$

This quadratic in $\sqrt{\alpha}/\sqrt{\beta}$ has equal roots if

$$lm\nu^2 = (l\nu + n\lambda)(m\nu + n\mu)$$

This condition reduces to

$$l/\lambda + m/\mu + n/\nu = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence (3) is the condition that the line (2) may touch the inscribed conic (1).

Note—This condition may also be deduced as a particular case of eq'n (6), § 417; the rationalized form of (1) being used.

§ **428.** *To find the equation of the inscribed circle.*

Let A′, B′, C′ be its points of contact with the sides.

Put $\alpha = 0$ in the equation

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0;$$

the co-ordinates of A′ ∴ satisfy the equation

$$n/m = \beta/\gamma$$

$$\begin{aligned} \therefore\quad n/m &= A'C \sin C/(A'B \sin B) \\ &= r \cot \tfrac{1}{2} C \sin C/(r \cot \tfrac{1}{2} B \sin B) \\ &= \cos^2 \tfrac{1}{2} C/\cos^2 \tfrac{1}{2} B \end{aligned}$$

A similar value is found for l/m; the equation of the inscribed circle is ∴

$$\cos \tfrac{1}{2} A \sqrt{\alpha} + \cos \tfrac{1}{2} B \sqrt{\beta} + \cos \tfrac{1}{2} C \sqrt{\gamma} = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Note—The equations of the escribed circles are similarly obtained. The escribed circle touching BC is

$$\cos \tfrac{1}{2} A \sqrt{-\alpha} + \sin \tfrac{1}{2} B \sqrt{\beta} + \sin \tfrac{1}{2} C \sqrt{\gamma} = 0 \quad . \quad . \quad . \quad (2)$$

Exercises

[The conic in these questions is the inscribed conic

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$$

The points of contact are denoted by A′, B′, C′.]

1. Prove that AA′, BB′, CC′ are concurrent.

2. If AA′, BB′, CC′ meet the conic again in A″, B″, C″, prove that the tangents at A″, B″, C″ meet the opposite sides in three points lying on the line

$$l\alpha + m\beta + n\gamma = 0$$

[*Note*—The equation of the conic may be written

$$n\gamma(n\gamma - 2l\alpha - 2m\beta) + (l\alpha - m\beta)^2 = 0$$

The tangent at C″ is ∴ (§ 353)

$$n\gamma - 2l\alpha - 2m\beta = 0.]$$

3. If the centre is $(\alpha\beta\gamma)$, prove that

$$\alpha/(n \sin B + m \sin C) = \beta/(l \sin C + n \sin A) = \gamma/(m \sin A + l \sin B)$$

[*Note*—If ϕ is the sinister of eq'n (2), § 426, the centre is determined (§ 417) by the equations

$$\frac{d\phi}{d\alpha}\Big/\sin A = \frac{d\phi}{d\beta}\Big/\sin B = \frac{d\phi}{d\gamma}\Big/\sin C$$

Otherwise thus: It is known that the joins of A to the mid point of B'C' and of B to the mid point of C'A' pass through the centre; form the eq'ns of these joins and determine $\alpha : \beta : \gamma$.]

4. Prove that the centre lies on the join of the mid points of AA', BC.

[*Note*—Use Ex. 3. The theorem also follows from Ex. 31, page 306, by supposing two sides of the quadrilateral to coincide.]

5. Prove that the conic is a parabola if

$$l/\sin A + m/\sin B + n/\sin C = 0$$

6. If one focus describes the straight line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$

the other describes the conic

$$\lambda/\alpha + \mu/\beta + \nu/\gamma = 0$$

[*Note*—If the foci are $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ then (§ 248, VII.)

$$\alpha'\alpha'' = \text{square of semi-minor axis} = \beta'\beta'' = \gamma'\gamma''.]$$

7. Prove that the equation of the chord joining $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ is

$$\sqrt{l}\,(\alpha - \alpha')/(\sqrt{\alpha'} + \sqrt{\alpha''}) + \ldots + \ldots = 0;$$

and the tangent at $(\alpha'\beta'\gamma')$ is

$$\alpha\sqrt{l/\alpha'} + \beta\sqrt{m/\beta'} + \gamma\sqrt{n/\gamma'} = 0$$

8. If A', B', C' are the mid points of the sides; the equation of the conic in areal co-ordinates is $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$

NINE-POINT CIRCLE

§ **429.** If the circle represented by eq'n (1), § 425, pass through the mid point of BC ($\alpha = 0$, $\beta \sin B = \gamma \sin C$), we must have

$$\sin A \sin B \sin C + 2 \sin B \sin C\,(m \sin C + n \sin B) = 0$$

$$\therefore\ 2\,m/\sin B + 2\,n/\sin C = -\sin A/(\sin B \sin C)$$

If the circle pass also through the mid points of CA, AB we have two similar equations. Solving these, we find

$$-2\,l/\sin A = (\sin^2 B + \sin^2 C - \sin^2 A)/(2 \sin A \sin B \sin C)$$

$$\therefore \quad 2\,l = -\cos A; \quad \text{and} \quad 2\,m = -\cos B, \quad 2\,n = -\cos C$$

The equation of the nine-point circle is $\therefore$

$$2\,(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)$$
$$-(\alpha \cos A + \beta \cos B + \gamma \cos C)(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0 \quad . \quad . \quad (1)$$

or

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C - 2\,\beta\gamma \sin A - 2\,\gamma\alpha \sin B$$
$$-2\,\alpha\beta \sin C = 0 \quad . \quad . \quad . \quad (2)$$

§ **430.** *The nine-point circle of a triangle touches the inscribed and escribed circles.*

(Feuerbach's Theorem.)

The following proof is by Prof′ Genese.

Let A′, B′, C′ be the mid points of the sides of the given triangle ABC; take A′B′C′ as triangle of reference.

The equation to the nine-point circle is thus

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The eq′ns to BC, CA, AB are

$$b\beta + c\gamma = 0, \qquad c\gamma + a\alpha = 0, \qquad a\alpha + b\beta = 0$$

Let the equation to a circle touching these be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) = 0 \quad . \quad . \quad (2)$$

Putting $b\beta + c\gamma = 0$, (2) becomes

$$(a\beta + b\alpha)\left(-\frac{b}{c}\beta\right) + c\alpha\beta - \left(l\alpha + m\beta - \frac{n}{c}b\beta\right)a\alpha = 0$$

This is to be a complete square

$$\therefore \quad 4\left(-\frac{ab}{c}\right)(-la) = \left(-\frac{b^2}{c} + c - am + \frac{anb}{c}\right)^2$$

or

$$\pm\, 2\sqrt{abc}\,\sqrt{al} = c^2 - b^2 + abn - cam$$

Similarly that (2) may touch CA, AB,

$$\pm\, 2\sqrt{abc}\,\sqrt{bm} = a^2 - c^2 + bcl - abn$$

$$\pm\, 2\sqrt{abc}\,\sqrt{cn} = b^2 - a^2 + cam - bcl$$

Adding we find $\qquad \pm\sqrt{al} \pm \sqrt{bm} \pm \sqrt{cn} = 0$

But this is the condition that the line

$$l\alpha + m\beta + n\gamma = 0$$

should touch (1), and this line is the radical axis of (1) and (2); ∴ (2) touches (1).

PASCAL'S THEOREM; OTHER GEOMETRICAL THEOREMS

§ 431. Pascal's Theorem. *If any hexagon is inscribed in a conic, the intersections of its three pairs of opposite sides are collinear.*

Let the hexagon be $AC'BA'CB'$; the reader can draw the figure.

Using abridged notation, let the equations of AC', $C'B$, BA', $A'C$, CB', $B'A$, and AA' be respectively

$$u = 0, \quad v = 0, \quad w = 0, \quad u' = 0, \quad v' = 0, \quad w' = 0, \quad \phi = 0$$

Since the conic circumscribes $AC'BA'$, $AB'CA'$, its equation may (§ 355) be written in the forms

$$uw - \lambda v \phi = 0, \qquad u'w' - \lambda' v' \phi = 0$$

$$\therefore \quad uw - \lambda v \phi \equiv \lambda'' (u'w' - \lambda' v' \phi) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

λ, λ', λ'' being numerical constants.

From (1) we infer

$$uw - \lambda'' u'w' \equiv \phi (\lambda v - \lambda'\lambda'' v')$$

This identity shows that the conic

$$uw - \lambda'' u'w' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is a line-pair, one of the lines being ϕ, and the other passing through the intersection of v and v'.

But the conic (2) passes through the intersections of u, u' and of w, w'; these points are ∴ collinear with (v, v'). Q.E.D.

Note (1)—Another proof is indicated in *Note*, Ex. 47, page 357; and a proof of the converse theorem in Ex. 48, page 357. We add an outline of a proof by trilinears.

Take ABC as triangle of reference; let the other points be $A'(\alpha_1 \beta_1 \gamma_1)$, $B'(\alpha_2 \beta_2 \gamma_2)$, $C'(\alpha_3 \beta_3 \gamma_3)$.

Express that A', B', C' are on the circumscribing conic

$$l/\alpha + m/\beta + n/\gamma = 0$$

and eliminate l, m, n. This gives

$$\begin{vmatrix} 1/\alpha_1 & 1/\beta_1 & 1/\gamma_1 \\ 1/\alpha_2 & 1/\beta_2 & 1/\gamma_2 \\ 1/\alpha_3 & 1/\beta_3 & 1/\gamma_3 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Next, form the equations of AC′, A′C, &c.; and find the co-ord′s of the intersections of the three pairs of opposite sides.

The condition that these points should be collinear reduces to (3). The reader should work this out.

Note (2)—The six points A, C′, ... may be taken in sixty different orders, thus forming sixty 'hexagons.' The theorem is applicable to each of these hexagons; so that there are sixty *Pascal Lines* corresponding to six points on a conic.

Note (3)—Many particular results are deducible from Pascal's Theorem. For example, suppose that A coincides with C′, and A′ with C; then AC′, A′C are the tangents at A, A′. We infer that—

The tangents at two opposite vertices of an inscribed quadrilateral meet on its third diagonal.

§ **432.** Brianchon's Theorem. *If a hexagon circumscribe a conic, its three diagonals co-intersect.*

Take the triangle formed by producing three alternate sides as triangle of reference; then the equation of the conic is (§ 426)

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$$

Let the sides of the hexagon respectively opposite to α, β, γ be

$$\lambda_1\alpha + \mu_1\beta + \nu_1\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\lambda_2\alpha + \mu_2\beta + \nu_2\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\lambda_3\alpha + \mu_3\beta + \nu_3\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have then (§ 427) three equations

$$l/\lambda_1 + m/\mu_1 + n/\nu_1 = 0, \quad \ldots, \quad \ldots$$

The elimination of l, m, n gives

$$\begin{vmatrix} 1/\lambda_1 & 1/\mu_1 & 1/\nu_1 \\ 1/\lambda_2 & 1/\mu_2 & 1/\nu_2 \\ 1/\lambda_3 & 1/\mu_3 & 1/\nu_3 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the condition that the six lines should touch a conic.

Again, one diagonal is the join of the inters′n of the line (2) with γ to the inters′n of the line (3) with β. Its equation is ∴

$$\begin{vmatrix} \alpha & \beta & \gamma \\ -\mu_2 & \lambda_2 & 0 \\ \nu_3 & 0 & -\lambda_3 \end{vmatrix} = 0, \quad \text{or} \quad \alpha + \mu_2\beta/\lambda_2 + \nu_3\gamma/\lambda_3 = 0$$

Similarly the other diagonals are

$$\lambda_1\alpha/\mu_1 + \beta + \nu_3\gamma/\mu_3 = 0, \qquad \lambda_1\alpha/\nu_1 + \mu_2\beta/\nu_2 + \gamma = 0$$

The diagonals $\therefore$ co-intersect if

$$\begin{vmatrix} 1 & \mu_2/\lambda_2 & \nu_3/\lambda_3 \\ \lambda_1/\mu_1 & 1 & \nu_3/\mu_3 \\ \lambda_1/\nu_1 & \mu_2/\nu_2 & 1 \end{vmatrix} = 0,$$ which is equivalent to (4). Q.E.D.

§ **433.** *If the six vertices of two triangles lie on a conic, their six sides touch a conic.*

Let the triangles be ABC, A′B′C′; take ABC as triangle of reference. Let the sides of A′B′C′ be represented by eq′ns (1), (2), (3), § 432.

The equation of a conic circumscribing A′B′C′ is

$$\begin{aligned} p\,(\lambda_2\alpha + \mu_2\beta + \nu_2\gamma)\,(\lambda_3\alpha + \mu_3\beta + \nu_3\gamma) \\ + q\,(\lambda_3\alpha + \mu_3\beta + \nu_3\gamma)\,(\lambda_1\alpha + \mu_1\beta + \nu_1\gamma) \\ + r\,(\lambda_1\alpha + \mu_1\beta + \nu_1\gamma)\,(\lambda_2\alpha + \mu_2\beta + \nu_2\gamma) = 0 \end{aligned}$$

This passes through A, B, C if

$$p\,\lambda_2\lambda_3 + q\,\lambda_3\lambda_1 + r\,\lambda_1\lambda_2 = 0, \quad \text{or} \quad p/\lambda_1 + q/\lambda_2 + r/\lambda_3 = 0; \ \ldots; \ \ldots$$

Eliminating p, q, r we have the condition that the six points may lie on a conic; this is the same as eq′n (4), § 432, which is the condition that the six sides may touch a conic. Q.E.D.

Cor′—If two conics are so related that one triangle can be inscribed in one and circumscribed to the other, there is an infinite number of such triangles.

Let ABC be the given triangle; let S, Σ be the two conics.

Let any tangent to Σ meet S in A′, B′; and let the tangents from A′, B′ to Σ meet in C′. Then the six sides of ABC, A′B′C′ touch Σ; and $\therefore$ by the preceding the six points A, B, C, A′, B′, C′ lie on a conic.

But A, B, C, A′, B′ lie on S; and five points determine a conic. $\therefore$ C′ lies on S. Q.E.D.

CONIC REFERRED TO A SELF-CONJUGATE TRIANGLE

§ **434.** The general conic [§ 414, eq′n (1)] meets the line α in the points determined by eq′n (1), § 426; if $f = 0$ these points divide BC harmonically.

The equations $g = 0$, $h = 0$ are similarly interpreted; and we see that each vertex of the triangle of reference is the pole of the opposite side with respect to the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is proved otherwise in § 356.

§ 435. The centre, being the pole of the line at infinity, is determined by the equations

$$l\,\alpha/\sin A = m\,\beta/\sin B = n\,\gamma/\sin C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Suppose that the conic is a circle; then the join of any point to the centre is perpendicular to the polar of the point. If ∴ O is the centre, AO, BO, CO are respectively perpendicular to BC, CA, AB ; or O is the ortho-centre.

The co-ord's of the ortho-centre, which are proportional to sec A, sec B, sec C, ∴ satisfy eq'ns (2) ; this determines $l : m : n$.

The equation of the circle with respect to which the triangle of reference is self-conjugate is ∴

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0$$

Note—This result may also be obtained by methods similar to those of §§ 424, 429.

Cor'—From eq'ns (2), (1), § 429, we see that the self-conjugate circle, the circum-circle and the nine-point circle are co-axal, the radical axis being

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0$$

§ 436. It may be proved as in § 423, or deduced from § 417, that the line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

touches the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if

$$\lambda^2/l + \mu^2/m + \nu^2/n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Note—In questions relating to four lines it is convenient to assume that their equations are

$$\lambda\alpha \pm \mu\beta \pm \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus each side of the quadrilateral (4) touches the conic if the single condition (3) is satisfied. The reader will find no difficulty in proving that the diagonal triangle of this quadrilateral is the triangle of reference.

In questions relating to four points it is convenient to assume that the points are

$$\frac{\alpha}{p} = \pm\frac{\beta}{q} = \pm\frac{\gamma}{r};$$

these points are on the conic (2) if

$$lp^2 + mq^2 + nr^2 = 0$$

§ 437. In questions relating to two conics it is convenient to choose the harmonic* triangle of the quadrangle formed by their four intersections (see *Cor'*, § 314) as triangle of reference; so that the equations of the two conics are

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0, \qquad l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0$$

Exercises

1. Find the locus of the pole of a given line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$

with respect to conics inscribed in the quadrilateral

$$p\alpha \pm q\beta \pm r\gamma = 0$$

Ans. The straight line $p^2\alpha/\lambda + q^2\beta/\mu + r^2\gamma/\nu = 0$

2. Find locus of centres of conics inscribed in this quadrilateral.

Ans. The straight line $p^2\alpha/\sin A + q^2\beta/\sin B + r^2\gamma/\sin C = 0$

[Compare the solution, page 307.]

3. Find the locus of the pole of a given line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$

with respect to conics through the four fixed points $(p, \pm q, \pm r)$.

Ans. The conic $\lambda p^2/\alpha + \mu q^2/\beta + \nu r^2/\gamma = 0$

4. Show that the triangle of reference is the harmonic triangle of the quadrangle whose vertices are $(p, \pm q, \pm r)$.

5. Find locus of centres of conics circumscribing this quadrangle.

Ans. The conic $p^2 \sin A/\alpha + q^2 \sin B/\beta + r^2 \sin C/\gamma = 0$

[Compare § 322.]

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* Let A, A′, B′, B be the vertices of a quadrangle (fig', page 284); the triangle OPQ whose vertices are the intersections of the three pairs of lines joining the points is called the *harmonic triangle* of the quadrangle (Casey); the triangle whose sides are the three diagonals AB′, A′B, OP is called the *diagonal triangle* (Cremona).

6. Find the conditions that the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

may be (1), a parabola; (2), a rectangular hyperbola.

Ans. $\sin^2 A/l + \sin^2 B/m + \sin^2 C/n = 0$; $l + m + n = 0$

7. Prove that a rectangular hyperbola passes through the centres of the inscribed and escribed circles of any triangle which is self-polar with respect to the hyperbola.

8. The polars of A, B, C with respect to a conic form a triangle A′B′C′; prove that AA′, BB′, CC′ co-intersect.

9. Prove that the locus of centres of rectangular hyperbolas passing through the vertices of a triangle is the nine-point circle.

10. Find the director circle of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

Ans. $l(m + n)\alpha^2 + \ldots + \ldots + 2\,mn\,\beta\gamma\cos A + \ldots + \ldots = 0$

[*Note*—The eq'n of the tangents from $(\alpha'\beta'\gamma')$ is (§ 415)

$$(l\alpha'^2 + m\beta'^2 + n\gamma'^2)(l\alpha^2 + m\beta^2 + n\gamma^2) = (l\alpha\alpha' + m\beta\beta' + n\gamma\gamma')^2$$

If these are at right angles the coeff's satisfy the cond'n for a rectangular hyperbola (§ 419).]

11. If the curve is a parabola, find the equation of its directrix and the co-ordinates of its focus.

Ans. $\dfrac{l(m+n)}{\sin A}\alpha + \dfrac{m(n+l)}{\sin B}\beta + \dfrac{n(l+m)}{\sin C}\gamma = 0$;

$$\frac{\alpha\sin A}{m+n} = \frac{\beta\sin B}{n+l} = \frac{\gamma\sin C}{l+m}$$

[*Note*—Use Ex. 10. The locus of intersection of rectangular tangents breaks up into factors, one of which is the directrix, and the other the line at infinity

$$\alpha\sin A + \beta\sin B + \gamma\sin C = 0$$

The focus is the pole of the directrix.]

CONIC REFERRED TO TWO TANGENTS AND THEIR CHORD OF CONTACT

§ 438. The equation of a conic touching the sides CA, CB of the triangle of reference at A, B is (§ 353)

$$\alpha\beta = k^2\gamma^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The results of § 354 are applicable if we write α, β, $k\gamma$ for L, M, R.

Ex. If the bisector of the angle C meet AB in R, and PQ is any chord through R; prove that PR, RQ subtend equal angles at C.

The eq'ns to CR, PQ are

$$\alpha - \beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\alpha - \beta + \lambda\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminate γ from (1), (3); this gives the equation to the pair of lines CP, CQ, viz.

$$k^2(\alpha - \beta)^2 - \lambda^2\alpha\beta = 0$$

The factors of this are of the form given, Ex. 13, page 370; and $\therefore$ &c.

§ **439.** If $\mu_1, \mu_2, \mu_3, \mu_4$ (see § 354) are four points on a conic, the cross ratio of the pencil passing through these points, and whose vertex is any fifth point on the conic, is constant (§ 355); this cross ratio is briefly designated 'the cross ratio of four points on a conic.'

Take A for the vertex of the pencil; its rays are (§ 354)

$$\beta = \mu_1 k\gamma, \qquad \beta = \mu_2 k\gamma, \ldots.$$

The cross ratio of the four points is $\therefore$ (§ 140)

$$\frac{(\mu_1 - \mu_3)(\mu_4 - \mu_2)}{(\mu_3 - \mu_2)(\mu_1 - \mu_4)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Four fixed tangents cut any fifth in a range whose cross ratio is constant, and equal to that of the four points of contact.

Let $\mu_1, \mu_2, \mu_3, \mu_4$ be the four points; μ the point of contact of the variable tangent.

Let the variable tangent meet the other tangents in P_1, P_2, P_3, P_4.

The elimination of γ from the equations of the tangents at μ and μ_1,

$$\mu^2\alpha - 2\mu k\gamma + \beta = 0, \qquad \mu_1^2\alpha - 2\mu_1 k\gamma + \beta = 0$$

gives

$$\mu\mu_1\alpha = \beta$$

This is the equation of CP_1.

Hence $$\{P_1P_2P_3P_4\} = \{C.P_1P_2P_3P_4\}$$

$$= \text{cross ratio of pencil } \mu\mu_1\alpha = \beta, \quad \mu\mu_2\alpha = \beta, \ldots;$$

the latter cross ratio, which is given by the formula of § 140, reduces to the preceding expression (1). Q.E.D.

§ **440.** In working with homogeneous equations we may replace the co-ordinates by any quantities proportional to them. Thus we may take $(1, \mu^2, \mu)$ as the co-ord's of a point on the conic

$$LM = R^2$$

Ex. 1. Prove that the tangents at μ, μ' meet at the point

$$[1,\ \mu\mu',\ (\mu + \mu')/2]$$

These co-ord's are obtained by solving for L : M : R the eq'ns of the tangents

$$\mu^2 L - 2\mu R + M = 0, \qquad \mu'^2 L - 2\mu' R + M = 0$$

§ **441.** Ex. 2. *If the three sides of a triangle touch a conic* S *and two of its vertices move on another conic* Σ, *the locus of the third vertex is a conic inscribed in the quadrilateral formed by the common tangents of* S *and* Σ.

Take two of the (real or imaginary) common tangents as two sides of the triangle of reference; so that the equations of S and Σ are

$$LM = R^2, \qquad LM = (l L + m M + 2 n R)^2$$

Let μ_1, μ_2, μ_3 be the points of contact; then expressing that the inters'ns of the tangents at μ_1, μ_3 and at μ_2, μ_3 lie on Σ, we see that μ_1 and μ_2 are the roots of the quadratic in μ

$$\mu\mu_3 = (l + m\,\mu\mu_3 + n\mu + n\mu_3)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We can ∴ write down the values of $\mu_1 + \mu_2$, $\mu_1\mu_2$.

But, by Ex. 1 these are $2\,R/L$ and M/L, where (L, M, N) is the inters'n of the tangents at μ_1, μ_2; hence $2\,R/L$ and M/L are expressed in terms of μ_3: μ_3 is easily eliminated; and we find for the locus an equation of the form

$$LM = (l' L + m' M + 2 n' R)^2$$

The locus is ∴ a conic touching L, M; but L, M are *any* two common tangents; ∴ &c.

Note—The reader should work out this solution in full, supplying the suppressed details. It will be seen that the locus becomes two coincident lines if $lm = n^2$; i.e. if the chord of contact of Σ with L, M touches S.

INVARIANTS

§ **442.** The following convenient notation is due to Prof' Cayley.

$$(a, b, c, d)\,(x, y)^3 \quad \text{means} \quad ax^3 + 3\,bx^2y + 3\,cxy^2 + dy^3$$

Thus

$$(a, b, c, f, g, h)\,(\alpha, \beta, \gamma)^2 \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta$$

and $\quad (a, b, c, f, g, h)\,(x, y, 1)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

§ **443.** Let the result of substituting the values of α, β, γ, given in § 398 in $\phi\,(\alpha, \beta, \gamma)$, be

$$(a, b, c, f, g, h)\,(\alpha, \beta, \gamma)^2 \equiv (a_1, b_1, c_1, f_1, g_1, h_1)\,(x, y, 1)^2$$

Let the discriminant of $\phi(\alpha, \beta, \gamma)$ be Δ. Also, as in § 323,

$$A = bc - f^2, \quad \&c.;$$

i.e. A, B, ... are the minors of the determinant Δ. We shall also put

$$\theta = a_1 b_1 - h_1^2, \qquad \theta' = a_1 + b_1$$

The conic is a parabola if $\theta = 0$, and a rectangular hyperbola if $\theta' = 0$; θ and θ' are the invariants of § 212.

Then by substituting the actual values of a_1, b_1, h_1 we shall obtain the following relations:

$$\theta \equiv a_1 b_1 - h_1^2 \equiv (A, B, C, F, G, H)(\sin A, \sin B, \sin C)^2 \quad . \quad . \quad (1)$$

$$\theta' \equiv a_1 + b_1 \equiv a + b + c - 2f\cos A - 2g\cos B - 2h\cos C \quad . \quad . \quad (2)$$

Ex. 1. Suppose that the conic is a circle; and let its centre be the origin of rectangular axes, so that

$$(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2 \equiv q(x^2 + y^2 - \rho^2)$$

$$\therefore \quad \theta = q^2, \qquad \theta' = 2q$$

The square of the tangent from any point (α, β, γ), or (x, y) is $\therefore$

$$2\phi(\alpha, \beta, \gamma)/\theta'$$

Ex. 2. *The circum-circle of any triangle which is self-polar with respect to an ellipse cuts its director circle orthogonally.* (*M. Faure.*)

Take the given triangle as triangle of reference; let the equation of the ellipse be

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

The equation of the director circle is given, Ex. 10, page 386; the square of the tangent from the centre of the circle $(R\cos A, R\cos B, R\cos C)$ is $\therefore$ (by Ex. 1)

$$\frac{2R^2[l(m+n)\cos^2 A + \ldots + \ldots + 2mn\cos A\cos B\cos C + \ldots + \ldots]}{l(m+n) + \ldots + \ldots - 2mn\cos^2 A - \ldots - \ldots}$$

We find by easy Trigonometrical reductions that this expression $= R$; this proves the theorem.

§ 444. Suppose that the axes of x and y are the axes of figure of the conic; so that

$$\phi(\alpha, \beta, \gamma) \equiv q(x^2/\rho^2 + y^2/\rho'^2 - 1) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then by § 443,

$$\theta = q^2/\rho^2\rho'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\theta' = q(1/\rho^2 + 1/\rho'^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Another equation which determines q may be deduced thus.

From the equation of the asymptotes [*Cor'*, § 420] we see that

$$\phi\,(\alpha, \beta, \gamma) - (\Delta/\theta)\,(D^2/R^2) \equiv q\,(x^2/\rho^2 + y^2/\rho'^2)$$

Subtract this from (1),

$$\therefore\quad -q = (\Delta/\theta)\,(D^2/R^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The relations (2), (3), (4) suffice to determine all constants connected with the conic. Thus we have at once expressions for $\rho^2 + \rho'^2$ and $\rho^2\rho'^2$; and we find that the semi-axes are the roots of the following equation in ρ:

$$R^4\,\theta^3\,\rho^4 + R^2\,D^2\,\Delta\,\theta\theta'\rho^2 + D^4\,\Delta^2 = 0$$

§ 445. If S, S′ are two conics, the conic through their intersections,

$$S + k\,S' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is a line-pair (§§ 118, 416) if its discriminant vanishes; i. e. if

$$\begin{vmatrix} a + ka' & h + kh' & g + kg' \\ h + kh' & b + kb' & f + kf' \\ g + kg' & f + kf' & c + kc' \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

This cubic in k is usually written

$$\Delta + \Theta\,k + \Theta'\,k^2 + \Delta'\,k^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The substitution of the roots of this cubic in (1) gives the line-pairs through the intersections of $S = 0$, $S' = 0$.

§ 446. If by a change of axes S and S′ become $\overline{S}$ and $\overline{S'}$, then $S + kS'$ becomes $\overline{S} + k\,\overline{S'}$; and if k is such that $S + kS'$ breaks up into factors, so does $\overline{S} + k\,\overline{S'}$.

Hence the roots of the preceding cubic (3), and ∴ the ratios of its coefficients, are independent of the axes of co-ordinates to which the two conics are referred. This conclusion leads to many interesting properties of a system of two conics; for these the student is referred to Salmon's *Conics*.

Exercises on Chapter XIV

[In these questions ABC is the triangle of reference.]

1. Find the trilinear co-ordinates of the centre of the nine-point circle.
Ans. $\frac{1}{2}R\cos(B - C)$, $\frac{1}{2}R\cos(C - A)$, $\frac{1}{2}R\cos(A - B)$

2. Prove that the in-centre, circum-centre, and ortho-centre of the triangle ABC are collinear if

$$\cos^2 A\,(\cos B - \cos C) + \cos^2 B\,(\cos C - \cos A) + \cos^2 C\,(\cos A - \cos B) = 0$$

3. If p, q, r are the perpendiculars from A, B, C on the join of the centroid and in-centre of ABC, prove that

$$p \cot \tfrac{1}{2} A + q \cot \tfrac{1}{2} B + r \cot \tfrac{1}{2} C = 0$$

4. Prove that the equation of the circle through the three centres of the escribed circles is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + (a + b + c)(\beta\gamma + \gamma\alpha + \alpha\beta) = 0$$

5. Prove that the equation of the nine-point circle may be written

$$a^2/(D - a\alpha) + b^2/(D - b\beta) + c^2/(D - c\gamma) = 0$$

[*Note*—Let A′, B′, C′ be mid points of sides. Prove that length of $\perp$ from $(\alpha'\beta'\gamma')$ on B′C′ is $\alpha' - D/a$; then form eq′n of circum-circle of A′B′C′.]

6. Find the radius of the circle

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 2k$$

Ans. $R\sqrt{1 - 4k/D}$

7. Through A′, B′, C′, the mid points of the sides of the triangle ABC, lines A′L, B′M, C′N are drawn perpendicular to the sides and proportional to them; prove that AL, BM, CN co-intersect in a point whose locus is

$$\beta\gamma \sin(B - C) + \gamma\alpha \sin(C - A) + \alpha\beta \sin(A - B) = 0$$

8. Find the equation of the circle whose diameter is BC.

Ans. $\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = (D/R)\, \alpha \cos A$

9. If $S = 0$ is the equation to the circum-circle of the triangle ABC, the equation of the circle through the three points $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$ is

$$\begin{vmatrix} S & \alpha\,\beta\,\gamma \\ S_1 & \alpha_1\beta_1\gamma_1 \\ S_2 & \alpha_2\beta_2\gamma_2 \\ S_3 & \alpha_3\beta_3\gamma_3 \end{vmatrix} = 0$$

10. The equation of the circum-circle of the triangle whose sides are $\alpha = 0$ and the bisectors of the angle A is

$$\sin(B - C)\{\beta\gamma \sin A + \ldots + \ldots\}$$
$$+ (\beta \sin C - \gamma \sin B)(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0$$

This circle and its two analogues are co-axal, the radical axis being

$$\alpha \sin(B - C) + \beta \sin(C - A) + \gamma \sin(A - B) = 0$$

(*Prof′ Purser.*)

11. The radical axis of the in-circle and nine-point circle of the triangle of reference is

$$a\alpha/(b - c) + b\beta/(c - a) + c\gamma/(a - b) = 0$$

12. Prove that the intersections of the lines

$$l\alpha + m\beta + n\gamma = 0, \qquad l'\alpha + m'\beta + n'\gamma = 0$$

with the circum-circle and self-conjugate circle respectively, lie on the same circle if

$$\begin{vmatrix} \cos A & \cos B & \cos C \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$

[*Note*—The radical axes of the three pairs of circles are concurrent.]

13. The equation $\alpha\beta = \gamma^2$ represents a circle if the two sides CA, CB of the triangle of reference are equal.

14. Show that the conic

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$$

is an ellipse, parabola, or hyperbola, according as

$$lmn\,(l/a + m/b + n/c) > = < 0$$

15. Show that the theorem, Ex. 3, page 311, is a particular case of Ex. 6, page 379.

16. Determine λ so that the equation

$$\alpha^2 = \lambda\beta\gamma$$

may represent a parabola; find the equation to its directrix.

Ans. $\lambda = 4\,bc/a^2$; $a\alpha\cos A = c\beta + b\gamma$

17. If p, q, r are the perpendiculars from A, B, C on any tangent to a parabola touching AB, AC at B, C, prove that

$$p^2 = qr$$

18. Prove that the normals to the conic

$$\beta\gamma\cot\tfrac{1}{2}A + \gamma\alpha\cot\tfrac{1}{2}B + \alpha\beta\cot\tfrac{1}{2}C = 0$$

at A, B, C co-intersect.

19. The locus of centres of rectangular hyperbolas touching the sides of a given triangle is the self-conjugate circle.

20. The locus of centres of rectangular hyperbolas having ABC for a self-conjugate triangle is the circum-circle.

21. The circum-centre of any triangle, self-conjugate with respect to a parabola, lies on the directrix.

22. A point describes a straight line; prove that the locus of the intersection of its polars with respect to two conics is a conic circumscribing their common self-conjugate triangle.

23. Find the relation between the perpendiculars p, q, r from A, B, C on any tangent; 1°, to the inscribed circle; 2°, to the ellipse touching the sides at their mid points.

Ans. 1°, $(s-a)\,qr + (s-b)\,rp + (s-c)\,pq = 0$

2°, $qr + rp + pq = 0$

[*Note*—Use eq'n (2), § 400, and eq'n (3), § 427.]

24. Through each vertex of the triangle of reference a parallel is drawn to the opposite side; these parallels form a triangle A′B′C′. If p, q, r are the perpendiculars from A, B, C on any tangent to the circum-circle of A′B′C′, prove that

$$\sin A\sqrt{q+r-p} + \sin B\sqrt{r+p-q} + \sin C\sqrt{p+q-r} = 0$$

25. The major axis of a conic inscribed in ABC passes through the point in which the external bisector of the angle A meets BC; prove that the locus of its foci is the conic

$$\alpha^2 = \beta\gamma$$

[*Note*—See *Note*, Ex. 6, page 379. If $(\alpha\beta\gamma)$ is one focus, $\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$ is the other; express cond'n that these are collinear with (0, 1, − 1).]

26. The minor axis of a conic inscribed in the triangle of reference is given $(=\rho)$; prove that the locus of its foci is the curve

$$\alpha\beta\gamma\,(a\alpha + b\beta + c\gamma) = \rho^2\,(a\beta\gamma + b\gamma\alpha + c\alpha\beta)$$

27. Prove that the locus of the foci of conics touching the four lines

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 0, \qquad l\alpha + m\beta + n\gamma = 0$$

is the cubic

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = \frac{l^2 + m^2 + n^2 - 2\,mn\cos A - 2\,nl\cos B - 2\,lm\cos C}{l\alpha + m\beta + n\gamma}$$

28. Prove that the six points in which tangents to any conic from the vertices of the triangle of reference meet the opposite sides lie on a conic.

[*Note*—Let the conic be represented by the general equation in *areal* * co-ordinates

$$\phi(x, y, z) = 0$$

The equation of the tangents from $A(1, 0, 0)$ is (§ 415)

$$a(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = (ax + hy + gz)^2$$

or

$$Cy^2 - 2Fyz + Bz^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The roots of this equation in y/z are the ratios in which BC is divided by the lines (1); the product of these ratios is $\therefore B/C$.

The result follows by Carnot's Theorem (Ex. 2, § 311).]

29. Two points are joined to the vertices of the triangle of reference. Prove that the six points in which the joins meet the sides lie on a conic. If the areal co-ordinates of the points are $(x'y'z')$ and $(x''y''z'')$, find the equation of the conic.

Ans. $\dfrac{x^2}{x'x''} + \ldots + \ldots = \left(\dfrac{1}{y'z''} + \dfrac{1}{y''z'}\right) yz + \ldots + \ldots$

30. A parabola drawn through the mid points of the sides of a triangle ABC meets the sides again in A'', B'', C''; prove that AA'', BB'', CC'' are parallel.

31. If lines are drawn through a point O parallel to the sides of a triangle, the six points in which the parallels meet the sides lie on a conic.

32. If a triangle be self-conjugate with respect to a parabola, prove that its nine-point circle passes through the focus.

33. CA, CB are tangents to a conic at A and B; P is any point on the curve. Any line through C meets AP, BP in Q, R. Prove that BQ, AR meet on the curve.

34. Having given five tangents to a conic, show how to determine their points of contact.

[*Note*—Let ABCDE be a circumscribed pentagon. Then AD, BE meet on the join of C to the point of contact of AE. This follows from Brianchon's Theorem, by supposing two sides of the hexagon to coincide.]

* Areal co-ordinates possess some advantages over trilinears. If we put $x = 0$ in the homogeneous equation of a curve in areal co-ordinates, the roots of the resulting equation in y/z are the ratios in which the curve divides BC. It is worth noticing that the point (xyz) is the centre of gravity of masses x, y, z placed at A, B, C.

35. A parabola inscribed in the triangle of reference touches the line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0;$$

prove that the equation to the directrix is

$$\alpha \cot A\left(\frac{1}{\mu} - \frac{1}{\nu}\right) + \beta \cot B\left(\frac{1}{\nu} - \frac{1}{\lambda}\right) + \gamma \cot C\left(\frac{1}{\lambda} - \frac{1}{\mu}\right) = 0$$

36. The four common tangents to two conics intersect two and two on the sides of their common conjugate triangle.

37. Two triangles are self-polar with respect to a conic; prove that their six vertices lie on a conic.

38. If tangents are drawn to a conic from any point of a straight line whose pole is P, the sum of their distances from a fixed point Q divided respectively by the distances of P from the same tangents is constant.

39. A transversal drawn through a fixed point O meets a conic in P, Q. Prove that the algebraic sum of the distances of P, Q from a fixed straight line divided respectively by the distances of P, Q from the polar of O is constant.

40. If the summits of three angles are collinear, their arms are the opposite sides of a hexagon whose vertices are on a conic.

[See Ex. 48, page 357. This is the converse of Pascal's Theorem.]

41. The three diagonals of the quadrilateral whose sides are

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 0, \qquad \lambda\alpha + \mu\beta + \nu\gamma = 0$$

are divided harmonically by the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 + r(\lambda\alpha + \mu\beta + \nu\gamma)^2 = 0$$

(*Hesse.*)

42. The equation to the isotropic lines through A is

$$\beta^2 + \gamma^2 + 2\beta\gamma \cos A = 0$$

43. The trilinear co-ordinates of the focoids are proportional to

$$\cos B + \sqrt{-1}\sin B, \qquad \cos A - \sqrt{-1}\sin A, \qquad -1$$

[*Note*—Solve for $\alpha : \beta : \gamma$ from equations to circum-circle and line at infinity.]

44. Find the circle of curvature at the point A on the conic

$$\lambda\alpha\beta = \gamma^2$$

Ans. $\sin A(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) = \lambda\beta(\alpha\sin A + \beta\sin B + \gamma\sin C)$

45. Given in position the three diagonals of a complete quadrilateral, and that one of its sides passes through a fixed point, show that the other sides pass through fixed points.

[See *Note*, § 436.]

46. Prove that the equation of the circle described on the side BC of the triangle of reference, and containing an angle λ, is

$$\sin\lambda\,(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C)$$
$$= \alpha\sin(\lambda - A)\,(\alpha\sin A + \beta\sin B + \gamma\sin C)$$

Hence, or otherwise, prove that the trilinear co-ordinates of a point at which the sides subtend angles λ, μ, ν are proportional to

$$\sin\lambda/\sin(A-\lambda), \qquad \sin\mu/\sin(B-\mu), \qquad \sin\nu/\sin(C-\nu)$$

[*Note*—Find co-ord's of point where circle meets AC; use result of Ex. 9.]

47. Find the areal co-ordinates of the radical centre of circles on AB, BC, CA as diameters.

Ans. $\cot B \cot C$, $\cot C \cot A$, $\cot A \cot B$

48. O is the ortho-centre, G the centroid of the triangle ABC. Find the equation to the circle on OG as diameter; and prove that this circle is co-axal with the circum-circle, the nine-point circle and the polar circle of the triangle.

Ans. $3(\beta\gamma\sin A + \ldots + \ldots) = 2(\alpha\sin A + \ldots + \ldots)(\alpha\cos A + \ldots + \ldots)$

49. Find the equation of the line passing through the mid points of the three diagonals of the quadrilateral whose sides are

$$\lambda\alpha \pm \mu\beta \pm \nu\gamma = 0$$

Ans. $\lambda^2\alpha/\sin A + \mu^2\beta/\sin B + \nu^2\gamma/\sin C = 0$

50. Find the equation to the asymptotes of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

Ans. $(l\alpha^2 + m\beta^2 + n\gamma^2)(a^2/l + b^2/m + c^2/n) = (a\alpha + b\beta + c\gamma)^2$

51. Find the locus of a point such that the tangents drawn from it to the conics

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0, \qquad l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0$$

form a harmonic pencil.

Ans. The conic $F \equiv ll'(mn' + m'n)\alpha^2 + \ldots + \ldots = 0$

[*Note*—Form eq'ns of pairs of tangents from $(\alpha'\beta'\gamma')$, (§ 415); putting $\alpha = 0$ we obtain the eq'ns of the pairs of conjugate rays joining A to the range in which the pencil formed by the four tangents meets BC; then use cond'n of § 141.]

52. Show that the eight points of contact of two conics with their common tangents lie on the conic F.

53. If two circles cut orthogonally, and the tangents from a point P to the circles form a harmonic pencil, the locus of P is a line-pair.

54. The polars of the mid points of the sides of the triangle ABC with respect to an inscribed conic form a triangle whose area is constant, and = D. (*Faure.*)

55. The director circles of conics touching four given lines form a co-axal system. Prove this by showing that if the four lines are

$$\lambda\alpha \pm \mu\beta \pm \nu\gamma = 0$$

the circles pass through the two fixed points determined by

$$\frac{\beta^2 + \gamma^2 + 2\beta\gamma\cos A}{\lambda^2} = \frac{\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B}{\mu^2} = \frac{\alpha^2 + \beta^2 + 2\alpha\beta\cos C}{\nu^2}$$

[*Note*—See *Note,* § 436, and Ex. 10, page 386. Another proof of the theorem will be given in Chapter XV. One conic of the system is a parabola; the directrix of this parabola is the radical axis of the system.]

56. The circles described on the three diagonals of a complete quadrilateral as diameters are co-axal.

57. The director circles of conics inscribed in a triangle are cut orthogonally by the self-conjugate circle.

58. Given three tangents to a conic and the sum of the squares of the axes; prove that the locus of its centre is a circle.

59. Find the equation of a parabola inscribed in the triangle of reference, and whose focus is $(\alpha'\beta'\gamma')$.

Ans. $\sin A\sqrt{\alpha/\alpha'} + \sin B\sqrt{\beta/\beta'} + \sin C\sqrt{\gamma/\gamma'} = 0$

60. Prove that the equation to the Simson's Line of a point $(\alpha'\beta'\gamma')$ on the circum-circle of the triangle of reference is

$$\alpha\left(\sin A - \frac{R\sin 2A}{\alpha'}\right) + \ldots + \ldots = 0$$

61. The locus of centres of conics inscribed in a triangle, and such that the normals at the points of contact co-intersect, is a cubic which passes through the vertices of the triangle, the centroid, the ortho-centre, the in- and ex-centres, and the mid points of the sides and perpendiculars.

62. Tangents are drawn from a fixed point P to parabolas inscribed in a given triangle; prove that the locus of the point of contact is a cubic.

63. If the triangle of reference is equilateral, find the equation to the axes of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

Ans. $l^2(m - n)\alpha^2 + \ldots + \ldots + 2mn(m - n)\beta\gamma + \ldots + \ldots = 0$

[*Note*—The axes are the loci of points whose polars with respect to the conic and its director circle are parallel.]

64. Obtain equations to determine the foci of the conic in the last question.

Ans. $l(m + n)\alpha^2 + mn(\beta + \gamma)^2 = m(n + l)\beta^2 + nl(\gamma + \alpha)^2$
$$= n(l + m)\gamma^2 + lm(\alpha + \beta)^2$$

[*Note*—The equation of the tangents from a focus satisfies the analytical conditions for a circle. The foci may also be obtained by the methods of §§ 368, 372, using the result of Ex. 43, page 395; the equations thus obtained are however unsymmetrical.]

65. If the general trilinear equation represent a circle whose centre is $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and radius r, prove that

$$r^2 = -2\phi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})/\theta'$$

[*Note*—See § 443. The square of the tangent from the centre is $-r^2$.]

66. If the general trilinear equation represent a pair of lines, prove that the angle between them is

$$\tan^{-1}\sqrt{-4\theta/\theta'^2}$$

67. The co-ordinates of the focoids are proportional to

$$e^{\theta\sqrt{-1}}, \quad e^{\phi\sqrt{-1}}, \quad e^{\psi\sqrt{-1}}$$

where θ, ϕ, ψ are the angles which the sides of the triangle of reference make with any line; these angles being measured round from the line in the same direction. (*Prof^r Genese.*)

68. Two concentric and similar conics are, one inscribed and the other circumscribed to a triangle. Prove that the locus of the common centre consists of two circles. (*Faure.*)

69. If θ, ϕ, ψ are the angles which the sides of the triangle of reference make with an axis of the conic represented by the general trilinear equation, prove that

$$a \sin 2\theta + b \sin 2\phi + c \sin 2\psi + 2f \sin(\phi + \psi) + 2g \sin(\psi + \theta) + 2h \sin(\theta + \phi) = 0$$

(*M. Combier.*)

70. The locus of foci of conics inscribed in the quadrilateral whose sides are

$$\lambda\alpha \pm \mu\beta \pm \nu\gamma = 0$$

is the cubic

$$(\alpha \sin A + \ldots + \ldots)(\lambda^2\alpha^2 \cot A + \ldots + \ldots) = (\beta\gamma \sin A + \ldots + \ldots)(\lambda^2\alpha/\sin A + \ldots + \ldots)$$

(*M. Koehler.*)

71. A conic is inscribed in a quadrilateral ABCD. If p_1, p_2, p_3, p_4 are the perpendiculars from A, B, C, D on any tangent, prove that

$$p_1 p_3 \propto p_2 p_4$$

72. The condition that five points A, B, C, P, Q may be on a parabola is

$$\sqrt{PBC \,.\, QBC \,.\, PQA} + \ldots + \ldots = 0$$

where PBC stands for the area of the triangle PBC, &c.

(*Prof' Genese.*)

[*Note*—Use areal co-ordinates ; take ABC as triangle of reference.]

73. Prove that the conics

$$\beta\gamma + \gamma\alpha + \alpha\beta = 0, \quad \sin\frac{A}{2}\sqrt{\alpha} + \sin\frac{B}{2}\sqrt{\beta} + \sin\frac{C}{2}\sqrt{\gamma} = 0$$

are confocal.

CHAPTER XV

ENVELOPES

§ **447.** If the equation to a line involve a parameter μ in the first degree, then we have seen (§ 130) that the line passes through a fixed point. Let

$$\mu^2 L + 2\mu R + M = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equation of a straight line involving a parameter μ in the second degree.

If a line of the system (1) pass through a given point, then

$$\mu^2 L' + 2\mu R' + M' = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where L', M', R' are the results of substituting the co-ordinates of the point in L, M, R.

The equation (2) is a quadratic in μ; hence two lines of the system can be drawn through a given point P. In other words, two tangents can be drawn from P to the envelope of the system (1). But if P is on the envelope these tangents coincide: hence the quadratic (2) has equal roots.

The envelope of the system of lines (1) is ∴ the conic

$$LM = R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

[Compare § 354.]

Cor'—If $L = 0$, $M = 0$, $R = 0$ represent any curves, then all curves of the system (1) touch the curve (3). This is proved by the same reasoning.

Ex. 1. Find the envelope of a line which moves so that the sum of its intercepts on the axes is constant ($= \delta$).

Let the intercept on the axis of x be a; the equation of the line is

$$\frac{x}{a} + \frac{y}{\delta - a} = 1$$

or
$$a^2 + (y - x - \delta)a + x\delta = 0$$

This equation involves the parameter a in the second degree; and the envelope is

$$(y - x - \delta)^2 = 4x\delta,$$

a parabola touching the axes.

Ex. 2. Find the envelope of a line which moves so that the product of the perpendiculars on it from two fixed points $(+c, 0)$, $(-c, 0)$ is constant $(= b^2)$.

Let the line be $$x\cos\alpha + y\sin\alpha - p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad (p + c\cos\alpha)(p - c\cos\alpha) = b^2 \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating p from (1), (2), we see that we have to find the envelope of

$$(x\cos\alpha + y\sin\alpha)^2 = b^2(\sin^2\alpha + \cos^2\alpha) + c^2\cos^2\alpha$$

or $$(x^2 - b^2 - c^2)\cos^2\alpha + 2xy\cos\alpha\sin\alpha + (y^2 - b^2)\sin^2\alpha = 0$$

Expressing that this quadratic in $\cos\alpha/\sin\alpha$ has equal roots, we see that the envelope is

$$(x^2 - b^2 - c^2)(y^2 - b^2) = x^2y^2$$

or $$x^2/(b^2 + c^2) + y^2/b^2 = 1$$

§ **448.** *If* ϕ *is indeterminate, the envelope of*

$$L\cos\phi + M\sin\phi = R \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is $$L^2 + M^2 = R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting $\mu = \tan\frac{1}{2}\phi$, (1) becomes

$$L(1 - \mu^2) + 2M\mu = R(1 + \mu^2)$$

The envelope is $\therefore$

$$(L + R)(R - L) = M^2; \text{ and } \therefore \text{ \&c.}$$

Ex. If CP, CD are conjugate semi-diameters of an ellipse, find the envelope of PD.

If θ is the eccentric angle of P, the equation of PD is found to be

$$(ay - bx)\sin\theta + (ay + bx)\cos\theta = ab$$

The envelope is $\therefore$ the ellipse

$$2a^2y^2 + 2b^2x^2 = a^2b^2$$

§ **449.** To find the envelope of the line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $$(a, b, c, f, g, h)(\lambda, \mu, \nu)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating ν from (1), (2), we see that the lines of the system which pass through a given point $(\alpha'\beta'\gamma')$ are determined by the equation

$$(a, b, c, f, g, h)\left(\lambda, \mu, -\frac{\lambda\alpha' + \mu\beta'}{\gamma'}\right)^2 = 0$$

Expressing the condition that this quadratic in λ/μ has equal roots, and reducing (as in § 323), we obtain for the equation of the envelope

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta = 0 \quad . \quad . \quad (3)$$

where A, B, &c. have the usual meanings.

§ 450. By writing x, y, 1 instead of α, β, γ, the preceding investigation becomes applicable to Cartesians. It is convenient to interchange the large and small letters A, a, &c. Hence the envelope of the line

$$\lambda x + \mu y + \nu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $$(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the conic $$(a, b, c, f, g, h)(x, y, 1)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $$a = BC - F^2, \qquad f = GH - AF, \quad \&c.$$

Since eq'n (2) is the condition that the line (1) should touch the conic (3), this result should be equivalent to that of § 323. This is verified by the identities

$$BC - F^2 = \Delta a, \qquad GH - AF = \Delta f, \quad \&c.$$

where the letters Δ, A, a, &c. have the same meanings as in § 323.

Cor' (1)—The centre of the envelope is $(G/C, F/C)$. [§ 323.]

Cor' (2)—The equation to the director circle of the envelope is given (§ 325). The envelope is a rectangular hyperbola if the radius of this circle is zero, i.e. if

$$G^2 + F^2 = C(A + B)$$

Cor' (3)—If the axis of x is a tangent to the envelope, the tangential equation (2) is satisfied by $\lambda = 0$, $\nu = 0$; hence $B = 0$. The envelope $\therefore$ touches the axes if

$$A = 0, \qquad B = 0$$

Cor' (4)—The envelope is a parabola (§ 287) if $C = 0$; this condition may also be deduced by noticing that if the line at infinity is a tangent the tangential equation is satisfied by

$$\lambda = 0, \qquad \mu = 0$$

Cor' (5)—If the origin is a focus, the tangential equation is satisfied by

$$\lambda = 1, \qquad \mu = \pm\sqrt{-1}, \qquad \nu = 0;$$

$$\therefore \quad A - B = 0, \qquad H = 0$$

§ **451.** Ex. 1. *The locus of centres of conics inscribed in a quadrilateral is a straight line.*

Take the asymptotes of one inscribed conic as axes; so that its tangential equation is [§ 450, *Cor's* (1), (3)]

$$\Sigma \equiv C\nu^2 + 2H\lambda\mu = 0$$

Let the tangential equation to another inscribed conic be

$$\Sigma' \equiv (A', B', C', F', G', H')(\lambda, \mu, \nu)^2 = 0$$

Then if the line $\lambda x + \mu y + \nu = 0$

touches both conics, we have

$$\Sigma = 0 \quad \text{and} \quad \Sigma' = 0;$$

$$\therefore\ \Sigma + k\Sigma' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence (1) is the tangential equation of conics inscribed in the quadrilateral formed by the common tangents of Σ and Σ'.

The centre of the conic (1) is [§ 450, *Cor'* (1)]

$$x = kG'/(C + kC'), \qquad y = kF'/(C + kC')$$

Eliminate k; the locus of centres is $\therefore$ the line

$$x/y = G'/F'$$

[Compare solution on page 307.]

Ex. 2. *The director circles of conics inscribed in a quadrilateral form a co-axal system.*

For the director circle of the conic whose tangential eq'n is (1) is (§ 325)

$$C(x^2 + y^2) + k[C'(x^2 + y^2) - 2G'x - 2F'y + A' + B'] = 0$$

This is the eq'n of a circle co-axal with the director circles of Σ and Σ'; and the radical axis is the line

$$2G'x + 2F'y = A' + B'$$

It follows from the result of Ex. 1 that the radical axis is perpendicular to the locus of centres.

TANGENTIAL CO-ORDINATES

§ **452.** The *Boothian tangential co-ordinates* of a straight line are the reciprocals of its intercepts on the axes.

Thus (l, m) are the tangential co-ordinates of the line

$$lx + my = 1$$

The equation to a point $(x'y')$ in this system is

$$lx' + my' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i.e. the co-ordinates of every line passing through the point are connected by the relation (1).

The envelope of a line is a conic if its co-ordinates l, m are connected by a relation of the second degree; for the results of § 450 become applicable to this system if we write l, m, -1 for λ, μ, ν.

Ex. Let the tangent to the ellipse (a, b) at the point whose eccentric angle is ϕ intercept lengths $1/l$, $1/m$ on its axes; then

$$1/l = a/\cos\phi, \qquad 1/m = b/\sin\phi$$

$$\therefore\quad a^2 l^2 + b^2 m^2 = 1$$

This is the Boothian tangential equation of an ellipse; i.e. it is the relation connecting the co-ordinates of every tangent to the curve.

§ 453. In another system of line co-ordinates the tangential co-ordinates of a line are the perpendiculars p, q, r on the line from the vertices A, B, C of the triangle of reference; these co-ordinates are connected by the relation of § 413.

The equation of the point whose areal co-ordinates are $(x'y'z')$ is (see § 410)

$$px' + qy' + rz' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i.e. the co-ordinates of every line passing through the point are connected by the relation (1).

The results of § 449 are applicable to this system if we substitute p sin A, q sin B, r sin C for λ, μ, ν [see eq'n (2), § 400]; or we may replace λ, μ, ν by p, q, r, and α, β, γ by x, y, z (areal co-ordinates).

We add some propositions on this system of co-ordinates.

§ 454. I. *The equation*

$$u + \lambda v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

represents a point dividing the join of the points $u = 0$, $v = 0$ *in the ratio* $\lambda : 1$.

Let $(x'y'z')$, $(x''y''z'')$ be the areal co-ord's of the latter points.

Written at full length, (1) becomes

$$px' + qy' + rz' + \lambda\,(px'' + qy'' + rz'') = 0$$

or

$$p\,(x' + \lambda x'') + q\,(y' + \lambda y'') + r\,(z' + \lambda z'') = 0$$

This is the equation to the point whose areal co-ord's are

$$\frac{x' + \lambda x''}{1 + \lambda}, \quad \frac{y' + \lambda y''}{1 + \lambda}, \quad \frac{z' + \lambda z''}{1 + \lambda}; \quad \text{and} \therefore \text{ \&c.}$$

II. *The co-ordinates of a line through the intersection of the lines* $(p'q'r')$, $(p''q''r'')$ *are proportional to*

$$p' - \lambda p'', \qquad q' - \lambda q'', \qquad r' - \lambda r''$$

The point equations of the given lines are

$$u \equiv p'x + q'y + r'z = 0, \qquad v \equiv p''x + q''y + r''z = 0$$

Hence (§ 128) the point equation of a line passing through their intersection, and dividing the angle between them into parts whose sines are in the ratio $\lambda : 1$, is

$$u - \lambda v = 0, \quad \text{or} \quad (p' - \lambda p'')x + (q' - \lambda q'')y + (r' - \lambda r'')z = 0;$$

the co-ord's of this line are in the ratios of the coeff's of x, y, z; and $\therefore$ &c.

III. The length of the perpendicular from the point $(x'y'z')$ on the line (pqr) is

$$px' + qy' + rz'$$

For the point equation of the line is

$$px + qy + rz = 0; \quad \text{and } \therefore \text{ \&c.} \qquad [\S\ 410.]$$

IV. Hence the tangential equation of a circle whose centre is $(x'y'z')$ and radius ρ is

$$px' + qy' + rz' = \rho$$

This may be written in the homogeneous form

$$4\,D^2\,(px' + qy' + rz')^2 = \rho^2\,\Omega,$$

where Ω is the sinister of the equation of § 413.

V. Let the tangential equation of a conic be

$$\phi\,(p, q, r) \equiv (a, b, c, f, g, h)\,(p, q, r)^2 = 0$$

Let $(p'q'r')$, $(p''q''r'')$ be two tangents to the conic.

The equation of the chord joining $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ is given in § 414. In this equation replace α, β, γ by p, q, r; * α', β', γ' by $\frac{p'r}{r'}, \frac{q'r}{r'}, \frac{r'r}{r'}$; and $\alpha'', \beta'', \gamma''$ by $\frac{p''r}{r''}, \frac{q''r}{r''}, \frac{r''r}{r''}$.

Then the resulting equation is satisfied if we substitute p', q', r' or p'', q'', r'' for p, q, r; and $\therefore$ the given lines are tangents to the envelope represented by the equation.

Further, when the equation is expanded the result is divisible by r; and the quotient is the equation of the point of intersection of the tangents $(p'q'r')$, $(p''q''r'')$.

Hence, as in § 414, we deduce the equation of the point of contact of $(p'q'r')$; this may be written in either of the forms

$$p'\frac{d\phi}{dp} + q'\frac{d\phi}{dq} + r'\frac{d\phi}{dr} = 0, \quad \text{or} \quad p\frac{d\phi'}{dp'} + q\frac{d\phi'}{dq'} + r\frac{d\phi'}{dr'} = 0$$

If $(p'q'r')$ is not a tangent, either of these equations represents its pole.

* It might seem simpler to replace α' by p', &c.; the result, however, when expanded is in that case of the first form given in IV; and is the equation to a circle.

VI. The conic is a parabola if its equation is satisfied by the co-ordinates of the line at infinity (1, 1, 1); i. e. if

$$a + b + c + 2f + 2g + 2h = 0$$

VII. The centre being the pole of the line at infinity, its equation is

$$\frac{d\phi}{dp} + \frac{d\phi}{dq} + \frac{d\phi}{dr} = 0$$

Exercises on Chapter XV

1. P is any point on the parabola

$$y^2 = 4ax;$$

PM, PN are perpendiculars on the axes. Find the envelope of MN.

Ans. The parabola $y^2 = -16ax$

2. Find the envelope of a line which forms with the axes a triangle of constant area k^2.

Ans. The hyperbola $2xy \sin\omega = k^2$

3. Two fixed lines meet in O; if P, Q are points on the lines such that

$$m\,OP + n\,OQ$$

is constant; the envelope of PQ is a parabola touching the lines.

4. If circles are described on double ordinates to the axis of a parabola as diameters, their envelope is an equal parabola.

5. Find the envelope of a line such that the sum of the squares of the perpendiculars on it from two fixed points (c, 0) and (– c, 0) is constant ($= 2k^2$).

Ans. The conic $x^2/(k^2 - c^2) + y^2/k^2 = 1$

6. Find the envelope if the difference of the squares of the perpendiculars $= 2k^2$.

Ans. The parabola $c^2y^2 = k^2(k^2 - 2cx)$

7. Two fixed lines intersect at O; a circle passing through O and another fixed point P meets the given lines in L, M. Show that the envelope of LM is a parabola touching the given lines.

8. The axes of an ellipse are given in position. If the product of the lengths of its semi-axes $= k^2$, find its envelope.

Ans. The hyperbolas $2\,xy = \pm k^2$

9. The normals at four points P, Q, R, S on an ellipse co-intersect: if PQ passes through a fixed point, prove that RS envelopes a parabola touching the axes of the ellipse.

[*Note*—Use the eq'ns of PQ, RS given at end of § 373.]

10. Being given the radius of the director circle of an ellipse, and two conjugate diameters in position: show that the ellipse touches four fixed straight lines.

11. Prove that the envelope of the polar of a given point P with respect to a system of confocals whose centre is C is a parabola whose directrix is CP.

12. A transversal meets the sides of a given triangle in P, Q, R. If PQ : QR is a given ratio, the envelope of the transversal is a parabola touching the sides of the triangle.

13. A line which revolves round a fixed point O meets a given line AB in P. Prove that the envelope of the bisector of the angle OPA is a parabola whose focus is O and directrix AB.

Prove also that the envelope of a line through P inclined at a constant angle to OP is a parabola which touches AB and has its focus at O.

14. The diagonals of a quadrilateral inscribed in a circle intersect at right angles in a fixed point. Prove that the sides of the quadrilateral touch a fixed conic, whose foci are the fixed point and the centre of the circle.

15. The envelope of chords of a conic which subtend a right angle at a fixed point O is a conic, whose focus is O and directrix the polar of O.

16. If a straight line is cut harmonically by two circles, its envelope is a conic whose foci are the centres of the circles.

17. A transversal is cut harmonically by two conics. Show that its envelope is a conic.

[*Note*—Refer the conics to their common self-conjugate triangle. Adopt the notation of Ex. 51, page 396; let the transversal be

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eliminating γ between (1) and the equation of each conic we obtain the equations of the line-pairs joining the vertex C of the triangle of reference to the intersections of the transversal (1) with the conics. Using the condition of § 141 we find that the transversal (1) is cut harmonically if

$$(mn' + m'n)\lambda^2 + (nl' + n'l)\mu^2 + (lm' + l'm)\nu^2 = 0 \quad . \quad . \quad (2)$$

The envelope of the line (1) subject to the condition (2) is (§ 449) the conic

$$\Phi \equiv \alpha^2/(mn' + m'n) + \beta^2/(nl' + n'l) + \gamma^2/(lm' + l'm) = 0.]$$

18. Prove that the eight tangents to two conics at their points of intersection touch the conic Φ.

19. The vertices A′, B′, C′ of a triangle move along fixed lines BC, CA, AB; two sides A′C′, B′C′ pass through fixed points Q, P. Prove that the envelope of the third side A′B′ is a conic touching AQ, BP.

[*Note*—Take ABC as triangle of reference; let P, Q be $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$.]

20. Given three points on a conic; if one asymptote pass through a fixed point, the other will envelope a conic touching the sides of the given triangle.

21. Two sides of a triangle inscribed in a conic pass through fixed points P, Q. Prove that the envelope of the third side is a conic touching the given conic at its intersections with PQ.

[*Note*—Let the given conic be

$$\alpha\beta = \lambda\gamma^2;$$

PQ being the line $\gamma = 0$.]

22. Two tangents making a constant angle are drawn to two given circles; prove that the join of the points of contact touches a fixed conic.

23. The envelope of a transversal on which two given circles intercept equal chords is a paraboia.

24. PQ is a variable diameter of a conic, and the chords PR and QR make equal angles with the tangent at R; prove that all the lines PR and QR touch the same conic.

25. Show that the foci of a conic touching the sides of a parallelogram lie on a rectangular hyperbola circumscribing the parallelogram.

[*Note*—See §§ 369, 451.]

26. The locus of foci of conics inscribed in a quadrilateral is in general a cubic.

27. If p, q, r are the perpendiculars on a line from three given points A, B, C: find the envelope of the line if

$$p^2 = q^2 + r^2$$

Ans. The conic $x^2 = y^2 + z^2$

28. The envelope of chords of the parabola

$$y^2 = 4ax$$

which subtend a constant angle α at the vertex is the conic

$$(x - 4a)^2 + 4y^2 + 4\cot^2\alpha\,(y^2 - 4ax) = 0$$

29. Conics are drawn with a fixed point within a given circle as one of their foci, and touching two fixed tangents to the circle; show that their chords of intersection with the circle envelope a second circle.

30. The envelope of a circle on a chord of a conic fixed in direction as diameter is a conic.

31. A straight line moves so that the sum of the squares of the perpendiculars on it from any number of fixed points is constant: prove that its envelope is a conic.

32. If the normals drawn to the ellipse (a, b) from any point on the normal at (hk) meet the ellipse in P, Q, R; prove that the sides of the triangle PQR touch the parabola

$$(xh/a^2 + yk/b^2 + 1)^2 = 4hkxy/(a^2b^2)$$

33. A triangle is inscribed in a conic S, and two sides touch another conic S′; prove that the envelope of the third side is a conic passing through the intersections of S, S′.

34. Interpret the tangential equations

$$pq + qr + rp = 0, \qquad p^2 = qr, \qquad p^2 = 4qr$$

Ans. (1) The ellipse touching the sides of the triangle of reference at their mid points.

(2) The parabola touching AB, AC at B, C.

(3) The ellipse touching AB, AC at B, C, and passing through the centroid of ABC.

35. Find the tangential equations to the nine-point circle and the self-conjugate circle of the triangle of reference.

Ans. $a\sqrt{q+r} + b\sqrt{r+p} + c\sqrt{p+q} = 0,$

$p^2 \tan A + q^2 \tan B + r^2 \tan C = 0$

36. Interpret the tangential equation

$$p\,(a_2 q + a_1 r) = a_3\,(p - q)\,(p - r)$$

Ans. The triangle of reference being ABC, the *form* of the equation shows that it represents a conic inscribed in the quadrilateral whose vertices are A, the mid points of AB, AC, and the point dividing BC in the ratio $a_1 : a_2$.

37. The tangential equation of a conic whose foci are $(x'y'z')$, $(x''y''z'')$ is

$$(x'p + y'q + z'r)\,(x''p + y''q + z''r) = \lambda\,\Omega$$

where Ω is the sinister of the equation of § 413, and λ is a constant.

38. If the equation to the normal at either extremity of a chord PQ of an ellipse be

$$\lambda x + \mu y + \nu = 0$$

prove that

$$(a^2 - b^2)\,\lambda\mu + \alpha\mu\nu - \beta\nu\lambda = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

the ellipse being referred to its axes, and (α, β) the pole of PQ.

(*Prof' Burnside, Educ' Times Reprint*, Vol. VI., p. 108.)

39. Hence prove that a parabola can be described touching the two normals, the chord PQ and the axes of the conic; the diameter conjugate to the chord being the directrix.

[*Note*—The preceding equation (1) is satisfied by

$$(\lambda = 0,\ \mu = 0), \qquad (\mu = 0,\ \nu = 0), \qquad (\lambda = 0,\ \nu = 0);$$

hence the curve of the second degree represented by this equation touches the line at infinity and the axes of the conic. It is also satisfied by

$$(\lambda = \alpha/a^2,\ \mu = \beta/b^2,\ \nu = -1);$$

hence this curve touches the chord PQ.

If

$$\lambda x + \mu y + \nu = 0$$

is the eq'n of one of the tangents from (α, β), then

$$\lambda\alpha + \mu\beta + \nu = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Eliminating ν between (1) and (2) we find that the sum of the coeff' of λ^2, μ^2 is zero, or the tangents to the parabola from (α, β) are at right angles. Hence (α, β) is a point on the directrix; also the centre is another point on the directrix, since the parabola touches the axes.]

CHAPTER XVI

METHODS OF TRANSFORMATION

RECIPROCAL POLARS

§ **455.** By taking the polars of all the points and the poles of all the lines in a plane figure A with respect to a fixed conic S, a new figure B is constructed; if B is treated in the same manner, the figure A is reproduced.

Two such figures are said to be *polar reciprocals.*

If p, q* are the lines in one figure corresponding to two points P, Q in the other, the intersection (p, q) corresponds to the line PQ [*Cor′* (3), § 306].

To a series of collinear points P, Q, R, ... in one figure corresponds a series of concurrent lines p, q, r, ... in the other [*Cor′* (4), § 306].

Thus from any theorem relating to the position of points and lines we can deduce another relating to lines and points.

Ex. The following theorem is the reciprocal of Ex. 7, p. 174.

If the three vertices of a triangle move one on each of three concurrent lines, and two of its sides pass through fixed points; then the third side passes through a fixed point.

§ **456.** Let P, Q, R, ... be a series of points on a curve Σ; their polars p, q, r, ... with respect to the fixed conic S envelope another curve Σ'.

* It is sometimes convenient to denote lines by single letters p, q, r, ... ; the point of intersection of two lines p, q is denoted by (p, q).

Also the points (p, q), (q, r), ... are the poles of the lines PQ, QR, ...

If now we suppose that the points P, Q are consecutive, then PQ becomes the tangent to Σ at P, and (p, q) becomes the point of contact of p with its envelope Σ'.

Hence Σ is the envelope of the polars of points on Σ' with respect to S; and the relation between the curves Σ, Σ' is reciprocal.

The relation between the reciprocal curves Σ, Σ' may also be stated thus: each curve is the locus of poles of tangents to the other with respect to the auxiliary conic S.

Cor' (1)—To a point on any curve and its tangent correspond a tangent to the reciprocal curve and its point of contact.

Cor' (2)—To a point of intersection of two curves corresponds a common tangent to the reciprocal curves.

Cor' (3)—If two curves touch, their reciprocals also touch.

For the original curves have a point and the tangent at that point common; the reciprocal curves have $\therefore$ a tangent and its point of contact common.

§ **457**. *The polar reciprocal of a conic is a conic.*

Refer the given conic Σ and the auxiliary conic S to their common self-conjugate triangle; let their equations be,

to S, $$L\alpha^2 + M\beta^2 + N\gamma^2 = 0$$

and to Σ, $$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

The reciprocal polar of Σ is the locus of poles of tangents to Σ with respect to S.

Expressing (§ 436) the condition that the polar of $(\alpha'\beta'\gamma')$ with respect to S, viz.

$$L\alpha'\alpha + M\beta'\beta + N\gamma'\gamma = 0$$

touches Σ, and writing α, β, γ for α', β', γ'; we find that the reciprocal polar of the conic Σ is the conic Σ',

$$L^2\alpha^2/l + M^2\beta^2/m + N^2\gamma^2/n = 0$$

§ **458.** Let P, Q be two points on Σ; PT, QT the tangents at these points. Also, let p, q be the points on Σ' corresponding to the tangents PT, QT; let the tangents at p, q meet in t.

Then T is the pole of pq and t is the pole of PQ with respect to the auxiliary conic S [*Cor'* (3), § 306.]

Hence—*To a line and its pole with respect to a conic correspond a point and its polar with respect to the reciprocal conic.*

§ **459.** Ex. 1. The following theorem is the reciprocal of Ex. 24, p. 305.

The locus of the pole of a given line with respect to conics inscribed in a given quadrilateral is a straight line.

Ex. 2. Reciprocating Ex. 25, p. 305, we find—

The envelope of the polars of a given point with respect to conics inscribed in a given quadrilateral is a conic.

Ex. 3. The theorem, Ex. 33, p. 409, is the reciprocal of Ex. 2, § 441.

Ex. 4. Reciprocate Pascal's Theorem (§ 431).

Let A, B, C, A′, B′, C′ be the vertices of the hexagon.

Their polars with respect to the auxiliary conic S are six tangents a, b, c, a′, b′, c′ to the reciprocal conic Σ'.

To the opposite sides AB, A′B′ of the first hexagon correspond opposite vertices (a, b), (a′, b′) of the reciprocal hexagon; and to the intersection of AB, A′B′ corresponds the join of the intersections (a, b), (a′, b′).

Similarly with reference to the other pairs of opposite sides of the given hexagon; and as the intersections of its three pairs of opposite sides are collinear, the joins of opposite vertices of the reciprocal hexagon cointersect. This is Brianchon's Theorem (§ 432).

Note—The learner will remark that in many cases the process of reciprocation reduces to a mechanical interchange of the words 'point' and 'line,' 'locus' and 'envelope,' 'inscribed' and 'circumscribed,' &c.

RECIPROCATION WITH RESPECT TO A CIRCLE

§ **460.** If the auxiliary conic S is a circle, the construction for the reciprocal polar of a given curve Σ may be stated thus.

Let O be the centre and δ the radius of the circle S. [See fig′, page 415.]

Let P be any point on the curve Σ; draw OM perpendicular to the tangent at P. On OM take a point p such that

$$\mathrm{OM} \, . \, \mathrm{O}p = \delta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then Σ' is the locus of p.

If δ changes, the curve Σ' will evidently remain similar to itself. As we are usually concerned only with the shape of Σ', it is unnecessary to mention the radius δ; and we may simply speak of Σ' as the reciprocal of Σ *with respect to the point* O. The point O may be called the *origin.*

Cor—We see from the relation (1) that—*The distance of any point from the origin varies inversely as the distance of its reciprocal therefrom.*

§ 461. Let the lines a, b be the reciprocals of the points A, B; then a is perpendicular to OA and b to OB.

Hence *the angle between the two lines* (a, b) *is equal to the angle* AOB, *which the join of their poles subtends at the origin.*

This principle is often useful.

§ 462. Returning to the construction of § 458, let the point T coincide with the origin O. Then pq, which is the reciprocal of the origin, is the line at infinity; t is the centre of the reciprocal conic Σ', and tp, tq are its asymptotes.

The centre and asymptotes of the reciprocal of a given conic Σ with respect to any origin O are ∴ determined by this construction—

Draw tangents OP, OQ *to the conic* Σ; *the centre of the reciprocal conic is the reciprocal of* PQ, *and its asymptotes are the reciprocals of the points of contact* P, Q.

Cor (1)—The angle between the asymptotes of Σ' is the supplement of the angle POQ (§ 461).

Cor (2)—Hence Σ' is a rectangular hyperbola if POQ is a right angle; i. e. if the origin O is on the director circle of the conic Σ.

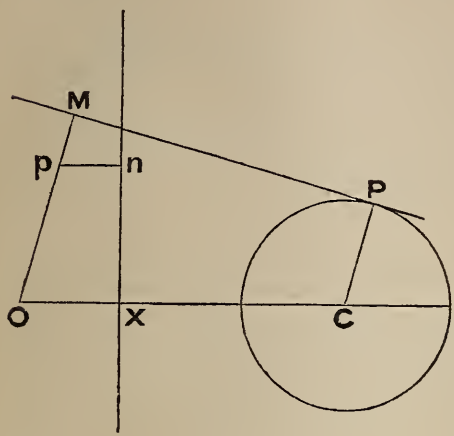
Cor (3)—As *either* Σ or Σ' may be regarded (§ 456) as the original conic, we see that—*The reciprocal of a rectangular hyperbola with respect to any point* O *is a conic whose director circle passes through* O.

Cor (4)—To find the condition that the reciprocal conic Σ' may be an hyperbola.

The points p, q in which it is met by the line at infinity must be real and different; the tangents OP, OQ must ∴ be real and different, i. e. the point O must be *outside* the conic Σ.

Similarly the reciprocal curve Σ' is an ellipse if the origin O is *inside* the conic Σ; and it is a parabola if the origin O is *on* the conic Σ.

§ **463.** *To find the polar reciprocal of a circle whose centre is* C *and radius* r *with respect to an origin* O.



Make the construction of § 460; let the polar of the point C with respect to O meet OC in X.

Draw pn perpendicular to the polar of C. Then PM is the polar of p, and nX is the polar of C.

$$\therefore\ Op : OC = pn : CP$$

[§ 202.]

$$\therefore\ Op : pn = OC : r$$

The reciprocal polar is ∴ a conic whose focus is O, directrix the polar of C, and eccentricity OC/r.

Cor^y (1)—The latus rectum of the conic varies inversely as r.

For it $= 2\,e\,OX = 2\,(OC/r)\,(\delta^2/OC) = 2\,\delta^2/r$

Cor^y (2)—The centre of the reciprocal conic is the reciprocal of the chord of contact of tangents from O to the circle (C).

This is proved in § 462.

§ **464.** Ex. 1. 'The product of the segments of chords of a circle drawn through a fixed point O is constant.' (Euclid III. 35, 36.)

The reciprocal of this is (*Cor^y*, § 460)—

The product of the perpendiculars from a focus of a conic on two parallel tangents is constant.

This is equivalent to the Theorem of § 248, VII.

Ex. 2. 'The vertices of a rectangle circumscribing a conic lie on the director circle.' If this is reciprocated with respect to a focus we obtain the theorem, Ex. 14, page 407.

§ **465.** It follows from § 463 that concentric circles reciprocate into conics having the same focus and directrix.

Ex. Two conics have the same focus and directrix; their eccentricities are are e and e′. If $e' = 2\,e$, prove that an infinite number of triangles can be inscribed in one conic and circumscribed to the other.

The reciprocal of this is—

If the radii of two concentric circles are r and r', then if $r = 2r'$ an infinite number &c. ; this is proved at once by Elementary Geometry.

§ 466. *The reciprocal of a system of co-axal circles with respect to either limiting point is a system of confocal conics.*

For the conics have one focus common, viz., the limiting point ; also, the limiting point having the same polar with respect to every circle of the system (Ex., § 189), the conics are concentric [*Cor*^y (2), § 463]. The second focus is $\therefore$ the same for all the conics.

Ex. The theorem of § 340 is the reciprocal of Ex. 31, page 161.

§ 467. If we are given the trilinear equation of a curve,

$$\phi(\alpha, \beta, \gamma) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

it is easy to obtain the tangential equation of its reciprocal with respect to any point $O(\alpha_1 \beta_1 \gamma_1)$.

Let δ be the radius of the reciprocating circle, $A'B'C'$ the triangle reciprocal to the triangle of reference ABC ; then A' is the pole of BC, and A is the pole of $B'C'$, &c. ; and

$$\alpha_1 . A'O = \beta_1 . B'O = \gamma_1 . C'O = \delta^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let p, q, r be the perpendiculars from A', B', C' on the reciprocal of a point $P(\alpha\beta\gamma)$ on the curve (1) ; this reciprocal is a tangent to the reciprocal of the curve (1). Then, by § 202,

$$A'O : PO = \perp \text{ from } A' \text{ on polar of } P : \perp \text{ from } P \text{ on polar of } A'$$
$$= p : \alpha$$
$$\therefore \quad \delta^2/\alpha_1 : PO = p : \alpha$$
$$\therefore \quad \alpha : p\alpha_1 = PO : \delta^2 = \beta : q\beta_1 = \gamma : r\gamma_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The tangential equation to the reciprocal curve is $\therefore$

$$\phi(p\alpha_1, q\beta_1, r\gamma_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, from (3), $\quad p : q : r = \alpha/\alpha_1 : \beta/\beta_1 : \gamma/\gamma_1$

If $\therefore$ the tangential equation of a curve referred to $A'B'C'$ is

$$\psi(p, q, r) = 0$$

the trilinear equation of its reciprocal referred to ABC is

$$\psi(\alpha/\alpha_1, \beta/\beta_1, \gamma/\gamma_1) = 0$$

Ex. The tangential equation of the circum-circle of $A'B'C'$ is

$$\sin A' \sqrt{p} + \sin B' \sqrt{q} + \sin C' \sqrt{r} = 0$$

[§§ 424, 423, 400.]

Now $$A' = 180^\circ - BOC, \quad \&c.;$$
and the reciprocal of the circle is a conic inscribed in ABC, and having its focus at O.

The equation of a conic inscribed in ABC and having its focus at $(\alpha_1 \beta_1 \gamma_1)$ is

$$\therefore \ \sin BOC \sqrt{\alpha/\alpha_1} + \sin COA \sqrt{\beta/\beta_1} + \sin AOB \sqrt{\gamma/\gamma_1} = 0$$

Exercises

1. Prove that the ellipses (a, b) and (a′, b′) are polar reciprocals with respect to the ellipse $(\sqrt{aa'}, \sqrt{bb'})$.

2. Find the equation of the reciprocal of the ellipse (a, b) with respect to a circle whose centre is $(x'y')$, and radius δ.

Ans. $(xx' + yy' + \delta^2)^2 = a^2x^2 + b^2y^2$

3. The reciprocal of a parabola with respect to a point on the directrix is a rectangular hyperbola.

4. Find the polar reciprocal of the conic
$$2l\alpha\beta = \gamma^2$$
with respect to the conic $$2m\alpha\beta = \gamma^2$$

Ans. $l\gamma^2 - 2m^2\alpha\beta = 0$

5. Prove by reciprocation the results of Exs. 1, 2, 4. § 330; Exs. 19, 20, 24, page 322; Ex. 26, page 323.

6. Two conics are described with given directrices and a given common focus S; the sum of the squares of the reciprocals of their latera recta is given.

Prove that their common tangents envelope a conic having one focus at S.

7. Prove that Ex. 29, page 323, reduces by reciprocation to the theorem:—If on a line which revolves round a fixed point and meets a given circle in P a point Q is taken so that OQ is to OP in a constant ratio, the locus of Q is a circle.

8. A conic touches three given lines, and its director circle passes through a given point; prove that the conic touches another fixed straight line.

[*Note*—This is the reciprocal of Ex. 2, § 315.]

9. Reciprocate the theorem, Ex. 1, § 315.

[See *Cor′* (3), § 462. The reciprocal is equivalent to Ex. 55, page 397.]

10. Reciprocate the theorems of § 183; (3), § 190; also Ex. 40, p. 162.

11. A circle is reciprocated with respect to a point O. The second focus of the conic is the reciprocal of the radical axis of the circle and the point O.

12. 'Rectangular tangents to a conic meet on a concentric circle.' The reciprocal of this is Ex. 15, page 407.

13. 'The sum of the focal distances of the extremities of a variable diameter is constant.' What is the reciprocal theorem?

14. A straight line is drawn across a rhombus so that its segments included between the arms of opposite angles subtend supplementary angles at the centre. Prove that it envelopes an inscribed conic.

[*Note*—Reciprocate with respect to centre of rhombus.]

15. Deduce the equation of Ex. 59, page 397, from that of Ex., § 467.

16. A triangle is inscribed in a given ellipse (a, b), so that the focus is the centre of its inscribed circle.

Prove that the radius of this circle is

$$b^2/(a + \sqrt{2a^2 - b^2})$$

PROJECTION

§ 468. The lines joining the points of a figure in a plane Π to a point V in space form a cone; the section of this cone by a plane π is a figure which is called the *projection* of the given figure. The point V is called the *vertex*; the plane π is called the *plane of projection.*

The projection of a point P is the point p where the join VP is cut by the plane π; we shall denote points in the plane Π by capital letters, and their projections by the corresponding small letters.

§ 469. The joins of the points in a straight line to V form a plane; this plane intersects the plane π in a straight line.

Hence *the projection of a straight line is a straight line.*

Cor' (1)—If a system of lines co-intersect in a point P, their projections co-intersect in p.

Cor' (2)—To a chord PQ of a curve corresponds a chord pq of its projection; and if P, Q are consecutive, so are p, q.

Hence *curves, tangents, and points of contact project into curves, tangents, and points of contact.*

§ **470.** *The projection of a conic is a conic.*

Let A, B, C, D be four fixed points, and P a moveable point on a conic; let the lines PA, PB, PC, PD be intersected by any line in the points A′, B′, C′, D′. Let the projections of these points be a, b, c, d, p, a′, b′, c′, d′. Then (§ 137)

$$\{P.ABCD\} = \{A'B'C'D'\} = \{V.A'B'C'D\} = \{a'b'c'd'\}$$
$$= \{p.abcd\}$$

But $\{P.ABCD\}$ is constant (§ 355);

$$\therefore \quad \{p.abcd\} \text{ is constant.}$$

The locus of p is ∴ a conic passing through a, b, c, d. Q.E.D.

Cor' (1)—The statement in § 213 that any section of a right circular cone by a plane is a conic is a particular case of the proposition just proved.

Cor' (2)—It is evident from the preceding proof that the cross ratios of ranges and pencils are unaltered by projection.

Cor' (3)—The properties of poles and polars with respect to a conic are projective: this follows evidently from the *Cor's* to § 469, and the preceding *Cor'* (2).

§ **471.** Let LM (see fig′, next §) be the line of intersection of the primitive plane Π with a plane through the vertex V parallel to the plane of projection π.

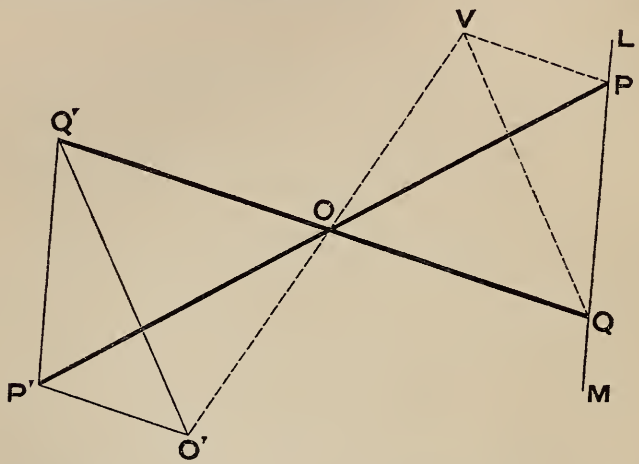
Then points on LM are projected to infinity.

Hence lines which cointersect in a point on LM are projected into lines which cointersect on the line at infinity in the plane π, i.e. into parallel lines.

Similarly, systems of parallels in the primitive plane are projected into systems of lines which co-intersect in points situated on the line of intersection of the plane of projection with a plane through V parallel to the primitive plane.

§ **472.** Let POQ be any angle in the primitive plane; LM the line which is projected to infinity, i. e. the line of intersection of the plane Π with a plane through V parallel to π.

Let OV, OP, OQ meet the plane π in O′, P′, Q′; then O′P′, O′Q′ are the projections of OP, OQ.



Now O′P′, VP are parallel; for these lines are the intersections of the parallel planes π and VLM by the plane VOP.

Similarly O′Q′, VQ are parallel.

$$\therefore \quad P'\hat{O}'Q' = P\hat{V}Q \qquad \text{[Euclid XI. 10.]}$$

i. e. $P\hat{O}Q$ is projected into an angle $= P\hat{V}Q$.

§ **473.** *Any line and any two angles being chosen in a plane; the line can be projected to infinity and the angles at the same time into given angles.*

Let LM be the line; let the arms of the angles meet LM in p, q and p′, q′ respectively.

Draw any plane through LM; and let segments of circles described in this plane on pq and p′q′, and containing angles respectively equal to the given angles, meet in V.

Take V as vertex, and any plane parallel to VLM as plane of projection; it follows from § 472 that the given figure will be projected in the manner described.

Cor'—The locus of V is a circle in a plane perpendicular to LM.

If this locus meet the primitive plane in ϕ, then *any* angle is projected into the angle which the intercept of its arms on LM subtends at ϕ. Thus by a plane construction we can determine the angle into which any given angle is projected.

§ 474. *A conic and a line in its plane being given; the line may be projected to infinity, and the conic at the same time into a circle.*

Let LM be the line, C its pole with respect to the conic.

Project (§ 473) LM to infinity, and the angles between two pairs of conjugate lines through C into right angles.

Then c, the projection of C, is the centre of the projected conic; and this conic has two pairs of conjugate diameters at right angles.

The projected conic is ∴ a circle.

Cor'—As the pole of the line which is sent to infinity becomes the centre, the theorem may be stated thus—

Given a conic and a point in its plane; the conic may be projected into a circle whose centre is the projection of that point.

§ 475. By projecting the figure, the truth of a general theorem of position can be inferred from that of a simpler particular case.

Ex. 1. Prove Pascal's theorem (§ 431).

Project the conic into a circle, and the join of the intersections of two pairs of opposite sides to infinity. The theorem is then—

If a hexagon inscribed in a circle has two pairs of opposite sides parallel, the remaining sides are parallel.

This is proved at once by Elementary Geometry.

Ex. 2. LOL', MOM', NON', ROR' *are four concurrent chords of a conic. If* P *is any point on the conic, then*

$$\{P.LMNR\} = \{P.L'M'N'R'\}$$

Project the conic into a circle whose centre is the projection of O.

Ex. 3. *Any quadrilateral can be projected into a square.*

Project the third diagonal to infinity: then the quadrilateral becomes a parallelogram. Project the angles included by two adjacent sides and by the diagonals into right angles: the projected figure is then a square.

Ex. 4. *Two conics can be projected into concentric conics.*

Project a side of their common self-conjugate triangle to infinity.

§ **476.** *Any two points can be projected into the focoids.*

Let the points be L, M; draw any conic through LM.

Project the conic into a circle, and LM to infinity (§ 474).

Cor′ (1)—*Conics passing through four fixed points can be projected into coaxal circles.*

Let L, M, N, R be the four points.

Project LM to infinity and one of the conics into a circle, i. e. project L, M into the focoids.

Then all the conics are projected into conics passing through the focoids, i. e. into circles; and these circles all pass through n, r, the projections of N, R.

Cor′ (2)—*Conics having double contact can be projected into concentric circles.*

This follows from *Cor′*, § 387. Let L, M be the points of contact; project LM to infinity and one of the conics into a circle.

Cor′ (3)—*Conics inscribed in a quadrilateral can be projected into confocal conics.*

This follows from *Cor′*, § 367; project the extremities of the third diagonal into the focoids.

Ex. 1. *Two sides of a triangle inscribed in a conic pass through fixed points* P, Q; *prove that the envelope of the third side is a conic having double contact with the given conic.*

Project the given conic into a circle, and PQ to infinity: then the theorem becomes—

If two sides of a triangle inscribed in a circle are parallel to fixed lines, the third side envelopes a concentric circle.

Ex. 2. To prove Desargues' Theorem (§ 391).

Project the conics into co-axal circles whose common chord is CD. Let any line meet CD in O and the circles in P, P′; Q, Q′; R, R′; &c. Then (Euclid III. 35, 36)

$$OP \,.\, OP' = OQ \,.\, OQ' = OR \,.\, OR' = \&c.$$

Hence O is the centre of an involution determined on the line by the circles; and the theorem is proved.

Ex. 3. *If the six sides of two triangles* ABC, A′B′C′ *touch a conic, their six vertices lie on a conic.*

Project B′, C′ into the focoids, then A′ becomes a focus; and the theorem is reduced to that of Ex. 3, § 330.

§ **477.** *The polars of four collinear points with respect to a conic form a pencil whose cross ratio is equal to that of the four points.*

Project the conic into a circle whose centre is the projection of a point on the line of collinearity; then the four polars become parallel. The four points in this case, and the four intersections of the polars with the line of collinearity, form an involution, whose focus is the centre. [§§ 143, 145.]

PROJECTION OF ANGLE PROPERTIES

§ **478.** Let i, j be the projections of the focoids I, J.

Then (§ 360) *two lines at right angles are projected into lines cutting the fixed segment* ij *harmonically.*

Also, (§ 362) two lines including an angle θ are projected into lines cutting the fixed segment ij in the cross ratio

$$\cos 2\theta + \sqrt{-1} \sin 2\theta$$

Ex. 1. 'If A, B are fixed points and P a variable point on a circle, the angle APB is constant.'

From this we infer—

If a, b, i, j are fixed points, and p a variable point on a conic, the cross ratio $\{p . aibj\}$ is constant. [Compare § 355.]

Ex. 2. 'Rectangular tangents to a parabola meet on the directrix.

The generalization of this by projection is—

The locus of the intersection of tangents to a conic cutting the segment determined by two fixed points i, j on a given tangent harmonically is a straight line; this line is the polar of the intersection of the other tangents from i, j to the conic.

PROJECTION OF METRICAL PROPERTIES

§ **479.** Lemma. *If* I, J *are fixed points in a plane, and* A, B, C, D *any other points in the plane, then if*

$$\phi(A, B) \equiv \sqrt{\text{area AIB . area AJB}}/(\text{area IAJ . area IBJ}),$$

$$\phi(A, B) \div \phi(C, D) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is unaltered by projection.

Let a, b, &c. be the projections of A, B, &c.; let P, p be the perpendiculars from the vertex V on the planes Π and π.

Then, by known theorems in Solid Geometry, the volumes of the pyramids subtended by AIB and aib at V are in the ratios

$$\mathrm{P}\,.\,\text{area AIB} : \mathrm{p}\,.\,\text{area aib}, \quad \text{and} \quad \mathrm{VA\,.\,VI\,.\,VB : Va\,.\,Vi\,.\,Vb}$$

$$\therefore \quad \text{area AIB} = \frac{\mathrm{p}}{\mathrm{P}}\,\frac{\mathrm{VA\,.\,VI\,.\,VB}}{\mathrm{Va\,.\,Vi\,.\,Vb}}\,(\text{area aib})$$

Similar values are obtained for the other areas AJB, &c.; and substituting these in (1), it reduces to

$$\phi\,(\mathrm{a},\ \mathrm{b}) \div \phi\,(\mathrm{c},\ \mathrm{d})$$

§ 480. Let us ascertain the value of ϕ (A, B) when I, J are the focoids.

Let the co-ord's of A, B referred to any axes in their plane be $(x_1\, y_1)$, $(x_2\, y_2)$; then I, J are $(\lambda,\ \pm\,\lambda\sqrt{-1})$, where $\lambda = \infty$.

Then

$$2\ \text{area AIB} = \begin{vmatrix} \lambda & \lambda\sqrt{-1} & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \lambda\,(y_1 - y_2) - \lambda\sqrt{-1}\,(x_1 - x_2)$$

$$2\ \text{area AJB} = \lambda\,(y_1 - y_2) + \lambda\sqrt{-1}\,(x_1 - x_2)$$

Similarly $\quad 2\ \text{area IAJ} = -\,2\,\lambda^2\sqrt{-1} = 2\ \text{area IBJ}$

$$\therefore \quad \phi\,(\mathrm{A},\ \mathrm{B}) = -\,\mathrm{AB}/(2\,\lambda^3)$$

We may $\therefore$ in any equation connecting the distances of points A, B, &c. replace each distance AB by ϕ (a, b); this gives the corresponding relation for the projected figure.

Ex. The following is the generalization of Ptolemy's Theorem.

If a, b, c, d, i, j are six points on a conic, and

$$\overline{12} \equiv \sqrt{\text{area aib}\,.\,\text{area ajb}}, \quad \&\text{c.};$$

then $\quad \overline{12}\cdot\overline{34} + \overline{23}\cdot\overline{14} + \overline{31}\cdot\overline{24} = 0$

§ 481. It may be similarly proved that from any homogeneous relation connecting the areas of triangles ABC, &c. the corresponding relation for the projected figure is obtained by replacing each area ABC by the function

$$\text{area abc}/(\text{area aij}\,.\,\text{area bij}\,.\,\text{area cij})$$

Exercises

1. A conic is inscribed in a triangle ABC; L, M are the points of contact of BC, AC. If any tangent meets BC, AC in P, Q, then

$$\{\mathrm{CBLP}\} = \{\mathrm{MACQ}\}$$

2. AOA′, BOB′, COC′, DOD′ are four concurrent chords of a conic. Conics passing through the two sets of five points A, B, C, D, O and A′, B′, C′, D′, O, touch at O.

3. Prove by projection that the diagonal triangle of a quadrilateral circumscribing a conic is self-conjugate with respect to the conic.

4. Prove by projection the theorems of Ex. 2, § 349; Exs. 2, 3, § 352; Ex. 9, page 369; Ex. 2, § 441.

5. Generalize by projection *Cor′*, § 322 and § 430.

6. The tangents drawn from any point to a series of conics inscribed in a quadrilateral form a pencil in involution.

7. If two pairs of conjugate rays determining a pencil in involution are at right angles, every pair is at right angles.

8. Deduce Gaskin's Theorem (Ex. 55, page 397) from Ex. 6, 7.

9. Show that any three angles may be projected into right angles.

10. Show that any quadrilateral may be projected into a square of given magnitude.

11. A triangle is circumscribed to a given conic, and two of its vertices move on fixed straight lines. Prove that the third vertex describes a conic having double contact with the given one.

12. The locus of points from which two co-planar conics can be projected into circles consists of six circles in a perpendicular plane.

[*Note*—This follows at once from *Cor′*, § 473.]

THE END

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